

Lieb-Robinson Bounds and the Speed of Light from Topological Order

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We apply the Lieb-Robinson bounds technique to find the maximum speed of interaction in a spin model with topological order whose low-energy effective theory describes light [see X.-G. Wen, *Phys. Rev. B* **68**, 115413 (2003)]. The maximum speed of interactions in two dimensions is bounded from above by less than e times the speed of emerging light, giving a strong indication that light is indeed the maximum speed of interactions. This result does not rely on mean field theoretic methods. In higher spatial dimensions, the Lieb-Robinson speed is conjectured to increase linearly with the dimension itself. The implications for the horizon problem in cosmology are discussed.

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Introduction.—The principle of locality is one of the most fundamental ideas of modern physics. It states that every physical system can be influenced only by those in its neighborhood. The concept of *field* is the outcome of taking this principle seriously: if object A causes a change on object B , there must be changes involving the points in between. The field is exactly what changes. In addition, if something is “happening” at all the intermediate points, then the interaction between the objects must propagate with a finite speed. Relativistic quantum mechanics is built by taking the locality principle as a central feature. In nonrelativistic quantum mechanics the situation is more subtle: signals can propagate at every speed, and quantum correlations are nonlocal in their nature. One can, in fact, send information over any finite distance in an arbitrarily small time [1]. However, the amount of information that can be sent decreases exponentially with the distance if the Hamiltonian of the system is the sum of local pieces. Specifically there is an effective light cone resulting from a finite maximum speed of the interactions in quantum systems. This is the essence of the Lieb-Robinson bounds [2]. This notion has recently attracted interest in the context of quantum information theory, condensed matter physics, and the creation of topological order [1,3–6].

The concept of topological order is one of the most productive recent ideas in condensed matter theory [7]. It provides explanations for phases of matter (for example, fractional quantum Hall liquids) that cannot be described by the paradigm of local order parameters and symmetry breaking. If local order parameters cannot describe such phenomena, then their order could be of topological nature [7]. Topological order gives rise to a ground state degeneracy that depends on the topology of the system and is robust against any local perturbations [8]. Because of this property, topologically ordered systems appear to be good

candidates for robust quantum memory and fault-tolerant quantum computation [9].

Not only can topological order explain exotic phases of matter but it offers a whole new perspective to the problem of elementary particles. There are particles that we regard as fundamental, like photons and fermions, and other particles that can be interpreted as collective modes of a crystal. For example, we can describe phonons in this way because of the symmetry of the crystal. The understanding of the phases of matter provides an explanation for the phonon and other gapless excitations. However, one can also ask whether photons, electrons, or gravitons are emergent phenomena too, not elementary particles. Let us consider the case of light. Photons are $U(1)$ gauge bosons, and they cannot correspond to the breaking of any local symmetry [10]. Nevertheless, they can be collective modes of a different kind of order, and this is the case of topological order. Indeed models with topological order can feature photons, fermions, and even gravitons as emerging collective phenomena [7,11].

Light emerges from topological order as the effective low-energy theory of a quantum spin system. The quantum spin system is built as a local bosonic model, namely, a system in which the principle of locality is enforced by the fact that the Hilbert space decomposes in a direct product of local Hilbert spaces and all the observables have to commute when far apart. Moreover, the Hamiltonian must be a sum of local operators. In the low-energy sector, and in the continuum limit, the effective theory can be described by the Lagrangian of electromagnetism. Therefore low-energy excitations behave like photons. Maybe this is what photons really are, collective excitations of a spin system on a lattice with Planck-scale distance. But then, why do we not see signals that are faster than light? There could be all sorts of interactions that can

propagate as fast as permitted by the coupling constants of the underlying spin model. A theory of light as an emergent phenomenon needs to explain why we do not see signals faster than light.

In this Letter, we exploit the Lieb-Robinson bounds to show that the maximum speed of the interactions is of the same order of magnitude as the speed c of emerging light. This answers why we can think of light as an emergent phenomenon and still not see any faster signals in this model. In the last part of the Letter, we argue that the maximum speed increases linearly with the dimension of the space and consider the implications for the horizon problem in cosmology.

