Combinatorics, Probability and Computing (2001) 10, 41–77. Printed in the United Kingdom
© 2001 Cambridge University Press

Bounds on the Complex Zeros of (Di)Chromatic Polynomials and Potts-Model Partition Functions

ALAN D. SOKAL†

Department of Physics, New York University,
4 Washington Place, New York, NY 10003, USA
(e-mail: sokal@nyu.edu)

Received 12 April 1999; revised 18 June 1999 and 25 October 2000

We show that there exist universal constants $C(r) < \infty$ such that, for all loopless graphs G of maximum degree $\leqslant r$, the zeros (real or complex) of the chromatic polynomial $P_G(q)$ lie in the disc $|q| < C(r)$. Furthermore, $C(r) \leqslant 7.963907r$. This result is a corollary of a more general result on the zeros of the Potts-model partition function $Z_G(q, \{v_e\})$ in the complex antiferromagnetic regime $|1 + v_e| \leqslant 1$. The proof is based on a transformation of the Whitney–Tutte–Fortuin–Kasteleyn representation of $Z_G(q, \{v_e\})$ to a polymer gas, followed by verification of the Dobrushin–Koteckỳ–Preiss condition for nonvanishing of a polymer-model partition function. We also show that, for all loopless graphs G of second-largest degree $\leqslant r$, the zeros of $P_G(q)$ lie in the disc $|q| < C(r) + 1$. Along the way, I give a simple proof of a generalized (multivariate) Brown–Colbourn conjecture on the zeros of the reliability polynomial for the special case of series-parallel graphs.

1. Introduction

The polynomials studied in this paper arise independently in graph theory and in statistical mechanics. It is appropriate, therefore, to begin by explaining each of these contexts. Specialists in these fields are warned that they will find at least one (and perhaps both) of these summaries excruciatingly boring; they can skip them.

Let $G = (V, E)$ be a finite undirected graph[1] with vertex set V and edge set E. For each positive integer q, let $P_G(q)$ be the number of ways that the vertices of G can be assigned 'colours' from the set $\{1, 2, \ldots, q\}$ in such a way that adjacent vertices always receive different colours. It is not hard to show (see below) that $P_G(q)$ is the restriction to $\mathbb{Z}_+$ of a polynomial in q. This (obviously unique) polynomial is called the *chromatic*

† Research supported in part by US National Science Foundation grants PHY–9520978 and PHY–9900769 and by UK Engineering and Physical Sciences Research Council grant GR/M 71626.

[1] In this paper a 'graph' is allowed to have loops and/or multiple edges unless explicitly stated otherwise.

polynomial of G, and can be taken as the *definition* of $P_G(q)$ for arbitrary real or complex values of q.[2]

The chromatic polynomial was introduced in 1912 by Birkhoff [13]. The original hope was that study of the real or complex zeros of $P_G(q)$ might lead to an analytic proof of the Four-Colour Conjecture [75, 88], which states that $P_G(4) > 0$ for all loopless planar graphs G. To date this hope has not been realized, although combinatorial proofs of the Four-Colour Theorem have been found [2, 3, 4, 84, 112]. Even so, the zeros of $P_G(q)$ are interesting in their own right and have been extensively studied. Most of the available theorems concern real zeros [14, 117, 128, 129, 56, 130, 113, 38], but there has been some study (mostly numerical) of complex zeros as well [52, 10, 12, 8, 9, 41, 6, 7, 82, 121, 17, 18, 19, 93, 94, 95, 96, 85, 97, 86, 114, 98, 99, 100, 90].

A more general polynomial can be obtained as follows. Assign to each edge $e \in E$ a real or complex weight v_e. Then define

$$Z_G(q, \{v_e\}) \;=\; \sum_{\{\sigma_x\}} \prod_{e \in E} \left[1 + v_e \delta(\sigma_{x_1(e)}, \sigma_{x_2(e)}) \right], \tag{1.1}$$

where the sum runs over all maps $\sigma\colon V \to \{1, 2, \ldots, q\}$, the δ is the Kronecker delta, and $x_1(e), x_2(e) \in V$ are the two endpoints of the edge e (in arbitrary order). It is not hard to show (see below) that $Z_G(q, \{v_e\})$ is the restriction to $q \in \mathbb{Z}_+$ of a polynomial in q and $\{v_e\}$. If we take $v_e = -1$ for all e, this reduces to the chromatic polynomial. If we take $v_e = v$ for all e, this defines a two-variable polynomial $Z_G(q, v)$ that was introduced implicitly by Whitney [124, 125, 126] and explicitly by Tutte [115, 116]; it is known variously (modulo trivial changes of variable) as the *dichromatic polynomial*, the *dichromate*, the *Whitney rank function* or the *Tutte polynomial* [123, 11].[3]

In statistical mechanics, (1.1) is known as the partition function of the *q-state Potts model*.[4] In the Potts model [131, 132], an 'atom' (or 'spin') at site $x \in V$ can exist in any one of q different states (where q is an integer $\geqslant 1$). The *energy* of a configuration is the sum, over all edges $e \in E$, of 0 if the spins at the two endpoints of that edge are unequal and $-J_e$ if they are equal. The *Boltzmann weight* of a configuration is then $e^{-\beta H}$, where H is the energy of the configuration and $\beta \geqslant 0$ is the inverse temperature. The *partition function* is the sum, over all configurations, of their Boltzmann weights. Clearly this is

[2] Two excellent reviews on chromatic polynomials are [81, 83]. An extensive bibliography on chromatic polynomials is [30].

[3] The Tutte polynomial $T_G(x, y)$ is conventionally defined as [123, p. 45] [11, pp. 73, 101]

$$T_G(x, y) \;=\; \sum_{E' \subseteq E} (x-1)^{k(E')-k(E)} \, (y-1)^{|E'|+k(E')-|V|},$$

where $k(E')$ is the number of connected components in the subgraph (V, E'). Comparison with (1.2) yields

$$T_G(x, y) \;=\; (x-1)^{-k(E)} \, (y-1)^{-|V|} \, Z_G((x-1)(y-1),\, y-1).$$

[4] The Potts model [80] was invented in the early 1950s by Domb (see [36]). The $q = 2$ case, known as the Ising model [54], was invented in 1920 by Lenz [70] (see [21, 63, 109]). The $q = 4$ case, which is a special case of the Ashkin–Teller model, was invented in 1943 by Ashkin and Teller [5].

just a rephrasing of (1.1), with $v_e = e^{\beta J_e} - 1$. A coupling J_e (or v_e) is called *ferromagnetic* if $J_e \geqslant 0$ ($v_e \geqslant 0$) and *antiferromagnetic* if $-\infty \leqslant J_e \leqslant 0$ ($-1 \leqslant v_e \leqslant 0$).

To see that $Z_G(q, \{v_e\})$ is indeed a polynomial in its arguments (with coefficients that are in fact 0 or 1), we proceed as follows. In (1.1), expand out the product over $e \in E$, and let $E' \subseteq E$ be the set of edges for which the term $v_e \delta_{\sigma_{x_1(e)}, \sigma_{x_2(e)}}$ is taken. Now perform the sum over configurations $\{\sigma_x\}$: in each connected component of the subgraph (V, E') the spin value σ_x must be constant, and there are no other constraints. Therefore,

$$Z_G(q, \{v_e\}) \;=\; \sum_{E' \subseteq E} q^{k(E')} \prod_{e \in E'} v_e, \tag{1.2}$$

where $k(E')$ is the number of connected components (including isolated vertices) in the subgraph (V, E'). The expansion (1.2) was discovered by Birkhoff [13] and Whitney [124] for the special case $v_e = -1$ (see also Tutte [115, 116]); in its general form it is due to Fortuin and Kasteleyn [58, 44] (see also [39]). We take (1.2) as the *definition* of $Z_G(q, \{v_e\})$ for arbitrary complex q and $\{v_e\}$.

In statistical mechanics, a very important role is played by the complex zeros of the partition function. This arises as follows [133]. Statistical physicists are interested in *phase transitions*, in other words in points where one or more physical quantities (*e.g.*, the energy or the magnetization) depend nonanalytically (in many cases even discontinuously) on one or more control parameters (*e.g.*, the temperature or the magnetic field). Now, such nonanalyticity is manifestly impossible in (1.1)/(1.2) for any finite graph G. Rather, phase transitions arise only in the *infinite-volume limit*. That is, we consider some countably infinite graph $G_\infty = (V_\infty, E_\infty)$ – usually a regular lattice, such as $\mathbb{Z}^d$ with nearest-neighbour edges – and an increasing sequence of finite subgraphs $G_n = (V_n, E_n)$. It can then be shown (under modest hypotheses on the G_n) that the (*limiting*) *free energy per unit volume*

$$f_{G_\infty}(q, v) \;=\; \lim_{n \to \infty} |V_n|^{-1} \log Z_{G_n}(q, v) \tag{1.3}$$

exists for all *nondegenerate physical* values of the parameters,[5] namely, either

(a) q integer $\geqslant 1$ and $-1 < v < \infty$ (using (1.1) – see, *e.g.*, [55, Section I.2]), or
(b) q real > 0 and $0 \leqslant v < \infty$ (using (1.2) – see [51, Theorem 4.1] and [50, 92]).

This limit $f_{G_\infty}(q, v)$ is in general a continuous function of v; but it can fail to be a real-analytic function of v, because complex singularities of $\log Z_{G_n}(q, v)$ – namely, complex zeros of $Z_{G_n}(q, v)$ – can approach the real axis in the limit $n \to \infty$. Therefore, the possible points of physical phase transitions are precisely the real limit points of such complex zeros. As a result, theorems that constrain the possible location of complex zeros of the partition function are of great interest. In particular, theorems guaranteeing that a certain complex domain is free of zeros are often known as *Lee–Yang theorems*.[6]

[5] Here ‘physical’ means that the weights are nonnegative, so that the model has a probabilistic interpretation; and ‘nondegenerate’ means that we exclude the limiting cases $v = -1$ in (a) and $q = 0$ in (b), which cause difficulties due to the existence of configurations having zero weight.

[6] The first such theorem, concerning the behaviour of the ferromagnetic Ising model for a complex magnetic field, was proved by Lee and Yang [69] in 1952. A partial bibliography (to 1980) of generalizations of this result can be found in [71].

The purpose of this paper is to prove an upper bound on the complex q-plane zeros of the Potts-model partition function $Z_G(q, \{v_e\})$, valid throughout the 'complex antiferromagnetic regime' $|1 + v_e| \leqslant 1$, under certain 'local' conditions on the weights $\{v_e\}$: for example, in terms of the quantity $\max_{x \in V} \sum_{e \ni x} |v_e|$. As a corollary, I obtain upper bounds on the zeros of the chromatic polynomial $P_G(q)$ in terms of the maximum degree of the graph G. More precisely, I show that there exist universal constants $C(r) < \infty$ such that, for all loopless graphs G of maximum degree $\leqslant r$, the zeros of $P_G(q)$ lie in the disc $|q| < C(r)$. This answers in the affirmative a question posed by Brenti, Royle and Wagner [17, Question 6.1], generalizing an earlier conjecture of Biggs, Damerell and Sands [12] limited to r-regular graphs. The constants $C(r)$ arise as the solution of an explicit minimization problem, and I prove that $C(r) \leqslant 7.963907r$. This linear dependence on r is best possible, as the example of the complete graph K_{r+1} shows that $C(r) \geqslant r$.

Furthermore, I show that the presence of *one* vertex of large degree cannot lead to large chromatic roots. More precisely, if all but one of the vertices of G have degree $\leqslant r$, then the zeros of $P_G(q)$ lie in the disc $|q| < C(r) + 1$. Note that a result of this kind *cannot* hold if 'all but one' is replaced by 'all but two', for in this case the chromatic roots can be unbounded, even when $r = 2$ and G is planar [103].

The proofs of these results are based on well-known methods of mathematical statistical mechanics. The first step is to transform the Whitney–Tutte–Fortuin–Kasteleyn representation (1.2) into a gas of 'polymers' interacting via a hard-core exclusion (Section 2). I then invoke the Dobrushin condition [34, 35] (or the closely related Kotecký–Preiss condition [65, 104]) for the nonvanishing of a polymer-model partition function (Section 3). Lastly, I verify these conditions for our particular polymer model, using a series of simple combinatorial lemmas, some of which may be of independent interest (Section 4); in particular, I give a simple proof of a generalized (multivariate) Brown–Colbourn conjecture on the zeros of the reliability polynomial for the special case of series-parallel graphs (Remark 3 in Section 4.1). The main results of this paper are contained in Section 5; some generalizations and extensions are in Section 6. I conclude with some conjectures and open questions (Section 7).

With a little more work, it should be possible to extend the arguments of this paper to prove the existence and analyticity of the limiting free energy per unit volume (1.3) for suitable regular lattices G_∞ and translation-invariant edge weights v_e, in the same region of complex q- and $\{v_e\}$-space where Z will be proved (in Section 5) to be nonvanishing uniformly in the finite subgraphs V_n ('uniformly in the volume' in statistical–mechanical language). In particular, this would provide a convergent expansion for the limiting free energy in powers of $1/q$. However, I have not worked out the details.

This paper would never have seen the light of day without the help and advice of Antti Kupiainen. During my visit to Helsinki in September–October 1997, I told Antti of my conjectures about $P_G(q)$ and $Z_G(q, \{v_e\})$ – conjectures that I had no good idea how to prove. He immediately saw that they ought to be provable by cluster (or Mayer) expansion. My reaction was, 'Ugh! You know how I *detest* the cluster expansion!'; indeed, I had resisted learning it for nearly 20 years and had devoted much of my work in mathematical physics to finding ways of circumventing it [102, 23, 24, 42]. Antti assured me that the cluster expansion is not so difficult, and he suggested that I study the excellent

review article of Brydges [22]. We also quickly figured out how to represent $Z_G(q, \{v_e\})$ as a polymer gas. Jean Bricmont then told me about the work of Kotecký and Preiss [65], and Roman Kotecký informed me of the work of Dobrushin [34]. Here, finally, was a version of the cluster expansion simple enough that even I could understand it! Nine months later, I figured out how to verify the Dobrushin (or Kotecký–Preiss) condition and thereby complete the proof.

2. Transformation of the Potts-model partition function to a polymer gas

Let $G = (V, E)$ be a finite undirected graph equipped with complex edge weights $\{v_e\}_{e\in E}$. If G contains a loop e (*i.e.*, an edge connecting a vertex to itself), this simply multiplies $Z_G(q, \{v_e\})$ by a factor $1 + v_e$; so we can assume without loss of generality that G is loopless, and we shall do so in this section in order to avoid unnecessary complications. Likewise, if G contains multiple edges $e_1, \ldots, e_n$ connecting the same pair of vertices, they can be replaced, without changing the value of Z, by a single edge e with weight $v_e = \prod_{i=1}^{n}(1 + v_{e_i}) - 1$. So we could assume without loss of generality, if we wanted, that G has no multiple edges. But this assumption would not simplify most of our subsequent arguments, so we shall usually refrain from making it. Note, however, that our numerical bounds frequently get better if multiple edges are replaced by a single equivalent edge.

So let G be loopless, and consider the Whitney–Tutte–Fortuin–Kasteleyn representation (1.2) of the Potts-model partition function $Z_G(q, \{v_e\})$. For each term in (1.2) we decompose the subgraph (V, E') into its connected components. Some of these components may consist of a single vertex and no edges; the remaining components are disjoint connected subgraphs $(S_1, E_1), \ldots, (S_N, E_N)$ with $|S_i| \geqslant 2$. The total number of components is

$$k(E') \;=\; N + \left(|V| - \sum_{i=1}^{N} |S_i|\right) \tag{2.1a}$$

$$\;=\; |V| - \sum_{i=1}^{N}(|S_i| - 1). \tag{2.1b}$$

Hence we obtain the following result.

