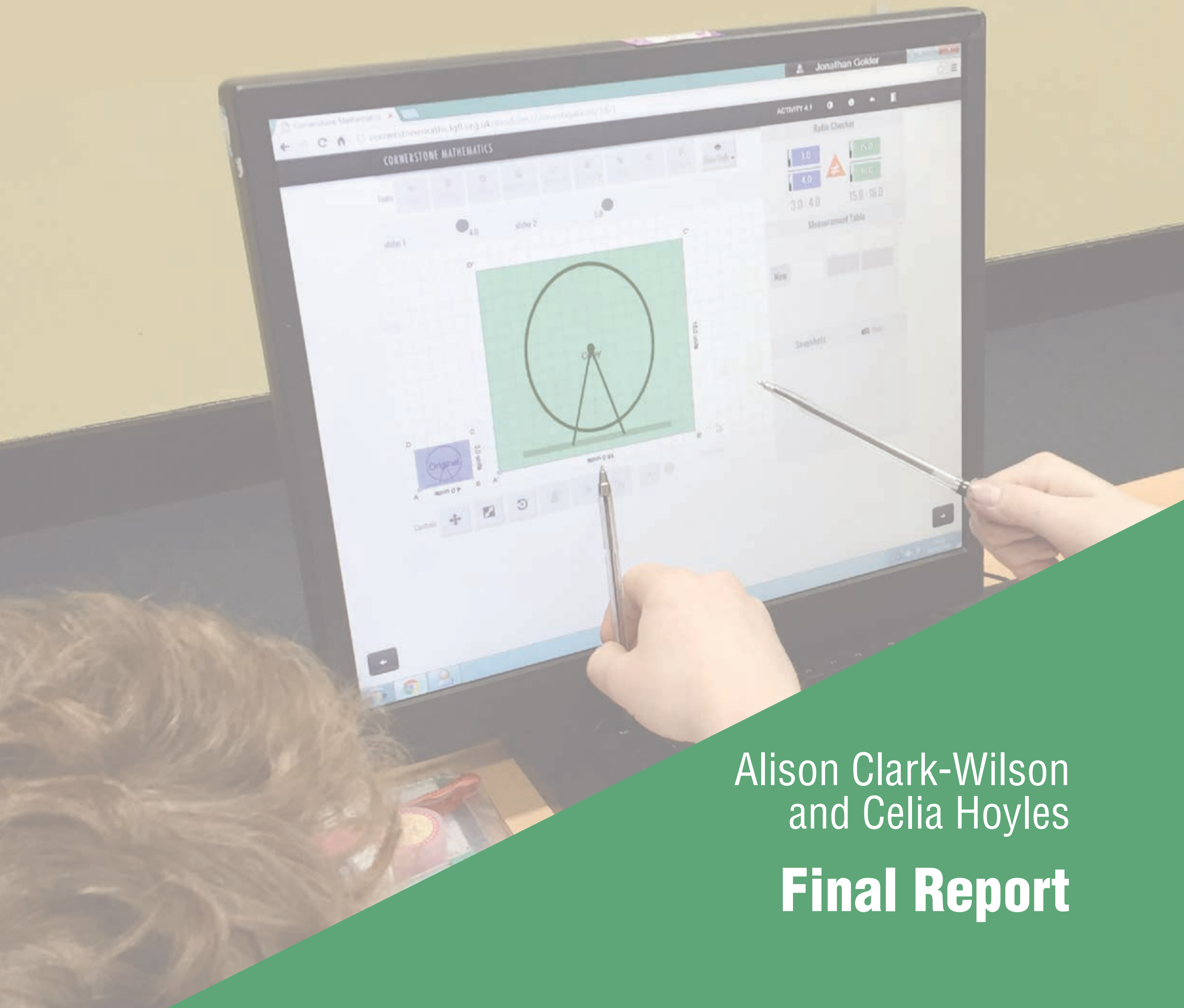
Dynamic Digital Technologies for Dynamic Mathematics

Implications for teachers' knowledge and practice



Alison Clark-Wilson
and Celia Hoyles

Final Report

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The Nuffield Foundation is an endowed charitable trust that aims to improve social well-being in the widest sense. It funds research and innovation in education and social policy and also works to build capacity in education, science and social science research. The Nuffield Foundation has funded this project, but the views expressed are those of the authors and not necessarily those of the Foundation. More information is available at www.nuffieldfoundation.org

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Terminology and acronyms used in the report

Cornerstone Maths (CM): A set of three curriculum units that combine dynamic mathematical technology, pupil and teacher materials and teacher professional development to address 'hard to teach topics' in key stage 3 mathematics (algebraic patterns and expressions, linear functions and geometric similarity).

Dynamic mathematical technology (DMT): Technology offering different mathematical representations (geometric shapes, graphs, tables, algebraic expressions) that teachers and pupils can manipulate and by doing so, engage with the underlying mathematical concepts and relationships.

Instrumental genesis: The process through which teachers' and pupils' mathematical knowledge shapes and is shaped by their interactions with the technology as they accomplish a particular mathematical task. These geneses are different for teachers and pupils as the mathematical task for teachers also involves teaching (Guin and Trouche, 1999; Guin and Trouche, 2002; Noss and Hoyles 1996).

Mathematical knowledge for teaching (MKT): This is described as 'knowing mathematics from the perspective of helping others to learn it and includes being mathematically ready to teach an idea, method, or other aspect' (Even and Loewenberg Ball, 2009).

Mathematics pedagogical practices (MPP): Teaching approaches employed by teachers as they support pupils to learn mathematics.

Mathematics pedagogical technology knowledge (MPTK): A theoretical frame that incorporates 'the principles, conventions and techniques required to teach mathematics through the [dynamic mathematical] technology' (Thomas and Palmer, 2014, 75).

National Centre for Excellence in Teaching Mathematics (NCETM): The government-funded body that provides a portal for supporting the professional development of mathematics teachers, which includes the facility to host open and closed online project communities for forum discussions and file sharing.

About the authors

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Executive Summary

1 Aims and Background

This project set out to research the impact on teachers' mathematical knowledge for teaching (MKT) and associated mathematics pedagogical practice (MPP) on their teaching of key stage 3 topics with dynamic technology. We define such technology as that offering various mathematical representations that teachers and pupils can manipulate and link, and by doing so engage with the underlying mathematical concepts and relationships. The context chosen for the research is Cornerstone Maths, an existing, extensively researched resource for teaching key stage 3 mathematics topics with dynamic mathematical technology (DMT). Our work led us to adopt a comprehensive framework through which to interpret teachers' knowledge and practice: Mathematical Pedagogical Technology Knowledge (MPTK) (Thomas and Hong, 2005).

Furthermore, the project aimed to develop a professional development 'toolkit' (PD Toolkit), an online resource to support teachers in London with some prior experience of Cornerstone Maths as they engage and help other colleagues to also develop their use of DMT.

Cornerstone Maths is a set of curriculum units that aims to support teachers to develop their pupils' uses of carefully designed dynamic mathematical technology for the learning of three 'hard-to teach' topics in key stage 3 mathematics. We define dynamic mathematical technology as technology offering various mathematical representations (such as geometric shapes, graphs, tables and algebraic expressions) that teachers and pupils can manipulate and by doing so, engage with the underlying mathematical concepts and relationships. The three topic areas are: algebraic generalisation; geometric similarity; and linear functions.

The design of the Cornerstone Maths curriculum units is underpinned by the following design features:

- A DMT in which pupils explore, make conjectures and solve problems within sequential guided structured activities that have 'realistic' contexts.
- The dynamic environment makes the links between representations explicit to highlight the underlying mathematical concepts and relationships.
- Accompanying professional development through face-to-face professional development and an online community.

2 Research questions and methodology

RQ1 What is the impact on teachers' MPTK around the mathematical concepts of algebraic generalisation, geometric similarity, and linear functions, of their engagement with cycles of professional development and associated teaching that embeds DMT?

RQ2 What MPTK is desirable for teachers to integrate DMT into their teaching of these concepts?

RQ3 What are the design features of professional development activities for key stage 3 mathematics teachers that support them to use DMT in ways that become embedded in their practice and lead to effective learning?

The project took place between January 2014 and November 2016. It involved 48 self-selecting secondary schools from 23 London Boroughs, which were given free access to Cornerstone Maths via London Grid for Learning. At least 209[1] teachers participated in the project, and the number of participating teachers in each school ranged from 1 to 16. The teachers were either self-selecting or nominated by their school leadership team.

The participating teachers engaged in up to three cycles of professional activity, with each cycle focusing on one curriculum topic during which they:

- completed a pre-survey that captured contextual data; a pre-assessment of their MKT and practices with DMT; and an ethical agreement to give their consent as a participant in the study;
- participated in a one-day PD event;
- joined the online project community, hosted on the National Centre for Excellence in Teaching Mathematics (NCETM) portal;
- developed a plan to teach a nominated landmark activity as a 'research lesson' to a chosen key stage 3 class, which was shared by uploading to the NCETM portal;
- engaged in (optional) online PD webinars and tasks;
- taught the landmark activity as a 'research lesson' to their chosen class;
- attended a half-day PD meeting to feed back on their experiences;
- completed a post-PD cycle survey.

In addition, at least 10 per cent of the teachers within each cycle were selected for classroom observation of their teaching of the research lesson, and were interviewed to gather their reflections. The sample was chosen across a range of classroom and school contexts.

The research data set comprised: teacher contextual data; teacher responses to pre- and post-MKT survey items; teacher responses to PD tasks; lesson plans for landmark activity (planned in pairs/trios) and associated reflections; lesson observations of landmark activities and associated reflections. The data was analysed using a mixed-methods approach that involved both qualitative (deductive and thematic) and quantitative (frequencies, measures of spread and effect size) techniques. Three teacher case studies were developed that triangulated multiple data sources to result in narrative accounts of the development of their MPTK.

1 We suspect that this is a minimum number as some schools did not inform us of their school-based PD activity.

During the second year of the study we worked with a group of project teachers to design a 'professional development toolkit' that would support teachers who had some knowledge of the CM approach in sustaining and scaling their use of the DMT in the longer term in their own school and/or in leading PD for teachers from other schools.

3 Summary of findings

3.1 Development of teachers' Mathematical Pedagogical Technology Knowledge

3.1.1 Algebraic variable

- **Defining and naming variables**. Teachers developed more precise mathematical responses of the meaning of a variable in terms of recognising its domain and range. Teachers required carefully mediated PD support to ensure that they fully appreciated the representation of the variable provided by the DMT, which was a slider. This mediation included drawing attention to the values on the slider and discussing the 'point' of dragging the slider.
- **Building a general expression**. There was limited improvement among one-third of the teachers in being able to relate the algebraic expression for a figural pattern to its geometrical structure. The remaining teachers were able to do this, as indicated by responses to the pre- and post-surveys. This indicates that, for some teachers, MPTK is resistant to change and that further PD cycles are needed, possibly in school.
- **Understanding letters as variables**. Teachers reported that their predominant teaching approach at key stage 3 centred on the treatment of 'letters as unknowns' rather than as variables. The pre- and post-survey data again suggest little change in teachers' responses, although some teachers did give fuller explanations in their later responses.

3.1.2 Linear functions

- **The meaning of m and c in y=mx+c**. The teachers' knowledge of the meaning of m and c in $y=mx+c$ showed improvements, particularly concerning the interpretation of the value of c as a variable that can take a negative value.
- **Defining a linear function**. Most teachers' initial definitions of a 'linear function' were partial and tended only to refer to the characteristics of the graphical output of the function, with few teachers making reference to the one-to-one nature of a function or its domain and range. The teachers' subsequent professional discussion was a significant factor in the teachers' refinement and expansion of their earlier responses.
- **Editing graphs of functions**. Teachers required carefully mediated PD support to ensure that they fully appreciated the rationale, role and effects of draggable 'hotspots' on the graph. Initially, only half of the teachers paid attention to the specific affordances of these 'hotspots', which controlled the resulting animation.
- **Interpreting multiple representations of motion.** Teachers used a limited number of strategies to read the multiple representations of the DMT when faced with a problem-solving task.

3.1.3 Geometric similarity

- **Properties of geometrically similar polygons**. The task to state the properties of geometrically similar shapes was challenging, with just over two-thirds of the teachers stating no, or only one, correct property (e.g. failing to mention angles, restricting definitions to triangles). By the end of the project many teachers could produce more general and complete responses.
- **Defining geometric similarity**. Teachers' definitions of geometrically similar shapes showed notable improvement, although not statistically significant. This improvement was also matched by an increase in their reported confidence in their definitions.
- **Within polygon and between polygon ratios**. The most significant improvement in the teachers' MPKT related to their understanding of the 'within polygon' invariant for geometrically similar polygons, an 'invisible' property that had been made visible by the DMT.

3.1.4 Teachers' lesson planning

- The teachers' lesson plans became increasingly detailed with respect to planning what they were going to do (especially with the DMT) and say during the landmark activity.
- The plans showed how teachers became more mindful of the need to provide support for pupils in making sense of the DMT, such that they could use it in mathematically productive ways beyond their initial experiences.
- Over time, the teachers' plans became more explicit about the inclusion of whole-class plenaries to focus on the mathematics at stake, with more teachers incorporating how they would use the DMT to support this.

3.1.5 Lesson observations of landmark activities

- The majority of teachers were initially reluctant to use the DMT 'live' during lessons, particularly during whole-class teaching. It is clear that this practice takes time to develop. When it was used effectively, the focus was on its use to: show counter-examples, extend the mathematical ideas, promote the use of appropriate vocabulary, and support or refute the pupils' mathematical predictions.
- Where teachers had taken the time to rehearse their 'curriculum script', they were noticeably more confident during whole-class use of the DMT and in scaffolding the pupils' work with the DMT. In such cases, the emphasis was on encouraging pupils to see the mathematical connections between the different representations and to describe the impact of the dynamic elements on these representations.
- In more effective lessons, the mathematical focus for the activities was maintained throughout. For example, in the *linear functions* landmark activity, the mathematical focus remained on the value of the coefficient of x, and the impact of using the DMT to change its value on the linked mathematical representations. In these lessons it was apparent that both teacher and pupils had sufficient prior knowledge and experience of the DMT to enable this to happen. Furthermore, teachers sustained the

predict–check–explain pedagogical approach, and encouraged the use of correct vocabulary to support pupils' oral and written explanations.

- Some teachers developed practices whereby the pupils took on key roles in the use of the DMT to share with others, for example by demonstrating or talking through their own strategies, which in turn widened the teachers' own knowledge and understanding of its use.

3.1.6 Teacher engagement

- Teachers overcame their initial apprehension about discussing the mathematical concepts in depth, reporting high levels of motivation and value in such professional discussions with colleagues, particularly when stimulated and supported by the PD resources embedding DMT.
- The alignment of the project goals to individual, departmental and school-level goals was a crucial factor with respect to both individual teachers' engagement and the potential sustainability of their use of the DMT.
- The participating teachers' roles and responsibilities were varied. An ideal pairing seemed to be the combination of a less-experienced teacher alongside a more-experienced teacher (in terms of general teaching experience, not necessarily the use of DMT) with some departmental responsibility with respect to the project goals.

3.2 Design features of the professional development toolkit (PD toolkit)

- Professional development to support teachers to implement DMT in their classrooms needs to blend face-to-face sessions that involve first-hand experiences with the DMT alongside PD tasks that deepen teachers' understandings of the mathematics involved and promote lesson planning for common activities. The adapted lesson study design, which offers a cyclical PD approach over a period of 6–8 weeks, was reported to be successful model by the participating teachers.
- The PD Toolkit resources extended beyond resources for PD sessions for other teachers of key stage 3 to include resources to support the wider communication of research findings that underpinned the use of DMT, and case studies of successful departmental implementations of DMT in various schools.
- Early data on the use of the PD Toolkit resources by schools suggests that there is sufficient content to enable further scaling within the CM project schools and to support the associated sustainability of new practices with DMT.

4 Discussion

4.1 Development of teachers' Mathematical Pedagogical Technology Knowledge

The project findings highlight how the process of integrating DMT within secondary mathematics teaching presents considerable potential for learning but also a significant challenge for teachers, as they rethink the underlying mathematical concepts, undergo their own instrumental genesis, develop their curriculum scripts and learn to support pupils' technological experiences.

Initially, the teachers reported very low level of use of DMT in both their whole-class teaching and by their pupils, and indicated a range of professional development needs in this respect. This suggests that both initial teacher education and in-service professional development opportunities are not currently enabling teachers to develop sufficient knowledge and use of DMT.

Teachers reported that their MPTK had developed in ways that directly supported their teaching of the topics within and beyond the Cornerstone Maths lessons. It was a key finding that teachers' mathematical knowledge for teaching algebraic variable and linear functions, as assessed by the items in the pre- and post-surveys, was resistant to change. However, as the majority of teachers performed well in these pre-survey items, it is possible that the intervention was too short to impact strongly on those teachers with weaker mathematical starting points. There was some evidence of more significant impact in relation to mathematical knowledge for teaching geometric similarity, which concerned more robust definitions of geometric similarity for a broader range of polygons and the appreciation of the invariant ratio property for pairs of corresponding sides within similar polygons.

The linear functions PD cycle impacted most on teachers' instrumental genesis, which we hypothesise is due to the more familiar set of representations (graphs, tables and equations), alongside the ease of the initial access to the context by playing the animation. Of the three topics, the highest proportion of teachers went on to teach this lesson and to disseminate this unit to their colleagues.

On the whole, teachers did not seem accustomed to discussing mathematical concepts in great depth but these opportunities were appreciated and perceived as worthwhile professional activity. This was further evidenced by teachers' requests to include some of the MPKT survey items and PD tasks within the PD Toolkit.

Teachers' lesson plans for the landmark activities improved in terms of their explicit focus and accompanying detail relating to the following eight quality features:

- includes teachers' actions and questions (not involving the DMT);
- includes pupils' actions on the DMT;
- provides support for pupils' instrumental genesis;
- maintains focus on the mathematical concept involved;
- privileges actions on representations to explore mathematical concepts;
- incorporates mathematical vocabulary;
- incorporates technical and/or contextual vocabulary; and
- includes planned plenary phases involving the DMT.

There was particular impact on the teachers' planning with respect to increased frequencies of the intended use of the DMT by teachers and pupils during the lesson. Furthermore, there was evidence that the practice of sharing plans within and across the different PD groups through the online community supported the teachers to learn from each other.

Teachers developed deeper appreciation of each of the three mathematical concepts (algebraic variable, linear functions and geometric similarity), which included greater rigour in definitions, representations, vocabulary, and relationships to other areas of mathematics. Enhanced teacher knowledge was more apparent where the teacher had engaged with two or more of the CM units of work.

The development of the teachers' MPTK happens over a significant amount of time (our hypothesis is 2–3 years), which is in part dependent on the need for teachers to assimilate use of the DMT into their personal mathematical experience, a necessary precursor to being able to incorporate it confidently within their classroom practices. Our case studies show that teachers' classroom practices developed over a 15-month period and improved with each PD cycle as they reflected on previous experiences and incorporated this learning into their future planning and practice.

The majority of teachers were initially very reluctant to use the DMT 'live' during lessons, particularly during whole-class teaching. It is clear that this practice takes time to develop. Where teachers had taken the time to rehearse their curriculum script, they were noticeably more confident during whole-class use of the DMT.

In the more effective lessons, the mathematical focus for the activities was maintained throughout. In these lessons, it was apparent that the teacher and pupils had sufficient prior knowledge and experience of the DMT to enable this to happen.

Some teachers developed practices whereby the pupils took on key roles in the use of the DMT, for example by demonstrating or talking through their own strategies, which in turn, widened the teachers' own knowledge and understanding of its use.

Maintaining a balance between pupils' uses of the DMT, their mathematical discussions and written recording was key to effective pedagogy.

The project revealed a number of effective classroom practices involving the use of the DMT, sometimes in conjunction with generic classroom management software available in school computer suites. Central to this was the use of the DMT 'live' during lessons, by both the teacher and pupils, in order to:

- support pupils to become familiar with the DMT for mathematical purposes;
- share, discuss and critique pupils' responses;
- highlight mathematical concepts through planned and thoughtful interactions with the DMT, such as dragging sliders and encouraging pupils to observe carefully and explain or justify the effects; and
- develop mathematical and technical vocabulary to support shared understandings.

Each curriculum topic revealed rich exemplar practices that were particular to the mathematical concept involved. This highlights the need for a mathematics-focused approach to professional development as the representations, mathematical connections and tangible objects are specific to the design of the DMT for each topic. Hence the nature of effective practice is at the level of the curriculum unit, although there are some more general uses of the technology that teachers need to master to establish familiar classroom routines.

Not all teachers wanted to be involved in all three PD cycles. Of the 27 who did, all have become PD Champions and are now leading the PD of other colleagues in their schools. However, we also have some teachers who only engaged in one PD cycle who chose to focus on embedding their first unit in their practice before engaging with another topic. A small number of schools (5) withdrew from the project altogether.

The alignment of the project goals to individual, departmental and school-level goals was a crucial factor for both individual teachers' engagement and the potential sustainability of their use of the DMT. The participating teachers' roles and responsibilities were varied. An ideal pairing seemed to be the combination of a less-experienced teacher alongside a more-experienced teacher (in terms of general teaching experience, not necessarily the use of DMT) with some departmental responsibility with respect to the project goals.

Our methodological decision to focus the PD cycles around nominated landmark activities as mediating constructs to reveal important mathematical knowledge and prompt teachers' reflections of their pedagogic practices proved to be a highly successful feature of the project. The landmark activities made the purpose of the DMT tangible for the teachers, although some compromises had to be made. For example, some teachers struggled to find sufficient time and access to the DMT to teach the lessons that preceded the landmark activity, resulting in both the teacher and pupils being less prepared to use the DMT. Other teachers were unable to teach the landmark activity during the period of the PD cycle, although during the second PD session in each cycle most teachers benefited from listening to the experiences of others even though they were yet to teach the lesson for themselves.

4.2 Design of professional development activities and support

The challenges associated with learning to engage with mathematics through the medium of a DMT, however well-designed, lead us to conclude that some initial face-to-face professional development is essential to support early activities such as discussion of the mathematics, hands-on experiences with the DMT, and preparation for classroom-based uses.

The project adopted a design-based methodology (which involved project teachers) to produce a PD Toolkit, accessible from the UCL-hosted website at http://ucl.ac.uk/cornerstone-maths.

The components and content of the PD Toolkit are:

Component	Content
Overview	› Video introduction to the CM Project.
Getting started	› Video overview of CM and its design principles. › Overview of the PD Toolkit and suggestions for how it might be used. › Technical information. › Project history and funders.
Curriculum units *This component is replicated for all three curriculum topics: algebraic variable; linear functions; and geometric similarity.*	PD resources › Video overview of the curriculum unit. › Pupil workbook and teacher guide. › Structured PD session, including: ○ MPKT items; ○ instrumentation tasks; ○ MPTK tasks to analyse pupils' digital productions; and ○ introduction to the landmark activity; leading to a: ○ lesson-planning task of the landmark activity. › Resources to support planning and reflection on the landmark activity: ○ video of the landmark activity; ○ video clips of 'typical' classroom enactments; and ○ examples of typical pupil responses. › Suggestions for assessing pupils' mathematical understanding.
Departmental case studies (embedding DMT)	› Outline of their approach for scaling/ sustainability. › Examples of within-school PD activities.
Evidence base	› Research summary for senior leaders and headteachers. › More detailed research summary for heads of department (including references to departmental case studies). › Links to published research.
Project community	› Link to the online project community – where teachers can share resources they create and discuss their implementations.

The take-up of the PD Toolkit in schools is still in its early stages but early data from CM Champions in three schools indicates that:

- it offers sufficient initial resource for the design of school-based PD in London schools that fits with the prevailing PD culture in the school concerning frequency and timing (from one 2-hour session to three 2-hour sessions spread over a few weeks).
- the overviews of the curriculum units (in video and document formats) were reported to be particularly useful.

5 Implications for policy and practice

We conclude by stating the implications of our findings, making our recommendations for both policy and practice and highlighting some next steps that would support both the dissemination of the project's outcomes and the future research agenda.

5.1 Implications

The mathematics education community does need to keep revisiting the role of technology within teaching and learning and the nature of its associated pedagogy. Although DMT has been around for over 20 years it is continually evolving and, while access to DMT may be easier, the main barrier is still opportunities for teachers' sustained PD that maintains a strong focus on the mathematics. Without this level of support for teachers to engage in cycles of PD that scaffold their uses of DMT, PD is unlikely to be effective.

The project outcomes make a substantial contribution to knowledge of more effective features of teachers' classroom practices with DMT that might underpin future large-scale PD initiatives in this area and offer a tighter framework within which such opportunities can be designed to be more effective and sustainable. This is even more important when teachers are choosing to move schools frequently or are leaving the profession early in their careers.

Initial teacher education seems an obvious starting point. However, as all teachers' first experiences with DMT are tentative, they need to be well supported (with realistic expectations of the time required) and involve productive collaboration with more-expert practitioners. All too often teachers reject DMT based on early lessons that they deem to have been unsuccessful, without necessarily having tools for reflection that enable them to learn from the experience.

Our experiences within the previous Cornerstone Maths-funded project phases alongside this one lead us to suggest that it takes at least two years for teachers to develop more confident teaching practices with DMT, which means that for a department, the timeline to 'embed' DMT across key stage 3 is likely to be four to five years. A key element of this is the commitment to sustain the department-wide efforts alongside other demands on departmental time and energy. Hence leadership at the school level is key. The PD Toolkit promises to play a major role in this respect, as it aims to improve the quality of in-school professional development activities and enable new teachers to become involved through the inclusion of PD resources that can be revisited over time.

5.2 Recommendations for policy

There is a role for government, its agencies and affiliated organisations to provide better sign-posting to more effective teaching practices with DMT in secondary mathematics. This role may fall within the remit of the National Centre for Excellence in Teaching Mathematics and the Department for Education-funded Maths Hub programme, in which case a targeted effort is needed over time to share the project's outcomes, develop capacity in the school system and disseminate successful models for school-based professional development.

The findings from this study have fed into the Joint Mathematical Council's ongoing work on 'developing mathematics-specific pedagogy in Initial Teacher Education' with respect to the use of digital resources has been disseminated to ITE providers in collaboration with the NCETM and the National College for School Leadership. This may lead to an opportunity to develop a specific publication aimed at disseminating models for the introduction of DMT within mathematics teacher training routes.

We reiterate the recommendation from the recent Advisory Committee on Mathematics Education report that 'personalised and sustained professional learning within a supportive professional environment, with time for self-reflection' is an essential opportunity for teachers of key stage 3 mathematics to enable them to deepen and strengthen the quality of their teaching with DMT (Advisory Committee on Mathematics Education, 2016). Furthermore, the severe teacher shortage and staff turnover rates for key stage 3 mathematics in some parts of the country present additional systemic challenges for which the CM PD Toolkit may offer a possible solution as schools are increasingly unable to release teachers for externally led professional development.

5.3 Recommendations for practice

There should be an expectation that teachers of secondary mathematics on training routes are supported to use a DMT from the outset and the expectation that such practice is continued in the early years of teaching. These early experiences should then be built upon such that a range of (research-informed) DMTs are experienced, ideally supported by reflective PD cycles in collaboration with more expert colleagues. While it is likely that a minority of these teachers may become innovative designers of new teaching tasks and approaches with DMT, it is more likely that they will draw on their earlier experiences to enable them to select appropriate DMT for their teaching purposes using a more research-informed set of criteria. It is important that both school leaders and institutional structures provide a sustainable set of conditions within which this can be achieved.

Main Report

1 Aims and Background

The project set out to research the impact on teachers' *mathematical knowledge for teaching* (MKT) and associated *mathematics pedagogical practice* (MPP) on their teaching of key stage 3 topics with dynamic technology. We define such technology as that offering various mathematical representations that teachers and pupils can manipulate and link, and by doing so engage with the underlying mathematical concepts and relationships. The research questions were modified from those in the original proposal and are given in Section 1.4.

