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We give effective lower bounds for $L(1, \chi)$ via Eisenstein series on $\Gamma_0(q) \setminus \mathbb{H}$. The proof uses the Maass–Selberg relation for truncated Eisenstein series and sieve theory in the form of the Brun–Titchmarsh inequality. The method follows closely the work of Sarnak in using Eisenstein series to find effective lower bounds for $\zeta(1+it)$.

1. Introduction

Let q be a positive integer, let χ be a Dirichlet character modulo q, and let

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the associated Dirichlet L-function, which converges absolutely for $\Re(s) > 1$ and extends holomorphically to the entire complex plane except when χ is principal, in which case there is a simple pole at $s = 1$. It is well known that Dirichlet's theorem on the infinitude of primes in arithmetic progressions is equivalent to showing that $L(1, \chi) \neq 0$ for every Dirichlet character χ modulo q. Of further interest is obtaining lower bounds for $L(1, \chi)$ in terms of q. By complex analytic means [Montgomery and Vaughan 2007, Theorems 11.4 and 11.11], one can show that if χ is complex, then

$$|L(1, \chi)| \gg \frac{1}{\log q},$$

while

$$L(1, \chi) \gg \frac{1}{\sqrt{q}}$$

if χ is quadratic. In both cases, the implicit constants are effective. For quadratic characters, the Landau–Siegel theorem states that

$$L(1, \chi) \gg_{\varepsilon} q^{-\varepsilon}$$

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for all $\varepsilon > 0$ [Montgomery and Vaughan 2007, Theorem 11.14], though this estimate is ineffective due to the possible existence of a Landau–Siegel zero of $L(s, \chi)$.

In this article, we give a novel proof of effective lower bounds for $L(1, \chi)$, albeit in slightly weaker forms.

Theorem 1.1. *Let $q \geq 2$ be a positive integer, and let χ be a primitive character modulo q. If χ is complex, then*

$$|L(1, \chi)| \gg \frac{1}{(\log q)^3},$$

while

$$L(1, \chi) \gg \frac{1}{\sqrt{q}(\log q)^2}$$

if χ is quadratic. In both cases, the implicit constants are effective.

Our proof of Theorem 1.1 makes use of the fact that $L(s, \chi)$ appears in the Fourier expansion of an Eisenstein series associated to χ on $\Gamma_0(q) \backslash \mathbb{H}$, together with sieve theory — specifically the Brun–Titchmarsh inequality — to find these lower bounds. As is well-known, improving the constant in the Brun–Titchmarsh inequality is essentially equivalent the nonexistence of Landau–Siegel zeroes; it is for this same reason that the lower bounds in Theorem 1.1 are weak for quadratic characters, as we discuss in Remark 4.7.

That one can use Eisenstein series to prove nonvanishing of L-functions is well known, first appearing in unpublished work of Selberg, but such methods were not shown to give good effective lower bounds for L-functions on the line $\Re(s) = 1$ until the work of Sarnak [2004]. He showed that

$$|\zeta(1+it)| \gg \frac{1}{(\log |t|)^3}$$

for $|t| > 1$ by exploiting the inhomogeneous form of the Maass–Selberg relation for the Eisenstein series $E(z, s)$ for the group $\mathrm{SL}_2(\mathbb{Z})$.

More precisely, for $t > 1$, Sarnak studied the integral

$$\mathcal{I} := \int_{1/t}^{\infty} \int_0^1 |\zeta(1+2it)|^2 \left| \Lambda^t \left(z, \frac{1}{2} + it \right) \right|^2 \frac{dx\, dy}{y^2},$$

involving a truncated Eisenstein series $\Lambda^T E(z, s)$ and found an upper bound up to a scalar multiple for this integral of the form

$$t (\log t)^2 |\zeta(1+2it)|$$

via the Maass–Selberg relation, and a lower bound up to a scalar multiple of the form

$$\frac{1}{t}\sum_{\frac{t^2}{8}\le m\le\frac{t^2}{4}}|\sigma_{-2it}(m)|^2$$

via Parseval's identity, where

$$\sigma_{-2it}(m):=\sum_{d\mid m}d^{-2it}.$$

By restricting the summation over m to primes, Sarnak was able to use sieve theory to show that

$$\sum_{\frac{t^2}{8}\le p\le\frac{t^2}{4}}|\sigma_{-2it}(p)|^2\gg\frac{t^2}{\log t},$$

from which the result follows. Indeed, the use of sieve theory to prove lower bounds for $\zeta(1+it)$ (and also $L(1+it,\chi)$) has its roots in work of Balasubramanian and Ramachandra [1976].

The chief novelty of Sarnak's work is to use the Maass–Selberg relation to obtain effective lower bounds for $\zeta(1+it)$; more precisely, it is the inhomogeneous nature of the Fourier expansion of the Eisenstein series $E(z,s)$, whose constant term involves $\zeta(2s-1)/\zeta(2s)$ and whose nonconstant terms involve $1/\zeta(2s)$. This method has been generalized by Gelbart and Lapid [2006] to determine effective lower bounds on the line $\Re(s)=1$ for L-functions associated to automorphic representations on arbitrary reductive groups over number fields, albeit with the lower bound being in the weaker form $C|t|^{-n}$ for some constants C, n depending on the L-function, for Gelbart and Lapid make no use of sieve theory in this generalized setting. More recently, Goldfeld and Li [2016] have succeeded in generalizing Sarnak's method to show that

$$|L(1+it,\pi\times\widetilde{\pi})|\gg_\pi\frac{1}{(\log|t|)^3}$$

for any cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ that is unramified and tempered at every place, with the implicit constant in the lower bound dependent on π.

All three of these results give lower bounds for L-functions on the line $\Re(s)=1$ in the height aspect, namely in terms of t. In this article, we give the first example of Sarnak's method being used to give lower bounds for L-functions on the line $\Re(s)=1$ in the level aspect, namely in terms of q.

2. Eisenstein series

We introduce Eisenstein series for the group $\Gamma_0(q)$ associated to a primitive Dirichlet character χ modulo q. Standard references for this material are [Deshouillers and Iwaniec 1982], [Duke et al. 2002], and [Iwaniec 2002].

Cusps. Let $\mathbb{H}$ be the upper half plane, upon which $\mathrm{SL}_2(\mathbb{R})$ acts via Möbius transformations $\gamma z = (az+b)/(cz+d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$. Let q be a positive integer, and let $\mathfrak{a}$ be a cusp of $\Gamma_0(q) \backslash \mathbb{H}$, where

$$\Gamma_0(q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod q \right\},$$

and we denote the stabilizer of $\mathfrak{a}$ by $\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma_0(q) : \gamma\mathfrak{a} = \mathfrak{a}\}$. This subgroup of $\Gamma_0(q)$ is generated by two parabolic elements $\pm\gamma_{\mathfrak{a}}$, where

$$\gamma_{\mathfrak{a}} := \sigma_{\mathfrak{a}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_{\mathfrak{a}}^{-1},$$

and the scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ is such that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \qquad \sigma_{\mathfrak{a}}^{-1}\Gamma_\infty\sigma_{\mathfrak{a}} = \Gamma_\infty,$$

where

$$\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma_0(q) : n \in \mathbb{Z} \right\}$$

is the stabilizer of the cusp at infinity. The scaling matrix is unique up to translation on the right.

Let χ be a primitive character modulo q. A cusp $\mathfrak{a}$ of $\Gamma_0(q) \backslash \mathbb{H}$ is said to be singular with respect to χ if $\chi(\gamma_{\mathfrak{a}}) = 1$, where $\chi(\gamma) := \chi(d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$. As χ is primitive, any singular cusp is equivalent to $1/v$ for a single unique divisor v of q satisfying $vw = q$ and $(v, w) = 1$, where w is the width of the cusp; when $v = q$, this cusp is equivalent to the cusp at infinity, while when $v = 1$, the cusp is equivalent to the cusp at zero. Note that if $q = 1$, so that χ is the trivial character, there is merely a single equivalence class of cusps, namely the cusp at infinity.

