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On steady linear diffusion-driven flow

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Wunsch (1970) and Phillips (1970) (*Deep-Sea Res.* vol. 17, pp. 293, 435) showed that a temperature flux condition on a sloping non-slip surface in a stratified fluid can generate a slow steady upward flow along a thin ‘buoyancy layer’. Their analysis is extended here to the more-general case of steady flow in a contained fluid where buoyancy layers may expel or entrain fluid from their outer edge. A compatibility condition that relates the mass flux and temperature gradient along that edge is derived, and this allows the fluid recirculation and temperature perturbation to be determined in the broader-scale ‘outer flow’ region. The analysis applies when the Wunsch–Phillips parameter R is small, in the linear case for which the density variations are dominated by a constant vertical gradient.

1. Introduction

In concurrent and closely related studies, Wunsch (1970) (hereinafter referred to as W) and Phillips (1970) (hereinafter referred to as P) demonstrated that flow is generated in an otherwise quiescent but linearly stratified fluid if any of the boundary surfaces is sloping. Both studies proposed that under suitable conditions the no-flux boundary condition on the temperature (or more generally, density) on a sloping boundary curved the isolines locally and induced a slow steady flow in a thin layer along the slope. P presented some experimental results that demonstrated the phenomenon, albeit as an early stage of an unsteady flow that would eventually become homogeneous and stagnant.

This paper aims to clarify the steady form of this problem in a closed container in the presence of ongoing forcing. Woods (1991) and Quon (1989), for example, examined that problem but inherited some of the simplifying assumptions made by W and P – in particular that the induced thin layer does not entrain or expel fluid from its outer edge when the boundary has a constant slope. That assumption is appropriate in the idealized semi-infinite situation but in a closed container it can overconstrain the solution. In contrast, this paper seeks a more general solution in the layer, from which it is possible to determine the steady broader-scale flow that can be induced in a contained fluid.

For simplicity of presentation, the temperature is used here as a proxy for density, but the analysis applies equally to other sources of stable vertical density variations. (Hereinafter the opposite direction to the gravitational force is referred to as ‘vertical’.) W and P show that the key parameter for these flows is $R=\sqrt{\nu^*\kappa^*}/N^*L^{*2}$, in terms of the viscous and temperature diffusivities, the Brunt–Väisälä frequency, and a typical length scale. As in the previous studies it is assumed here that $R \ll 1$ but further it is assumed that the induced temperature variations in the fluid are much smaller than those of the background temperature profile. W noted the analogy between this configuration and the two-dimensional flow of a rapidly rotating fluid, as described by Veronis (1967) for example, and used that to assist in his analysis.

For steady 'diffusion-driven flows' in a closed container it is shown here that there are three main flow regions. The first, and most important, region is the so-called 'buoyancy layer' (Peacock, Stocker & Aristoff 2004) that was originally described by W and P in the context of a semi-infinite self-similar flow along a sloping planar boundary. Their analysis is extended here to account for leading-order temperature-gradient variations at the outer edge of the layer, and a 'compatibility condition' between the mass and temperature fluxes at that edge is derived, using a similar approach to that introduced by Jacobs (1964) for rotating flows. This provides an important link between the temperature flux boundary condition and the broader-scale flow, via the buoyancy-layer flow.

The second region, referred to as the 'outer flow' here, occupies the bulk of the container and the motion in it is vertical to leading order. For the self-similar flow considered by W and P it was assumed that the temperature gradient was constant in this region, with no induced flow – as is appropriate for the early stages of the unsteady contained flow. Subsequently, Quon (1983) considered a similar steady regime in a closed container under the same conditions, with the constant temperature gradient being maintained by specifying the temperature on part of the boundary. Woods (1991) recognized that the leading-order temperature gradient may vary within the outer flow, and accounted for some aspects of the mass flux closure, but effectively assumed that the vertical mass flux was constant in the outer region and therefore did not fully link its temperature-gradient variations to the mass flux into and out of the buoyancy layer.

In some circumstances the buoyancy layer may be required to gain or shed fluid over a small distance, for example when there are sudden changes in boundary conditions. This effectively creates a 'point' sink or source on the edge of the outer flow, the fluid from which is redistributed across the container via the third important region, referred to as an '$R^{1/3}$ layer' here. This layer connects that mass flux with that required for the outer flow, and is equivalent to the Stewartson $E^{1/3}$ layer in the rotating-flow context. As occurs in that situation, 'jump conditions' can be specified on the outer-flow variables across the layer, which enable the outer flow to be determined uniquely.

The scaling and governing equations for this problem are outlined in § 2, based on W and for the case of a flow dominated by a steady constant background temperature gradient. The three key regions of a steady diffusion-driven flow in a closed container are then described in § 3 based on an asymptotic analysis for $R \ll 1$. The analysis is illustrated in § 4 by considering the flow in a tilted square container, similar to that in Quon (1983) but with slightly different boundary conditions. The results of the analysis are then compared with numerical solutions of the full governing equations in § 5.