Proposition 2.1 (jointly with Antti Kupiainen). *Let $G = (V, E)$ be a loopless finite undirected graph equipped with edge weights $\{v_e\}_{e\in E}$. Then*

$$Z_G(q, \{v_e\}) \;=\; q^{|V|} Z_{\mathrm{polymer},G}(q, \{v_e\}), \tag{2.2}$$

where

$$Z_{\mathrm{polymer},G}(q, \{v_e\}) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{S_1,\ldots,S_N \text{ disjoint}} \prod_{i=1}^{N} w(S_i) \tag{2.3}$$

and

$$w(S) \;=\; \begin{cases} q^{-(|S|-1)} \displaystyle\sum_{\substack{\widetilde{E} \subseteq E \\ (S,\widetilde{E}) \text{ connected}}} \prod_{e\in\widetilde{E}} v_e, & \text{if } |S| \geqslant 2, \\ 0, & \text{if } |S| \leqslant 1. \end{cases} \tag{2.4}$$

The sum in (2.3) *runs over pairwise disjoint subsets* $S_1, \ldots, S_N$ *of* V, *and the term* $N = 0$ *in* (2.3) *is understood to contribute* 1. □

Note, in particular, that $w(S) = 0$ if S is disconnected (*i.e.*, if the induced subgraph (S, E_S) is disconnected).

The 'polymer model' (2.3)–(2.4) has the form of a grand-canonical gas (see Section 3 for the precise definition)

$$Z_{\text{polymer},G}(q, \{v_e\}) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{S_1,\ldots,S_N} \prod_{i=1}^{N} w(S_i) \prod_{1\leqslant i<j\leqslant N} W(S_i, S_j) \tag{2.5}$$

with single-particle state space $\mathscr{P}_*(V)$ (the set of all nonempty subsets of V), fugacities $w(S)$, and two-particle Boltzmann factor given by a hard-core exclusion

$$W(S, S') \;=\; \begin{cases} 1, & \text{if } S \cap S' = \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Graph theorists will recognize the right-hand side of (2.5) as the generating function, in the variables $w(S)$, for independent subsets of vertices of the intersection graph of $\mathscr{P}_*(V)$.

The usefulness of (2.2)–(2.6) comes from the fact that the fugacities $w(S)$ are all suppressed by powers of q^{-1}, hence are small for large $|q|$. Moreover, if the sum over $\widetilde{E}$ in (2.4) can be controlled, one expects that $w(S)$ will be exponentially decaying in $|S|$ when $|q|$ is large enough. This raises the hope that the Mayer expansion [119], which is an expansion of $\log Z_{\text{polymer},G}$ in powers of the fugacities $w(S)$, might converge for sufficiently large $|q|$. If so, this would imply that $Z_{\text{polymer},G} \neq 0$ in the region of convergence. That is what we go about proving in the following sections – but in the opposite order.

3. Dobrushin and Koteck´y–Preiss conditions for the nonvanishing of Z

In statistical mechanics, a *grand-canonical gas* is defined by a *single-particle state space* X (here assumed for simplicity to be finite), a *fugacity vector* $w = \{w_x\}_{x\in X} \in \mathbb{C}^X$, and a *two-particle Boltzmann factor* $W(x, y)$ (a symmetric function $W\colon X \times X \to \mathbb{C}$). The (grand) partition function $Z(w, W)$ is then defined to be the sum over ways of placing $N \geqslant 0$ 'particles' on 'sites' $x_1, \ldots, x_N \in X$, with each configuration assigned a 'Boltzmann weight' given by the product of the corresponding factors w_{x_i} and $W(x_i, x_j)$:

$$Z(w, W) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{x_1,\ldots,x_N\in X} \prod_{i=1}^{N} w_{x_i} \prod_{1\leqslant i<j\leqslant N} W(x_i, x_j), \tag{3.1}$$

where the $N = 0$ term is understood to contribute 1. Under very mild conditions on W (*e.g.*, $|W(x, y)| \leqslant 1$ for all x, y is more than sufficient), $Z(w, W)$ is an entire analytic function of w. Our goal is to find a sufficient condition for $Z(w, W)$ to be nonvanishing in a polydisc $D_R = \{w\colon |w_x| < R_x\}$. This would imply, in particular, that $\log Z(w, W)$ is an analytic function of w in D_R.

We say that W is

- *physical* if $0 \leqslant W(x, y) < +\infty$ for all $x, y \in X$,
- *repulsive* if $|W(x, y)| \leqslant 1$ for all $x, y \in X$,
- *physical and repulsive* if $0 \leqslant W(x, y) \leqslant 1$ for all $x, y \in X$,
- *hard-core* if $W(x, y) = 0$ or 1 for all $x, y \in X$,
- *hard-core self-repulsive* if $W(x, x) = 0$ for all $x \in X$.

An important special case is when W is hard-core and hard-core self-repulsive: then $Z(w, W)$ is the generating function for independent sets of vertices of the graph $\widetilde{G} = (X, E)$ defined by placing an edge between each pair of vertices $x \neq y$ for which $W(x, y) = 0$.

Dobrushin [34, 35] has given an elegant sufficient condition for the nonvanishing of Z in a polydisc D_R, whenever W is hard-core and hard-core self-repulsive. His proof is astoundingly simple, avoiding all the combinatorial complication that has given cluster expansions such a reputation for difficulty. Here I shall present a slight extension of Dobrushin's theorem, in which the condition of hard-core interaction is replaced by the weaker assumption that the interaction is physical and repulsive; moreover, the conclusion of the theorem is slightly strengthened. (We won't really need this extension – the original Dobrushin theorem would suffice for our purposes – but the stronger result is no more difficult, and it gives a bit more insight into the method of proof.) The hard-core *self*-repulsion is, however, essential both in Dobrushin's version and in my own: it guarantees that each 'site' $x \in X$ can be occupied by at most one 'particle' x_i. It follows that the partition function can be rewritten as a sum over subsets:

$$Z(w, W) \;=\; \sum_{X' \subseteq X} \prod_{x \in X'} w_x \prod_{\langle xy \rangle \in X'} W(x, y), \tag{3.2}$$

where the second product runs over unordered pairs $x, y \in X'$ $(x \neq y)$ with each pair counted once.

Let us define, for each subset $\Lambda \subseteq X$, the restricted partition function

$$Z_\Lambda(w, W) \;=\; \sum_{X' \subseteq \Lambda} \prod_{x \in X'} w_x \prod_{\langle xy \rangle \in X'} W(x, y). \tag{3.3}$$

Of course this notation is redundant, since the same effect can be obtained by setting $w_x = 0$ for $x \in X \setminus \Lambda$, but it is useful for the purposes of the inductive proof. We have the following result.

Theorem 3.1. *Let X be a finite set, and let W satisfy*

(a) $0 \leqslant W(x, y) \leqslant 1$ *for all* $x, y \in X$,
(b) $W(x, x) = 0$ *for all* $x \in X$.

Suppose there exist constants $R_x \geqslant 0$ and $0 \leqslant K_x < 1/R_x$ satisfying

$$K_x \;\geqslant\; \prod_{y \neq x} \frac{1 - W(x, y) K_y R_y}{1 - K_y R_y} \tag{3.4}$$

for all $x \in X$. *Then, for each subset* $\Lambda \subseteq X$, $Z_\Lambda(w, W)$ *is nonvanishing in the closed polydisc* $\bar{D}_R = \{w \in \mathbb{C}^X : |w_x| \leqslant R_x\}$ *and satisfies there*

$$\left|\frac{\partial \log Z_\Lambda(w, W)}{\partial w_x}\right| \leqslant \begin{cases} \dfrac{K_x}{1 - K_x|w_x|}, & \text{for all } x \in \Lambda, \\ 0, & \text{for all } x \in X \setminus \Lambda. \end{cases} \tag{3.5}$$

Moreover, if $w, w' \in \bar{D}_R$ *and* $w'_x/w_x \in [0, +\infty]$ *for each* $x \in \Lambda$, *then*

$$\left|\log \frac{Z_\Lambda(w', W)}{Z_\Lambda(w, W)}\right| \leqslant \sum_{x \in \Lambda} \left|\log \frac{1 - K_x|w'_x|}{1 - K_x|w_x|}\right| \tag{3.6}$$

where on the left-hand side we take the standard branch of the log, i.e., $|\operatorname{Im} \log \cdots| \leqslant \pi$.

Remarks. 1. It follows from (3.4) that $K_x \geqslant 1$ and hence that $R_x < 1$.

2. The conclusion of Dobrushin's theorem [34, 35] is the special case of (3.6) in which some of the w'_x are equal to w_x and others are equal to 0, and in which only the *real part* of the logarithm on the left-hand side is handled.

Proof. Note first that (3.5) for any given Λ implies (3.6) for the same Λ, by integration.

The proof is by induction on the cardinality of Λ. If $\Lambda = \emptyset$ the claims are trivial. So let us assume that (3.5) (and hence also (3.6)) holds for all sets of cardinality $< n$, and let a set Λ of cardinality n be given. Let x be any element of Λ, and let $\Lambda' = \Lambda \setminus \{x\}$. It follows from (3.3) that

$$Z_\Lambda(w, W) = Z_{\Lambda'}(w, W) + w_x Z_{\Lambda'}(\widetilde{w}, W) \tag{3.7}$$

where

$$\widetilde{w}_y = W(x, y)\, w_y; \tag{3.8}$$

here the first term on the right-hand side of (3.7) covers the summands $X' \not\ni x$, while the second covers $X' \ni x$. Note that $\widetilde{w} \in \bar{D}_R$ since $|W(x, y)| \leqslant 1$. From (3.7) we have

$$\frac{\partial}{\partial w_x} \log Z_\Lambda(w, W) = \frac{k(w)}{1 + k(w) w_x} \tag{3.9}$$

where

$$k(w) = \frac{Z_{\Lambda'}(\widetilde{w}, W)}{Z_{\Lambda'}(w, W)}. \tag{3.10}$$

Now by the inductive hypothesis (3.6) for Λ', and using the fact that $\widetilde{w}_y/w_y = W(x, y) \geqslant 0$, we have

$$|k(w)| \leqslant \prod_{y \in \Lambda'} \frac{1 - W(x, y) K_y |w_y|}{1 - K_y |w_y|} \leqslant \prod_{y \in X \setminus \{x\}} \frac{1 - W(x, y) K_y |w_y|}{1 - K_y |w_y|}, \tag{3.11}$$

which is $\leqslant K_x$ by the hypothesis (3.4). This proves (3.5) for Λ, and hence completes the induction. □

Let us now return to the special case of a hard-core interaction. If $W(x, y) = 0$ (resp. 1), we say that x and y are *incompatible* (resp. *compatible*) and write $x \not\sim y$ (resp. $x \sim y$). Note that in our convention $x \not\sim x$, in agreement with some authors' conventions [65, 91, 101] and contrary to others' [34, 35]. The hypothesis (3.4) is then equivalent to the existence of constants $c_x \geqslant 0$ such that

$$R_x \;\leqslant\; (e^{c_x} - 1)\, \exp\left(-\sum_{y \not\sim x} c_y \right) \tag{3.12}$$

for all $x \in X$ (set $c_x = -\log(1 - K_x R_x)$). This is the Dobrushin [34, 35] condition. Slightly stronger, and more convenient to check, is the Kotecký–Preiss [65, 104] condition

$$R_x \;\leqslant\; c_x \, \exp\left(-\sum_{y \not\sim x} c_y \right) . \tag{3.13}$$

Let us now consider the important special case in which the single-particle state space X can be partitioned as $X = \bigcup_{n=1}^{\infty} X_n$ in such a way that

$$\sum_{y \in X_n \colon y \not\sim x} R_y \;\leqslant\; A_n m, \quad \text{for all } x \in X_m, \tag{3.14}$$

for suitable constants $\{A_n\}_{n=1}^{\infty}$. (This typically arises when X is some set of nonempty subsets of a finite set V, and $x \not\sim y$ means $x \cap y \neq \emptyset$; we will then take X_n to be the sets of cardinality n, and will prove (3.14) by proving

$$\sum_{y \in X_n \colon y \ni i} R_y \;\leqslant\; A_n, \quad \text{for all } i \in V, \tag{3.15}$$

which is manifestly stronger than (3.14).) Let us take

$$c_x \;=\; e^{\alpha n} R_x, \quad \text{for all } x \in X_n, \tag{3.16}$$

with some suitably chosen $\alpha > 0$. Then, for (3.14) to imply the Kotecký–Preiss condition (3.13), it suffices that

$$\sum_{n=1}^{\infty} e^{\alpha n} A_n \;\leqslant\; \alpha. \tag{3.17}$$

We have therefore proved the following.

Proposition 3.2. *Suppose that* $X = \bigcup_{n=1}^{\infty} X_n$ (*disjoint union*) *and that there exist constants* $\{A_n\}_{n=1}^{\infty}$ *and* α *such that*

(a) $\sum_{y \in X_n \colon y \not\sim x} R_y \;\leqslant\; A_n m$ *for all* m, n *and all* $x \in X_m$,

(b) $\sum_{n=1}^{\infty} e^{\alpha n} A_n \;\leqslant\; \alpha$.

Then the Kotecký–Preiss condition (3.13) *holds with the choice* $c_x = e^{\alpha n} R_x$ *for* $x \in X_n$. □

Remarks. **1.** Suppose we try the more general ansatz $c_x = b_n R_x$ for $x \in X_n$. Then (3.14) implies the Kotecký–Preiss condition (3.13) in case $b_n \geqslant e^{\alpha n}$ where $\alpha \equiv \sum_{n=1}^{\infty} b_n A_n$. But in that case $b'_n \equiv e^{\alpha n} \geqslant e^{\alpha' n}$ where $\alpha' \equiv \sum_{n=1}^{\infty} b'_n A_n$. So there is no loss of generality in restricting attention to $b_n = e^{\alpha n}$ for some α.

2. Since the state space X is finite, only finitely many of the A_n are nonzero. Nevertheless, we often have occasion to consider simultaneously an infinite *family* of problems – for example, in this paper, all loopless graphs G of maximum degree $\leqslant r$ and arbitrarily many vertices – and it is natural to seek bounds that are *uniform* over the family. So it is useful to forget that only finitely many of the A_n are nonzero. (Moreover, similar methods can be applied to problems with an infinite state space X, in which case $\{A_n\}$ is a genuinely infinite sequence.) This leads to two further remarks.

3. For the condition

$$\exists \alpha > 0 \text{ such that } \sum_{n=1}^{\infty} e^{\alpha n} A_n \;\leqslant\; \alpha \tag{3.18}$$

to hold, it is necessary that the sequence $\{A_n\}_{n=1}^{\infty}$ have *some* exponential decay (*i.e.*, $A_n \leqslant Ce^{-\epsilon n}$ for some $\epsilon > 0$), but there is no minimum required rate of decay. Indeed, if $\{A_n\}_{n=1}^{\infty}$ has any exponential decay at all, then by modifying finitely many of the A_n one can make (3.18) hold. It can thus be valuable in applications to work hard on estimating the first few coefficients A_n (see [59] for an example).[7]

4. Let $\delta = \liminf_{n\to\infty}(-\log A_n)/n$. Then $F(\alpha) = \alpha^{-1}\sum_{n=1}^{\infty} e^{\alpha n} A_n$ is finite-valued and continuous (in fact, real-analytic) on $0 < \alpha < \delta$, left-continuous (as a map into the extended real line) as $\alpha \uparrow \delta$, and identically $+\infty$ for $\alpha > \delta$. In particular, the infimum of $F(\alpha)$ is attained, so (3.18) is equivalent to

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=1}^{\infty} e^{\alpha n} A_n \;\leqslant\; 1. \tag{3.19}$$

Important final remark. The results in this section provide an extraordinarily simple proof of the convergence of the Mayer expansion for a grand-canonical gas with physical and repulsive two-particle interactions. To see what is at issue, let us first trivially rewrite the partition function (3.1) as

$$Z(w, W) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{x_1,\ldots,x_N \in X} \prod_{i=1}^{N} w_{x_i} \sum_{G \in \mathscr{G}_N} \prod_{\langle ij\rangle \in G} F(x_i, x_j), \tag{3.20}$$

where $\mathscr{G}_N$ is the set of all (simple loopless undirected) graphs on the vertex set $\{1, \ldots, N\}$, and

$$F(x, y) \;=\; W(x, y) - 1 \tag{3.21}$$

is called the *two-particle Mayer factor*. Then standard combinatorial arguments [119] show that

$$\log Z(w, W) \;=\; \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{x_1,\ldots,x_N \in X} \prod_{i=1}^{N} w_{x_i} \sum_{G \in \mathscr{C}_N} \prod_{\langle ij\rangle \in G} F(x_i, x_j) \tag{3.22}$$

at least in the sense of formal power series in w, where $\mathscr{C}_N \subseteq \mathscr{G}_N$ is the set of *connected* graphs on $\{1, \ldots, N\}$. This is the Mayer expansion; the principal problem is to prove its

[7] The emphasis in [28, 91, 22, 65, 101] on the special case $A_n = Ce^{-\epsilon n}$ with $C = 1$ is thus somewhat misleading, inasmuch as it suggests that there is a minimum allowed rate of decay ϵ.

convergence in some specified polydisc. The usual approach to proving convergence of the Mayer expansion [79, 91, 28, 22, 25, 27, 101, 26, 104] is to explicitly bound the terms in (3.22); this requires some rather nontrivial combinatorics (for example, Proposition 4.1 below together with the counting of trees). Once this is done, an immediate consequence is that Z is nonvanishing in any polydisc where the series for $\log Z$ is convergent. Dobrushin's brilliant idea [34, 35] was to prove these two results in the opposite order. First one proves, by an elementary induction on the cardinality of the state space, that Z is nonvanishing in some specified polydisc (Theorem 3.1); it then follows immediately that $\log Z$ is analytic in that polydisc, and hence that its Taylor series (3.22) is convergent there. It is an interesting open question to know whether this approach can be made to work without the assumption of hard-core self-repulsion.

