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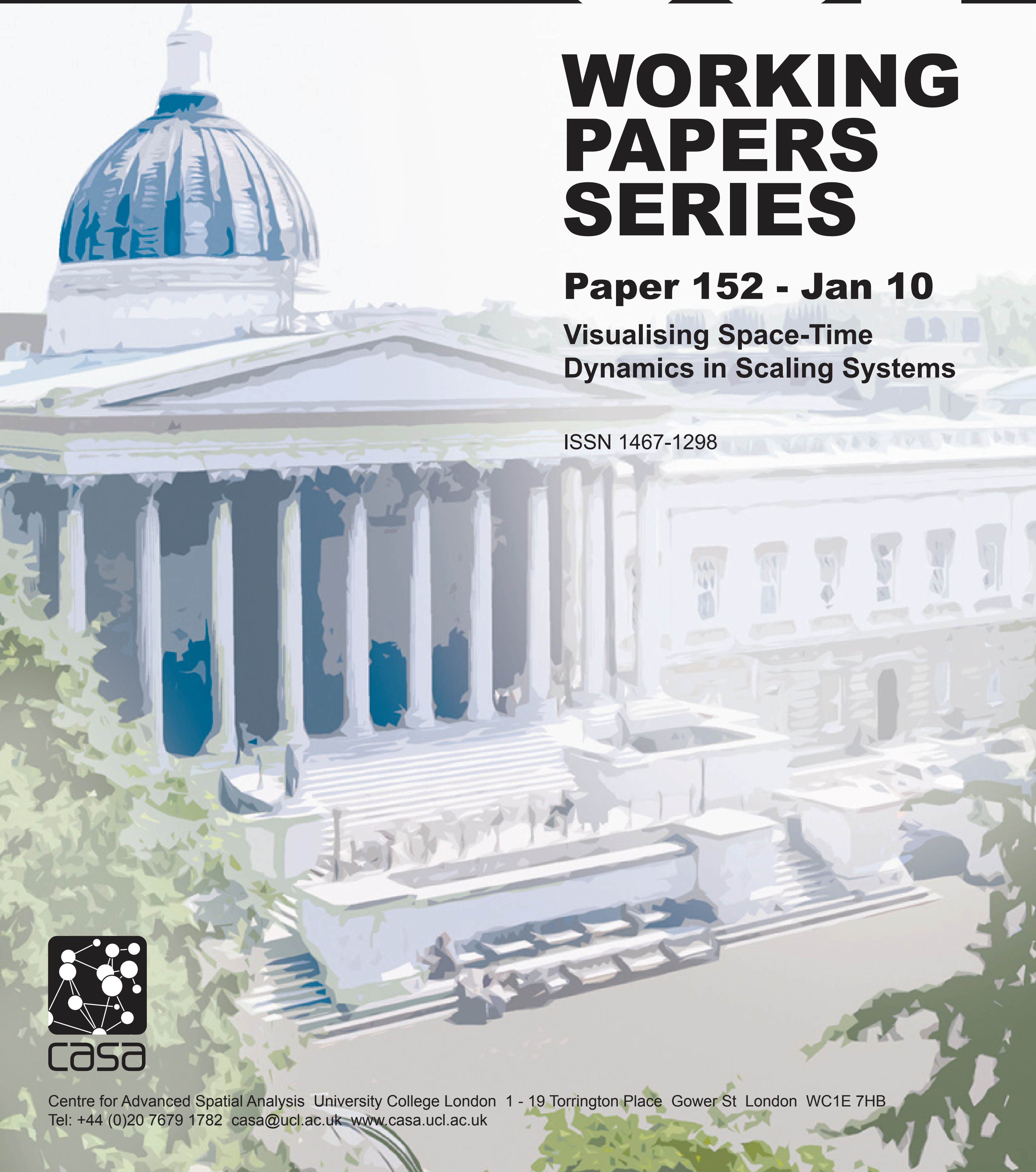

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Visualising Space-Time Dynamics in Scaling Systems

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Abstract

The signature of scaling in human systems is the well-known power law whose key characteristic is that the size distributions of the elements or objects that comprise such systems, display self-similarity in space and time. In fact, in many of the systems such as cities, firms, and high buildings which we use as examples, power laws represent an approximation to the fat or heavy tails of their rank-size distributions, appearing to be stable in time showing little sign of changes in their scaling over tens or even hundreds of years. However when we examine the detailed dynamics of how their ranks shift in time, there is considerable volatility with the objects in such distributions not often persisting for longer than about 50-100 years. To explore this kind of micro-volatility, we introduce a number of measures of rank shift over space and time and visualise size distributions using the idea of the 'rank clock'. We use the example of changes in the populations of Italian towns between 1300 and 1861 to introduce these ideas and then compare this analysis with city-size distributions for the World from 430BCE, the US from 1790, the UK from 1901, and Israel from 1950. The morphologies of growth and change displayed by these clocks are all quite different. When we compare these to the distribution of US firms from 1955 in the Fortune 500 and to the distribution of high buildings in New York City and the World from 1909, we generate a panoply of different visual morphologies and statistics. This provides us with a rich portfolio of space-time dynamics that adds to our understanding of how different systems can display stability and regularity at the macro level with a very different dynamics at the micro.

Scaling in Complex Systems

The objects or entities that define many complex systems often scale with respect to the frequency at which they occur in space and/or in time. Such scaling provides an order to the system which is manifest in the fact that patterns recur over different spatial or temporal scales, revealing what is called in fractal geometry, self-similarity. In spatial terms, such self-similarity is best visualised as some configuration of system entities that appear the same, or at least as a transformation of a standard pattern, from one scale to another. The exemplar par excellence is the dendrite whose branches mirror the way rivers drain a landscape, crystals solidify and liquids of different viscosity penetrate one another, all the way to how energy is delivered to the human body and how organisations arrange themselves in overlapping hierarchies (Pumain, 2006).

Formally the most general scaling law which captures the frequency $f(x)$ with which elements of different size x recur, is a power law defined as $f(x) \sim x^{-\alpha}$ where α is the scaling parameter of the distribution. Such frequencies scale in the sense that if x is multiplied by λ, then $f(\lambda x) \sim (\lambda x)^{-\alpha} = \lambda^{-\alpha} x^{-\alpha} \sim f(x)$. One essential feature of such power laws, apart from their very wide occurrence in natural and physical systems, is that fact that they have no length scale, being unbounded with respect to the size of an event or its frequency. Systems that are so defined have the potential to generate events of any size on any scale, having applicability to situations that are characterised by extreme events. In this sense, the signatures of systems that have the potential for uncapacitated growth, often reveal themselves in power laws.

In systems that scale, the focus can either be upon the frequency of events of different size or on the size of the events themselves. In one sense, this is only a matter of changing perspective but it does lead to confusion in terminology with respect to the shape of distributions. Examining the frequency $f(x) \sim x^{-\alpha}$, as events get larger, their frequency gets less and vice versa. In fact in popular usage, those events that are less frequent form the 'long tail' of the distribution with events of a smaller size being ever more frequent. In these systems which range from word frequencies to the sale of popular music, size is less important for each event is considered as unique (Anderson, 2006). In this sense, frequency is often taken as size. But in other systems such as those we deal with here, the larger the event, the less frequent this is, as for example in city-size or firm-size distributions. To deal with these systems, the usual practice is to transform the frequency distribution representing this as the counter-cumulative which gives the rank of the event or object in question according to its size, with the largest event being rank 1, the second largest rank 2 and so on. By integrating $f(x)$ from some x to $x_{\max}$, we define the rank as $r(x) = F_{x \to x_{\max}}(x) \sim x^{-\alpha+1}$ which can then be transformed with respect to size x as $x \sim r(x)^{1/1-\alpha}$ (Adamic, 2002). This is what Zipf (1949) called the rank-size rule which in its purest form exists when $\alpha = 2$. In this case, $x \sim r^{-1}$ and it is easy to show that when $r = 1$, then $x = x_{\max}$ and thus $x = x_{\max} r^{-1}$. Here the long tail is sometimes referred to as being the tail characterising the smallest sizes, with the heavy or fat tail being the one describing the largest (and often most significant) events of which there are far fewer. We will adopt this definition of scaling henceforth.

