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STABILITY OF THE PRÉKOPA–LEINDLER INEQUALITY

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Abstract. We prove a stability version of the Prékopa–Leindler inequality.

§1. *The problem.* The main theme of this paper is the Prékopa–Leindler inequality, due to Prékopa [16] and Leindler [14]. Soon after its proof, the inequality was generalized by Prékopa [17, 18], Borell [7] and Brascamp and Lieb [8]. Various applications were provided and surveyed in the works of Ball [1], Barthe [5] and Gardner [13]. The following version from [1] is often more useful and is more convenient for our purposes.

THEOREM 1.1 (Prékopa–Leindler). *If m, f and g are non-negative integrable functions on $\mathbb{R}$ satisfying $m((r+s)/2) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} m \geq \sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g}.$$

Dubuc [9] characterized the equality case under the restriction that the integrals of f, g and m above be positive. To explain this characterization, we need to recall that a non-negative real function h on $\mathbb{R}$ is log-concave if for any $x, y \in \mathbb{R}$ and $\lambda \in (0, 1)$ we have

$$h((1-\lambda)x + \lambda y) \geq h(x)^{1-\lambda} h(y)^{\lambda},$$

that is, if the support of h is an interval and $\log h$ is concave on the support. Now, in [9], it was proved that equality holds in the Prékopa–Leindler inequality if and only if there exist $a > 0$, $b \in \mathbb{R}$ and a log-concave h with positive integral on $\mathbb{R}$ such that for almost every $t \in \mathbb{R}$ we have

$$m(t) = h(t),$$
$$f(t) = a \cdot h(t+b),$$
$$g(t) = a^{-1} \cdot h(t-b).$$

In addition, for all $t \in \mathbb{R}$, we have $m(t) \geq h(t)$, $f(t) \leq a \cdot h(t+b)$ and $g(t) \leq a^{-1} \cdot h(t-b)$.

Our goal is to prove the following stability version of the Prékopa–Leindler inequality.

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THEOREM 1.2. *There exists an absolute positive constant c with the following property. If m, f and g are non-negative integrable functions with positive integrals on $\mathbb{R}$ such that m is log-concave, $m((r+s)/2) \geq \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$ and*

$$\int_{\mathbb{R}} m \leq (1+\varepsilon)\sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g} \quad \textit{for } \varepsilon > 0,$$

then there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$\int_{\mathbb{R}} |f(t) - am(t+b)|\, dt \leq c \cdot \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3} \cdot a \cdot \int_{\mathbb{R}} m(t)\, dt,$$
$$\int_{\mathbb{R}} |g(t) - a^{-1}m(t-b)|\, dt \leq c \cdot \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3} \cdot a^{-1} \cdot \int_{\mathbb{R}} m(t)\, dt.$$

Remark 1.3. The statement also holds if the condition of m being log-concave is replaced by the condition that both f and g be log-concave. The reason is that the function

$$\tilde{m}(t) = \sup\left\{\sqrt{f(r)g(s)} : t = \frac{r+s}{2}\right\}$$

will be log-concave in this case.

Remark 1.4. Most probably, the optimal error estimate in Theorem 1.2 is of order ε. However, this cannot be proved using the method of this note, i.e. by proving first an estimate on the quadratic transportation distance.

The paper is organized as follows. In §2 we establish the main properties of log-concave functions that we need, and we introduce the transportation map in §3. After translating the hypothesis $\int_{\mathbb{R}} m \leq (1+\varepsilon)\sqrt{\int_{\mathbb{R}} f \cdot \int_{\mathbb{R}} g}$ into an estimate for the transportation map, we estimate the quadratic transportation distance between our two functions in §4. Based on this, we estimate the L_1 distance between f and g in §5, which leads to the proof of Theorem 1.2 in §6. We note that the upper bound in §5 for the L_1 distance between two log-concave probability distributions in terms of their quadratic transportation distance is close to being optimal.

Another way to prove the Prékopa–Leindler inequality on $\mathbb{R}$ is by using the "one-dimensional Brunn–Minkowski inequality", namely, that the outer Lebesgue measure of $X + Y$ is at least the sum of the measures of the two Lebesgue-measurable $X, Y \subset \mathbb{R}$. For this proof, one assumes that the two functions on $\mathbb{R}$ have the same bounded supremum; the one-dimensional Brunn–Minkowski inequality is then applied to the level sets. Unlike the transportation argument (see §3), this proof works for any pair of bounded functions, but we see no way that it could lead to a stability version of the Prékopa–Leindler inequality on $\mathbb{R}$.

Remark 1.5. It is not clear whether the condition in Theorem 1.2 that m be log-concave is necessary for there to be a stability estimate.

Remark 1.6. Given $\alpha, \beta \in (0, 1)$ with $\alpha + \beta = 1$, we also have the following version of the Prékopa–Leindler inequality: if m, f and g are non-negative integrable functions on $\mathbb{R}$ satisfying $m(\alpha r + \beta s) \geq f(r)^\alpha g(s)^\beta$ for $r, s \in \mathbb{R}$, then

$$\int_{\mathbb{R}} m \geq \left(\int_{\mathbb{R}} f\right)^\alpha \left(\int_{\mathbb{R}} g\right)^\beta.$$

The method of this note also yields the corresponding stability estimate, except that the c in the new version of Theorem 1.2 would depend on α. For this statement, the formula

$$\frac{1 + T'(x)}{2\sqrt{T'(x)}} = 1 + \frac{(1 - T'(x))^2}{2\sqrt{T'(x)}(1 + \sqrt{T'(x)})^2}$$

used widely in this note should be replaced by Koebe's estimate

$$\frac{\alpha + \beta T'(x)}{T'(x)^\beta} \geq 1 + \frac{\min\{\alpha, \beta\}(1 - T'(x))^2}{T'(x)^\beta(1 + \sqrt{T'(x)})^2}$$

as long as $T'(x)$ is "not too large" or, if $T'(x)$ is "large", by the estimate $(\alpha + \beta T'(x))/T'(x)^\beta > \beta T'(x)^\alpha$.

Remark 1.7. The Prékopa–Leindler inequality also holds in $\mathbb{R}^n$ for $n \geq 2$. One possible approach to finding a higher-dimensional analogue of the stability statement is to use Theorem 1.2 and a suitable stability version of the injectivity of the Radon transform on log-concave functions; here the difficulty is caused by the fact that the Radon transform is notoriously unstable even on the space of smooth functions. Another possible approach is to combine Theorem 1.2 with the recent stability version of the Brunn–Minkowski inequality due to Figalli *et al* [**11, 12**], which improves on the result of Esposito *et al* [**10**]. This approach has been successfully applied by the authors in [**4**], at least for even functions. Actually, the Brunn–Minkowski inequality is equivalent to the Prékopa–Leindler inequality (see, for example, Ball [**3**] or Barthe [**5**]). A third possible approach to obtaining a stability version of the Prékopa–Leindler inequality in $\mathbb{R}^n$ is to use mass transportation as in Figalli *et al* [**11, 12**]. Unfortunately, the fact that the corresponding functions are not constant on their supports makes the problem much more complicated for a transportation argument than when the Brunn–Minkowski inequality is used.