4. Some combinatorial lemmas

4.1. Reduction to trees

The weight $w(S)$ involves a sum (2.4) over connected subgraphs $(S, \widetilde{E})$ of the induced subgraph (S, E_S). The trouble is that there may be 'too many' connected subgraphs. It is remarkable, therefore, that this sum can sometimes be bounded by a sum over a much smaller set of graphs, namely *spanning trees*. The following proposition underlines the special role played by the 'complex antiferromagnetic regime' $A \equiv \{v \in \mathbb{C}\colon\ |1+v| \leqslant 1\}$.

Proposition 4.1 (Penrose [79]). *Let $G = (V, E)$ be a finite undirected graph equipped with complex edge weights $\{v_e\}_{e \in E}$ satisfying $|1 + v_e| \leqslant 1$ for all e. Then*

$$\left| \sum_{\substack{E' \subseteq E \\ (V,E') \text{ connected}}} \prod_{e \in E'} v_e \right| \leqslant \sum_{\substack{E' \subseteq E \\ (V,E') \text{ tree}}} \prod_{e \in E'} |v_e|. \tag{4.1}$$

Penrose [79] proved this when G is the complete graph K_n; the result then follows for all graphs without loops or multiple edges (it suffices to set $v_e = 0$ on the nonexistent edges). Here I present a minor modification of Penrose's proof that permits loops and multiple edges.

Proof. We can assume without loss of generality that G is connected, since otherwise both sides of the inequality are zero. Let $\mathscr{C}$ (resp. $\mathscr{T}$) be the set of subsets $E' \subseteq E$ such that (V, E') is connected (resp. is a tree). Clearly $\mathscr{C}$ is an increasing family of subsets of E with respect to set-theoretic inclusion, and the minimal elements of $\mathscr{C}$ are precisely those of $\mathscr{T}$ (*i.e.*, the spanning trees). It is a nontrivial but well-known fact ([15, Sections 7.2 and 7.3], [134, Section 8.3]) that the (anti-)complex $\mathscr{C}$ is *partitionable*: that is, there exists a map $\mathbf{R}\colon \mathscr{T} \to \mathscr{C}$ such that $\mathbf{R}(T) \supseteq T$ for all $T \in \mathscr{T}$ and $\mathscr{C} = \biguplus [T, \mathbf{R}(T)]$ (disjoint union), where $[E_1, E_2]$ denotes the Boolean interval $\{E'\colon\ E_1 \subseteq E' \subseteq E_2\}$. In fact, many alternative choices of $\mathbf{R}$ are available ([15, Sections 7.2 and 7.3], [47, Sections 2 and 6], [11, Proposition 13.7 *et seq.*]), and none of the subsequent arguments will depend on a specific choice of **R**. Nevertheless, for completeness, we shall give at the end of this proof a concrete construction of one possible **R**.

Given the existence of **R**, we have the immediate identity

$$\sum_{\substack{E' \subseteq E \\ (V,E') \text{ connected}}} \prod_{e \in E'} v_e \;=\; \sum_{\substack{T \subseteq E \\ (V,T) \text{ tree}}} \prod_{e \in T} v_e \sum_{T \subseteq E' \subseteq \mathbf{R}(T)} \prod_{e \in E' \setminus T} v_e$$

$$= \sum_{\substack{T \subseteq E \\ (V,T) \text{ tree}}} \prod_{e \in T} v_e \prod_{e \in \mathbf{R}(T) \setminus T} (1+v_e). \tag{4.2}$$

In particular, if $|1+v_e| \leqslant 1$ for all e, then (4.1) follows.

We now indicate a construction of **R** that is a slight variant of the one used by Penrose [79] (he orders the vertices, while I order the edges). Choose (arbitrarily) a vertex $x \in V$ and call it the root; and choose (arbitrarily) a numbering of the edges. For each $E' \in \mathscr{C}$ and $y \in V$, let $depth_{E'}(y)$ be the length of the shortest path in E' connecting y to the root. For each $y \in V \setminus \{x\}$, let $e(y)$ be the lowest-numbered edge in E' connecting y to a vertex y' with $depth_{E'}(y') = depth_{E'}(y) - 1$. And finally, let $\mathbf{S}(E') = \{e(y)\colon\ y \in V \setminus \{x\}\}$. Then trivially $\mathbf{S}(E') \subseteq E'$; moreover, it is easy to see that $(V, \mathbf{S}(E'))$ is a tree and that $depth_{\mathbf{S}(E')}(y) = depth_{E'}(y)$ for all $y \in V$. Conversely, given a spanning tree (V,T), it is not hard to see that $\mathbf{S}(E') = T$ if and only if $T \subseteq E' \subseteq \mathbf{R}(T)$, where $\mathbf{R}(T)$ is obtained from T by adjoining all edges $e \in E$ that

(a) connect two vertices of equal $depth_T$ (this includes loops, if any), or
(b) connect a vertex y to a vertex y' having $depth_T(y') = depth_T(y) - 1$ where e is higher-numbered than the edge already in T that connects y to a vertex y'' having $depth_T(y'') = depth_T(y) - 1$.

This completes the proof. □

Remarks. **1.** The identity (4.2) and the inequality (4.1) generalize to matroids. Indeed, for any matroid M, the independent sets of M form a simplicial complex $IN(M)$, called a *matroid complex*; moreover, every matroid complex is shellable, and every shellable complex is partitionable [15, Theorem 7.3.3 and Proposition 7.2.2]. For the cographic matroid $M^*(G)$, the independent sets are the complements of connected subgraphs, and the bases are complements of spanning trees, so we recover the situation of Proposition 4.1. I thank Criel Merino for teaching me about matroids and for helping me notice a silly error in my original proof of Proposition 4.1. Earlier, Dave Wagner had informed me that analogues of (4.2) hold for shellable simplicial complexes (see [107, Sections 0.3 and III.2] for the definition). There is a long history of identities related to (4.2): see, for example, [87, 73, 22, 25, 27, 1, 26] and the references cited therein.

2. I conjecture that (4.1) can be strengthened so that on the right-hand side the absolute value is put outside the sum rather than inside. (This would be useful in case the $\{v_e\}$ do not all have the same phase.) In fact, I conjecture more. Let

$$c(E') \;=\; |E'| - |V| + k(E') \tag{4.3}$$

be the cyclomatic number of the subgraph (V, E'), and define the generalized connected sum

$$C_G(\lambda, \{v_e\}) \;=\; \sum_{\substack{E' \subseteq E \\ (V,E')\ \text{connected}}} \lambda^{c(E')} \prod_{e \in E'} v_e \tag{4.4a}$$

$$= \; \lambda^{1-|V|}\, C_G(1, \{\lambda v_e\}) \tag{4.4b}$$

$$= \; \lim_{q \to 0} q^{-1} \lambda^{-|V|} Z_G(\lambda q, \{\lambda v_e\}). \tag{4.4c}$$

In particular, $\lambda = 0$ corresponds to the tree sum and $\lambda = 1$ to the connected sum. Then I conjecture that

(a) if $|1 + v_e| \leqslant 1$ for all e, then $|C_G(\lambda, \{v_e\})|$ is a decreasing function of λ on $0 \leqslant \lambda \leqslant 1$.

I had originally conjectured a stronger result, namely,

(a'_r) if $|r + v_e| \leqslant r$ for all e, then $(-1)^n\,(d^n/d\lambda^n)\,|C_G(\lambda, \{v_e\})|^2 \geqslant 0$ on $0 \leqslant \lambda \leqslant 1$, for all $n \geqslant 0$,

either for $r = 1$ or, failing that, for $r = \frac{1}{2}$; but this is in fact false for all $r > 0$, even for the second derivative evaluated at $\lambda = 0$ with equal edge weights $v_e = v$. Indeed, if we write

$$C_G(\lambda, v) \;=\; v^{|V|-1}\left[a_0 + a_1 v \lambda + a_2 v^2 \lambda^2 + \cdots\right] \tag{4.5}$$

where a_j is the number of spanning subgraphs of G having j cycles, then

$$(d^2/d\lambda^2)\,|C_G(\lambda, v)|^2 \Big|_{\lambda=0} \geqslant 0$$

holds for all v if $a_1^2 \geqslant 2a_0a_2$, but fails for v in a wedge near the imaginary axis if $a_1^2 < 2a_0a_2$. Now the complete bipartite graph $K_{3,4}$ has $a_0 = 432$, $a_1 = 612$, $a_2 = 456$ and hence provides a counterexample. Nevertheless, (a'_r) might be true for some interesting subclasses of graphs G.

For *real* $v_e \in [-1, 0]$, by contrast, I am able to prove a result even stronger than ($\mathrm{a}'_{1/2}$), namely,

(b) if $-1 \leqslant v_e \leqslant 0$ for all e, then $(-1)^{n+|V|-1}\,(d^n/d\lambda^n)\,C_G(\lambda, \{v_e\}) \geqslant 0$ on $0 \leqslant \lambda \leqslant 1$, for all integers $n \geqslant 0$.

Indeed, by (4.2) and (4.4b), we have

$$C_G(\lambda, \{v_e\}) \;=\; \sum_{\substack{T \subseteq E \\ (V,T)\ \text{tree}}} \prod_{e \in T} v_e \prod_{e \in \mathbf{R}(T) \setminus T} (1 + \lambda v_e) \tag{4.6}$$

and hence

$$\frac{d^n}{d\lambda^n} C_G(\lambda, \{v_e\}) \;=\; \sum_{\substack{T \subseteq E \\ (V,T)\ \text{tree}}} \sum_{\substack{\widetilde{T} \subseteq \mathrm{R}(T) \setminus T \\ |\widetilde{T}| = n}} \prod_{e \in T \cup \widetilde{T}} v_e \prod_{e \in \mathbf{R}(T) \setminus (T \cup \widetilde{T})} (1 + \lambda v_e), \tag{4.7}$$

which has the claimed sign whenever $0 \leqslant \lambda \leqslant 1$ and $-1 \leqslant v_e \leqslant 0$ for all e.

3. $C_G(1,\{v_e\})$ is equal, up to a prefactor, to the reliability polynomial $R_G(\{p_e\})$ [32], where p_e is the probability that edge e is operational and $v_e = p_e/(1-p_e)$:

$$R_G(\{p_e\}) \;=\; \left[\prod_{e\in E}(1-p_e)\right] C_G(1,\{p_e/(1-p_e)\}). \tag{4.8}$$

Now the Brown–Colbourn conjecture [20, 120] states that, for any connected graph G (loops and multiple edges are allowed), $R_G(p) \neq 0$ whenever $|p-1| > 1$. A more general conjecture is that $R_G(\{p_e\}) \neq 0$ whenever $|p_e - 1| > 1$ for all edges e, or, equivalently, that $C_G(1,\{v_e\}) \neq 0$ whenever $0 < |1+v_e| < 1$ for all e. But this generalized Brown–Colbourn conjecture is an immediate consequence of conjecture (a): for if we had $C_G(1,\{v_e\}) = 0$ with $|1+v_e| < 1$ for all e, then we could choose $\epsilon > 0$ such that $v'_e \equiv (1+\epsilon)v_e$ satisfy $|1+v'_e| < 1$ for all e, and we would have $C_G(\lambda,\{v'_e\}) = 0$ for $\lambda = 1/(1+\epsilon)$ (but not, of course, identically for $1/(1+\epsilon) \leqslant \lambda \leqslant 1$).

Note also that, if the generalized Brown–Colbourn conjecture holds for a graph G, then it holds also for any graph that can be obtained from G by a sequence of doublings of edges ('parallel expansions') and/or subdivisions of edges ('series expansions'). This follows from the formulae [32, p. 35]

$$R_{G'}(\{p_e, p_1, p_2\}) \;=\; R_G(\{p_e, 1-(1-p_1)(1-p_2)\}) \tag{4.9}$$

$$R_{G'}(\{p_e, p_1, p_2\}) \;=\; [1-(1-p_1)(1-p_2)]\, R_G\left(\left\{p_e, \frac{p_1p_2}{p_1+p_2-p_1p_2}\right\}\right) \tag{4.10}$$

where G' is obtained from G by parallel (resp. series) expansion of an edge e_0 into a pair of edges e_1, e_2. It suffices to note that if $|1-p_i| > 1$ for $i = 1,2$, then the same inequality holds for $p_\parallel \equiv 1-(1-p_1)(1-p_2)$ and for $p_{\text{series}} \equiv p_1p_2/(p_1+p_2-p_1p_2)$; the former is obvious, and the latter follows by observing that the series-expansion formula corresponds to addition of $1/v = 1/p - 1$ and that $|1-p| > 1$ corresponds to $\text{Re}(1/v) < -1/2$. In particular, since the generalized Brown–Colbourn conjecture manifestly holds for trees, it also holds for all connected graphs without a K_4 minor, as these are precisely the graphs that can be obtained from trees by a sequence of series and parallel expansions [37, 72, 122, 76]. The (original) Brown–Colbourn conjecture for series-parallel graphs was first proved by Wagner [120], by a vastly more complicated method.

4.2. Connected subgraphs containing a specified vertex

Let $G = (V,E)$ be a finite or countably infinite undirected graph equipped with edge weights $\{v_e\}_{e\in E}$, and let $x \in V$. Let us define the weighted sum over connected subgraphs $G' = (V', E') \subseteq G$ containing n vertices, one of which is x, and m edges:

$$C_{n,m}(G,\{v_e\},x) \;=\; \sum_{\substack{G'=(V',E')\subseteq G \\ G' \text{ connected} \\ V' \ni x \\ |V'|=n \\ |E'|=m}} \prod_{e\in E'} |v_e|. \tag{4.11}$$

Special cases are the tree sum

$$T_n(G, \{v_e\}, x) \;=\; C_{n,n-1}(G, \{v_e\}, x) \tag{4.12}$$

and the edge-counted sum

$$C_{\bullet,m}(G, \{v_e\}, x) \;=\; \sum_{n=1}^{m+1} C_{n,m}(G, \{v_e\}, x). \tag{4.13}$$

When the edge weights v_e are all equal to 1, we shall optionally omit them from the notation; note in particular the obvious bound

$$C_{n,m}(G, \{v_e\}, x) \;\leqslant\; C_{n,m}(G, x) \left(\sup_{e \in E} |v_e| \right)^m . \tag{4.14}$$

In this subsection we shall obtain a variety of upper bounds on $C_{n,m}(G, \{v_e\}, x)$ in terms of 'local' information about the graph G and the weights $\{v_e\}$.