The scaling matrix $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ for a singular cusp $\mathfrak{a} \sim 1/v,\ v \neq q$, can be chosen to be

$$\sigma_{\mathfrak{a}} := \begin{pmatrix} \sqrt{w} & 0 \\ v\sqrt{w} & 1/\sqrt{w} \end{pmatrix},$$

while for the cusp at infinity, we simply take σ_∞ to be the identity.

The Bruhat decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}}$ [Iwaniec 2002, Theorem 2.7] states that

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a}\mathfrak{b}}\Omega_\infty \sqcup \bigsqcup_{c>0} \bigsqcup_{d \,(\mathrm{mod}\, c)} \Omega_{d/c},$$

where $\delta_{\mathfrak{a}\mathfrak{b}} = 1$ if $\mathfrak{a} \sim \mathfrak{b}$ and 0 otherwise, and

$$\Omega_\infty := \Gamma_\infty \omega_\infty, \qquad \omega_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}},$$

$$\Omega_{d/c} := \Gamma_\infty \omega_{d/c} \Gamma_\infty, \qquad \omega_{d/c} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}} \quad \text{with } c > 0,$$

and c, d run over all real numbers such that $\sigma_{\mathfrak{a}}^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}}$ contains $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$. In particular, for the cusp at infinity we have the Bruhat decomposition

$$\sigma_\infty^{-1} \Gamma_0(q) \sigma_\infty = \Gamma_\infty \sqcup \bigsqcup_{\substack{c=1 \\ c \equiv 0 \ (\mathrm{mod}\ q)}}^{\infty} \ \bigsqcup_{\substack{d \ (\mathrm{mod}\ c) \\ (c,d)=1}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty.$$

For $\mathfrak{a} \sim \infty$ and $\mathfrak{b} \sim 1/v$ a nonequivalent singular cusp with $1 \le v < q$, v dividing q, $vw = q$, and $(v, w) = 1$, and for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$, we have that

$$\sigma_\infty^{-1} \gamma \sigma_{\mathfrak{b}} = \begin{pmatrix} (a + bv)\sqrt{w} & b/\sqrt{w} \\ (c + dv)\sqrt{w} & d/\sqrt{w} \end{pmatrix},$$

and so

$$\text{(2.1)} \quad \sigma_\infty^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}} = \left\{ \begin{pmatrix} a\sqrt{w} & b/\sqrt{w} \\ c\sqrt{w} & d/\sqrt{w} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \right.$$
$$\left. c \equiv 0 \ (\mathrm{mod}\ v),\ d \equiv c/v \ (\mathrm{mod}\ w),\ (c, d) = 1,\ (c, w) = 1 \right\}.$$

So the Bruhat decomposition in this case can be explicitly written in the form

$$\text{(2.2)} \qquad \sigma_\infty^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}} = \bigsqcup_{\substack{c=1 \\ (c,w)=1 \\ c \equiv 0 \ (\mathrm{mod}\ v)}}^{\infty} \ \bigsqcup_{\substack{d \ (\mathrm{mod}\ cw) \\ (cw,d)=1 \\ d \equiv c/v \ (\mathrm{mod}\ w)}} \Gamma_\infty \begin{pmatrix} * & * \\ c\sqrt{w} & d/\sqrt{w} \end{pmatrix} \Gamma_\infty.$$

Eisenstein series. Given a primitive Dirichlet character χ modulo q and a singular cusp $\mathfrak{a}$ of $\Gamma_0(q) \backslash \mathbb{H}$, we define the Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ for $z \in \mathbb{H}$ and $\Re(s) > 1$ by

$$E_{\mathfrak{a}}(z, s, \chi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(q)} \bar{\chi}(\gamma) j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-\kappa} \Im(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s,$$

where $\kappa \in \{0, 1\}$ is such that $\chi(-1) = (-1)^\kappa$, and for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$,

$$j_\gamma(z) := \frac{cz + d}{|cz + d|} = e^{i \arg(cz+d)}.$$

The Eisenstein series associated to a singular cusp $\mathfrak{a}$ is independent of the choice of representative of $\mathfrak{a}$ and of the scaling matrix $\sigma_{\mathfrak{a}}$. For fixed $z \in \mathbb{H}$, the Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ converges absolutely for $\Re(s) > 1$ and extends meromorphically to the entire complex plane with no poles on the closed right half-plane $\Re(s) \geq \frac{1}{2}$ except at $s = 1$ when $q = 1$, so that χ is the trivial character.

For any $z \in \mathbb{H}$ and $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$, the j-factor satisfies the cocycle relation

$$j_{\gamma_1\gamma_2}(z) = j_{\gamma_2}(z) j_{\gamma_1}(\gamma_2 z), \tag{2.3}$$

while the Eisenstein series satisfies the automorphy condition

$$E_{\mathfrak{a}}(\gamma z, s, \chi) = \chi(\gamma) j_{\gamma}(z)^{\kappa} E_{\mathfrak{a}}(z, s, \chi) \tag{2.4}$$

for any $\gamma \in \Gamma_0(q)$.

For any singular cusps $\mathfrak{a}$, $\mathfrak{b}$ of $\Gamma_0(q)$, one can show using the Bruhat decomposition that there exists a function $\varphi_{\mathfrak{ab}}(s, \chi)$ such that the constant term in the Fourier expansion for the function $j_{\sigma_{\mathfrak{b}}}(z)^{-\kappa} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi)$ is

$$c_{\mathfrak{ab}}(z, s, \chi) := \int_0^1 j_{\sigma_{\mathfrak{b}}}(z)^{-\kappa} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi)\, dx = \delta_{\mathfrak{ab}} y^s + \varphi_{\mathfrak{ab}}(s, \chi) y^{1-s}.$$

The functions $\varphi_{\mathfrak{ab}}(s, \chi)$ are the entries of the scattering matrix associated to χ. We will calculate $\varphi_{\mathfrak{ab}}(s, \chi)$ when $\mathfrak{a} \sim \infty$ for each nonsingular cusp $\mathfrak{b}$ of $\Gamma_0(q)$ with respect to χ, and also find the rest of the Fourier coefficients of $E_{\infty}(z, s, \chi)$.

Fourier expansion of $E_{\infty}(z, s, \chi)$.

Lemma 2.5. *Let χ be a primitive character modulo q. For $m \neq 0$ and $c \equiv 0 \pmod q$,*

$$\sum_{\substack{d \pmod c \\ (c,d)=1}} \chi(d) e\left(\frac{md}{c}\right) = \chi(\operatorname{sgn}(m)) \tau(\chi) \sum_{d \mid \left(|m|, \frac{c}{q}\right)} d \overline{\chi}\left(\frac{|m|}{d}\right) \chi\left(\frac{c}{dq}\right) \mu\left(\frac{c}{dq}\right).$$

Here, as usual, we define $e(x) := e^{2\pi i x}$ for $x \in \mathbb{R}$.