§2. *Some elementary properties of log-concave probability distributions on* $\mathbb{R}$. Let h be a log-concave probability distribution on $\mathbb{R}$. In this section we discuss some useful elementary properties of h. Many of these properties are explicitly or implicitly used in various parts of the paper.

First, assuming that $h(t_0) = a \cdot b^{t_0}$ for $a, b > 0$ and $t_1 < t_0 < t_2$, we have:

$$\begin{aligned} &\text{if } h(t_1) \geq a \cdot b^{t_1}, \text{ then } h(t_2) \leq a \cdot b^{t_2}; \\ &\text{if } h(t_2) \geq a \cdot b^{t_2}, \text{ then } h(t_1) \leq a \cdot b^{t_1}. \end{aligned} \tag{1}$$

Next, we write w_h and μ_h for the median and mean of h, respectively; that is,

$$\int_{-\infty}^{w_h} h = \int_{w_h}^{\infty} h = \frac{1}{2} \quad \text{and} \quad \mu_h = \int_{\mathbb{R}} x h(x)\, dx.$$

Our first goal is to describe how a log-concave probability distribution is concentrated around its median; this is done in Proposition 2.2 below.

PROPOSITION 2.1. *Suppose that f and g are positive, θ is an increasing function on (a, b) and there exists $c \in (a, b)$ such that $f(t) \le g(t)$ if $t \in (a, c)$ while $f(t) \ge g(t)$ if $t \in (c, b)$. If $\int_a^b g(t)\, dt = \int_a^b f(t)\, dt$, then*

$$\int_a^b \theta(t) g(t)\, dt \le \int_a^b \theta(t) f(t)\, dt.$$

Proof. Since both factors of $(\theta(t) - \theta(c))(f(t) - g(t))$ change sign at c, the product is non-negative. Therefore $\int_a^b g(t)\, dt = \int_a^b f(t)\, dt$ yields

$$\int_a^b \theta(t) f(t)\, dt - \int_a^b \theta(t) g(t)\, dt = \int_a^b (\theta(t) - \theta(c))(f(t) - g(t))\, dt \ge 0. \quad \square$$

PROPOSITION 2.2. *If h is a log-concave probability distribution on $\mathbb{R}$, then for $w = w_h$ and $\mu = \mu_h$ we have:*

(i) $h(w) \cdot |w - \mu| \le \ln \sqrt{e/2}$;

(ii) $h(w) \cdot e^{-2h(w)|x-w|} \le h(x) \le h(w) \cdot e^{2h(w)|x-w|}$ *if* $|x - w| \le \ln 2/(2h(w))$;

(iii) $h(x) \le 2h(w)$ *for* $x \in \mathbb{R}$;

(iv) *if* $x > w$, *then* $\int_x^\infty h \le h(x)/(2h(w))$;

(v) *if* $x > w$ *and* $\int_x^\infty h = \nu > 0$, *then*

$$\int_x^\infty (t - w) h(t)\, dt \le \frac{\nu}{4h(w)} \cdot (1 - \ln 2\nu),$$

$$\int_x^\infty (t - w)^2 h(t)\, dt \le \frac{\nu}{8h(w)^2} \cdot [(\ln 2\nu)^2 - 2 \ln 2\nu + 2].$$

Remark. All the above estimates are optimal.

Proof. After replacing h by $a \cdot h(a(t - w))$ for $a = 1/2h(w)$, we may assume that $w = 0$ and $h(w) = 1/2$. It is natural to compare h near 0 to the probability distribution

$$\varphi(x) = \begin{cases} \frac{1}{2} \cdot e^{-x} & \text{if } x \ge -\ln 2, \\ 0 & \text{if } x < -\ln 2, \end{cases}$$

which satisfies $w_\varphi = 0$ and $\varphi(0) = h(0)$. We observe that $\log \varphi$ is linear and h is a log-concave function on $[-\ln 2, \infty]$, hence the set of all $x \in [-\ln 2, \infty]$ with $h(x) > \varphi(x)$ is an interval (possibly empty). Since $\varphi(0) = h(0)$ and $\int_0^\infty h = \int_0^\infty \varphi$, there exists some $v > 0$ such that

$$\begin{aligned} h(x) &\ge \varphi(x) \text{ provided } x \in [0, v]; \\ h(x) &\le \varphi(x) \text{ provided } x \ge v \text{ or } x \in [-\ln 2, 0]. \end{aligned} \tag{2}$$

In particular, the equalities $\int_{-\infty}^0 h = \int_{-\infty}^0 \varphi$ and $\int_0^\infty h = \int_0^\infty \varphi$ together with Proposition 2.1 yield

$$\begin{aligned}-\ln\frac{\mathrm{e}}{2} &= \int_{-\infty}^0 x\varphi(x)\,dx + \int_0^\infty x\varphi(x)\,dx\\ &\le \int_{-\infty}^0 xh(x)\,dx + \int_0^\infty xh(x)\,dx = \mu.\end{aligned}$$

Comparing h with $\varphi(-x)$ shows that $\mu \le \ln(\mathrm{e}/2)$, and, in turn, we deduce (i).

Turning to (ii), the upper bound follows directly from (2) and its consequence $h(x) \le \varphi(-x)$ for $x \in [0, \ln 2]$ by symmetry. To prove the lower bound, we may assume that $x > 0$. According to $h(0) = \frac{1}{2}$ and the log-concavity of h, it is enough to check the case where $x = \ln 2$. Therefore, we suppose that

$$h(\ln 2) < 1/4$$

and seek a contradiction. Since h is log-concave, there exists some $a \in \mathbb{R}$ such that

$$h(x) < \tfrac{1}{4}\,\mathrm{e}^{-a(x-\ln 2)} \quad \text{for } x \in \mathbb{R}.$$

Here, $h(0) = \frac{1}{2}$ gives $a > 1$.

We observe that $\frac{1}{4}\,\mathrm{e}^{-a(x_0-\ln 2)} = \frac{1}{2}\,\mathrm{e}^{x_0}$ for $x_0 = ((a-1)/(a+1))\ln 2$, and upon applying the analogue of (2) to $\varphi(-x)$ we obtain that $h(x) \le \frac{1}{2}\,\mathrm{e}^x$ for $x \in [0, x_0]$. In particular,

$$\int_0^\infty h < \int_0^{x_0} \frac{1}{2}\,\mathrm{e}^x\,dx + \int_{x_0}^\infty \frac{1}{4}\,\mathrm{e}^{-a(x-\ln 2)}\,dx = \left(\frac{1}{a}+1\right)2^{-2/(a+1)} - \frac{1}{2}.$$

Differentiation shows that the last expression is first strictly decreasing and then strictly increasing in $a \ge 1$. Since the value of this last expression is $\frac{1}{2}$ both at $a = 1$ and at $a = \infty$, we deduce that $\int_0^\infty h < \frac{1}{2}$. But this is absurd, and therefore we have proved (ii).