MKT is defined as 'knowing mathematics from the perspective of helping others to learn it and includes being mathematically ready to teach an idea, method, or other aspect' (Even and Loewenberg Ball, 2009). It is widely acknowledged that this knowledge needs to be redefined within the context of using dynamic, mathematical technologies (Clark-Wilson, Robutti and Sinclair, 2014; Heid and Blume, 2008; Hoyles and Lagrange, 2009). Our hypothesis is that, by engaging teachers with mathematics in these new ways, their MKT will develop as it makes them rethink their knowledge and classroom routines, ultimately resulting in improved outcomes for the pupils they teach.

MPPs are teaching approaches employed by teachers as they support pupils to learn mathematics, which, for this project, are mediated by dynamic mathematical technology (DMT).

The subsidiary objectives of the project were:

- A clear articulation of the desirable MKT and MPP that enable teachers to maximise their pupils' learning within the context of teachers' and pupils' uses of DMT.
- Analytical descriptions of good classroom practices as sets of case studies that provide exemplification of the desirable MPP.
- Identification of 'landmark activities', which we define as those that indicate a rethinking of the mathematics or an extension of previously held ideas.

It was anticipated that teachers would take up the opportunity to be involved in the project as a means to develop their abilities and confidence to plan and teach using dynamic technologies and, through supported reflective activities, learn from their own and their colleagues' experiences. In the longer term, we anticipated that teachers would develop mathematical pedagogic approaches that are, at least to some extent, 'transferable' to lessons with other digital technologies and will have more confidence to use these in their classroom practices.

Cornerstone Maths constitutes an example of a DMT embedded within a set of research-informed curriculum units that support teachers to develop their pupils' understanding of three 'hard-to-teach' topics in key stage 3 mathematics: algebraic generalisation; geometric similarity; and linear functions. Each unit includes DMT, pupil workbooks, teacher guides and professional development; and so offers teachers a context and opportunities to re-think and re-engage with mathematics, while introducing them to new pedagogies involving pupil use of technology, for example, 'predict–check–explain'.

1.1 Brief history of Cornerstone Maths

Cornerstone Maths (CM) and its accompanying resources is used as an exemplification of a curriculum that embeds DMT. The CM project began in June 2011 (funded by the Li Ka Shing Foundation and Hutchison Whampoa Europe Ltd) and its overarching aims have been to build systematic knowledge and understanding of the process and content of scaling an educational technology innovation within key stage 3 mathematics education.

Digital technology offers new:

- Representations of mathematics and links between them (Kaput, 1986; Laborde and Laborde, 1995; Noss and Hoyles, 1996; Papert, 1980).
- Pupil–teacher–mathematics interactions (Clark-Wilson, Aldon, *et al*., 2014).
- Connections between: procedures and concepts; and concepts and their applications (Roschelle, Tatar and Kaput, 2008; Tall, 1991).

Studies across the world indicate that using digital technology effectively can lead to more pupils succeeding in mathematics and 'getting the point' (Roschelle, 2016). However, empirical and anectdotal research reveals that it is still widely underused in mathematics (Bretscher, 2014; Ofsted, 2012).

Against this setting, the CM approach was to exploit the dynamic and visual nature of digital technology to stimulate engagement with mathematical ways of thinking and to improve mathematics attainment by:

- Focusing on 'big mathematical ideas' that are core to the key stage 3 curriculum.
- Making links between key representations.
- Providing an environment for pupils to explore, make conjectures and solve problems within guided structured activities (outlined in the pupil and teacher materials).
- Embedding activities within 'realistic' contexts.

In the first phase of CM research (2011–14) three curriculum units were designed, piloted and evaluated with 15 teachers from 8 'design' schools before scaling to 258 teachers from 124 schools across England. An accompanying professional development approach was simultaneously designed and refined for scaling. The CM project has evolved over several phases since 2011, during which the focus has shifted. Initially the central concern was the design and evaluation of replacement units and establishing the learning gains for a diverse sample of pupils (Hoyles, Noss, Vahey and Roschelle, 2013). Subsequent project phases focused on identifying the factors that influenced how teachers and schools sustained CM after researcher support had faded, and also stimulated spreading the innovation to more schools and teachers (Clark-Wilson, Hoyles, Noss, Vahey and Roschelle, 2015). This project takes a new step in researching teachers' knowledge and practice.

1.2 Research methodology

1.2.1 Sample of schools and teachers

The project took place between January 2014 and November 2016. It involved 48 secondary schools from 23 London boroughs where the CM software was freely available, as the schools were already subscribed to the London Grid for Learning network.[2] The schools were self-selecting and were loosely affiliated to the network of six Department for Education (DfE) funded London Maths Hubs that came into existence in July 2014, through which we organised initial recruitment events and project publicity. However, the Maths Hubs lacked capacity to be more involved in the project, despite the human and financial resource that we were able to offer. Furthermore, it was quickly apparent that teachers paid little attention to Maths Hubs' geographical reach as they attended project meetings that were most convenient for them by date or venue location. Consequently, the UCL Knowledge Lab team retained the overall organisation and leadership of the project meetings, although we were able to maintain some level of collaboration with the Maths Hubs, as shown in Appendix A.

A total of 209 teachers participated in the project and for each school, the number of participating teachers ranged between 1 and 16.[3] The teachers were either self-selecting or nominated by their school leadership team. The breakdown of teachers' involvement in the different CM curriculum units is shown in Figure 1.

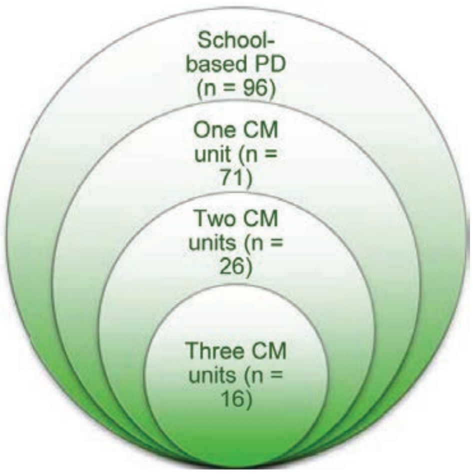


Figure 1: Numbers of teachers involved in the project and their level of involvement (n = 209)

2 Some 95% of London schools are subscribed to the LGfL network, which provides both broadband services and curriculum content.

3 We suspect 209 is a minimum number as, in our experience some schools are less inclined to inform us of their school-based PD activity.

1.2.2 The professional development cycles

The participating teachers engaged in up to three cycles of professional activity, with each cycle focusing on one curriculum topic during which they:

- completed a pre-survey that captured contextual data, a pre-assessment of their MKT/MPP, and an ethical agreement to give their consent as a participant in the study;
- participated in a one-day PD event at a local venue;
- joined the online project community, hosted on the National Centre for Excellence in Teaching Mathematics (NCETM) portal;
- developed a plan to teach the landmark activity as a 'research lesson' to a chosen key stage 3 class, which was shared by uploading to the NCETM portal;
- engaged in (optional) online PD webinars and tasks;
- taught the research lesson to their chosen class;
- attended a half-day PD meeting to feed back on their experiences; and
- completed a post-survey that captured feedback on teaching and assessment of their MKT/MPP.

In addition, at least 10% of teachers within each cycle were selected for classroom observation of their teaching of the research lesson, and were also interviewed on their reflections. The sample was chosen across a range of classroom and school contexts. The detailed lesson observation methodology is provided in the project Technical Report (Clark-Wilson and Hoyles 2017).

The complete data set is shown in Table 1.

Data type		Patterns and expressions	Linear functions	Geometric similarity
Teacher contextual data	111[4]			
Teacher MKT/MPP responses		73	60	40
Teacher PD Task responses		73	63	39
Lesson plan for landmark activity (planned in pairs/trios)		28	42	21
Lesson observation of landmark activity		7	15	9

Table 1: Summary of research data

1.2.3 Landmark activities

Landmark activities originate from Winograd and Flores' concept of cognitive breakdown, or 'situation[s] of non-obviousness' (1986, 165), in which established routines are replaced by conflict, disagreement or doubt (Hoyles, Noss and Kent, 2010). Thus, we define landmark activities as those

4 Of the 209 teachers, 111 were involved in complete PD cycles and gave ethical consent for their data to be used within the research.

that provoke a rethinking of the mathematics or an extension of previously held ideas – the ‘aha!’ moments that show surprise – and provide evidence of pupils’ developing appreciation of the underlying concept (Clark-Wilson, Hoyles and Noss, 2015).

For each of the three maths foci, the research team selected *one landmark activity*, which became the focus for teachers’ planning, teaching and subsequent reflection. The choice was guided by the criteria for landmark activities. They should:

- introduce and consolidate a significant aspect of mathematical knowledge, a ‘big mathematical idea’;
- include design features of the DMT that offer new dynamic ways to engage with the mathematics, the design principles; and
- be selected from mid-way through the curriculum unit, so that teachers and pupils would have some prior experience of the mathematics, technology and context.

The three selected landmark activities are described in Figure 2, Figure 3 and Figure 4.[5]

Context: *Create, analyse and algebraically model the LED light show.*

Pupil activity:

Big mathematical ideas:

- The geometric structure of the pattern – seeing the general in the particular.
- Algebraic equivalence modelled through the different ways of seeing a pattern leading to identification of variables that can be named and linked.

DMT design principles:

- Dynamic graphical representation of figural patterns in ‘player’ and ‘designer’ windows.
- Dynamic linking between representations, particularly graphical and algebraic, controlled by the slider.
- Variables conceived as ‘unlocked numbers’ that can vary dynamically.

Figure 2: Landmark activity: Patterns and expressions

5 Brief introductory video clips of the three landmark activities are available at: www.ucl.ac.uk/cornerstone-maths

Context: *Let's work on a game with robots. We need to set up the mathematics to make our robots move at different speeds.*

Pupil activity:

Big mathematical ideas

› Coordinating algebraic, graphical, and tabular representations.

› Speed as a context to introduce rates of change.

› $y = mx+c$ as a model of constant velocity motion – the meaning of m and c in the motion context.

DMT design principles

› dynamic simulation and linking between representations.

› drive the simulation from the graph (through draggable 'hotspots') or the function (by editing values).

› show/hide representations, as appropriate.

Figure 3: Landmark activity: Linear functions

Context: *The Graphics Department is using a software programme that makes enlargements (mathematically similar copies) by using scale factor. Our staff are unclear about how the software works. You can help us by investigating how the scale factor works.*

Pupil activity:

Big 'mathematical' ideas

› The variants and invariants in shapes that are mathematically similar, including identification of scale factor of enlargements.

› The one-to-one correspondence of sides and vertices within mathematically similar polygons.

DMT design principles

› dynamic measurements and comparisons, driven by angle and scale factor sliders.

› structuring recording within tables.

› linking geometrical manipulation with values in tables and ratio checker.

Figure 4: Landmark activity: Geometric similarity

1.2.4 Adapted lesson study approach

The project adopted a professional development approach (Figure 5) developed within another Nuffield-funded project, 'Lessons in Mathematical Problem Solving' (LeMaPS), which was being conducted concurrently (Foster, Swan, and Wake, 2014; Wake, Swan, and Foster, 2016).

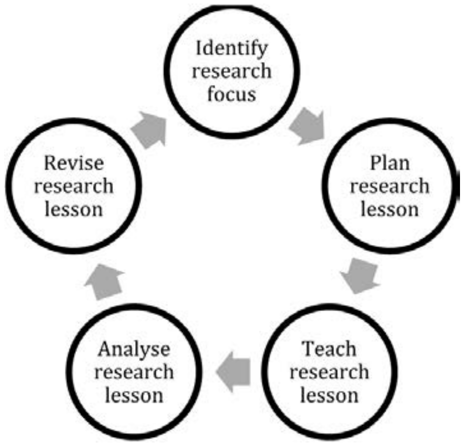


Figure 5: Adapted lesson study approach

We adapted the LeMaPS approach in the following ways:

- We agreed a common research focus for the landmark activity (our 'research lesson'), which was adopted by each group of teachers. These were:
  - Algebraic patterns and expressions: To develop pupils' appreciation of an algebraic variable as a dynamic concept.
  - Linear functions: To develop pupils' appreciation of the multiple representations of linear function as a dynamic concept.
  - Geometric similarity: To develop pupils' appreciation of geometric similarity as a dynamic concept.
- The teachers' lesson plans were created to a common proforma and shared electronically via the project's NCETM community.
- In their plans, the teachers were encouraged to think and plan explicitly for their mediating role relating to the use of the DMT – i.e. their words and actions when supporting whole-class and pupils' independent work *that involved the DMT*.

1.3 Conceptualising mathematical knowledge for teaching with technology (MPTK)

Early in the project we set out a more elaborated notion of MKT, incorporating mathematics teachers' knowledge for teaching with technology, adopting a framework developed by Thomas and colleagues, called '[mathematics] pedagogical technology knowledge' (Thomas and Hong, 2013; Thomas and Palmer, 2014), henceforth MPTK, as shown in Figure 6.

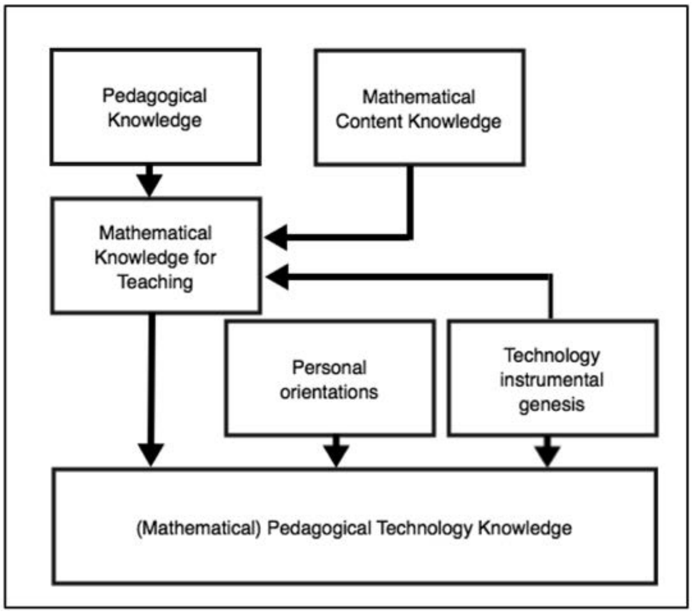


Figure 6: Components of [Mathematical] Pedagogical Technology Knowledge (Thomas and Hong, 2013)[6]

This is a theoretical construct with the following components:

- **Pedagogical knowledge**: First suggested by Shulman, this is a teacher's knowledge of the 'broad principles and strategies of classroom management and organization that appear to transcend subject matter' (Shulman 1987, 8).
- **Mathematical content knowledge**: A teacher's own knowledge of mathematics.
- **Mathematical Knowledge for Teaching (MKT)**: This combination of a teacher's pedagogical knowledge alongside their mathematical content knowledge was first defined as MKT by Ball, Hill and Bass (2005). Further research in this area has articulated a more detailed understanding of the 'mathematical work of teaching' that involves the use of mathematical representations (Selling *et al.*, 2016). We have expanded this idea to incorporate technological representations, resulting in the following processes that a teacher might describe in a lesson plan or enact during a lesson:
  - Connecting or matching representations as expressed with the digital tools.
  - Analysing representations by identifying correct or misleading representations in a text, talk, written or technological work.
  - Selecting, creating, or evaluating different representations as expressed by the digital tools.
  - Verbalising the meaning of representations as expressed by digital tools and how they are connected to key ideas.

6 Thomas and Hong do not include 'Mathematical' in their description of PTK. We surmise that their 'overarching world' is mathematical, so for clarity we add the word to make MPTK.

› **Personal orientations:** The teachers' affective variables – that is, their goals, attitudes, dispositions, beliefs, values, tastes and preferences, as described by Schoenfeld (2011, 29), also incorporating their perceptions of the nature of mathematical knowledge and how it should be learned (with and without technology).

› **Technology instrumental genesis:** This is the process through which the teachers' mathematical knowledge shapes and is shaped by their interactions with the technology as they accomplish a particular mathematical task. Furthermore, for teachers, this genesis incorporates the development of the teachers' understanding of pupils' processes of instrumental genesis, whereby pupils become familiar with the affordances of the technology and can begin to use it in mathematically productive ways.

This framework was operationalised as shown in Table 2, which summarises how each of the above components was researched within the study. The detailed methodology associated with the design of the tools and their analyses is contained in the accompanying Technical Report (Clark-Wilson and Hoyles, 2017).

Component of MPTK	Methodological tools used in the study
1. **Pedagogical knowledge**	a) Lesson plans for landmark activities. b) Teachers' reflections on lesson outcomes. c) Lesson observations and interviews.
2. **Mathematical content knowledge**	a) Teachers' highest mathematical qualifications. b) Content knowledge items in surveys and PD tasks. c) Lesson plans for landmark activities.
3. **Mathematical knowledge for teaching**	a) MKT/MPP items in surveys and PD tasks. b) Lesson plans for landmark activities. c) Lesson observations and interviews. d) Teachers' reflections on lesson outcomes.
4. **Personal orientations**	a) Prior experiences with technology (surveys). b) Confidence levels to teach landmark activity (PD Evaluation). c) Lesson plans for landmark activities. d) Lesson observations and interviews.
5. **Technology instrumental genesis**	a) PD tasks. b) Lesson plans for landmark activities. c) Lesson observations and interviews.

Table 2: Components of MPTK and the methodological tools employed within the study

The teachers' lesson plans for the landmark activities were a key methodological tool to provide insight into every category of their MPTK as above. From our previous research we hypothesised a set of eight features of teachers' lesson plans (see Table 3) that might result in strong adherence to the original design principles of an intervention embedding DMT.

Feature 1:	Describes teachers' actions and questions that do not involve the DMT.
Feature 2:	Describes pupils' actions on DMT.
Feature 3:	Supports pupils in their developing understanding and use of the DMT (their 'instrumental genesis'), as appropriate to the activities.
Feature 4:	Refers specifically to the mathematical concept at stake (i.e. variables, functions, and geometric objects).
Feature 5:	Describes acting on and connecting mathematical representations.
Feature 6:	Uses mathematical vocabulary.
Feature 7:	Uses technological/contextual vocabulary.
Feature 8:	Includes planned teacher use of the DMT.

Table 3: Eight desirable features of lesson plans

Our *a priori* view of how these lesson features might reveal aspects of a teacher's MPTK is shown in Figure 7.

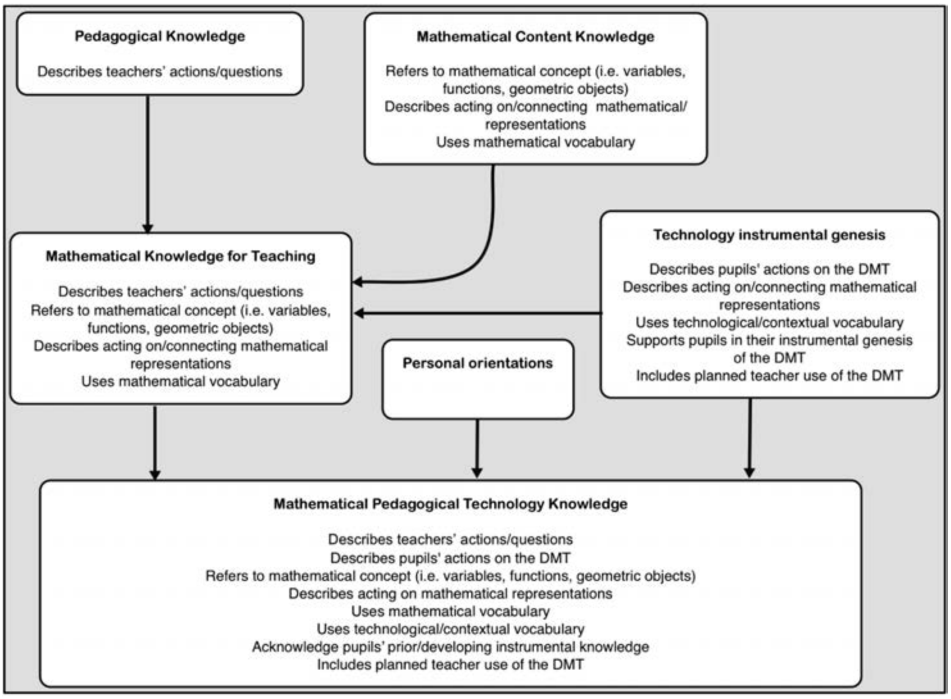


Figure 7: A mapping of the features of landmark activity lesson plans to provide insight into a teacher's MPTK

1.3.1 Analysing lesson plans to gain insight into MPTK

Initially, teachers worked in pairs or trios to plan the research lesson for the landmark activity using the proforma provided. This was to encourage them to discuss what they considered to be the essential components. Several teachers then decided to focus on a particular class of their own and adapt this plan to create a more personalised one. Consequently, it is not possible to attribute the named plan wholly to individual teachers and they were analysed as a representative data source at the teacher cohort level.[7] Moreover, the teachers' prior experience with the use of DMT suggested that for most of them, this was their first (or an early) plan for this type of lesson. Each lesson plan was analysed and awarded a score of 1 mark for each of the eight features listed in Table 3, resulting in a total score of 0–8 marks and enabling a quantitative evaluation of the plans to be made for each of the three topic areas. The lesson plans also provided an insight into how teachers were adopting the 'predict–check–explain' pedagogy.

1.3.2 Research ethics

The research project was granted approval by the UCL Institute of Education Research Ethics Committee. We consider our participants to be those who attended the formal project meetings and engaged in the project surveys and PD Tasks. This amounted to 111 teachers of the 209 that we know to have been involved overall. The remaining 63 teachers did not give their ethical consent for their data to be used in the research, although they may have been present at some project meetings and school-based professional development events. All participants were provided with information about the project that outlined their role and responsibilities, and were asked to give their ethical consent. There were no refusals. A copy of the ethical consent form is included in the project Technical Report (Clark-Wilson and Hoyles, 2017). Schools have not been named in this report and teachers' names are pseudonyms.

1.4 Research questions

In our original research questions, we conceived MKT and MPP as two distinct aspects of teachers' knowledge and practice. However, as we began to analyse our data set, and in discussion with our advisory group, it seemed that in order to better understand the situation, this distinction was not helpful. Hence the theoretical construct of MPTK, which combines all aspects of a teachers' knowledge and practice, offered a more integrated model, and we revised our research questions thus:

RQ1 What is the impact on teachers' MPTK around the mathematical concepts of algebraic generalisation, geometric similarity, and linear functions of their engagement with cycles of professional development and associated teaching that embeds DMT?

RQ2 What MPTK is desirable for teachers to integrate DMT in their teaching of these concepts?

7 The teachers who were observed were asked to provide a copy of their personal lesson plan prior to the lesson observation.

RQ3 What are the design features of professional development activities for key stage 3 mathematics teachers that support them to use DMT in ways that become embedded in their practice and lead to effective learning?

1.5 Teacher sample and their classroom settings

Given that teachers are the focus for the study, we start by summarising their professional and mathematical backgrounds, followed by some insight into their classroom settings.

1.5.1 The professional and mathematical backgrounds of the teachers

We report data from the 111 teachers who gave their ethical consent to the study.

The participant cohort was 59% female, 41% male, and 85% of the teachers reported that they were teaching a key stage 3 class during the 2015–16 school year. The proportion with a leadership role in mathematics in their school (Head/Assistant head of mathematics department) was 14%, and 11% had overall responsibility for the key stage 3 curriculum in their school. The cohort reflected a higher proportion of younger teachers than that reported in the most recent secondary school workforce survey.[8] Some 41% were under 30 when they became involved in the project, which contrasts with 25% nationally. Alongside this, 56% of the teachers had between 1 and 5 years' teaching experience and 9% were training to teach (participating in their placement school).

Teachers holding a BA/BSc degree in Mathematics, Statistics or Operational Research accounted for 75% of the total, close to the nationally reported figure of 73.7%. Of those who had not achieved this threshold qualification, 13% held GCE A level mathematics as their highest qualification, 2% had GCSE O level mathematics and 2% held no formal qualification in mathematics. Overall, 86% of the teachers considered themselves to be mathematics specialists. With respect to the teachers' qualifications in mathematics education, 75% held a postgraduate certificate in mathematics education, 7% held a Master's degree in mathematics education, 8% held a PGCE in a subject other than mathematics and 9% did not yet hold a formal qualification. The breakdown of the teachers' routes to qualification is shown in Table 4.

8 *School workforce in England*: *November 2015* (www.gov.uk/government/statistics/school-workforce-in-england-november-2015).

Route to Qualified Teacher Status		
Certificate in Education (Pre-1983)	1	1%
PGCE (University based)	79	71%
School based route (School Direct, SCITT etc.)	8	7%
TeachFirst	9	8%
Other	14	13%

Table 4: Teachers' routes to qualified teacher status (n = 111)

1.5.2 Teachers' classroom settings

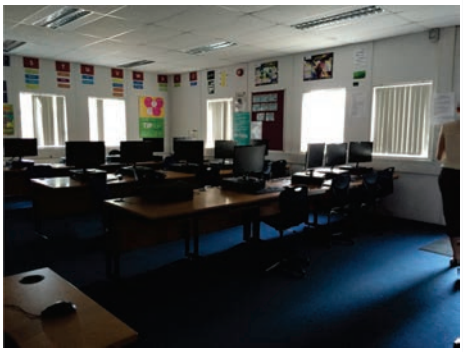
The teachers reported two types of classroom setting. The first type was a classroom designated for the teaching of mathematics, which was often (but not always) the normal teaching room for the class and, in some cases, there were fixed computers available in the room (see Figures 8a and 8b).

Alternatively, the class and teacher moved to generic IT teaching room settings, which varied in their configuration (see Figures 8c and 8d).

a) Mathematics classroom with computers around the perimeter.

c) Mathematics classroom with laptops in a trolley for distribution to tables.

b) IT classroom, arranged in rows with pupils facing the front of the class.



d) IT classroom, arranged in rows with pupils' screens facing the front of the class (i.e. their backs were towards the teacher).

Figure 8: Contrasting classroom arrangements (photos all taken from the 'front' of the classroom)

2 Main findings

The findings are presented to highlight the trajectory of development of teachers' MPTK for each of the example curriculum areas from their starting points ('Pre-MPTK'), through their planned DMT-mediated lesson activity ('Espoused MPTK') and, where we had lesson observation and interview data, their actual MPTK ('Enacted MPTK').

This is followed by the cross-topic analysis of teachers' trajectories in the form of case studies of three teachers who engaged in all three topics – from PD to enactment.

2.1 Teachers' MPTK prior to their classroom use of dynamic mathematical technology

2.1.1 The teachers' orientations towards dynamic mathematical technology

The pre-survey captured information about the teachers' prior uses of DMT using any software such as Geogebra, Autograph, Geometer's Sketchpad and Cornerstone Maths.

We surveyed the teachers on their personal use of DMT for whole-class teaching (Table 5) and their pupils' use (Table 6).

Never	Occasionally	Fairly regularly	Whenever I can
41	50	10	10
37%	45%	9%	9%

Table 5: Frequency of teachers' self-reported planned use of DMT for whole-class teaching in KS3 mathematics (n = 111)

Never	Occasionally	Fairly regularly	Whenever I can
49	47	9	6
44%	42%	8%	5%

Table 6: Teachers' self-reported planned use of DMT ***by their KS3 pupils*** (n = 111)

These usage figures are remarkably low. This pattern of usage is partly explained by the teachers' varying years of teaching experience. However, when the data were probed further, teachers across the experience range exhibited similar usage patterns.

We then analysed their self-reported levels of confidence to use DMT for both whole class teaching (Table 7) and pupils' independent use (Table 8).

Not at all	A little	Confident	Very confident
25	49	27	10
23%	44%	24%	9%

Table 7: Teachers' self-reported levels of confidence to use DMT during whole-class teaching (n = 111)

Not at all	A little	Confident	Very confident
12	56	37	6
11%	50%	33%	5%

Table 8: Teachers' self-reported levels of confidence to allow ***their KS3 pupils*** to use DMT (n = 111)

Fewer than one in five teachers reported that they were frequent users of dynamic mathematics technology in their key stage 3 teaching and one in three felt confident enough to do so. With respect to pupils' use of DMT, fewer than one in eight teachers reported frequent use, although a greater proportion of teachers felt more confident in this setting.

Contrasting these results, it seems that the teachers were more confident overall to use DMT for pupils' independent work than for whole class-teaching – although levels of confidence are universally low.