Proof. For m positive, this is [Miyake 1989, Lemma 3.1.3]. The result for m negative follows by replacing m with $|m|$ and χ with $\overline{\chi}$, then taking complex conjugates of both sides and using the fact that $\overline{\tau(\overline{\chi})} = \chi(-1)\tau(\chi)$. □

Proposition 2.6 (cf. [Iwaniec 2002, Theorem 3.4]). *The Eisenstein series associated to the cusp at infinity has the Fourier expansion*

$$E_{\infty}(z,s,\chi) = y^s + \varphi_{\infty\infty}(s,\chi) y^{1-s} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \rho_{\infty}(m,s,\chi) W_{\operatorname{sgn}(m)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi |m| y) e(mx),$$

where $W_{\alpha,\nu}(y)$ is the Whittaker function,

$$\varphi_{\infty\infty}(s,\chi)=\begin{cases}\sqrt{\pi}\dfrac{\Gamma\big(s-\frac{1}{2}\big)}{\Gamma(s)}\dfrac{\zeta(2s-1)}{\zeta(2s)} & \text{if } q=1,\\ 0 & \text{if } q\geq 2,\end{cases}$$

and for $m\neq 0$,

$$\rho_{\infty}(m,s,\chi)=\frac{\chi(\operatorname{sgn}(m))i^{-\kappa}\tau(\overline{\chi})\pi^{s}|m|^{s-1}}{q^{2s}\Gamma\big(s+\operatorname{sgn}(m)\frac{\kappa}{2}\big)L(2s,\overline{\chi})}\sigma_{1-2s}(|m|,\chi),$$

where $\tau(\chi)$ is the Gauss sum of χ and

$$\sigma_s(m,\chi):=\sum_{d\mid m}d^{s}\chi\Big(\frac{m}{d}\Big).$$

Note in particular that if $\kappa=0$, so that χ is even, the Whittaker function is simply

$$W_{0,s-\frac{1}{2}}(4\pi|m|y)=\sqrt{4|m|y}\,K_{s-\frac{1}{2}}(2\pi|m|y),$$

where $K_\nu(y)$ is the K-Bessel function. On the other hand, if $\kappa=1$, so that χ is odd, and we set $s=\frac{1}{2}$, then

$$W_{\operatorname{sgn}(m)\frac{\kappa}{2},0}(4\pi|m|y)=\begin{cases}\sqrt{4\pi|m|y}\,e^{-2\pi|m|y} & \text{if } m>0,\\ \sqrt{4\pi|m|y}\,e^{2\pi|m|y}\int_{4\pi|m|y}^{\infty}e^{-u}/u\,du & \text{if } m<0.\end{cases}$$

Proof. Via the Bruhat decomposition (2.2), $E_\infty(z,s,\chi)$ is equal to

$$y^{s}+\sum_{\substack{c=1\\ c\equiv 0 \pmod q}}^{\infty}\sum_{\substack{d \pmod c\\ (c,d)=1}}\overline{\chi}(d)\sum_{n=-\infty}^{\infty}\left(\frac{c(z+n)+d}{|c(z+n)+d|}\right)^{-\kappa}\frac{y^{s}}{|c(z+n)+d|^{2s}}.$$

So if $m=0$, the zeroth Fourier coefficient of $E_\infty(z,s,\chi)$ is

$$y^{s}+\sum_{\substack{c=1\\ c\equiv 0 \pmod q}}^{\infty}\sum_{\substack{d \pmod c\\ (c,d)=1}}\overline{\chi}(d)\int_{-\infty}^{\infty}\left(\frac{cz+d}{|cz+d|}\right)^{-\kappa}\frac{y^{s}}{|cz+d|^{2s}}\,dx$$

$$=y^{s}+y^{1-s}\int_{-\infty}^{\infty}\left(\frac{t+i}{|t+i|}\right)^{-\kappa}\frac{1}{|t+i|^{2s}}\,dt\sum_{\substack{c=1\\ c\equiv 0 \pmod q}}^{\infty}\frac{1}{c^{2s}}\sum_{\substack{d \pmod c\\ (c,d)=1}}\overline{\chi}(d)$$

by the change of variables $x\mapsto yt-d/c$. From [Gradshteyn and Ryzhik 2007, (8.381.1)], we have that

$$\int_{-\infty}^{\infty}\left(\frac{t+i}{|t+i|}\right)^{-\kappa}\frac{1}{|t+i|^{2s}}\,dt=i^{-\kappa}\sqrt{\pi}\frac{\Gamma\big(\frac{1}{2}(2s-1+\kappa)\big)}{\Gamma\big(\frac{1}{2}(2s+\kappa)\big)},$$

while for $c \equiv 0 \pmod{q}$, the fact that χ is primitive implies that

$$\sum_{\substack{c=1 \\ c\equiv 0 \pmod{q}}}^{\infty} \frac{1}{c^{2s}} \sum_{\substack{d \pmod{c} \\ (c,d)=1}} \overline{\chi}(d) = \begin{cases} \displaystyle\sum_{c=1}^{\infty} \frac{\varphi(c)}{c^{2s}} = \frac{\zeta(2s-1)}{\zeta(2s)} & \text{if } q = 1, \\ 0 & \text{if } q \geq 2. \end{cases}$$

If $m \neq 0$, on the other hand, then the m-th Fourier coefficient is

$$\sum_{\substack{c=1 \\ c\equiv 0 \pmod{q}}}^{\infty} \sum_{\substack{d \pmod{c} \\ (c,d)=1}} \overline{\chi}(d) \int_{-\infty}^{\infty} \left(\frac{cz+d}{|cz+d|}\right)^{-\kappa} \frac{y^s}{|cz+d|^{2s}} e(-mx)\, dx$$
$$= y^{1-s} \int_{-\infty}^{\infty} \left(\frac{t+i}{|t+i|}\right)^{-\kappa} \frac{e(-myt)}{|t+i|^{2s}}\, dt \sum_{\substack{c=1 \\ c\equiv 0 \pmod{q}}}^{\infty} \frac{1}{c^{2s}} \sum_{\substack{d \pmod{c} \\ (c,d)=1}} \overline{\chi}(d) e\left(\frac{md}{c}\right)$$

again by the change of variables $x \mapsto yt - d/c$. Moreover, [Gradshteyn and Ryzhik 2007, (3.384.9)] implies that

$$\int_{-\infty}^{\infty} \left(\frac{t+i}{|t+i|}\right)^{-\kappa} \frac{e(-myt)}{|t+i|^{2s}}\, dt = \frac{i^{-\kappa} \pi^s |m|^{s-1} y^{s-1}}{\Gamma\left(s + \operatorname{sgn}(m)\frac{\kappa}{2}\right)} W_{\operatorname{sgn}(m)\frac{\kappa}{2}, s-\frac{1}{2}}(4\pi |m| y),$$

and via Lemma 2.5,

$$\sum_{\substack{c=1 \\ c\equiv 0 \pmod{q}}}^{\infty} \frac{1}{c^{2s}} \sum_{\substack{d \pmod{c} \\ (c,d)=1}} \overline{\chi}(d) e\left(\frac{md}{c}\right)$$
$$= \chi(\operatorname{sgn}(m)) \tau(\overline{\chi}) \sum_{d \mid |m|} d \chi\left(\frac{|m|}{d}\right) \sum_{\substack{c=1 \\ c\equiv 0 \pmod{dq}}}^{\infty} \frac{\overline{\chi}\left(\frac{c}{dq}\right) \mu\left(\frac{c}{dq}\right)}{c^{2s}}$$
$$= \chi(\operatorname{sgn}(m)) \frac{\tau(\overline{\chi})}{q^{2s}} \sum_{d \mid |m|} d^{1-2s} \chi\left(\frac{|m|}{d}\right) \sum_{n=1}^{\infty} \frac{\overline{\chi}(n) \mu(n)}{n^{2s}}$$
$$= \chi(\operatorname{sgn}(m)) \frac{\tau(\overline{\chi})}{q^{2s} L(2s, \overline{\chi})} \sigma_{1-2s}(|m|, \chi)$$

where we have let $c = dqn$. We thereby obtain the desired identity. □

Proposition 2.7. *Suppose that $q \geq 2$. Then $\varphi_{\infty\mathfrak{b}}(s, \chi)$ vanishes unless $\mathfrak{b} \sim 1$, in which case*

$$\varphi_{\infty 1}(s, \chi) = \frac{\overline{\tau(\chi)}}{q^s} \frac{\Lambda(2-2s, \chi)}{\Lambda(2s, \overline{\chi})}, \tag{2.8}$$

where

$$\Lambda(s,\chi) := \left(\frac{\pi}{q}\right)^{-\frac{s+\kappa}{2}} \Gamma\left(\frac{s+\kappa}{2}\right) L(s,\chi), \tag{2.9}$$

is the completed Dirichlet L-function. In particular,

$$\left|\varphi_{\infty 1}\left(\tfrac{1}{2}+it,\chi\right)\right| = 1. \tag{2.10}$$

