

PARALLELIZABLE MANIFOLDS AND THE FUNDAMENTAL GROUP

F. E. A. JOHNSON AND J. P. WALTON

§0. *Introduction.* Low-dimensional topology is dominated by the fundamental group. However, since every finitely presented group is the fundamental group of some closed 4-manifold, it is often stated that the effective influence of π_1 ends in dimension three. This is not quite true, however, and there are some interesting border disputes. In this paper, we show that, by imposing the extra condition of parallelizability on the tangent bundle, the dominion of π_1 is extended by an extra dimension. First, we prove

THEOREM A. *For each $n \geq 5$, any finitely presented group may occur as the fundamental group of a smooth closed parallelizable n -manifold.*

The proof may be compared with the usual proof that all finitely presented groups are realized as fundamental groups of closed 4-manifolds, but is nevertheless more delicate, because of the need to avoid the obstruction to parallelizability. In dimension 4, the obstruction cannot be avoided. In fact, a smooth closed almost parallelizable manifold M^4 is parallelizable when both its signature $\text{Sign}(M)$ and Euler characteristic $\chi(M)$ vanish. In particular, the vanishing of the Euler characteristic severely restricts the groups which can occur as fundamental groups of smooth closed parallelizable 4-manifolds. In fact, for many fundamental groups the Euler characteristic bounds the signature (see, for example, [4], [16]), but the relationship is neither universal nor straightforward.

To describe our results, it helps to partition the class of finitely presented groups as

$$\Pi(4) \cup C(4),$$

where $\Pi(4)$ is the class of fundamental groups of smooth closed parallelizable 4-manifolds, and $C(4)$ is the complementary class. An exact algebraic description of this partition seems very unlikely. Nevertheless it is possible to give some sufficient conditions for membership; the first involves membership of $\Pi(4)$.

THEOREM B. *If Γ admits a presentation of the form*

$$\Gamma = \langle X_1, \dots, X_g; W_1(X), \dots, W_r(X) \rangle$$

in which $r < g$, then $\Gamma \in \Pi(4)$.

Our condition for membership of $C(4)$ is given in terms of low dimensional homological restrictions on a discrete group Γ ; here k is a field.

$$\mathcal{A}(k): 2 \dim_k H^1(\Gamma, k) \leq 1 + \dim_k H^2(\Gamma, k);$$

$$\mathcal{B}(k): \dim_k H^1(\Gamma, k) \leq \dim_k H^2(\Gamma, k) \quad \text{and} \quad H^4(\Gamma, k) = 0.$$

THEOREM C. $C(4)$ contains all finitely presented groups which satisfy some condition $\mathcal{A}(k)$ where k is a field, and all those which satisfy some condition $\mathcal{B}(k)$ where k is a field of characteristic other than 2.

We conclude by discussing the $\Pi(4)$ - $C(4)$ partition with reference to some familiar infinite groups, and give some applications to combinatorial/geometric group theory.

The 4-dimensional results of this paper date from March 1991, and have circulated in a number of versions since then. Some thanks are long overdue, notably to Professor Hirzebruch and the staff of the Max-Planck Institut für Mathematik for their hospitality when the first-named author was visiting the Max-Planck Institut für Mathematik, Bonn in 1991, to Thomas Delzant, Dieter Kotschick, Ralph Strebel who saw the earliest version and made some useful comments, and to Hansjörg Geiges and Charles Thomas for comments on the material in Section 1.

§1. *Framed surgery.* We review briefly the theory of framed surgery on stably parallelizable manifolds. The standard reference for such matters is [9]. Throughout we shall assume that all manifolds are smooth, closed and connected; we denote by $M_1 \# M_2$ the connected sum of manifolds M_1, M_2 . A manifold M^n is *parallelizable* when its tangent bundle τ_M is isomorphic, as a vector bundle, to the product bundle $\tau_M \cong M \times \mathbf{R}^n$; M^n is *stably parallelizable* when τ_M becomes trivial upon addition of a trivial line bundle $\tau_M \oplus \varepsilon \cong M \times \mathbf{R}^{n+1}$; finally, M is *almost parallelizable* when $M - \{pt\}$ is parallelizable. We note the following, which is essentially a restatement of Lemma (3.4) of [9].