2. Configuration and governing equations

Consider a steady two-dimensional flow of a viscous stratified fluid in a closed container with a typical length scale L^*. A Cartesian coordinate system (x^*, y^*, z^*) is defined so that gravitational acceleration g^* is aligned with the negative z^*-direction. The velocity components in these coordinates are denoted as (u^*, v^*, w^*), and it is assumed both that $v^* = 0$ everywhere and that u^* and w^* are both independent of y^*. The overall temperature in the fluid is denoted by $T^*(x^*, z^*)$ and the Boussinesq approximation is used, based upon a constant background density ρ^*_{00}. The coefficient of thermal expansion α^*, thermal diffusivity κ^* and kinematic viscosity ν^* are all taken to be constant.

To keep the analysis linear, it is assumed that the temperature profile is dominated by a steady, stable and constant temperature gradient which is maintained by applying boundary conditions around the container that ensure that the background stratification is not reduced over time through diffusion. In effect, this assumption was also made by W and P for their semi-infinite flow region outside of the buoyancy layer. In particular, it is assumed that the background stratification throughout the container is only slightly perturbed by the induced motion, with temperature variations of relative size $\epsilon \ll 1$.

As in Veronis (1967) and W, lengths are scaled using L^*, so $(x, z) = (x^*, z^*)/L^*$, and the temperature scale $\Delta T^* = \frac{1}{4} L^* \mathrm{d}T_0^*/\mathrm{d}z^*$ is based upon the linear background stratification $T_0^*(z^*) = T_{00}^* + 4\Delta T^*(z^*/L^*)$. A non-dimensional temperature perturbation $T(x, z)$ is then defined through

$$T^*(x^*, z^*) = T_0^*(z^*) + 2(\Delta T^*)\epsilon\sqrt{\sigma}\ T(x, z), \tag{2.1}$$

where $\sigma = \nu^*/\kappa^*$ is the Prandtl number. The velocity is non-dimensionalized using the Brunt-Väisälä frequency $N^* = (g^*\alpha^*\Delta T^*/L^*)^{1/2}$ and length scale L^*, and is scaled by the perturbation magnitude ϵ so that $(u, w) = (u^*, w^*)/\epsilon N^* L^*$. Outside the buoyancy layer the pressure is predominantly hydrostatic so variations from the hydrostatic background pressure are non-dimensionalized using $\epsilon\sqrt{\sigma}\rho_{00}^*(N^*L^*)^2$, to yield a scaled pressure p.

As noted by W, the key dynamical parameter that arises from the above scaling is

$$R = \sqrt{\nu^*\kappa^*}/N^* L^{*2}, \tag{2.2}$$

and this is taken to be small here, with the Prandtl number σ assumed to be $O(1)$ with respect to R. The magnitude of ϵ is chosen to ensure that the background temperature gradient remains constant to leading order and that the governing equations are linear everywhere, which requires that $\epsilon\sqrt{\sigma} \ll 1$.

Under these assumptions, the governing equations are identical to those in W, namely

$$0 = -p_x + R\nabla^2 u, \quad -2T = -p_z + R\nabla^2 w \quad \text{and} \quad 2w = R\nabla^2 T, \tag{2.3a–c}$$

with the continuity equation $u_x + w_z = 0$. A streamfunction ψ can also be defined with

$$u = \partial\psi/\partial z \qquad \text{and} \qquad w = -\partial\psi/\partial x \tag{2.4}$$

so that continuity is automatically satisfied. The similarity between these equations and those for a two-dimensional rapidly-rotating flow at zero Rossby number was noted by Veronis (1967). That analogy also extends to some key regions of the flow for $R \ll 1$, with R corresponding to the Ekman number E in a rotating flow.

In this paper, specifying $O(1)$ values of $T_n = \partial T/\partial n$ at all boundary points provides the 'driving force' for the steady flow, where n is the outward normal. Non-slip boundary conditions are used for the velocity components (u, w), which in turn implies (via (2.3c)) that the average value of $\partial T/\partial n$ around the boundary must be zero. The average value of T around the boundary is also set to zero. In contrast, Quon (1983) considered a mixture of boundary conditions on T with, effectively, $T = 0$ on some parts of the boundary.

Note that the boundary conditions impose no separate requirements on the magnitude of ϵ so its size is determined solely by the assumption above that the linearized equations (2.3) remain valid throughout the entire flow. That limitation $\epsilon \ll 1$ does however mean that the fully insulating condition $\partial T^*/\partial n^* = 0$ cannot be applied on any boundary since that would require $\epsilon\sqrt{\sigma} = O(1)$ – except in the special case when T is constant throughout the bulk of the container. As a result, a non-zero overall temperature flux $\partial T^*/\partial n^*$ is effectively applied all around the boundary here as a means of both maintaining the steady constant background temperature gradient and providing a small ongoing driving force for the buoyancy-layer flow.