The reasons we have for referring to these distributions with respect to their tails is due to the fact that often power laws are used to best approximate certain portions of the distribution such as the fat tails. In fact this raises the general notion that although scaling distributions represent our starting point here, many distributions that define complex systems can only be so approximated and at the present time, it appears that lognormal distributions represent a wider generic class describing the way such systems are structured. This relates directly to the fact that lognormal distributions emerge from extremely simple models of competitive growth involving growth and decline as well as births and deaths of new system elements, and under certain conditions, such growth models generate power laws (Simon, 1955; Blank and Solomon, 2000; Gabaix, 1999).

In the sequel, we will first define growth mechanisms based on models of competition that lead to systems whose elements change in size. These types of generic mechanism define the population systems that form our concern here. These are cities, buildings and firms, which in general change in size over decades or centuries but are subject to a more volatile dynamics over very long time periods such as millennia. Our analysis shows that although the envelopes defining their distributions are extremely stable and with respect to their upper or heavy tails are essentially scaling, there is considerably more volatility over short periods of time measured in decades where cities, firms and buildings rise and fall in size. To provide some sense of these dynamics, we introduce a powerful method for visualising how these elements of complex systems change relative to each other over time. We present the idea of the rank clock (Batty, 2006a) which encapsulates the nature of the space-time dynamics while at the same time, revealing how different systems have very different kinds of dynamics and growth regime dependent on the type of system, its spatial scales and the temporal periods over which it is examined. After introducing the generic clock, we examine different city-size distributions, moving to non-spatial systems such as firm sizes, and thence to one-off growth as in the construction of high buildings. Finally we draw together the implications of these different growth regimes which characterise the systems in question and speculate on how these visual methods might be extended to deal with other manifestations of scaling in the same kinds of system. This sets an agenda for further research which we indicate by way of conclusions.

The Dynamics of Competition

The simplest model of a growth process assumes no births or deaths but starts at time t with a fixed number n of elements $x_i(t)$ and grows each one (i) by applying a random growth ε_i to their existing size so that $x_i(t+1)=(1+\varepsilon_i)x_i(t)$. Simple experimentation with this process for a fixed number of objects such as firms or cities, say, indicates that it is increasingly unlikely for all the objects to grow at the greatest rate, and that ultimately one of these will dominate. In the process, the size of the objects will go up and down but eventually one will dominate and there will be some scaling such that the frequency of the largest objects is much less than the smallest. If the growth rate can also become negative, then some objects will ultimately become too small and will be deemed to have died introducing the idea that objects might also be born. Generalising the process in this way and basing the entire model on random proportionate growth, generates distributions which are essentially lognormal as Gibrat (1931) first illustrated. The long tail of the lognormal frequency distribution might be considered the fat tail of the rank-size distribution and it is this that is often approximated by a power function, essentially cutting off this tail and assuming that it applies to the most significant part of the system. If this model is further generalised to repel objects from becoming too small, various researchers (Blank and Solomon, 2000; Gabaix, 1999; Sornette and Cont, 1997) have shown that a power law distribution can be generated over the whole range of sizes.

This is one of the most parsimonious ways of generating a size distribution that scales but it does not admit any interactions between the elements. However, when adapted to a network where the size of the node grows using proportionate effect and adding network links from which such effects emanate, similar scaling distributions for the size of the hubs emerge (Batty, 2006b). Indeed, Barabasi's (2005) models which depend on preferential attachment can be seen as a subclass of these kinds of growth model, all falling under a wider class of models that emphasise cumulative mutual advantage.

There are several types of growth process that can be modelled using this generic mechanism. In terms of cities, populations that are measured by their size can grow or decline. Cities grow from the smallest settlements to the point where they 'become' cities and their growth is thus asymmetric in that to be a large city, one must first be a small city. This is characteristic of all growth processes of course. Because of difficulties of defining what a city is at the lowest population level but more because of the lack of data, we will only work with the largest cities, taking the top 100, 200 and so on, thus establishing immediately that the analysis and visualisation is relative to the size of the system in terms of numbers of objects chosen. In such cases, births of cities occur when they enter the top ranks – say the top 100 – and cities die when they leave this ranking. Cities are of course regularly spaced according to a spatially competitive hierarchy originally formulated by Christaller (1933) under the banner of central place theory, and the random proportionate growth model appears to be consistent with this logic.

There are of course difficulties in defining the spatial extent of cities but generally these can be dealt with. It is much harder to ensure that firms are defined appropriately as mergers and acquisitions can destroy any size grouping immediately for there are no constraints as in cities where spatial extent determines size. In terms of buildings, although these have generally got taller as cities have grown, certainly over the last one hundred years since skyscrapers were first constructed, buildings do not usually grow *per se*. They are usually constructed afresh and thence demolished and only small fraction of buildings are modified and grown or reduced in size *in situ*. It is easier to see how firms and cities compete to increase in size than it is buildings. Nevertheless, bigger buildings usually do occur in places where there are already high buildings, and it is easy to consider the random proportionate growth model being adapted to take account of copy-cat like behaviours as ever higher buildings are proposed and constructed. In the case of buildings then, we might expect a rather different dynamics from that which is characteristic of city and firm sizes. In the period in question – the last 100 years or so – we will disregard the very small number of tall buildings that have been demolished concentrating only on those which have been constructed anew, which do not grow *per se*, and which are still within their planned lifetime.

In the analysis that follows, we will first examine the scaling that is implicit in the space-time dynamics of cities. This will provide us with evidence that very strong scaling exists across different time spans and at different spatial scales from the world to the nation to the region. But within this, there is substantial evidence of volatility and it is this that we need to explain, visualise and understand here. To this end, we will introduce the idea of the rank clock and illustrate this for city-size distributions before we continue by examining firm sizes and then high buildings. In this way, we will provide a spectrum of spatial and temporal scales which will illustrate the complexity of the systems we are dealing with on many different levels.

The Space-Time Dynamics of Cities

We begin with a modest example involving the growth of cities in the Italian peninsula from the 14th to the 19th century (Bosker, Brakman, Garretsen, de Jong, Schramm, 2007; Malanima, 1998)[1]. The space-time dynamics is relatively uncomplicated in that the main cities – Bologna, Firenze, Genova, Milano, Napoli, Padova, Palermo, Roma, Venezia, and Verona – were established by the 14th century and the period until Italian unification when our analysis ends (in 1861) did not see dramatic growth or radical shifts in the ranking of the key centres. We have a manageable seven time instants – 1300, 1400, 1500, 1600, 1700, 1800, and 1861 – for which we have population size data on the 555 towns that existed during the period, and we can thus rank the towns by population size and examine any changes in rank rather easily. This core of towns has remained in the top 15 by population size since the beginning of the Renaissance. In the subsequent analysis, we will only examine the rankings of the top 100 towns by size at each of the seven time instances although over the temporal period, a total of 195 distinct towns enter the analysis. Of course some enter the top 100 and then leave, only to enter again by the end of the period, and our analysis cannot account for towns that move up and down the rankings during the inter-temporal periods for which we do not have data. 360 towns that exist in the data set in fact never enter the top 100 and are thus really villages or at least settlements that never reach significance. We need to note that although we are dealing with the top 100 towns by population size in each of the seven time periods, in the second period ($t = 1400$), the total number of towns falls to 95 while in all other periods, the number is slightly greater than 100 due to ties in the rank orders.