Topological order and artificial light.—If we want to impose the principle of locality in a strong sense, we must consider local bosonic models [10]. Fermionic models are not really local because fermionic operators do not generally commute even at distance. A local bosonic model is a theory where the total Hilbert space is the tensor product of local Hilbert spaces, local physical operators are finite products acting on nearby local Hilbert spaces, and the Hamiltonian is a sum of local physical operators. Thus local physical operators must commute when they are far apart. If we restrict ourselves to the case of a discrete number of degrees of freedom and finite-dimensional local Hilbert spaces, we have a quantum spin model. A quantum spin model can be therefore defined as follows. To every vertex x in a graph G we associate a finite-dimensional Hilbert space \mathcal{H}_x . The total Hilbert space of the theory is $\mathcal{H} = \otimes_{x \in G} \mathcal{H}_x$. To every finite subset of vertices $X \subset G$, we associate the local physical operators with support in X as the algebra $\mathcal{B}(\mathcal{H}_X)$ of the bounded linear operators over the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$. The Hamiltonian will have the form $H_{\text{local}} = \sum_{X \subset G} \Phi_X$, where to every finite subset $X \subset G$ we associate a Hermitian operator Φ_X with support in X . An example of a local bosonic model is given by a spin 1/2 system on a lattice. To every vertex x in the lattice we associate a local Hilbert space $\mathcal{H}_x \cong \mathbb{C}^2$. Local physical operators are finite tensor products of the Pauli matrices at every vertex.

The bosonic model we consider is a lattice of quantum rotors. Its low-energy effective theory is a $U(1)$ lattice gauge theory whose deconfined phase contains emergent light. Consider a square lattice whose vertices are labeled by \mathbf{i} , with angular variable $\hat{\theta}_{\mathbf{i}\mathbf{j}}$ and angular momentum $S_{\mathbf{i}\mathbf{j}}^z$ on its links. The Hamiltonian for the quantum rotor model is given by

$$H_{\text{rotor}} = U \sum_{\mathbf{i}} \left(\sum_{\alpha} S_{\mathbf{i},\alpha}^z \right)^2 + J \sum_{\mathbf{i},\alpha} (S_{\mathbf{i},\alpha}^z)^2 + \sum_{\mathbf{i},\{\alpha_1,\alpha_2\}} (t_{\langle \alpha_1 \alpha_2 \rangle} e^{i(\theta_{\mathbf{i}+\alpha_1} - \theta_{\mathbf{i}+\alpha_2})} + \text{H.c.}),$$

where $\alpha = \pm 1/2(1, 0), \pm 1/2(0, 1)$ are the vectors of length 1/2 pointing towards the lattice axes [10] and the t 's are coupling constants. In the limit $t, J \ll U$, the first

term of the Hamiltonian H_{rotor} behaves like a local constraint and makes the model a local gauge theory. Defining $g := 2/U(t_{12}t_{-1-2} + t_{2-1}t_{-21})$, the effective low-energy theory becomes the spin Hamiltonian

$$H_{\text{eff}} = J \sum_{\langle \mathbf{i}\mathbf{j} \rangle} (S_{\mathbf{i}\mathbf{j}}^z)^2 - g \sum_{\mathbf{p}} \frac{W_{\mathbf{p}} + \text{H.c.}}{2}, \quad (1)$$

where now $\theta_{ij} = S_{ij}^z$ and $W_{\mathbf{p}} = S_{12}^+ S_{23}^- S_{34}^+ S_{41}^-$ is the operator that creates a string around the plaquette \mathbf{p} (see Fig. 1). The smaller S , the smaller is the energy at which emergent light emerges [10]. In the following we assume $S = 1$. Although a lattice gauge theory is not a local bosonic model, this does not violate locality because H_{eff} is just an effective theory. The fundamental theory is local and H_{eff} is still a sum of local terms. In the large g/J limit, the continuum theory for the Hamiltonian H_{eff} is the Lagrangian of electromagnetism

$$L = \int d^2 \mathbf{x} \left(\frac{1}{4J} \mathbf{E}^2 - \frac{g}{2} \mathbf{B}^2 \right),$$

with speed of light given by $c = \sqrt{2gJ}$.

