LARGE CHARACTER SUMS: BURGESS'S THEOREM AND ZEROS OF L-FUNCTIONS

ANDREW GRANVILLE AND KANNAN SOUNDARARAJAN

Abstract. We study the conjecture that $\sum_{n\le x}\chi(n)=o(x)$ for any primitive Dirichlet character $\chi \pmod q$ with $x\ge q^\epsilon$, which is known to be true if the Riemann Hypothesis holds for $L(s,\chi)$. We show that it holds under the weaker assumption that '100%' of the zeros of $L(s,\chi)$ up to height $\frac14$ lie on the critical line. We also establish various other consequences of having large character sums; for example, that if the conjecture holds for χ^2 then it also holds for χ.

1. Introduction

A central quest of analytic number theory is to estimate the character sum

$$S(x,\chi)=\sum_{n\le x}\chi(n), \tag{1.1}$$

where $\chi \pmod q$ is a primitive character. We would like to show that

$$S(x,\chi)=o(x) \tag{1.2}$$

in as wide a range for x as possible, and in particular whenever $x\ge q^\epsilon$ for any fixed positive ϵ (which implies Vinogradov's conjecture that the least quadratic non-residue mod q is $\ll_\epsilon q^\epsilon$). In [8] we showed that (1.2) holds when $\log x/\log\log q\to\infty$, assuming the Riemann Hypothesis for $L(s,\chi)$, and proved unconditionally that this range is the best possible. Burgess [3, 4] gave the best unconditional result, now more than fifty years old, that (1.2) holds for all $x\ge q^{1/4+\epsilon}$ if q is assumed to be cube-free, and slightly weaker variants for general q. The main results of this paper give further connections between large values of character sums and zeros of the corresponding L-function.

Before describing our two main theorems, we give the following corollaries which give a qualitative feel for what is established.

Corollary 1.1. *Let χ be a primitive quadratic character $\pmod q$, let $\epsilon>(\log q)^{-\frac13}$ be real. If the region $\{s:\ Re(s)\ge\frac34,\ |Im(s)|\le\frac14\}$ contains no more than $\epsilon^2(\log q)/1600$ zeros of $L(s,\chi)$, then for all $x\ge q^\epsilon$ we have*

$$\Big|\sum_{n\le x}\chi(n)\Big|\ll\frac{x}{(\log x)^{\frac{1}{100}}}.$$

Date: May 22, 2017.

2010 *Mathematics Subject Classification.* Primary: 11M26. Secondary: 11L40, 11M20.

Key words and phrases. Bounds on character sums, zeros of Dirichlet L-functions, multiplicative functions.

Andrew Granville was partially supported by an NSERC Discovery Grant, a Canadian Research Chair, and an ERC Advanced Grant. Kannan Soundararajan was partially supported by NSF grant DMS 1001068, and a Simons Investigator grant from the Simons Foundation.

There are $\ll \log q$ zeros $\beta + i\gamma$ of $L(s,\chi)$ with $0 < \beta < 1$ and $|\gamma| \le \frac{1}{4}$, and we expect these zeros to satisfy the Riemann hypothesis $\beta = \frac{1}{2}$. Our result can therefore be paraphrased as stating that if (1.2) is false for $x = q^\epsilon$ then a positive proportion ($\gg \epsilon^2$) of the zeros of $L(s,\chi)$, up to height 1, lie off the $\frac{1}{2}$-line. We believe that the method could be adapted to increase this proportion to $\gg \epsilon^{1+\delta}$ for any $\delta > 0$, but we do not pursue this here. A similar result holds for arbitrary primitive characters.

Corollary 1.2. *Let $\chi \pmod q$ be a primitive character. Let ϵ and T be real numbers with $1 \le T \le (\log q)^{\frac{1}{200}}$ and $\epsilon \ge (\log q)^{-\frac{1}{3}}$. Suppose that for every real ϕ with $|\phi| \le T$ the region $\{s: \ Re(s) \ge \frac{3}{4}, \ |Im(s) - \phi| \le \frac{1}{4}\}$ contains no more than $\epsilon^2(\log q)/1440$ zeros of $L(s,\chi)$. Then for all $x \ge q^\epsilon$ we have*

$$\Big|\sum_{n\le x}\chi(n)\Big| \ll \frac{x}{T}.$$

We now state our main theorems, from which the corollaries above follow as special cases.