3. Flow regions

There are three key flow regions for a steady 'diffusion-driven flow' in a closed container. On vertical or sloping boundaries there are thin 'buoyancy layers', on horizontal lines there can be thin '$R1/3$ layers', while the remainder is described as the 'outer flow'.

3.1. *The buoyancy layer*

This is arguably the most important region as it provides the driving force behind the flow, via the specified flux condition on T. It has thickness $O(R^{1/2})$ and can form on any surface that is not horizontal, including a planar surface with constant slope α. It can also exist on any vertical surface, where it may act as a means of mass redistribution.

As in W, the flow above a sloping planar surface is considered when the surface is at an angle α, measured here as anticlockwise from horizontal. A rotated coordinate system $(\hat{x}, \hat{z})$ can be defined, with $\hat{x} = x\cos\alpha + z\sin\alpha$ and $\hat{z} = -x\sin\alpha + z\cos\alpha$, and corresponding velocity components $(\hat{u}, \hat{w})$. The equations of motion (2.3) become

$$-2T\sin\alpha = -p_{\hat{x}} + R\hat{\nabla}^2\hat{u}, \qquad -2T\cos\alpha = -p_{\hat{z}} + R\hat{\nabla}^2\hat{w}, \qquad (3.1a, b)$$

$$2(\hat{u}\sin\alpha + \hat{w}\cos\alpha) = R\hat{\nabla}^2 T, \qquad (3.2)$$

with $\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0$. At $\hat{z} = 0$ the boundary conditions $\hat{u} = \hat{w} = 0$ are applied, along with a steady forcing condition $\partial T/\partial\hat{z} = -T_n(\hat{x})$ for a given function T_n.

For $R \ll 1$ and $\alpha \neq 0$ an expansion of the solution is sought of the form

$$p = \hat{p}_0(\hat{x}, \zeta) + R^{1/2}\hat{p}_1(\hat{x}, \zeta) + \cdots, \qquad \hat{u} = R^{1/2}\hat{u}_1(\hat{x}, \zeta) + R\hat{u}_2(\hat{x}, \zeta) + \cdots, \quad (3.3a)$$

$$T = \hat{T}_0(\hat{x}, \zeta) + R^{1/2}\hat{T}_1(\hat{x}, \zeta) + \cdots, \qquad \hat{w} = R\hat{w}_2(\hat{x}, \zeta) + R^{3/2}\hat{w}_3(\hat{x}, \zeta) + \cdots, \quad (3.3b)$$

using the boundary-layer coordinate $\zeta = \hat{z}/R^{1/2}$. The inclusion in this expansion of the zeroth-order terms in p and T will be justified *a posteriori* when the outer flow is considered below, but otherwise this form of solution is equivalent to that used by W. From the leading-order terms in (3.1*a*) and (3.1*b*), both $\hat{p}_0$ and $\hat{T}_0$ are functions of $\hat{x}$ only. At the next order, integration of (3.1*b*) yields $\hat{p}_1(\hat{x}, \zeta) = 2\zeta\ \hat{T}_0(\hat{x})\cos\alpha + \hat{p}_{10}(\hat{x})$, where $\hat{p}_{10}(\hat{x})$ is yet to be determined, while (3.1*a*) and (3.2) give

$$-2\hat{T}_1\sin\alpha = -\frac{\partial\hat{p}_1}{\partial\hat{x}} + \frac{\partial^2\hat{u}_1}{\partial\zeta^2} \quad \text{and} \quad 2\hat{u}_1\sin\alpha = \frac{\partial^2\hat{T}_1}{\partial\zeta^2}. \qquad (3.4a, b)$$

The solutions of these that satisfy the boundary condition $\hat{u}_1 = 0$ on $\zeta = 0$, and have no exponential growing terms for large ζ, are

$$\hat{u}_1(\hat{x}, \zeta) = \hat{U}_1(\hat{x})\exp(-\zeta\sqrt{|\sin\alpha|})\sin(\zeta\sqrt{|\sin\alpha|}), \qquad (3.5)$$

$$\hat{T}_1(\hat{x}, \zeta) = \operatorname{sgn}\alpha\ \hat{U}_1(\hat{x})\exp(-\zeta\sqrt{|\sin\alpha|})\cos(\zeta\sqrt{|\sin\alpha|}) + \frac{1}{2}\frac{\partial\hat{p}_1}{\partial\hat{x}}(\hat{x}, \zeta)\operatorname{cosec}\alpha. \quad (3.6)$$

Key differences from the solutions in W (apart from the sign of α) are the $\hat{x}$-dependence of $\hat{U}_1(\hat{x})$ and the $\partial \hat{p}_1/\partial \hat{x}$ term in (3.6), which allows for a more-general form for $\hat{T}_0(\hat{x})$.