Lieb-Robinson bounds and the speed of sound in spin systems.—Here we review the proof of the standard Lieb-Robinson bounds [2] in the variant first proven in [4] and also exposed in [3]. We consider a Hamiltonian of the form $H := \sum_{X \subset G} \Phi_X$. Now consider an operator O_Y with sup-

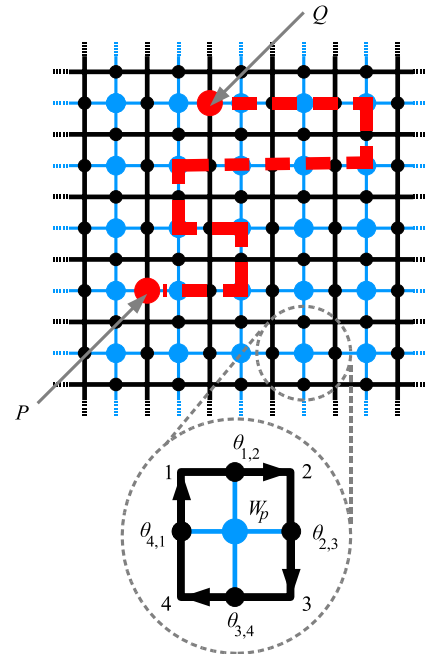


FIG. 1 (color online). A 2D-dimensional rotor lattice. To every plaquette p is associated a rotor operator W_p as a function of the variables θ_{ij} . The graph G is the one drawn in thin black lines. The graph G' is the graph with black and blue (lighter, bigger) dots as vertices and blue thin lines as edges. The red dashed line shows a path of length $n = 22$ from the point P to the point Q which are at a distance $2d(P, Q) = 8$ on G' or $d(P, Q) = 4$ on G . These paths contain alternating link and plaquette operators.

port in a set $Y \subset G$. The time evolution for this operator under the unitary induced by H is $O_Y(t) = e^{itH} O_Y e^{-itH}$.

The Lieb-Robinson bound is an estimate of an upper bound of the commutator of two operators $O_P(t)$, $O_Q(t')$ with support in different regions P and Q and at different times t and t' . If the interaction map Φ_X couples only nearest-neighbor degrees of freedom, the Hamiltonian can be written as $H = \sum_{\langle ij \rangle} h_{ij}$ and the Lieb-Robinson bound reads

$$\begin{aligned} \|[O_P(t), O_Q(0)]\| &\leq 2\|O_P\| \|O_Q\| \sum_{n=0}^{\infty} \frac{(2|t|h_{\max})^n}{n!} N_{PQ}(n) \\ &\leq 2\|O_P\| \|O_Q\| C \exp\{-a[d(P, Q) - vt]\}, \end{aligned}$$

where $h_{\max} = \max_{\langle ij \rangle \in G} \|h_{ij}\|$ and $N_{PQ}(s)$ is the number of paths of length $s/2$ between the regions P , Q at distance $d(P, Q)$ in G [1]. The constants C , a , v have to be determined in order to get the tightest possible bound. This bound is loose for several reasons: the crude maximization over $\|h_{ij}\|$, the overlook about the Hamiltonian's details, and the fact that all interactions are summed in modulus instead of amplitude, so that destructive interference is not taken in account. Note that, although we implement the principle of locality strongly by considering local bosonic models, Lieb-Robinson bounds also exist for fermionic models [12].

Lieb-Robinson bound for the emergent $U(1)$ model.—What do the Lieb-Robinson bounds tell us about the model H_{eff} with emergent light? Is the maximum speed of the interactions something like the speed of the emergent light or something completely different? As we have seen, this is of great importance if we want to take seriously the theory of light as an emergent phenomenon.