In terms of the relative stability of population during this period, total population of all the 195 towns that enter the top 100 shows that a population of 2.1m(illion) in 1300 declines to 1.71m by 1500 (probably due to the Black Death and war). It only recovers to 2.22m in 1700 and then rises to 3.97 by 1861, indicating the early beginnings of political unification and industrialisation. As we will see, this comparative stability is complemented by highly stable rank-size relations. However, this is only true of the core towns as there is considerable volatility in the smallest towns in terms of their ranks which show the essential struggle to compete in the face of the predominance of the core towns of the Italian city states.

[1] The data has been compiled by Paolo Malanima and is available at http://www.issm.cnr.it/

This is a good case for it lets us introduce a number of tools for analysing and visualising space-time dynamics using a particularly tractable and easy-to-understand example. We first graph the seven rank-size relations as logarithmic transformations of $x_i \sim r(x_i)^{1/1-\alpha}$ which give $\log x_i = \Phi - \beta\, r(x_i)$ where the slope β is related to the scaling parameter $1/(1-\alpha)$. We show these distributions in Figure 1(a) where their similarity is even clearer when they are collapsed onto one another. There are various ways of estimating the scaling parameter. The most traditional which in fact is the most biased is to estimate β using ordinary least squares (*OLS*) from which we can then compute the scaling parameter as $\hat{\alpha}_{OLS} = 1 + \beta^{-1}$. A less biased way however is to use maximum-likelihood which has been adapted to power functions by Newman (2005) and consists of solving the likelihood equation for each distribution as

$$\hat{\alpha}_{ml} = 1 \quad + \quad n\left[\sum_i \log \frac{x_i}{x_{\min}}\right]^{-1} \qquad (1)$$

where $\hat{\alpha}_{ml}$ is an estimate of the scaling parameter, n is the number of observations in the city data set (which can vary for each time period), and $x_{\min}$ is the minimum population defining the lower bound of each city-size distribution. Because the conventional argument is that such size distributions are more likely to be in their steady state in the fat or upper tail, then the lower tail should be truncated at some minimum and as Clauset, Shalizi, and Newman (2009) suggest, this might be a matter for experimentation. We do indeed explore this in the software available for these visualizations and having estimated $\hat{\alpha}_{ml}$ for different K minimum values x_k, we form an average as

$$\overline{\alpha} = \frac{1}{K}\sum_k \left\{1 \quad + \quad n\left[\sum_i \log \frac{x_i}{x_k}\right]^{-1}\right\} \quad . \qquad (2)$$

In Table 1, we present these estimates whose values are very close to one another. Moreover we speculate that these values are close enough to suggest that there have been no major transitions in top ranked cities during this 500 year period.

Visualising Scaling: The Rank Clock

There are two obvious ways in which to further explore the space-time dynamics in these distributions. First we can examine the shift in ranks between two periods. For any two different time instants, we can examine the rank shift which can be visualized by plotting one of the size distributions using the ranks associated with the other distribution. In Figure 1(b), we show this shift for the Italian city sizes, plotting the distributions in 1400 and 1861, where shift is based on plotting the 1400 sizes using the 1861 ranks. The picture is one of greater volatility than we have seen so far. We can plot all the shifts for every time instant in rank space if we trace the rank and size of each city through their evolution as we do in Figure 1(c). Here the colours are set as follows: the largest and most ancient town is coloured red throughout and towns that are smaller and/or enter the top rank later are coloured according the spectrum red-orange-yellow-green to blue. We also use this colour map for the rank clocks that we use to visualize these and other distributions introduced later in this paper. This visualisation in rank space however confuses the picture even more. Whereas Figure 1(b) does imply considerable shifts in rank, Figure 1(c) tends to play these down for the balance of colour shows that the oldest and largest ranks from early in the evolution of the city system tend to remain in place throughout. What we need is something a little more visually intrusive to pull out any deviations that are significant and to this end, we introduce the idea of the rank clock.

Table 1: Estimates of Rank-Size

Year	$\hat{\alpha}_{ml}$	$\overline{\alpha}$	$\hat{\alpha}_{OLS}$	r^2
1300	2.493	2.699	2.447	0.886
1400	2.836	2.696	2.393	0.851
1500	2.606	2.522	2.283	0.858
1600	2.500	2.438	2.262	0.873
1700	2.631	2.506	2.277	0.871
1800	2.698	2.594	2.403	0.893
1861	2.582	2.562	2.365	0.899

The rank clock focuses entirely on changes in the ranks with the trajectory of each object – in this example a town or city – representing its rank order in a circular space. At any time, the top ranked object is always at the centre, the lowest rank at the circumference or the furthest point to the edge if the circumference has not been reached. Time is arrayed in a clockwise direction from the usual north-noon point of the clock. The time period over which the analysis takes place is marked by the circumference whose length is 2π , in short the complete clock. To compare two rank clocks, it must be assumed that the temporal circularity is directly comparable between examples and it is thus only relevant to do this, if one assumes that the temporal behaviours of two or more different systems are comparable within the sweep of the circumference. Moreover, systems with different number of objects – are usually not directly comparable although a suitable scaling of the clock might generate some comparability. As the examples introduced below will show, different clock configurations define very different kinds of scaling system, thus representing a new kind of morphology for analysing and visualising space-time dynamics.

We show the clock in Figure 1(d) where two features are immediately clear. First the towns that are top ranked in the late Middle Ages and early Renaissance retain their dominance to the middle of the 19th century (and casual observation suggests that this is still the case in the early 21st century). This is clearly seen in the concentration of the colour red around the pole of the clock. Second, most of the volatility where colours diverge markedly from circularity, occurs towards the edge of the clock. Here we see little evidence of towns that are large and become small leaving the pole of the clock, spiralling out to the edge, or of towns that rise in rank dramatically spiralling into the clock. Unlike our later cases, there are only a handful of such examples in the Italian city system. Torino does not enter the picture (the top 100 ranks) until 1500 and then it rises swiftly to rank 4 by 1861. Siena on the other hand occupies rank 7 in 1300 and drops to rank 45 in 1861. It is hard to make these out in the clock in Figure 1(d) but the software that is available enables this to be illustrated quite easily as we will show for other examples (http://www.casa.ucl.ac.uk/software/rank.asp and Batty, 2006a).

The clock also focuses us on other measures of change. Changes in rank from time period to time period are clearly seen as changes in the distance travelled around the clock as this kind of morphology is rooted in a circular geometry. For the rank $r_i(t)$ of each object i at time t, we can define an individual distance – a first order difference – which can be plotted on the clock. We do not show the distance clock here (see Batty, 2006b) but distance is defined as

$$d_i(t) = | r_i(t) - r_i(t-1) | \quad . \tag{3}$$

Aggregate distances can be defined over all cities for each time period as

$$d(t) = \sum_i | r_i(t) - r_i(t-1) | / n(t) \tag{4}$$

where $n(t)$ is the number of cities in the distribution at each time period. If cities are unique in terms of measures of their size, then this might be set as $n(t) = 100, \forall t$ although in the Italian example because of ties in size, the number varies slightly at each time. This gives a measure of the average changes in rank that occur for all cities that remain in the set. In fact second, third, fourth and greater order changes can be computed if so required as can related measures of cumulative change such as

$$d(\tau) = \sum_{i,t=1,\ldots,\tau} | r_i(t) - r_i(t-1) | / n(t) \quad . \tag{5}$$

The average switch in ranks over all cities and all time periods, in this example from 1300 to 1861, and over all 195 cities that appear in the top ranks, is

$$d = \sum_t d(t)/T \quad . \tag{6}$$

where T is the total time over which the change takes place; in this example, it is 561 years. We show these measures for the Italian system in Table 2 where it is clear that on average, a typical city shifts about 15 ranks (d) over 100 years, consistent with shifts $(d(t))$ taking place over each 100 year interval which range from 12 to 17.