Proposition 4.2. *Let $G = (V, E)$ be a finite or countably infinite loopless undirected graph of maximum degree $\leqslant r$, equipped with edge weights $\{v_e\}_{e \in E}$; and let $x \in V$. Let $\mathbf{T}_r$ be the infinite r-regular tree, and let y be any vertex in $\mathbf{T}_r$. Then*

$$C_{\bullet,m}(G, x) \;\leqslant\; C_{\bullet,m}(\mathbf{T}_r, y) \;=\; T_{m+1}(\mathbf{T}_r, y) \;\equiv\; t^{(r)}_{m+1} \;=\; r\, \frac{[(r-1)(m+1)]\,!}{m!\, [(r-2)m + r]\,!} \tag{4.15}$$

and hence

$$C_{\bullet,m}(G, \{v_e\}, x) \;\leqslant\; t^{(r)}_{m+1} \left(\sup_{e \in E} |v_e| \right)^m . \tag{4.16}$$

In particular,

$$T_n(G, \{v_e\}, x) \;\leqslant\; t^{(r)}_{n} \left(\sup_{e \in E} |v_e| \right)^{n-1} . \tag{4.17}$$

Proof. We can assume without loss of generality that $G = (V, E)$ is connected. Let $U = (\widetilde{V}, \widetilde{E})$ be the universal covering graph of G, with covering map $f \colon U \to G$; and let $\widetilde{x}$ be a vertex of U such that $f(\widetilde{x}) = x$. (The universal covering graph of a connected loopless graph G can be constructed as follows. Fix a base vertex x of G, and let the vertices of U be the walks in G (of finite length) that begin at x and do not contain any 'doublebacks', *i.e.*, two consecutive uses of the same edge in opposite directions. Two vertices of U are defined to be adjacent if one of them is a one-step extension of the other, and $f \colon U \to G$ maps each walk onto its final vertex. We take $\widetilde{x}$ to be the zero-step walk starting at x.) It is easy to see that U is a tree (in general countably infinite even when G is finite). Moreover, since G has maximum degree $\leqslant r$, U is a subtree of $\mathbf{T}_r$, from which it follows trivially that $C_{\bullet,m}(U, \widetilde{x}) \leqslant C_{\bullet,m}(\mathbf{T}_r, \widetilde{x})$. Let us prove, then, that $C_{\bullet,m}(G, x) \leqslant C_{\bullet,m}(U, \widetilde{x})$.

Fix an arbitrary total order on E, and choose arbitrarily for each edge $e \in E$ a distinguished direction. Now let H be a connected m-edge subgraph of G that contains x. Let S be the lexicographically first (with respect to the chosen total order on E) spanning tree of H. Then S based at x has a unique lifting to a subgraph $\widetilde{S}$ of U based at $\widetilde{x}$: it is defined by mapping each vertex s of S to the unique path in S from x to s.

Now, for each edge e of H not belonging to S, there is a unique edge $\widetilde{e}$ of U such that $f(\widetilde{e}) = e$ and $\widetilde{e}$ is incident with the image in $\widetilde{S}$ of the vertex of S from which e is directed. The addition of these edges to $\widetilde{S}$ produces a connected m-edge subgraph $\widetilde{H}$ of U that contains $\widetilde{x}$. Moreover, the map $H \mapsto \widetilde{H}$ is injective, since $H = f(\widetilde{H})$. This completes the proof that $C_{\bullet,m}(G, x) \leqslant C_{\bullet,m}(\mathbf{T}_r, \widetilde{x})$.

We conclude by calculating the numbers $t^{(r)}_{m+1} \equiv C_{\bullet,m}(\mathbf{T}_r, \widetilde{x})$.[8] Let $\mathbf{U}_r$ be the infinite tree in which all vertices have degree r except for one vertex y which has degree $r - 1$, and let $u^{(r)}_{m+1} = C_{\bullet,m}(\mathbf{U}_r, y)$. Then define, as formal power series, the generating functions

$$T_r(z) = \sum_{n=1}^{\infty} t^{(r)}_n z^n, \tag{4.18}$$

$$U_r(z) = \sum_{n=1}^{\infty} u^{(r)}_n z^n. \tag{4.19}$$

The recursive structure of r-regular rooted trees easily implies the functional equations

$$T_r(z) = z[1 + U_r(z)]^r, \tag{4.20}$$

$$U_r(z) = z[1 + U_r(z)]^{r-1}. \tag{4.21}$$

We now use the Lagrange Implicit Function Theorem for formal power series [48, Theorem 1.2.4], which states that for formal power series $f(u) = \sum_{n=0}^{\infty} f_n u^n$ and $g(u) = \sum_{n=0}^{\infty} g_n u^n$ with $g_0 \neq 0$, the functional equation $U(z) = zg(U(z))$ has a unique solution $U(z)$, and for all $n \geqslant 1$ one has

$$[z^n]f(U(z)) = \frac{1}{n}\,[u^{n-1}](f'(u)g(u)^n), \tag{4.22}$$

where $[z^n]P(z)$ denotes the coefficient of z^n in the formal power series $P(z)$. Applying this with $f(u) = (1 + u)^r$ and $g(u) = (1 + u)^{r-1}$ yields

$$t^{(r)}_n = \frac{r}{n-1}\binom{(r-1)n}{n-2} = r\,\frac{[(r-1)n]\,!}{(n-1)!\,[(r-2)n+2]\,!}, \tag{4.23}$$

$$u^{(r)}_n = \frac{1}{n}\binom{(r-1)n}{n-1} = (r-1)\,\frac{[(r-1)n-1]\,!}{(n-1)!\,[(r-2)n+1]\,!}. \tag{4.24}$$

□

Remarks. 1. The proof presented here is a simplification of my original proof, based on independent suggestions by Paul Seymour, Dave Wagner and an anonymous referee.

2. *A posteriori*, we learn from Proposition 4.3(c) and (d) below that the power series (4.18)/(4.19) in fact define analytic functions in the disc $|z| < (r-2)^{r-2}/(r-1)^{r-1}$.

3. I suspect that Proposition 4.2 is known somewhere in the graph-theory literature, but I do not know any reference. A weaker version of Proposition 4.2 can be found in [35, Lemma 5.4].

[8] For similar computations, see, *e.g.*, [43].

Let us also collect some properties of the numbers $t_n^{(r)}$ that arise in Proposition 4.2.

Proposition 4.3. *The quantities*

$$t_n^{(r)} \;=\; r\,\frac{[(r-1)n]\,!}{(n-1)\,!\,[(r-2)n+2]\,!}, \tag{4.25}$$

defined for integers $n, r \geqslant 1$, have the following properties.

(a) $t_n^{(1)} = \begin{cases} 1, & \textit{for } n = 1, 2, \\ 0, & \textit{for } n \geqslant 3. \end{cases}$

(b) $t_n^{(2)} = n$.

(c) *As $n \to \infty$ at fixed $r \geqslant 3$,*

$$t_n^{(r)} \;=\; \frac{r\,(r-1)^{1/2}}{\sqrt{2\pi}\,(r-2)^{5/2}} \left(\frac{(r-1)^{r-1}}{(r-2)^{r-2}}\right)^{n} n^{-3/2}\left[1 \,+\, \frac{\frac{1}{r-1} - \frac{37}{r-2} - 1}{12n} \,+\, O\!\left(\frac{1}{n^2}\right)\right]. \tag{4.26}$$

(d) *For all n and all $r \geqslant 3$,*

$$t_n^{(r)} \;\leqslant\; \left(\frac{(r-1)^{r-1}}{(r-2)^{r-2}}\right)^{n-1} \;\leqslant\; [e(r-\tfrac{3}{2})]^{n-1}. \tag{4.27}$$

(e) *As $r \to \infty$ at fixed $n \geqslant 1$,*

$$t_n^{(r)} \;=\; \frac{(rn)^{n-1}}{n!}\left[1 \,-\, \frac{3(n-1)(n-2)}{2nr} \,+\, O\!\left(\frac{1}{r^2}\right)\right]. \tag{4.28}$$

(f) *For all $r, n \geqslant 1$,*

$$t_n^{(r)} \;\leqslant\; \frac{(rn)^{n-1}}{n!}. \tag{4.29}$$

Proof. Parts (a) and (b) are trivial, while parts (c) and (e) follow from Stirling's formula. Part (f) is trivial for $r = 1$, while for $r \geqslant 2$ it follows immediately from

$$\frac{[(r-1)n]\,!}{[(r-2)n+2]\,!} \;=\; \frac{(rn-n)\,!}{(rn-2n+2)\,!} \;\leqslant\; (rn)^{n-2}. \tag{4.30}$$

The first inequality in part (d) is obvious for $n = 1$, so assume $n \geqslant 2$. We have

$$\begin{aligned} t_n^{(r)} \;&=\; \frac{rn}{[(r-2)n+1]\,[(r-2)n+2]}\binom{(r-1)n}{n} \\ &\leqslant\; \frac{r}{(r-2)^2 n}\binom{(r-1)n}{n} \\ &\leqslant\; \frac{r(r-1)^{1/2}}{\sqrt{2\pi}(r-2)^{5/2}}\, n^{-3/2} \left(\frac{(r-1)^{r-1}}{(r-2)^{r-2}}\right)^{n} \\ &=\; \frac{r(r-1)^{r-\frac{1}{2}}}{\sqrt{2\pi}(r-2)^{r+\frac{1}{2}}}\, n^{-3/2} \left(\frac{(r-1)^{r-1}}{(r-2)^{r-2}}\right)^{n-1}, \end{aligned} \tag{4.31}$$

where the second inequality uses Lemma 4.4 below. Then straightforward calculus shows that the function $F(r) = r(r-1)^{r-\frac{1}{2}}/[\sqrt{2\pi}(r-2)^{r+\frac{1}{2}}]$ is decreasing on $r > 2$; and we have $F(3) = 12/\sqrt{\pi} < 4^{3/2}$, $F(4) = (27/8)\sqrt{3/\pi} < 3^{3/2}$ and $F(5) = (1280/243)\sqrt{2/(3\pi)} < 2^{3/2}$.

So the first inequality in (d) follows except for the cases $(r,n) = (3,2)$, $(3,3)$ and $(4,2)$, which can be checked by hand. The final inequality in (d) follows from

$$\begin{aligned}
\log \frac{\sigma^\sigma}{(\sigma-1)^{\sigma-1}} &= \log\sigma + (\sigma-1)\log\frac{\sigma}{\sigma-1} \\
&= \log\sigma + 1 - \sum_{k=1}^{\infty} \frac{\sigma^{-k}}{k(k+1)} \\
&\leqslant \log\sigma + 1 - \sum_{k=1}^{\infty} \frac{\sigma^{-k}}{k2^k} \\
&= \log\sigma + 1 + \log\left(1 - \frac{1}{2\sigma}\right), \qquad (4.32)
\end{aligned}$$

where the sums are convergent for $\sigma > 1$, so that $\sigma^\sigma/(\sigma-1)^{\sigma-1} \leqslant e(\sigma - \frac{1}{2})$. □

Lemma 4.4. *Let $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$ be integers. Then*

$$\binom{n}{k} < \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} \sqrt{\frac{n}{2\pi k(n-k)}}. \qquad (4.33)$$

Proof. We use the following strong form of Stirling's formula [31, pp. 45–46]: for integer $n \geqslant 1$,

$$\log n! = (n + \tfrac{1}{2})\log n - n + \log\sqrt{2\pi} + \epsilon_n \qquad (4.34)$$

with

$$\frac{1}{12n+1} < \epsilon_n < \frac{1}{12n}. \qquad (4.35)$$

(The proof in [31] is valid only for $n \geqslant 2$, but $\epsilon_1 = 1 - \log\sqrt{2\pi} \approx 0.08106$ clearly satisfies $1/13 < \epsilon_1 < 1/12$.) Then

$$\begin{aligned}
\epsilon_n - \epsilon_k - \epsilon_{n-k} &< \frac{1}{12n} - \frac{1}{12k+1} - \frac{1}{12(n-k)+1} \\
&= \frac{-144[n^2 - k(n-k)] - 12n + 1}{12n\,(12k+1)\,[12(n-k)+1]} \\
&< 0. \qquad (4.36)
\end{aligned}$$

□

Proposition 4.2 clearly gives the best possible bound for $C_{\bullet,m}(G,x)$ and $T_n(G,x)$ in terms of the maximum degree of G, since it is sharp when $G = \mathbf{T}_r$. On the other hand, Proposition 4.2 is somewhat unnatural for general (unequal) edge weights $\{v_e\}$, since adding an edge of small weight v_e makes little change in $C_{n,m}(G,\{v_e\},x)$ but can cause the bound to jump (in case it increases the maximum degree). It is of interest, therefore, to find alternative bounds that depend 'smoothly' on the weights $\{v_e\}$. We shall now give two such bounds (Propositions 4.5 and 4.6). Unfortunately, both of them are strictly weaker than Proposition 4.2 when the edge weights are equal, and neither one is strictly stronger than the other.

Proposition 4.5. *Let $G = (V, E)$ be a finite or countably infinite loopless undirected graph equipped with edge weights $\{v_e\}_{e \in E}$. Then for any $x \in V$,*

$$C_{\bullet,m}(G, \{v_e\}, x) \;\leqslant\; \frac{(m+1)^m}{(m+1)!} \left(\sup_{i \in V} \sum_{e \ni i} |v_e| \right)^m \tag{4.37a}$$

$$\leqslant\; \left(e \sup_{i \in V} \sum_{e \ni i} |v_e| \right)^m . \tag{4.37b}$$

(*The e in front of the sup in* (4.37b) *denotes, of course, the base of natural logarithms.*)

Proof. As in the proof of Proposition 4.2, we pass to the universal covering graph $U = (\widetilde{V}, \widetilde{E})$ of G with covering map $f \colon U \to G$; and we define the weight $v_{\tilde{e}}$ of an edge $\tilde{e} \in \widetilde{E}$ to be the weight v_e of its image $e = f(\tilde{e})$. It then follows, as in Proposition 4.2, that $C_{\bullet,m}(G, \{v_e\}, x) \leqslant C_{\bullet,m}(U, \{v_{\tilde{e}}\}, \tilde{x})$.

Let us now define, for each vertex $x \in \widetilde{V}$, the formal generating function

$$C_x(z) \;=\; \sum_{m=0}^{\infty} C_{\bullet,m}(U, \{v_{\tilde{e}}\}, x)\, z^m. \tag{4.38}$$

Then the recursive structure of rooted trees implies that

$$C_x(z) \;\preceq\; \prod_{y \sim x} [1 + |v_{xy}| z C_y(z)] \tag{4.39a}$$

$$\preceq\; \prod_{y \sim x} e^{|v_{xy}| z C_y(z)} \tag{4.39b}$$

where $y \sim x$ denotes that y is adjacent to x, xy denotes the (unique) corresponding edge, and $\preceq$ denotes coefficientwise inequality at all orders in z; the second inequality holds because $1 + \alpha z \preceq e^{\alpha z}$ for $\alpha \geqslant 0$. It then follows, by induction on the power of z, that $C_x(z) \preceq \bar{C}(z)$ for all x, where $\bar{C}(z)$ is determined by the equation

$$\bar{C}(z) \;=\; e^{\mu z \bar{C}(z)} \tag{4.40}$$

with

$$\mu \;=\; \sup_{x \in \widetilde{V}} \sum_{y \sim x} |v_{xy}| \;=\; \sup_{i \in V} \sum_{e \ni i} |v_e|. \tag{4.41}$$

The coefficients of $\bar{C}(z)$ can be determined by applying the Lagrange Implicit Function Theorem to $\bar{U}(z) = z\bar{C}(z)$, and we have the well-known (*e.g.*, [62, p. 392]) result

$$\bar{C}(z) \;=\; \sum_{m=0}^{\infty} \frac{(m+1)^m}{(m+1)!} \mu^m z^m. \tag{4.42}$$

Finally, the inequality $e^m \geqslant (m+1)^m/(m+1)!$ is trivial for $m = 0, 1$, and for $m \geqslant 2$ it follows from $e^x \geqslant x^n/n!$ by setting $x = n = m + 1$. □

Remarks. 1. The proof presented here is a simplification of my original proof, based on the suggestions of an anonymous referee.