To prove (iii), we may assume that $x > 0$ and $h(x) \ge 1$; hence (ii) yields that $x \ge \ln 2$. Since $h(t) \ge \frac{1}{2}\,\mathrm{e}^{(t/x)\ln 2h(x)}$ for $t \in [0, x]$ by the log-concavity of h and the fact that $h(0) = 1/2$, we have

$$\frac{1}{2} \ge \int_0^x h \ge \int_0^x \frac{1}{2}\,\mathrm{e}^{(t/x)\ln 2h(x)}\,dt = \frac{x(2h(x)-1)}{2\ln 2h(x)}.$$

As $(s-1)/\ln s > 1/\ln 2$ for $s > 2$, we conclude that $h(x) \le 1$.

For (iv), recall that $2h(w) = 1$. In particular, (iv) holds if $h(x) \ge \frac{1}{2}$, as $\int_0^x h < \frac{1}{2}$. Thus we assume that $h(x) < \frac{1}{2}$, and hence $h(x) = \frac{1}{2}\,\mathrm{e}^{-x_0}$ for some $x_0 > 0$. If $x \ge x_0$, then the log-concavity of h and the fact that $h(0) = \frac{1}{2}$ yield

$$\int_0^x h(t)\,dt \ge \int_0^x \frac{1}{2}\,\mathrm{e}^{-tx_0/x}\,dt = \frac{x}{x_0}\int_0^{x_0} \frac{1}{2}\,\mathrm{e}^{-t}\,dt \ge \int_0^{x_0} \frac{1}{2}\,\mathrm{e}^{-t}\,dt,$$

and therefore

$$\int_x^\infty h(t)\,dt \le \int_{x_0}^\infty \frac{1}{2}\,\mathrm{e}^{-t}\,dt = h(x).$$

On the other hand, if $x < x_0$, then $h(x) = \frac{1}{2}\,\mathrm{e}^{-ax}$ for $a = x_0/x > 1$. It follows from the log-concavity of h and $h(0) = \frac{1}{2}$ that $h(t) \le \frac{1}{2}\,\mathrm{e}^{-at}$ for $t > x$. Therefore

$$\int_x^\infty h(t)\,dt \le \int_x^\infty \frac{1}{2}\,\mathrm{e}^{-at}\,dt = h(x)/a < h(x).$$

Finally, we prove (v). Let $x_1 = -\ln 2\nu$; then x_1 satisfies $\int_x^\infty h(t)\,dt = \int_{x_1}^\infty \frac{1}{2}\,\mathrm{e}^{-t}\,dt$. It follows from (2) that $x_1 \ge x$. We define two functions f and g on $[x, \infty)$: let $f(t) = \frac{1}{2}\,\mathrm{e}^{-t}$ if $t \ge x_1$ and $f(t) = 0$ if $t \in [x, x_1)$; let $g = h|_{[x,\infty)}$. These two functions satisfy the conditions in Proposition 2.1; therefore, for $\alpha \ge 0$ we have

$$\int_x^\infty t^\alpha h(t)\,dt = \int_x^\infty t^\alpha g(t)\,dt \le \int_x^\infty t^\alpha f(t)\,dt = \int_{x_1}^\infty \frac{t^\alpha \mathrm{e}^{-t}}{2}\,dt.$$

Evaluating the last integral for $\alpha = 1$ and $\alpha = 2$ yields (v). □

Next, we discuss various consequences of Proposition 2.2.

COROLLARY 2.3. *Let h be a log-concave probability density function on $\mathbb{R}$, and let $\int_x^\infty h = \nu \in (0, \frac{1}{2}]$. Then:*

(i) $h(x)\cdot \mathrm{e}^{-h(x)|t-x|/\nu} \le h(t) \le h(x)\cdot \mathrm{e}^{h(x)|t-x|/\nu}$ *if* $|t-x| \le \nu \ln 2/h(x)$*;*

(ii) *if* $\nu \in (0, \frac{1}{6})$*,* $w = w_h$ *and* $\mu = \mu_h$*, then*

$$\int_x^\infty |t-\mu| h(t)\,dt \le \frac{\nu}{2h(w)}\cdot|\ln \nu|$$
$$\int_x^\infty |t-\mu|^2 h(t)\,dt \le \frac{5\nu}{4h(w)^2}\cdot(\ln \nu)^2.$$

Remark. The order of all estimates is optimal, as shown by the example of $h(t) = \mathrm{e}^{-|t|}/2$.

Proof. To prove (i), let $|t-x| \le \nu \ln 2/h(x)$. There exists a unique $\lambda \in \mathbb{R}$ such that for the function

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \ge x,\\ \min\{h(t),\ h(x)\cdot \mathrm{e}^{\lambda(t-x)}\} & \text{if } t \le x,\end{cases}$$

we have $\int_{-\infty}^x \tilde{h} = \nu$. We note that $\tilde{h}$ is log-concave and that $\lambda \ge -h(x)/\nu$. In particular, $\tilde{h}/(2\nu)$ is a log-concave probability distribution whose median is x, and hence Proposition 2.2(ii) yields $h(t) \ge \tilde{h}(t) \ge h(x)\cdot \mathrm{e}^{-h(x)|t-x|/\nu}$. Since for $s = 2x - t$ we have $h(s) \ge h(x)\cdot \mathrm{e}^{-h(x)|s-x|/\nu}$, we conclude (i) from (1).

For (ii), we may assume that $h(w) = \frac{1}{2}$; hence Proposition 2.2(i) yields $|w-\mu| \le \ln(\mathrm{e}/2)$. Since $\ln 2\nu \le -1$, we deduce from Proposition 2.2(v) that

$$\int_x^\infty |t-\mu| h(t)\,dt \le \int_x^\infty [|t-w| + |w-\mu|] h(t)\,dt$$
$$\le \nu\cdot(-\ln 2\nu) + \nu\cdot\ln\frac{\mathrm{e}}{2} < \nu\cdot|\ln \nu|.$$

In addition,

$$\int_x^{\infty}(t-\mu)^2h(t)\,dt \le \int_x^{\infty} 2[(t-w)^2+(w-\mu)^2]h(t)\,dt$$
$$\le \nu\cdot 5(\ln 2\nu)^2+\nu\cdot 2\left(\ln\frac{\mathrm{e}}{2}\right)^2 < 5\nu\cdot(\ln\nu)^2. \qquad \square$$

§3. *The transportation map for log-concave probability distributions and the Prékopa–Leindler inequality.* Let f and g be log-concave probability distributions on $\mathbb{R}$, and let I_f and I_g denote the open intervals that are the supports of f and g, respectively. We define the transportation map $T : I_f \to I_g$ by the identity

$$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{T(x)} g(t)\,dt. \tag{3}$$

Among other things, T is monotone increasing, bijective and differentiable on I_f, and for any $x \in I_f$ we have

$$f(x) = g(T(x))T'(x). \tag{4}$$

Remark. Using (3), the transportation map $T : \mathbb{R} \to \mathbb{R}$ can be defined for any two probability distributions f and g, and T is naturally monotone increasing. In addition, (4) holds for almost all x provided that the product of two numbers of which one is zero and the other is undefined is understood to be zero.