To elicit the reasons why teachers reported such low levels of use and confidence concerning pupils' use of such technologies, we asked the teachers to imagine a scenario where they and their KS3 pupils had unlimited access to DMT in lessons and to report the barriers they faced in making the most of this opportunity. The results are shown in Table 9.

Barrier (select all that apply)	Total	%
I don't know enough about suitable dynamic mathematical technologies	51	46%
I'd need more time to develop my knowledge and practice (on my own)	67	60%
I'd need more time to work with colleagues to develop lesson resources	63	57%
I'd need better access to suitable lesson resources	57	51%
I'd need to build my confidence in letting the pupils use the technology	31	28%
Other barriers (please list)	7	6%

Table 9: Teachers' self-reported barriers to possible unlimited access to DMT by their KS3 pupils (n = 111)

Most of the responses indicated a professional development need of one form or another. 'Other barriers' reported by teachers concerned: the institutional demand for paper-based evidence of pupils' learning during assessment (n = 2); potential inappropriate use of technology by pupils (n = 1); potential technical issues (n = 2); and that pupils' might lose focus or become complacent (n = 2).

For those teachers who had responded that they *did* use some form of DMT in their teaching, the most commonly cited resources by the teachers are shown in Table 10, although only one teacher provided any insight into *how* they were used, saying: 'GeoGebra (once for studying the circle theorems) and Excel (for teaching some topics in statistics and algebra such as solving equations by trial and improvement, moving average, and statistical diagrams).'

Resource	Frequency	% of cohort (n = 111)
Dynamic geometry software (Geogebra, Cabri-Geometry, Geometer's Sketchpad)	41	37%
Graphing software (Autograph, Omnigraph)	30	27%
Spreadsheet software (Excel)	29	26%
Other (Desmos, Euklid)	2	<1%

Table 10: Main DMT resources used by some teachers (n = 63, or 57% of cohort)

Summary of findings: Teachers' initial orientations

- The teachers' uses of DMT both in whole-class teaching and by their pupils was low.
- Teachers reported a range of professional development needs to enable them to increase their own and their pupils' uses of DMT. These included an increased knowledge of suitable DMTs; better access to suitable resources; and personal time and time with colleagues to review and develop suitable lessons.

2.1.2 Teachers' mathematical pedagogical technology knowledge

The teachers' MPTK was researched in relation to the three mathematical topics. The findings are presented accordingly. For each topic, we describe the main aspects of the topic that were researched and state the methodological tool adopted (e.g. survey item, PD task). The nature of each topic necessitated slightly different methodological tools. Some were offered before and after the PD cycle, others were initially offered as a pre-survey item and then developed within a task during the face-to-face PD.

2.1.2.1 Algebraic variable

The teachers' conception of algebraic variable was researched along three dimensions, in the following ways:

- Defining and naming algebraic variables: Pre-survey item and PD Task;
- Building a general expression: Pre- and post-survey item;
- Understanding letters as variables: Pre- and post-survey item.

Defining and naming algebraic variables

The pre-survey item 'What is an algebraic variable?' revealed a diversity of responses that indicated a wide range of understanding. The responses were initially categorised according to Küchemann (1981), 'Letter as unknown' and 'Letter as variable'. The latter category was expanded to include 'Letter as variable with acknowledgement of range and/or domain'; to take account of the audience of teachers not pupils. The findings are shown in Table 11.

Classification of response	Example	Frequency of response (%, n = 73)
Letter as unknown (LaU)	An algebraic variable is normally represented by a letter e.g. 'x' where the value is unknown. Different variables can be denoted by different letters, e.g. 'y', 'z' etc.	19%
Letter as variable – a number that can change (LaV)	Something that doesn't have a standard value (unknown) and can take values from a set of numbers.	62%
Letter as variable with acknowledgement of range and/or domain (LaVr/d)	An algebraic variable is something that can represent any number on a number line. It could, in fact, be considered to be the number line itself. It only takes specific values once an equation/inequality sign is introduced or the domain needs to be restricted in order to make the function defined.	10%
Incorrect response		5%
Unclear/ambiguous		4%

Table 11: Teachers' definitions of variable (n = 73)

Initially, 62% of the teachers (n = 73) defined an algebraic variable as 'a number that can change' with a further 10% offering definitions that took account of the range and/or domain. Immediately following this question, the teachers were asked to indicate their level of confidence in their previous response. Nearly half of the teachers indicated that they were only 'Quite confident' or 'Not at all confident'. The subsequent PD discussion in the face-to-face session supported all teachers to improve the clarity and mathematical rigour of their definitions and, alongside this, they reported increased confidence.

The teachers' naming of variables was examined during PD Task 1 where teachers were introduced to the *Patterns and expressions* dynamic software, which they used to create a dynamic figural pattern and define an algebraic variable (see Figure 9).

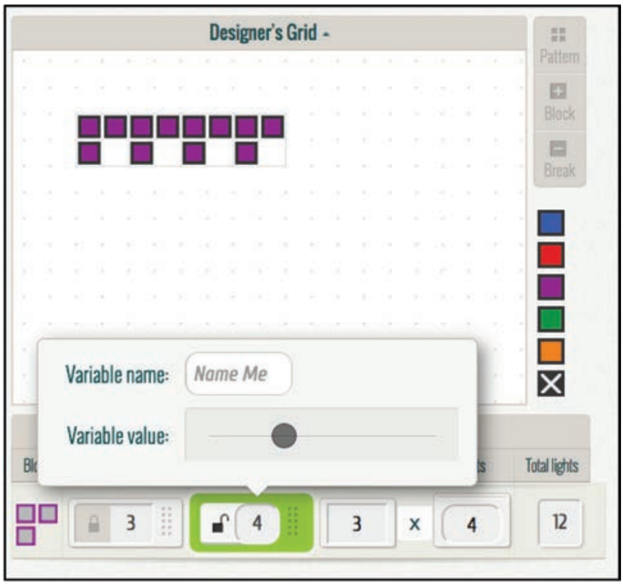


Figure 9: Software prompt to name an algebraic variable to represent the 'pattern number', showing the slider that can be dragged to change its value

The teachers' choices of names were classified as shown in Table 12.

Name of variable	Frequency of response (%, n = 74)
N or n	38%
a word	34%
B or b	11%
x	3%
another letter	15%

Table 12: Teachers' choices of names for the variable

Faced with the open question to name a variable that represented the number of blocks, 38% of the teachers chose the letter 'n' or 'N', 34% chose an actual word (i.e. a common or proper noun) and 11% chose 'B' or 'b' (n = 74).

Having defined their variable, the teachers were then asked to give a written response to the question 'what does the slider do?', which required them to interact with the software. Their descriptions were categorised and the results given in Table 13.

Feature of description	Frequency of response (%, n = 74)
changed the value of the variable (without further explanation)	84%
changed the **appearance** of the figural pattern	27%
increased the value of the variable	39%
decreased the value of the variable	3%
indicated the domain (i.e. integer values)	3%
indicated the range of values (from zero to ten)	7%
referenced the context (referred to lights and lighting patterns)	5%
explicitly referenced other representations (e.g. the impact of the slider's movement on particular representations)	5%

Table 13: Features of teachers' descriptions of 'What does the slider do?'

The teachers' responses were scored 1 mark for each of the above features mentioned within their descriptions. For example, the following response scored 5 marks: 'it **changes the value of the variable** to **integer** values **increasing** on right of the slider to **10** and **decreasing** to **1** on the left'. A less-developed response such as 'changes the number of blocks in the pattern' scored 2 marks. The box and whisker plot shown in Figure 10 gives the distribution of scores for the teachers' descriptions.

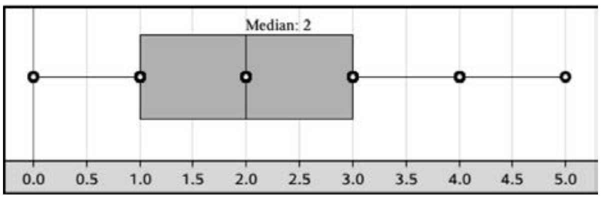


Figure 10: Distribution of scores (n = 74, $\overline{x}$ = 3.0, SD = 1.25)

Almost half the teachers gave responses stating that the effect of the slider was to change the value of the variable, without any further descriptive analysis. This suggests that teachers' early experiences with mathematical dynamic sliders were not necessarily sufficient to lead to a more detailed awareness of the slider's inherent mathematical features, with fewer than 7% of teachers paying attention to aspects such as range, domain and directional effects (the point of dragging to the right and to the left).

Building a general expression

The item 'Lisa's tiles', used previously by Hoyles and Healy (2003) in the Longitudinal Proof Project, was included in the pre- and post-survey (see Figure 11). The responses were analysed using a method adapted from that used by Hoyles and Healy to capture the nuances of the more detailed responses given by the teachers and allowing for an online testing environment that did not support diagrammatic responses.

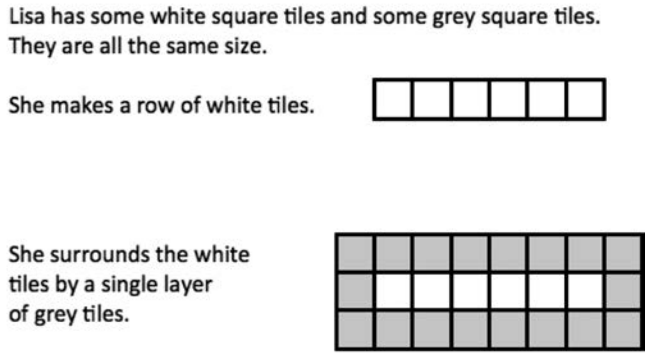


Figure 11: Lisa's tiles (Hoyles and Healy, 2003)

(a) How many grey tiles does she need to surround a row of 60 white tiles?
(b) Explain how you obtained your answer.
(c) Write an expression for the number of grey tiles needed to surround a row of n white tiles.

The data collected in response to this item is summarised in Table 14.

Response code	Frequency of response Pre-survey (n = 74)	Frequency of response Post-survey (n = 49)
Correct numeric response that included correct expression and correct reference to geometric structure (3 marks)	62%	63%
Correct numeric response that included correct expression or correct reference to geometric structure (2 marks)	26%	22%
Correct numeric response only (1 mark)	1%	6%
Incorrect response (0 marks)	11%	8%

Table 14: Teachers' responses to Lisa's tiles

Two-thirds of the teachers gave fully correct explanations in both the pre- and post-survey, which might suggest some stability in their knowledge.

Understanding letters as variables

The item 'Which is larger, 2n or n+2?', which was originally developed for the Concepts in Secondary Mathematics and Science Project (CSMS) (Küchemann, 1985, 14), probes teachers' MKT in relation to their flexibility to interpret *n* as a variable and to give a complete justification for their response that takes account of the domain of values for *n*. The question was expressed to the teachers as shown in Figure 12.

n is an integer. Which is larger, 2*n* or *n*+2? Explain your answer.

Figure 12: Which is larger, 2*n* or *n*+2?

The analyses of the teachers' responses are shown in Tables 15, 16 and 17.

Table 15: Analysis of teachers' responses to 'Which is larger, 2***n*** or n+2?'

Never	Pre-survey (n = 73)	Post-survey (n = 46)
Complete correct response (Code 1 complete)	51%	53%
Partial correct response (Code 1 Correct partial)	23%	22%
Incorrect (Code 8 Premature closure)	15%	16%
Incorrect (Code 2 Ambiguous)	4%	7%
Incorrect (Code 9 Other)	7%	2%

Table 16: Summary statistics for quality of justification (all teachers)

Quality score	Pre-survey (n = 73)	Post-survey (n = 46)
0 marks	28%	26%
1 marks	6%	6%
2 marks	63%	68%
3 marks	4%	0%
	$\bar{x}$ = 1.49 SD = 0.94	$\bar{x}$ = 1.82 SD = 1.11

Table 17: Summary statistics for quality of justification (matched teachers)

Quality of explanation	Pre-survey (n = 45)	Post-survey (n = 45)
0 marks	21%	26%
1 mark	21%	6%
2 marks	49%	68%
3 marks	4%	0%
	$\bar{x}$ = 1.79 SD = 1.33	$\bar{x}$ = 1.74 SD = 1.31

There was no significant change in the overall quality of teachers' explanations as indicated by the quantitative data in Table 17. However, ten teachers did noticeably improve their responses. For example, a teacher who responded initially, 'Depends: if n = 1 2n < n+2. If n = 3 2n > n+2', which was classified as Code 8 (Premature closure) improved his justification to provide a complete description, 'If n<2, 2n<n+2. If n = 2, 2n = n+2. If n>2, 2n>n+2.'

In the pre-survey, and subsequent project reflections, many teachers commented that their predominant teaching approach at key stage 3 treated 'letters as unknowns' and that pupils had little tangible experience of 'letters as variables'. In addition, the teachers themselves were very familiar with the generation of sequences in relation to figural patterns but had little familiarity with how the geometric structure of figural patterns related to the algebraic expressions, or how figural patterns enabled the important concept of mathematical equivalence to be introduced at key stage 3.

Summary of findings: Algebraic variable

- **Defining and naming variables:** Teachers' developed more precise mathematical responses of the meaning of a variable in terms of recognising its domain and range. Teachers required carefully mediated PD support to ensure that they fully appreciated the representation of variable provided by the DMT – that is, a slider. This mediation included drawing attention to the values on the slider and discussing the 'point' of dragging the slider.
- **Building a general expression:** There was limited improvement among one-third of the teachers in being able to relate the algebraic expression for a figural pattern with its geometrical structure. The remaining teachers were able to do this, as indicated by responses to the pre- and post-surveys. This indicates that their MPTK is resistant to change and that further PD cycles are needed, possibly in school.
- **Understanding letters as variables:** Teachers reported that their predominant teaching approach at key stage 3 centred on the treatment of letters as unknowns rather than as variables. The pre- and post-survey data again suggest little change in teachers' responses, although some teachers did give fuller explanations for their responses.

2.1.2.2 Linear functions

The teachers' conception of linear functions was researched along four dimensions, in the following ways:

- The meaning of m and c in $y = mx+c$: pre- and post-survey items.
- Defining a linear function: PD Task.
- Editing graphs of functions: PD Task.
- Interpreting multiple representations of motion: PD Task.

The meaning of *m* and *c* in *y=mx+c*

Three items were administered in the pre- and post-survey to research teachers' knowledge of the meaning of m and c in the equation $y = mx + c$ (Runners,[9] Mrs Kingston, and Wendella the dog).

a) Runners

This item probed teachers' understanding of $y = mx+c$ as a model for (ideal) linear motion and the meaning of m and c in this context.

9 These items were designed for an earlier study to assess teachers' MKT relating to linear functions (Shechtman *et al.*, 2010).

A teacher asked her class to compare two runners starting at the same time running along the same route. The runners' motions are described as: (a) $y = 3x$ (b) $y = 3x + n$, where x is the number of minutes they have been running, y is the distance from the starting line and n is an integer. Which of the following would be true statements about the runners? Choose *all* that apply. A: (b) is always ahead. B: You cannot tell who is ahead. C: (b) must be going faster. D: (a) and (b) are travelling at the same speed. E: You cannot determine which runner is faster.

Figure 13: Runners

Never	Pre-survey (n = 55)	Post-survey (n = 35)
Complete correct response – B **and** D only.	36%	46%
Partial correct response – B **or** D only.	27%	20%
Incorrect response – A and (C or E) selected.	36% (27%)	34% (18%)
	$\bar{x} = 0.93$ SD = 0.85	$\bar{x} = 1.11$ SD = 0.89

Table 18: Analysis of teachers' responses to Runners

This item reveals a slight improvement in the proportion of correct answers although the overall effect is not significant. The percentage of teachers who chose response A in the pre-survey (27%) highlights that a notable number of teachers did not perceive that the value of c could be negative, possibly indicating a fixed view of the letter representing a (positive) unknown rather than any integer. This percentage reduced to 18%, indicating some development of this important knowledge.

b) Mrs Kingston

This item probed teachers' understanding of the role of c in the model $y = mx+c$ in different contexts.

Mrs Kingston is discussing with her class the use of the equation $y = mx + c$ to model everyday situations. She wants to illustrate the role of c. Which are reasonable contexts in which c might vary? Choose ALL that apply. A: Three students run a race. All three start in the same place and run at different speeds. B: Three students run a race. They each start at different places but run at the same speeds. C: You need to buy custom T-shirts for the school football team. Two different manufacturers offer T-shirts at £6 each, but one charges a larger initial 'design' fee. D: You want to take a number of friends to the cinema and need to choose which time showing to attend. The admission price is more in the evening than in the afternoon. E: None of the above.

Figure 14: Mrs Kingston

Never	Pre-survey (n = 55)	Post-survey (n = 35)
Complete correct response – B **and** C only. (2 marks)	60%	72%
Partial correct response – B **or** C only. (1 mark)	25%	11%
Incorrect response – A **or** D **or** E selected. (0 marks)	15%	17%
	$\bar{x} = 1.45$ SD = 0.73	$\bar{x} = 1.55$ SD = 0.76

Table 19: Analysis of teachers' responses to Mrs Kingston

This item reveals a slight improvement in the proportion of complete correct answers but the overall effect is not statistically significant, which implies that the teachers' knowledge is resistant to change.

The 'role of *c*' in linear functions was explored further in the post-survey through teachers' responses to the prompt 'Create a context of your own that could also be used to illustrate the role of *c* in linear functions'. Of the 33 responses, 30 offered correct suggestions (91%) of which 3 related to the context of motion, for example 'Two pupils are on their way to school. Pupil *a* lives further away from school than pupil *b* but pupil *a* and pupil *b* travel to school at the same speed'. The remaining 27 offered contexts that indicated a 'transfer' of the concept to a context other than motion. For example, 'Two electricity companies charge the same amount of money per unit but one has a standing charge whilst one does not' and 'Plumber A charges £5 per hour and a call-out fee of £20. Plumber B charges £3 per hour with a call out-fee of £30. Using linear functions, describe who is cheaper.' The high success rate for this item suggests that the teachers' knowledge had been consolidated through their project engagement.

c) Wendella the dog

The item 'Wendella the dog' presented the teachers with a short video clip of an animation and graph for a position–time graph. When the video clip was played, the dog (Wendella) moved position and, simultaneously, the dog's position–time graph was plotted. The video paused after 6 minutes and the teachers were prompted to 'describe the rest of the dog's journey as shown on the graph so that it can be reproduced exactly'.

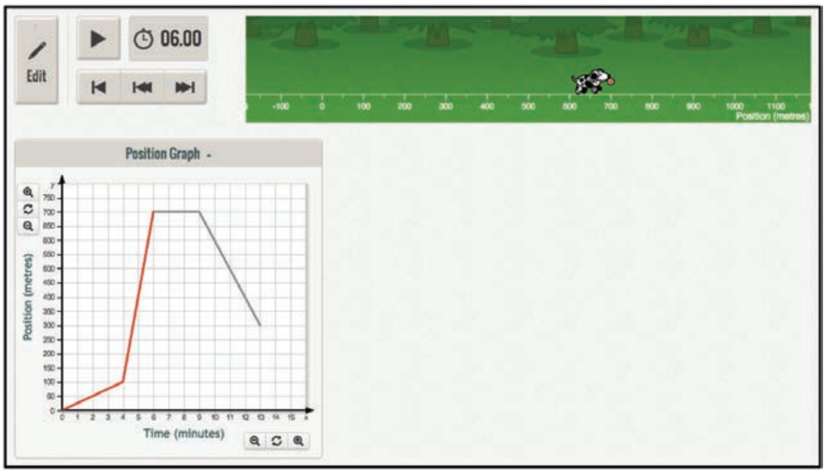


Figure 15: The paused animation from 'Wendella the Dog' (t = 6 minutes)

The teachers' written descriptions of the remaining part of Wendella's journey were classified and scored according to the story's reference to: units of measurement; a statement that Wendella 'stops'; that Wendella is stationary for 3 minutes; Wendella's speed (expressed as 400m in 4min or 100 m/min); and Wendella's backward movement. The maximum score was 5 marks. The distribution of total scores is shown in Figure 16 and the mean score and standard deviation were 3.52 marks and 1.37 marks respectively.

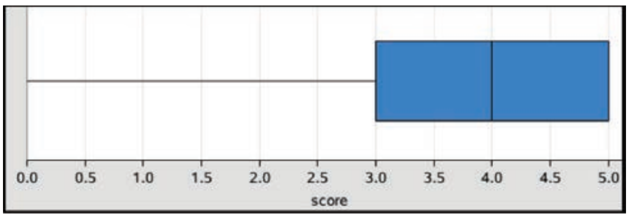


Figure 16: Distribution of scores for 'Wendella the Dog' (n = 55)

This indicates that the teachers appeared to have a good understanding of this general context, although only 45% included Wendella's speed in their description. A typical example that scored 5 marks was: 'Doesn't move for 3 minutes then moves back 400 metres for 4 minutes'. A weaker response, scoring only 2 marks, was: '0–4 speed is smaller than during 4–6, during 6–9 rest, then goes back'.

Defining a linear function

In an initial PD task during the first PD session, the teachers worked individually to write a definition of a linear function as if they were talking to another mathematics teacher and following this, rate their confidence in their definition. This topic was then opened up for discussion. Most notably, only 6 of the 55 teachers felt very confident about their response, which is surprising given the mathematical backgrounds of the cohort and that this is a key topic in both the key stage 3 and 4 curricula. Most definitions were partial and referred to the characteristics of the graphical output of a linear function (i.e. referring to gradient and intercept properties), or defined it by saying that it could be expressed in the form $y = mx+c$. Only two teachers included that a linear function is a one-to-one mapping of a relationship and no teachers mentioned the number domain.

Editing graphs of functions

This task was undertaken as an introduction for the teachers to the *Linear functions* DMT. The teachers were presented with the scenario of a race between two cars and invited to edit the graph to meet a given set of criteria. They were then asked to describe what each of the 'hotspots' on the graph did, i.e. how each of the draggable points or hotspots (labelled 1, 2 and 3 in Figure 17) affected the simulation of each car's motion.

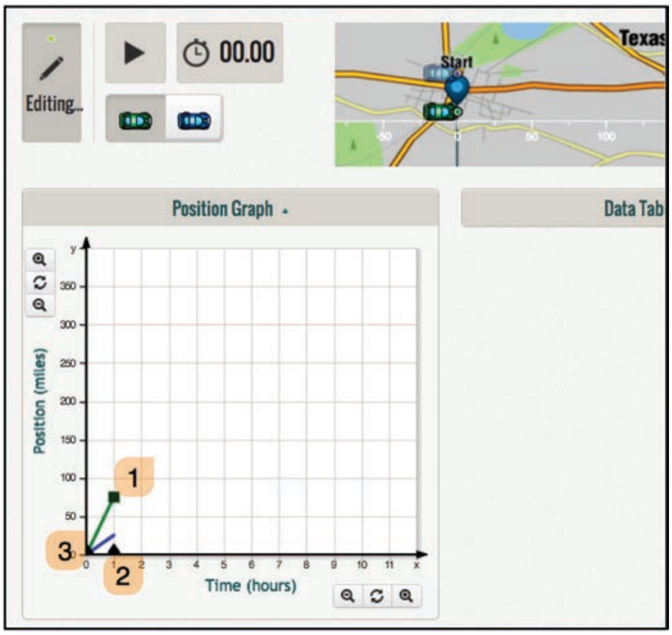


Figure 17: Opening screen for PD Task 2, showing the three 'hotspots', labelled 1, 2 and 3

Even though we emphasised these hotspots during our introduction to the teachers, only half of the teachers paid attention to the hotspots in their description. They tended to use general language such as 'we moved the green to 250 miles at 5 hours and the blue to 250 miles at 6 hours' or 'edit the blue car to win', with no detail about which hotspot they were dragging and how. Alternatively, descriptions such as 'Dragged hotspot (A) for blue to 200 while "C" is at 1' implied that the teacher had labelled the hotspots (A, B and C) and was using these as references.

This suggests that, if teachers had not articulated how their own dragging actions using the technology had impacted mathematically on the context, they might struggle to support their pupils to do the same. Furthermore, the absence of clear vocabulary to describe such actions (and their outcomes) might also hinder subsequent classroom discussion.

Teachers' initial engagement with the DMT prompted much curiosity and interest and, although many teachers struggled initially to makes sense of the dynamic aspects, in particular the mathematical nuances of dragging the hotspots, nearly all saw value in the potential afforded by the technology. The motion context, supported by the animation, enabled teachers to make the important connections between the value of the coefficient of x, the gradient of the line segment on the graph and the multiplicative and additive relationships inherent within the table in ways that suggested they had rethought their prior understanding.

Interpreting multiple representations of motion

The teachers were presented with a screen shot from the *Linear functions* DMT of a 'race' between a blue and a green car (Figure 18) and were asked during PD Task 3 to work individually and 'use the information to work out the distance travelled by the blue car after 5 hours' in as many ways as they could. They recorded their response in a table with three columns by stating: the representation used; what they did (their working out, explanation and solution) and the 'underlying mathematical idea'. The task was to set a challenge with the expectation to 'find as many successful strategies as possible', as it sought to reveal the depth of the teachers' mathematical knowledge in terms of the range of representations that could be used to obtain a correct solution.

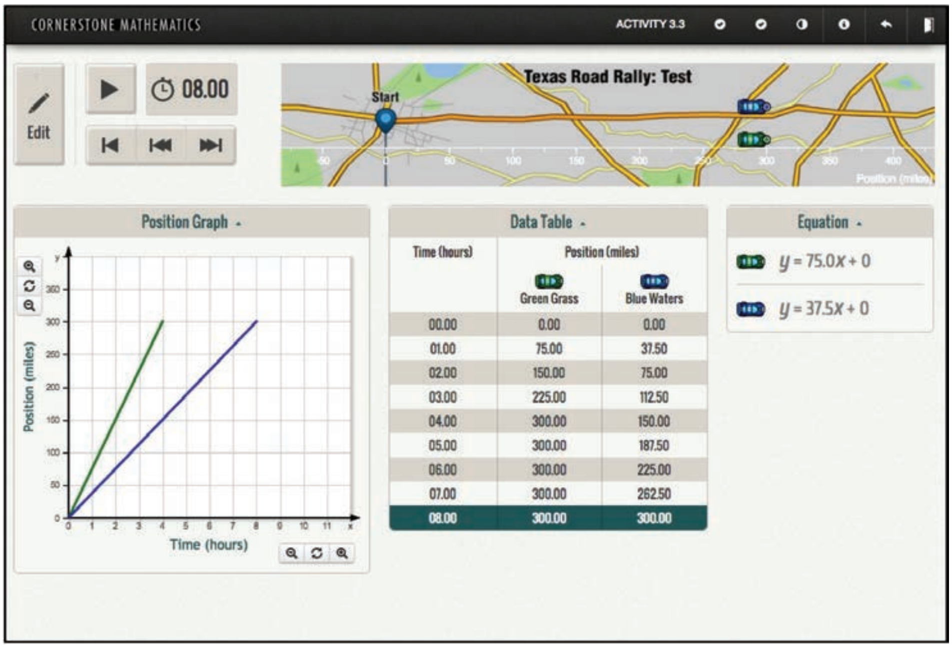


Figure 18: Interpreting multiple representations of motion (PD Task 3)

Almost all of the teachers' first choice of strategy was to make an estimated reading of the distance from the graph (57%) or to read the distance when Time = 5 hours directly from the table (37%). Overall, 80% of the teachers included a graph-reading strategy and 97% used the table as one of their cited strategies. Use of the equation to substitute x = 5 hours to calculate a value for the distance travelled was used by 93% of the teachers overall, but this was most commonly their third-cited strategy after a graph-reading and table-reading approach.

The following strategies were used by far fewer teachers:

- a proportional reasoning strategy based on the graph, for example using the distance travelled after 4 hours to 'scale up' and achieve an accurate answer (22%);
- a similar proportional reasoning strategy based on the simulation (27%); and
- a proportional reasoning strategy from the graph, table or simulation, based on finding a unit rate (8%).

The number of distinct correct strategies used by the teachers is shown in Table 20, which indicates that 84% of the teachers had two or three successful approaches.

Number of correct strategies used	Frequency	%
1	3	5%
2	2	3%
3	31	52%
4	19	32%
5	5	8%

Table 20: Number of correct strategies used (n = 60)

It was notable that few teachers had made the mathematical connection between the gradient property of graphical representations of linear functions with the variant and invariant properties of similar right-angled triangles (see Figure 19).