(1.1) *Let M be a smooth closed connected manifold. Then*

$$\begin{aligned} M \text{ is parallelizable} &\Rightarrow M \text{ is stably parallelizable} \\ &\Rightarrow M \text{ is almost parallelizable.} \end{aligned}$$

The following two propositions are straightforward.

(1.2) *Let M_1, M_2 be closed smooth manifolds. If M_1 is stably parallelizable and M_2 is parallelizable, then $M_1 \times M_2$ is parallelizable.*

(1.3) *Let M_1, M_2 be closed smooth stably parallelizable manifolds. Then $M_1 \# M_2$ is also stably parallelizable.*

Now let $\Gamma = \langle X_1, \dots, X_g; W_1(X), \dots, W_r(X) \rangle$ be a finitely presented group. Put

$$M_1 = \underbrace{(S^1 \times S^{n-1}) \# (S^1 \times S^{n-1}) \# \dots \# (S^1 \times S^{n-1})}_g.$$

Then $\pi_1(M_1)$ is a free group of rank g . We may identify the canonical generators of $\pi_1(M_1)$ with X_1, \dots, X_g . Observe that M_1 is stably parallelizable, since $S^1 \times S^{n-1}$ is parallelizable. We may represent each relator $W_j(X)$ by a closed path $\lambda_j: S^1 \rightarrow M_1$. By a standard general position argument, we may assume that $\lambda_1, \dots, \lambda_r$ are all imbeddings, with empty mutual intersections. Since M_1 is stably parallelizable, the normal bundle to any λ_j is trivial. Let M be obtained from M_1 by framed surgery on $\lambda_1, \dots, \lambda_r$. Then we obtain the classical result

(1.4) *Let $\Gamma = \langle X_1, \dots, X_g; W_1(X), \dots, W_r(X) \rangle$ be a finitely presented group. Then, for each $n \geq 4$, there exists a smooth closed stably parallelizable n -manifold M^n such that $\pi_1(M^n) \cong \Gamma$.*

If M^n is a closed almost parallelizable n -manifold, then the obstruction to parallelizability is a single element

$$\psi(M) \in H^n(M; \pi_{n-1}(SO(n))) \cong \pi_{n-1}(SO(n)).$$

Let $\sigma: \pi_{n-1}(SO(n)) \rightarrow \pi_{n-1}(SO)$ denote the stabilization map, and put

$$\mathcal{K}_n = \text{Ker}(\sigma: \pi_{n-1}(SO(n)) \rightarrow \pi_{n-1}(SO)).$$

Then $\sigma(\psi(M))$ represents the obstruction to the almost parallelizable manifold M being stably parallelizable, so that, when M is stably parallelizable, $\psi(M) \in H^n(M; \mathcal{K}_n) \cong \mathcal{K}_n$. We can rephrase matters as follows.

(1.5) *Let M^n be a smooth closed connected stably parallelizable n -manifold. Then there is a single obstruction $\psi(M) \in H^n(M; \mathcal{K}_n) \cong \mathcal{K}_n$ which vanishes precisely when M is parallelizable.*

The groups \mathcal{K}_n have been computed by Kervaire [8], using earlier work of Paechter [14]. When n is even, $\mathcal{K}_n \cong \mathbb{Z}$, and, in this case, the obstruction $\psi(M)$ to parallelizability of the stably parallelizable manifold M^n may be identified with the Euler characteristic $\chi(M)$. When n is odd, \mathcal{K}_n is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, except in three special dimensions $n = 1, 3, 7$, where one has $\mathcal{K}_1 \cong \mathcal{K}_3 \cong \mathcal{K}_7 \cong 0$; in these dimensions, stably parallelizable manifolds are necessarily parallelizable.

Proof of Theorem A. The first three cases, $n = 5, 6, 7$, are essentially different, and we give separate arguments for each. By means of (1.4), we may first construct a smooth closed stably parallelizable n -manifold M_1^n such that $\pi_1(M_1) \cong \Gamma$. Then M_1 is orientable and so satisfies Poincaré Duality with rational coefficients.