For log-concave probability distributions f and g and an integrable function m on $\mathbb{R}$ satisfying $m((r+s)/2) \ge \sqrt{f(r)g(s)}$ for $r, s \in \mathbb{R}$, one proof of the Prékopa–Leindler inequality runs as follows:

$$\begin{aligned}
1 = \int_{\mathbb{R}} f &= \int_{I_f} \sqrt{f(x)}\cdot\sqrt{g(T(x))T'(x)}\,dx\\
&\le \int_{I_f} m\left(\frac{x+T(x)}{2}\right)\sqrt{T'(x)}\,dx\\
&\le \int_{I_f} m\left(\frac{x+T(x)}{2}\right)\cdot\frac{1+T'(x)}{2}\,dx\\
&= \int_{(I_f+I_g)/2} m(x)\,dx \le \int_{\mathbb{R}} m.
\end{aligned}$$

The basic fact that we will exploit is the following. If we know that $\int_{\mathbb{R}} m \le 1+\varepsilon$, then, using (4) in the last inequality, we have

$$\begin{aligned}
\varepsilon &\ge \int_{I_f} m\left(\frac{x+T(x)}{2}\right)\cdot\left(\frac{1+T'(x)}{2}-\sqrt{T'(x)}\right)dx\\
&\ge \int_{I_f} \sqrt{f(x)}\cdot\sqrt{g(T(x))T'(x)}\left(\frac{1+T'(x)}{2\sqrt{T'(x)}}-1\right)dx\\
&= \int_{I_f} f(x)\cdot\frac{(1-\sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}}\,dx.
\end{aligned} \tag{5}$$

As long as T' is not too large, the integrand is at least about $f(x)(1-T'(x))^2$ and, using a Poincaré inequality for the density f, we can bound the integral of this expression from below by the transportation cost $\int f(x)(x-T(x))^2$. The main technical issue is to handle the places where T' is large.

§4. *The quadratic transportation distance.* Let f and g be log-concave probability distributions on $\mathbb{R}$ with zero mean, that is,

$$0=\int_{\mathbb{R}} xf(x)\,dx=\int_{\mathbb{R}} yg(y)\,dy.$$

In this section, we show that (5) yields an upper bound for the quadratic transportation distance

$$\int_{I_f} f(x)(T(x)-x)^2\,dx$$

between f and g.

LEMMA 4.1. *If f and g are log-concave probability distributions on $\mathbb{R}$ with zero mean and (5) holds for $\varepsilon\in(0,\frac{1}{48})$, then*

$$\int_{I_f} f(x)(T(x)-x)^2\,dx \le 2^{20} f(w_f)^{-2}\cdot\varepsilon|\ln\varepsilon|^2,$$

where w_f is the median as mentioned earlier.

Remark. The optimal power of ε in Lemma 4.1 is most likely to be ε^2 (cf. Example 7.1). To improve the estimate, we should improve on inequality (6) with $R(x)=T(x)-x$ where T is the transportation map for another log-concave probability distribution. One could use the fact that $T(x)-x$ is of at most logarithmic order.

Proof. The main tool in the proof of Lemma 4.1 is the Poincaré inequality for log-concave measures, which can be found in Bobkov [**6**, (1.3) and (4.2)]. This guarantees that if h is a log-concave probability distribution on $\mathbb{R}$ and the function R on $\mathbb{R}$ is locally Lipschitz with expectation $\mu=\int_{\mathbb{R}} h(x)R(x)\,dx$, then

$$\begin{aligned}\int_{\mathbb{R}} h(x)(R(x)-\mu)^2\,dx &= \int_{\mathbb{R}} h(x)R(x)^2\,dx-\mu^2\\ &\le h(w_h)^{-2}\cdot\int_{\mathbb{R}} h(x)R'(x)^2\,dx. \qquad (6)\end{aligned}$$

By symmetry we may assume that $g(w_g)\le f(w_f)$, and by scaling that $f(w_f)=\frac{1}{2}$. Let T be the transportation map from f to g, and let S be its inverse; then, for $x\in I_f$ and $y\in I_g$, we have

$$f(x)=g(T(x))T'(x) \quad\text{and}\quad g(y)=f(S(y))S'(y). \qquad (7)$$

Suppose that for some $x\in\mathbb{R}$ with $\int_x^\infty f=\nu\in(0,\frac{1}{2}]$, we have $g(T(x))\le f(x)/16$. If $x\le t\le x+\nu\ln 2/f(x)$, then Corollary 2.3(i) yields $f(t)\ge f(x)\cdot \mathrm{e}^{-f(x)(t-x)/\nu}\ge f(x)/2$. On the other hand, the log-concavity of g and Proposition 2.2(iii) yield that if $x\le t<x+\nu\ln 2/f(x)$, then $g(t)<2g(x)\le f(t)/4$. In particular, $T'(t)>4$ by (7), and hence (cf. (5)) we have

$$\varepsilon \geq \int_{\mathbb{R}} \frac{(1-\sqrt{T'(t)}\,)^2}{2\sqrt{T'(t)}} f(t)\,dt > \int_x^{x+\nu \ln 2/f(x)} \frac{f(x)}{4} \cdot \mathrm{e}^{-f(x)(t-x)/\nu}\,dt = \frac{\nu}{8}.$$

A similar argument for $f(-x)$ and $g(-x)$ shows that if $\int_{-\infty}^x f = \nu$ and $g(T(x)) \leq f(x)/16$, then $\nu < 8\varepsilon$.

We define x_1, x_2, y_1 and y_2 by

$$\int_{-\infty}^{x_1} f = \int_{x_2}^{\infty} f = \int_{-\infty}^{y_1} g = \int_{y_2}^{\infty} g = 8\varepsilon < \frac{1}{6}.$$

The argument above yields that if $x \in (x_1, x_2)$, then $T'(x) \leq 16$ and $g(T(x)) \geq f(x)/16$, and hence $g(w_g) \geq \frac{1}{32}$. As the means of f and g are zero, we deduce from Corollary 2.3(ii) and (7) that

$$\int_{\mathbb{R}\setminus[x_1,x_2]} |x| f(x)\,dx \leq 2^4 \varepsilon\,|\ln \varepsilon|, \tag{8}$$

$$\int_{\mathbb{R}\setminus[x_1,x_2]} |T(x)| f(x)\,dx = \int_{\mathbb{R}\setminus[y_1,y_2]} |y| g(y)\,dy \leq 2^8 \varepsilon\,|\ln \varepsilon|, \tag{9}$$

$$\int_{\mathbb{R}\setminus[x_1,x_2]} x^2 f(x)\,dx \leq 2^7 \varepsilon (\ln \varepsilon)^2, \tag{10}$$

$$\int_{\mathbb{R}\setminus[x_1,x_2]} T(x)^2 f(x)\,dx = \int_{\mathbb{R}\setminus[y_1,y_2]} y^2 g(y)\,dy \leq 2^{15} \varepsilon (\ln \varepsilon)^2. \tag{11}$$

Since $(T(x)-x)^2 \leq 2[T(x)^2 + x^2]$, we have

$$\int_{\mathbb{R}\setminus[x_1,x_2]} (T(x)-x)^2 f(x)\,dx \leq 2^{17} \varepsilon (\ln \varepsilon)^2. \tag{12}$$