Theorem 1.3. *Let $\chi \pmod q$ be a primitive character, and let $\exp(\sqrt{\log q}) \le x \le \sqrt{q}$ be such that $|S(x,\chi)| = x/N$ where $1 \le N \le (\log x)^{\frac{1}{100}}$. There exists an absolute positive constant $c > 0$ such that for some real number ϕ with $|\phi| \le cN$, and any parameter $(\log x)/2 \ge L \ge cN^6$, the region*

$$\Big\{ s: \ |s - (1+i\phi)| < L\frac{\log q}{(\log x)^2} \Big\}$$

contains at least $L/360$ zeros of the Dirichlet L-function $L(s,\chi)$.

When the character χ has small order, we give the following variant which removes the parameter ϕ in Theorem 1.3.

Theorem 1.4. *Let $\chi \pmod q$ be a primitive Dirichlet character of order k, and let x, and N be as in Theorem 1.3. There exists an absolute constant $c > 0$ such that for any parameter L in the range $(\log x)/2 \ge L \ge (cN)^{2k^2}$, the region*

$$\Big\{ s: \ |s - 1| < L\frac{\log q}{(\log x)^2} \Big\}$$

contains at least $L/400$ zeros of $L(s,\chi)$.

Our first corollaries showed that large character sums produced many violations to the GRH. Our next corollary shows that large character sums force some zeros of $L(s,\chi)$ to lie very close to the 1-line (refining an old result of Rodosskiĭ [13] who treated the related problem of determining the smallest prime p with $\chi(p) \ne 1$, for characters of small order; see [12] for a lucid exposition).

Corollary 1.5. *Let $1 \ge \epsilon \ge (\log q)^{-\frac{1}{200}}$, and suppose there exists $x \ge q^\epsilon$ with $|S(x,\chi)| \ge \epsilon x$. Then there is an absolute constant $c > 0$ such that there is at least one zero of $L(s,\chi)$ inside the region*

$$\Big\{ s: \ Re(s) \ge 1 - \frac{c}{\epsilon^8 \log q}, \ |Im(s)| \le \frac{c}{\epsilon} \Big\}.$$

If χ has order k and $\epsilon \ge (\log q)^{-\frac{1}{4k^2}}$ then there is a zero in the region $|s-1| \le c/(\epsilon^{2k^2+2}\log q)$.

A classical argument of Backlund (see Theorem 13.5 of [15]) could be adapted to show that if almost all the zeros of $L(s,\chi)$ in intervals of length 1 have real part $\le 1/2+\epsilon$ then the Lindelöf hypothesis for $L(s,\chi)$ would follow. From such a bound, one could obtain strong estimates for character sums. Our results provide a sharper version of such ideas of Backlund and Rodosskiĭ, by finding zeros even closer to the 1-line, and localizing their imaginary parts. A key ingredient in our argument is work on mean-values of multiplicative functions, in particular the feature that such mean values vary slowly (see Lemma 3.3 below).

By a compactness argument, another consequence of our work is that if (1.2) fails for $x \ge q^{\epsilon}$ for infinitely many characters of bounded order, then one can find a sequence of L-functions with arbitrarily many pinpointed zeros near the 1-line.

Theorem 1.6. *Fix an integer $k \ge 2$ and a constant $\eta > 0$. Suppose there is an infinite sequence of distinct primitive characters $\chi_j \pmod{q_j}$ of order k for which $|\sum_{n\le x}\chi_j(n)| \ge \eta x$ for some $x \ge q_j^{\eta}$. There exists an infinite sequence $z_1,\ z_2,\ \ldots,$ of complex numbers with $|z_n|+1 \le |z_{n+1}|$ with the following property: There is a sequence of primitive characters $\psi_j \pmod{r_j}$ (in fact a subsequence of the original sequence χ_j) with $L(s_\ell,\psi_j)=0$ for $1\le \ell \le j$ and some s_ℓ satisfying*

$$s_\ell = 1 + \frac{z_\ell + o(1)}{\log r_j},$$

and the $o(1)$ term tends to zero as $r_j \to \infty$.