If $n = 5$, the obstruction to parallelizability $\psi(M)$ can be identified with the mod 2 semi-characteristic of M , namely

$$\psi(M) = \sum_{r=0}^2 \dim_{\mathbb{F}_2} H_r(M; \mathbb{F}_2) \pmod{2},$$

where \mathbb{F}_2 is the field with two elements. This follows easily from (9.2) of [7], or rather from the equivalent statement in cohomology.

Thus, if $\psi(M_1) \neq 0$, then $\psi(M_1 \# (S^2 \times S^3)) = 0$. Hence either M_1 or $M_1 \# (S^2 \times S^3)$ is parallelizable, so that the desired result follows, since $\pi_1(M_1 \# (S^2 \times S^3)) \cong \pi_1(M_1) \cong \Gamma$.

If $n = 6$, then the Euler characteristic of M_1 is even. In fact, we have

$$\chi(M_1) = 2(1 - \beta_1 + \beta_2) - \beta_3,$$

where β_k is the k th (rational) Betti number. However, the middle dimensional intersection form is here skew-symmetric and non-degenerate, so that β_3 is also even. Thus we may write $\chi(M_1) = 2k$. If $k = 0$, put $M = M_1$. Then $\psi(M) = \chi(M) = 0$, and so M is parallelizable. If $k > 0$, put

$$M = M_1 \# \underbrace{(S^3 \times S^3) \# \cdots \# (S^3 \times S^3)}_k,$$

whilst if $k > 0$, put

$$M = M_1 \# \underbrace{(S^2 \times S^4) \# \cdots \# (S^2 \times S^4)}_k.$$

Either way, M is stably parallelizable with $\pi_1(M) \cong \pi_1(M_1) \cong \Gamma$. However $\psi(M) = \chi(M) = 0$, so that M is parallelizable as desired.

If $n = 7$, there is nothing left to do, since $\chi_7 = 0$ and M_1 is automatically parallelizable.

If $n > 7$, then we may write $n = m + 3p$, where $m = 5, 6, 7$. By (1.3), we can find a smooth closed stably parallelizable m -manifold X^m with $\pi_1(X) \cong \Gamma$. Put

$$M^n = X^m \times \underbrace{S^3 \times S^3 \times \cdots \times S^3}_p.$$

Then M is parallelizable by (1.2), and $\pi_1(M) \cong \Gamma$. This completes the proof.

§2. Proofs of Theorems B and C. The above discussion shows that, for stably parallelizable 4-manifolds, the Euler characteristic may be identified with the obstruction to parallelizability of M^4 . The problem of finding a smooth closed 4-manifold with given fundamental group and vanishing Euler characteristic was previously considered by Hausmann and Weinberger [2]; see also the article by Kotschick [10].

Proof of Theorem B. Let $\langle X_1, \dots, X_g; W_1(X), \dots, W_r(X) \rangle$ be a finite presentation of the group Γ for which $r < g$. Let M_0 be the smooth 4-manifold obtained by attaching g 1-handles to S^4 ; then $\pi_1(M_0)$ is a free group F_g of rank g , and the cores of the attached 1-handles may be identified with a set of free generators X_1, \dots, X_g . Moreover,

$$\chi(M_0) = -2(g - 1).$$

Using general position, we may represent $W_1(X), \dots, W_r(X)$ by a collection of r imbedded frame 1-spheres $\Sigma_1, \dots, \Sigma_r$. Let M_1 be formed by framed surgery on $\Sigma_1, \dots, \Sigma_r$. Then M_1 is a smooth closed stably parallelizable 4-manifold with $\pi_1(M_1) = \Gamma$, and

$$\chi(M_1) = -2(g - r - 1).$$

Observe that $\chi(M_1) \leq 0$, since $(r + 1) \leq g$. If $r + 1 = g$, we are finished on putting $M = M_1$, since $\chi(M_1) = 0$. If $r + 1 < g$, put

$$M = \underbrace{M_1 \# (S^2 \times S^2) \# \dots \# (S^2 \times S^2)}_{g-r-1}.$$

Then again $\chi(M) = 0$, and M is a closed parallelizable 4-manifold with $\pi_1(M) = \Gamma$. In either case, it follows that $\Gamma \in \Pi(4)$.