If we apply naively the Lieb-Robinson bounds to the Hamiltonian of the $U(1)$ lattice gauge theory, we see that the speed v is proportional to the strongest of the coupling constants, $v \propto g$. Since light only exists in the phase $g \gg J$, we would have $v \gg \sqrt{gJ}$. Fortunately, the bound can be made much tighter by examining the details of the Hamiltonian and the specific way the interactions propagate. Since we consider H_{eff} as an effective gauge theory, we will only consider commutators of operators that are gauge invariant, that is, that preserve the low-energy sector. Consider the function $f(t) := [O_P(t), O_Q(0)]$. Then consider the set $Z_1 := \{Z \subset G: [\Phi_Z, O_P] \neq 0\}$, the support of the complement of the commutant of O_P in the set of interactions. It turns out [3] that $f(t)$ obeys the differential equation

$$f'(t) = -i \sum_{Z \subset Z_1} ([f(t), \Phi_Z(t)] + [O_P(t), [\Phi_Z(t), O_Q(0)]]), \quad (2)$$

where $\Phi_Z(t) = e^{iHt} \Phi_Z e^{-iHt}$. From the above equation, and using the norm-preserving property of unitary evolutions, the following bound can be established [3,4]:

$$\begin{aligned} \|[O_P(t), O_Q(0)]\| &\leq \|[O_P, O_Q]\| \\ &\quad + 2\|O_P\| \int_0^{|t|} \left\| \sum_{Z \subset Z_1} [\Phi_Z(t), O_Q(0)] \right\|. \end{aligned}$$

Now we want to build an iteration from the above formula, exploiting the details of H_{eff} . For the sake of simplicity, consider the operators S_P^z, S_Q^z on the regions P, Q consisting of only one point each. The interactions propagate only when the operators do not commute. The only noncommuting operators in H_{eff} are W_p and S_i^z when they have a vertex in common (see Fig. 1). We therefore define $Z_{i+1} := \{Z \subset G: [\Phi_Z, \Phi_{Z' \subset Z_i}] \neq 0\}$, with $\Phi_{Y_{i(n)}} = S_{P(Q)}^z$. Iterating Eq. (2), we obtain

$$\|[S_P^z(t), S_Q^z(0)]\| \leq \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n, \quad (3)$$

where

$$a_n := \sum_{Y_i \subset Z_i} \prod_{i=1}^n \|\Phi_{Y_i}\|. \quad (4)$$

The meaning of the above expression is the following. Every element of the sum is a product of the type $\prod_i \|\Phi_i\|$ such that $[\Phi_i, \Phi_{i-1}] \neq 0$ for every i . If each Φ_i is a local bosonic operator, every one of those products is a path on the lattice. Therefore, a path in (4) will consist of steps from a plaquette to any of the four links bordering it, alternated with steps from a link to any of the two incident plaquettes. Any such path is then a path drawn with dashed edges in Fig. 1 on the lattice G' . To every path of length n on G' will then correspond an operator whose norm is bounded by $(gJ)^{n/2}$. Therefore, denoting by $N'_{PQ}(n, d)$ the number of paths of length n on G' from P to a given point Q at a distance $2d$, we obtain the following bound: $a_n \leq N'_{PQ}(n, d)(gJ)^{n/2}$. A gross bound is given by $N'_{PQ}(n, d) \leq 2\sqrt{8}e^{\kappa(n-2d+4)}n$, for every $\kappa > 0$. This is because there are eight ways to do a succession of two steps on G' : four choices from a blue vertex and two from a black one (see Fig. 1), and there must be at least one path of length n between P and Q . So for every $\kappa > 0$ we have

$$\|[S_P^z(t), S_Q^z(0)]\| \leq 4e^{4\kappa} e^{-2\kappa[d - \sqrt{2}(\sqrt{2gJ}e^{\kappa}/\kappa)|t|]}.$$

Optimizing for κ we get $v_{\text{LR}} = \sqrt{2}e\sqrt{2gJ} \equiv \sqrt{2}ec$. Numerically we can find a slightly better approximation. An exact combinatorial formula for $N'(n, d)$ is (we drop the subscript PQ from now on)

$$N'(n, d) = \sum_{k=1}^{\lfloor (n-d)/2-1 \rfloor} \binom{n-2k}{\frac{n-2k-d}{2}} \binom{n-2k}{\frac{n-2k}{2}} \binom{n}{2k} 4^{2k}.$$