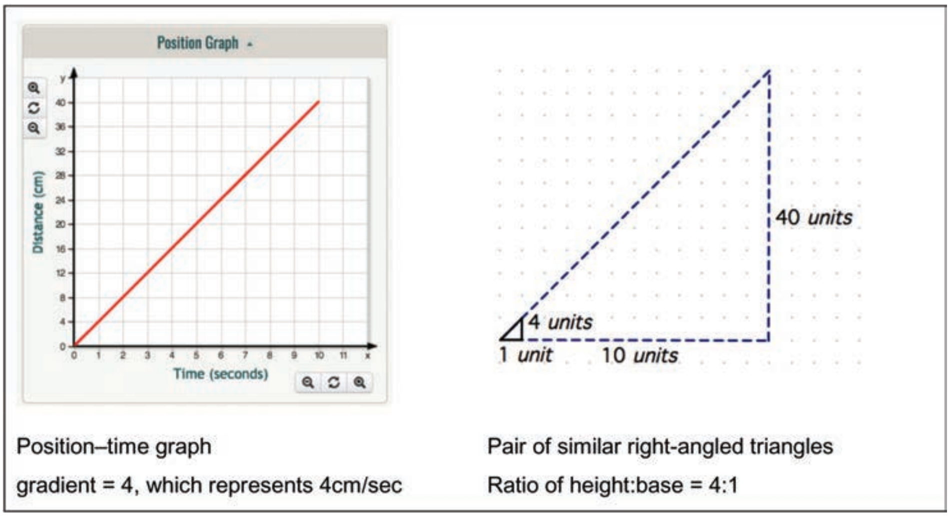


Figure 19: Algebraic and geometric interpretations of the gradient property of linear functions

Following this exercise, the PD session continued with group discussions during which the teachers were asked to: share their strategies; seek to understand those that they had not used; and record a written reflection on their underlying thoughts. Their notable comments included:

- ‘bringing geometry into this didn’t seem obvious once we’re conditioned how to think and what to apply’;
- [I] ‘wouldn’t have linked similarity to a graph’.

Furthermore, the teachers’ reflective comments provided clear evidence of how the dynamic motion context was supporting them to rethink their own mathematical knowledge as they related the different strategies to their existing models. For example, in Figure 20 the teacher is attempting to connect the formula that connects speed, distance and time to the linear function, and is still unsure of what *m* and *c* actually represent.

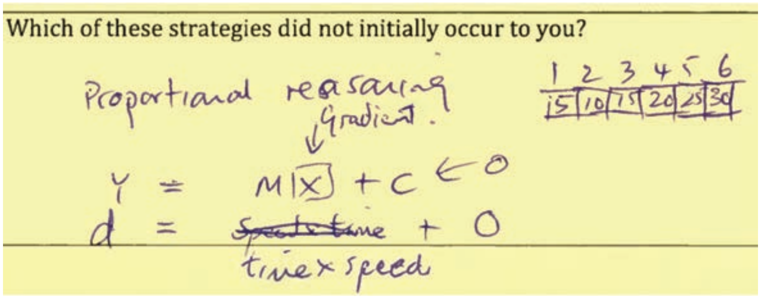


Figure 20: A teacher's response to PD Task 4

Summary of findings: Linear functions

- **The meaning of m and c in $y = mx+c$**: The teachers' knowledge of the meaning of m and c in $y = mx+c$ showed improvements, particularly concerning the interpretation of the value of c as a variable that can take a negative value.
- **Defining a linear function**: Most teachers' initial definitions of a 'linear function' were partial and tended only to refer to the characteristics of the graphical output of the function, with few teachers making reference to the one-to-one nature of a function or its domain and range. The teachers' subsequent professional discussion was a significant factor in their refinement and expansion of their earlier responses.
- **Editing graphs of functions:** Teachers required carefully mediated PD support to ensure that they fully appreciated the rationale for, role and effects of draggable 'hotspots' on the graph. Initially, only half the teachers paid attention to the specific affordances of these 'hotspots', which controlled the resulting animation.
- **Interpreting multiple representations of motion**: Teachers used a limited number of strategies to read the multiple representations of the DMT when faced with a problem-solving task.

2.1.2.3 Mathematical (geometric) similarity

The teachers' conception of geometric similarity was researched along three dimensions, in the following ways:

- properties of geometrically similar polygons (pre- and post-survey item);
- defining geometric similarity (pre- and post-survey item, with discussion in PD Task); and
- within-polygon and between-polygon ratios (pre-survey item and PD Task).

Properties of geometrically similar polygons

Teachers were invited to list the properties of geometrically similar polygons. These were then coded and scored 1 mark for each reference to the following properties that featured:

- same number of sides;
- same set of angles (accounting for order);

- corresponding angles are equal;
- all pairs of corresponding sides are in the same ratio;
- enlarged by same scale factor;
- congruency is when the scale factor equals 1;
- any pair of corresponding length measurements are in same ratio (i.e. perimeter, diagonals); and
- areas are in ratio of (length scale factor)^2.

Table 21: Properties of geometrically similar polygons: Frequency of stated properties (n = 40)

Response	Frequency	%
Enlarged by same scale factor	7	18%
Corresponding sides in same ratio	4	10%
Corresponding angles are equal	10	25%
Polygons have same no. of sides	3	8%
Insufficient/incorrect	17	43%

Table 22: Properties of geometrically similar polygons: Quality of response (n = 40)

Quality score	Frequency	%
No correctly stated property	17	43%
One correctly stated property	10	25%
Two correctly stated properties	7	18%
Three correctly stated properties	5	13%
Four correctly stated properties	1	3%

Further analysis of the insufficient/incorrect responses revealed:

- 8 responses (20%) stated relevant key words without defining the characteristic, i.e. gave a response such as 'angles and sides'.
- 4 responses (10%) refer to the idea of 'same shape different size' without further explanation.

It is also notable that 11 responses (28%) make reference to the concepts of enlargement/scale factor, which indicates a tautology. No responses made reference to the apparent *invisible property* of the invariant ratio between pairs of corresponding sides within the shape. There were also a few responses that indicate that the teacher's conceptualisation of geometric similarity existed within a limited view of polygons, for example by implying particular quadrilaterals: 'Both the length and width scale factors MUST be the same (the length can't increase by a different scale factor to the width and vice versa).'

Still on the theme of the properties of geometrically similar shapes, the teachers gave their responses to a set of seven statements in accordance with whether they were always, sometimes or never true.

1. Similar shapes are the same size.
2. Similar shapes have equal corresponding angles.
3. Shapes with the same set of internal angles are mathematically similar.
4. Congruent shapes are similar.
5. Similar shapes are congruent.

6. The lengths of corresponding sides in congruent shapes are equal.
7. The scale factor between the lengths of corresponding sides for congruent shapes is one.

This item was given in both the pre- and post-surveys, and each correct response scored 1 mark. In the pre-survey, 9 teachers (23%) gave fully correct responses. A notable number of teachers gave *incorrect* responses to the following two statements:

Statement 3 *Shapes with the same set of internal angles are mathematically similar* (57% incorrect), with all of the incorrect answers stating 'Always', which implies that the order of the sets of angles was not important.

Statement 7 *The scale factor between the lengths of corresponding sides for congruent shapes is one* (25% incorrect), possibly indicating that teachers' knowledge of congruency is fragmented and we hypothesise that congruency is not a generalised idea as it is only encountered when working with triangles.

The overall success rate in the post-survey did improve to 33% (n = 26), although the improvement was not significant, as shown in Table 23.

Number of correct statements	Pre-survey (n = 40)	Post-survey (n = 26)
2	3%	4%
3	5%	4%
4	13%	17%
5	28%	16%
6	30%	38%
7 (all correct)	23%	33%
	$\overline{x}$ = 5.5 SD = 1.24	$\overline{x}$ = 5.8 SD = 1.29

Table 23: Properties of geometrically similar polygons (always-sometimes-never): Analysis of teachers' responses

The pre-survey also revealed teachers' incomplete knowledge regarding the concept of corresponding sides in geometrically similar shapes. Again, this seemed to be constrained by the domain of triangles, with many definitions only holding true in this context. Furthermore, few teachers considered the importance of the angles formed at each end of the pairs of corresponding sides. The teachers' revised definitions captured in the post-survey revealed less dependency on triangles but they still struggled to express this particular property in words concisely.

Defining geometric similarity

Having listed the properties of geometrically similar polygons, the teachers were then asked to state a definition – that is, to determine the minimum set of properties that should be true.

During the first PD session, the teachers discussed and critiqued each other's definitions and were given the opportunity to say whether they would revise their original words and if so, how. This prompted reflections such as 'that ALL corresponding sides are proportional in *the same ratio*' and 'include the idea that mathematically similar shapes can be the same'. It is again notable that no revisions were suggested that take account of the invariant property of the ratio of the side lengths for any corresponding *pair of sides within* two similar polygons.

The summary of the teachers' initial responses and their espoused confidence level associated with their response is shown in Tables 24 to 27.

Table 24: Definitions of geometric similarity (Initial, before discussion) (n = 39)

Definition		
Fully correct	27.5%	11
Insufficient	72.5%	29

Table 25: Teachers' espoused confidence level (Initial, before discussion) (n = 39)

Definition		
Not at all confident	12.5%	5
Quite confident	42.5%	17
Confident	37.5%	15
Very confident	7.5%	3

Table 26: Definitions of geometric similarity (Final, at end of PD cycle) (n = 26)

Definition		
Fully correct	81%	21
Insufficient	19%	5

Table 27: Teachers' espoused confidence level (Final, at end of PD cycle) (n = 26)

Definition		
Not at all confident	0%	0
Quite confident	38.5%	10
Confident	46%	12
Very confident	15%	4

Although the second survey was completed by only 26 of the 40 participants in the *Geometric similarity* PD cycle, there was still a noticeable improvement in the quality of teachers' responses, indicating a growth in their subject knowledge. For example, one teacher's definition developed from 'they are similar if their angles are the same' to 'Two shapes are similar if their corresponding angles are the same and their pairs of corresponding sides are in proportion'.

Finally, the teachers were asked to comment on where the concept of geometric similarity was introduced to pupils in their department's mathematics schemes of work. There was great diversity in their responses, which addressed both the timing (e.g. Year 7), the curriculum topic area (e.g. enlargements) and the resources that were used (e.g. textbooks).

Following the teaching of the *Geometric similarity* research lesson the teachers were asked to comment on the key ideas that pupils at key stage 3 could be introduced to. Most teachers' responses indicated a rethinking of the concept and suggested that they saw value in placing more emphasis on its teaching, with and without the use of DMT.

For example:

> What constitutes a similar shape, there are two types of ratios for similar shapes, the idea of a scale factor and the link to ratio.
>
> The ideas of finding a true definition of mathematically similar for themselves and making the learning visual. The definitions, keywords and why shapes are similar and how they are mathematically similar. It is more visual and the benefits that are attached with investigations/discovering for themselves versus being told by myself.

And in relation to teaching the topic with a DMT:

> I did not realise that without introducing the students to the word trigonometry they were already doing trigonometry. For mathematical similarity I was used to just relating corresponding sides but relating sides within the same shape can also be helpful. I have gained more confidence in using dynamic technologies and hope to persuade others to use it but I struggled to keep up with the scheme of works and also have time to spend on getting a deeper understanding of maths using dynamic technologies. I am enthusiastic and hope I can persuade others.

Within-polygon and between-polygon ratios

The pre-survey included an item that invited teachers to find two missing sides within a set of three similar parallelograms as shown in Figure 21.

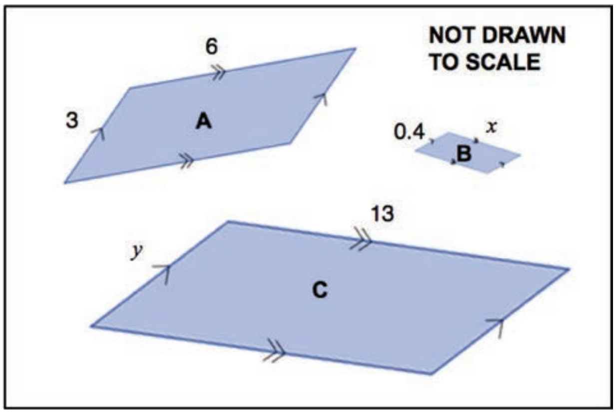


Figure 21: Geometric similarity: Similar parallelograms

The teachers' responses were classified correct/incorrect and the correct solution strategies, which formed 83% of the cohort (n = 40), were further classified using the criteria shown in Table 28.

Solution strategy	Frequency	% (n = 27)
Scale factor method ('between-shape' ratio)	32	90%
Internal scaling ('within-shape' ratio)	4	10%

Table 28: Teachers' initial responses to 'Similar parallelograms' item[10]

This item was designed to directly assess the breadth of teachers' knowledge of invariant properties of geometrically similar shapes since, if the teachers' dominant strategy involved a scale factor method, the accompanying calculation was tedious, as shown by a teacher's written workings Figure 22.

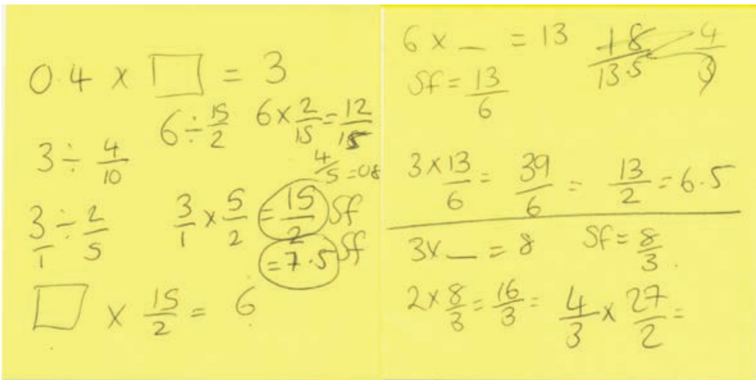


Figure 22: A teacher's written response to 'Similar parallelograms'

However, if the teachers had knowledge of the property that, for geometrically similar shapes, the ratios of the side lengths for any pair of corresponding sides within the shape is also invariant, then the solution strategy is trivial as the longer side is twice the shorter side in each case. We refer to this property as the 'within-polygon ratio' in the remainder of this report.

Seven teachers noticed this property while in the process of responding to this item – and commented accordingly, for example:

> I initially looked for a scale factor by calculating 6/0.4 and then multiplying by 3 to get x. Then I realised that the longer side was double the shorter side and used this to work out y as it was quicker.

The teachers' awareness of within ratios was probed further during two tasks in the initial PD session.

In the first of these, the teachers watched a 40-second screen recording of an activity within the *Geometric similarity* DMT in which ratio sliders

10 Unfortunately, a technical issue at two of the PD venues resulted in the image for this question not being displayed within the final online survey, so there was insufficient data for analysis to enable a worthwhile comparison.

were used to highlight this same invariant property. (Two static images are shown in Figure 23 and the video clip can be viewed at https://mediacentral.ucl.ac.uk/Play/5417)

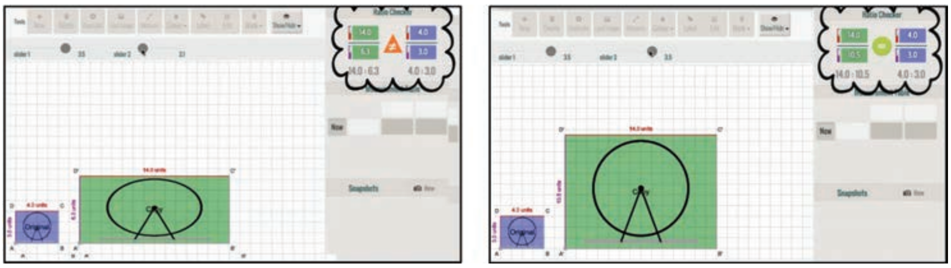
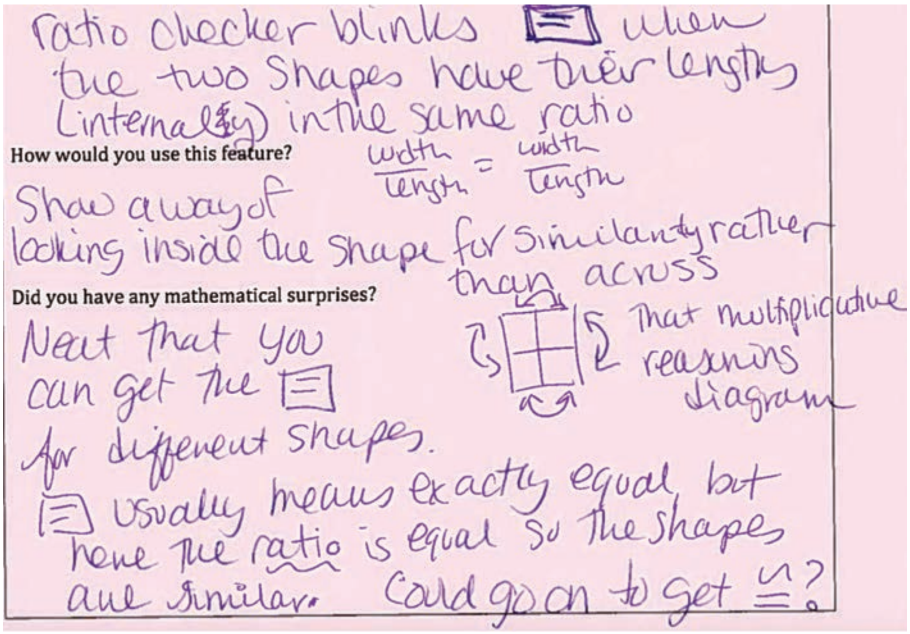


Figure 23: Geometric similarity: PD Task 2, 'What did you notice?'

All teachers noticed the equals sign appear on the screen. However, fewer than 25% of teachers looked closely enough to notice the specific length properties of the two shapes that were being compared within the (dynamic) on screen 'ratio checker'.



ratio checker blinks $\boxed{=}$ when the two shapes have their lengths (internally) in the same ratio

How would you use this feature?

$\frac{\text{width}}{\text{length}} = \frac{\text{width}}{\text{length}}$

Show a way of looking inside the shape for similarity rather than across

Did you have any mathematical surprises?

Neat that you can get the $\boxed{=}$ for different shapes.

That multiplicative reasoning diagram

$\boxed{=}$ usually means exactly equal but here the ratio is equal so the shapes are similar. Could go on to get $\cong$?

Figure 24: Geometric similarity: A teacher's response to PD Task 2

The second PD task was even more explicit in its design to prompt the teachers to think deeply about the 'within ratio' invariant property by presenting them with two 'correct' screen shots of pupils' work using the *Geometric similarity* DMT, which exemplified typical responses that the teachers might come across in their classroom (see Figure 25). The teachers worked in pairs to: individually recreate each screen using the DMT; discuss and note the differences; and state the aspects of geometric similarity represented in each of the two statements in the ratio checker.

Figure 25: Geometric similarity: PD Task 3, 'Analysing representations'

This activity was very well received by the teachers, who had animated discussions in which they seemed to connect strongly with this apparently 'new' idea, and resolved some initial struggle to analyse these new representations; and articulate the underlying mathematical ideas.

Summary of findings: Geometric similarity

- **Properties of geometrically similar polygons:** The task to state the properties of geometrically similar shapes was challenging with just over two-thirds of the teachers stating no or only one correct property (e.g. failing to mention angles, restricting definitions to triangles). By the end of the project many teachers could produce more general and complete responses.
- **Defining geometric similarity:** Teachers' definitions of geometrically similar shapes showed notable improvement, although not statistically significant. This improvement was also matched by an increase in their reported confidence in their definitions.
- **Within-polygon and between-polygon ratios:** The most significant improvement in the teachers' MPTK related to their understanding of the 'within-polygon' invariant for geometrically similar polygons, an 'invisible' property that had been made visible by the DMT.

2.2 Teachers' espoused MPTK as seen through their lesson plans for landmark activities

The teachers planned their lessons according to a common proforma, which was designed to encourage:

- a focus on the teachers' actions during the lesson;
- general pedagogy such as managing pupil activity, questioning and adoption of the predict–check–explain pedagogical approach with DMT;
- emphasis on the key variant/invariant properties that underpin the mathematics of the lesson;
- teacher and pupil use of the technology; and
- use of vocabulary.

The teachers worked in pairs/groups during the planning task. Consequently, the plans cannot necessarily be attributed directly to individual teachers, although they provide a source for triangulation of data related to the observed lessons. In addition, for most teachers, it was apparent that these were their first ever plans for a lesson in which pupils were using a *DMT*.

The lesson plans were analysed individually and scored by awarding one mark for each of the following features that had been included:

1. It describes teachers' actions and questions that do not involve the DMT.
2. It describes pupils' actions on DMT.
3. It supports pupils in their developing understanding and use of the DMT (their 'instrumental genesis'), as appropriate to the activities.
4. It refers specifically to the mathematical concept at stake (variables, functions, geometric objects).
5. It describes acting on and connecting mathematical representations.
6. It uses mathematical vocabulary.
7. It uses technological/contextual vocabulary.
8. It includes planned teacher use of the DMT.

Hence each plan was awarded an overall 'quality score' of between 0 and 8 marks. The detailed process and examples of lesson plans that scored 1 and 8 respectively are included in the project Technical Report (Clark-Wilson and Hoyles, 2017).

This analysis of the teachers' lesson plans provided an insight into their MPTK as they prepared to teach the lessons. The enactment of these lessons was then explored further during the lesson observations and subsequent teacher interviews and feedback sessions, and this is reported later in Section 2.3.

We present our findings on teachers' espoused MPTK with respect to the three CM curriculum topics, followed by the cross-topic analysis that led to a more general set of outcomes.

2.2.1 Algebraic variable

Twenty-eight lesson plans that had been produced in pairs and trios by 74 teachers were analysed and the frequencies of each feature is shown in Table 29.

Feature of lesson plan	Frequency	% (n = 28 plans)
1. Explicit descriptions of teachers' actions/ questions that do not involve the DMT	16	57%
2. Explicit descriptions of pupils' actions on DMT during the lesson	9	32%
3. Appreciation of pupils' instrumental knowledge	12	43%
4. Explicit reference to variables (creating, naming, acting on)	11	39%
5. Explicit reference to acting on representations (dragging/moving sliders)	17	61%
6. Explicit use of mathematical vocabulary	18	64%
7. Explicit use of technological/contextual vocabulary	18	64%
8. Planned plenary phases that involved teacher use of software	5	18%

Table 29: Algebraic variable: Summary of lesson plan analysis (28 lesson plans)

The distribution of the quality scores for each plan is shown in Figure 26.

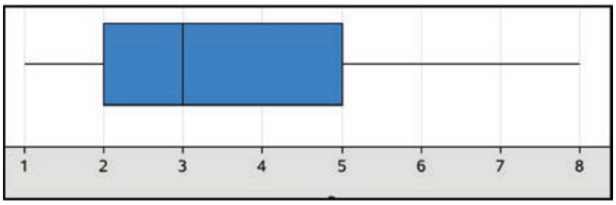


Figure 26: Algebraic variable: Distribution of quality scores for lesson plans (n = 27, $\bar{x} = 3.9$ SD = 1.8)

The plans were of a highly variable quality and it was notable that only five plans included six or more of the desirable features, which suggests that the teachers had very little prior experience of a lesson-planning approach that emphasised their own actions and words, rather than solely a plan of what their pupils would be expected to do. For at least two-thirds of the teachers, their plan for the landmark activity was their first to involve pupil use of a DMT and, unsurprisingly, the plans lacked a detailed 'curriculum script'[11] concerning the use of the dynamic slider to validate the algebraic models for the figural pattern. Although it was implicit in many of the plans that the sliders *would* be used, these plans did not state who would use them and how. As this was the first of the CM curriculum units to be taught by the teachers, it is not possible to infer whether this lack of detail was due to

11 'curriculum script' is one of the five 'structuring features of classroom practice' with digital technologies, as described by Ruthven and Hennessy (2002).

a lack of classroom experience of using dynamic sliders or related to their underlying knowledge of how dynamic sliders support the development of understanding of variable, or both.

An exemplification of high-quality planning for the algebraic variable research lesson in relation to each of the desirable features (taken from the complete set of lesson plans) is provided in Table 30.

Feature of lesson plan	Exemplification from teachers' plans
1. Explicit descriptions of teachers' actions/ questions	'Encourage pupils to play the pattern again and ask does it correspond to your pattern if you change the number of blocks?'
2. Explicit descriptions of pupils' actions on DMT during the lesson	'Encourage students to use slider – ask them how you can make both sliders move at the same time. What will they need to do the variables?'
3. Appreciation of pupils' instrumental knowledge	'Remind students how to use the software – recap Investigation 1, i.e. blocking and patterning. (Lock student screens)'
4. Explicit reference to variables (creating, naming, acting on)	'Ensure all pupils start to introduce a variable for their blocks ("unlock" the no of blocks column)'.
5. Explicit reference to acting on representations (dragging/moving sliders)	'[Ask] What is the purpose of the slider? What impact is it having when you slide along the bar?'
6. Explicit use of mathematical vocabulary	'[Ask] How can we check if our orange and green blocks increase in the same way?'
7. Explicit use of technological/ contextual vocabulary	'Ask students to create a table snapshot, starting from 1 block. What do students notice about the total number of lights?'
8. Includes planned plenary phases that involved teacher use of software	'Demonstrate how the Blocks and Pattern should have been made. What does the slider do?'

Table 30: Algebraic variable: Exemplification of the features of high-quality lesson plans

The weaker plans tended to lack detail, particularly in relation to the teacher's actions and questions during the landmark activity. They often stated what the pupils should do, i.e. 'pupils open software and complete question 1C', without stating how the teacher would instigate pupils' work. In other cases, the plan lists a teacher action such as 'Discuss question 1C', neither mentioning the focus or necessary vocabulary for the discussion, nor how the DMT might be used as a support.

2.2.2 Linear functions

Forty-two lesson plans that had been produced in pairs and trios by 65 teachers were analysed and the frequencies of each feature is shown in Table 31.

Feature of lesson plan	Frequency	% (n = 42 plans)
1. Explicit descriptions of teachers' actions/ questions	29	69%
2. Explicit descriptions of pupils' actions on DMT during the lesson	19	45%
3. Appreciation of pupils' instrumental knowledge	16	38%
4. Explicit reference to meaning of functions (relating to other representations)	24	57%
5. Explicit reference to acting on representations to change speed	11	26%
6. Explicit use of mathematical vocabulary	26	62%
7. Explicit use of technological/contextual vocabulary	13	31%
8. Planned plenary phases that involved teacher use of software	10	24%

Table 31: Linear functions: Summary of lesson plan analysis (42 lesson plans)

The analysis of the lesson plans revealed that just over one-third of plans included five or more of the desirable features, which in most cases excluded those that directly involved the technology (i.e. features 2, 3, 5, 7 and 8). Approximately one-half of the plans were explicit about the way in which the pupils would use the DMT (Feature 2).

A quarter of the plans indicated an intention to use the DMT during a whole-class plenary (i.e. Feature 8). In many cases this was to play the original animation to the class. Very few plans went further to include explicit discussion or demonstration of the editable 'hot spots' on the graph that controlled the animation in relation to the mathematical purpose that each served (varying the speed, start position and overall journey time). This was surprising since the initial PD session had allowed the teachers almost an hour to explore, discuss and make sense of this fundamental aspect of the DMT. It is as though their personal struggle had been forgotten as soon as they had made sense of the hot spots, such that they did not imagine that their pupils would need substantial time and support to reach the same level of fluency.

The distribution of the quality scores is shown in Figure 27.

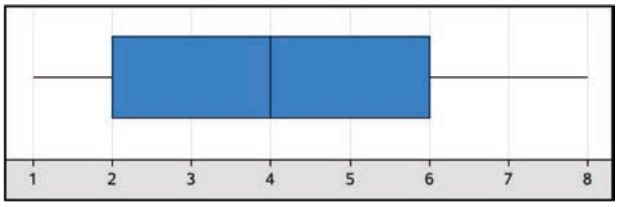


Figure 27: Linear functions: Distribution of quality scores for lesson plans (n = 42, $\overline{x}$ = 4.2, SD = 2.1)

To provide further insight, Table 32 gives an exemplification of high-quality planning for the linear functions research lesson in relation to each of the desirable features.