Proof. The fact that $\varphi_{\infty\mathfrak{b}}(s,\chi)=0$ when $\mathfrak{b}$ is the cusp at infinity follows from Proposition 2.6. For the entries of the scattering matrix at other cusps, we use (2.3) to write

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = j_{\sigma_{\mathfrak{b}}}(z)^{\kappa} \sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}}} \overline{\chi}(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}) j_\gamma(z)^{-\kappa}\Im(\gamma z)^s.$$

The singular cusp $\mathfrak{b}$ is equivalent to $1/v$ for some divisor v of q with $v<q$, $vw=q$, and $(v,w)=1$. Given a matrix

$$\gamma = \begin{pmatrix} a\sqrt{w} & b/\sqrt{w} \\ c\sqrt{w} & d/\sqrt{w} \end{pmatrix}$$

in $\sigma_\infty^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}}$ as in (2.1), we have that

$$\sigma_\infty\gamma\sigma_{\mathfrak{b}}^{-1} = \begin{pmatrix} a-bv & b \\ c-dv & d \end{pmatrix},$$

and so as $d\equiv c/v \pmod{w}$,

$$\overline{\chi}(\sigma_\infty\gamma\sigma_{\mathfrak{b}}^{-1}) = \overline{\chi_v}(d)\,\overline{\chi_w}\left(\frac{c}{v}\right),$$

where we have decomposed the primitive character χ modulo q into the product of primitive characters χ_v modulo v and χ_w modulo w. From this and (2.2), we see that $j_{\sigma_{\mathfrak{b}}}(z)^{-\kappa}E_\infty(\sigma_{\mathfrak{b}}z,s,\chi)$ is equal to

$$\sum_{\substack{c=1\\(c,w)=1\\c\equiv 0 \ (\mathrm{mod}\ v)}}^{\infty} \overline{\chi_w}\left(\frac{c}{v}\right) \sum_{\substack{d \ (\mathrm{mod}\ cw)\\(cw,d)=1\\d\equiv c/v \ (\mathrm{mod}\ w)}} \overline{\chi_v}(d) \times \sum_{n=-\infty}^{\infty}\left(\frac{c(z+n)\sqrt{w}+d/\sqrt{w}}{|c(z+n)\sqrt{w}+d/\sqrt{w}|}\right)^{-\kappa}\frac{y^s}{|c(z+n)\sqrt{w}+d/\sqrt{w}|^{2s}},$$

and so integrating from 0 to 1 with respect to x, making the change of variables $x \mapsto yt - d/(cw)$, and dividing by y^{1-s} yields

$$\varphi_{\infty\mathfrak{b}}(s,\chi) = \frac{1}{w^s}\int_{-\infty}^{\infty}\left(\frac{t+i}{|t+i|}\right)^{-\kappa}\frac{1}{|t+i|^{2s}}\,dt \sum_{\substack{c=1\\(c,w)=1\\c\equiv 0 \pmod{v}}}^{\infty}\frac{\overline{\chi_w}(c/v)}{c^{2s}}\sum_{\substack{d \pmod{cw}\\(cw,d)=1\\d\equiv c/v \pmod{w}}}\overline{\chi_v}(d).$$

From [Gradshteyn and Ryzhik 2007, (8.381.1)], the integral is equal to

$$i^{-\kappa}\sqrt{\pi}\,\frac{\Gamma\left(\frac{1}{2}(2s-1+\kappa)\right)}{\Gamma\left(\frac{1}{2}(2s+\kappa)\right)}.$$

To evaluate the sum over d, we write $d = \bar{v}c + wd'$, where $\bar{v}v \equiv 1 \pmod{w}$ and $(d', c) = 1$. This allows us to replace the sum over d with a sum over d' modulo c with $(c, d') = 1$, so that

$$\sum_{\substack{d \pmod{cw}\\(cw,d)=1\\d\equiv c/v \pmod{w}}}\overline{\chi_v}(d) = \overline{\chi_v}(w)\sum_{\substack{d' \pmod{c}\\(c,d')=1}}\overline{\chi_v}(d')$$

by the fact that $c \equiv 0 \pmod{v}$.

If $\overline{\chi_v}$ is nonprincipal, this sum vanishes, and as χ is a primitive character, $\overline{\chi_v}$ can only be the principal character if $v = 1$; consequently, $\varphi_{\infty\mathfrak{b}}(s,\chi)$ vanishes if $\mathfrak{b}$ is inequivalent to the cusp at 1.

If $\mathfrak{b} \sim 1$, so that $v = 1$ and $w = q$, then this sum over d' is merely $\varphi(c)$, and so

$$\sum_{\substack{c=1\\(c,w)=1\\c\equiv 0 \pmod{v}}}^{\infty}\frac{\overline{\chi_w}(c/v)}{c^{2s}}\sum_{\substack{d \pmod{cw}\\(cw,d)=1\\d\equiv c/v \pmod{w}}}\overline{\chi_v}(d) = \sum_{c=1}^{\infty}\frac{\varphi(c)\overline{\chi}(c)}{c^{2s}} = \frac{L(2s-1,\overline{\chi})}{L(2s,\overline{\chi})}.$$

Using the definition of the completed Dirichlet L-function together with the fact that it satisfies the functional equation

$$\Lambda(s,\chi) = \frac{\tau(\chi)}{i^{\kappa}\sqrt{q}}\Lambda(1-s,\overline{\chi}),$$

we see that we may write

$$\varphi_{\infty 1}(s,\chi) = \frac{i^{-\kappa}}{q^{s-\frac{1}{2}}}\frac{\Lambda(2s-1,\overline{\chi})}{\Lambda(2s,\overline{\chi})} = \frac{\overline{\tau(\chi)}}{q^s}\frac{\Lambda(2-2s,\chi)}{\Lambda(2s,\overline{\chi})}.$$

As $\overline{\Lambda(s,\chi)} = \Lambda(\bar{s},\overline{\chi})$ and $|\tau(\chi)| = \sqrt{q}$, the result follows. □

3. Maass–Selberg relation

For $z \in \mathbb{H}$ and $T \geq 1$, we define the truncated Eisenstein series

$$\Lambda^T E_{\mathfrak{a}}(z,s,\chi) := E_{\mathfrak{a}}(z,s,\chi) - \sum_{\mathfrak{c}} \sum_{\substack{\gamma \in \Gamma_{\mathfrak{c}} \backslash \Gamma_0(q) \\ \Im(\sigma_{\mathfrak{c}}^{-1}\gamma z) > T}} \overline{\chi}(\gamma) j_{\sigma_{\mathfrak{c}}^{-1}\gamma}(z)^{-\kappa} c_{\mathfrak{a}\mathfrak{c}}(\sigma_{\mathfrak{c}}^{-1}\gamma z, s, \chi), \tag{3.1}$$

where the summation over $\mathfrak{c}$ is over all singular cusps of $\Gamma_0(q) \backslash \mathbb{H}$. It is not difficult to see that $\Lambda^T E_{\mathfrak{a}}(z,s,\chi)$ satisfies the automorphy condition

$$\Lambda^T E_{\mathfrak{a}}(\gamma z, s, \chi) = \chi(\gamma) j_\gamma(z)^\kappa \Lambda^T E_{\mathfrak{a}}(z,s,\chi) \tag{3.2}$$

for any $\gamma \in \Gamma_0(q)$. We will show that, unlike $E_{\mathfrak{a}}(z,s,\chi)$, the function $\Lambda^T E_{\mathfrak{a}}(z,s,\chi)$ is square-integrable on $\Gamma_0(q) \backslash \mathbb{H}$, and give an explicit expression for the resulting integral.