Our last measure of space-time dynamics involves the speed at which cities enter or leave the top ranked set of cities at different times. It is easy enough to count the number of cities composing the top set of ranks at time t which are still in the new top set at some time later or earlier than t, say $t+\tau$ or $t-\tau$. We define this number as $L(t, t+\tau)$ for the cities that existed at time t in the top ranks and are still in the top ranks at $t+\tau$; and $L(t-\tau, t)$, the number of cities that were in the top ranks at time

$t-\tau$ and are still in the top ranks at time t. A little reflection suggests symmetry of these counts, that is $L(t,t+\tau)=L(t-\tau,t)$. We can work out the average number of cities that perpetuate with respect to a given time period t as

$$L(t)=\sum\nolimits_{m\neq t} L(t,m)/(n-1) \qquad , \qquad (7)$$

while the average number of cities that perpetuate from the top ranks at any time over all other time periods is

$$L=\sum\nolimits_{\ell\neq t,m\neq\tau} L(\ell,m)/(n-1)^2 \qquad . \qquad (8)$$

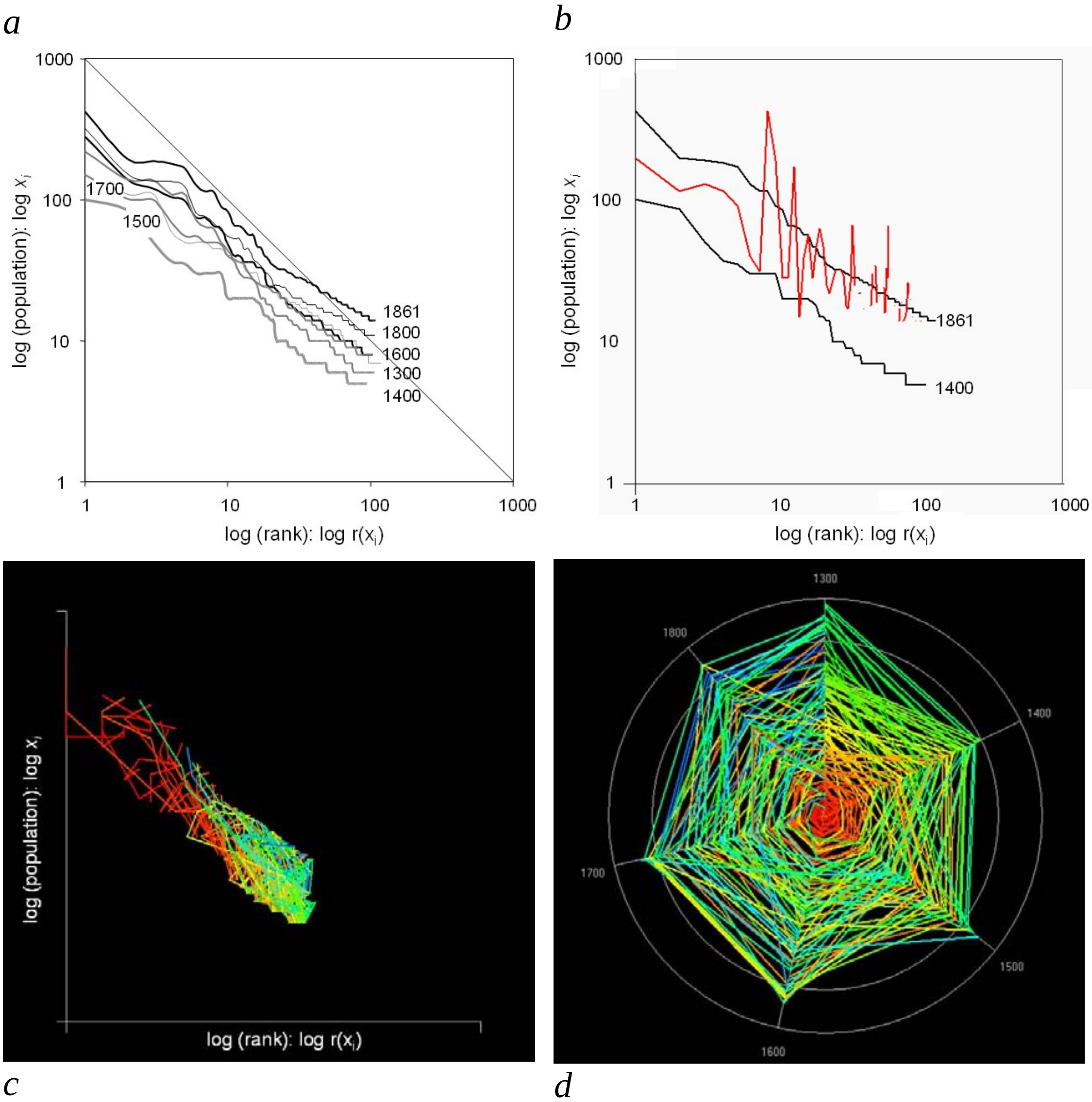

Figure 1: Visualising Space-Time Dynamics in Terms of Rank Shift

a) Zipf Plots for the Seven Time Instants b) Shifts in Rank Between 1400 and 1861 c) Shifts in Ranks of all Towns Throughout the Seven Time Periods in Rank Space, and d) the Rank Clock. The colours are chosen from red through blue according to the chronology of appearance of towns from 1300 and according to their rank. The first town the top rank at 1300 is coloured red; the last town is the last rank to enter at the latest time period and is coloured blue.

Table 2: Distance Measures: Changes in Rank for All Cities over All Time Periods

Time Period	$d(t)$	$d(\tau)$
1300-1400	16.949	16.949
1400-1500	16.797	21.612
1500-1600	17.064	23.775
1600-1700	11.780	25.371
1700-1800	13.063	23.940
1800-1861	14.518	21.446
Average Distance d	15.519	

These measures assume that the number of cities is fixed and need to be modified if these vary by time period as in the Italian example; proportionate or probability measures can also be computed if required.

The half life has to be defined with respect to the entire time period T. We do not have explicit functions that describe the process by which cities persist in the top ranks. Thus we need to figure out these half lives by inspection and interpolation where the time intervals are usually not fine enough to compute these half lives exactly. We can do this for the whole series, of course, or we can do it for each time instant where it will vary. Essentially for each time t, we need to solve the equation $L(t,t)/2 = L(\tau,t) = L(t,\tau)$. Because no generic formal function for this persistence exists, we need to examine the matrix $L(t,t+\tau)$ but assuming exponential decay, it is likely that the number of cities in the top ranks will decline through time from the point at which they are considered. If they do not, then this is evidence of extreme persistence and regularity in the system and the degree to which the half life approaches the maximum number of ranks is a measure of this stability.

Table 3: Number of Cities $L(\ell,m)$ that Persist from Times ℓ to m

Time ℓ,m	**1300**	**1400**	**1500**	**1600**	**1700**	**1800**	**1861**
1300	*118*	81	80	71	70	67	65
1400	81	*94*	81	65	62	64	59
1500	80	81	*106*	82	76	73	70
1600	71	65	82	*104*	85	80	76
1700	70	62	76	85	*103*	89	79
1800	67	64	73	80	89	*107*	*86*
1861	65	59	70	76	79	86	109
$L(t)$	72	69	77	77	77	77	73
L	74						

Our Italian example is useful for the number of time periods is small enough to enable a casual examination of the structure of the matrix elements $L(t, t+\tau)$ which we present in Table 3. Although the maximum number of towns in any one distribution is 118, 195 different towns appear in the matrix from 1300 to 1861. There is in fact a remarkable persistence of the core set of towns with an average of about 74 appearing to perpetuate throughout the period. In fact in the remaining towns, there is considerable volatility in the rankings. So in this system which at first sight seems simple and robust with little change amongst the largest towns, there is considerable movement in the smaller towns which continually and aggressively compete for pride of place. From Table 3, it is clear that the half life in all cases is perhaps a little longer than the 561 time periods over which these distributions of towns are observed, implying very little spatial restructuring and rather low levels of growth. For the 118 towns in 1300, some 65 or 55% of these still exist in the top 109 in 1861. An average half life of around 600 years would be a good guess but this is much larger than the other examples of city systems that we will now examine.