The boundary condition $\partial \hat{T}_1/\partial \zeta = -T_n$, along with $\hat{p}_1(\hat{x}, \zeta)$, determines $\hat{U}_1(\hat{x})$ in (3.6) in terms of both $T_n(\hat{x})$ and $\hat{T}_0'(\hat{x})$. Introducing a scaled streamfunction $\hat{\psi}_2$ such that $\psi = R\hat{\psi}_2(\hat{x}, \zeta) + O(R^{3/2})$ and $\hat{u}_1 = \partial \hat{\psi}_2/\partial \zeta$, with $\hat{\psi}_2 = 0$ at $\zeta = 0$, the $O(R)$ mass flux $\hat{\psi}_2(\hat{x}, \infty)$ of the layer can be expressed in terms of $\hat{U}_1(\hat{x})$ from (3.5). Together, these give

$$2\hat{\psi}_2(\hat{x}, \infty)\sin\alpha - \hat{T}_0'(\hat{x})\cot\alpha = T_n(\hat{x}) \tag{3.7}$$

for the buoyancy-layer flow. This is equivalent to the 'Ekman compatibility condition' originally derived by Jacobs (1964), which has been used extensively for rapidly rotating flows (see, for example, Stewartson 1966). The advantage of this condition is that it connects mass and temperature fluxes at the outer edge of the buoyancy layer to the boundary condition T_n, without requiring the buoyancy-layer flow to be evaluated.

If T_n is constant along a sloping boundary then both $\hat{T}_0'$ and $\hat{\psi}_2$ can be independent of $\hat{x}$ along the outer edge of the layer, as in W and P, but generally they will both vary with $\hat{x}$. To determine them uniquely the mass flux in the broader-scale outer flow must be considered, which depends in turn on the flow at the other boundaries of the container.

In the original variables (x, z), (3.7) implies that the values of ψ and T at the outer edge of the buoyancy layer are related through

$$\psi = \tfrac{1}{2}R\left[\frac{\partial T}{\partial x}\cot^2\alpha + \frac{\partial T}{\partial z}\cot\alpha + T_n \operatorname{cosec}\alpha\right] + O(R^{3/2}), \tag{3.8}$$

where T_n is the given boundary condition. This 'compatibility condition' extends the mass-flux analysis in § 3 of P to the case where the external temperature gradient $\partial T/\partial z$ is non-zero and also T_n is not determined by requiring a zero overall temperature flux at the boundary. It can be simplified further when $\partial T/\partial x = 0$ to leading order, as below.

The condition (3.8) also applies on vertical boundaries ($|\alpha| = \pi/2$), in which case the flux along the buoyancy layer is proportional to T_n, with no flow if $T_n = 0$. It applies for $\alpha < 0$ as well, but not on horizontal boundaries, or indeed for $\alpha = O(R^{1/3})$. It can also be used when α varies on an $O(1)$ length scale, and for unsteady flows that evolve over a sufficiently-long time scale (Page & Johnson 2008).

3.2. *The outer flow*

The flux condition (3.8) indicates that, under some conditions at least, an $O(R)$ flux can be generated at the outer edge of the buoyancy layer. This in turn would induce $O(R)$ velocities (u, w) in the region abutting the buoyancy layer, along with temperature perturbations T of $O(1)$. If (x, z) are $O(1)$ in this 'outer flow' region then (2.3) reduce to

$$0 = -p_x, \quad -2T = -p_z \quad \text{and} \quad 2w = R\nabla^2 T \tag{3.9a–c}$$

to leading order. The solution can be expanded as

$$p = p_0(x, z) + R^{1/2}p_1(x, z) + \cdots, \quad T = T_0(x, z) + R^{1/2}T_1(x, z) + \cdots, \tag{3.10a}$$

$$w = Rw_2(x, z) + R^{3/2}w_3(x, z) + \cdots, \quad \psi = R\psi_2(x, z) + R^{3/2}\psi_3(x, z) + \cdots, \tag{3.10b}$$

where ψ is given by (2.4) and, from (3.9a,b), both p_0 and T_0 are independent of x. The leading-order solution can then be written in terms of two functions $f(z)$ and

$g(z)$ as

$$T_0 = f(z), \quad w_2 = \tfrac{1}{2} f''(z) \quad \text{and} \quad \psi_2 = -\tfrac{1}{2} x f''(z) + g(z). \tag{3.11a–c}$$

W and P take $T_0 = 0$, giving $f'' = 0$. Further, as $T_n = 2\cos\alpha/\epsilon\sqrt{\sigma}$ is constant in their case, (3.8) implies that ψ_2 is also constant. As a result, no motion is induced in the semi-infinite outer flow for their configuration. In a closed container, however, at each value of z there are two boundaries in x, with conditions (3.8) to be satisfied at each. This pair of constraints determines both f' and g at each z, at least to within one arbitrary constant, and other forms of T_0 and ψ_2 are possible, as in § 4.2.