Next, we consider the log-concave probability distribution

$$\tilde{f}(t) = \begin{cases} (1-16\varepsilon)^{-1} f(t) & \text{if } t \in [x_1, x_2], \\ 0 & \text{if } t \in \mathbb{R}\setminus[x_1, x_2]. \end{cases}$$

To estimate $\tilde{f}(w_{\tilde{f}})$, we define $z_1 = w_f - \ln 2$ and $z_2 = w_f + \ln 2$. Since $f(w_f) = \frac{1}{2}$, Proposition 2.2(ii) applied to f yields

$$\int_{\mathbb{R}\setminus[z_1,z_2]} \tilde{f}(x)\,dx \leq (1-16\varepsilon)^{-1} \left(1 - 16\varepsilon - \int_{z_1}^{z_2} \frac{\mathrm{e}^{-|x-w_f|}}{2}\,dx\right) < \frac{1}{2}.$$

It follows that $|w_{\tilde{f}} - w_f| < \ln 2$, and hence we deduce again by using Proposition 2.2(ii) that

$$\tilde{f}(w_{\tilde{f}}) > \tfrac{1}{4}.$$

For the expectation

$$\mu = \int_{\mathbb{R}} (T(x)-x) \tilde{f}(x)\,dx,$$

we have the estimate

$$|\mu| = (1-16\varepsilon)^{-1}\left|\int_{\mathbb{R}\setminus[x_1,x_2]} (T(x)-x)f(x)\,dx\right| \le 2^{10}\varepsilon\,|\ln\varepsilon|.$$

If $x \in (x_1, x_2)$, then $T'(x) \le 16$, and thus the expression in (5) satisfies

$$\begin{aligned}\frac{(1-\sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}} &= \frac{(T'(x)-1)^2}{2(1+\sqrt{T'(x)}\,)^2\sqrt{T'(x)}}\\ &\ge \frac{(T'(x)-1)^2}{200} > 2^{-8}(T'(x)-1)^2.\end{aligned}$$

We deduce, using (6) and (5), that

$$\begin{aligned}\int_{[x_1,x_2]} (T(x)-x)^2 f(x)\,dx &\le \int_{\mathbb{R}} (T(x)-x)^2\tilde{f}(x)\,dx\\ &\le \mu^2 + \tilde{f}(w_{\tilde{f}})^{-2}\int_{\mathbb{R}}(T'(x)-1)^2\tilde{f}(x)\,dx\\ &\le 2^{20}\varepsilon^2\,|\ln\varepsilon|^2 + 2^{13}\int_{x_1}^{x_2}\frac{(1-\sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}}f(x)\,dx\\ &\le 2^{20}\varepsilon^2\,|\ln\varepsilon|^2 + 2^{13}\varepsilon. \qquad (13)\end{aligned}$$

Therefore, combining (12) and (13) completes the proof of Lemma 4.1. □

§5. *The L_1 and quadratic transportation distances.* Our goal is to estimate the L_1 distance between two log-concave probability distributions f and g in terms of their quadratic transportation distance. In this section, T will always denote the transportation map $T : I_f \to I_g$ satisfying

$$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{T(x)} g(t)\,dt.$$

We prepare for the proof of Theorem 5.3 by establishing Propositions 5.1 and 5.2. While the ideas for Propositions 5.1 and 5.2 are rather simple, they still lead to the essentially optimal (up to a logarithmic factor) estimate in Theorem 5.3.

For expressions A and B, when we write $A \ll B$ we shall mean that $|A| \le c \cdot B$ where $c > 0$ is an absolute constant that is independent of all the quantities occurring in A and in B. In addition, $A \approx B$ means that $A \ll B$ and $B \ll A$.

PROPOSITION 5.1. *Let f and g be log-concave probability distributions on $\mathbb{R}$ satisfying $\int_{-\infty}^{z} f \ge \nu$ and $\int_{z}^{\infty} f \ge \nu$ for $\nu \in (0, \frac{1}{2}]$ and $z \in \mathbb{R}$. If either $\int_{-\infty}^{z} g \le \nu/2$ or $\int_{z}^{\infty} g \le \nu/2$, then*

$$\int_{z-\nu/f(z)}^{z+\nu/f(z)} (T(x)-x)^2 f(x)\,dx \gg \frac{\nu^3}{f(z)^2}.$$

Proof. We may assume that $\int_z^\infty g \le \nu/2$. It follows from Corollary 2.3(i) that if $z < x \le z + \nu \ln(3/2)/f(z)$, then

$$\int_z^x f \le \int_z^{z+\nu \ln(3/2)/f(z)} f(z)\mathrm{e}^{f(z)|t-z|/\nu}\, dt = \nu/2$$

and hence $T(x) \le z$. Therefore

$$\begin{aligned}\int_{z+\nu \ln(5/4)/f(z)}^{z+\nu \ln(3/2)/f(z)} (T(x)-x)^2 f(x)\, dx &\gg \int_{z+\nu \ln(5/4)/f(z)}^{z+\nu \ln(3/2)/f(z)} \left(\frac{\nu \ln \frac{5}{4}}{f(z)}\right)^2 \frac{f(z)}{2}\, dx \\ &\gg \frac{\nu^3}{f(z)^2}. \end{aligned}$$

□

PROPOSITION 5.2. *Let f and g be log-concave probability distributions on $\mathbb{R}$ satisfying $\int_{-\infty}^z f \ge \nu$, $\int_z^\infty f \ge \nu$ and $\int_{-\infty}^z g \ge \nu/2$, $\int_z^\infty g \ge \nu/2$ for $\nu > 0$ and $z \in \mathbb{R}$. If $g(z) \ne f(z)$ and $\Delta = \nu \ln 2/3f(z) \cdot \min\{|\ln(g(z)/f(z))|,\ 3\}$, then*

$$\int_{z-\Delta}^{z+\Delta} (T(x)-x)^2 f(x)\, dx \gg \frac{\nu^3}{f(z)^2} \cdot \min\left\{\left|\ln \frac{g(z)}{f(z)}\right|, 3\right\}^4.$$

Remark. If, in addition, $\mathrm{e}^{-3} f(z) \le g(z) \le \mathrm{e}^3 f(z)$, then the arguments in cases 2 and 3 below show that the interval $[z-\Delta, z+\Delta]$ of integration can be replaced by $[z-\Delta/150, z+\Delta/150]$, and if $x \in [z-\Delta/150, z+\Delta/150]$, then

$$\frac{1}{3}\left|\ln \frac{g(z)}{f(z)}\right| \le \left|\ln \frac{g(x)}{f(x)}\right| \le \frac{5}{3}\left|\ln \frac{g(z)}{f(z)}\right|.$$

Proof. According to Corollary 2.3(i), if $z - \Delta \le x \le z + \Delta$, then

$$f(z)/2 \le f(z) \cdot \mathrm{e}^{-f(z)|x-z|/\nu} \le f(x) \le f(z) \cdot \mathrm{e}^{f(z)|x-z|/\nu} \le 2f(z). \tag{14}$$

Similarly, if $z - \nu \ln 2/2g(z) \le x \le z + \nu \ln 2/2g(z)$, then

$$g(z)/2 \le g(z) \cdot \mathrm{e}^{-2g(z)|x-z|/\nu} \le g(x) \le g(z) \cdot \mathrm{e}^{2g(z)|x-z|/\nu} \le 2g(z). \tag{15}$$

We may assume

$$T(z) \le z.$$

For the rest of the argument, we distinguish four cases.