2. Proposition 4.5 holds also for graphs with loops, but they impose slight technical complications.

An alternative estimate is due to Campanino, Capocaccia and Tirozzi [29, p. 129] (see also [28, p. 522] and [101, pp. 463–464]):

Proposition 4.6. *Let $G = (V, E)$ be a finite or countably infinite undirected graph equipped with edge weights $\{v_e\}_{e \in E}$. Define the matrix $M = (M_{xy})_{x,y \in V}$ by*

$$M_{xy} \;=\; \sum_{e:\, e \text{ connects } x \text{ to } y} |v_e|^{1/2}. \tag{4.43}$$

Then, for any $x \in V$,

$$C_{\bullet,m}(G, \{v_e\}, x) \;\leqslant\; (M^{2m})_{xx} \;\leqslant\; \left(\sup_{i \in V} \sum_{e \ni i} |v_e|^{1/2} \right)^{2m}. \tag{4.44}$$

Proof. Let $G' = (V', E')$ be a connected subgraph of G having m edges. Then, for any vertex $x \in V'$, there exists a path on G' starting and ending at x that uses each edge $e \in E'$ exactly twice. (Proof: the multigraph formed by doubling each edge of G' is Eulerian. Alternate proof: by induction on m.[9]) Conversely, every path on G starting and ending at x corresponds in this way to at most one subgraph G'. The claim follows. □

Let us conclude by examining the relative sharpness of these bounds when G is an r-regular graph and the edge weights v_e are equal. Then the tree bound $t_n^{(r)}$ of Proposition 4.2 grows as $n \to \infty$ at an exponential rate $(r-1)^{r-1}/(r-2)^{r-2}$: this is less than $e(r - \frac{3}{2})$ for all r, and behaves as $e[r - \frac{3}{2} - O(1/r)]$ as $r \to \infty$. The bound of Proposition 4.5 is slightly weaker: it grows at exponential rate er. Finally, the bound of Proposition 4.6 grows at exponential rate r^2, which is vastly weaker for large r but is slightly better when $r = 2$.

In particular, when G is a regular lattice, it can be shown by supermultiplicativity arguments [60, 61, 127, 57] that the limits

$$\lambda_o(G) \;=\; \lim_{n \to \infty} T_n(G, x)^{1/n}, \tag{4.45}$$

$$\lambda_b(G) \;=\; \lim_{m \to \infty} C_{\bullet,m}(G, x)^{1/m} \tag{4.46}$$

exist. For the simple hypercubic lattice $\mathbb{Z}^d$ with nearest-neighbour bonds, these growth constants have been computed (non-rigorously) in a large-d asymptotic expansion [45, 77] (see also [78, 53, 33] for related rigorous results):

$$\log \lambda_o(\mathbb{Z}^d) \;=\; \log \sigma \,+\, 1 \,-\, \frac{1}{2}\sigma^{-1} \,-\, \frac{8}{3}\sigma^{-2} \,-\, \cdots, \tag{4.47}$$

[9] See, *e.g.*, [101, Lemma V.7.A.2].

$$\log \lambda_b(\mathbb{Z}^d) \;=\; \log\sigma + 1 - \frac{1}{2}\sigma^{-1} - \left(\frac{8}{3} - \frac{1}{2e}\right)\sigma^{-2} - \cdots, \tag{4.48}$$

where $\sigma = r-1 = 2d-1$. Let us compare this with the tree bound of Proposition 4.2:

$$\log\frac{(r-1)^{r-1}}{(r-2)^{r-2}} \;=\; \log\sigma + 1 - \frac{1}{2}\sigma^{-1} - \frac{1}{6}\sigma^{-2} - \cdots. \tag{4.49}$$

Thus, the latter bound is very close to sharp for $G = \mathbb{Z}^d$ in high dimension d, confirming the intuition that high-dimensional regular lattices are 'like trees' to leading order in $1/d$.

5. Application to the Potts-model partition function

We are now ready for the main theorem of this paper.

Theorem 5.1. *Let* $G = (V, E)$ *be a loopless finite undirected graph equipped with complex edge weights* $\{v_e\}_{e\in E}$ *satisfying* $|1+v_e| \leqslant 1$ *for all e. Let* $Q = Q(G, \{v_e\}) > 0$ *be the smallest number for which*

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n}\, Q^{-(n-1)} \max_{x\in V} T_n(G, \{v_e\}, x) \;\leqslant\; 1. \tag{5.1}$$

(*Note that* Q *is automatically finite, since* $T_n(G, \{v_e\}, x) = 0$ *for* $n > |V|$.) *Then all the zeros of* $Z_G(q, \{v_e\})$ *lie in the disc* $|q| < Q$.

Proof. Starting from the polymer-gas representation (2.2)–(2.4) of $Z_G(q, \{v_e\})$, we apply Theorem 3.1 and Proposition 3.2 with the choice $R_S = |w(S)|$. We verify hypothesis (a) of Proposition 3.2 by verifying (3.15) with

$$A_n \;=\; \max_{x\in V} \sum_{\substack{S \ni x \\ |S| = n}} |w(S)|. \tag{5.2}$$

Now we use Proposition 4.1 to conclude that $w(S)$ can be bounded by a sum over trees:

$$|w(S)| \;\leqslant\; |q|^{-(|S|-1)} \sum_{\substack{\widetilde{E} \subseteq E \\ (S,\widetilde{E}) \text{ tree}}} \prod_{e\in\widetilde{E}} |v_e|. \tag{5.3}$$

Inserting this into (5.2), we get

$$A_n \;\leqslant\; |q|^{-(n-1)} \max_{x\in V} T_n(G, \{v_e\}, x). \tag{5.4}$$

If $|q| \geqslant Q$, hypothesis (b) of Proposition 3.2 holds (recall Remark 4 following that proposition) and hence $Z_G(q, \{v_e\}) \neq 0$. □

In applying Theorem 5.1 we are of course free to use any convenient upper bound on $\max_{x\in V} T_n(G, \{v_e\}, x)$. In particular, when G has maximum degree $\leqslant r$, Proposition 4.2 provides such a bound. Recall that

$$t_n^{(r)} \;=\; r\,\frac{[(r-1)n]!}{(n-1)!\,[(r-2)n+2]!}, \tag{5.5}$$

and let $C = C(r) > 0$ be the smallest number for which

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n}\, C^{-(n-1)}\, t_n^{(r)} \;\leqslant\; 1. \tag{5.6}$$

The following is then an immediate consequence of Theorem 5.1 and Proposition 4.2.

Corollary 5.2. *Let $G = (V, E)$ be a loopless finite undirected graph of maximum degree $\leqslant r$, equipped with complex edge weights $\{v_e\}_{e\in E}$ satisfying $|1 + v_e| \leqslant 1$ for all e. Let $v_{\max} = \max_{e\in E} |v_e|$. Then all the zeros of $Z_G(q, \{v_e\})$ lie in the disc $|q| < C(r)v_{\max}$.* □

And for the chromatic polynomials we deduce the following result.

Corollary 5.3. *Let $G = (V, E)$ be a loopless finite undirected graph of maximum degree $\leqslant r$. Then all the zeros of $P_G(q)$ lie in the disc $|q| < C(r)$.* □

Table 1 lists rigorous upper bounds on $C(r)$ for $2 \leqslant r \leqslant 20$, proved (with the assistance of MATHEMATICA) as follows. After computing numerically an approximate value of $C(r)$,[10] I added 10^{-6} and rounded it upwards to a rational number $p/10^6$. (Thus, the value reported in Table 1 exceeds my best estimate of $C(r)$ by at most 2×10^{-6}.) I likewise approximated the numerically found α by a rational number $p'/10^6$. Thereafter I did all computations in exact rational arithmetic. First I computed a rational upper bound on e^{α} (differing from the true e^{α} by at most 2×10^{-10}) by truncating the Taylor series for $e^{-\alpha}$ at odd order (here ninth or eleventh) to obtain a lower bound on $e^{-\alpha}$. Finally, I computed an upper bound on (5.6) by summing the terms explicitly through $n =$ some n_0 and bounding the tail of the series ($n \geqslant n_0 + 1$) using Proposition 4.3(d); I systematically increased n_0 until the inequality (5.6) was verified. For $r = 2$, of course, I just summed the series exactly.

As $r \to \infty$ we have the following.

Proposition 5.4. *Let $K \approx 7.963906\ldots$ be the smallest number for which*

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n}\, K^{-(n-1)}\, \frac{n^{n-1}}{n!} \;\leqslant\; 1. \tag{5.7}$$

Then $C(r) \leqslant Kr$ for all r, and $\lim_{r\to\infty} C(r)/r = K$. Moreover, we have the rigorous bound $K \leqslant 7.963907$.

Proof. Clearly $\widetilde{C}(r) \equiv C(r)/r$ is the smallest number for which

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n}\, \widetilde{C}^{-(n-1)}\, \frac{t_n^{(r)}}{r^{n-1}} \;\leqslant\; 1. \tag{5.8}$$

[10] Using the generating function $f(z) = \sum_{n=2}^{\infty} t_n^{(r)} z^n$ where $z = e^{\alpha}/C$, I solve simultaneously the equations $f(z) = (\log z + \log C)/C$ and $f'(z) = 1/(Cz)$ by solving numerically $f(z) = -zf'(z)\log f'(z)$ and then plugging back in to determine $C = 1/[zf'(z)]$ and $\alpha = -\log f'(z)$.

Table 1 Upper bounds on $C(r)$ for $2 \leqslant r \leqslant 20$; they differ from my best estimate of the true $C(r)$ by at most 2×10^{-6}. The third column gives a value of α for which (5.6) is proved to be $\leqslant 1$. The fourth column gives the upper bound on e^{α} employed in this proof. The fifth column gives the number of terms explicitly summed in the series

r	$C(r)$	α	e^{α}	n_0
2	13.234367	0.453177	1.5733026366	∞
3	21.144294	0.436884	1.5478765131	15
4	29.081607	0.428653	1.5351882318	18
5	37.029702	0.423694	1.5275940786	20
6	44.983130	0.420382	1.5225430561	22
7	52.939585	0.418013	1.5189404206	23
8	60.897921	0.416236	1.5162436603	24
9	68.857505	0.414852	1.5141466306	25
10	76.817961	0.413745	1.5124713976	25
11	84.779049	0.412839	1.5111017191	26
12	92.740610	0.412084	1.5099612679	26
13	100.702534	0.411445	1.5089967109	26
14	108.664743	0.410898	1.5081715154	27
15	116.627179	0.410423	1.5074553040	27
16	124.589800	0.410008	1.5068298399	28
17	132.552573	0.409641	1.5062769348	28
18	140.515473	0.409315	1.5057859685	28
19	148.478479	0.409024	1.5053478486	28
20	156.441575	0.408761	1.5049519941	28

It follows from Proposition 4.3(f) that $\widetilde{C}(r) \leqslant K$ for all r.

Now suppose that we were to have $\liminf_{r\to\infty} \widetilde{C}(r) \leqslant K - \epsilon < K$. Then there would exist infinite sequences $\{r_i\} \uparrow \infty$ and $\{\alpha_i\}$ such that

$$\alpha_i^{-1} \sum_{n=2}^{\infty} e^{\alpha_i n} (K-\epsilon)^{-(n-1)} \frac{t_n^{(r_i)}}{r_i^{n-1}} \leqslant 1 \tag{5.9}$$

for all i. Now the finiteness of (5.9) implies that the α_i are bounded (*e.g.*, from Proposition 4.3(c) we have $e^{\alpha_i} \leqslant (K-\epsilon) r_i (r_i-2)^{r_i-2}/(r_i-1)^{r_i-1} \leqslant \frac{3}{4}(K-\epsilon)$ whenever $r_i \geqslant 3$). So we can extract a subsequence of $\{\alpha_i\}$ that converges to some value α_*. Then Proposition 4.3(e,f) and the dominated convergence theorem imply that

$$\alpha_*^{-1} \sum_{n=2}^{\infty} e^{\alpha_* n} (K-\epsilon)^{-(n-1)} \frac{n^{n-1}}{n!} \leqslant 1, \tag{5.10}$$

which contradicts the definition of K.

Finally, it is easy to prove that $K \leqslant 7.963907$, by a computer-assisted method similar to that used above for $C(r)$. For the tail of the series ($n \geqslant n_0 + 1$), it suffices to use the crude bound $n^{n-1}/n! \leqslant e^{n-1} \leqslant 3^{n-1}$. The proof succeeds with the choices $\alpha = 0.403774$, $e^{\alpha} \leqslant 1.4974655$ and $n_0 = 32$. □

If we employ Proposition 4.5 in place of Proposition 4.2, Theorem 5.1 yields the following.

Corollary 5.5. *Let $G = (V, E)$ be a loopless finite undirected graph equipped with complex edge weights $\{v_e\}_{e\in E}$ satisfying $|1 + v_e| \leqslant 1$ for all e. Then all the zeros of $Z_G(q, \{v_e\})$ lie in the disc $|q| < K \max_{i\in V} \sum_{e\ni i} |v_e|$, where K is the constant defined in Proposition 5.4.* □

Remarks. 1. Since $Z_G(q, \{v_e\})/q$ for any graph G is the product of the same quantity over the blocks of G, it is legitimate to apply Theorem 5.1 and its corollaries separately to each block. This can lead to large improvements (consider trees, for instance).

2. What happens if we drop the assumption that $|1 + v_e| \leqslant 1$? Because we can no longer use Proposition 4.1 to reduce the sum to trees, we need to consider all n-vertex connected subgraphs of G containing a given vertex x. But the number m of *edges* in such a subgraph could be as large as $\lfloor rn/2 \rfloor$ (where r is the maximum degree of G). Therefore, the factor $t_n^{(r)} v_{\max}^{n-1}$ coming from (4.17) has to be replaced by a factor $\sum_{m=n-1}^{\lfloor rn/2 \rfloor} t_{m+1}^{(r)} v_{\max}^m$ coming from (4.16) (or the analogue from Proposition 4.5). As a consequence, the radius of the q-plane disc containing all the zeros of $Z_G(q, \{v_e\})$ will scale as $\max[v_{\max}, v_{\max}^{r/2}]$ rather than simply $v_{\max}$. And this is not simply an artifact of the method of proof: for the q-state Potts *ferromagnet* ($v > 0$, $q > 0$) on the the simple hypercubic lattice $\mathbb{Z}^d$ with nearest-neighbour bonds, the first-order phase-transition point v_t indeed behaves as

$$v_t(q) \;=\; q^{1/d}\,[1 + O(1/q)] \tag{5.11}$$

as $q \to +\infty$ [74, 67, 64, 66, 16]. In the Whitney–Tutte–Fortuin–Kasteleyn polynomial (1.2), this reflects the coexistence at $v = v_t$ (for all $q \gg 1$) between a phase with a low density of occupied edges and a phase with a high density of occupied edges.