Woods (1991) described some properties of the flow in this 'interior region' but much of his analysis assumed that ψ_2 was constant on each boundary, via his condition (2.2), and therefore his vertical velocity (2.5) differs from that above. For the situations considered by Quon (1983), where $T = 0$ on one boundary, the adjacent buoyancy-layer flow has $\hat{T}_0 = 0$ and hence $T_0 = 0$ in the outer flow. Details of this case are given in Page (2008) but in essence ψ_2 for the outer flow is determined by T_n at the other boundary.

As noted in § 2, the only requirement on ϵ for the steady analysis above to remain valid is that $\epsilon\sqrt{\sigma} \ll 1$. Once that condition is violated the left-hand side of (3.9c) must be modified, as described in Page & Johnson (2008).

3.3. *The $R^{1/3}$ layer*

Veronis (1967), W and Quon (1983) noted that 'Stewartson layers' of thickness $O(R^{1/3})$ and $O(R^{1/4})$ can exist in these flows under some circumstances, by analogy with the equivalent rotating-flow layers considered originally by Stewartson (1957). Perhaps surprisingly, for the boundary conditions considered here the outer layer in § 3.2 represents the equivalent of the Stewartson $E^{1/4}$ layer, even though it has $O(1)$ thickness.

Thin horizontal layers that are equivalent to Stewartson $E^{1/3}$ layers can occur in the current context when there is a fluid source (or sink) on a vertical (or sloping) boundary. For example, Koh (1966) showed that a two-dimensional horizontal 'jet-like' flow will develop in a viscous stratified fluid near a line mass sink, with a similarity structure close to the sink. Owing to the linearity of the equations, these results apply also to a source.

Moore & Saffman (1969) undertook a detailed analysis of $E^{1/3}$ layers for a rapidly-rotating fluid and analysed the strength and type of singularities that are allowable in such layers. They confirmed the conjecture by Stewartson (1966) that some outer-flow variables can be continuous across the layer. Details of the equivalent analysis in the current context of an $R^{1/3}$ layer are given in Page (2008), where it is demonstrated that the layers enable a source (or sink) of fluid on the boundary to be redistributed throughout the container along lines of constant z. It is also shown that

$$\text{both } T \text{ and } \partial T/\partial z \text{ must be continuous across the } R^{1/3} \text{ layer} \tag{3.12}$$

but that w may be discontinuous. Along with the flux condition (3.8) at x boundaries, this is sufficient to determine the outer flow uniquely when $R^{1/3}$ layers are present.

4. Flow in a tilted square container

As an illustration of the solutions and principles outlined in § 3, the outer flow in a square container that has been tilted 45° is considered. This is similar to one of the cases considered by Quon (1983), and it has the advantage of being a simple geometry for which all boundaries are sloping. Specifically, the square here is taken to have sides of length $\sqrt{2}$ so that the boundaries are at $z = 1 \pm (1 - |x|)$ for $|x| \leqslant 1$.

The flow is forced by using two different cases of the boundary condition $\partial T/\partial n = T_n$ around the container, with further cases considered in Page (2008). The non-slip boundary conditions $u = w = 0$ are imposed and, as noted in § 2, for consistency this requires that T_n must have an average value of zero. The temperature T is also taken to have an average value of zero around the boundary, in order to fix the arbitrary additive constant. The boundary conditions are chosen to ensure antisymmetry of T and ψ about $z = 1$ and therefore only the solution for $z < 1$ is described, with solutions for $z > 1$ obtained by replacing z with $(2 - z)$ and changing the signs of T and ψ.

4.1. *Linear temperature variation*

An interesting set of boundary conditions to consider are those which are locally equivalent to the problem posed by W and P in a semi-infinite fluid. From § 2, the overall scaled temperature in the fluid is $4z + 2\epsilon\sqrt{\sigma}T$, and the normal gradient of that can be set to zero on all four boundaries by considering the problem with $T_n = 1$ on the two boundaries with $z < 1$ (and $T_n = -1$ for $z > 1$). A suitable steady solution to this problem is that the internal temperature gradient T_0' is constant everywhere, and hence through (3.11*c*) that there is no outer flow, but it is still instructive to use this case to illustrate the analysis in § 3, as well as to confirm that the solution is unique.