Feature of lesson plan	Exemplification from teachers' plans
1. Explicit descriptions of teachers' actions/ questions	'Draw their attention to the screen – is there anything different from the previous day?'
2. Explicit descriptions of pupils' actions on DMT during the lesson	'Come back together and project chosen students' screens onto the main screen and discuss answers'
3. Appreciation of pupils' instrumental knowledge (i.e. prior skills with software, progression of skills in lesson)	'[Ask] How do we make Shakey go slowly? reminding them of the use of control buttons in controlling the time taken in the simulations'
4. Explicit reference to meaning of functions (relating to other representations)	'Students highlight hops on their graph and table which is the same as the gradient in their equation'
5. Explicit reference to acting on representations to change speed	'let the students explore changing the steepness of the line and whether this means slower or faster'
6. Explicit use of mathematical vocabulary	'discuss where time, distance and speed is on the graph table and equation'
7. Explicit use of technological/contextual vocabulary	'Students hit the edit button and experiment with moving the graph (Ask how to make it steeper/ less steep etc.) what hotspots do you need to select?'
8. Includes planned plenary phases that involved teacher use of software	'Play Shakey simulation ask students: How fast is Shakey going?'

Table 32: Linear functions: Exemplification of the features of high-quality lesson plans

For some teachers involved in the Linear functions PD cycle, this was their second lesson-planning experience for a landmark activity involving a DMT. Their levels of engagement in the task, which drew on their classroom experiences from the algebraic patterns and expressions curriculum unit, had a strong influence on the new teachers to the project. Furthermore, the visibility of the previous lesson plans within the NCETM project community had a positive impact as teachers noted particularly helpful pedagogies and adapted these within their plans.

2.2.3 Geometric similarity

Twenty-one lesson plans that had been produced in pairs and trios by 41 teachers were analysed and the frequencies of each feature appearing in each plan are shown in Table 33.

Feature of lesson plan	Frequency	% (n = 21 plans)
1. Explicit descriptions of teachers' actions/ questions	20	95%
2. Explicit descriptions of pupils' actions on technology during the lesson	18	86%
3. Appreciation of pupils' instrumental genesis (i.e. prior skills with software, progression in lesson)	7	33%
4. Explicit reference to meaning of similarity (relating to representations)	15	71%
5. Explicit reference to acting on representations to vary lengths and angles	13	62%
6. Explicit use of mathematical vocabulary	18	86%
7. Explicit use of technological/contextual vocabulary	15	71%
8. Includes planned plenary phases that involve teacher use of software	9	43%

Table 33: Geometric similarity: Summary of lesson plan analysis (21 lesson plans)

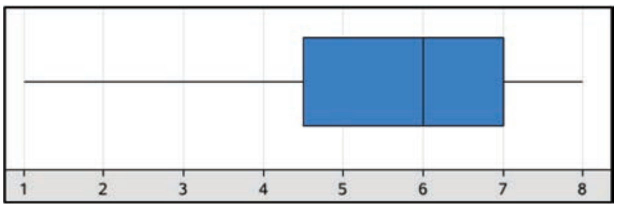


Figure 28: Geometric similarity: Distribution of quality scores for lesson plans (n = 21, $\overline{x} = 5.5$, SD = 1.8)

Two-thirds of the lesson plans for the landmark activity scored more than 5 marks. Strong features across all of the plans were: the inclusion of explicit teachers' actions and questions; and high levels of attention to the inherent mathematical focus for the lesson and its associated vocabulary. The inclusion of planned plenary phases that involved teacher use of the software was only apparent in just under one-half of the plans. Attention to the need to support pupils' instrumental genesis was only apparent in a third of the lesson plans.

The lesson plans tended not to articulate in great detail how the dynamic features of the DMT would be used during the lesson, in particular the angle sliders and ratio checker. However, the lesson plans did include much greater attention to the use of mathematical, technological and contextual vocabulary than in the previous two topics. The teachers seemed to have grasped the need to involve the pupils in both the definition and use of important terms such as 'corresponding', 'congruent' and 'similar'.

An exemplification of high-quality planning for the geometric similarity research lesson in relation to each of the desirable features (taken from the complete set of lesson plans) is provided in Table 34.

Feature of lesson plan	Exemplification from teachers' plans
1. Explicit descriptions of teachers' actions/ questions	'Get pupils to discuss in groups, why copy 7 is not mathematically similar. After a minute or two of discussing, I will ask pupils to write down their thoughts.'
2. Explicit descriptions of pupils' actions on DMT during the lesson	'Encourage pupils to rotate the shape to make it easier to compare.'
3. Appreciation of pupils' instrumental knowledge (i.e. prior skills with software, progression of skills in lesson)	'remind the class of how to use the measure tools and state that they may want to use these tools today.'
4. Explicit reference to similarity (relating to representations)	'[Ask] What can we say about the relationship between corresponding angles in mathematically similar shapes?'
5. Explicit reference to acting on representations to vary lengths and angles	'Get students to answer 1B and have a go at using the sliders. Identify what each slider does.'
6. Explicit use of mathematical vocabulary	'[Ask] is the relation between the lengths of sides sufficient factor for two shapes to be mathematically similar?'

Feature of lesson plan	Exemplification from teachers' plans
7. Explicit use of technological/contextual vocabulary	'[Ask] What did you notice when you moved the scale factor slider?'
8. Includes planned plenary phases that involved teacher use of software	'On the IWB direct the pupils to look at the ratio checker and as a class use it to verify that "copy 1" is mathematically similar to the original.'

Table 34: Geometric similarity: Exemplification of the features of high-quality lesson plans

Furthermore, the teachers' responses to the PD tasks in which they had used the DMT and engaged with typical pupil work also provided an insight into how they envisaged the use of the dynamic sliders and the ratio checker to highlight important geometric invariants of similar polygons. Most teachers commented that they would use this dynamic functionality in some way, for example 'to explore about what should change or stay similar' or 'to show if 2 polygons are similar', but far fewer were specific about how. One teacher did specify and stated 'To *demonstrate* ratio of length:width of sides are equal for two similar shapes', implying teacher use of the DMT in a whole-class setting.

2.2.4 Development of teachers' lesson plans over time

One pair of teachers produced high-quality lesson plans for all three of the PD cycles, including seven or eight features. The pair comprised an experienced second in department (with no prior experience of using a DMT with key stage 3 pupils) working with a newly qualified teacher (her mentee) who had been introduced to DMT within her PGCE course. It was clear that the pair were used to working together on lesson-planning tasks to a high level of detail with respect to the teachers' actions within a lesson alongside thoughtful hypotheses of the resulting pupils' actions/interpretations (and the possible follow-up teacher responses).

In the earlier PD cycles, the majority of planning pairs/trios approached the task pragmatically and made many assumptions that the pupils would learn through their experiences with the DMT with little or no intervention or support from the teacher. The weakest plans made no reference to any actions on the DMT nor did they make reference to the mathematical concept at stake or its related vocabulary. As the project progressed, the experienced teachers demonstrated more thoughtful planning and this influenced the newer teachers as they both took the task more seriously and picked up on useful pedagogical approaches that involved the DMT more explicitly. The role and nature of vocabulary by teachers and pupils became a key element of planning as teachers realised the need to support pupils to move between the technological/contextual vocabulary of the curriculum unit and the mathematical language of the curriculum.

The frequency of the eight lesson plan quality features for each curriculum topic is provided in Table 35, which highlights aspects of the teachers' development of MPT.

	Teachers' actions and questions	Pupils' actions on DMT	Supports for pupils' instrumental genesis	Focus on mathematical concept	Actions on representations to explore mathematical concepts	Uses maths vocabulary	Uses technical and/or contextual vocabulary	Planned plenary involving DMT
Algebraic variable (n = 28)	57%	43%	32%	39%	61%	64%	64%	18%
Linear functions (n = 42)	69%	38%	45%	57%	26%	62%	31%	24%
Geometric similarity (n = 21)	95%	33%	86%	71%	62%	86%	71%	43%

Table 35: Comparison of lesson plan quality features by topic

The quality of the teachers' lesson plans improved across the three curriculum units with respect to both the number of quality features and the increased frequency of particular quality features (see Section 2.2). The mean average 'quality scores' increased over time, from $\overline{x}$ = 3.9 (Algebraic variable) to $\overline{x}$ = 4.2 (Linear functions) and $\overline{x}$ = 5.5 (Geometric similarity). In addition, the frequency of important elements of the plan also increased in relation to teachers' planned uses of the DMT, from 18% (Algebraic variable) to 24% (Linear functions) and 43% (Geometric similarity). As 95% of the teachers who were involved in the Geometric similarity PD cycle had previously taught one or two other curriculum units, this does strongly suggest that time and prior experience are important factors when supporting teachers to develop more comprehensive and thoughtful lesson plans for the use of DMT in their teaching.

The nature of the individual landmark activities did provoke a need for teachers to plan in ways that might privilege particular features, for instance the geometric similarity landmark activity required increasingly more detailed definitions, which would, by necessity, privilege the use of mathematical language. However, key features such as the inclusion of explicit teacher actions and questions, and the focus on the mathematical concept showed notable increases in their frequency of inclusion.

Summary of findings: Teachers' lesson planning

- Over time, the teachers' lesson plans became increasingly more detailed with respect to planning what they were going to do (especially with the DMT) and say during the landmark activity.
- The plans evidenced how teachers became more mindful of the need to provide support for pupils to make sense of the DMT such that they could use it in mathematically productive ways beyond their initial experiences.
- Over time, the teachers' plans became more explicit about the inclusion of planned whole-class plenaries to focus on the mathematics at stake, with more teachers incorporating how they would use the DMT to support this.

2.3 Teachers' MPTK made visible through lesson observations/interviews for the landmark activities

Twenty-four teachers were observed and interviewed as they taught the landmark activity from one of the curriculum units over the course of the three PD cycles. This represented 11% of the total number of teachers involved in the project (n = 209) and 21% of the teachers who gave their ethical consent to the project (n = 111). The observation focus was to gain an insight into their MPTK, for example through:

- teachers' actions and questions that referred to the DMT, for example gesturing to the software on their own or pupils' screens (as opposed to actual uses);
- teachers actions on the DMT, and their accompanying 'curriculum script' (Ruthven and Hennessy, 2002);
- teachers' prompts and instructions that resulted in pupils' actions on the DMT;
- teachers' prompts and instructions to support pupils in their instrumental genesis of the DMT, as appropriate to the activities; and
- teachers' references to the mathematical concept at stake (variables, functions, geometric objects).

Central to all of the above teaching activity was the vocabulary that concerned the mathematics, technology and context inherent in each landmark activity.

The criteria for the selection of the teachers for observation was based on:

- range of school types (by GCSE mathematics results (%A*–C) and % free school meals, 2014 validated data);
- range of experience with DMT;
- range of confidence with DMT;
- willingness to develop practice with DMT; and
- the mutual availability of individual teachers and researchers.

The complete methodology for the lesson observations and analysis along with the anonymised list of teachers observed is included in the project Technical Report (Clark-Wilson and Hoyles, 2017).

2.3.1 Algebraic variable: Lesson observations

Seven lesson were observed in June–July 2015. The timing of this particular PD cycle, to coincide with the post-GCSE exam period in schools, was convenient in terms of teacher release. However, it proved problematic for some teachers to organise access to the technology and complete the teaching of the research lesson prior to the end of term. In addition, teachers experienced changes to classes/timetables due to trips and cross-curricular activities, which made a higher number of observations problematic to arrange.

Effective features from the observed lessons included:

- Concise introductions to the activity to maximise the time and opportunity for pupils to engage with the DMT. These included short recaps of the previous activity that involved teacher or pupil use of the DMT to remind pupils of essential steps. For example, a short card-sort activity that was then 'tested out', or a pupil demonstration to highlight aspects such as 'creating a block' and defining a figural pattern.
- The use of computer management software to support short plenary session to share pupils' screens with the whole class and use them to provoke discussion or for the teacher to demonstrate particular instrumentation skills such as 'blocking and patterning'.
- A high focus on the full range of vocabulary needed to connect the mathematical, technological and contextual aspects of the landmark activity. In some cases, this vocabulary emanated from the pupils: 'you need to *block it* Miss!'
- Tight classroom management to ensure that pupils discussed their ideas, recorded their predictions and reflected on these in the light of their activity with the DMT.
- Most crucially, at the key point in the landmark activity when pupils are required to struggle to work out how to link their two variables, the more effective practice prompted, recognised and then resolved the struggle in ways that enabled pupils to have the 'aha' or 'light bulb' moment – and then resolve the task to a successful outcome.

2.3.2 Linear functions: Lesson observations

Fourteen lessons were observed in November 2015–March 2016. In contrast to the previous CM topic, teachers seemed more enthusiastic about teaching the linear functions DMT and were able to organise its teaching in a timely manner. In addition, many schools' key stage 3 schemes of work included algebra topics in the autumn term, another factor that facilitated the teachers to teach this particular CM unit.

Effective features from the observed lessons included:

- High emphasis placed on the need for pupils to make sense of how the graph is edited. This was often accompanied by agreeing some vocabulary to support pupils to be able to describe to each other what they were doing.

- Attention to the graphs' axes, and careful discussion of the effect of changing the scale.
- Emphasising the multiplicative and additive relationships in the table as a means to justify the meaning of the equation, in some cases highlighting the invisible variant m within the table for equations with a non-zero value of c.
- Strategies to gather back the pupils' multiple responses to the landmark activity as particular cases in order to support the overarching generalisation that the greater the value of m, the faster that Shakey will move, supported by shared display and use of the DMT.
- Extending Shakey's journey time such that his final position could not be read from the graph nor the table, to provoke pupils to use the equation to calculate his position after a given time and thus highlighting the power of the mathematical equation as a generalisation.

2.3.3 Geometric similarity: Lesson observations

Five lessons were observed in the period February–March 2016 and two further observations took place in July 2016.

Effective features from the observed lessons included:

- Consistent focus on the mathematical purpose of the activity, with an emphasis on the side and angle properties of shapes that remain mathematically similar.
- Use of the ratio checker as a local check for the side properties of (potentially) mathematically similar shapes.
- The involvement of a pupil to carry out technological tasks on behalf of the teacher,[12] i.e. to use the DMT to create particular screens for discussion with the whole class, thus alleviating the teacher of this responsibility and enabling the teacher to focus on their mathematical discussion with the pupils.
- The use of counter-examples, constructed using the DMT to challenge pupils' developing ideas.

2.4 Lesson observations of landmark activities

The major finding of the project that most teachers were initially very reluctant to use the DMT 'live' during whole-class teaching, which was probed during interviews and post-lesson discussions at the second PD session, is possibly explained by two factors:

First, the teachers' own instrumental geneses had predominantly involved their personal or classroom computer (albeit a tablet/laptop or fixed machine in their classroom) and very few teachers had practised using the DMT on the particular IT setup in the classroom in which the lesson took place.

12 This idea, referred to as a 'sherpa' pupil (i.e. carrying the load for the teacher) was first conceived by Guin and Trouche in an early study on the use of DMT in classrooms (1999).

Second, most teachers preferred to stand at the front of the classroom beside the computer display. Given their general unfamiliarity with using the DMT on their interactive whiteboard, several then moved to the computer connected to the whiteboard to operate the software from there. In many cases, these computers were poorly positioned and teachers found themselves in an unfamiliar place to lead a whole class discussion involving the DMT. A few teachers overcame this by involving pupils as demonstrators of the DMT, a role that Guin and Trouche named a 'sherpa at work' (2002).

Our findings show that effective pedagogies involving the CM DMT require teachers and/or pupils to interact with the software by dragging sliders and hotspots and moving fluently between representations to highlight the variant/invariant properties of the important mathematical concept at stake. However, it was not initially natural for teachers to model or support these types of interactions. In some observations, the pupils were more inquisitive and it was their actions that prompted the teachers to rethink, providing clear evidence of the teachers' growth of MPTK.

A fundamental aspect of all mathematical lessons is the vocabulary and resulting discourse that is a key mediator of the pupils' mathematics learning. Lessons that involve pupil use of DMT are no different in this respect. However, new vocabulary emerges that is a blending of the mathematical terms legitimised by the curriculum (and its assessment) alongside the language of the DMT and its associated context (for instance within the linear functions unit, the narrative of game design and 'Shakey the robot'). The lesson observations revealed how, where teachers were mindful to this full range of vocabulary by using it themselves as they interacted with the software, the pupils were more able to articulate their emerging mathematical understandings both orally and in their recorded work. In some cases, the teachers took their cue from language that the pupils developed (e.g. in algebraic variable, 'you need to block it'). In other cases, the teacher 'gave' or reminded pupils of the words that might be useful to them. In the more effective practice, the teachers enacted their plans to highlight the mathematical language that was fundamental to the landmark activity. For example, within the linear functions lesson, introducing the word 'coefficient' was crucial for pupils to be able to express the more common, colloquial description 'the number in front of the x' within the linear function modelled by $y = mx+c$. The teacher's articulation of this term was vital to enable pupils to give clear descriptions of how the value of the coefficient affected Shakey's speed and the related mathematical representations.

Summary of findings: Lesson observations of landmark activities

- The majority of teachers were initially reluctant to use the DMT 'live' during lessons, particularly during whole-class teaching. It is clear that this practice takes time to develop. When it was used effectively, the focus was on its use to: show counter-examples, extend the mathematical ideas; promote the use of appropriate vocabulary; and to support or refute the pupils' mathematical predictions.
- Where teachers had taken the time to rehearse their own 'curriculum script', they were noticeably more confident during whole-class use of the DMT and in scaffolding the pupils' work with the DMT. In such cases, the emphasis was on encouraging pupils to see the mathematical connections between the different representations and to describe the impact of the dynamic elements on these representations.
- In more effective lessons, the mathematical focus for the activities was maintained throughout. For example, in the *linear functions* landmark activity, the mathematical focus remained on the value of the coefficient of x, and the impact of using the DMT to change its value on the linked mathematical representations. In these lessons, it was apparent that both the teacher and pupils had sufficient prior knowledge and experience of the DMT to enable this to happen. Furthermore, teachers sustained the predict–check–explain pedagogical approach and encouraged the use of correct vocabulary to support pupils' oral and written explanations.
- Some teachers developed practices whereby the pupils took on key roles in the use of the DMT to share with others, for example by demonstrating or talking through their own strategies, which in turn, widened the teachers' own knowledge and understanding of its use.

2.5 General comments on lesson organisation and management

There were some distinct advantages to the class and teacher remaining in a familiar classroom as the normal classroom behaviours and routines were not disrupted and the teachers were more familiar with their classroom display technology. In most cases, laptop and/or tablet computers were brought to the classroom and distributed quickly with little disruption.

A number of teachers developed pedagogies with the generic technologies available to them that greatly enhanced both their confidence and the quality of their whole-class plenaries using the DMT. In particular, the use of computer classroom management software such as Impero enabled the pupil's screen to be controlled in ways that highlighted aspects of the pupils' instrumental geneses and supported rich mathematical discourse (Appendix C, Case study 2).

2.6 Case studies of teachers

Our findings have outlined the components of teachers' MPTK that are desirable for the integration of DMT in the teaching of the three topic areas. We also report the detailed case studies of teachers to provide more detailed insights into trajectories of development of MPTK during the project.

The three cases have been selected to highlight:

- A confident and experienced teacher who has little prior experience of using dynamic technology in the classroom (Case Study 1: Sasha).
- An inexperienced teacher who has little prior experience of using dynamic technology in the classroom (Case Study 2: Phoebe).
- A confident and experienced teacher who is also a confident user of technology, but is new to DMT (Case Study 3: Chris).

The case studies, which have been anonymised through the use of pseudonyms, draw on data from the pre- and post-surveys, responses to PD tasks during face-to-face sessions, lesson plans and classroom observations of the 'landmark activity' for each curriculum unit and teachers' post-lesson reflections and feedback (see Appendix C).

2.7 Factors affecting teachers' engagement

We draw on data from the teacher surveys, observations of the PD sessions (and related teacher feedback), and the case study teachers' lesson observations and interviews in order to report findings on the main factors that seem to have influenced teachers' engagement in the project's activities.

2.7.1 The focus on mathematics

Initially many teachers seemed reluctant or lacked confidence to discuss the mathematical topics in great depth. It was necessary to have strategies to maintain this focus, which included: expecting teachers to work individually on mathematics tasks prior to opening the discussion; encouraging the sharing of definitions, strategies and solutions; and modelling an open classroom culture whereby teachers' views and opinions could be shared, while striving for mathematical rigour, if appropriate. Some teachers commented that they did not commonly discuss mathematical concepts (and related teaching) to this depth within their departments. Teachers' evaluations of the PD sessions regularly commented on how they found these discussions very valuable and some mentioned how their general approaches to teaching the topics (that is, without the use of a DMT) had also been rethought as a result.

2.7.2 Alignment with personal goals and ambitions

Teachers were motivated differently to become involved in the project. Some teachers had learnt about the project and gained their school agreement to participate as it met with their personal interest in technology. Others were nominated by their school because they held school responsibilities for the key stage 3 curriculum or in overseeing how the department integrated technology into teaching. A small minority were nominated by their school and arrived to the first meeting with little or no understanding of why they were there.

Furthermore, we observed some teachers who, having valued their experiences and outcomes in the project, grasped the opportunity to develop their leadership skills by committing to disseminate the project more widely both within their department and beyond their school. By leading a short initiative in their own department, they were able to build confidence and experience that in a few cases contributed to their promotion within the profession.

2.7.3 School roles and responsibilities

As previously mentioned, the teachers held a range of different roles in their schools. Where senior leaders had assigned teachers a clear leadership role with respect to the school's engagement in the project, or the teacher held a curriculum responsibility (as a key stage 3 coordinator), this often resulted in wider departmental dissemination activities and clearer plans for future sustainability of new teaching approaches with DMT. Although there were a few individual teachers who exhibited great personal commitment and involvement, a lack of school-level understanding of the project's overarching goals resulted in lower levels of impact at the department level.

The subject leaders' direct participation in the project was often helpful, but not necessarily sufficient to develop the department's capacity to sustain their involvement. In some cases, this was related to a lack of personal alignment with the project's goals, but was also related to the more general demands on departmental time during a period of intensive curriculum and assessment reform.

2.7.4 Departmental culture

We learnt through our observations and interviews that the teachers were working in mathematics departments that had very different cultures. For some teachers (for example, Sasha – see Appendix C, Case study 1), the teachers regularly planned collaboratively in small groups, creating teaching resources that were subsequently shared with other colleagues in the department. In this case, the teacher participating in the PD cycles arrived knowing that they would be intended to share what they produced, which provided additional focus and incentive when they were planning their lessons.

For other teachers, and particularly those who had participated in the project as a sole representative of their department, although they might be individually keen and self-motivated, their department culture was not conducive for them to share their experiences and practices with colleagues. This more fragmented approach is likely to hinder the longer-term development and sustainability of more widespread classroom practices with DMT in the school. In these case, it may need a second teacher to undergo a similar PD opportunity to bolster the knowledge and experience with DMT in the department; and to revitalise the initial teacher's interest.[13]

Summary of findings: Teacher engagement

- Teachers overcame their initial apprehension about discussing the mathematical concepts in depth and reported high levels of motivation and value in such professional discussions with colleagues, particularly when stimulated and supported by the PD resources embedding DMT.
- The alignment of the project goals to individual, departmental and school-level goals was a crucial factor with respect to both individual teachers' engagement and the potential sustainability of their use of the DMT.
- The participating teachers' roles and responsibilities were varied. An ideal pairing seemed to be the combination of a less-experienced teacher alongside a more-experienced teacher (in terms of teaching experience, not necessarily the use of DMT) with some departmental responsibility with respect to the project goals.

3 The professional development toolkit (PD Toolkit)

The CM PD Toolkit is accessible from the UCL-hosted website at http://ucl.ac.uk/cornerstone-maths. The website pages are maintained by the Institute of Education website team under the full editorial control of the UCL Knowledge Lab research team.

3.1 Design features of the PD Toolkit

The challenges associated with learning to engage with mathematics through the medium of a DMT (however well designed) lead us to conclude that some initial face-to-face professional development is essential to support early activities such as, discussion of the mathematics, hands-on experiences with the DMT, and preparation for classroom-based uses. Appendix D provides a more detailed set of desirable early PD activities for teachers in this respect.

We adopted a design-based methodology (Cobb *et al.*, 2003) to develop the PD Toolkit. Our first set of design criteria was based on past research that concluded a set of processes for optimal, high-fidelity scaling and sustainability of the innovation:

- School-devised methods to evaluate pupils' outcomes.
- Development of school-based PD.
- Support to embed the DMT within local schemes of work.

13 Our long-term data on schools' involvement with CM has indicated that this is quite common, particularly given the high levels of mathematics teachers' movement in and out of in London secondary schools.

- Development of a lead practitioner (who may be the subject leader).
- Development of peer-support for participating teachers.

(Clark-Wilson, Hoyles, Noss, Vahey and Roschelle, 2015)

These processes are interdependent; for example, the development of school-based PD is often the (formal or informal) responsibility of a lead practitioner or 'champion' who, by definition, then becomes a 'PD lead' for a group of colleagues.

We drew from pre-survey data (reported earlier in Table 9) that provided insights into the teachers' typical professional development needs alongside data from eight 'CM champions' who had been highly engaged within the project and had indicated that they wanted to spread the innovation in their school. During a one-day meeting with the CM champions they completed a pre-survey in which they indicated the different types of resource that they would expect to find within the toolkit. Six of the eight champions mentioned additional teaching resources to support bridging or transfer to the more traditional exam-style questions and other resources related to assessing mathematical learning. Other resources that were mentioned included: audio/video of other teachers talking about their experiences (n = 2); summary statistics of prior research findings (n = 1); and an 'overview of CM' video (n = 1). It was a surprise to us that, even though this group of teachers had experienced a total of 8 hours of PD activity related to each curriculum unit, none of the teachers' initial comments mentioned any PD activities that had addressed the fundamental mathematical ideas within the unit or introduced them to the software for the first time. We hypothesise that the nature of such activities is 'alien' to the norms of within-school professional activity. This is supported by the CM champions' responses when asked to describe the sort of PD activities they had already initiated in their schools concerning CM, which tended to be demonstrating the CM software to colleagues and sharing the outcomes of CM lessons with them.

The components and content of the resulting PD Toolkit are shown in Table 36.

Each component is then described in more detail.

Component	Content
Overview	› Video introduction to the CM Project.
Getting started	› Video overview of CM and its design principles. › Overview of the PD toolkit and suggestions for how it might be used. › Technical information. › Project history and funders.
Curriculum units *This component is replicated for all three curriculum topics: Algebraic variables; Linear functions; and Geometric similarity.*	PD resources › Video overview of the curriculum unit. › Pupil workbook and teacher guide. › Structured PD session that includes: ◦ MPTK items. ◦ Instrumentation tasks. ◦ MPTK tasks to analyse pupils' digital productions. ◦ Introduction to the landmark activity. *Leading to a …* ◦ Lesson planning task of the landmark activity. › Resources to support planning and reflection on the landmark activity: ◦ Video of the landmark activity. ◦ Video clips of 'typical' classroom enactments. ◦ Examples of typical pupil responses. › Suggestions for assessing pupils' mathematical understanding.
Departmental case studies (Embedding DMT)	› Outline of their approach for scaling/ sustainability. › Examples of within-school PD activities.
Evidence base for different audiences	› Research summary for senior leaders and headteachers. › More detailed research summary for heads of department (including references to departmental case studies). › Links to published research.
Project community	› Link to the online project community, where teachers can share resources they create and discuss their implementations.

Table 36: Design components and content of the PD Toolkit for London schools

3.1.1 Getting started

The core audience for the PD Toolkit is existing CM project schools that have some knowledge and classroom expertise with at least one curriculum unit. However, as teachers introduce the resource to others, our findings indicate that there is a need to support the communication of the overarching vision of the innovation along with insight into the curriculum and its core pedagogy; and to highlight the resources in the PD Toolkit that would support the realisation of this vision. We believe these to be key elements that are necessary, but not sufficient, for implementation fidelity. The CM champions were clear that the limited time that they would have to work with colleagues meant that much of this would be better communicated through short videos than by reading web pages and documents.