Lemma 3.3. *Let $\mathfrak{b}$ and $\mathfrak{c}$ be singular cusps of $\Gamma_0(q) \backslash \mathbb{H}$, and let $\gamma \in \sigma_{\mathfrak{c}}^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}}$. Then for any $z = x + iy \in \mathbb{H}$, we have that $\Im(z)\Im(\gamma z) \leq 1$ if $\mathfrak{b}$ and $\mathfrak{c}$ are inequivalent or if $\mathfrak{b}$ and $\mathfrak{c}$ are equivalent but $\gamma \notin \Gamma_\infty\omega_\infty$. If $\mathfrak{b}$ and $\mathfrak{c}$ are equivalent and $\gamma \in \Gamma_\infty\omega_\infty$, then $\Im(\gamma z) = \Im(z)$.*

Proof. We deal with the cases where neither $\mathfrak{b}$ nor $\mathfrak{c}$ are equivalent to the cusp at infinity; when $\mathfrak{b} \sim \infty$ or $\mathfrak{c} \sim \infty$, the proof is similar but simpler. Let $\mathfrak{b} \sim 1/v$ and $\mathfrak{c} \sim 1/v'$, $1 \leq v, v' < q$, with w, w' such that $vw = v'w' = q$. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)$, we have that

$$\sigma_{\mathfrak{c}}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sigma_{\mathfrak{b}} = \begin{pmatrix} (a+bv)\sqrt{w/w'} & b/\sqrt{w'w} \\ (c - av' + dv - bv'v)\sqrt{w'w} & (d - bv')\sqrt{w'/w} \end{pmatrix}.$$

So for

$$\gamma = \begin{pmatrix} * & * \\ C\sqrt{w'w} & D\sqrt{w'/w} \end{pmatrix} \in \sigma_{\mathfrak{c}}^{-1}\Gamma_0(q)\sigma_{\mathfrak{b}},$$

where $C = c - av' + dv - bv'v$ and $D = d - bv'$ are integers, we have that

$$\Im(\gamma z) = \frac{1}{w'w} \frac{y}{(Cx + Dw^{-1})^2 + C^2y^2}.$$

By the Bruhat decomposition, if $\mathfrak{b}$ and $\mathfrak{c}$ are inequivalent, then $C\sqrt{w'w}$ must be nonzero, and so $C^2 \geq 1$. In particular, if $\mathfrak{b}$ and $\mathfrak{c}$ are inequivalent, then

$$\Im(z)\Im(\gamma z) \leq \frac{1}{w'w} \leq 1.$$

If $\mathfrak{b}$ and $\mathfrak{c}$ are equivalent and $\gamma \notin \Gamma_\infty\omega_\infty$, then again $C\sqrt{w'w} \neq 0$, and the same result holds. Finally, if $\mathfrak{b}$ and $\mathfrak{c}$ are equivalent and $\gamma \in \Gamma_\infty\omega_\infty$, then it is clear that $\Im(\gamma z) = \Im(z)$. □

Corollary 3.4. *If $\Im(z) > T \geq 1$, then for any singular cusp $\mathfrak{b}$, we have that*

$$\Lambda^T E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) - j_{\sigma_{\mathfrak{b}}}(z)^{\kappa} c_{\mathfrak{a}\mathfrak{b}}(z, s, \chi).$$

Proof. From the definition of $\Lambda^T E_{\mathfrak{a}}(z, s, \chi)$ and (2.3), we must show that for any singular cusp $\mathfrak{c}$ and $\gamma \in \Gamma_{\mathfrak{c}} \backslash \Gamma_0(q)$ that the inequalities $\Im(z) > T$ and $\Im(\sigma_{\mathfrak{c}}^{-1} \gamma \sigma_{\mathfrak{b}} z) > T$ are simultaneously satisfied only when $\mathfrak{c} \sim \mathfrak{b}$ and $\gamma = \omega_\infty$. This is equivalent to showing that if $\gamma \in \Gamma_\infty \backslash \sigma_{\mathfrak{c}}^{-1} \Gamma_0(q) \sigma_{\mathfrak{b}}$ is such that $\Im(z) > T$ and $\Im(\gamma z) > T$, then $\mathfrak{c} \sim \mathfrak{b}$ and $\gamma = \omega_\infty$, which follows immediately from Lemma 3.3. $\square$

With these results in hand, we can prove the following Maass–Selberg relation.

Proposition 3.5. *For any two singular cusps $\mathfrak{a}, \mathfrak{b}$, $T \geq 1$, and $s \neq \bar{r}$, $s + \bar{r} \neq 1$,*

$$\int_{\Gamma_0(q)\backslash\mathbb{H}} \Lambda^T E_{\mathfrak{a}}(z, s, \chi) \overline{\Lambda^T E_{\mathfrak{b}}(z, r, \chi)} \, d\mu(z) = \overline{\varphi_{\mathfrak{b}\mathfrak{a}}(r, \chi)} \frac{T^{s-\bar{r}}}{s - \bar{r}} + \varphi_{\mathfrak{a}\mathfrak{b}}(s, \chi) \frac{T^{\bar{r}-s}}{\bar{r} - s} + \delta_{\mathfrak{a}\mathfrak{b}} \frac{T^{s+\bar{r}-1}}{s + \bar{r} - 1} + \sum_{\mathfrak{c}} \varphi_{\mathfrak{a}\mathfrak{c}}(s, \chi) \overline{\varphi_{\mathfrak{b}\mathfrak{c}}(r, \chi)} \frac{T^{1-s-\bar{r}}}{1 - s - \bar{r}},$$

where the sum is over singular cusps $\mathfrak{c}$. Here $d\mu(z) = dx\, dy/y^2$ is the $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathbb{H}$.

Proof. We initially assume that $\Re(s), \Re(r) > 1$ with $\Re(s) - \Re(r) > 1$; the identity then extends to all $s, r \in \mathbb{C}$ with $s \neq \bar{r}$ and $s + \bar{r} \neq 1$ by analytic continuation.

We first show that

$$\int_{\Gamma_0(q)\backslash\mathbb{H}} \Lambda^T E_{\mathfrak{a}}(z, s, \chi) \left(\overline{\Lambda^T E_{\mathfrak{b}}(z, r, \chi)} - \overline{E_{\mathfrak{b}}(z, r, \chi)} \right) d\mu(z) = 0.$$

Indeed, the left-hand side is equal to

$$\sum_{\mathfrak{c}} \int_{\Gamma_0(q)\backslash\mathbb{H}} \Lambda^T E_{\mathfrak{a}}(z, s, \chi) \sum_{\substack{\gamma \in \Gamma_{\mathfrak{c}} \backslash \Gamma_0(q) \\ \Im(\sigma_{\mathfrak{c}}^{-1} \gamma z) > T}} \overline{\chi(\gamma) j_{\sigma_{\mathfrak{c}}^{-1} \gamma}(z)^{-\kappa} c_{\mathfrak{b}\mathfrak{c}}(\sigma_{\mathfrak{c}}^{-1} \gamma z, r, \chi)} \, d\mu(z),$$

which, by (2.3) and (3.2), is equal to

$$-\sum_{\mathfrak{c}} \int_{\Gamma_0(q)\backslash\mathbb{H}} \sum_{\substack{\gamma \in \Gamma_{\mathfrak{c}} \backslash \Gamma_0(q) \\ \Im(\sigma_{\mathfrak{c}}^{-1} \gamma z) > T}} \overline{c_{\mathfrak{b}\mathfrak{c}}(\sigma_{\mathfrak{c}}^{-1} \gamma z, r, \chi)} j_{\sigma_{\mathfrak{c}}}(\sigma_{\mathfrak{c}}^{-1} \gamma z)^{-\kappa} \Lambda^T E_{\mathfrak{a}}(\gamma z, s, \chi) \, d\mu(z),$$

and this integral can be unfolded to yield

$$-\sum_{\mathfrak{c}} \int_T^\infty \int_0^1 \overline{c_{\mathfrak{b}\mathfrak{c}}(z, r, \chi)} j_{\sigma_{\mathfrak{c}}}(z)^{-\kappa} \Lambda^T E_{\mathfrak{a}}(\sigma_{\mathfrak{c}} z, s, \chi) \frac{dx\, dy}{y^2}.$$