Case 1. $g(z) \ge \mathrm{e}^3 f(z)$.

In this case, $\Delta = \nu \ln 2/f(z)$. We note that

$$\frac{\ln 2}{2 \cdot \mathrm{e}^3} < \frac{\ln 2}{10} < \frac{3 \ln 2}{10} < \ln \frac{5}{4}. \tag{16}$$

Since $\nu \ln 2/(2g(z)) < \Delta/10$, (15) yields that if $x \ge z + \Delta/10$, then

$$\int_z^x g > \frac{\nu}{4}. \tag{17}$$

However, (14) and (16) imply that if $z < x \le z + 3\Delta/10$, then

$$\int_z^x f < \frac{\nu}{4}. \tag{18}$$

Since $T(z) \le z$, (17) and (18) yield that if $z + 2\Delta/10 \le x \le z + 3\Delta/10$, then $T(x) \le z + \Delta/10$. In particular,

$$\int_{z+2\Delta/10}^{z+3\Delta/10} (T(x) - x)^2 f(x)\, dx \ge \int_{z+2\Delta/10}^{z+3\Delta/10} \left(\frac{\Delta}{10}\right)^2 \frac{f(z)}{2}\, dx \gg \Delta^3 f(z).$$

Case 2. $f(z) < g(z) \le e^3 f(z)$.

Let $\lambda = (f(z)/g(z))^{1/3} \ge 1/e$. Since $2g(z) \le 2e^3 f(z) < 50 f(z)$ and $\Delta = (\nu \ln 2/(3f(z))) \ln(g(z)/f(z))$, if $z \le x \le z + \Delta/50$, then (14) and (15) yield

$$\lambda \cdot f(z) \le f(x) \le \lambda^{-1} \cdot f(z) \quad \text{and} \quad \lambda \cdot g(z) \le g(x) \le \lambda^{-1} \cdot g(z).$$

In particular, if $z \le s, t \le z + \Delta/50$, then $f(s)/g(t) \le \lambda$. We deduce that if $z < x \le z + \Delta/150$, then

$$\int_z^x f \le \int_z^{z+\lambda(x-z)} g.$$

Thus $T(x) \le z + \lambda(x - z)$ by $T(z) \le z$, and hence

$$x - T(x) \ge (1 - \lambda)(x - z) = \lambda\left(\frac{1}{\lambda} - 1\right)(x - z) \ge \frac{x - z}{3e} \cdot \ln \frac{g(z)}{f(z)}.$$

It follows that

$$\int_{z+\Delta/300}^{z+\Delta/150} (T(x) - x)^2 f(x)\, dx \gg \Delta^3 f(z) \ln \frac{g(z)}{f(z)}.$$

Case 3. $e^{-3} f(z) \le g(z) < f(z)$.

Let $\lambda = (f(z)/g(z))^{1/3} \le e$. Since $\Delta = (\nu \ln 2/(3f(z))) \ln(f(z)/g(z))$, if $z - \Delta/2 \le x \le z$, then (14) and (15) yield

$$\lambda^{-1} \cdot f(z) \le f(x) \le \lambda \cdot f(z) \quad \text{and} \quad \lambda^{-1} \cdot g(z) \le g(x) \le \lambda \cdot g(z).$$

In particular, if $z - \Delta/2 \le s, t \le z$, then $f(s)/g(t) \ge \lambda$. We deduce that if $z - \Delta/(2e) < x \le z$, then

$$\int_x^z f \ge \int_{z-\lambda(z-x)}^z g.$$

Thus $T(x) \le z - \lambda(z - x)$ by $T(z) \le z$, and hence

$$x - T(x) \ge (\lambda - 1)(z - x) \ge \frac{z - x}{3} \cdot \ln \frac{f(z)}{g(z)}.$$

It follows that

$$\int_{z-\Delta/150}^{z-\Delta/300} (T(x) - x)^2 f(x)\, dx \gg \Delta^3 f(z) \ln \frac{f(z)}{g(z)}.$$

Case 4. $g(z) \le e^{-3} f(z)$.

Since $\Delta = \nu \ln 2/f(z)$, if $z - \Delta \le x \le z$, then (14) and (15) yield $f(x) \ge f(z)/2$ and $g(x) \le 2g(z)$. In particular, if $z - \Delta \le s, t \le z$, then $f(s) \ge 2g(t)$. We deduce that if $z - \Delta/2 < x \le z$, then

$$\int_x^z f \ge \int_{z-2(z-x)}^z g.$$

Thus $T(x) \le z - 2(z-x)$ by $T(z) \le z$, and hence $x - T(x) \ge z - x$. It follows that

$$\int_{z-\Delta/2}^{z-\Delta/4} (T(x)-x)^2 f(x)\,dx \gg \Delta^3 f(z). \qquad \square$$

THEOREM 5.3. *If f and g are log-concave probability distributions on $\mathbb{R}$ and $\int_{I_f} f(x)(T(x)-x)^2\,dx = \varepsilon \cdot f(w_f)^{-2}$ for $\varepsilon \in (0,1)$, then*

$$\int_{\mathbb{R}} |f(x) - g(x)|\,dx \ll \sqrt[3]{\varepsilon}\,|\ln \varepsilon|^{2/3}.$$

Remark. According to Example 7.2, the exponent $\frac{1}{3}$ of ε in Theorem 5.3 is optimal.

Proof. It is enough to prove the statement for $\varepsilon < \varepsilon_0$, where $\varepsilon_0 \in (0, \frac{1}{2})$ is an absolute constant to be specified later. We may assume that $f(w_f) = 1$; then $f(x) \le 2$ for any $x \in \mathbb{R}$ by Proposition 2.2(iii), and for the inverse S of T we have that

$$\int_{I_f} f(x)(T(x)-x)^2\,dx = \int_{I_g} g(y)(S(y)-y)^2\,dy \le \varepsilon.$$

For $x \in \mathbb{R}$, we define

$$\nu(x) = \min\left\{\int_{-\infty}^x f,\ \int_x^\infty f\right\},$$
$$\tilde{\nu}(x) = \min\left\{\int_{-\infty}^x g,\ \int_x^\infty g\right\}.$$

First we estimate g. Since $\nu(w_f) = \frac{1}{2}$, if ε_0 is small enough, then Propositions 5.1 and 5.2 yield $\tilde{\nu}(w_f) > \frac{1}{4}$ and $g(w_f) \le 2$, respectively. We conclude from Proposition 2.2(ii) that $g(w_g) \le 4$, and hence $g(x) \le 8$ for any $x \in \mathbb{R}$ by Proposition 2.2(iii).