3.1.2 Curriculum units

This area provides the gateway to the DMT and pupil resources, while also offering a bank of resources to support within-school PD activities. The CM Champions requested short video introductions or introductory PowerPoint slides for each curriculum unit, along with overviews of the activities and mathematical goals. This highlighted that the CM Champions possibly saw themselves in the role of project disseminators as conveyers of professional knowledge, which aligns with predominant cultures of 'within-school' knowledge exchange.

However, during our focus group session with the CM Champions, we distributed the original MPTK items and PD tasks for each unit and invited the group to try to return to a position of 'no knowledge' of the curriculum unit, the related software and pupil activities so as to better understand new teachers' perspectives in relation to the innovation. This shifted the CM Champions' mind-sets as evidenced by follow-up requests for information about our rationale for the design of the face-to-face PD and for the inclusion of the MPTK items and PD tasks.

In each curriculum unit, we highlight the concept of the landmark activity, and provide resources to support teachers to plan, teach and reflect upon this activity using the adapted lesson study approach described previously. Some short (< 3min) video clips of classroom implementations of the landmark activity are also offered that seek to capture the range of implementations and illustrate some of the more common mutations. Each clip is accompanied by focusing questions to stimulate deeper analysis and focused viewing. Teachers are also prompted to review other teachers' existing lesson plans for the landmark activity, which were available in the project's online community.

Finally, as all teachers are required to assess and report pupil attainment data to the school management on a regular basis, we direct teachers to key questions within the pupil material that might support assessment of mathematical understanding. We also invite teachers to share their school-devised assessment, which often take the form of 'end of unit' tests within the project online community.

3.1.3 Departmental case studies: Embedding DMT

Discussions with the CM champions, school leaders and others who were keen to support within-school scaling highlighted the importance of between-school knowledge exchange concerning the pragmatic but also the epistemic aspects of 'embedding' the innovation. It is our view that if the DMT has truly scaled to a school, there would be strong evidence that it is embedded in the localised scheme of learning within that school, and a critical mass of the teachers of lower secondary maths will have experienced some form of school-initiated PD to support them to teach the CM unit(s). Alongside this, and to avoid lethal mutations in implementation, the need for individual teachers to reflect on their own approaches and how far they align with the original design principles of the innovation: that is, they themselves recognise the mutations they have instigated. Furthermore, the departmental discourse had focused on the mathematics of the curriculum unit; supporting the colleagues to understand and more deeply articulate the concepts, representations and dynamic links.

The case studies are structured to include the important pragmatic aspects of a departments' CM implementation such as:

- Which CM curriculum unit did the department focus on first?
- Which (new) teachers were involved and how?
- How was the CM unit aligned to the mathematics scheme of work?
- How did the department organise access to the technology?
- What types of technology were used?
- How did the department assess pupils' learning to satisfy school-imposed data collection?

However, as we identified in our departmental case studies, for those who claimed to have begun to embed one or more CM units it became apparent that the relevant CM Champions had shown some appreciation of at least some of the epistemic aspects of the CM innovation, as evidenced by:

- The within-school PD including some discussion of the development of mathematics in relation to the specific CM unit.
- Strategies to assess pupils' learning that aligned to the CM vision and goals.
- Support for the ongoing development of teachers' mathematical pedagogic practices with technology, aligned to the CM vision and goals.
- A recognition of the time that it takes for participating teachers to make sense of the epistemic affordances of CM: our data suggests 2–3 years.

Hence, in structuring the format of the Case Studies within the PD toolkit, it was necessary to highlight both pragmatic and epistemic aspects, communicated in a range of forms and presenting an honest picture of the complexities of the process of within-school scaling of CM. Furthermore, in our future work, we see it as our role to enhance these Case Studies by annotating them with concise links between the individual school's case, the more general research findings of the CM project as a whole, and wider related research.

3.1.4 Evidence base for different audiences

Project teachers reported in our surveys that they wanted concise quantitative data (i.e. one-page 'infographics'), 'to hear from other teachers', and 'videos of real classrooms', whereas heads of departments wanted knowledge that 'there are research papers (even if we don't read them!)'. Only 3 per cent of teachers requested access to the full-text published research and all of these had achieved or were studying for a master's degree in mathematics education.

In the light of this pressure, we present our research evidence in the toolkit in a range of forms, essentially re-versioning the research findings for different audiences and purposes. Hence we include:

- A one-page infographic about Cornerstone Maths.
- 'Vox-pops', videos of CM teachers, school leaders and mathematics consultants/advisers giving a concise (< 1 minute) and immediate personal reaction to the CM innovation.
- Longer videos (≈ 3 minutes) of CM project teachers giving a personal evaluation of impacts of CM.
- Published research reports, papers and conference presentations.
- References to CM being mentioned in the public media: Education Minister's speeches, newspaper articles and other media coverage of the project.

3.1.5 Project community

Early in the development of the toolkit, as we spoke with our CM Champions and project teachers, it became obvious that there was a demand for an area of the toolkit that supported between-school knowledge exchange and community support. This is accomplished using an online project community, mediated within the UK government-funded National Centre for Excellence in Teaching Mathematics portal, where teachers can:

- Participate in forum discussions.
- Access digital resources from the documents area.
- Contribute digital resources (including lesson plans and resources) by uploading them to the documents area.

Finally, the online community provides an opportunity for teachers to stay abreast of future developments of CM in addition to providing a window on its implementation fidelity as the CM innovation scales more widely.

3.2 Evaluation of the PD Toolkit within school-based settings in London

We conducted a small-scale evaluation of the PD toolkit within three self-nominating CM schools that had taken part in the project. The PD toolkit was evaluated with respect to:

› If and how it met the CM Champions' expectations as they selected resources to design school-based PD for colleagues in their own schools.
› The pathways that the CM Champions chose through the PD Toolkit resources.
› The CM Champions' perceptions of the usefulness of the resources and their ease of access.

In all cases the CM Champions reported that they had found it very difficult to organise time in school to lead a PD session for their colleagues in the period January–March 2017. This was directly related to school pressures to prepare pupils for the revised GCSE in mathematics, which was due for first examination in June 2017. Only one school was able to convene a departmental PD session in this period. The remaining two school reported that they planned to convene PD sessions in the second half of the summer term, after the school exam period. For this, however, three interviews were arranged and the outcomes of the evaluation are given in Appendix E.

Summary of findings: PD Toolkit

› Professional development to support teachers to implement DMT in their classrooms needs to blend face-to-face sessions that involve first-hand experiences with the DMT alongside PD tasks that deepen teachers' understandings of the mathematics at stake and promote lesson planning for common activities. The adapted lesson study design, which offers a cyclical PD approach over a period of 6–8 weeks, was a model reported to be successful by the participating teachers.
› The PD Toolkit resources extended beyond only resources for PD sessions for other teachers of key stage 3. They included resources to support the wider communication of research findings that underpinned the use of DMT and case studies of successful departmental implementations of DMT in different schools.
› Early evaluations of the use of the PD Toolkit resources by London schools suggest that there is sufficient content to enable further scaling within the CM project schools and to support the associated sustainability of new practices with DMT.

4 Discussion

The project findings highlight how the process of integrating dynamic technology within secondary mathematics teaching presents considerable potential for learning but also a significant challenge for teachers as they rethink the underlying mathematical concepts, undergo their own instrumental genesis, develop their curriculum script and learn to support pupils' technological experiences.

Geometric similarity proved to be the richest topic for the development of teachers' mathematical knowledge for teaching, in part because, we hypothesise, most teachers' knowledge of this topic is derived from teaching – that is, the rules and conventions associated with particular domains or aspects of the curriculum, such as that similarity is only to do with triangles. Hence, many teachers are unaware of the important connections within the school mathematics curriculum that might enable pupils to have a more fluent understanding. Most notable developments in subject knowledge concerned more robust definitions of geometric similarity for a broader range of polygons and the appreciation of the invariant ratio property for pairs of corresponding sides within similar polygons.

The linear functions curriculum unit proved to be the richest of the three CM topics with respect to the development of the teachers' instrumental geneses. This was most probably due to the more familiar set of representations (graph, table, equation) and the easy initial access to a dynamic scenario by playing the animation. Of the three topics, it was the one that most teachers went on to teach, and several teachers reported that they had subsequently disseminated this particular unit to colleagues in their schools.

The lesson observations confirmed teachers' general reluctance or lack of confidence to use the DMT 'live' during the lesson to support the pupils to fully understand the important mathematical knowledge at stake. Whilst most of the observed lessons included a teacher-led plenary that focused on the mathematical concept at stake, few teachers used (or invite pupils to use) the DMT to demonstrate, model or stimulate pupils' mathematical thinking. Post-lesson discussions highlighted how unprepared teachers admitted they felt when it came to having an appropriate 'script' to accompany their actions involving the DMT. Most teachers encountered some surprises with respect to pupils' instrumental genesis of the DMT, which in some cases led to unplanned plenaries to highlight particular issues, and drawing on the teachers' contingent knowledge.

Easy access to the technology is necessary but not sufficient and ongoing, sustained support over time is needed to address issues of mathematical knowledge and pedagogy by:

- Exploiting the dynamic features of the software in
  - whole class teaching to highlight key concepts; and
  - interactions with pupils to probe and extend their emergent understandings.
- Developing a mathematical language that supports the classroom discourse in relation to the new dynamic mathematical objects within the technology (sliders, 'hotspots', dynamic images) and being able to engage in descriptive and analytical conversations stimulated by these objects.
- Exhibiting the perseverance that is required to go beyond 'first lessons', and seek out time and support to move past the challenges above to develop more confident classroom practices.

The study did not reveal large effect sizes with respect to the development of teachers' MKT as measured by the designed items that were administered pre- and post-teaching of the landmark activities for the three curriculum topics (see Sections 2.3.1, 2.3.2 and 2.3.3). It seems that teachers' mathematical knowledge might be more resistant to development or that our intervention was not sufficient in time or content to achieve more significant effects. However, while there were a few notable individual gains in measured MKT knowledge relating to geometric similarity, the project had more impact on the development of teachers' MPTK.

It is important for teachers to pay close attention to their actions on the DMT, particularly when first engaging with the dynamic and multi-representational features. For example, which variables or representations can (or cannot) be changed and why (or why not). These aspects may be constrained by:

- the mathematics itself (i.e. the independent and dependent variables);
- technical constraints related to the design of the DMT; or
- mathematical or pedagogical choices influencing the design of the DMT (and related tasks).

Hence it is important that teachers are curious about the DMT and take time to play with it in purposeful ways to support their instrumental genesis while simultaneously remembering their own learning journey, such that they remain aware that their pupils will also need similar opportunities and time.

We now highlight how teachers' participation in the CM PD cycles has impacted on the components of their MPTK. Our findings in this respect have led us to amend the original diagram (Figure 6) to indicate a bi-directional connection between 'mathematical knowledge for teaching' and 'technology instrumental genesis' (shown in green below) to capture the process through which a teacher's existing MKT impacts on the knowledge that they bring to their early technology experiences.

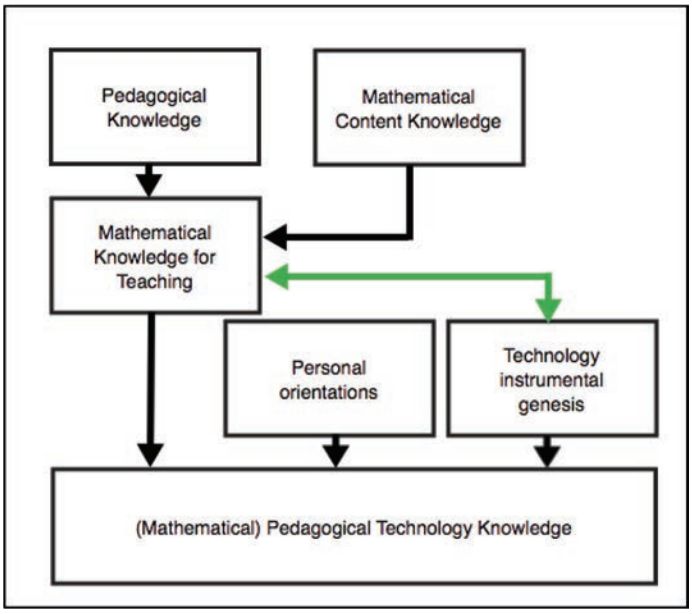


Figure 29: Revised components of (Mathematical) Pedagogical Technology Knowledge (Thomas and Hong, 2013)

4.1 Pedagogical knowledge

Underpinning CM was a pedagogic approach to the use of DMT 'predict–check–explain', whereby the DMT is used by pupils to test their mathematical conjectures and provide a tangible experience through which they can reflect upon and make sense of the outcomes. In such scenarios, from the pupils' perspective the technology acts as another potential source of mathematical knowledge in the classroom in addition to the teacher. Consequently, teachers are required to reconsider their role and develop new approaches to support their pupils to makes sense of their experiences with the DMT.

Our findings suggest that the teachers were paying increased attention to the 'predict–check–explain' pedagogy and the importance of monitoring that pupils adhere to this, particularly when discussing and recording their predictions and explanations. In addition, some teachers developed new general pedagogies for supporting and managing pupils' technological work. For example, by exploiting generic computer screen management software to support pupil learning in various ways, including the monitoring of whole-class work, selecting and focusing on particular pupils' screens, demonstrating use of the DMT, and establishing common classroom language.

Finally, familiar 'successful' pedagogical approaches were further developed to support pupils' work involving the DMT. For example, the use of card sort tasks (see Appendix C, Case study 2: Phoebe), the focus on key vocabulary, and the sharing of lesson objectives.

The following final reflection comments from teachers indicate their awareness of their developing pedagogical knowledge:

> [I am now] Using dynamic software to check their understanding on what they have learnt. Showing them different examples using dynamic software so that the pupils can explain what is happening.
>
> I am thinking more about deeper, probing questioning. The type of questions you ask, how you respond to pupils' answers/follow-up questions and who to target these to.

4.2 Mathematical Knowledge and Mathematical Knowledge for Teaching

We address two components, mathematical knowledge and mathematical knowledge for teaching, together as we found that it was neither possible nor helpful to treat them separately, since our study concerned the development of teachers' knowledge *within* the context of their professional work.

The teachers' mathematical knowledge for teaching showed particular growth in two key areas, namely, mathematical language and mathematical connections. In both cases, although the DMT prompted discussion and reflection, the nature of this discourse was not necessarily related to use of the DMT. For example, in the *Linear functions* curriculum unit, the dynamically linked representations highlighted the mathematical connections for teachers in ways that prompted them to revisit how the gradient and intercept properties were visible in each representation and how these related to the motion context.

First, teachers' overall attention to their mathematical language became more precise through their engagement in the project. We conjecture that giving time early on in each PD cycle for teachers to express and enhance mathematical definitions that related to each of the curriculum topics was key to this development.

Second, the emphasis on the different representations enabled the teachers to make important connections both within the respective knowledge domain and in relation to the context or application. For example, most of the teachers who participated in the *Geometric similarity* unit expressed genuine surprise as they realised that the trigonometric ratios were merely special cases of the invariant property of 'within-shape ratios' applied to similar right-angled triangles. Again, although this realisation had been prompted by their engagement with the DMT, this new knowledge has the potential to influence their teaching of trigonometry with and without the use of DMT.

4.3 Personal orientations

It is clear that an important first step in the development of MPTK is to offer a boost to teachers' confidence in trying to use DMT with pupils for the first time. The project opportunity provided the supportive environment in which there was a permission for teachers to take risks, which most importantly had been underwritten by their senior leaders' commitment to the project's goals. However, a further aspect of teachers' orientations concerned the level of their commitment to sustain their teaching of the curriculum unit as evidenced by trying again with another class or by continuing to teach the later activities beyond the research lesson.

Evidence from the teachers who engaged in two or more PD cycles indicated that their growth in confidence had been supported by both listening and contributing to the post-lesson discussions during which they learned from each other's challenges and successes. In addition, as aspects of mathematics lessons with DMT became more normalised by themselves and their pupils, so certain classroom routines became more second-nature.

We noticed that it did seem important that the teachers had some interest or curiosity concerning DMT and how it might impact on teaching and learning. That is, they had a flexible enough view of mathematics and its teaching and learning to imagine that it might offer a useful tool to their pedagogic armoury. Alongside this, they needed to be prepared to let the pupils have control of the technology and to pace lessons such that (the majority of) pupils could experience the 'aha' or 'light bulb' moments.

Another important aspect of teachers' orientations was that they were not too fixed in their views about their pupils' abilities and they could be open to the possibility that by accessing the mathematics through the technology, previous learning and attainment hierarchies might change.

We do report a number of teachers who were negatively oriented towards the DMT, which manifested itself in different ways. In four cases, the teacher attended the initial face-to-face PD event but withdrew from the project soon after, without giving any reason. In two such cases the teachers left the school at the end of the term. In the other two cases, the school was facing particular staffing problems that meant that it could no longer be involved. Three teachers engaged in the complete PD cycle for the unit but concluded that they did not intend to continue with the project as they had not been convinced by their experiences and/or their pupils' outcomes.

The vast majority of teachers commented that their engagement in the project had been a positive experience and, in particular, how it had supported their confidence to grow. In some cases, the teachers' reflections intimated that they were more prepared to use other DMT, for example, 'I think it has given me a huge confidence increase and I can't wait to finally use geogebra with my students in an interactive way, rather than me just showing examples on the whiteboard'.

4.4 Technology Instrumental Genesis

The development of the teachers' technology instrumental genesis was critical for this study, which was offering the majority of teachers their first professional development experience on the use of DMT in their classrooms with pupils.

This genesis is twofold in that it has both a personal and a professional dimension:

- Teachers' personal instrumental genesis: the processes through which the teachers learnt how to use the functionality of the DMT in mathematically productive ways to accomplish the tasks as presented to pupils.

which precedes:

- Teachers' professional instrumental genesis: the processes through which the teachers developed their use of the DMT as a pedagogic tool, which includes considerations of how to support their pupils' personal instrumental geneses.

Across the three CM curriculum units, it was essential that the teachers had sufficient time to engage with the DMT within the context of the tasks for pupils before beginning wider discussions that related to the development of classroom practices. During the PD sessions, it was often challenging for teachers to maintain their focus on completing the PD Tasks as an individual and some seemed content to watch another teacher's actions with the DMT rather than have a go for themselves. It is not possible to begin the process of personal instrumental genesis without a hands-on experience of the DMT, which highlights an essential feature of any professional development activity for teachers of mathematics.

4.5 Mathematical Pedagogical Technology Knowledge

The following important facets of teachers' MPTK were found, which concerned their personal engagement, instrumental geneses and classroom experiences. The teachers' personal engagement with the DMT resulted in:

- Deep(er) understanding of the mathematics that underpinned each activity, with an emphasis on the dynamic features and representational forms.
- The development of particular repertoires of use of the DMT that supported confidence and competence to be developed.
- Recognition of key aspects of their own process of learning to use the technology in order to engage with the mathematics.
- The emergence of vocabulary that was particular to the DMT and/or the mathematical context that supported the communication of their mathematical activities and outcomes.

The teachers' professional instrumental geneses began with:

- An awareness of the potential difficulties or issues that pupils might encounter, both mathematically and with the DMT.
- The need to develop a curriculum script for early lessons that attend to:
  - the importance of vocabulary – mathematical, technical and contextual; and
  - a knowledge of dynamic metaphors that might support pupils' interactions and enable them to explain and justify their actions and outcomes in a form of 'situated abstraction' that bridges the machine and paper-and-pencil mathematical worlds.

As the teachers' experience with the classroom use of DMT increased so they:

- became more aware of the need to debug pupils' work, which is inherently difficult in early lessons but is a key factor in the process of their professional learning (Clark-Wilson, 2010);
- increased their planned use of the dynamic features of the software during the lesson;
- were more confident to lead unplanned uses of dynamic features of the software in response to pupils' needs; and
- became more proficient in their highlighting of dynamic features in ways that supported pupils learning.

The combination of the growth of the above facets of a teacher's knowledge results in a teacher who is more confident to take risks with a DMT and 'have a go' by trying it out with a group of pupils. For all teachers, their project experience marked the beginning of a professional journey, even for those who had initially expressed some confidence with more generic technologies or with DMT as a demonstration tool. Furthermore, the teacher is more mindful of potential contingent moments prompted by the technology use,[14] which would suggest that the more knowledgeable a teacher is, the less contingency is needed.

While the teacher's personal orientations directly influence their motivations, desire and resilience with respect to their developing practice with the DMT, it is the bi-directional relationship between the teacher's instrumental genesis and their mathematical knowledge for teaching that is key to their knowledge growth. The 27 teachers who completed all three PD cycles were most impacted by the project. All of them wrote highly positive comments that related to their increased confidence and how their experiences had enabled them to see the impact of pupil use of DMT on mathematical learning. All of them expressed their ambition to continue to work on this aspect of their practice.

As the case studies show, engaging with several PD cycles that have a central focus on mathematics has been shown to be pivotal to individual teachers' development of MPTK.

5 Implications for policy and practice

We conclude by stating the implications of our findings, making our recommendations for both policy and practice and highlighting some next steps that would support both the dissemination of the project's outcomes and the future research agenda.

5.1 Implications

The mathematics education community does need to keep revisiting the role of dynamic mathematical technology (DMT) within teaching and learning and the nature of its associated pedagogy. Although DMT has been around for over 20 years it is continually evolving and, while access to DMT may be easier, the main barrier is still opportunities for teachers' sustained PD that maintains a strong focus on the mathematics. Without this level of support for teachers to engage in cycles of PD that scaffold their uses of DMT, PD is unlikely to be effective.

The project outcomes make a substantial contribution to knowledge of more effective features of teachers' classroom practices with DMT that might underpin future large-scale PD initiatives in this area and offer a tighter framework within which such opportunities can be designed to be more effective and sustainable within and across schools. This is even more important when teachers are choosing to move schools frequently or are leaving the profession early in their careers.

Initial teacher education seems an obvious starting point. However, as all teachers' first experiences with DMT are tentative, they need to be well-supported (with realistic expectations of the time required) and involve productive collaboration with more expert practitioners. All too often teachers reject DMT based on early lessons that they deem to have been

14 These contingent moments or 'hiccups' have been shown to be key in the development of teachers' knowledge and practice for teaching with DMT (Clark-Wilson, 2010).

unsuccessful, without necessarily having tools for reflection that enable them to learn from the experience.

Our experiences with Cornerstone Maths as well as this project lead us to suggest that it takes at least 2 years for teachers to develop more confident teaching practices with DMT. This means that for a department, the timeline to 'embed' DMT across key stage 3 is likely to be 4–5 years. A key element of this is the commitment to sustain the department-wide efforts alongside other demands on departmental time and energy. Hence leadership at the school level is key. The PD Toolkit promises to play a major role in this respect, at least in London schools, as it aims to improve the quality of the in-school professional development activities and enable new teachers to become involved through the inclusion of PD resources that can be revisited over time.

5.2 Recommendations for policy

There is a role for government, its agencies and affiliated organisations to provide better sign-posting to more effective teaching practices with DMT in secondary mathematics. This role may fall within the remit of the National Centre for Excellence in Teaching Mathematics and the Department for Education funded Maths Hub programme, in which case a targeted effort is needed over time to share the project's outcomes, develop capacity in the school system and disseminate successful models for school-based professional development.

The findings from this study have fed into the Joint Mathematical Council's ongoing work relating to 'Developing mathematics-specific pedagogy in Initial Teacher Education' with respect to the use of digital resources, which has been disseminated to ITE providers in collaboration with the NCETM and National College for School Leadership. This may lead to an opportunity to develop a specific publication aimed at disseminating models for the introduction of DMT within mathematics teacher training routes.

We reiterate the recommendation from the recent Advisory Committee on Mathematics Education report that 'Personalised and sustained professional learning within a supportive professional environment, with time for self-reflection' is an essential opportunity for teachers of key stage 3 mathematics to enable them to deepen and strengthen the quality of their teaching with DMT (ACME, 2016). Furthermore, the severe teacher shortage and staff turnover rates for key stage 3 mathematics in some parts of the country present additional systemic challenges for which the CM PD Toolkit may offer a possible solution as schools are increasingly unable to release teachers for externally led professional development.

5.3 Recommendations for practice

There should be an expectation that teachers of secondary mathematics on training routes are supported to use a DMT from the outset and the expectation that such practice is continued in the early years of teaching. These early experiences should then be built upon such that a range of (research-informed) DMTs are experienced, ideally supported by reflective PD cycles in collaboration with more expert colleagues, using resources from the PD Toolkit. The PD Toolkit needs to be more widely evaluated, particularly outside London, as a vehicle to support within-school scaling. While it is likely that a minority of these teachers may become innovative designers of new teaching tasks and approaches with DMT, it is more likely that they will draw on their earlier experiences to enable them to select appropriate DMT for their teaching purposes using a more research-informed set of criteria. It is important that both school leaders and institutional structures provide a sustainable set of conditions within which this can be achieved.

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Appendix A: Links with DfE-funded Maths Hubs

DfE-funded Maths Hub	Activity
London South East[15]	• Advertised the project to contacts. • Promoted the project at its launch event (Mar. 2016). • Publicised the project through newsletters.
London South West	• Advertised the project to contacts. • Promoted the project at its launch event (Feb. 2015). • Publicised the project through newsletters. • Facilitated the hosting of the project meetings for two CM PD cycles at a partner school.
London Central and West	• Advertised the project to its contacts. • Facilitated the hosting of a taster session at a partner school.[16] • Facilitated the hosting of project meetings for two CM PD cycles at a partner school.
London Central and North West	• Advertised the project to its contacts. • Publicised the project through newsletters.
London North East	• Advertised the project to its contacts. • Publicised the project through newsletters. • Facilitated the hosting of a taster session at a partner school. • Facilitated the hosting of the project meetings for all three CM PD cycles at a partner school.
London Thames	• Advertised the project to its contacts. • Facilitated the hosting of a taster session at two Maths Hub events (Dec. 2014 and Mar. 2015). • Hosted the project meetings for all three CM PD cycles at a partner school. • Provided a CM Hub Lead for one CM unit.

15 The designated school acting as the DfE-funded SE Maths Hubs changed in Summer 2015.
16 The school is also part of the UCL IOE Research Network and the SW Teaching School Alliance.

Appendix B: Project Advisory Group

The project advisory group, which included the two principal investigators, was established for the duration of the project to:

- Provide critical feedback on research design, methodological tools and analyses, the PD toolkit design and content, and reports of the project's outcomes.
- Support the wider dissemination of the project activities and outcomes.
- Provide other timely advice as appropriate.

Name	Role	Organisation
Hilary Povey	Invited academic	Sheffield Hallam University
Tim Rowland	Invited academic	University of Cambridge
Cosette Crisan	ITE Programme (PGCE)	UCL Institute of Education
Andrew Labinjoh	Head of Mathematics in school that had embedded CM technology in KS3 maths curriculum	Langdon Park School, LB Tower Hamlets
Teresa Smart	CM PD leader	UCL Institute of Education
Josh Hilman (circulation of docs only)	Project funder	Nuffield Foundation

The project advisory group met five times during the project (thee face-to-face and two online meetings). In addition, members of the group played key roles during the final project conference by leading discussion groups and workshops.

Appendix C: Case studies of three teachers

These case studies have been constructed to report outcomes in relation to teachers' MPTK. They directly address the following research question:

> **RQ2** What MPTK is desirable for teachers to integrate dynamic visual technologies in their teaching of these concepts?

The case studies draw on data from: the pre- and post-surveys; responses to PD tasks during face-to-face sessions; lesson plans and classroom observations of the 'landmark activity' for each curriculum unit; and teachers' post-lesson reflections and feedback.

The three cases have been selected to highlight:

- A confident and experienced teacher who has little prior experience of using dynamic technology in the classroom (case study 1: Sasha).
- An inexperienced teacher who has little prior experience of using dynamic technology in the classroom (case study 2: Phoebe).
- A confident and experienced teacher who is also a confident user of technology, but is new to DMT (case study 3: Chris).

Case study 1: Sasha

Sasha participated in all three PD cycles and she also taught all three of the related landmark activities to different key stage 3 classes. Her case study is structured in the order that she encountered the curriculum topics, drawing on the components of her MPTK to highlight the trajectory of her growth in knowledge.