But $\overline{c_{\mathfrak{bc}}(z,r,\chi)}$ is independent of x, while for $\Im(z) > T \geq 1$, the zeroth Fourier coefficient of the function $j_{\sigma_\mathfrak{c}}(z)^{-\kappa}\Lambda^T E_\mathfrak{a}(\sigma_\mathfrak{c} z, s, \chi)$ vanishes via Corollary 3.4, and so this vanishes. Consequently,

$$\int_{\Gamma_0(q)\backslash\mathbb{H}} \Lambda^T E_\mathfrak{a}(z,s,\chi)\overline{\Lambda^T E_\mathfrak{b}(z,r,\chi)}\,d\mu(z) = \int_{\Gamma_0(q)\backslash\mathbb{H}} \Lambda^T E_\mathfrak{a}(z,s,\chi)\overline{E_\mathfrak{b}(z,r,\chi)}\,d\mu(z).$$

The right-hand side can be written as

$$\begin{aligned}
&\int_{\Gamma_0(q)\backslash\mathbb{H}}\Bigg(\sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma_0(q)} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-\kappa}\Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s\,\overline{E_\mathfrak{b}(z,r,\chi)}\\
&-\sum_{\mathfrak{c}}\sum_{\substack{\gamma\in\Gamma_\mathfrak{c}\backslash\Gamma_0(q)\\ \Im(\sigma_\mathfrak{c}^{-1}\gamma z)>T}} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{c}^{-1}\gamma}(z)^{-\kappa} c_{\mathfrak{ac}}(\sigma_\mathfrak{c}^{-1}\gamma z,s,\chi)\overline{E_\mathfrak{b}(z,r,\chi)}\Bigg)d\mu(z)\\
&\quad= \int_{\Gamma_0(q)\backslash\mathbb{H}} \sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma_0(q)\\ \Im(\sigma_\mathfrak{a}^{-1}\gamma z)\leq T}} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-\kappa}\Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s\,\overline{E_\mathfrak{b}(z,r,\chi)}\,d\mu(z)\\
&\qquad+\int_{\Gamma_0(q)\backslash\mathbb{H}} \sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma_0(q)\\ \Im(\sigma_\mathfrak{a}^{-1}\gamma z)>T}} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-\kappa}\varphi_{\mathfrak{aa}}(s,\chi)\Im(\sigma_\mathfrak{a}^{-1}\gamma z)^{1-s}\,\overline{E_\mathfrak{b}(z,r,\chi)}\,d\mu(z)\\
&\qquad\quad-\sum_{\mathfrak{c}\neq\mathfrak{a}}\int_{\Gamma_0(q)\backslash\mathbb{H}} \sum_{\substack{\gamma\in\Gamma_\mathfrak{c}\backslash\Gamma_0(q)\\ \Im(\sigma_\mathfrak{c}^{-1}\gamma z)>T}} \overline{\chi}(\gamma) j_{\sigma_\mathfrak{c}^{-1}\gamma}(z)^{-\kappa} c_{\mathfrak{ac}}(\sigma_\mathfrak{c}^{-1}\gamma z,s,\chi)\overline{E_\mathfrak{b}(z,r,\chi)}\,d\mu(z).
\end{aligned}$$

By (2.3) and (2.4), the first term is

$$\int_{\Gamma_0(q)\backslash\mathbb{H}} \sum_{\substack{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma_0(q)\\ \Im(\sigma_\mathfrak{a}^{-1}\gamma z)\leq T}} \Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s\,\overline{j_{\sigma_\mathfrak{a}}(\sigma_\mathfrak{a}^{-1}\gamma z)^{-\kappa} E_\mathfrak{b}(\gamma z,r,\chi)}\,d\mu(z),$$

and upon unfolding the integral, this becomes

$$\begin{aligned}
\int_0^T\int_0^1 y^s\,\overline{j_{\sigma_\mathfrak{a}}(z)^{-\kappa}E_\mathfrak{b}(\sigma_\mathfrak{a} z,r,\chi)}\,\frac{dx\,dy}{y^2} &= \int_0^T y^s\,\overline{c_{\mathfrak{ba}}(z,r,\chi)}\,\frac{dy}{y^2}\\
&= \delta_{\mathfrak{ab}}\frac{T^{s+\bar{r}-1}}{s+\bar{r}-1} + \overline{\varphi_{\mathfrak{ba}}(r,\chi)}\,\frac{T^{s-\bar{r}}}{s-\bar{r}}.
\end{aligned}$$

Similarly, the second term is

$$\int_T^\infty \varphi_{\mathfrak{aa}}(s,\chi)y^{1-s}\,\overline{c_{\mathfrak{ba}}(z,s,\chi)}\,\frac{dy}{y^2} = \delta_{\mathfrak{ab}}\varphi_{\mathfrak{ab}}(s,\chi)\frac{T^{\bar{r}-s}}{\bar{r}-s} + \varphi_{\mathfrak{aa}}(s,\chi)\overline{\varphi_{\mathfrak{ba}}(r,\chi)}\,\frac{T^{1-s-\bar{r}}}{1-s-\bar{r}},$$

and the third term is

$$-\sum_{\mathfrak{c}\neq\mathfrak{a}}\int_T^\infty c_{\mathfrak{a}\mathfrak{c}}(z,s,\chi)\overline{c_{\mathfrak{b}\mathfrak{c}}(z,r,\chi)}\,\frac{dy}{y^2} = (1-\delta_{\mathfrak{a}\mathfrak{b}})\varphi_{\mathfrak{a}\mathfrak{b}}(s,\chi)\frac{T^{\bar r-s}}{\bar r-s}+\sum_{\mathfrak{c}\neq\mathfrak{a}}\varphi_{\mathfrak{a}\mathfrak{c}}(s,\chi)\overline{\varphi_{\mathfrak{b}\mathfrak{c}}(r,\chi)}\frac{T^{1-s-\bar r}}{1-s-\bar r}.$$

Combining these identities yields the result. □

Corollary 3.6. *For $T\geq 1$ and $t\in\mathbb{R}$, we have that*

$$\int_{\Gamma_0(q)\backslash\mathbb{H}}\left|\Lambda^T E_\infty\left(z,\tfrac12+it,\chi\right)\right|^2 d\mu(z) = 2\log T-\Re\left(\frac{\varphi_{\infty 1}'}{\varphi_{\infty 1}}\left(\frac12+it,\chi\right)\right).$$

Proof. We take $\mathfrak{a}\sim\mathfrak{b}\sim\infty$ and $s=r=\frac12+it+\varepsilon$ with $\varepsilon>0$ in the Maass–Selberg relation to obtain

$$\int_{\Gamma_0(q)\backslash\mathbb{H}}\left|\Lambda^T E_\infty\left(z,\frac12+it+\varepsilon,\chi\right)\right|^2 d\mu(z)=\frac{T^{2\varepsilon}}{2\varepsilon}-\left|\varphi_{\infty 1}\left(\frac12+it+\varepsilon,\chi\right)\right|^2\frac{T^{-2\varepsilon}}{2\varepsilon}.$$