6. Some generalizations

The following generalization of the Whitney–Tutte–Fortuin–Kasteleyn polynomial (1.2) is motivated by some work of Tutte [115, 118], Farrell [40] and Stanley [106, 108] as well as by the statistical–mechanical application to be discussed below. Let us replace the single complex number q by a map $\mathsf{q}\colon \mathscr{P}_*(V) \to \mathbb{C}$, and define

$$Z_G(\mathsf{q}, \{v_e\}) \;=\; \sum_{E'\subseteq E} \left(\prod_{i=1}^{k(E')} \mathsf{q}(V_i) \right) \left(\prod_{e\in E'} v_e \right), \tag{6.1}$$

where $(V_1, E_1), \ldots, (V_{k(E')}, E_{k(E')})$ are the connected components of (V, E'). We immediately deduce an analogue of Proposition 2.1: the identity (2.2) is replaced by

$$Z_G(\mathsf{q}, \{v_e\}) \;=\; \left(\prod_{x\in V} \mathsf{q}(\{x\}) \right) Z_{\text{polymer},G}(\mathsf{q}, \{v_e\}), \tag{6.2}$$

and the fugacities $w(S)$ are now given by

$$w(S) \;=\; \begin{cases} \dfrac{\mathsf{q}(S)}{\prod\limits_{x\in S}\mathsf{q}(\{x\})} \displaystyle\sum_{\substack{\widetilde{E}\subseteq E \\ (S,\widetilde{E})\text{ connected}}} \prod_{e\in\widetilde{E}} v_e, & \text{if } |S|\geqslant 2, \\ 0, & \text{if } |S|\leqslant 1. \end{cases} \tag{6.3}$$

The proof of Theorem 5.1 then goes through without change, and yields the following.

Theorem 6.1. *Let $G=(V,E)$ be a loopless finite undirected graph equipped with complex edge weights $\{v_e\}_{e\in E}$ satisfying $|1+v_e|\leqslant 1$ for all e, and let $\mathsf{q}\colon \mathscr{P}_*(V)\to\mathbb{C}$. Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of positive numbers satisfying*

$$\inf_{\alpha>0}\alpha^{-1}\sum_{n=2}^{\infty} e^{\alpha n}\,Q_n^{-1}\,\max_{x\in V} T_n(G,\{v_e\},x) \;\leqslant\; 1, \tag{6.4}$$

and assume that

$$\left|\frac{\mathsf{q}(S)}{\prod\limits_{x\in S}\mathsf{q}(\{x\})}\right| \;\leqslant\; Q_{|S|}^{-1} \tag{6.5}$$

for all nonempty subsets $S\subseteq V$. Then $Z_G(\mathsf{q},\{v_e\})\neq 0$. □

The following special case is of particular interest. Fix an integer $N\geqslant 0$, and for each $x\in V$ choose a vector $u_x=(u_x^{(1)},\ldots,u_x^{(N)})\in\mathbb{C}^N$. Then define

$$\mathsf{q}(S) \;=\; q - N + \sum_{i=1}^{N}\prod_{x\in S}(1+u_x^{(i)}), \tag{6.6}$$

where q is a fixed complex number. This corresponds to a q-state Potts model in a magnetic field $h_x=(h_x^{(1)},\ldots,h_x^{(N)})$ in the first N spin directions, where $u_x^{(i)}=\exp(h_x^{(i)})-1$. To see this, we first define, for each integer $q\geqslant N$, the partition function for the q-state Potts model in a magnetic field, generalizing (1.1):

$$Z_G(q,\{v_e\},\{u_x^{(i)}\}) \;=\; \sum_{\{\sigma_x\}}\prod_{e\in E}\Big[1+v_e\delta(\sigma_{x_1(e)},\sigma_{x_2(e)})\Big]\prod_{x\in V}\prod_{i=1}^{N}\Big[1+u_x^{(i)}\delta(\sigma_x,i)\Big]. \tag{6.7}$$

Now expand out the product over $e\in E$, and let $E'\subseteq E$ be the set of edges for which the term $v_e\delta_{\sigma_{x_1(e)},\sigma_{x_2(e)}}$ is taken. Then perform the sum over configurations $\{\sigma_x\}$: in each connected component of the subgraph (V,E') the spin value σ_x must be constant, and there are no other constraints. The sum over possible spin values in a connected component with vertex set S yields (6.6). It follows that, for any integer $q\geqslant N$, the partition function $Z_G(q,\{v_e\},\{u_x^{(i)}\})$ equals $Z_G(\mathsf{q},\{v_e\})$ with weights (6.6). We then take the latter, which is a polynomial in q, $\{v_e\}$ and $\{u_x^{(i)}\}$, as the *definition* of $Z_G(q,\{v_e\},\{u_x^{(i)}\})$ for general complex q.

The following lemma gives a sufficient condition for the applicability of Theorem 6.1 to this situation.

Lemma 6.2. *Let* $\mathsf{q}(S)$ *be defined by* (6.6).

(a) *If* $N = 1$ *and each* $u_x^{(i)}$ *equals either* 0 *or* -1, *then*

$$\left| \frac{\mathsf{q}(S)}{\prod\limits_{x \in S} \mathsf{q}(\{x\})} \right| \;\leqslant\; \min(|q|, |q-1|)^{-(|S|-1)}. \tag{6.8}$$

(b) *If* $-1 \leqslant u_x^{(i)} \leqslant 0$ *for all* x, i *and* $|q| > N$, *then*

$$\left| \frac{\mathsf{q}(S)}{\prod\limits_{x \in S} \mathsf{q}(\{x\})} \right| \;\leqslant\; (|q| - N)^{-(|S|-1)}. \tag{6.9}$$

(c) *If* $|1 + u_x^{(i)}| \leqslant 1$ *for all* x, i *and* $|q - N| > N$, *then*

$$\left| \frac{\mathsf{q}(S)}{\prod\limits_{x \in S} \mathsf{q}(\{x\})} \right| \;\leqslant\; \frac{|q-N| + N}{(|q-N| - N)^{|S|}}. \tag{6.10}$$

The proof of Lemma 6.2 is deferred to the end of this section.

We can exploit this example to obtain new results for the ordinary (zero-field) Potts-model partition function $Z_G(q, \{v_e\})$ and in particular for the chromatic polynomial $P_G(q)$, by employing a variant of the 'ghost spin' trick of Suzuki [110] and Griffiths [49]. Given a finite graph $G_0 = (V_0, E_0)$ and an integer $N \geqslant 1$, we define G to be the join of G_0 with the complete graph on N vertices. Thus, the vertex set of G is $V = V_0 \bigcup \{y_1, \ldots, y_N\}$ (disjoint union) and the edge set is $E = E_0 \bigcup \{\langle x y_i \rangle\}_{x \in V_0,\, 1 \leqslant i \leqslant N} \bigcup \{\langle y_i y_j \rangle\}_{1 \leqslant i < j \leqslant N}$. We allow the edge weights $\{v_e\}_{e \in E_0}$ and $\{v_{\langle x y_i \rangle}\}_{x \in V_0,\, 1 \leqslant i \leqslant N}$ to be arbitrary complex numbers, but we require that $v_{\langle y_i y_j \rangle} = -1$ for $1 \leqslant i < j \leqslant N$ (this condition is crucial). We then have the identity

$$Z_G(q, \{v_e, v_{\langle x y_i \rangle}, v_{\langle y_i y_j \rangle}\}) \;=\; q_{\langle N \rangle}\, Z_{G_0}(q, \{v_e\}, \{u_x^{(i)}\}), \tag{6.11}$$

where $q_{\langle N \rangle} = q(q-1) \cdots (q-N+1)$ is the Nth 'falling factorial' polynomial and $u_x^{(i)} = v_{\langle x y_i \rangle}$. This is most easily proved in the Potts spin representation (1.1)/(6.7): let q be an integer $\geqslant N$, and let us compute the left-hand side of (6.11). There are $q_{\langle N \rangle}$ admissible ways to colour the vertices $\{y_1, \ldots, y_N\}$, all of which are equivalent modulo permutations of $\{1, \ldots, q\}$; and with any such colouring fixed, the sum over colourings of V_0 yields precisely $Z_{G_0}(q, \{v_e\}, \{u_x^{(i)}\})$ with $u_x^{(i)} = v_{\langle x y_i \rangle}$. Since both sides of (6.11) are polynomials in q and the equality holds for infinitely many values of q, it must hold identically.

By applying Theorem 6.1 to the graph G_0, we can obtain new results for the ordinary Potts-model partition function of the graph G. In particular, given *any* graph $G = (V, E)$ and any vertex $y \in V$, we can interpret G as the join of $G_0 \equiv G \setminus y$ (the graph obtained from G by deleting y and all edges incident on it) and K_1. (Any edge $\langle xy \rangle$ that was not originally present in G can be introduced and given $v_{\langle xy \rangle} = 0$.) More generally, given any N-clique $y_1, \ldots, y_N$ of G, we can interpret G as the join of $G_0 \equiv G \setminus \{y_1, \ldots, y_N\}$ and K_N; however, for $N > 1$ we must require that $v_{\langle y_i y_j \rangle} = -1$ for each pair $i \neq j$. Theorem 6.1, Lemma 6.2(b,c) and Proposition 4.2 then yield an extension of Corollary 5.2. To state it, we first define $\widetilde{C} = \widetilde{C}(r, N, \bar{v})$ to be the smallest number for which

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n} \frac{\widetilde{C}+N}{(\widetilde{C}-N)^n} \, \bar{v}^{n-1} \, t_n^{(r)} \;\leqslant\; 1. \tag{6.12}$$

We then have the following.

Theorem 6.3. *Let $G = (V, E)$ be a loopless finite undirected graph in which all vertices have degree $\leqslant r$ except perhaps for an N-clique $y_1, \ldots, y_N$. Let G be equipped with complex edge weights $\{v_e\}_{e \in E}$ satisfying $|1 + v_e| \leqslant 1$ for all e and $v_{\langle y_i y_j \rangle} = -1$ for all $i \neq j$. Let $v_{\max} = \max_{e \in E_0} |v_e|$, where E_0 is the set of edges not incident on any of the vertices $y_1, \ldots, y_N$. Then,*

(a) *all the zeros of $Z_G(q, \{v_e\})$ lie in the disc $|q - N| < \widetilde{C}(r, N, v_{\max})$,*
(b) *if, in addition, all the edges e incident on any of the vertices $y_1, \ldots, y_N$ satisfy $-1 \leqslant v_e \leqslant 0$, then all the zeros of $Z_G(q, \{v_e\})$ lie in the disc $|q| < C(r)v_{\max} + N$.* □

And for the chromatic polynomials, we have, using Lemma 6.2(a), the following result.

Corollary 6.4. *Let $G = (V, E)$ be a loopless finite undirected graph in which all vertices, except perhaps one, have degree $\leqslant r$. Then all the zeros of $P_G(q)$ lie in the union of the discs $|q| < C(r)$ and $|q - 1| < C(r)$. In particular, they all lie in the disc $|q| < C(r) + 1$.* □

Thus, the zeros of $P_G(q)$ can be bounded in terms of the *second-largest* degree of a vertex in G. Such a result was recently conjectured by Shrock and Tsai [99]; see Section 7 for further discussion.

Let us note that the phrase 'except perhaps one' in Corollary 6.4 *cannot* be replaced here by 'except perhaps two', not even in the case $r = 2$. Indeed, I have elsewhere [103] constructed a family of planar graphs in which all but two vertices have degree 2 and whose chromatic roots are together dense in $\{q \in \mathbb{C}\colon |q - 1| \geqslant 1\}$. Modifications of these graphs show also [103] that the condition $v_{\langle y_i y_j \rangle} = -1$ for $i \neq j$ in Theorem 6.3 (when $N > 1$) cannot be relaxed.

Let us now give the proof of Lemma 6.2. We will need the following elementary fact.

Lemma 6.5. *Let z and a be complex numbers. Then*

$$|z + \lambda a|^2 \;\geqslant\; |z + a| \, (|z| - |a|) \tag{6.13}$$

whenever $0 \leqslant \lambda \leqslant 1$.

Proof. Simple calculus shows that

$$\min_{0 \leqslant \lambda \leqslant 1} |z + \lambda a|^2 \;=\; \begin{cases} |z|^2, & \text{if } \operatorname{Re}(z^* a) \geqslant 0, \\ |z|^2 - \dfrac{\operatorname{Re}(z^* a)^2}{|a|^2}, & \text{if } -|a|^2 \leqslant \operatorname{Re}(z^* a) \leqslant 0, \\ |z + a|^2, & \text{if } \operatorname{Re}(z^* a) \leqslant -|a|^2. \end{cases} \tag{6.14}$$

In the first two cases we clearly have

$$\min_{0\leqslant\lambda\leqslant 1} |z+\lambda a|^2 \;\geqslant\; |z|^2-|a|^2 \;=\; (|z|+|a|)(|z|-|a|) \;\geqslant\; |z+a|\,(|z|-|a|), \tag{6.15}$$

while in the third case we have

$$\min_{0\leqslant\lambda\leqslant 1} |z+\lambda a|^2 \;=\; |z+a|^2 \;\geqslant\; |z+a|\,(|z|-|a|). \tag{6.16}$$

□

Proof of Lemma 6.2. We use the shorthand $w(S) = \mathsf{q}(S)/\prod_{x\in S}\mathsf{q}(\{x\})$.

(a) Let $|S| = n$ and suppose that the sequence $(u_x^{(1)})_{x\in S}$ consists of m -1s and $n-m$ 0s. Then

$$w(S) \;=\; \begin{cases} q^{-(n-1)}, & \text{if } m=0,\\ q^{-(n-m)}(q-1)^{-(m-1)}, & \text{if } 1\leqslant m\leqslant n, \end{cases} \tag{6.17}$$

from which (6.8) immediately follows.

(b) Let $S = \{x_1,\dots,x_n\}$; we then have $\mathsf{q}(\{x_j\}) = q+u_j$ and $\mathsf{q}(S) = q+\bar u$ with $-N \leqslant \bar u \leqslant u_1,\dots,u_n \leqslant 0$. Now apply Lemma 6.5 with $z=q$, $a=\bar u$ and $\lambda = u_j/\bar u$ for $j=1,2$: we have

$$\left|\frac{\mathsf{q}(S)}{\mathsf{q}(\{x_1\})\,\mathsf{q}(\{x_2\})}\right| \;\leqslant\; \frac{1}{|q|-|\bar u|} \;\leqslant\; \frac{1}{|q|-N}, \tag{6.18}$$

and hence

$$|w(S)| \;\leqslant\; \frac{1}{|q|-N}\prod_{j=3}^{n}\frac{1}{|q+u_j|}, \tag{6.19}$$

which implies (6.9).

(c) This bound is trivially obtained by bounding the numerator and denominator separately. □

Remarks. **1.** For simplicity, I have not bothered to exploit the full strength of (6.19), which is quite a bit sharper than (6.9).

2. I am not entirely happy with Lemma 6.2, and I suspect that it can be improved. In particular, it is disconcerting that (6.9) is not uniformly stronger than (6.10), even though the corresponding hypothesis on the $u_x^{(i)}$ *is* strictly stronger.

7. Some conjectures and open questions

The bounds in this paper are, of course, far from sharp, and it is of some interest to speculate on what the best-possible results might be. Let us define

$$C_{\text{opt}}(r) \;=\; \max\{|q|:\, P_G(q)=0 \text{ for some loopless graph } G \text{ of maximum degree } r\}. \tag{7.1}$$

Table 2 The chromatic root of largest modulus for the complete bipartite graphs $K_{r,r}$ for $2 \leqslant r \leqslant 20$

r	q	$\lvert q \rvert$
2	$1.500000 \pm 0.866025\,i$	1.732051
3	$2.140640 \pm 1.948682\,i$	2.894772
4	$2.802489 \pm 3.097444\,i$	4.177093
5	$3.469365 \pm 4.291184\,i$	5.518221
6	$4.138450 \pm 5.516667\,i$	6.896404
7	$4.808805 \pm 6.765768\,i$	8.300616
8	$5.480007 \pm 8.033190\,i$	9.724331
9	$6.151830 \pm 9.315289\,i$	11.163316
10	$6.824136 \pm 10.609446\,i$	12.614641
11	$7.496833 \pm 11.913711\,i$	14.076186
12	$8.169855 \pm 13.226591\,i$	15.546358
13	$8.843156 \pm 14.546915\,i$	17.023928
14	$9.516697 \pm 15.873744\,i$	18.507925
15	$10.190450 \pm 17.206318\,i$	19.997566
16	$10.864391 \pm 18.544006\,i$	21.492211
17	$11.538501 \pm 19.886280\,i$	22.991328
18	$12.212764 \pm 21.232697\,i$	24.494469
19	$12.887165 \pm 22.582876\,i$	26.001256
20	$13.561693 \pm 23.936489\,i$	27.511362

The example of the complete graph K_{r+1} shows that $C_{\mathrm{opt}}(r) \geqslant r$. It is easy to see that $C_{\mathrm{opt}}(1) = 1$ and $C_{\mathrm{opt}}(2) = 2$; and there is some evidence that $C_{\mathrm{opt}}(3) = 3$.[11] But, at least for $r \geqslant 4$, $C_{\mathrm{opt}}(r)$ must in fact be strictly larger than r, as is shown by numerical computations on the complete bipartite graph $K_{r,r}$ (see Table 2).[12] Indeed, Gordon Royle (private communication) has conjectured that, for $r \geqslant 4$, $K_{r,r}$ is the graph of maximum degree r having the largest chromatic roots (in modulus). It would be useful to have a better understanding of the chromatic zeros of the complete bipartite graphs $K_{m,n}$. In particular, it would be useful to have a *proof* that $K_{r,r}$ has chromatic roots of magnitude $> r$ for all $r \geqslant 4$; and it would be valuable to understand the asymptotic behaviour of the chromatic roots of $K_{m,n}$ as $m, n \to \infty$ in various ways (*e.g.*, with $\alpha = m/n$ fixed).