Proof of Theorem C. Let M be a smooth closed orientable 4-manifold for which $\chi(M) = 0$, and let $c: M \rightarrow K(\Gamma, 1)$ be a map inducing an isomorphism of fundamental groups. Write $h_n(\Gamma, k) = \dim_k H^n(\Gamma, k)$, where k is a field. Observe that the induced map on cohomology $c^*: H^i(\Gamma, k) \rightarrow H^i(M, k)$ is an isomorphism for $i = 1$, and is injective for $i = 2$.

Writing $\dim_k H^2(M, k) = h_2(\Gamma, k) + e$ with $e \geq 0$, we have

$$2 - 2h_1(\Gamma, k) + h_2(\Gamma, k) \leq 2 - 2h_1(\Gamma, k) + h_2(\Gamma, k) + e = \chi(M) = 0,$$

from which it follows that $2 + \dim_k H^2(\Gamma, k) \leq 2 \dim_k H^1(\Gamma, k)$. Hence we see that, if Γ belongs to $\Pi(4)$, then Γ fails to satisfy conditions $\mathcal{A}(k)$ for any field k .

Suppose, in addition, that k has characteristic different from 2, and that $\dim_k H^1(\Gamma, k) \leq \dim_k H^2(\Gamma, k)$ and $H^4(\Gamma, k) = 0$. Let $x, y \in H^2(\Gamma, k)$; since $H^4(\Gamma, k) = 0$ we see that $x \cup y = 0 \in H^4(\Gamma, k)$. However, $c^*(x \cup y) = c^*(x) \cup c^*(y) = 0$, so that c^* maps $H^2(\Gamma, k)$ isomorphically to a subspace of $H^2(M, k)$ which is totally isotropic with respect to the cup product form. Thus $2h_2(\Gamma, k) \leq \dim_k H^2(M, k)$.

Writing $\dim_k H^2(M, k) = 2h_2(\Gamma, k) + e$ with $e \geq 0$, we see that

$$\chi(M) = 2 - 2h_1(\Gamma, k) + 2h_2(\Gamma, k) + e = 2 + 2(h_2 - h_1) + e.$$

Since $h_1 \leq h_2$, we see that $\chi(M) > 0$, which is a contradiction. Thus, when Γ belongs to $\Pi(4)$, Γ fails to satisfy $\mathcal{B}(k)$ for any field of characteristic other than 2.

One should point out that the $\mathcal{A}(k)$ conditions are related to an inequality which Epstein ([1]) attributes to Philip Hall.

§3. *The $\Pi(4)$ - $C(4)$ partition.* We begin with some general observations about the class $C(4)$. Let Γ denote a finitely presented group. The abelianisation Γ^{ab} may be identified with $H_1(\Gamma; \mathbb{Z})$, and when Γ^{ab} is finite then $H^1(\Gamma; \mathbb{Q}) \cong H_1(\Gamma; \mathbb{Q}) = 0$, so that Γ satisfies the condition $\mathcal{A}(\mathbb{Q})$, that is,

$$(3.1) \text{ If } \Gamma \text{ is finitely presented and } \Gamma^{ab} \text{ is finite, then } \Gamma \in C(4).$$

It follows from the condition $\mathcal{B}(k)$ that

(3.2) *If Γ satisfies (orientable) 3-dimensional Poincaré Duality over a field k of characteristic other than 2, then $\Gamma \in C(4)$.*

Next consider products. It follows easily from the Künneth Theorem that

(3.3) *If $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ all satisfy $H^1(\Gamma_i; \mathbb{Q}) \neq 0$, then $\Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n \in C(4)$, provided that $n \geq 5$.*

It follows from (3.1) and (3.3) that, for any finitely presented group Γ , the n -fold direct product $\Gamma^{(n)}$ belongs to $C(4)$ for all $n \geq 5$. It is tempting to conjecture that $C(4)$ is closed with respect to direct products.