City Systems at Different Scales

We now have a small arsenal of tools to explore the space-time dynamics of scaling systems and it is worth briefly summarising what we have presented. First, a good measure of stability with respect to size is the Zipf plot, specifically the scaling parameter α or its equivalent β which is the slope of the plot. This gives the degree of competition between the elements with $\beta = 1$ and $\alpha = 2$, the pure Zipf case where $x_i \sim r(x_i)^{-1}$. When $\beta < 1$ or $\alpha > 2$, then the larger objects in the distribution are closer in size to the smaller implying less competition and vice versa. The measure of fit of the power law to the Zipf plot of course and how these change through time is also a measure of how close these plots are. The two methods we have suggested – ML and OLS – are standard, notwithstanding various known biases (Newman, 2005; Clauset et al., 2009). Second, the best graphic is the rank clock rather than any measure associated with the Zipf plot for individual objects can be traced in terms of their trajectories and the whole set of objects can be visualised as a morphology. The clock is composed of distinct trajectories, five of which we visualise in Figure 2, although our more general quest is to compile different examples of these morphologies which will ultimately provide some sense of the dynamics of different kinds of scaling systems. We will attempt this now although we will show different morphologies for cities later in this section and for firms and tall buildings in the next.

In the clock in Figure 2, objects that remain at the same rank are marked by exact circles around the clock which close on themselves. Objects which rise inexorably in their rank, spiral into the clock while objects that decline systematically spiral out. Objects can of course enter and leave the clock as many times as is possible and there are obvious extreme cases: if an object were to enter at t, leave at $t+1$, enter again at $t+2$ and so on, then this pattern would simply appear as a dot every other time period at the rank where it entered. Note that we do not have the rank of the object before it enters the top ranks so objects quite literally appear on the clock when they enter. If we did have these ranks, then we could compute their actual trajectories. One way of doing this would be to construct the clock for a much bigger system of objects and then reduce this to a lesser number of top ranks. We also show in Figure 2, objects that display such oscillations showing one that enters and then leaves to re-enter again and its opposite which is one that leaves, re-enters and leaves again. The pole of the clock is significant for in many scaling systems, one of the objects often remains the biggest. In city systems, this is quite common: for example, for the last 200 years, New York City has been at rank number 1 in the US urban system, London at rank 1 for the UK, Paris for France, and so on.

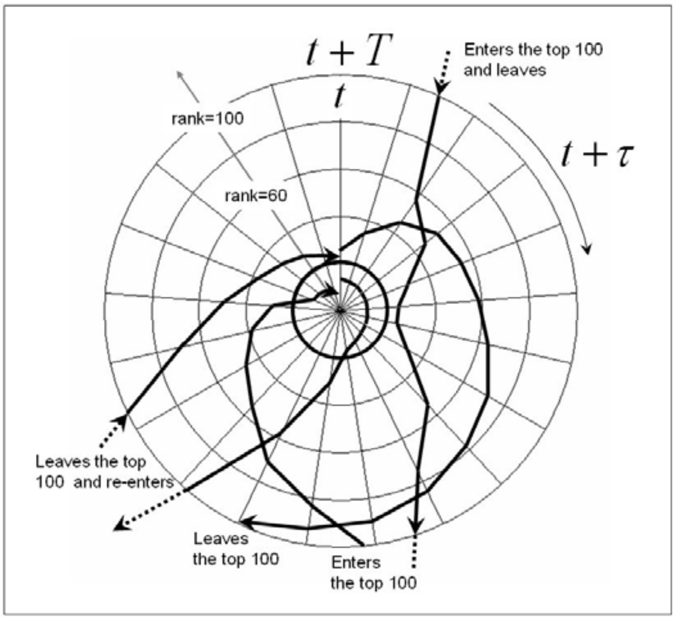

Figure 2: Possible Trajectories Defining the Morphology of the Rank Clock

The two measures of difference, the first based on distance and the second based on guesstimates of the half life, are useful in measuring actual shifts in rank. The distance of course depends on the time periods in question and represents the shift which takes place in numbers of ranks over one standard time period. In the case of the Italian data, this is 100 years (with 61 years appropriately adjusted for the statistics in the seventh period). Examining Table 2, we see that the average rank shift is about 15 over 100 years and this suggests that over 500 years the shift would be about 75. In fact the second statistic is the half life which we guessed at about 600 years where about 50-60 would shift out of the top ranks entirely during this period. As cities enter and leave the rankings, then simply taking the average distance d and scaling it by 5 or 6 will give an overestimate of the shift but it suggests that judicious use of half lives and distance measures provide a rich picture of these dynamics The cumulative distance $d(\tau)$ in fact is probably a better measure of shift because this does take into account cities that enter and re-enter the rankings as the system evolves.

We are now in a position to select and track some very different city systems which exist at very different scales. We have analysed city-size distributions for the top 50 cities in the world from 430BCE to 2000AD with variable time intervals using Chandler's (1987) data base which is the largest scale and longest time period that we have dealt with. At the next scale down, the continental, we have examined changes in rank-size for the top 100 cities in the US from 1790 to 2000 at ten year intervals (from the US Census Bureau, Gibson, 1998), then for country level, the top 100 in the

UK from 1901 to 2001 also at ten year intervals[2], and finally for a much smaller region, the top 172 towns in Israel from 1950 to 2005 with variable time intervals (Benguigui, Blumenfeld-Lieberthal, and Batty, 2009). Israel is more like a metropolitan region with respect to scale although each of these examples have very specific geo-political and cultural characteristics which add to the variety of this selection.

For each of these systems (including the Italian example), we need to choose either a fixed set of top ranks or simply let the number of cities in the wider database condition the number of cities being ranked at any time instant. For example in the Israeli example, there are 172 towns which exist at some time over the 55 year period from 1950 to 2005. In 1950, there are 34 towns defined while by 2005 there are 164. However 8 of the towns appear at some point in the rankings for the 13 time instants, entering and then leaving the space by 2005. In this sense, we do not constrain the Israeli example whereas for the UK example, we have 458 towns that do not change throughout the entire 100 year period. This is complicated by the fact that we only examine the top 50 towns at any of the 10 time instants, and in this case, 70 of the 458 towns are considered but at each point in the rankings, there are only 50 towns. To be able to make good comparisons, we need to be clear about these limits and how the statistics of rank shift are affected by changes in numbers of cities and times.

The World system of cities is the most volatile. There are no cities in the top 50 in 430BCE that are still in the top 50 by 2000, and there are only 6 that still exist in the top 50 from the Fall of Constantinople in 1453. In fact the half life of the original set of cities is about 200 years which reduces to about 100 years by the 20th century. In short, the rise and fall of civilizations, particularly Greece and Rome, the coming of the Dark Ages, the parallel growth and decline and growth again of China, and the explosion of cities in the developing world can all be gleaned from the trajectories of cities in this data base. The morphology of the rank clock is shown in Figure 3(a) and here it is quite clear that there is no sense in which there is a group of persistent core cities as in the Italian example. The longest lived example of a city in the top 50 is Suzhou in China which exists for 2158 years of the 2430 years covered and even this city is no longer in the top 50 (although it is growing fast and could re-enter the list unless it is absorbed in Greater Shanghai). The half life of cities in the World system is clearly reducing fast and is now no more than 75 years, falling at the rate of 20 years for every additional 25 years of time. The system nearest this rate of change is the US system which we plot in Figure 3(b) which displays all the features of the examples in Figure 2. New York City remains at the centre of the clock unchanging since 1790, while cities like Chicago, Los Angeles and Houston spiral into the top ranks as the population has diffused to the Mid-West, California and the South West over the last 150 years. Cities such as Charleston in the old colonial east spiral out of the clock as the US begins to industrialise from the mid 19th century on.