Sasha's background and starting points

Sasha held a first-class Bachelor's degree in mathematics, a Postgraduate Certificate in Education (mathematics) and had been teaching for fewer than five years when she joined the project. She taught key stage 3, 4 and 5 mathematics in a large outer London mixed community school and held the role of the key stage 3 mathematics coordinator.

Sasha worked with three other colleagues from her school during all of the PD sessions and they produced their lesson plans together, supplemented by sets of Smart NoteBook interactive whiteboard slides for the lessons, which they shared and adapted for their respective classes.

Sasha reported that she had never used any DMT, either in her whole-class teaching or for use by her pupils, and that she felt 'not at all confident' to do so. This was despite the fact that she taught in a mathematics classroom that had a half-class set of fixed computers around its perimeter. Sasha reported her main barriers to increased use of DMT by her pupils were that she did not 'know enough about suitable dynamic mathematical technologies' and needed 'more time to develop my knowledge and practice (on my own)'.

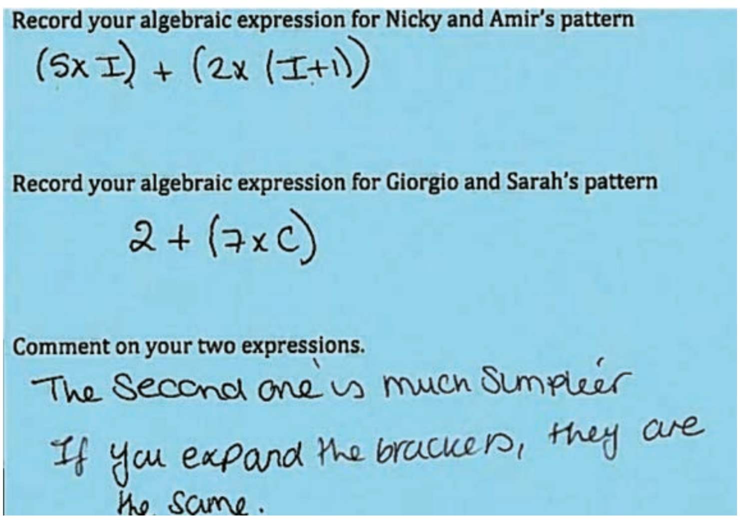
Algebraic patterns and expression

Sasha defined an algebraic variable thus: 'A term such as x or y. Or a multiple of that such as 3x or 4y' that shows little understanding and appreciation of the generality of naming variables. She also commented that she was 'Not at all confident' about her response. She answered the 'Lisa's tiles' item correctly (126, 2n+6), giving the explanation '60 X 2 plus 6 (three for each end)', which implied that she had obtained her solution by analysing the structure of the figural pattern. Sasha answered the item 'Which is larger, 2n or n?' with the explanation 'Neither, it depends on the value of n. If n = 5 then 2n is bigger. If n = -5, then n+2 is bigger', a partially correct response as she failed to notice the equality when n = 2.

During PD Task 1, which coincided with Sasha's first experiences using the DMT, she named her variable 'N', and described her variable as meaning the 'Number of blocks being used'. She identified the following important mathematical ideas that PD Task 1 had revealed as: 'n^{th} term; substitution; using algebra to represent variables and patterns linked to graphs'.

During PD Task 2, which required teachers to work though the landmark activity and note the steps that they took, Sasha's response was brief: 'Patterned the trees then the leaves; Put them together; Named the two blocks the same.' However, it can be implied that she had her own meaning for the technological steps (instrumentation) needed to 'pattern' and 'name' successfully. Her reflection on her own learning from this particular PD task was pragmatic. 'How to link variables', which aligned to a technological goal rather than the underlying mathematical concept (the epistemic goal).

Sasha completed PD Task 3 successfully, although she chose to name the variables differently for each pattern within the DMT, which meant that she did not immediately notice the equivalence of the two expressions (Figure 30).


Record your algebraic expression for Nicky and Amir's pattern

(5x I) + (2x (I+1))

Record your algebraic expression for Giorgio and Sarah's pattern

2 + (7xC)

Comment on your two expressions.

The Second one is much Simpleer
If you expand the brackers, they are the same.

Figure 30: Algebraic variable: Sasha's response to PD Task 3

Sasha was unsure about how she would respond to the pupils' mathematical dilemma, commenting 'use compare?'[17] During the PD session she expressed frustration as she hoped to be told straight away how to respond to the above hypothesised teaching scenario. She also noted this in her evaluation of the first PD session when she commented that she would have liked 'more showing us how to teach the module'.

Sasha's lesson plan included questions that she intended to ask. For example, 'but if both [variables] can change separately, it won't always form my pattern (show example), how could you ensure that they move at the same rate?', which implied that she had thought carefully about how to convene a whole-class discussion that might involve the DMT so as to focus pupils' attention to the mathematical knowledge at stake. Her Smart NoteBook interactive whiteboard slides included hyperlinks to the DMT 'images from the pupil workbook' and screen grabs of the DMT (which included some original patterns that Sasha and her colleagues had created using the DMT). The slides also supported the pupils' instrumental genesis by including the steps needed to construct a figural pattern (Figure 31).

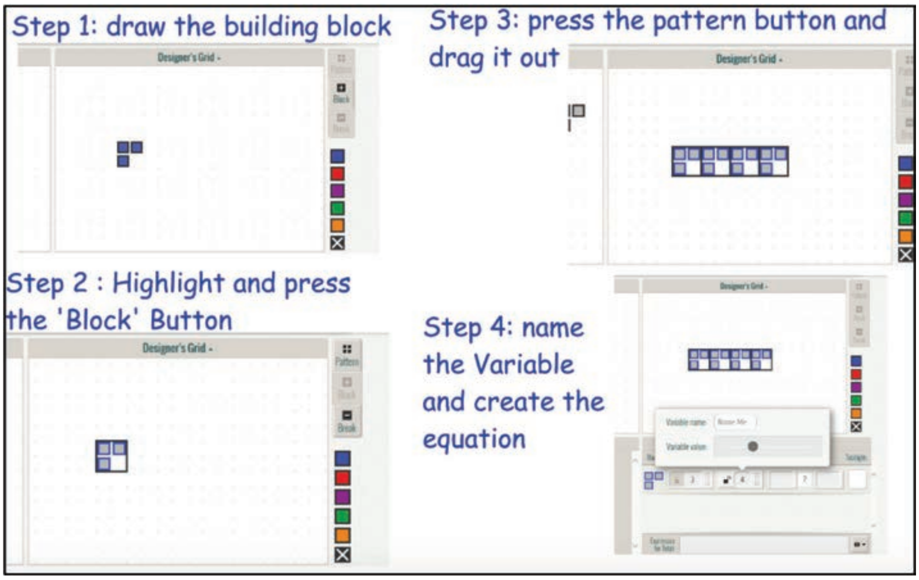


Figure 31: Algebraic variable: Sasha's presentation slide to support pupils' instrumentation

Sasha taught the lesson to her Year 8 class (a 'middle set') whom she considered to be 'highly disaffected students'. In her post-lesson reflection, she commented that most of her pupils had noticed the need for the two variables to be linked and they had relied on vocabulary such as 'link, join and move together' to try to express their reasoning. Following her teaching, Sasha redefined an algebraic variable as 'something that can change and does not have a fixed value', indicating a growth in her knowledge when compared to her earlier definition. In her final reflection on her work during the PD Cycle, Sasha commented on how she had rethought her teaching of

17 'Compare' was software functionality that enabled different representations of the two patterns to be viewed side by side.

the topic, saying ‘keywords are important and [the need to] make students more aware of variables and how this links to n^{th} term’. She also commented positively on the impact of the pupils’ use of the DMT, saying that it ‘made them more engaged’.

Linear functions

Sasha defined a linear function as ‘A straight-line graph. A relationship between x and y that results in a straight line’, a response that acknowledged the relationship and a property, but was not rigorous. She gave an incorrect response to the ‘Runners’ item as she included ‘You want to take a number of friends to the cinema and need to choose which time showing to attend. The admission price is more in the evening than in the afternoon’ as one of her answer choices.

Sasha indicated that linear functions would normally be introduced in Year 7 at her school and she would ‘Start with looking at co-ordinates and then look at using a table of values to draw a straight-line graph. We would also look at distance time graphs, describing what is happening and drawing from descriptions.’ Sasha reported that she would use Autograph and her interactive whiteboard to teach this topic, although, given that she had previously reported that she had never used a DMT in her whole-class teaching or with pupils, it is not entirely clear what she meant by this.

During her first experience with the DMT during PD Task 2, Sasha wrote a series of steps that paid attention to the on-screen functionality of the DMT, writing ‘Edit Green car – Choose distance travelled by moving A up’, which indicated that she had interpreted hotspot A as controlling the overall distance. (Hotspot A is shown in Figure 32.)

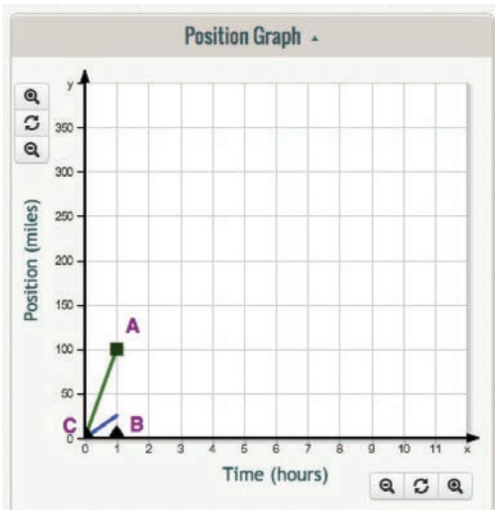


Figure 32: Linear functions: Labelling the hotspots

Sasha did not mention the gradient of the line and its relationship to the speed of the character until later in the PD Task, when the equation was visible on the screen.

Sasha did not prepare a lesson plan using the project template, choosing instead to devise a set of Smart NoteBook interactive whiteboard slides with a colleague. As before, these comprised screen shots of graphs that they had created using the DMT; they also included images from the pupil workbook.

During the observed lesson, Sasha introduced the landmark activity to the pupils and then moved around the room observing the pupils as they were working and posing questions to them. She particularly encouraged the pupils to record all of the information that was required and to give their reasoning in their written responses. Sasha did not display or use the DMT during the lesson (a static image of it was on display in her Smart NoteBook slides) and she did not convene a whole-class plenary to discuss the pupils' findings with respect to 'noticing' that the coefficient of x in the equation was related to the Shakey's speed.

In her post-lesson reflection, Sasha commented:

> More detail and structure was needed in the questions about a faster and slower Shakey before asking students to fill in the tables about where to notice the speed/distance and time. Asking them to find the speed at this stage and write it down and discussing the link to the equation may have made the tables easier to fill in.
>
> I also think an exercise about substituting values into the equation to get other distances would have been a useful way to help students realise how time and distance were [represented] in the equation. We did actually do this in the next lesson, as a class on the whiteboard and it helped students.

Sasha's overarching reflection on this particular PD cycle was very positive,

> The use of technology made this topic a lot clearer and students were able to understand a lot easier how to get speed, how to make it faster/slower and later, the linking to the equation. This is normally a topic that students struggle with, so using the software made teaching it a lot better. Next time, I would also have some more consolidation tasks, on paper, to ensure all students are on the right track. It has enhanced their understanding of this topic and I hope it will also help them to remember it more.

Geometric similarity

Sasha's initial definition of geometric similarity was 'polygons whose sides are in proportion to each other and angles are equal', which did not mention the word corresponding (which she included in her subsequent definition) nor that the number of sides in both polygons should be the same. She said that she was confident with her response. Her definition of the corresponding sides of two polygons was 'sides in proportion to each other' without attending to any angle properties. Sasha's response to the item 'Similar parallelograms' was particularly interesting because her written method was extensive and the nature of her calculations and subsequent checking strongly suggest that she was surprised when she discovered the invariant 'within-shape ratio' of the corresponding sides of the parallelograms (Figure 22).

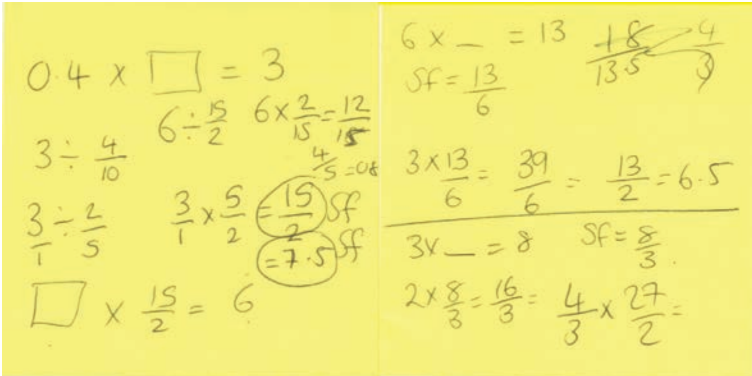


Figure 33: Sasha's written response to 'Similar parallelograms'

Sasha reported when and how she would normally introduce geometric similarity to pupils as follows. 'Introduced as enlargement and identifying similar and congruent triangles when constructing triangles. Similar shapes will then not come up until year 9'. She added that she would use her interactive whiteboard and 'sheets' as resources.

During PD Task 3, Sasha's response showed that she had recognised the differences in the pupils' strategies, but her explanations imply that she was unable to make an accurate distinction between the underlying mathematical property that was being revealed.

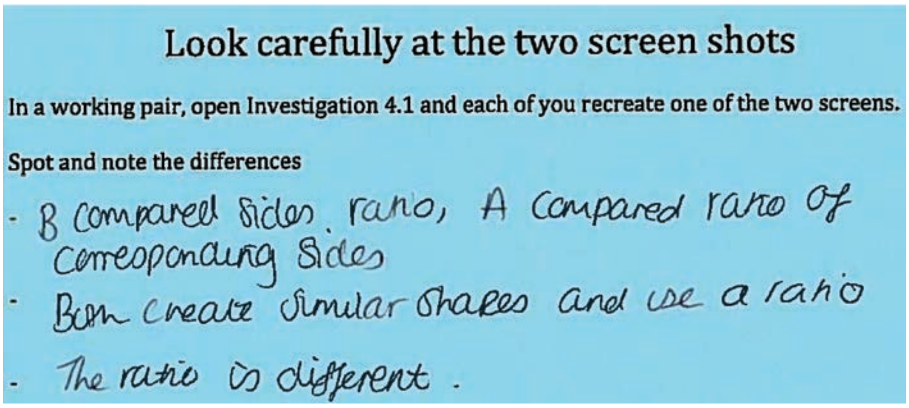


Figure 34: Geometric similarity: Sasha's response to PD Task 3

At the end of the first PD session, Sasha commented that she had found the most helpful aspect of the day the 'discussions to open our eyes to aspects of geometric similarity'.

Sasha did not produce a lesson plan for the landmark activity using the project proforma. Instead she created a set of Smart NoteBook interactive whiteboard slides. In the observed lesson, which she taught to a 'middle set' year 9 class, Sasha began with a starter activity she had created using screenshots from the DMT that required the pupils to calculate scale factors of similar rectangles and included a pair of congruent rectangles. She then posed a question to the class: 'If there is the same scale factor between corresponding sides does that mean that the two shapes are similar?', which the class discussed before moving to begin the landmark activity. The pupils worked on the landmark activity in pairs and whilst they were doing this, Sasha displayed the DMT on the interactive whiteboard and measured sides and angles in preparation for a whole-class plenary. This was the first time she was observed using the DMT 'live' in this way. She proceeded to question the class and, at the same time, dragged the angle slider to prompt them to think carefully.

In her lesson reflection, Sasha commented that she thought that her 'students [had] struggled with the language of corresponding angles and so this needed more clarification', but that somehow they already knew this meant that the angles were the same. She concluded that the DMT had 'made it easier being able to move the slider and not having to draw it [the enlarged shape] each time. Therefore, students understood quicker.'

Summary

Sasha was supported by the fact that she had worked with four other colleagues during the period of the project and as a result, they had been able to organise collaborative planning meetings and shared the resources they had created with each other. Furthermore, their Head of Department had provided the institutional support to enable the group to observe each other teach each landmark activity and facilitated one of the research team to attend their reflection meetings.

Sasha's departmental culture favoured the shared development of resources, which took the form of interactive whiteboard presentation slides and 'card sort' style tasks for pupils. There was a sense that, once these had been developed, they offered the 'departmental way to deliver lessons', which other teachers would happily follow, perhaps implying that there was a 'right way' to teach.

In her final project reflection, Sasha reported that she now felt confident to use the CM DMT with her pupils in her classroom. She has since become a CM Champion and has, with a colleague, co-led professional development sessions for other colleagues who teach key stage 3 mathematics in her school.

Case study 2: Phoebe

Phoebe participated in all three PD cycles and she taught two of the related landmark activities to different key stage 3 classes. Her case study is structured in the order that she encountered the curriculum topics, drawing on the components of her MPTK to highlight the trajectory of her growth in knowledge.

Phoebe's background and starting points

Phoebe held a Bachelor's degree in Chemical Engineering and a Master's degree in Mathematical Sciences. She considered herself to be a secondary mathematics specialist as her engineering degree 'contained a high level of mathematical content'. Phoebe held a Postgraduate Certificate in Education (mathematics) and joined the project in the summer term of her NQT year. She taught key stage 3, 4 and 5 mathematics in an inner London boys academy.

Phoebe worked with a more senior colleague during all of the PD sessions and they produced three consistently high-quality lesson plans, which were highly detailed exemplars in all respects, as each featured seven of the eight quality criteria.

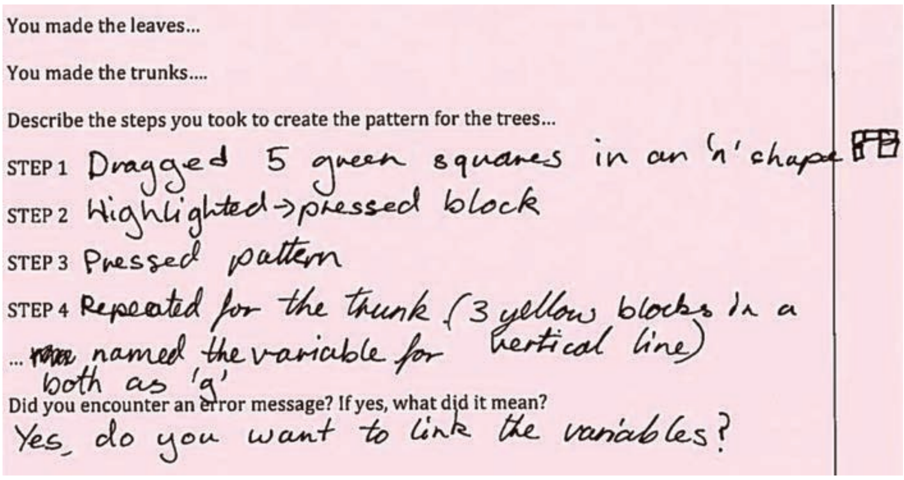
Phoebe reported that she occasionally used DMT such as Geogebra, Autograph and Excel in her whole-class teaching and felt confident to do so. In addition, she also occasionally enabled her pupils to use DMT and she would 'walk around, observe their results and ask key questions' whilst they were working. Phoebe commented that her main barrier to increased use of DMT by her pupils was 'better access to suitable lesson resources'.

Algebraic patterns and expression

Phoebe's definition: 'An algebraic variable is represented by a letter and can either have one specific unknown value, a specific range or set of possible values or it can be any real number', indicated an understanding of 'Letter as variable with attention to domain and range'. She answered 'Lisa's tiles' item correctly (126, 2n+6), giving the explanation 'There are two rows of grey tiles for every one row of white tiles and an additional six tiles on the ends; 2 x 60 = 120; 120 + 6 = 126', which indicated a clear understanding of figural pattern. She also answered the item 'Which is larger, 2n or n?' fully correctly with the explanation '2n is larger, when n > 2; when n<2 for example n = 1, 2(1) = 2 and (1) + 2 = 3 the latter is larger. However, when n = 2 both expressions are equal.'

During PD Task 1, where Phoebe interacted with the DMT for the first time, she named her variable 'n', and described it as meaning 'How many times the pattern (building block) is repeated'. She identified the following important mathematical idea that PD Task 1 had revealed as 'representing a sequence with an algebraic expression'.

During PD Task 2, which required teachers to work though the landmark activity and note the steps that they took, Phoebe wrote a detailed response that indicated that she was attending carefully to what she was doing as she was becoming personally familiar (instrumented) with the DMT (Figure 35).


You made the leaves...

You made the trunks....

Describe the steps you took to create the pattern for the trees...

STEP 1 Dragged 5 green squares in an 'n' shape

STEP 2 Highlighted → pressed block

STEP 3 Pressed pattern

STEP 4 Repeated for the trunk (3 yellow blocks in a vertical line)

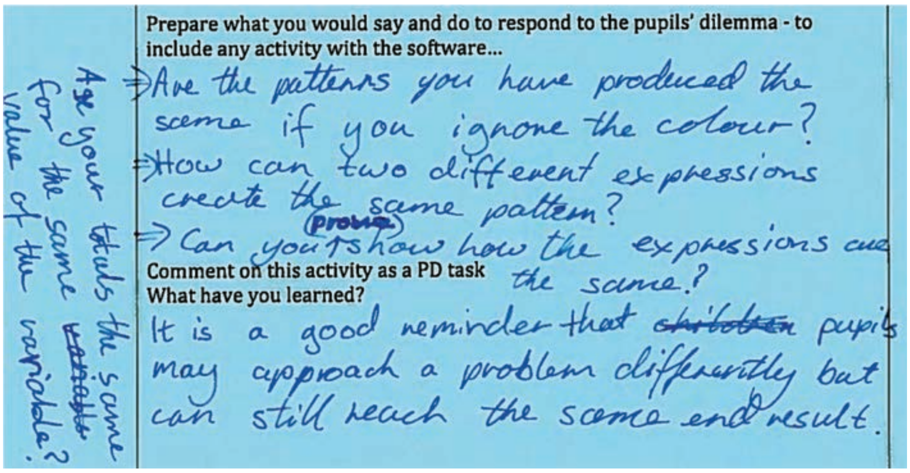
... named the variable for both as 'a'

Did you encounter an error message? If yes, what did it mean?

Yes, do you want to link the variables?

Figure 35: Algebraic variable: Phoebe's response to PD Task 2

Phoebe completed PD Task 3 successfully and wrote a detailed curriculum script in the form of questions that she would ask her pupils (Figure 36).


Prepare what you would say and do to respond to the pupils' dilemma - to include any activity with the software...

→ Are the patterns you have produced the same if you ignore the colour?

→ How can two different expressions create the same pattern?

→ Can you (prove) show how the expressions are the same?

Comment on this activity as a PD task

What have you learned?

It is a good reminder that ~~children~~ pupils may approach a problem differently but can still reach the same end result.

Are your totals the same for the same value of the variable?

Figure 36: Algebraic variable: Phoebe's response to PD Task 3

Phoebe's lesson plan (prepared with her colleague) was one of the most detailed produced during this particular PD cycle, with the following exemplary features:

- It began with a 'card sort' starter activity that required pupils to recap their knowledge of how to create an algebraic pattern in the software and model it algebraically, which showed an appreciation of the pupils' instrumental genesis.
- It included direct instructions to the pupils to check their algebraic models using the slider, exemplifying an important component of the predict–check–explain pedagogy.

In the observed lesson (in which Phoebe's colleague was also present), Phoebe placed a high emphasis on the use of vocabulary, both by herself and in her encouragement to her pupils to express their oral and written reasoning in complete sentences. She displayed and used the DMT on the interactive whiteboard and modelled the situation whereby the trunks and tops of the trees had been created separately, saying 'What do you think happens when we move the slider?' as a prompt for the pupils to resolve the mathematical discrepancy.

Phoebe's post-lesson reflection indicated that she thought that the lesson plan had been most helpful where she and her colleague had planned how to respond to pupils who did not immediately see the need to link the two variables. However, she also noted that the plan did not have sufficient emphasis on asking the pupils to make a prediction before they started to build the pattern using the software, indicating that reinforcing this particular aspect of the pedagogy was an area for her to work on. Phoebe commented on her surprise that 'one pupil said that linking the variables synchronised the movement of the two different building blocks'.

In Phoebe's final reflections, made at the completion of the PD cycle, she commented that 'It was a good reminder that in order to teach all algebra successfully pupils need to be exposed to it in many different situations and given varied activities. Also, the importance of allowing pupils to have the time to make connections or discover the maths for themselves.'

Linear functions

Phoebe defined a linear function as 'the process that is applied to an input to create an output. A linear function can be represented in many ways: algebraically, with words, a mapping, a diagram or graphically.' This was a typical teacher response that described the properties of a linear function without actually defining it. Phoebe gave fully correct responses to the 'Runners' item and the 'Mrs Kingston' item in both the pre- and post-teaching surveys.

Phoebe indicated when the linear functions would normally be introduced during key stage 3, citing the national curriculum and stating:

> 'recognise, sketch and produce graphs of linear functions of one variable using equations in x and y and the Cartesian plane' was introduced in the first term of year 7. Dependent on the ability of the class both of the following will either be introduced in year 7 or year 8: 'calculate and interpret gradients and intercepts of linear functions numerically, graphically and algebraically, using $y = mx+c$' and 'begin to model simple contextual and subject-based problems algebraically'.

She mentioned resources that she would use in her teaching of this mathematical content, such as: 'Card sort; Sticker Sort; Software on notebook that plots linear functions that change as you change the function and can plot more than one to compare; Worksheets'.

In her response to PD Task 2, Phoebe was again very detailed. She referred to the suggested labels for the hotspots (A, B and C, as in Figure 32).

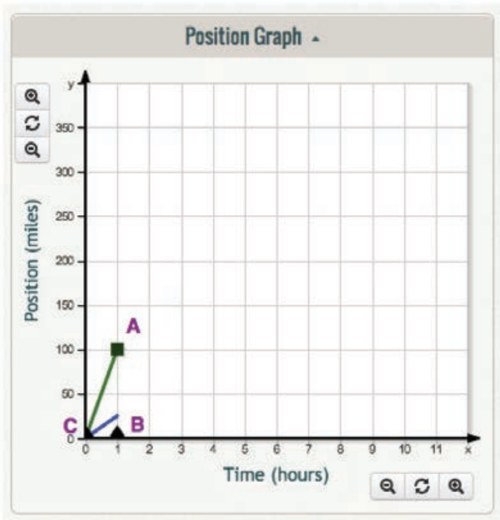


Figure 37: Linear functions: Labelling the hotspots

Phoebe described the effect of dragging B thus: 'B changes the length of the line (restricting the domain)' and that dragging A 'extends position and changes gradient'.

In her lesson plan, Phoebe was open to the pupils' exploration of the software ('Will they be able to edit the equation (unlikely) or move the graph so that it has a lower gradient'), whilst also leaving it open for her pupils to record the equation in different ways ('Will the students copy down the equation as it is written on the screen or will they simplify it to y = 4x instead of y = 4.0x+0 ?'). No other teachers involved in the study paid attention to the pupils' instrumental genesis to this level of detail.

During the lesson, Phoebe introduced the landmark activity by playing Shakey's original animation to the class before setting the class to work in pairs on laptops. However, as one pair were having problems with their laptop, Phoebe invited them to leave their seats and work on the classroom interactive whiteboard. However, this inadvertently affected the path of the lesson in two ways:

- Phoebe did not take back control of the interactive whiteboard in order to convene further classroom plenaries as she had originally planned.
- Phoebe began a conversation with the pupils who were working on the interactive whiteboard that other pupils then stopped to listen to. This became a whole-class discussion regarding the mathematical notation for a linear equation and whether y = 4.0x+0 was the same as y = 4x.

In her post-lesson reflection, Phoebe made the following comment, 'The comparison between the equations definitely helped to highlight any misconceptions and enabled pupils to better understand what the notation represented. Throughout the lesson, it would have been better to bring the class together to observe and discuss each stage of the work, so that conclusions can be drawn at each stage before they move on.'