Theorem 1.6 generalizes and gives a soft version of an unpublished observation of Heath-Brown. Heath-Brown observed that if there is an infinite sequence of primes q for which the least quadratic non-residue mod q is $\ge q^{1/4\sqrt{e}+o(1)}$ then one can locate precisely many zeros of $L(s,(\frac{\cdot}{q}))$. A precise version of his result, as described in Appendix 2 of [6], is as follows. Consider the zeros of

$$H(z) = \frac{2}{z}\int_{1/\sqrt{e}}^{1}(1-e^{-zu})\frac{du}{u}.$$

These zeros lie in the half plane $\mathrm{Re}(z)<0$, and occur in conjugate pairs. Let z_k denote the sequence of these zeros with positive imaginary parts, and arranged in ascending order of the imaginary part. For each $k\ge 1$, if q is sufficiently large (and the least quadratic non-residue is as large as $q^{\frac{1}{4\sqrt{e}}+o(1)}$), then there is a zero of $L(s,(\frac{\cdot}{q}))$ at

$$s = 1 + \frac{4z_k + o(1)}{\log q},$$

and at its complex conjugate. In this situation, one can also describe the zeros z_k precisely: arguing as in Lemma 2 of [6] gives that

$$z_k = -\log(\pi k) + 2\pi i(k+\tfrac{1}{4}) + o(1), \tag{1.3}$$

which corresponds well to the data given at the end of [6].

Recently Banks and Makarov [2] generalized Heath-Brown's observation, and showed that if there is a sequence of quadratic characters with a certain prescribed smooth way in which (1.2) fails, then one can pinpoint the zeros near 1 of the corresponding L-functions. The smoothness hypothesis that they assume, permits them to locate the zeros in a form similar to (1.3) (see Proposition 3.1 of [2]). In contrast, our Theorem 1.6 is softer but holds more generally; it would be interesting if some more precise version of Theorem 1.6 incorporating behavior as in (1.3) could be established. We note here the recent interesting work of Tao [14] relating Vinogradov's conjecture to the Elliott-Halberstam conjectures on the distribution of primes (and more general sequences) in progressions.

We also take this opportunity to record some other observations on large character sums. In [11], we proved that if χ_1, χ_2, χ_3 are three (not necessarily distinct) characters $\pmod q$ which have large maximal character sums (that is, if $\max_x |\sum_{n\le x}\chi_j(n)| \gg \sqrt{q}\log q$ for $j=1,2,3$, which is the largest size permitted by the Polya-Vinogradov theorem) then there exists some x for which

$$\Big|\sum_{n\le x_0}(\chi_1\chi_2\chi_3)(n)\Big| \gg \sqrt{q}\log q.$$

We will prove an analogous (but much easier) result with respect to (1.2).

Corollary 1.7. *Suppose χ_1, and χ_2 are Dirichlet characters $\pmod q$, such that for some x_1, x_2 and some $\eta>0$ we have (for $j=1,\,2$)*

$$\Big|\sum_{n\le x_j}\chi_j(n)\Big| \ge \eta x_j.$$

Then, with $\xi = c\eta^6$ for a suitable absolute constant $c>0$, there exists $x \ge (\min(x_1,x_2))^\xi$ with

$$\Big|\sum_{n\le x}(\chi_1\chi_2)(n)\Big| \ge \xi x.$$

Corollary 1.7 implies, for example, that if $\chi \pmod p$ is a character of order 4 for which (1.2) fails for $x \ge p^\epsilon$, then (1.2) also fails for the Legendre symbol $\pmod p$ for some suitably large x. We discuss Corollary 1.7 and related results in Section 6 below.

Given a prime q, Burgess's theorem guarantees that there are $\sim x/2$ of quadratic residues and $\sim x/2$ quadratic non-residues $\pmod q$ up to x, provided $x \ge q^{\frac14+o(1)}$. One of the main results in [9] shows that if x is large enough, then at least 17.15% of the integers below x are quadratic residues $\pmod q$ (uniformly for all primes q). In the 'Vinogradov range' $q^{1/4\sqrt{e}+o(1)} \le x \le q$, Banks *et al* (see Theorem 2.1 of [1]) showed that a positive proportion of the integers below x are quadratic non-residues $\pmod q$. We give the following strengthening of their work (as mentioned in §4 of [1]).