Our two other examples are as different again. In the case of the Great Britain (England, Scotland and Wales), the half life is about twice that of the US and the current World system, at about 150 years, and it shows little sign of changing. This is because by 1901, all the key settlements were established – the core cities had

[2] Census Dissemination Unit (CDU) http://census.ac.uk/cdu/, accessed 06/01/10

developed during the 19th century and suburbanisation in the 20th has not made much impression on the overall pattern of urban development. The clock is shown in Figure 3(c) where the core stands out as in the Italian system. The Israeli system however is quite different in that it is almost impossible to guess the half life as new settlements have been established and grown continually during the last half century during the time when the country has developed and consolidated. In such a growth situation, the half life is effectively an order of magnitude longer than the time during which development has taken place. A crude guess would be between 50 and 70 years but in a growth situation where all 34 of the original settlements in 1950 are still in the top ranks in 2005, this is hard to assess for the number of settlements dropping out of these ranks is very small. The half life is really a forward looking measure notwithstanding the symmetry of the flow matrix $L(\ell, m)$. Better measures involve distances which we will now discuss. The rank clock for the Israeli system is shown in Figure 3(d).

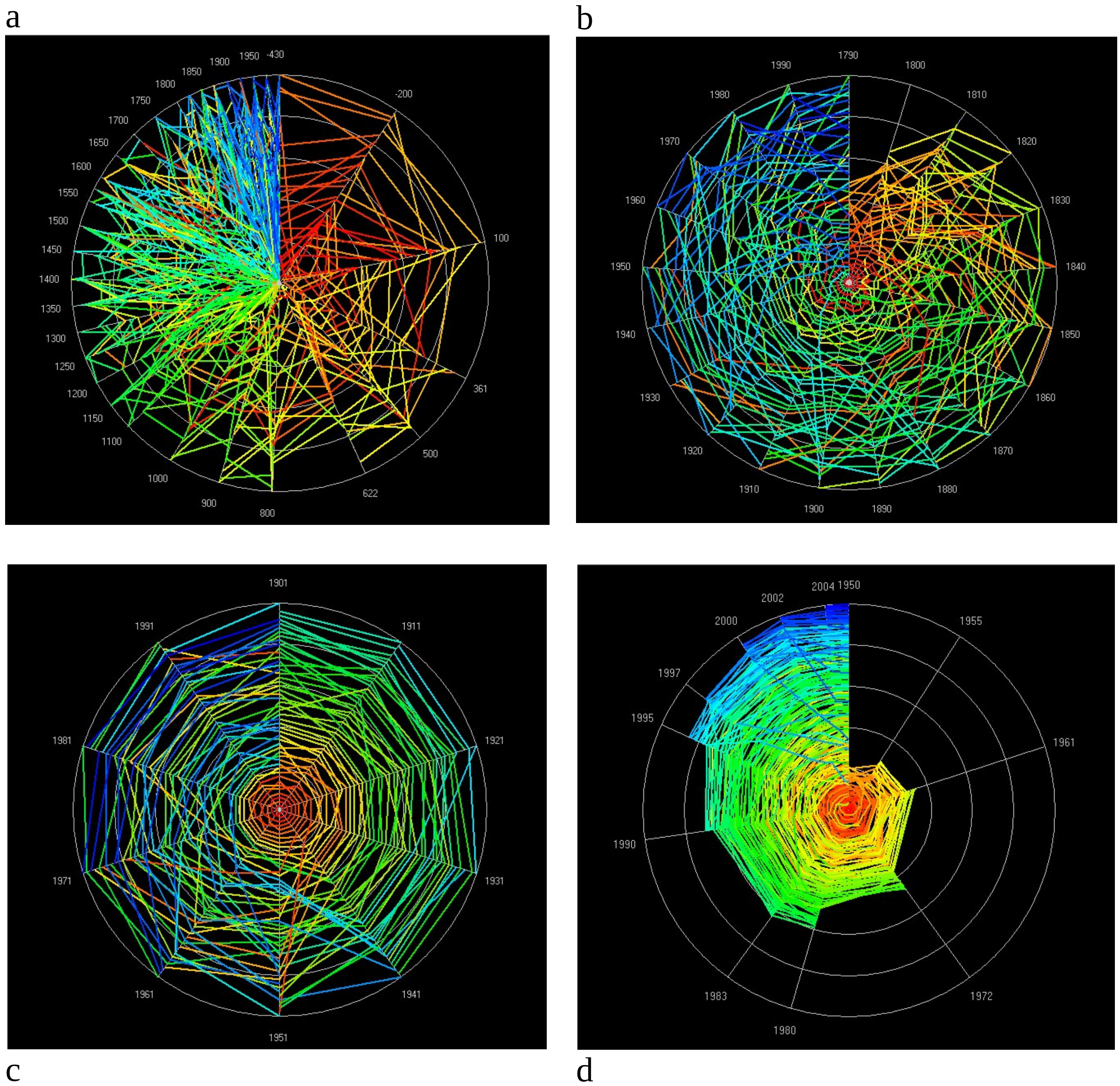

Figure 3: Rank Clocks for the World (a), US (b), Great Britain (c), and Israel (d)

The fit of the rank-size functions to the data for all city systems over all time periods is shown in Table 4 and the proportion of the variance explained ranges from 85% to 95%. The scaling parameters all tend to be greater than 2 suggesting that the largest cities tend to be less dominant than in the pure Zipf case ($\alpha \sim 2$) which is widely regarded as being the situation when the system has adjusted to its ultimate steady state (see Gabaix, 1999). The average distance shift in ranks must be interpreted in the light of the average time periods over which the size trajectories are observed. Interestingly for the city systems since industrialisation (the US, UK and Israeli), these show shifts of rank between 4 and 8 over ten year periods whereas the shift in rank for the World system is about 8 over periods of some 80 years. This distance measure is much less reliable as the time periods vary so massively while in the case of the Italian system, the shift is about 16 over 80 years which pro rata gives a shift of some 5 ranks for very ten years, similar in fact to the other national city systems.

Table 4: Comparative Measures of Rank Shift for Five City Systems

City Systems	**Number** n	**Time** T **yrs**	$t \rightarrow t+1$ **yrs**	**Min** $\hat{\alpha}_{ml}$	**Max** $\hat{\alpha}_{ml}$	**Min** r^2	**Max** r^2	**Min** $d(t)$	**Max** $d(t)$	d	L
World Cities	47-50 (390)	2430	80 (25-260)	2.127	3.236	0.906	0.945	4.914	10.973	7.785	14
US Cities	24-100 (266)	210	10	1.952	2.674	0.911	0.944	2.242	7.559	4.667	25
UK Cities	50 (458)	100	10	2.746	4.062	0.948	0.951	1.941	9.857	4.220	41
Israeli Cities	34-164 (172)	55	4 (1-11)	1.748	2.035	0.782	0.850	1.578	5.917	4.032	83
Italian Cities	94-118 (555)	561	80 (61-100)	2.493	2.836	0.851	0.899	11.780	17.064	15.519	74

The last point we need to make is based on a visual comparison of the rank clocks. Clearly the US and Israeli clocks begin with less than 100 top cities as these systems do not have 100 cities at their start. The US clock then reaches 100 in 1840 and then we keep these top ranks stable. The Israeli clock illustrates the entire growth trajectory with no constraints on numbers. Both these clocks show growth with the Israeli example showing the persistence of core cities and massive growth of new ones. The US system has much less of a focus on the core cities as it consistent with the rapid diffusion of large cities in the west and south. The British picture is one of more classic slow growth with many trajectories showing circularity, that is ranks remaining similar over time and the core being maintained. The World system is by far the least stable in that cities of the ancient era are clearly quite different from those of the middle ages, the Chinese empire, and the modern world.