To deduce the outer flow, consider first the case of $T_n = 1$ along $z = -x$ for $-1 < x < 0$, where the compatibility condition (3.8) with $\alpha = -\frac{1}{4}\pi$ gives that

$$\psi_2(-z, z) = \tfrac{1}{2}\left[T_0'(z)\cot\left(-\tfrac{1}{4}\pi\right) + (1)\operatorname{cosec}\left(-\tfrac{1}{4}\pi\right)\right] = -\tfrac{1}{2}[f'(z) + \sqrt{2}]. \tag{4.1}$$

Along the boundary at $z = x$ for $0 < x < 1$, of slope $\alpha = \frac{1}{4}\pi$, the same condition gives

$$\psi_2(z, z) = \tfrac{1}{2}\left[T_0'(z)\cot\tfrac{1}{4}\pi + (1)\operatorname{cosec}\tfrac{1}{4}\pi\right] = \tfrac{1}{2}[f'(z) + \sqrt{2}]. \tag{4.2}$$

As noted in § 2, the outer-flow vertical velocity w_2 is independent of x. Using the two conditions above it is therefore given by

$$w_2(z) = -\frac{\partial\psi_2}{\partial x} = -\frac{\psi_2(z, z) - \psi_2(-z, z)}{z - (-z)} = -\tfrac{1}{2}[f'(z) + \sqrt{2}]/z. \tag{4.3}$$

From (3.11*b*), w_2 is equal to $\frac{1}{2}f''(z)$ and so it follows that f' must satisfy $zf'' + f' = -\sqrt{2}$ for $0 < z < 1$. This has general solution $f'(z) = -\sqrt{2} + c/z$ for any constant c but for T to be finite at the bottom corner, where $z = 0$, then c must be zero.

To complete the determination of the outer flow, the unknown function $g(z)$ in (3.11*c*) can be found using either of the two boundary conditions. For example (4.2) gives

$$\psi_2(z, z) = -\tfrac{1}{2}(z)f''(z) + g(z) = \tfrac{1}{2}[f'(z) + \sqrt{2}] \tag{4.4}$$

and since $zf'' + f' = -\sqrt{2}$ then $g(z) = 0$ over $0 < z < 1$.

The unique solution with a zero average temperature on the boundary is therefore

$$T_0(z) = \sqrt{2}(1 - z) \quad \text{and} \quad \psi_2(x, z) = 0. \tag{4.5}$$

Generalizing this result, when the imposed temperature flux T_n on the boundaries of any closed container is in balance with a linear temperature profile T_0, then the only steady solution for the outer flow has no motion. Further, there is no mass flux along the buoyancy layer and hence no upslope flow.

These boundary conditions, when combined with the background stratification, yield an overall normal temperature gradient of zero on all four sides when $\epsilon = \sqrt{2/\sigma}$. While that case is beyond the strict applicability of the theory here, it is anticipated that the solution will be the same as above so that the overall density field is homogeneous everywhere with no motion. When comparing this to the experimental observations of P and others, it is important to note that their observed upslope flows were unsteady and will disappear once the problem has fully diffused and reached equilibrium. Despite that, non-trivial steady solutions for small values of ϵ can be generated for other choices of T_n.

4.2. *Piecewise constant* T_n

When the imposed temperature flux T_n varies around the boundary then it may not always be in balance with the corresponding value of T_0' in the outer flow, and hence a non-zero outer flow can be generated by the varying efflux from the buoyancy layer, via (3.8). Conversely, once there is vertical motion in the outer flow then (3.11*c*) implies that f'' must be non-zero and hence that T_0 must deviate from a linear variation in z. A example of this situation is when $T_n = 0$ on the lower part of the boundary for $z < \frac{1}{2}$, with $T_n = 1$ for $\frac{1}{2} < z < 1$, $T_n = -1$ for $1 < z < \frac{3}{2}$, and $T_n = 0$ for $z > \frac{3}{2}$ (to maintain antisymmetry). These conditions will induce outer-flow circulation since were T_0' to balance T_n in each separate region of the outer flow, as occurs in §4.1, then T_0' would be discontinuous across $z = \frac{1}{2}, \frac{3}{2}$ and vertical flow would be generated.

A similar analysis to §4.1 shows that $zf'' + f' = 0$ for $0 < z < \frac{1}{2}$ and so $f'(z) = 0$ to avoid a singularity at $z = 0$. For $\frac{1}{2} < z < 1$, f' satisfies $zf'' + f' = -\sqrt{2}$ and the solution is $f'(z) = -\sqrt{2} + c_2/z$, as in §4.1, but c_2 need not be zero in this case. Instead, in accordance with the conditions (3.12) across an $R^{1/3}$ layer, c_2 is determined by the requirement that $f'(z)$ is continuous at $z = \frac{1}{2}$, so that $c_2 = 1/\sqrt{2}$.