Geometric similarity

Phoebe's initial definition of geometric similarity was 'A polygon is a shape with 3 or more sides. If two polygons are mathematically similar then all corresponding angles are equal, all corresponding sides are proportional and the areas are proportional', which relied on an inherent definition of what was meant by 'corresponding' sides and angles. Following discussion, she revised this definition to include the 'same number of sides'. Her definition of the corresponding sides of two polygons relied on a definition of what she called 'equivalent angles', although she offered a dynamic way of recognising corresponding angles by implying that the shapes could be re-orientated: 'The corresponding sides of two similar polygons are sides that are between the equivalent angles and all corresponding sides of two similar shapes will have the same scale factor. Practically, you can identify the corresponding angles and then orientate the shapes.' She expanded on this later, when responding to the assessment item 'Corresponding sides', thus: 'By eye I could identify that angle BAC and angle ZYX are equal. Then I orientated the "copy" in my head and could see which where equivalent. I then verified that I was correct by counting the squares and ensuring that the scale factor for all corresponding sides was the same.'

Phoebe reported that she would normally introduce geometric similarity to pupils 'After transformations of shapes, however it is nice to make the link that enlargement is the only transformation that produces similar shapes' – a statement suggesting that Phoebe's understanding of 'similar' might be limited to a particular context. She added that 'the scheme of work doesn't depict what order you complete the topics for that half term, it is the teacher's choice, and that would then vary on ability and year group'. Phoebe mentioned that she would use resources such as a 'Worksheet, IWB and card sort'.

During PD Task 3, Phoebe struggled to express the equality of the two sets of correct statements in the ratio checker (Figure 38).

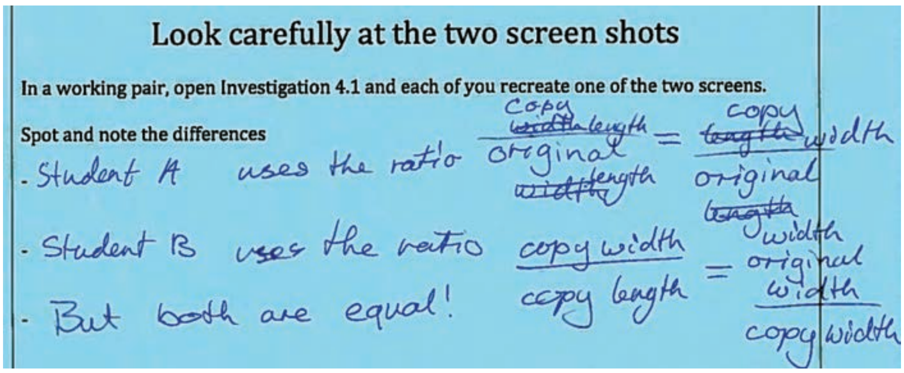


Figure 38: Geometric similarity: Phoebe's response to PD Task 3

She then expressed surprise in the final result: 'But both are equal!'

Phoebe commented at the end of the first PD session that she had 'developed her confidence with being able to define what geometric similarity is' and noted that it was 'a great reminder that it is important to expose pupils to the same maths in lots of different situations'.

Phoebe's lesson plan (developed with the same colleague as in the previous two PD cycles) was again highly detailed in its attention to its quality features. It was particularly explicit about the actions the teacher would take (with and without the technology), the questions that would be posed, and its directions to pupils concerning the use of the DMT. It also included explicit mathematical, technological and contextual vocabulary throughout. Furthermore, the plan recognised the need to be explicit about how the pupils and teacher would engage with the important features of the DMT (the angle slider and ratio checker). Although Phoebe did not have the opportunity to teach her planned lesson within the time period of the PD cycle, she planned to do as soon as the topic was next taught within the department's schemes of work for a key stage 3 class that she taught.

Summary

Phoebe's participation in the project was in close collaboration with her more senior colleague, Marja (the second in department) who, although more experienced as a mathematics teacher by four years, had little experience and confidence to use DMT in her whole-class teaching and had never enabled her pupils to use DMT in her teaching. However, as their department had recently acquired a class set of Google ChromeBook laptops, the department had been motivated to use the opportunity of the project to develop their classroom practice and accompanying schemes of work. Marja held a Master's degree in mathematics education and was also interested in researching the pupils' outcomes during the project. Marja observed both of Phoebe's research lessons and captured video data of the pupils' computer work, which she and Phoebe reviewed closely with great interest and enthusiasm.

Phoebe was very self-critical of her own lesson outcomes and she reflected deeply on how she would amend her original plans to learn from what she considered to be the weaknesses. However, the trajectory of Phoebe's development of MPTK was very fast as she not only put what she had learned into practice but also, through her discussions with Marja, supported Marja to plan her own teaching of the research lessons. Furthermore, Phoebe became a CM Champion and had already begun to disseminate her classroom work to colleagues with a view to formalising their school-based professional development plans to involve further colleagues in the subsequent school year.

Case study 3: Chris

Chris participated fully in all three PD cycles and he taught all of the related landmark activities to key stage 3 classes, in some cases teaching a curriculum unit to more than one class. His case study is structured in the order that he encountered the curriculum topics, drawing on the components of his MPTK to highlight the trajectory of his growth in knowledge.

Chris's background and starting points

Chris held an undergraduate Master's degree in Mechanical Engineering with Financial Management, and he considered himself to be a secondary mathematics specialist as his degree 'used maths to a high level'. Chris held a Postgraduate Certificate in Education (mathematics) achieved via a Diploma of Secondary Teaching that was awarded in New Zealand. He had been teaching for between 1 and 5 years and during the time of the project, he taught key stage 3 and 4 mathematics in an inner London mixed voluntary-aided (Roman Catholic) school.

Chris's lesson planning style was focused mainly on the posing of questions, which have been inferred as questions that he intended to ask during the lesson. For example, 'What does the expression represent in all of the designs?' (algebraic variables landmark activity) and 'What can we say about the relationship between corresponding angles in mathematically similar shapes?' (geometric similarity landmark activity).

Chris reported that he used DMT fairly regularly in his whole-class teaching and was confident to do so, citing 'Mathsworkout' as an example of a DMT that he used. This choice of example (an online resource that comprises exercises and maths games) indicated that Chris had a narrow interpretation of a DMT. Chris also commented that he planned for his pupils to use DMT 'Fairly regularly', was confident to do this and, while the pupils worked independently on the DMT his reported pedagogical action was to 'circulate'. Chris cited that his main barrier to developing his use of DMT was 'more time to work with colleagues on resources' also commenting that he would also need 'book-work evidence of progress'.

Algebraic patterns and expression

Chris defined an algebraic variable as 'a number which we don't know yet that could have any value', indicating an understanding of 'letter as variable'. Chris answered the 'Lisa's tiles' item correctly (126, 2n+6), giving the explanation 'found by using the visual pattern' but with no reference to domain or range. He also answered the item 'Which is larger, 2n or n?' with the explanation: 'depends on value of n. If n is 1 or less (including negatives) 2n is smaller than n+2. For 2 and above, 2n is bigger", which was only partially correct as he did not mention the equality when n = 2.

During PD Task 1, which coincided with his introduction to the DMT, Chris named his variable 'b', and changed his mind over what his variable represented.

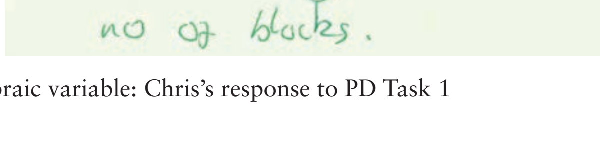


Figure 39: Algebraic variable: Chris's response to PD Task 1

He identified the following important mathematical ideas that PD Task 1 had revealed as 'creating an expression; substitution and number patterns' and commented that it was a 'good tool'.

During PD Task 2, which required teachers to work though the landmark activity and note the steps that they took, Chris wrote very brief responses 'build leaves, build trunks, name variables, link variables', indicating a pragmatic approach to the use of the software, with little need to describe the steps in more detail. However, in PD Task 3, which Chris completed successfully, he demonstrated that he had appreciated multiple ways to represent the same pattern algebraically and had also began to consider potential pupil difficulties that he might need to be aware of in the classroom, commenting that PD Task3 had 'helped me to see the wrong turns students could make (patterns growing in opposite directions)', which represents early evidence of how his own instrumental genesis was influencing his MPTK.

The classroom observation of Chris's teaching of the landmark activity to a year 9 class provided further evidence of his professional development as he reflected on how 'showing errors/solutions students had made to the class' (by taking control of the pupils' computer screens) had been a particularly useful pedagogic approach, while still commenting that he felt that his lesson needed a more effective plenary to ensure that all pupils had met his lesson objectives.

Following his teaching of the landmark activity, Chris redefined an algebraic value as 'a number whose value can change'. Chris's explanation to support his (again) correct response for 'Lisa's tiles' was more detailed than previously, 'Need to double 60 for the top and bottom rows, minus the ends. Need 3 again at each end so plus 6.' However, his response to the item 'Which is larger, 2n or n?' was less detailed than previously, 'depends on value of n, for n<2, n+2 is bigger' and still omitted to mention the equality at n = 2.

Chris concluded that he would like to incorporate the software into his teaching as it had given his pupils 'experience of a variable used in a real context and how expressions can be created'. Commenting on the impact of the PD cycle on his confidence to use DMT with key stage 3 classes, Chris said that he 'was already confident but gave me time to work with one and really get to know it. Helped me think about how to build learning around a good one'. He added 'An excellent resource – loved being involved. Pupils got a lot out of it. Is at odds with some people's expectations of explicit learning outcomes met quickly – however, I believe it gives deep and lasting learning and is very valuable.'

Linear functions

Chris defined a linear function as 'a relationship between two variables where there is a constant rate of change'. In the 'Runners' item he gave a complete response in that '(a) and (b) are travelling at the same speed' and 'you cannot tell which runner is ahead', while his written justification clearly indicated that he appreciated that the value of *c* could also vary: 'Running at same speed shown by coefficient of *x*. Constant term dictates starting point, which *could be same, ahead, or behind*. The starting point therefore dictates who is ahead since are moving at same speed.' In his later response to the same question, Chris again gave a complete correct response and a more detailed justification: 'The rate of three is the same for both. *n* could be a negative so *a* could be ahead. Both graphs would be parallel, travelling at the same speed but with one or the other starting ahead (or even starting at same position if n = 0).' Chris's response to the item 'Mrs Kingston' was correctly answered in both the pre- and post-survey. In the post-survey, he offered a question context that was both correct and indicated a transfer to a context other than motion: 'Buying tickets for a concert – you have to pay a booking fee for an order then a fixed price per individual ticket.'

Chris indicated that he would normally introduce the topic in key stage 3: 'Function machines is first intro to this topic. Happens in yr 7 for all students. Then depending on pathway (we have 3 depending on achievement level) they will look at linear functions in more detail as an extension/application of second function machines unit in yr 7. For fast track students. Yr 8 for medium and slow track.' He mentioned resources such as 'Worksheets. Textbook. Video of a machine that makes chickens into chicken nuggets etc. active inspire.'

Chris's response to PD Task 2 indicated that he had made some sense of the graph editing functionality, describing the draggable hotspots by their effects, i.e. '[it] change[s] distance travelled (speed car moves at)'. During his teaching of the landmark activity, he was mindful of the need for his Year 8 pupils to understand how the process of editing the graph affected the other mathematical representations. In his lesson, he asked questions such as 'How do we know that he [Shakey the robot] is going at 4 centimetres per second?' and, having projected a student's work to the class, 'How do we know if this person has made Shakey go slower?' He also reminded the pupils of the ways that they could edit the graph to change Shakey's motion by demonstrating how to drag the hotspots and also used the 'step' functionality within the DMT to highlight how they could 'see the speed'.

Following the lesson, Chris expressed concern that his pupils had not made sufficient connections between the mathematical ideas, but planned to work further on this in the subsequent lesson. He reflected that his biggest challenge was 'getting students to articulate their thinking effectively' while also concluding that his pupils (a lower-attaining class) had 'made good progress with [the] difficult concepts of rates, gradient, and made good links to learning on linear functions – they were engaged and interested and made powerful insights'.

Geometric similarity

Chris's initial definition of geometric similarity relied on a 'scaling' model: '2 polygons otherwise identical other than size has been increased or decreased in proportion; side lengths have been changed by a constant multiplier', which neglected to state any conditions relating to corresponding sides and did not mention any angle properties. His definition of the corresponding sides of two polygons was limited to a particular type of polygon (right-angled triangles), stating 'the equivalent side of the corresponding side, e.g. hypotenuse to hypotenuse'.

Chris reported that he would normally introduce geometric similarity to pupils in 'Year 7/8 after symmetry, reflection and rotation' using resources such as, 'mathsworkout and dynamic pictures. I often use a photo or image in Word which I drag, initially dragging out one side, asking if I have enlarged correctly, then after teasing out what required dragging corner so sides change in proportion.' This again indicates that Chris's prevailing pedagogic approach for this topic was related to the teaching of enlargements within transformational geometry.

During PD Task 2, in which Chris first encountered the dynamic use of the ratio checker, which revealed the invariant property of the 'within-shape ratio' for geometrically similar shapes, he commented on his surprise that the 'fraction and ratio shows relationship of one side to other side of *same* shape', another indication of his developing MPTK.

Chris's lesson plan for the landmark activity was not particularly detailed, although it scored highly in that it addressed seven of the eight quality features. His teaching of the research lesson revealed much more subtle and intuitive use of the DMT alongside the computer management software (Impero) that let him manage, take control and project the pupils' screens, which he used to good effect in the lesson. He used the dynamic features of the DMT several times, which included instructing a pupil (whose screen was on display to the class) to drag the angle slider, whilst asking another pupil to comment on the outcome and a third to explain further.

Summary

Chris joined the project with both a keen interest in how DMT might influence his classroom practice and a full commitment to the PD cycles, attending all of the meetings, facilitating 4 school visits by the researchers and, later in the project, encouraging his Head of Department and two further teachers to become involved. It was helpful that he taught in a mathematics classroom that was also the department's IT suite and had already established ways of working in this setting that made use of the computer management software. He was also a confident and proficient user of his interactive whiteboard and interacted with the DMT at his board from the outset.

Chris's school was considered to be a challenging one and it was noticeable that the pupils did not engage in many mathematical discussions between themselves, nor were they encouraged to do so. In all three of the observed lessons Chris used the computer management software to signal the different phases of the lesson, retaining tight control of the class. However, Chris balanced this by involving the pupils quite centrally in the critique of each other's work using the DMT by both projecting screens and photographs of pupils' written justifications in their workbook.

Chris was becoming a proficient user of the DMT and was developing his practice to enable the pupils' mathematical learning through the technology. Although he acknowledged that he still wanted to incorporate strategies to link the new learning with the pupils' prior understanding, he had moved beyond a focus on how pupils were using the technology to home in on important mathematical features of the multiple representations and the links between them. He was beginning to foreground the mathematics rather than the technology. Chris's engagement in the project prompted him to begin a Master's degree in mathematics education, with an initial focus on the use of technology.

Appendix D: Essential PD activities to use DMT for dynamic mathematics

The challenges associated with learning to engage with mathematics through the medium of a DMT, (however well-designed) lead us to conclude that some initial face-to-face professional development is essential to support early activities such as:

- Discussion of the mathematics that underpins the concept being exemplified through the DMT. The discussion should cover definitions, representations, mathematical progression, current teaching approaches, common student difficulties, and connections with other concepts. This implies that the PD is focused on a particular mathematical task that is an exemplification of the DMT and that forms part of a curriculum unit (a sequence of lessons) for that concept.
- Focused PD tasks that give **all** teachers a hands-on experience with the DMT and focus primarily on understanding the dynamic mathematical objects, their representations in the DMT, and how these might be manipulated in mathematically productive ways.
- Engagement with the early activities in the curriculum unit by completing them 'as learners' prior to discussion of how they might be taught.
- Hypothesised teaching scenarios involving the use of the DMT to support the development of a curriculum script.
- Detailed lesson planning for a landmark activity, selected from the curriculum unit, that is explicit about the teachers' actions and questions, the use of the DMT and the development of vocabulary, with the intention that this lesson will be taught soon after the PD event.
- Introduction to the online teacher community and other 'at distance' support.

Following this, follow-up PD sessions are necessary to enable:

- Teacher reflection on the outcomes of their teaching of the landmark activity (individually and in groups).
- Discussion of common classroom challenges (beyond smooth access to the DMT) and how they might be overcome.
- Engagement with the later activities in the curriculum unit by completing them 'as pupils' prior to discussion of how they might be taught.
- A revisit of the mathematics of each curriculum unit to support an overarching reflection on how the DMT has impacted on the related definitions, representations and mathematical progression, current

teaching approaches, common student difficulties, connections with other concepts, and how the concept can be developed further.

- Discussion of different strategies to involve other colleagues in the school.

Appendix E: Evaluation of the PD toolkit in three schools

We conducted a small-scale evaluation of the PD toolkit within three self-nominating project schools. The PD toolkit was evaluated with respect to:

› if and how it met the CM Champions' expectations as they selected resources to design school-based PD for colleagues in their own schools;
› the pathways that the CM Champions chose through the PD Toolkit resources;
› the CM Champions perceptions of the usefulness of the resources and their ease of access.

School A

School A joined the project in June 2015, nominating two teachers to participate in the *Algebraic patterns and expressions* curriculum unit. One teacher, who emerged to become the CM Champion in the school, continued to participate in both of the subsequent PD cycles.

The CM Champion was able to organise a one-hour professional development session for the whole department as part of a scheduled departmental meeting in January 2017, piloting resources from the prototype PD Toolkit. He chose to focus on the *Linear functions* curriculum unit and to adapt the PD session presentation slides. His aim was to support the maths team to teach the curriculum unit to an appropriate class in the second half of the spring half term 2017. This involved some room changes to enable classes to access the department's dedicated computer suite, alongside a timetabled schedule for laptop access.

The CM Champion presented his colleagues with a document that mapped the *Linear functions* unit to the school's key stage 3 and 4 schemes of work, which included Year 7 (sets 1 and 2), Year 8 (sets 1 to 5), Year 9 (sets 3 to 7) and Year 10 (sets 6 and 7) classes. His rationale was to maximise the opportunities for all teachers to identify an appropriate class with which to trial the unit.

The session included one of the PD tasks (PD Task 1: What is a linear function?) followed by the invitation for teachers to work through the *Linear functions* landmark activity for themselves (Investigation 4: Controlling characters with equations), followed by the lesson planning task. The CM Champion reported that he found the session challenging to lead, commenting: 'Teachers are such awful students and don't follow instructions! They just wanted to play with it rather than follow through the steps of Activity 4.1 as asked …'

The CM Champion provided the following detailed feedback about the PD Toolkit:

Overall I think that the resources are quite comprehensive of what teachers need to know and are well-scaffolded for the facilitator.

I wonder if there could be some more explicit discussion of the 'style of learning' that the Units are designed for: exploratory and conceptual rather than a transmission approach. In reality, I think for many teachers coming to this, it is quite radically different and explicitly addressing this could be useful. I see this as being a key barrier to colleagues going for it. They just don't get that learning can be slow and deep in this way and that this is actually better than 'delivering content'. The standards units have a good introduction that does this; could this be a model? Perhaps a short video that explains this and gives an overview of why it is a good approach. Particularly if it is in a succinct video that people would watch?

If I were going to run a session using this I would use it pretty much as-is. How it was done would depend on time available. In a perfect world do it over 4 weeks beginning with a 2-hour session then shorter 1-hour sessions in weeks 2, 3 and 4 as follows:

I would begin by discussing what a linear function is, then get teachers to explore the function (PD Task 1) and then do PD Task 2. I would see these as being essential learning. If I had time I would then do PD task 3 which I think is a great task but could be cut if time was very short.

Week 2: (PD task 3 could be a starter for this session if not done before, or perhaps even after the session when they have experienced the representations more). Teachers then do the early investigations as guided by the presentation in their own time or perhaps in a collaborative tutorial-type session when they work together with minimal guidance.

Week 3: Do Investigation 4 and start to plan in tutorial-type session as above.

Week 4: Meet up to discuss planned lessons. Share and comment on plans.

My comments about the [PD Session] PowerPoint:

- I like that the slides have been beefed out a little over the ones available on NCETM, there is more guidance for the facilitator for example on how to have a discussion around 'what is a linear function?'
- I like that the learning is exploratory and mirrors the learning from Cornerstone itself, giving teachers experience of learning in this way (alien to many!).
- Good guidance to explore the critical functions of the software: how to edit, the links between the representations.
- I like the explanation of the rationale behind the design of the software e.g. variation theory and the fact that each hotspot controls one variable.

- I like the top tips for planning, I found this section of the initial PD perhaps too 'open' and I think it is therefore useful to identify some critical elements of the lesson on which to focus.
- Post-lesson reflections are useful, again good guidance on how to effectively facilitate the PD which I would find very helpful.
- I like the videos which show well how to question around the tasks to facilitate learning.

School B

School B joined the project in 2015 and identified two teachers to lead the project in the school. The two teachers piloted all three of the Cornerstone Maths curriculum units and also supported three further teachers, who also attended the face-to-face PD with them. Back in school, the project teachers observed and supported each other's lessons, and following what they concluded to be a worthwhile development of their key stage 3 scheme, led PD in summer 2016 (prior to the design of the PD Toolkit) for other members of their large department focusing on the linear functions curriculum unit.

One of the CM Champions reported:

> We have already shared this module with our department last year and will be reminding them of it tomorrow as they will be starting it with year 7 over the next few weeks.
>
> When we first introduced it last year, we spoke about Cornerstone as an overview and what it aims to do and then we gave them each a pupil booklet and a computer and got them to work through it in pairs.
>
> We were given an hour in total and so we felt this was the best way for staff to get to grips with what they were doing and what the project was about.
>
> We then asked staff to go away and look through it in more detail before doing it with students and gave them the teacher guides.
>
> As a reminder tomorrow, we will again give teacher guides, show them the programme, remind them of the need for students to be discussing this and show them the students work that you have just sent the link too.
>
> I will also give them the overview of each investigation so they can read that.

The CM Champion added, 'the examples of students' work are very useful and the overview of each investigation is also good' but that in her school, she was 'not sure we would use all of the materials as we would not be given that length of time to share with the department'. She envisaged that the PD Toolkit 'will be useful for us to get staff started, although they will not have the luxury of a day on it as we did, so we will not have time to share all of the toolkit but share a link to it so staff can read in more detail before their lessons'.

School C

School C was one of the original pilot schools for Cornerstone Maths and so began piloting the curriculum units in 2011. The school has now embedded all three curriculum units in their key stage 3 schemes of work and the units are taught by all teachers of key stage 3 mathematics. They have also developed their own approach to assessment and have an ongoing plan to support trainees and new teachers to the school to become confident to teach the Cornerstone Maths units. The CM Champion (who also held the role of joint Head of Department) offered the perspective of a school that is seeking to sustain its use of Cornerstone Maths over time by considering the particular PD needs for teachers new to the school.

> The website is great! Really easy to use and I will definitely be using the PD resources to train new members in the faculty (see a rough outline of how I might use them attached). I think the video which explains linear functions acts as a nice introduction to the unit and the teacher videos and student responses are rich resources which will open up discussions about the pedagogy. I am also pleased that you've included some 'PD tasks' to get people reflecting on the mathematics and pedagogy.

The CM Champion proposed her draft plan for a one-day PD session using the PD Toolkit.

Preparing to teach linear functions

Aim of the PD: By the end of the PD teachers will be confident to teach linear functions using the Cornerstone Maths software with their Year 7 class.

Time frame: 2 departmental meetings before teaching and ongoing reflection post teaching.

Session 1: Introduction – time given 1 hour as part of faculty meeting time

Timings	*Activity*	*Rough outline*
10 mins	*PD task 1*	*Teachers to complete PD task 1 and then to feed back. Facilitate a group discussion to come to an agreement for the definition of a linear function.*
4 mins	*Introduction video* www.ucl.ac.uk/ioe/research/projects/cornerstone-maths/units-pd-toolkit/linear-functions	*Everyone to watch the video, which gives an overview of the unit on linear functions.*
5 mins	*Yari the yellow school bus*	*Hand out the student booklets.* *Show the video of Yari and complete together as a group, giving people time to look through the booklets.*
10 mins	*PD Task 2*	*Project on IWB activity 3.3 and play the demo a few times. Get teachers to focus on different dynamic aspects each time.* *Get teachers in groups onto the software and ask them to complete PD Task 2.* *Feedback and facilitate discussion.*
10 mins	*PD Task 3*	*Project slide 14 of 'linear functions initial PD training' and ask everyone to complete the PD task 3.* *Again, facilitate a discussion.*
20 mins	*Hands-on, up to and including investigation 3.3*	*Hand out the teacher booklet.* *In pairs or as a three, teachers to work through the first four investigations and filling out the student book.* *Stop at relevant moments to discuss the mathematical and pedagogical intensions of each task.* *After the meeting teachers are then expected to work through the rest of the investigations, up to number 7, ready for the next departmental meeting.* *(teachers might start to teach the first few investigations before the next meeting)*

Session 2: The Mathematical and pedagogical journey

Teachers are to come to this session having tried out investigation 1–7 with the student and teacher booklet.

Timings	*Activity*	*Rough outline*
10 mins	*Feedback*	*Give everyone the opportunity to briefly feed back on what they either initially think about the resources or describe any lessons where they have taught the first couple of investigations – use post-it notes if stuck for time.*
15 mins	*Mathematical journey*	*Divide into two groups* *Brainstorm collaboratively the mathematical journey pupils go on through in the first 7 investigations.* *Give just 8 minutes for discussion and then 2 minutes per group to feedback.* *Show slide 18 showing the mathematical progression and compare.*
10 mins	*Investigation 4 – landmark activity.* *Observing others*	*Give everyone time on the laptops to remind themselves of the activity.* *Show the two videos of other teachers teaching this part of the lesson and discuss as a group. Encourage teachers to think about when and how THEY could use the dynamic software as well as the students and to consider the questions they might ask.*
25 mins	*Planning investigation 4*	*Hand out the planning proforma and get teachers to complete in pairs or threes.* *Use the 'Linear functions landmark pupil work' PowerPoint to encourage people to further think about how students might react to some of the key questions and to discuss how they would deal with this.* *After this session teachers to go and teach investigation 4 and the rest of the unit when possible.*

Ongoing reflection

After all teachers have had a chance to teach get them to feed back and share some of the students' work. Discuss what went well for everyone and what were the challenges. If possible, pair people up to observe each other teach investigation 4.

Appendix F: List of project publications

Clark-Wilson, A. (2017) 'Transforming mathematics teaching with digital technologies: A community of practice perspective'. In Marcus-Quinn, A. and Hourigan, T. (eds) *Handbook on Digital Learning for K-12 Schools*. Cham: Springer, 45–57.

Clark-Wilson, A. (forthcoming) 'The multi-faceted role of technology to develop teachers' professional knowledge and practice to use dynamic technology in their lower secondary mathematics teaching'. In Kaiser, G. (ed.) *Proceedings of the 13th International Congress on Mathematical Education (ICME-13)*. Springer.

Clark-Wilson, A. and Hoyles, C. (2016) 'Designing a web-based professional development toolkit for supporting the use of dynamic technology in lower secondary mathematics'. *Proceedings of the ERME Topic Conference: Mathematics teaching, resources and teacher professional development, Humboldt-Universität zu Berlin, 5–7 October 2016*. Berlin: Humboldt-Universität zu Berlin.

Clark-Wilson, A. and Noss, R. (2015) 'Hiccups within technology mediated lessons: A catalyst for mathematics teachers' epistemological development'. *Research in Mathematics Education*, 17 (2), 92–109.

This report summarises the outcomes of the Nuffield Foundation funded 2014–17 project ‘Developing teachers’ mathematical knowledge for teaching and classroom use of technology through engagement with key mathematical concepts using dynamic digital technology’. The Nuffield Foundation is an endowed charitable trust that aims to improve social well-being in the widest sense. It funds research and innovation in education and social policy and also works to build capacity in education, science and social science research.

The Executive Summary of this report is also available as a separate publication, ISBN 9781782772248.

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Professor Dame Celia Hoyles holds a first class honours degree in mathematics and a Masters and Doctorate in mathematics education. She taught mathematics in London schools before moving into higher education, and is now Professor of Mathematics Education at the UCL Institute of Education. She was the UK Government’s Chief Adviser for Mathematics (2004–07) and Director of the National Centre for Excellence in the Teaching of Mathematics (2007–13). She was awarded an OBE in 2004 and made a Dame Commander of the Order of the British Empire in 2014.

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