Finally, observe that a finite covering of a parallelizable manifold is parallelizable. Thus $\Pi(4)$ is closed with respect to taking subgroups of finite index; dually

(3.4) *$C(4)$ is closed with respect to extension by finite groups.*

We conclude by showing how some familiar groups are situated with respect to the $\Pi(4)$ – $C(4)$ partition; we confine our attention to infinite groups since, by (3.1), all finite groups belong to $C(4)$.

The most familiar infinite groups are perhaps the free abelian groups \mathbb{Z}^n with $n \geq 1$. By (3.2) and (3.3), these belong to $C(4)$, with $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^4$ as the only possible exceptions. In fact, these are genuine exceptions, being the fundamental groups of the parallelizable 4-manifolds $S^1 \times S^3, S^1 \times S^1 \times S^2$ and $S^1 \times S^1 \times S^1 \times S^1$, respectively.

By a direct argument on the connected sum of parallelizable 4-manifolds, after adding on a copy of $S^2 \times S^2$ to kill the Euler characteristic, one may prove

(3.5) *$\Pi(4)$ is closed under free products.*

It follows that the non-abelian free groups \mathbb{F}_n , with $n \geq 2$, all belong to $\Pi(4)$. More generally

(3.6) *$\Pi(4)$ contains all Fuchsian groups.*

This follows from Theorem B, by the canonical description of Fuchsian groups as 1-relator groups [6].

Since the classical Wirtinger presentation of a knot group has deficiency equal to one, we see also that any knot group belong to $\Pi(4)$. The referee of a previous version of this article pointed out that, by Artin spinning, any knot group is the fundamental group of a ribbon 2-knot, so that there is a more general and geometrically natural observation that one can make, namely

(3.7) *The fundamental group of any 2-knot complement belongs to $\Pi(4)$.*

To see this, one does framed surgery on an embedded 2-sphere in S^4 , to obtain a stably parallelizable 4-manifold M^4 with $\chi(M) = 0$. However, by Van

Kampen’s Theorem, $\pi_1(M) \cong \pi_1(S^4 \setminus S^2)$; this line of argument seems to go back to Hausmann and Weinberger [2].

Next consider lattices in semisimple Lie groups. The simplest examples are lattices in $\mathrm{PSL}_2(\mathbb{R})$, that is, Fuchsian groups, and so belong to $\Pi(4)$ as already noted. By contrast, “most” semisimple lattices belong to $C(4)$. One may argue as follows. A lattice Γ in a connected linear semisimple Lie group contains a subgroup Γ_0 of finite index which is a direct product

$$\Gamma_0 \cong \Gamma_1 \times \cdots \times \Gamma_n,$$

where Γ_i is an irreducible lattice in a linear semisimple Lie group G_i . If each G_i has real rank at least 2, then, by a result of Kazhdan [5], Matsushima [13] and Wang [15], the abelianisation of each Γ_i is finite; thus Γ^{ab} is also finite, so that $\Gamma \in C(4)$ by (3.1).

The remaining cases, where some of the irreducible components have rank 1, are more obscure. Even so, one can still say something; for example, if Γ_i is Fuchsian for at least two indices i , then, after passing to subgroups of finite index for which $H_1(\Gamma_i; \mathbb{Q})$ is suitably large, a Künneth argument shows that Γ_0 satisfies condition $\mathcal{A}(\mathbb{Q})$, hence $\Gamma \in C(4)$. Note also that a torsion free uniform lattice in $\mathrm{PSL}_2(\mathbb{C})$ satisfies 3-dimensional Poincaré Duality, and so belongs to $C(4)$.

By applying the $\mathcal{A}(k)$ conditions, we also obtain

(3.8) *If Γ is a finitely presented group satisfying 4-dimensional Poincaré Duality over a field k of arbitrary characteristic, and if $\chi(\Gamma) > 0$, then $\Gamma \in C(4)$.*

From the examples given above, one might be forgiven for assuming that $\Pi(4)$ was a small and reasonably tractable class. This is far from the case, however; from Theorem B, we obtain

(3.9) *Let Γ be a finitely presented group. Then $\Gamma * \mathbb{F}_m \in \Pi(4)$ for some large enough integer m .*

In particular, every finitely presented group imbeds as a subgroup of an element of $\Pi(4)$. In fact, taking Γ to be Higman’s universal finitely presented group [3], and observing that Γ is contained in $\Gamma * \mathbb{F}_m$, we see that