It follows from $f(x) \le 2$ and Proposition 5.1 that there exists a positive constant c_1 such that if $\nu(x) \ge c_1\sqrt[3]{\varepsilon}$, then $\tilde{\nu}(x) \ge \nu(x)/2$. Now, upon applying Proposition 5.1 to g and possibly increasing c_1, we have that if $\nu(x) \ge c_1\sqrt[3]{\varepsilon}$, then $\tilde{\nu}(x) \le 2\nu(x)$. Finally, possibly after increasing c_1 further, we obtain that if $\nu(x) \ge c_1\sqrt[3]{\varepsilon}$, then $|\ln(g(x)/f(x))| \le \ln 2$ by Proposition 5.2. We choose ε_0 small enough to satisfy $2c_1\sqrt[3]{\varepsilon_0} < \frac{1}{2}$.

For $z \in \mathbb{R}$, we define $\Delta(z) = (\nu \ln 2/(450 f(z))) \cdot |\ln(g(z)/f(z))|$. We assume that $\nu(z) \geq c_1 \sqrt[3]{\varepsilon}$, and hence $1/2 \leq g(z)/f(z) \leq 2$. It follows from Corollary 2.3(i) that $f(x) \geq f(z)/2$ and $\nu(x) \leq 2\nu(z)$ if $x \in [z - \Delta(z), z + \Delta(z)]$. We deduce, by using Proposition 5.2 and its remark, that there exists an absolute constant c_2 such that, assuming $g(z) \neq f(z)$, we have

$$\int_{z-\Delta(z)}^{z+\Delta(z)} \frac{\nu(x)^2}{f(x)} \cdot \left|\ln \frac{g(x)}{f(x)}\right|^3 dx \leq c_2 \int_{z-\Delta(z)}^{z+\Delta(z)} (T(x) - x)^2 f(x)\, dx. \tag{19}$$

We define $z_1 < z_2$ by the properties $\nu(z_1) = \nu(z_2) = 2c_1 \sqrt[3]{\varepsilon}$. We observe that if $g(z) \neq f(z)$ and some $x \in [z - \Delta(z), z + \Delta(z)]$ satisfies $\nu(x) \geq 2c_1 \sqrt[3]{\varepsilon}$, then $\nu(z) \geq c_1 \sqrt[3]{\varepsilon}$. It is not hard to show, based on (19), that

$$\int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \cdot \left|\ln \frac{g(x)}{f(x)}\right|^3 dx \leq c_2 \int_{\mathbb{R}} (T(x) - x)^2 f(x)\, dx.$$

Since $f(x) \leq 2$ and $|f(x) - g(x)|/f(x) \leq 4|\ln(g(x)/f(x))|$ for $x \in [z_1, z_2]$, we deduce that

$$\begin{aligned}\int_{z_1}^{z_2} \frac{\nu(x)^2 |f(x) - g(x)|^3}{f(x)^2}\, dx &= 4 \int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \left(\frac{|f(x) - g(x)|}{f(x)}\right)^3 dx \\ &\leq 4^4 \int_{z_1}^{z_2} \frac{\nu(x)^2}{f(x)} \left|\ln \frac{g(x)}{f(x)}\right|^3 dx \leq 4^4 c_2 \varepsilon.\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}\int_{z_1}^{z_2} |f(x) - g(x)|\, dx &= \int_{z_1}^{z_2} \frac{\nu(x)^{2/3} |f(x) - g(x)|}{f(x)^{2/3}} \cdot \frac{f(x)^{2/3}}{\nu(x)^{2/3}}\, dx \\ &\leq \left[\int_{z_1}^{z_2} \frac{\nu(x)^2 |f(x) - g(x)|^3}{f(x)^2}\, dx\right]^{1/3} \\ &\quad \times \left[\int_{z_1}^{z_2} \frac{f(x)}{\nu(x)}\, dx\right]^{2/3}.\end{aligned}$$

Here $f(x) = |\nu'(x)|$, and therefore

$$\begin{aligned}\int_{z_1}^{z_2} |f(x) - g(x)|\, dx &\leq (4^4 c_2 \varepsilon)^{1/3} \left[\int_{z_1}^{w_f} \frac{\nu'(x)}{\nu(x)}\, dx + \int_{w_f}^{z_2} \frac{-\nu'(x)}{\nu(x)}\, dx\right]^{2/3} \\ &= (4^4 c_2 \varepsilon)^{1/3} \left[2 \cdot \ln \frac{1}{2} - 2 \cdot \ln(2c_1 \sqrt[3]{\varepsilon})\right]^{2/3} \ll \sqrt[3]{\varepsilon} |\ln \varepsilon|^{2/3}.\end{aligned}$$

On the other hand, the fact that $\tilde{\nu}(x_i) \leq 2\nu(x_i) = 4c_1 \sqrt[3]{\varepsilon}$ for $i = 1, 2$ yields

$$\int_{-\infty}^{z_1} |f(x) - g(x)|\, dx \leq 6c_1 \sqrt[3]{\varepsilon} \quad \text{and} \quad \int_{z_2}^{\infty} |f(x) - g(x)|\, dx \leq 6c_1 \sqrt[3]{\varepsilon},$$

and, in turn, we conclude Theorem 5.3. □

§6. *The proof of Theorem 1.2.* For a non-negative, bounded, not identically zero function h on $\mathbb{R}$, its log-concave hull is

$$\tilde{h}(x) = \inf\{p(x) : p \text{ is a log-concave function with } h(t) \leq p(t) \text{ for } t \in \mathbb{R}\}.$$

This function $\tilde{h}$ is log-concave, and $h(t) \leq \tilde{h}(t)$ for all $t \in \mathbb{R}$; therefore we may take the minimum in the definition. Next, we present a definition of $\tilde{h}$ in terms of $\ln h$. Let J_h be the set of all $x \in \mathbb{R}$ with $h(x) > 0$, and let

$$C_h = \{(x, y) \in \mathbb{R}^2 : x \in J_h \text{ and } y \leq \ln h(x)\}.$$

This set C_h is convex if and only if h is log-concave. Moreover, $J_{\tilde{h}}$ is the convex hull of J_h, and the interior of $C_{\tilde{h}}$ is the interior of the convex hull of C_h. We also observe that for any unit vector $u \in \mathbb{R}^2$,

$$\sup\{\langle u, v\rangle : v \in C_h\} = \sup\{\langle u, v\rangle : v \in C_{\tilde{h}}\}. \tag{20}$$

Let f, g and m be the functions in Theorem 1.2. The condition of the Prékopa–Leindler inequality is equivalent to

$$\tfrac{1}{2}(C_f + C_g) \subset C_m, \tag{21}$$

where $C_f + C_g$ is the Minkowski sum of the two sets. Choose $x_0, y_0 \in \mathbb{R}$ such that $f(x_0) > 0$ and $g(y_0) > 0$. For any $x \in \mathbb{R}$, we have $m((x + x_0)/2) \geq \sqrt{f(x_0)g(x)}$ and $m((x + y_0)/2) \geq \sqrt{f(x)g(y_0)}$, and hence

$$f(x) \leq \frac{m((x + y_0)/2)^2}{g(y_0)} \quad \text{and} \quad g(x) \leq \frac{m((x_0 + x)/2)^2}{f(x_0)}.$$