Disaggregate Populations: Firm Sizes and Building Heights

Our last two examples involve rather different human systems. The first size distributions of firms from 1955 to 1994 for the US are taken from the Fortune 500 data and the second size distributions for tall buildings (skyscrapers) for the World and for New York City from the Emporis buildings database[3]. The firm-size distribution is in fact non-spatial (other than pertaining to the US) while the distribution of skyscraper heights is at a level of spatial detail far below that of the city-size distributions for which we first developed these visualisations. In terms of firms, we have focussed not on all 500 but on 100 as by now it must be clear that as the number of objects increases, the rank clock is more and more impressionistic in that it requires zoom capability to see individual trajectories. Firms can be ranked, as can cities, by any measure of their size and in this case, we have revenue and the profit/earnings ratios which give a measure of how well the firm is doing. We will not show this latter set of rankings here (however see Batty, 2007) but in Figure 4(a), we plot the revenue clock and it is immediately clear that there is large but regular volatility in the rankings of firms.

a *b*

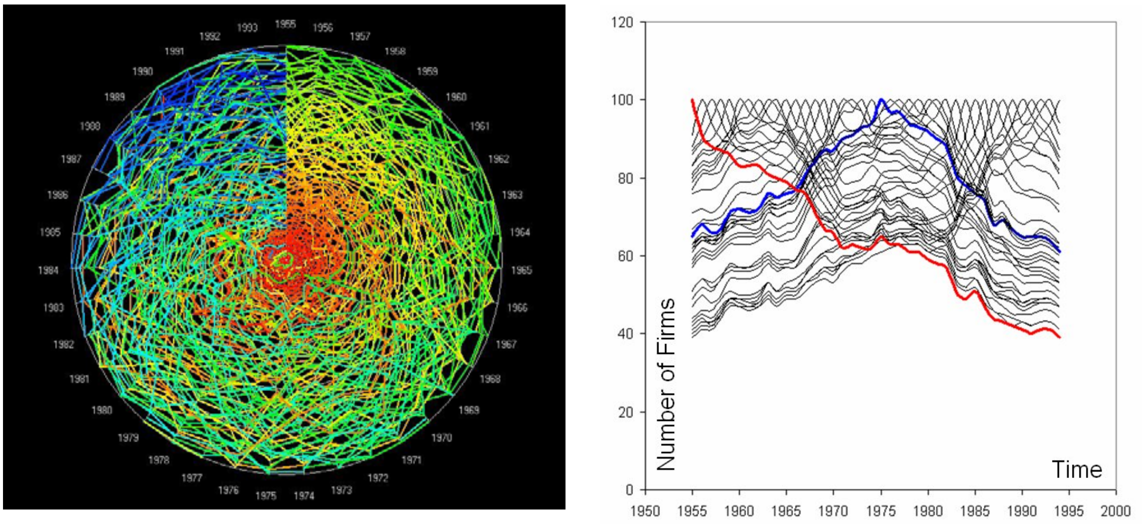

Figure 4: The Fortune 100 Rank Clock (a) and the Persistence-Decline of Firms by Rank 1955-1994 (b)

Essentially a core of firms do stand out but there is considerable erosion in their ranks. The rate of this erosion is rather regular and this is best seen in Figure 4(b) which is a visual plot of $L(\ell, m)$ showing the number of firms that stay in the top 100 rankings as time elapses. We can measure the half life from this which is about 25 years and Figure 4(a) marks out the firms that stay in the top rankings from the beginning year 1955 and the firms that enter or leave the rankings pivoted around the mid year 1975. Figure 5 shows the trajectories of six of these firms where quite clearly steel and heavy industry such as rubber spiral out of the rankings while high tech (e.g. IBM) spiral in. American car manufacturers like American Motors have mixed fortunes but the largest like Chrysler more or less maintain their rank with General Motors (not shown) ranked as number one throughout the 40 years. The picture is very different

[3] Fortune 500 data from 1955 to 2005 is available for each year from the CNN Money web site. http://money.cnn.com/magazines/fortune/fortune500_archive/full/1955/index.html. We only use the data from 1955 to 1994 because the series appears to have been redefined in 1995. The high buildings data is from the Emporis global data base http://www.emporis.com/en/bu/sk/

now in 2010 and the analysis needs to be brought up to date so that the relevance of these statistics of change can be explored further. In Table 5, the rank-size relations show extremely good fits with r^2 near 99% for all distributions. The change in rankings from the distance measures are based on years and indicate quite rapid change compared to all the city-size distributions.

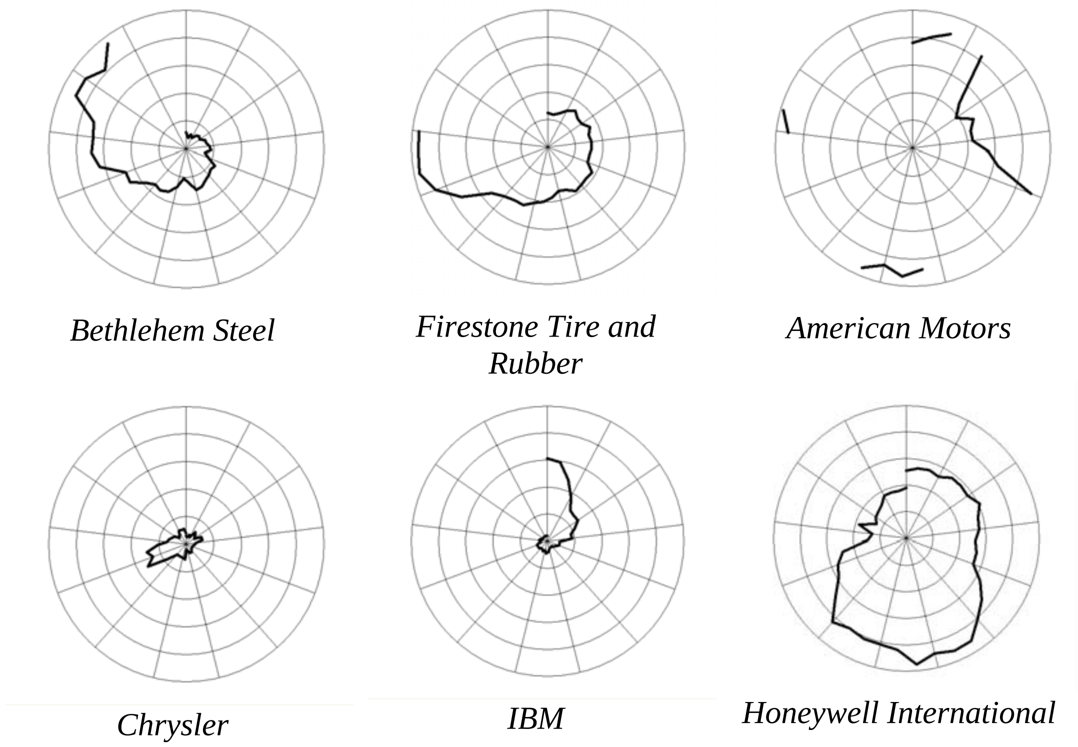

Figure 5: Individual Rank Trajectories for Selected Fortune 500 Firms 1955-1994

Our second example is dramatically different as we indicated earlier in the paper. So far all the tall buildings greater than 12 stories or 40 metres in height in our data sets, have not changed in height since their construction. Only in places of very rapid growth like Hong Kong are tall buildings now being systematically demolished and rebuilt and it might be argued that such instant changes in rank which such demolition and reconstruction occasions is tantamount to the construction of new buildings. This means that buildings do not rise in rank, they simply appear at a particular rank and never get any higher than their original rank. Buildings simply do not get bigger in this characterisation of the data. If the system is in the steady state, buildings may stay at the same rank but generally, once a building appears at particular rank, it will then decline in rank as taller buildings are constructed at later periods. We examine two data sets here: the first for New York City which is the archetypal high rise city and the second from the 60,000 high buildings in the Emporis global database.