(3.10) *There exists an element of $\Pi(4)$ which contains every other finitely presented group as a subgroup.*

The above results have an application to combinatorial group theory which is worth pointing out. Say that a finitely presented group Γ is *overdetermined* when every finite presentation for Γ has at least as many relations as generators; then Theorem B implies

(3.11) *If $\Gamma \in C(4)$ then Γ is overdetermined.*

In particular, we obtain

(3.12) *Any irreducible lattice in a non-compact semisimple Lie group of rank at least 2 is overdetermined.*

This does not seem to be well known, although we are informed that the question has recently been considered in the preprint by J. Lott [11]; the corresponding question for lattices in the rank 1 group $\mathrm{PSL}_2(\mathbb{C})$ has previously been considered, for example in [12].

Other classes of groups are also shown to be overdetermined by this method; for example, finite groups, PD^3 -groups, and PD^4 -groups Γ with $\chi(\Gamma) > 0$. The first two of these have more elementary proofs, but the result for PD^4 groups seems to be not only new, but, since $\mathbb{Z}^4 \in \Pi(4)$ and $\chi(\mathbb{Z}^4) = 0$, also best possible; it is not known if there exists a PD^4 -group Γ with $\chi(\Gamma) < 0$.

References

1. D. B. A. Epstein. Finite presentations of groups and 3-manifolds. *Quart. J. Math.*, 12 (1961), 205–212.
2. J. C. Hausmann and S. Weinberger. Caractéristiques d'Euler et groupes fondamentaux des variétés de dimension 4. *Comment. Math. Helv.*, 60 (1985), 139–144.
3. G. Higman. Subgroups of finitely presented groups. *Proc. Roy. Soc. (Ser. A)*, 262 (1961), 455–475.
4. F. E. A. Johnson and D. Kotschick. On the signature and Euler characteristic of certain four-manifolds. *Math. Proc. Camb. Phil. Soc.*, 114 (1993), 431–438.
5. D. A. Kazhdan. On the connection between the dual space of a group and the structure of its closed subgroups. *Function. Anal. Appl.*, 1 (1967), 63–65.
6. S. Katok. *Fuchsian Groups* (University of Chicago Press, Chicago, 1992).
7. M. A. Kervaire. Relative characteristic classes. *Amer. J. Math.*, 79 (1957), 517–558.
8. M. A. Kervaire. Some non-stable homotopy groups of Lie groups. *Illinois J. Math.*, 4 (1960), 161–169.
9. M. A. Kervaire and J. W. Milnor. Groups of homotopy spheres I. *Ann. Math.*, 77 (1963), 504–537.
10. D. Kotschick. Four-manifold invariants of finitely presented groups. Preprint (University of Basle, 1993).
11. J. Lott. Deficiencies of lattice subgroups of Lie groups. Preprint (University of Michigan, 1997).
12. A. Lubotzky. Group presentation, p -adic analytic groups and lattices in $\mathrm{SL}_2(\mathbb{C})$. *Ann. Math.*, 118 (1983), 115–130.
13. Y. Matsushima. On the first Betti number of compact quotient spaces of higher dimensional symmetric spaces. *Ann. Math.*, 75 (1962), 312–330.
14. G. F. Paechter. The groups $\pi_r(V_{n,m})$. *Quart. J. Math. (Ser. 2)*, 7 (1956), 249–268.
15. S. P. Wang. The dual space of semisimple Lie groups. *Amer. J. Math.*, 91 (1969), 921–937.
16. H. E. Winkelnkemper. Un teorema sobre variedades de dimensión 4. *Acta Mexicana Ci. Tecn.*, 2 (1968), 88–89.

Professor F. E. A. Johnson,
Department of Mathematics,
University College London,
Gower Street,
London, WC1E 6BT.

57M50: *MANIFOLDS AND CELL COMPLEXES: Low-dimensional topology; Geometric structures on low-dimensional manifolds.*

Dr. J. P. Walton,
Department of Mathematics,
University College London,
Gower Street,
London, WC1E 6BT.

Received on the 5th November, 1998.