As in §4.1, the condition (3.8) on $x = z$ can be used to find $g(z)$ in (3.11), since

$$\psi_2(z, z) = \tfrac{1}{2}[f'(z) + \sqrt{2}T_n] = -\tfrac{1}{2}(z)f''(z) + g(z), \qquad (4.6)$$

and for both cases of T_n this yields $g(z) = 0$, using the differential equations for f' in each region. The solution overall is therefore

$$T_0(z) = (1 - \ln 2)/\sqrt{2} \quad \text{and} \quad \psi_2(x, z) = 0 \quad \text{for } 0 < z < \tfrac{1}{2}, \qquad (4.7a)$$

$$T_0(z) = \sqrt{2}(1 - z) + \ln z/\sqrt{2} \quad \text{and} \quad \psi_2(x, z) = x/(2\sqrt{2}z^2) \quad \text{for } \tfrac{1}{2} < z < 1. \qquad (4.7b)$$

Clearly there is no flow in either the buoyancy layer or the outer flow for $0 < z < \frac{1}{2}$. For $\frac{1}{2} < z < 1$, however, the temperature gradient in the outer flow decreases with z and reaches a minimum at $z = 1^-$. The vertical velocity w_2 is negative throughout the outer flow, including above $z = \frac{1}{2}$, and decreases with z, without however reaching zero at $z = 1^-$. Correspondingly, the buoyancy layers on both boundaries are entraining fluid from the outer flow, via the negative value of w_2 at their outer edge, but note that they already contain a non-zero flux at their starting position of $z = \frac{1}{2}$.

The form of solution above differs from that in Woods (1991), which included leading-order density variations but effectively assumed that $w_2 \propto 1/z$ in this case (via his (2.3)) and also deduced that $zT_0'(z)$ is constant (see his (2.12)) for a steady flow. In contrast, $w_2 \propto 1/z^2$ here and T_0 is more complicated (although it does also include a log term). The differences arise from the varying entrainment into the buoyancy layer.

The buoyancy layers start with non-zero flux and gain fluid as they move up the sloping surfaces for $z < 1$. Antisymmetry requires that $\psi = 0$ at $z = 1$ and so the layers expel all their fluid over a short distance as $z \to 1$. This is exactly the situation discussed in relation to the $R^{1/3}$ layers in § 3.3, and therefore such a layer is expected along $z = 1$, over $-1 < x < 1$. This layer will expel fluid at $z = 1^-$ and feed the uniform downward flow w_2 in the outer flow for $\frac{1}{2} < z < 1$. Similarly, near $z = \frac{1}{2}$ another $R^{1/3}$ layer will entrain the remaining downward flux from above and redistribute it into the start of both buoyancy layers at $x = \pm\frac{1}{2}$. These $R^{1/3}$ layers complete the leading-order structure for the steady flow field and enable the fluid which is forced up the buoyancy layers to recirculate throughout the closed container.

The boundary conditions here have been chosen to demonstrate the three key regions of the flow. They do however also allow a qualitative interpretation of oceanic shelf circulation, as most of the flow field would be unchanged if the region $z < \frac{1}{2}$ were replaced by deep water with vertical sides, along with a suitable surface condition at $z = 1$.

5. Numerical results for the tilted square container

To substantiate the form of the recirculation indicated by the analysis of § 4, some simple numerical calculations were performed with the full equations (2.3*a–c*) solved on a uniform square grid with up to $n_x = n_z = 200$ grid intervals in each direction. Standard second-order finite-difference equations were used for the derivatives and Laplacian terms in the equivalent streamfunction–vorticity formulation. For fewer than 80 grid intervals the resulting system of coupled linear equations were solved directly using MATLAB on a general-purpose desktop PC. Although this is not the most efficient approach, the run times were modest and the key features of the flow were readily apparent. For very small values of R a finer grid was needed and a time-dependent iterative approach was used.

The results of these calculations were sufficiently well-resolved for small values of R to clearly indicate the key features of the two types of layers and the broader-scale outer flow. The streamfunction field for the case in § 4.2 is shown in figure 1(*a*), based on numerical solution for $R = 0.001$, and the corresponding outer-flow solution (4.7) is shown in figure 1(*c*). Clearly the latter represents the bulk of the flow, while the $R^{1/3}$-layer circulation is apparent in the former, with mass distributed between the ends of the bouyancy layer and the outer flow. A surface plot of the steady temperature field is shown for $R = 0.001$ in figure 1(*b*) and clearly it is independent of x in most of the container. In figure 1(*d*) the cross-sectional values of $\partial T/\partial z$ at $x = 0$ are shown for various values of R, along with the values for the outer-flow solution, and the latter is approached as R decreases. Further details of the numerical method, as well as results for other boundary conditions, are provided in Page (2008).

6. Conclusions

The analysis presented here demonstrates a consistent flow structure for the recirculation of a steady contained flow that is generated through the diffusion-driven mechanism originally outlined by W and P. The analysis allows the buoyancy layer to entrain or expel fluid at its outer edge, rather than having constant mass flux, and this is linked to temperature-gradient variations in the broader-scale flow.