Since m is a log-concave function with finite integral, it is bounded; thus f and g are bounded as well. Therefore we may define the log-concave hulls $\tilde{f}$ and $\tilde{g}$ of f and g, respectively. It follows that $\tilde{f}(x) \geq f(x)$ and $\tilde{g}(y) \geq g(y)$. Since m is log-concave, (20) and (21) yield $m((x + y)/2) \geq \sqrt{\tilde{f}(x)\tilde{g}(y)}$ for $x, y \in \mathbb{R}$. We may assume that $\tilde{f}$ and $\tilde{g}$ are probability distributions with zero mean, and that $\tilde{f}(w_{\tilde{f}}) = 1$. It follows that

$$\int_{\mathbb{R}} f \geq 1 - \varepsilon, \qquad \int_{\mathbb{R}} g \geq 1 - \varepsilon, \qquad \int_{\mathbb{R}} m \leq 1 + \varepsilon. \tag{22}$$

Next, upon applying (5), Lemma 4.1 and Theorem 5.3 to $\tilde{f}$ and $\tilde{g}$, we conclude that

$$\int_{\mathbb{R}} |\tilde{f}(t) - \tilde{g}(t)|\, dt \ll \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3}. \tag{23}$$

In addition, (22) yields

$$\int_{\mathbb{R}} |\tilde{f}(t) - f(t)|\, dt \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}} |\tilde{g}(t) - g(t)|\, dt \leq \varepsilon. \tag{24}$$

Therefore, to complete the proof of Theorem 1.2, all we have to do is estimate $\int_{\mathbb{R}} |m(t) - \tilde{g}(t)|\, dt$. For this, let $T : I_{\tilde{f}} \to I_{\tilde{g}}$ be the transportation map satisfying

$$\int_{-\infty}^{x} \tilde{f}(t)\, dt = \int_{-\infty}^{T(x)} \tilde{g}(t)\, dt.$$

We note that $R(x) = (x + T(x))/2$ is an increasing and bijective map from $I_{\tilde{f}}$ into $\frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$. We define the function $h : \mathbb{R} \to \mathbb{R}$ as follows. If $x \notin \frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$, then $h(x) = 0$, and if $x \in I_{\tilde{f}}$, then

$$h\left(\frac{x + T(x)}{2}\right) = \sqrt{\tilde{f}(x)\tilde{g}(T(x))}.$$

We have $h(x) \le m(x)$, and the proof in §3 of the Prékopa–Leindler inequality using the transportation map shows that $\int_{\mathbb{R}} h \ge 1$. We deduce from (22) that

$$\int_{\mathbb{R}} |m(t) - h(t)|\, dt \le \varepsilon. \tag{25}$$

To compare h with $\tilde{g}$, we note that $\int_{\mathbb{R}} h \le 1 + \varepsilon$ implies

$$\int_{\mathbb{R}} h(t) - \tilde{g}(t)\, dt \le \varepsilon. \tag{26}$$

Let $B \subset \mathbb{R}$ be the set of all $t \in \mathbb{R}$ for which $\tilde{g}(t) < h(t)$; then $B \subset \frac{1}{2}(I_{\tilde{f}} + I_{\tilde{g}})$. In addition, let $A = R^{-1}B \subset I_{\tilde{f}}$. If $t = (x + T(x))/2 \in B$ for $x \in A$, then, as $\tilde{g}$ is log-concave and $\tilde{f}(x) = \tilde{g}(T(x))T'(x)$, we have

$$\begin{aligned}
&[h(R(x)) - \tilde{g}(R(x))] \cdot R'(x) \\
&\quad \le \left[\sqrt{\tilde{f}(x)\tilde{g}(T(x))} - \sqrt{\tilde{g}(x)\tilde{g}(T(x))}\right] \cdot \frac{1 + T'(x)}{2} \\
&\quad \le (\tilde{f}(x) - \tilde{g}(x)) \cdot \frac{\sqrt{\tilde{g}(T(x))}}{\sqrt{\tilde{f}(x)}} \cdot \frac{1 + T'(x)}{2} \\
&\quad = (\tilde{f}(x) - \tilde{g}(x)) \cdot \left(1 + \frac{(1 - \sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}}\right).
\end{aligned}$$

In particular, $\tilde{g}(x) < \tilde{f}(x)$ for $x \in A$. It follows from (5) and (23) that

$$\begin{aligned}
\int_B h(t) - \tilde{g}(t)\, dt &= \int_A [h(R(x)) - \tilde{g}(R(x))] \cdot R'(x)\, dx \\
&\le \int_{I_{\tilde{f}(x)}} |\tilde{f}(x) - \tilde{g}(x)| + \tilde{f}(x) \cdot \frac{(1 - \sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}}\, dx \\
&\ll \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3}.
\end{aligned}$$

It follows from (26) that $\int_{\mathbb{R}} |h(t) - \tilde{g}(t)|\, dt \ll \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3}$. Therefore, combining this estimate with (24) and (25) leads to $\int_{\mathbb{R}} |m(t) - g(t)|\, dt \ll \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3}$. In turn, we deduce that $\int_{\mathbb{R}} |m(t) - f(t)|\, dt \ll \sqrt[3]{\varepsilon}\, |\ln \varepsilon|^{4/3}$ by (23) and (24). □

Remark 6.1. A careful check of the argument shows that the estimate for $\int_{\mathbb{R}} |m(t) - f(t)|\,dt$ and $\int_{\mathbb{R}} |m(t) - g(t)|\,dt$ is of the same order as the estimate for $\int_{\mathbb{R}} |\tilde{f}(t) - \tilde{g}(t)|\,dt$. Therefore, to improve on the estimate in Theorem 1.2, all one needs to improve is (23).

§7. *Examples.*

Example 7.1. If f is an even log-concave probability distribution, $g(x) = (1+\varepsilon)\cdot f((1+\varepsilon)x)$ and $m(x) = (1+\varepsilon)\cdot f(x)$, then we have (5), and

$$\int_{I_f} f(x)(T(x)-x)^2\,dx = \frac{\varepsilon^2}{(1+\varepsilon)^2}\int_{\mathbb{R}} x^2 f(x)\,dx.$$

Example 7.2. Let f be the constant 1 on $[-\frac{1}{2}, \frac{1}{2}]$, and let g be a modification such that if $|x| \ge \frac{1}{2} - \varepsilon$, then

$$g(x) = \mathrm{e}^{-(|x|-1/2+\varepsilon)/\varepsilon}.$$

In addition, let

$$m(x) = \begin{cases} 1 & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ \mathrm{e}^{-(|x|-1/2)/\varepsilon} & \text{otherwise.} \end{cases}$$

In this case, $\int_{\mathbb{R}} m = 1 + \varepsilon$,

$$\int_{\mathbb{R}} f(x)\cdot\frac{(1-\sqrt{T'(x)}\,)^2}{2\sqrt{T'(x)}}\,dx \approx \varepsilon \quad \text{and} \quad \int_{\mathbb{R}} |f(x)-g(x)|\,dx \approx \varepsilon.$$

Moreover,

$$\int_{\mathbb{R}} f(x)(T'(x)-1)^2\,dx = \infty \quad \text{and} \quad \int_{\mathbb{R}} f(x)(T(x)-x)^2\,dx \approx \varepsilon^3.$$

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