Using the Dobrushin uniqueness theorem [46, 101], it can be proved [89] that for a countable graph G of maximum degree r, the q-state Potts-model Gibbs measure on G is unique for all integer $q > 2r$ whenever $-1 \leqslant v_e \leqslant 0$ for all edges e. Uniqueness of the

[11] Biggs, Damerell and Sands [12] have verified that the chromatic roots of all 3-regular graphs with $\leqslant 10$ vertices, as well as those of ladders ('prisms') and Möbius ladders of arbitrary length, lie in $|q| \leqslant 3$. Read and Royle [82] have extended this verification to all 3-regular graphs with $\leqslant 16$ vertices, as well as to some larger graphs.

[12] Recall [111, 68] that

$$P_{K_{m,n}}(q) \;=\; \sum_{k=0}^{m} S(m,k)\, q_{\langle k \rangle}\, (q-k)^n,$$

where $S(m,k)$ is the Stirling number of the second kind (the number of ways of partitioning a set of m elements into k nonempty subsets) [105, pp. 33–38] and $q_{\langle k \rangle} = q(q-1)\cdots(q-k+1)$. See Woodall [128, pp. 219–220] and Brown [18] for some properties of the chromatic zeros of the $K_{m,n}$.

Gibbs measure is one of several (inequivalent) notions of 'absence of phase transition' [46, 101]. It does not imply the analyticity of the free energy, but it does make it plausible.[13] Likewise, a result that holds for *integer* $q > q_0$ need not hold for all *real* $q > q_0$, much less for a complex neighbourhood of that real semi-axis; but it does suggest that such a result might be true. It is not unreasonable, therefore, to conjecture that there is a complex domain D_r containing the interval $(2r, \infty)$ of the real axis, such that $Z_G(q, \{v_e\}) \neq 0$ whenever $q \in D_r$, $-1 \leqslant v_e \leqslant 0$ for all edges e, and G has maximum degree $\leqslant r$. Indeed, it is quite possible that $D_r = \{q\colon |q| > 2r\}$ works; this would be a slight extension of the conjecture that $C_{\mathrm{opt}}(r) \leqslant 2r$.

We can pose these questions more generally as follows. Let $\mathscr{G}$ be a class of finite graphs, and let $\mathscr{V}$ be a subset of the complex plane. Then we can ask about the sets

$$S_1(\mathscr{G}, \mathscr{V}) = \bigcup_{G \in \mathscr{G}} \bigcup_{v \in \mathscr{V}} \{q \in \mathbb{C}\colon Z_G(q, v) = 0\}, \tag{7.2}$$

$$S_2(\mathscr{G}, \mathscr{V}) = \bigcup_{G \in \mathscr{G}} \bigcup_{\{v_e\}\colon v_e \in \mathscr{V}\ \forall e} \{q \in \mathbb{C}\colon Z_G(q, \{v_e\}) = 0\}. \tag{7.3}$$

Among the interesting cases are the chromatic polynomials $\mathscr{V} = \{-1\}$, the antiferromagnetic Potts models $\mathscr{V} = [-1, 0]$, and the complex antiferromagnetic Potts models $\mathscr{V} = A \equiv \{v \in \mathbb{C}\colon |1 + v| \leqslant 1\}$. Indeed, one moral of this paper is that some questions concerning chromatic polynomials are most naturally studied in the more general context of antiferromagnetic or complex antiferromagnetic Potts models (with not necessarily equal edge weights). In Corollary 5.2 we have shown that the set $S_2(\mathscr{G}_r, A)$ is bounded, where $\mathscr{G}_r$ is the set of all loopless graphs of maximum degree $\leqslant r$; and in Theorem 6.3 we have extended this to $S_2(\mathscr{G}'_r, A)$, where $\mathscr{G}'_r$ is the set of all loopless graphs of second-largest degree $\leqslant r$. But it would be interesting to examine in more detail the location of all these sets in the complex plane, and to prove sharper bounds.

Another direction in which the results of this paper could be extended is by finding a criterion *weaker* than bounded maximum degree (or bounded second-largest degree) under which the zeros of $P_G(q)$ and $Z_G(q, \{v_e\})$ could be shown to be bounded. An interesting idea was suggested very recently by Shrock and Tsai [99], who studied a variety of families of graphs and arrived at a conjecture that can be rephrased as follows. For $G = (V, E)$ and $x, y \in V$, define

$$\lambda(x, y) = \max \# \text{ of edge-disjoint paths from } x \text{ to } y \tag{7.4a}$$

$$= \min \# \text{ of edges separating } x \text{ from } y, \tag{7.4b}$$

and

$$\Lambda(G) = \max_{x \neq y} \lambda(x, y). \tag{7.5}$$

Clearly $\lambda(x, y) \leqslant \min[\deg(x), \deg(y)]$ and hence $\Lambda(G) \leqslant$ second-largest degree of G. Now let $\mathscr{G}_r^{\Lambda}$ be the set of all loopless graphs with $\Lambda(G) \leqslant r$. Then the conjecture is that the set

[13] Indeed, it was by meditating on possible extensions of the theorem in [89] that I was led to conjecture the results in this paper.

$S_2(\mathscr{G}_r^\Lambda, \mathscr{V})$ is bounded, where $\mathscr{V} = \{-1\}$ or $[-1, 0]$ or perhaps even A.[14] More generally, one could define $\lambda(x, y; \{v_e\})$ to be the maximum flow from x to y when $|v_e|$ is taken to be the capacity of edge e, and likewise $\Lambda(G, \{v_e\})$; this *might* lead to the appropriate extension of Corollary 5.5. This possible connection of chromatic-polynomial and Potts-model problems with max-flow problems is intriguing. Note that $\Lambda(G)$ and $\Lambda(G, \{v_e\})$ possess a 'naturalness' property that maximum degree and its relatives lack: namely, for any graph G with blocks $G_1, \ldots, G_b$, we have $\Lambda(G, \{v_e\}) = \max_{1 \leqslant i \leqslant b} \Lambda(G_i, \{v_e\})$; contrast this with Remark 1 after Corollary 5.5.

Acknowledgements

I wish to thank Norman Biggs, Jean Bricmont, Jason Brown, Roberto Fernández, Ira Gessel, David Jackson, Roman Kotecký, Antti Kupiainen, Neal Madras, Gordon Royle, Jesús Salas, Paul Seymour, Robert Shrock, Gordon Slade, Richard Stanley, Dave Wagner and Stu Whittington for valuable conversations and/or correspondence. I also wish to thank Tony Guttmann, Elliott Lieb and Nick Wormald for putting me in contact with some of these people. Finally, I wish to thank an anonymous referee for many helpful comments, particularly regarding the proofs of Propositions 4.2 and 4.5; and to thank Criel Merino for helping me notice a silly error in my original proof of Proposition 4.1.

This work was begun during a visit to Helsinki sponsored by the University of Helsinki and the Finnish National Board of Education. It was continued during a sabbatical in London where I had the pleasure of affiliation with both Imperial College and the London School of Economics. The final revisions were made during a visiting fellowship at All Souls College, Oxford. I wish to thank all these institutions – and most especially Antti Kupiainen, Pekka Elo, Chris Isham, Helena Cronin and John Cardy – for their warm hospitality.

References

[1] Abdesselam, A. and Rivasseau, V. (1995) Trees, forests and jungles: A botanical garden for cluster expansions. In *Constructive Physics* (V. Rivasseau, ed.), Vol. 446 of *Lecture Notes in Physics*, Springer, Berlin, pp. 7–36.

[2] Appel, K. and Haken, W. (1977) Every planar map is four colorable: Part 1, Discharging. *Illinois J. Math.* **21** 429–490.

[3] Appel, K., Haken, W. and Koch, J. (1977) Every planar map is four colorable: Part 2, Reducibility. *Illinois J. Math.* **21** 491–567.

[4] Appel, K. and Haken, W. (1989) Every planar map is four colorable. *Contemp. Math.* **98**. American Mathematical Society, Providence, RI.

[14] Shrock and Tsai [99] studied only the chromatic-polynomial case $\mathscr{V} = \{-1\}$, and proposed an even stronger result, based on the quantity

$$\Lambda_{\text{non-adj}}(G) = \max_{\substack{x \neq y \\ x, y \text{ not adjacent}}} \lambda(x, y).$$

But this cannot work for $v \neq -1$: a counterexample is obtained [103] by gluing together n copies of the cycle C_k (any fixed $k \geqslant 3$) along a single common edge and then taking $n \to \infty$.

[5] Ashkin, J. and Teller, E. (1943) Statistics of two-dimensional lattices with four components. *Phys. Rev.* **64** 178–184.

[6] Baxter, R. J. (1986) q colourings of the triangular lattice. *J. Phys. A* **19** 2821–2839.

[7] Baxter, R. J. (1987) Chromatic polynomials of large triangular lattices. *J. Phys. A* **20** 5241–5261.

[8] Beraha, S. and Kahane, J. (1979) Is the four-color conjecture almost false? *J. Combin. Theory Ser. B* **27** 1–12.

[9] Beraha, S., Kahane, J. and Weiss, N. J. (1980) Limits of chromatic zeros of some families of maps. *J. Combin. Theory Ser. B* **28** 52–65.

[10] Berman, G. and Tutte, W. T. (1969) The golden root of a chromatic polynomial. *J. Combin. Theory* **6** 301–302.

[11] Biggs, N. (1993) *Algebraic Graph Theory*, 2nd edn, Cambridge University Press, Cambridge/New York.

[12] Biggs, N. L., Damerell, R. M. and Sands, D. A. (1972) Recursive families of graphs. *J. Combin. Theory Ser. B* **12** 123–131.

[13] Birkhoff, G. D. (1912) A determinantal formula for the number of ways of coloring a map. *Ann. Math.* **14** 42–46.

[14] Birkhoff, G. D. and Lewis, D. C. (1946) Chromatic polynomials. *Trans. Amer. Math. Soc.* **60** 355–451.

[15] Björner, A. (1992) The homology and shellability of matroids and geometric lattices. In *Matroid Applications* (N. White, ed.), Vol. 40 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, Chapter 7, pp. 226–283.

[16] Borgs, C., Kotecký, R. and Miracle-Solé, S. (1991) Finite-size scaling for Potts models. *J. Statist. Phys.* **62** 529–551.

[17] Brenti, F., Royle, G. F. and Wagner, D. G. (1994) Location of zeros of chromatic and related polynomials of graphs. *Can. J. Math.* **46** 55–80.

[18] Brown, J. I. (1998) On the roots of chromatic polynomials. *J. Combin. Theory Ser. B* **72** 251–256.

[19] Brown, J. I. (1998) Chromatic polynomials and order ideals of monomials. *Discrete Math.* **189** 43–68.

[20] Brown, J. I. and Colbourn, C. J. (1992) Roots of the reliability polynomial. *SIAM J. Discrete Math.* **5** 571–585.

[21] Brush, S. G. (1967) History of the Lenz–Ising model. *Rev. Mod. Phys.* **39** 883–893.

[22] Brydges, D. C. (1986) A short course on cluster expansions. In *Phénomènes Critiques, Systèmes Aléatoires, Théories de Jauge Critical Phenomena, Random Systems, Gauge Theories* (K. Osterwalder and R. Stora, eds), Les Houches summer school, Session XLIII, 1984, Elsevier/North-Holland, Amsterdam, pp. 129–183.

[23] Brydges, D. C., Fröhlich, J. and Sokal, A. D. (1983) The random-walk representation of classical spin systems and correlation inequalities, II: The skeleton inequalities. *Commun. Math. Phys.* **91** 117–139.

[24] Brydges, D. C., Fröhlich, J. and Sokal, A. D. (1983) A new proof of the existence and nontriviality of the φ_2^4 and φ_3^4 quantum field theories. *Commun. Math. Phys.* **91** 141–186.

[25] Brydges, D. C. and Kennedy, T. (1987) Mayer expansions and the Hamilton–Jacobi equation. *J. Statist. Phys.* **48** 19–49.

[26] Brydges, D. C. and Martin, Ph. A. (1999) Coulomb systems at low density: A review. *J. Statist. Phys.* **96** 1163–1330; cond-mat/9904122 at `xxx.lanl.gov`.

[27] Brydges, D. C. and Wright, J. D. (1988) Mayer expansions and the Hamilton–Jacobi equation, II: Fermions, dimensional reduction formulas. *J. Statist. Phys.* **51** 435–456; erratum **97** 1027 (1999).

[28] Cammarota, C. (1982) Decay of correlations for infinite range interactions in unbounded spin systems. *Commun. Math. Phys.* **85** 517–528.

[29] Campanino, M., Capocaccia, D. and Tirozzi, B. (1979) The local central limit theorem for a Gibbs random field. *Commun. Math. Phys.* **70** 125–132.

[30] Chia, G. L. (1997) A bibliography on chromatic polynomials. *Discrete Math.* **172** 175–191.

[31] Chow, Y.-S. and Teicher, H. (1997) *Probability Theory: Independence, Interchangeability, Martingales*, 3rd edn, Springer, New York.

[32] Colbourn, C. J. (1987) *The Combinatorics of Network Reliability*, Oxford University Press, New York/Oxford.

[33] Derbez, E. and Slade, G. (1998) The scaling limit of lattice trees in high dimensions. *Commun. Math. Phys.* **193** 69–104.

[34] Dobrushin, R. L. (1996) Estimates of semi-invariants for the Ising model at low temperatures. In *Topics in Statistical and Theoretical Physics*, *American Mathematical Society Translations*, Ser. 2, **177** 59–81.

[35] Dobrushin, R. L. (1996) Perturbation methods of the theory of Gibbsian fields. In *Lectures on Probability Theory and Statistics* (P. Bernard, ed.), Ecole d'Eté de Probabilités de Saint-Flour XXIV – 1994, Vol. 1648 of *Lecture Notes in Mathematics*, Springer, Berlin, pp. 1–66.

[36] Domb, C. (1974) Configurational studies of the Potts models. *J. Phys. A* **7** 1335–1348.

[37] Duffin, R. J. (1965) Topology of series-parallel graphs. *J. Math. Anal. Appl.* **10** 303–318.

[38] Edwards, H., Hierons, R. and Jackson, B. (1998) The zero-free intervals for characteristic polynomials of matroids. *Combin. Probab. Comput.* **7** 153–165.

[39] Edwards, R. G. and Sokal, A. D. (1988) Generalization of the Fortuin–Kasteleyn–Swendsen–Wang representation and Monte Carlo algorithm. *Phys. Rev. D* **38** 2009–2012.

[40] Farrell, E. J. (1979) On a general class of graph polynomials. *J. Combin. Theory Ser. B* **26** 111–122.

[41] Farrell, E. J. (1980) Chromatic roots: some observations and conjectures. *Discrete Math.* **29** 161–167.