We numerically studied the quantity $\sum_{n=0}^{\infty} (2|t|)^n a_n/n!$ because that is the one that enters the bound Eq. (3). The factorial at the denominator makes the series converge rapidly, and we obtain, together with Eq. (3),

$$\|[S_P^z(t), S_Q^z(0)]\| \leq Ae^{-[(d-vt)/\xi]}. \quad (5)$$

The speed v is estimated numerically as $v \approx e\sqrt{2gJ} \equiv v_{\text{LR}} = ec$. Let us try to understand this result. Equation (5) establishes that all the observables that are outside of the effective light cone centered on P with speed of light v_{LR} will have an exponentially small commutator with the observables in P . This result sets a limit to the speed of interactions in the spin system. It proves that any signal outside of a light cone generated with a speed that is of the same order of magnitude (and with the same dependence on coupling constants) of light will be exponentially suppressed. This constitutes a strong indication that the maximum speed of signals is light. So the theory of emerging light explains why its speed is also the maximum speed for any signal at low energies. If we were able to probe energies of order U , we could still find faster signals.

The cosmological horizon problem.—The isotropy of the cosmic microwave background presents us with the horizon problem: how is it possible that regions that were never causally connected have the same temperature? The horizon problem arises from the stipulation that interactions cannot travel faster than a finite speed, which defines a causal cone. Inflation solves the horizon problem by introducing an exponentially fast early expansion which allows for initial causal contact and thermalization of the observable Universe [13]. Other solutions require a time-dependent speed of light [14] or a bimetric theory [15]. Dynamically emerging light could also resolve the horizon problem.

Let us first understand how the speed v_{LR} depends on the dimension D of the space. Consider a hypercubic D -dimensional lattice. The number of paths of length n on a hypercubic D -dimensional lattice will be $N_D(n) \sim [4D(D-1)]^{n/2}$. If the two-dimensional case is any indication, a good enough approximation for $N_D(n, d)$, the number of paths of length n between two points P and Q at distance $2d$ apart will thus be $N_D(n, d) \sim [4D(D-1)]^{n/2} e^{\kappa(n-2d)}$, which implies a speed $v_{\text{LR}}(D)$ growing linearly with D [16]. Now consider a model of the Universe in which we start with an extremely connected graph that evolves towards a less and less connected graph, for instance, a hypercubic lattice of dimension $D(t) = D_{\text{in}}(1 - \alpha t)$. This type of situation has been hypothesized in quantum spin models of the Universe like quantum graphity [17]. In such a system, the maximum speed of interactions will decrease linearly with the dimension D of the space and hence, in time, will provide a possible explanation to the horizon problem in cosmology. Light cones then have parabolic sides and allow correlations at early times without violating causality since the distance of correlated points is of the order of $\alpha(t_i - t_f)^2$.

Conclusions.—In this Letter, we applied the technique of the Lieb-Robinson bounds to estimate the maximum speed of interactions for a quantum spin model with topological order and emerging $U(1)$ gauge symmetry. The importance of this model is that the low-energy excitations are photons. Light can be regarded as an emergent phenomenon,

and photons can be seen as collective modes instead of elementary particles [10]. This theory poses the problem of why we do not see other excitations that are faster than light. The technique of the Lieb-Robinson bounds, in the variation presented here, shows that the maximum speed of excitations in the model has the same order of magnitude as the speed of light. Of course, it is easy to construct a different model where there is emerging light *and* other faster particles. One of the fundamental questions of physics is to explain why this does not seem to happen in nature. In order to address this question, the Lieb-Robinson technique could prove useful if it could be modified to find a tight bound for a frustrated model, where destructive interference prohibits other signals potentially faster than light. In perspective, we think that this technique can prove useful to find exact results in $2D$ condensed matter models where there is scarcity of results that are not just numerical.

Finally, we have discussed the implications of the finite speed of signals for cosmology and the horizon problem.

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