The first high building in each data set is taken as being constructed in 1909 although there are skyscrapers built before then. In New York, there is a rapid increase in the number of such buildings being constructed in the 1920s and 1930s. Due to ties in height, the top 100 contain some 119 in 1916 which means that the particular rank clock shown in Figure 6(a) reaches out to 120 on its radius rather than 100 before falling back to 100 or 101. The clock is based on a complete spiralling out and downwards of buildings from the time they enter the top ranks. The early buildings before the late 1920s are coloured red, orange and yellow and these are blanked out by the flurry of construction shown in green colour in the early 1930s. Some of these buildings stay in the top 100 for the rest of the time period but in general, later buildings are greater in height and to produce a clearer picture, it is necessary to zoom into the clock and explore these trajectories. This is not possible on the printed page but the interactive software is crucial to using these tools effectively. Figure 6(b) shows the clock for the World data where it is quite clear that globally, some high buildings constructed in earlier eras persist at high ranks into recent times. The picture of growth is considerably later than in the New York case as high building has diffused around the world as countries in the developing world have grown. If one takes a cross-section across these clocks at any time, this gives a picture of the time when a building was constructed and its rank and needs to be compared to the persistence matrix $L(\ell, m)$.

a *b*

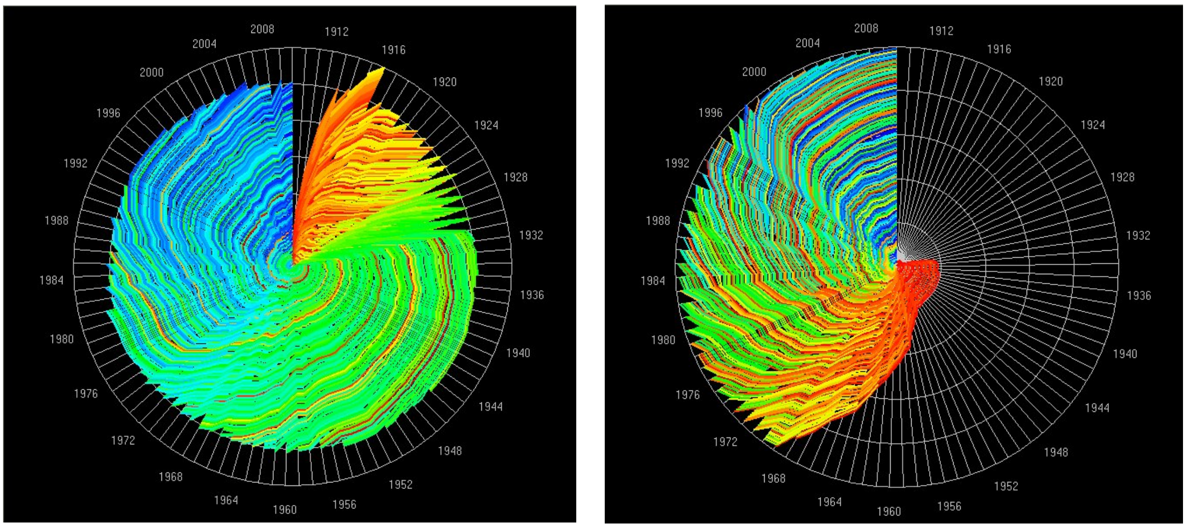

Figure 6: Rank Clocks of the Top 100 High Buildings in the New York City (a) and the World (b) from 1909 until 2010

Table 5: Comparative Measures of Rank Shift for Firms and Tall Buildings

Scaling Systems	**Number** n	**Time** T **yrs**	$t \rightarrow t+1$ **yrs**	**Min** $\hat{\alpha}_{ml}$	**Max** $\hat{\alpha}_{ml}$	**Min** r^2	**Max** r^2	**Min** $d(t)$	**Max** $d(t)$	d	L
Fortune 100 Firms	100 (343)	41	yearly	2.205	2.739	0.987	0.995	3.397	7.988	5.158	56
NY City Skyscrapers	12-119 (516)	101	yearly	3.048	6.259	0.873	0.959	–	–	–	39
World Skyscrapers	1-101 (500)	101	yearly	3.997	7.462	0.627	0.962	–	–	–	19

Numerical statistics are presented for these two examples in Table 5. The rank-size functions for New York fit with reasonably good approximations as the r^2 statistics show with later distributions having better fits. The same is true of the World data although the performance earlier in the time series is not as good. It ultimately becomes a lot better and is similar to the other distributions we have dealt with here. One issue which makes these distributions very different is the degree of competition as reflected in the scaling parameter α. The values are much higher than 2 meaning that the slope of the rank-size curves are considerably flatter than the pure Zipf case where $\beta = -1$. This is of interest in that it suggests much less competition between the construction of high buildings in an intraurban context but also that the growth dynamics is quite different from that which characterises cities and firms which do actually grow in the biological sense due to accretion. This entire dynamic requires considerably more research so we can explain the genesis of high buildings using models of copy-cat and fashion effects. In terms of the distance statistics, these are tricky to compute due to the fact that all the time periods are not distinct; in earlier periods, buildings persist but new ones are not constructed and thus all ranks remain the same. Moreover, the distance measures all point in one direction – downwards in terms of rank shift. This suggests again that we need some modifications to these visualisations to account for systems where objects do not grow *per se* but do lose their position in the rankings due to the appearance (growth) of other objects. Half lives are also tricky to estimate in this context. It appears that in the 1920s these are about 10 years rising to 20 years in the 1970s but they are impossible to estimate form this data for later times.

Next Steps and Future Research

Scaling systems involving human populations clearly display a form of regularity at the macro level which masks dynamic volatility at the micro level. We measure this volatility by the extent to which objects change their position in relation to one another through their rank-size distributions. We do not yet have a detailed understanding of the way these dynamics play out but we do know that competition between these various elements is intrinsic to the way they capture growth from one another. In fact, in city systems, traditional theories where central places grow from the bottom up, gradually deriving their functions as they get larger and serving ever large populations, go a long way to explaining how cities scale at the macro level. Random proportionate growth plays a part in the way individual cities grow relative to one another but as yet, we do not have good models of precisely how individual cities compete (but see Rozenfeld et al., 2008). When it comes to the growth of firms, the logic is more complicated by the process of mergers and acquisitions while in the case of systems that are manufactured such as buildings, a very different substrata of dynamics is at work, dictated by the way developers define the need for high buildings, much complicated by the construction process and by investment decisions.

Our visualisation of these changes suggests that different scaling growth regimes display different morphologies and we have made a start at classifying these as rank clocks. But we still need much better methods for making statistical comparisons between such systems where we find it hard to control the number of objects or the number and length of time intervals over which such distributions are observed and measured. In particular, the concept of half lives needs further refinement as do the various measures of distance that we have introduced here. The implicit notion that the size of the scaling parameter relates to the degree of competition between the elements comprising any particular distribution needs to be made much more definite while links between distributions through their individual elements need to be tracked in a more synthetic manner. Trajectories on the rank clock help to focus this tracking but our analysis does not yet link individual elements together.

In this sense, network interpretations of the way the elements in these distributions connect to one another are promising and much of what we have learned about how different objects grow and change relative to one another can probably be unpacked into changes occasioned by the development and growth of new network links. We need to pursue at least three lines of inquiry: first to develop a more robust set of space-time statistics for scaling systems, building on what we have introduced here; second to generate network equivalents of such scaling systems that will add a richness to the analysis by, for example, expanding city-size distributions to the networks that support interactions, trade, and migration between cities; and last but not least, we need more examples at different spatial scales where we can draw clear links between cities, buildings and firms which are all manifestations of how populations agglomerate in achieving economies of scale.

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