[42] Fernández, R., Fröhlich, J. and Sokal, A. D. (1992) *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, Springer, Berlin.

[43] Fisher, M. E. and Essam, J. W. (1961) Some cluster size and percolation problems. *J. Math. Phys.* **2** 609–619.

[44] Fortuin, C. M. and Kasteleyn, P. W. (1972) On the random-cluster model, I: Introduction and relation to other models. *Physica* **57** 536–564.

[45] Gaunt, D. S., Peard, P. J., Soteros, C. E. and Whittington, S. G. (1994) Relationships between growth constants for animals and trees. *J. Phys. A* **27** 7343–7351.

[46] Georgii, H.-O. (1988) *Gibbs Measures and Phase Transitions*, De Gruyter, Berlin.

[47] Gessel, I. M. and Sagan, B. E. (1996) The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions. *Electronic J. Combinatorics* **3**, no. 2, #R9.

[48] Goulden, I. P. and Jackson, D. M. (1983) *Combinatorial Enumeration*, Wiley, New York.

[49] Griffiths, R. B. (1967) Correlations in Ising ferromagnets, II: External magnetic fields. *J. Math. Phys.* **8** 484–489.

[50] Grimmett, G. (1978) The rank polynomials of large random lattices. *J. London Math. Soc.* **18** 567–575.

[51] Grimmett, G. (1995) The stochastic random-cluster process and the uniqueness of random-cluster measures. *Ann. Probab.* **23** 1461–1510.

[52] Hall, D. W., Siry, J. W. and Vanderslice, B. R. (1965) The chromatic polynomial of the truncated icosahedron. *Proc. Amer. Math. Soc.* **16** 620–628.

[53] Hara, T. and Slade, G. (1995) The self-avoiding-walk and percolation critical points in high dimensions. *Combin. Probab. Comput.* **4** 197–215.

[54] Ising, E. (1925) Beitrag zur Theorie des Ferromagnetismus. *Z. Phys.* **31** 253–258.

[55] Israel, R. B. (1979) *Convexity in the Theory of Lattice Gases*, Princeton University Press, Princeton, NJ.

[56] Jackson, B. (1993) A zero-free interval for chromatic polynomials of graphs. *Combin. Probab. Comput.* **2** 325–336.
[57] Janse van Rensburg, E. J. (1992) On the number of trees in $\mathscr{Z}^d$. *J. Phys. A* **25** 3523–3528.
[58] Kasteleyn, P. W. and Fortuin, C. M. (1969) Phase transitions in lattice systems with random local properties. *J. Phys. Soc. Japan* **26** (Suppl.) 11–14.
[59] Kennedy, T., Lieb, E. H. and Tasaki, H. (1988) A two-dimensional isotropic quantum antiferromagnet with unique disordered ground state. *J. Statist. Phys.* **53** 383–415.
[60] Klarner, D. A. (1967) Cell growth problems. *Can. J. Math.* **19** 851–863.
[61] Klein, D. J. (1981) Rigorous results for branched polymer models with excluded volume. *J. Chem. Phys.* **75** 5186–5189.
[62] Knuth, D. E. (1973) *The Art of Computer Programming*, Vol. 1, 2nd edn, Addison-Wesley, Reading, MA.
[63] Kobe, S. (1997) Ernst Ising: physicist and teacher. *J. Statist. Phys.* **88** 991–995.
[64] Koteckÿ, R., Laanait, L., Messager, A. and Ruiz, J. (1990) The q-state Potts model in the standard Pirogov–Sinai theory: Surface tensions and Wilson loops. *J. Statist. Phys.* **58** 199–248.
[65] Koteckÿ, R. and Preiss, D. (1986) Cluster expansion for abstract polymer models. *Commun. Math. Phys.* **103** 491–498.
[66] Laanait, L., Messager, A., Miracle-Solé, S., Ruiz, J. and Shlosman, S. (1991) Interfaces in the Potts model, I: Pirogov–Sinai theory of the Fortuin–Kasteleyn representation. *Commun. Math. Phys.* **140** 81–91.
[67] Laanait, L., Messager, A. and Ruiz, J. (1986) Phases coexistence and surface tensions for the Potts model. *Commun. Math. Phys.* **105** 527–545.
[68] Laskar, R. and Hare, W. R. (1975) The chromatic polynomial of a complete r-partite graph. *Amer. Math. Monthly* **82** 752–754.
[69] Lee, T. D. and Yang, C. N. (1952) Statistical theory of equations of state and phase transitions, II: Lattice gas and Ising model. *Phys. Rev.* **87** 410–419.
[70] Lenz, W. (1920) Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern. *Z. Phys.* **21** 613–615.
[71] Lieb, E. H. and Sokal, A. D. (1981) A general Lee–Yang theorem for one-component and multicomponent ferromagnets. *Commun. Math. Phys.* **80** 153–179.
[72] Liu, P. C. and Geldmacher, R. C. (1976) Graph reducibility. In *Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory, and Computing* (F. Hoffman *et al.*, eds), Louisiana State University, 1976. *Congressus Numerantium* **47** 433–445.
[73] Malyshev, V. A. (1979) Uniform cluster estimates for lattice models. *Commun. Math. Phys.* **64** 131–157.
[74] Martirosian, D. H. (1986) Translation invariant Gibbs states in the q-state Potts model. *Commun. Math. Phys.* **105** 281–290.
[75] Ore, O. (1967) *The Four Color Problem*, Academic Press, New York.
[76] Oxley, J. (1986) Graphs and series-parallel networks. In *Theory of Matroids* (N. White, ed.), Cambridge University Press, Cambridge, Chapter 6, pp. 97–126.
[77] Peard, P. J. and Gaunt, D. S. (1995) $1/d$-expansions for the free energy of lattice animal models of a self-interacting branched polymer. *J. Phys. A* **28** 6109–6124.
[78] Penrose, M. D. (1994) Self-avoiding walks and trees in spread-out lattices. *J. Statist. Phys.* **77** 3–15.
[79] Penrose, O. (1967) Convergence of fugacity expansions for classical systems. In *Statistical Mechanics: Foundations and Applications* (T. A. Bak, ed.), Benjamin, New York/Amsterdam, pp. 101–109.
[80] Potts, R. B. (1952) Some generalized order-disorder transformations. *Proc. Cambridge Philos. Soc.* **48** 106–109.
[81] Read, R. C. (1968) An introduction to chromatic polynomials. *J. Combin. Theory* **4** 52–71.

[82] Read, R. C. and Royle, G. F. (1991) Chromatic roots of families of graphs. In *Graph Theory, Combinatorics, and Applications* (Y. Alavi *et al.*, eds), Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs, Western Michigan University, 1988, Vol. 2, Wiley, New York, pp. 1009–1029.

[83] Read, R. C. and Tutte, W. T. (1988) Chromatic polynomials. In *Selected Topics in Graph Theory 3* (L. W. Beineke and R. J. Wilson, eds), Academic Press, London, pp. 15–42.

[84] Robertson, N., Sanders, D. P., Seymour, P. D. and Thomas, R. (1997) The four-colour theorem. *J. Combin. Theory Ser. B* **70** 2–44.

[85] Roček, M., Shrock, R. and Tsai, S.-H. (1998) Chromatic polynomials for families of strip graphs and their asymptotic limits. *Physica A* **252** 505–546; cond-mat/9712148 at `xxx.lanl.gov`.

[86] Roček, M., Shrock, R. and Tsai, S.-H. (1998) Chromatic polynomials for $J(\prod H)I$ strip graphs and their asymptotic limits. *Physica A* **259** 367–387; cond-mat/9807106 at `xxx.lanl.gov`.

[87] Rota, G.-C. (1964) On the foundations of combinatorial theory, I: Theory of Möbius functions. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **2** 340–368.

[88] Saaty, T. L. and Kainen, P. C. (1977) *The Four-Color Problem: Assaults and Conquest*, McGraw-Hill, New York.

[89] Salas, J. and Sokal, A. D. (1997) Absence of phase transition for antiferromagnetic Potts models via the Dobrushin uniqueness theorem. *J. Statist. Phys.* **86** 551–579; cond-mat/9603068 at `xxx.lanl.gov`.

[90] Salas, J. and Sokal, A.D. (2000) Transfer matrices and partition-function zeros for antiferromagnetic Potts models, I: General theory and square-lattice chromatic polynomial. Preprint, cond-mat/0004330 at `xxx.lanl.gov`; to appear in *J. Statist. Phys.*

[91] Seiler, E. G. (1982) *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Vol. 159 of *Lecture Notes in Physics*, Springer, Berlin/New York.

[92] Seppäläinen, T. (1998) Entropy for translation-invariant random-cluster measures. *Ann. Probab.* **26** 1139–1178.

[93] Shrock, R. and Tsai, S.-H. (1997) Asymptotic limits and zeros of chromatic polynomials and ground state entropy of Potts antiferromagnets. *Phys. Rev. E* **55** 5165–5178; cond-mat/9612249 at `xxx.lanl.gov`.

[94] Shrock, R. and Tsai, S.-H. (1997) Families of graphs with chromatic zeros lying on circles. *Phys. Rev. E* **56** 1342–1345; cond-mat/9703249 at `xxx.lanl.gov`.

[95] Shrock, R. and Tsai, S.-H. (1997) Families of graphs with $W_r(\{G\}, q)$ functions that are nonanalytic at $1/q = 0$. *Phys. Rev. E* **56** 3935–3943; cond-mat/9707096 at `xxx.lanl.gov`.

[96] Shrock, R. and Tsai, S.-H. (1997) Lower bounds and series for the ground state entropy of the Potts antiferromagnet on Archimedean lattices and their duals. *Phys. Rev. E* **56** 4111–4124; cond-mat/9707306 at `xxx.lanl.gov`.

[97] Shrock, R. and Tsai, S.-H. (1998) Ground state entropy of Potts antiferromagnets on homeomorphic families of strip graphs. *Physica A* **259** 315–348; cond-mat/9807105 at `xxx.lanl.gov`.

[98] Shrock, R. and Tsai, S.-H. (1998) Ground state degeneracy of Potts antiferromagnets: Cases with noncompact W boundaries having multiple points at $1/q = 0$. *J. Phys. A* **31** 9641–9665; cond-mat/9810057 at `xxx.lanl.gov`.

[99] Shrock, R. and Tsai, S.-H. (1999) Ground state degeneracy of Potts antiferromagnets: Homeomorphic classes with noncompact W boundaries. *Physica A* **265** 186–223; cond-mat/9811410 at `xxx.lanl.gov`.

[100] Shrock, R. and Tsai, S.-H. (1999) Ground state entropy of the Potts antiferromagnet on cyclic strip graphs. *J. Phys. A* **32** L195–L200; cond-mat/9903233 at `xxx.lanl.gov`.

[101] Simon, B. (1993) *The Statistical Mechanics of Lattice Gases*, Princeton University Press, Princeton NJ.

[102] Sokal, A. D. (1982) An alternate constructive approach to the φ_3^4 quantum field theory, and a possible destructive approach to φ_4^4. *Ann. Inst. Henri Poincaré A* **37** 317–398.

[103] Sokal, A. D. (2000) Chromatic roots are dense in the whole complex plane. Preprint, cond-mat/0012369 at `xxx.lanl.gov`.

[104] Sokal, A. D. (2000) Tree limit of Mayer expansion. In preparation.

[105] Stanley, R. P. (1986) *Enumerative Combinatorics*, Vol. I, Wadsworth and Brooks/Cole, Monterey, CA.

[106] Stanley, R. P. (1995) A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.* **111** 166–194.

[107] Stanley, R. P. (1996) *Combinatorics and Commutative Algebra*, 2nd edn, Birkhäuser, Boston.

[108] Stanley, R. P. (1998) Graph colorings and related symmetric functions: ideas and applications: A description of results, interesting applications, and notable open problems. *Discrete Math.* **193** 267–286.

[109] Stutz, C. and Williams, B. (1999) Ernst Ising (obituary). *Phys. Today* **52** (3) 106–108 (March).

[110] Suzuki, M. (1965) A theorem on the exact solution of a spin system with a finite magnetic field. *Phys. Lett.* **18** 233–234.

[111] Swenson, J. R. (1973) The chromatic polynomial of a complete bipartite graph. *Amer. Math. Monthly* **80** 797–798.

[112] Thomas, R. (1998) An update on the four-color theorem. *Notices Amer. Math. Soc.* **45** 848–859; `http://www.ams.org/notices/199807/thomas.pdf`.

[113] Thomassen, C. (1997) The zero-free intervals for chromatic polynomials of graphs. *Combin. Probab. Comput.* **6** 497–506.

[114] Tsai, S.-H. (1998) End graph effects on chromatic polynomials for strip graphs of lattices and their asymptotic limits. *Physica A* **259** 349–366; cond-mat/9807107 at `xxx.lanl.gov`.

[115] Tutte, W. T. (1947) A ring in graph theory. *Proc. Cambridge Philos. Soc.* **43** 26–40.

[116] Tutte, W. T. (1954) A contribution to the theory of chromatic polynomials. *Can. J. Math.* **6** 80–91.

[117] Tutte, W. T. (1970) On chromatic polynomials and the golden ratio. *J. Combin. Theory* **9** 289–296.

[118] Tutte, W. T. (1976) The dichromatic polynomial. In *Proceedings of the Fifth British Combinatorial Conference* (C. St.J. A. Nash-Williams and J. Sheehan, eds), University of Aberdeen, 1975. *Congressus Numerantium* **15** 605–635.

[119] Uhlenbeck, G. E. and Ford, G. W. (1962) The theory of linear graphs with applications to the theory of the virial development of the properties of gases. In *Studies in Statistical Mechanics* (J. de Boer and G. E. Uhlenbeck, eds), Vol. I, North-Holland, Amsterdam, pp. 119–211.

[120] Wagner, D. G. (2000) Zeros of reliability polynomials and f-vectors of matroids. *Combin. Probab. Comput.* **9** 167–190.

[121] Wakelin, C. D. and Woodall, D. R. (1992) Chromatic polynomials, polygon trees, and outerplanar graphs. *J. Graph Theory* **16** 459–466.

[122] Wald, J. A. and Colbourn, C. J. (1983) Steiner trees, partial 2-trees, and minimum IFI networks. *Networks* **13** 159–167.

[123] Welsh, D. J. A. (1993) *Complexity: Knots, Colourings, and Counting*, Vol. 186 of *London Mathematical Society Lecture Notes*, Cambridge University Press, Cambridge/New York.

[124] Whitney, H. (1932) A logical expansion in mathematics. *Bull. Amer. Math. Soc.* **38** 572–579.

[125] Whitney, H. (1932) On the coloring of graphs. *Ann. Math.* **33** 688–718.

[126] Whitney, H. (1933) A set of topological invariants for graphs. *Amer. J. Math.* **55** 231–235.

[127] Whittington, S. G. and Soteros, C. E. (1990) Lattice animals: Rigorous results and wild guesses. In *Disorder in Physical Systems* (G. R. Grimmett and D. J. A. Welsh, eds), Clarendon Press, Oxford, pp. 323–335.

[128] Woodall, D. R. (1977) Zeros of chromatic polynomials. In *Combinatorial Surveys: Proceedings*

of the Sixth British Combinatorial Conference (P. J. Cameron, ed.), Academic Press, London, pp. 199–223.

[129] Woodall, D. R. (1992) A zero-free interval for chromatic polynomials. *Discrete Math.* **101** 333–341.

[130] Woodall, D. R. (1997) The largest real zero of the chromatic polynomial. *Discrete Math.* **172** 141–153.

[131] Wu, F. Y. (1982) The Potts model. *Rev. Mod. Phys.* **54** 235–268. Erratum **55** (1983) 315.

[132] Wu, F. Y. (1984) Potts model of magnetism (invited). *J. Appl. Phys.* **55** 2421–2425.

[133] Yang, C. N. and Lee, T. D. (1952) Statistical theory of equations of state and phase transitions, I: Theory of condensation. *Phys. Rev.* **87** 404–409.

[134] Ziegler, G. M. (1995) *Lectures on Polytopes*, Springer, New York.