The result then follows by taking the limit as ε tends to zero and using the Taylor expansions

$$T^{2\varepsilon}=1+2\varepsilon\log T+O(\varepsilon^2),$$
$$\varphi_{\infty 1}\left(\tfrac12+it+\varepsilon,\chi\right)=\varphi_{\infty 1}\left(\tfrac12+it,\chi\right)+\varepsilon\varphi_{\infty 1}'\left(\tfrac12+it,\chi\right)+O(\varepsilon^2).$$

together with (2.10). □

Remark 3.7. This proof of the Maass–Selberg relation is via unfolding as in Section 4 of [Arthur 1980], and makes use of the Arthur truncation $\Lambda^T E_{\mathfrak{a}}(z,s,\chi)$ of the Eisenstein series $E_{\mathfrak{a}}(z,s,\chi)$ given by (3.1); compare Section 1 of the same work. One can instead prove the Maass–Selberg relation without recourse to the automorphy of the truncated Eisenstein series by only defining $\Lambda^T E_{\mathfrak{a}}(z,s,\chi)$ within a fundamental domain of $\Gamma_0(q)\backslash\mathbb{H}$. Let

$$\mathcal{F}\supset\{z\in\mathbb{H}:0<\Re(z)<1,\ \Im(z)\geq 1\}$$

be the usual fundamental domain of $\Gamma_0(q)\backslash\mathbb{H}$, and for each singular cusp $\mathfrak{a}$, we define the cuspidal zone

$$\mathcal{F}_{\mathfrak{a}}(T):=\{z\in\mathcal{F}:0<\Re(\sigma_{\mathfrak{a}}^{-1}z)<1,\ \Im(\sigma_{\mathfrak{a}}^{-1}z)\geq T\}$$

for $T\geq 1$; note that any two cuspidal zones will be disjoint provided that T is sufficiently large. Then from Lemma 3.3, we have that for $T \ge 1$,

$$\Lambda^T E_{\mathfrak{a}}(z,s,\chi) = \begin{cases} E_{\mathfrak{a}}(z,s,\chi) & \text{if } z \in \mathcal{F} \setminus \bigcup_{\mathfrak{c}} \mathcal{F}_{\mathfrak{c}}(T), \\ E_{\mathfrak{a}}(z,s,\chi) \\ \quad - \sum_{\mathfrak{c}\in A} j_{\sigma_{\mathfrak{c}}^{-1}}(z)^{-\kappa}(\delta_{\mathfrak{a}\mathfrak{c}} \Im(\sigma_{\mathfrak{c}}^{-1}z)^s + \varphi_{\mathfrak{a}\mathfrak{c}}(s,\chi)\Im(\sigma_{\mathfrak{c}}^{-1}z)^{1-s}) & \text{if } z \in \bigcap_{\mathfrak{c}\in A} \mathcal{F}_{\mathfrak{c}}(T), \end{cases}$$

where A is any subset of the set of singular cusps. The Maass–Selberg relation may then be proved using Green's theorem along the same lines as the proof of [Iwaniec 2002, Proposition 6.8].

4. Upper bounds and lower bounds for the integral $\mathcal{I}$

For $\eta \le 1$, we consider the integral

$$\mathcal{I} = \mathcal{I}(\chi,\eta,T) := \int_\eta^\infty \int_0^1 \left| \Lambda^T E_\infty\left(z, \frac{1}{2}, \chi\right)\right|^2 \frac{dx\,dy}{y^2}.$$

Our goal is to find upper and lower bounds for this integral: upper bounds via the Maass–Selberg relation and lower bounds via Parseval's identity and the Brun–Titchmarsh inequality. Combining these bounds will yield lower bounds for $L(1,\chi)$.

Upper bounds for $\mathcal{I}$.

Proposition 4.1. *For $\eta \ll 1/q$ and $T \ge 1$, we have that*

$$\mathcal{I} \ll \frac{\log q \log qT}{q\eta |L(1,\chi)|}.$$

Proof. By folding the integral, one can write

$$\mathcal{I} = \int_{\Gamma_0(q)\backslash\mathbb{H}} N_q(z,\eta) \left|\Lambda^T E_\infty\left(z, \tfrac{1}{2}, \chi\right)\right|^2 d\mu(z),$$

where for $\eta \le 1$,

$$N_q(z,\eta) := \#\{\gamma \in \Gamma_\infty \setminus \Gamma_0(q) : \Im(\gamma z) > \eta\}.$$

The Maass–Selberg relation then implies the upper bound

$$\mathcal{I} \le \sup_{z\in\Gamma_0(q)\backslash\mathbb{H}} N_q(z,\eta)\left(2\log T - \Re\left(\frac{\varphi'_{\infty 1}}{\varphi_{\infty 1}}\left(\frac{1}{2},\chi\right)\right)\right).$$

From [Iwaniec 2002, Lemma 2.10], we have the bound

$$N_q(z,\eta) < 1 + \frac{10}{q\eta}.$$

By taking logarithmic derivatives of (2.8),

$$\frac{\varphi'_{\infty 1}}{\varphi_{\infty 1}}(s,\chi) = -\log q - 2\frac{\Lambda'}{\Lambda}(2-2s,\chi) - 2\frac{\Lambda'}{\Lambda}(2s,\overline{\chi}).$$

Taking logarithmic derivatives of (2.9) and letting $s = \frac{1}{2}$ then shows that

$$\frac{\varphi'_{\infty 1}}{\varphi_{\infty 1}}\Big(\frac{1}{2},\chi\Big) = -4\Re\Big(\frac{L'}{L}(1,\chi)\Big) - 2\log q + \log 8\pi + \gamma_0 + (-1)^\kappa \frac{\pi}{2},$$

where γ_0 denotes the Euler–Mascheroni constant, and we have used the fact that

$$\frac{\Gamma'}{\Gamma}\Big(\frac{1+\kappa}{2}\Big) = -\log 8 - \gamma_0 - (-1)^\kappa \frac{\pi}{2}.$$

So if $\eta \ll 1/q$,

$$\mathcal{I} \ll \frac{(|L(1,\chi)| \log qT + |L'(1,\chi)|)}{q\eta |L(1,\chi)|}.$$

The desired upper bound then follows from the bounds

$$|L(1,\chi)| \ll \log q, \qquad |L'(1,\chi)| \ll (\log q)^2,$$

which are both easily shown via partial summation. See, for example, [Montgomery and Vaughan 2007, Lemma 10.15] for the former estimate; the latter follows by a similar argument. □

Lower bounds for $\mathcal{I}$.

Proposition 4.2. *If $T \geq 1$ and $\eta = 1/T$, we have the lower bound*

$$\mathcal{I} \gg \frac{1}{q|L(1,\chi)|^2} \sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2.$$

Proof. If $\eta = 1/T$, then Lemma 3.3 implies that

$$\Lambda^T E_\infty(z,s,\chi) = \begin{cases} E_\infty(z,s,\chi) & \text{if } 1/T < \Im(z) \leq T, \\ E_\infty(z,s,\chi) - c_{\infty\infty}(z,s,\chi) & \text{if } \Im(z) > T. \end{cases}$$

It follows that the nonzero Fourier coefficients of $\Lambda^T E_\infty(z,s,\chi)$ coincide with those of $E_\infty(z,s,\chi)$ for $\Im(z) > 1/T$. So by Parseval's identity, using the fact that $|\tau(\chi)| = \sqrt{q}$, and making the change of variables $y \mapsto y/|m|$ in the integral, we have that

$$\mathcal{I} \gg \begin{cases} \dfrac{1}{q|L(1,\chi)|^2} \displaystyle\sum_{m=1}^{\infty} |\sigma_0(m,\chi)|^2 \int_{m/T}^{\infty} |K_0(2\pi y)|^2 \frac{dy}{y} & \text{if } \kappa = 0, \\ \dfrac{1}{q|L(1,\chi)|^2} \displaystyle\sum_{m=1}^{\infty} |\sigma_0(m,\chi)|^2 \int_{m/T}^{\infty} e^{-4\pi y} \frac{dy}{y} & \text{if } \kappa = 1. \end{cases}$$

If we simply consider the contribution of the positive integers m for which $m/T \asymp 1$ — say $T \leq m \leq 2T$ — then we find that

$$\mathcal{I} \gg \frac{1}{q|L(1,\chi)|^2} \sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2,$$

as desired. □

Combining the upper and lower bounds for $\mathcal{I}$, we derive the following inequality for $L(1, \chi)$:

Corollary 4.3. *For all $T \geq q$, we have that*

$$|L(1,\chi)| \gg \frac{1}{T(\log T)^2} \sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2.$$

So to obtain lower bounds for $|L(1, \chi)|$, we must find lower bounds for

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2. \tag{4.4}$$

Sieve methods. For quadratic characters, lower bounds for (4.4) follow by restricting the sum to perfect squares.