The linearization $\epsilon \ll 1$ restricts the present analysis from applying directly to the closed-container version of the problem considered by W and P. The varying mass

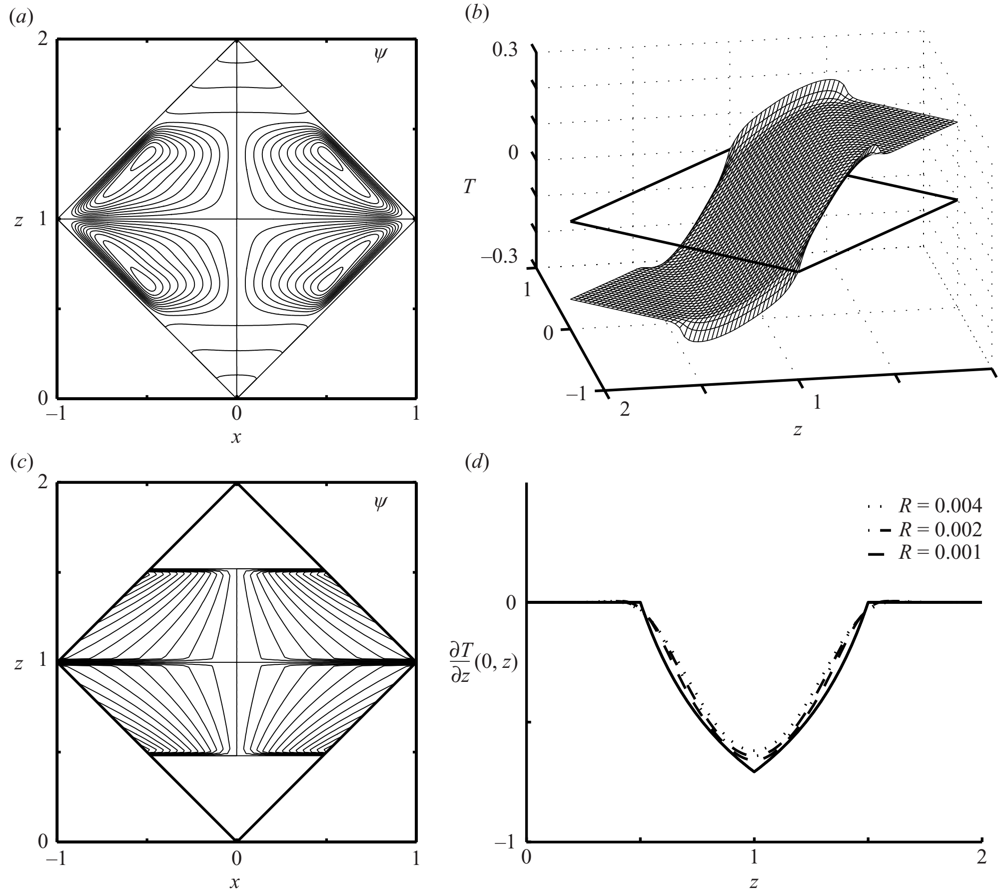

FIGURE 1. (a) The streamfunction ψ and (b) temperature T in a tilted square container for the boundary conditions in §4.2, based on numerical solutions of (2.3) for $R = 0.001$. (c) The outer-flow solution (4.7) with the $R^{1/3}$ layers at $z = 1/2, 1, 3/2$ shown thickened. (d) Values of $\partial T/\partial z(0, z)$ for $R = 0.004$, 0.002 and 0.001 compared with the outer-flow solution (solid line).

flux in the buoyancy layer generates an $O(1)$ temperature gradient in the outer flow, which means that w in (2.3c) will be multipied by a function of z. This nonlinear case is examined in Page & Johnson (2008), where it is shown that the buoyancy layer has non-constant thickness but that the structure of the flow field is similar to that for $\epsilon \ll 1$.

The discussion here has concentrated on the effect of the finite horizontal extent of the container in forcing an outer recirculating flow but it is actually the finite vertical extent, or depth, that causes the buoyancy layer to depend on z (and so x). In particular, Page & Johnson (2008) analyse the flow in a wide container and show that the sidewall layers are independent of each other when the width to depth ratio is large compared to R^{-1}.

As noted in §1, the experiments that demonstrate this phenomenon have in effect observed it during the initial stages of an unsteady linearly stratified flow. That flow would eventually settle down to a homogenous density field with no motion – as happens for the case in §4.1 when $\epsilon = \sqrt{2/\sigma}$. A time-dependent analysis of that situation is described in Page & Johnson (2008), including for the case of a closed container with a variably sloping bottom used in Peacock *et al.* (2004).

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