Lemma 4.5. *If χ is a quadratic character*, *then*

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \geq (\sqrt{2}-1)\sqrt{T}.$$

Proof. We restrict the sum over m to perfect squares and use the fact that $\sigma_0(m, \chi) \geq 1$ whenever m is a perfect square in order to find that

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \geq \sum_{T \leq m^2 \leq 2T} |\sigma_0(m^2,\chi)|^2 \geq (\sqrt{2}-1)\sqrt{T}. \qquad \square$$

For complex characters, we instead restrict the sum in (4.4) to primes and use the Brun–Titchmarsh inequality to show that there are sufficiently many primes for which $\bar{\chi}(p)$ is not close to -1, so that $|\sigma_0(p, \chi)|^2$ is not small. This is a result of Balasubramanian and Ramachandra [1976, Lemma 4], who combine it with an identity of Ramanujan together with a complex analytic argument to obtain lower bounds for $L(1+it, \chi)$, and consequently derive zero-free regions for $L(s, \chi)$. We reproduce a proof of this result here for the sake of completeness.

Lemma 4.6 [Balasubramanian and Ramachandra 1976, Lemma 4]. *There exists a large constant $K \geq 2$ such that for all complex characters χ modulo q with $q \geq 2$ and for $T = q^K$,*

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \gg_K \frac{T}{\log T}.$$

Proof. We restrict the sum over m to primes p in order to find that

$$\sum_{T\le m\le 2T} |\sigma_0(m,\chi)|^2 \ge \sum_{T\le p\le 2T} |1+\chi(p)|^2 = 2\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} (1+\Re(\chi(a)))(\pi(2T;q,a)-\pi(T;q,a)),$$

where $\pi(x;q,a) := \#\{p\le x : p\equiv a \pmod q\}$.

Let Q be the order of the Dirichlet character χ; this divides $\varphi(q)$, and as χ is complex, $Q\ge 3$. For any integer M between 0 and $\lfloor Q/2\rfloor$, we have that

$$\sum_{T\le m\le 2T} |\sigma_0(m,\chi)|^2 \ge 2\Big(1+\cos\frac{2\pi M}{Q}\Big)(\pi(2T)-\pi(T)) -2\Big(1+\cos\frac{2\pi M}{Q}\Big)\sum_{\substack{a\in(\mathbb{Z}/q\mathbb{Z})^\times\\ \Re(\chi(a))<\cos\frac{2\pi M}{Q}}} (\pi(2T;q,a)-\pi(T;q,a)).$$

For the former sum, we have that for fixed $\delta>0$ to be chosen,

$$\pi(2T)-\pi(T)\ge(1-\delta)\frac{T}{\log T}$$

for all sufficiently large T dependent on δ. See, for example, [Diamond and Erdős 1980]; in particular, this does not require the full strength of the prime number theorem.

For the latter sum, we first observe that there are $\varphi(q)/Q$ reduced residue classes a modulo Q for which $\chi(a)=e^{2\pi i m/Q}$ for each integer m between 0 and $Q-1$, and so the number of reduced residue classes modulo q for which $\Re(\chi(a))<\cos(2\pi M/Q)$ is

$$\frac{\varphi(q)}{Q}\#\{M<m<Q-M\} = \varphi(q)\frac{Q-2M-1}{Q}.$$

To find an upper bound for $\pi(2T;q,a)-\pi(T;q,a)$, we use the Brun–Titchmarsh inequality, which states that for $(q,a)=1$, $x\ge 2$, and $y\ge 2q$,

$$\pi(x+y;q,a)-\pi(x;q,a)\le\frac{2y}{\varphi(q)\log y/q}\Big(1+\frac{8}{\log y/q}\Big).$$

We take $x=y=T$, assuming that $T\ge 2q$, in order to obtain

$$\sum_{\substack{a\in(\mathbb{Z}/q\mathbb{Z})^\times\\ \Re(\chi(a))<\cos\frac{2\pi M}{Q}}} (\pi(2T;q,a)-\pi(T;q,a))\le\frac{2(Q-2M-1)}{Q}\frac{T}{\log T/q}\Big(1+\frac{8}{\log T/q}\Big).$$

We take $T = q^K$ with $K \geq 2$ sufficiently large and dependent on δ but not on q, such that

$$\frac{1}{\log T/q}\left(1 + \frac{8}{\log T/q}\right) \leq (1+\delta)\frac{1}{\log T}.$$

Combined, these estimates imply that for $T = q^K$ with $K \geq 2$ a sufficiently large constant,

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \geq 2(1-\cos\pi X)\left(1-\delta-2(1+\delta)X + \frac{2(1+\delta)}{Q}\right)\frac{T}{\log T}$$

for $X = (Q-2M)/Q$.

For $Q \geq 3$, we may choose

$$\delta = \frac{1}{10}, \qquad M = \left\lfloor \frac{1+4\delta}{2(1+\delta)}\frac{Q}{2} + \frac{1}{2}\right\rfloor,$$

so that

$$X = \frac{1-2\delta}{2(1+\delta)} - \frac{1}{Q} + \frac{2}{Q}\left\{\frac{1+4\delta}{2(1+\delta)}\frac{Q}{2} + \frac{1}{2}\right\},$$

and hence

$$1-\delta-2(1+\delta)X + \frac{2(1+\delta)}{Q} = \delta + \frac{4(1+\delta)}{Q}\left(1 - \left\{\frac{1+4\delta}{2(1+\delta)}\frac{Q}{2} + \frac{1}{2}\right\}\right) \geq \delta.$$

Moreover, the fact that $\delta = \frac{1}{10}$ and $Q \geq 3$ implies that $1 \leq M \leq \lfloor Q/2 \rfloor$ and $\frac{1}{33} \leq X \leq \frac{23}{33}$. So

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \gg_K \frac{T}{\log T}. \qquad \square$$

Remark 4.7. If χ is quadratic, so that the order of χ is $Q = 2$, then

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \geq 2(\pi(2T) - \pi(T)) - 2 \sum_{\substack{a \in (\mathbb{Z}/q\mathbb{Z})^\times \\ \chi(a) = -1}} (\pi(2T;q,a) - \pi(T;q,a)).$$

The Brun–Titchmarsh inequality is insufficient to show that the first term on the right-hand side dominates the second term; in its place, we would require a strengthening of the Brun–Titchmarsh inequality of the form

$$\pi(x+y;q,a) - \pi(x;q,a) \leq \frac{(2-\delta)y}{\varphi(q)\log y/q}(1+o(1)) \tag{4.8}$$

for some $\delta > 0$. With this in hand, we would then be able to show that

$$\sum_{T \leq m \leq 2T} |\sigma_0(m,\chi)|^2 \gg \frac{T}{\log T},$$

so that

$$L(1,\chi) \gg \frac{1}{(\log q)^3},$$

which would imply the nonexistence of a Landau–Siegel zero for $L(1,\chi)$. Of course, the fact that the strengthened Brun–Titchmarsh inequality (4.8) implies (and is in fact equivalent to) the nonexistence of Landau–Siegel zeroes is well known.

5. Proof of Theorem 1.1

With these upper and lower bounds established, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. If χ is quadratic, we have via Corollary 4.3 and Lemma 4.5 that for $T \geq q$,

$$L(1,\chi) \gg \frac{1}{\sqrt{T}(\log T)^2},$$

and so taking $T = q$ yields the desired lower bound.

If χ is complex, we have via Corollary 4.3 and Lemma 4.6 that for $T = q^K$,

$$|L(1,\chi)| \gg_K \frac{1}{(\log T)^3} \gg_K \frac{1}{(\log q)^3}. \qquad \square$$

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