Corollary 1.8. *Let q be a large prime, and suppose $1/\sqrt{e} \le u \le 1$. The number of quadratic non-residues $\pmod q$ up to $x = q^{u/4}$ is at least*

$$\Big(\min\Big(\delta_0, \frac14 - (\log u)^2\Big) + o(1)\Big)x,$$

where

$$\delta_0 = 1 - \log(1+\sqrt{e}) + 2\int_1^{\sqrt{e}} \frac{\log t}{t+1}dt = 0.1715\ldots.$$

2. Mean values of Multiplicative functions

In this section, we recall some results from the theory of mean-values of multiplicative functions. Let f be a multiplicative function for which each $|f(n)| \le 1$, and write $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$. Define the (square of the) "distance" between two such functions f and g,

$$\mathbb{D}(f,g;x)^2 := \sum_{p\le x}\frac{1 - \text{Re } f(p)\overline{g(p)}}{p},$$

and this distance function satisfies the triangle inequality

$$\mathbb{D}(f,g;x)+\mathbb{D}(g,h;x)\geq \mathbb{D}(f,h;x). \tag{2.1}$$

Further, the distance function is related to the Dirichlet series $F(s)$ via the relation

$$F\Big(1+\frac{1}{\log x}+it\Big)\asymp \log x\exp(-\mathbb{D}(f,n^{it};x)^2). \tag{2.2}$$

Given x, let $\phi=\phi_f(x)$ be a real number in the range $|t|\leq \log x$ where $|F(1+1/\log x+it)|$ attains its maximum. Put $M:=M_f(x)=\mathbb{D}(f,n^{i\phi};x)^2$. The first fact that we need is Halász's Theorem (see, e.g., Theorem 2b of [10]), which gives

$$\frac{1}{x}\sum_{n\leq x}f(n)\ll \frac{(M+1)e^{-M}}{1+|\phi|}+\frac{1}{(\log x)^{2-\sqrt{3}+o(1)}}. \tag{2.3}$$

Define $f_\phi(n):=f(n)/n^{i\phi}$. We next need a relation between the mean value of f and the mean value of f_ϕ. From Lemma 7.1 of [10], we quote the relation

$$\frac{1}{x}\sum_{n\leq x}f(n)=\frac{x^{i\phi}}{1+i\phi}\cdot\frac{1}{x}\sum_{n\leq x}f_\phi(n)+O\Big(\frac{\log\log x}{\log x}\exp\Big(\sum_{p\leq x}\frac{|1-f(p)|}{p}\Big)\Big). \tag{2.4}$$

Finally from Theorem 4 of [10] (refining work of Elliott [7]) we require the following Lipschitz estimate showing that the mean values of f_ϕ vary slowly: for any $\sqrt{x}\leq z\leq x^2$, we have

$$\frac{1}{x}\sum_{n\leq x}f_\phi(n)-\frac{1}{z}\sum_{n\leq z}f_\phi(n)\ll\Big(\frac{1+|\log x/z|}{\log x}\Big)^{1-\frac{2}{\pi}+o(1)}. \tag{2.5}$$

3. Large character sums and zeros off the critical line

Let $\chi \pmod q$ denote a primitive character. We shall make use of the Hadamard factorization formula (see [5])

$$\xi(s,\chi)=\Big(\frac{q}{\pi}\Big)^{\frac{s+\mathfrak{a}}{2}}\Gamma\Big(\frac{s+\mathfrak{a}}{2}\Big)L(s,\chi)=e^{A+Bs}\prod_\rho\Big(1-\frac{s}{\rho}\Big)e^{s/\rho}, \tag{3.1}$$

where $\mathfrak{a}=(1-\chi(-1))/2$, and ρ ranges over the non-trivial zeros of $L(s,\chi)$ and A and B are constants (depending on χ) with

$$\text{Re } B=-\sum_\rho \text{Re }\frac{1}{\rho}. \tag{3.2}$$

Lemma 3.1. *Let $\frac12\geq\lambda>0$ be a real number, and let t be a real number. Then*

$$|L(1-\lambda+it,\chi)|\ll\frac{1}{\lambda}\exp\Big(\sum_\rho\frac{2\lambda^2}{|1+\lambda+it-\rho|^2}\Big).$$

Proof. Put $s_0=1+\lambda+it$, and $s_1=1-\lambda+it$. Applying (3.1) and (3.2) with $s=s_0$ and $s=s_1$, and invoking Stirling's formula, we see that

$$\Big|\frac{L(s_1,\chi)}{L(s_0,\chi)}\Big|\asymp(q(1+|t|))^\lambda\Big|\frac{\xi(s_1,\chi)}{\xi(s_0,\chi)}\Big|=(q(1+|t|))^\lambda\prod_\rho\frac{|s_1-\rho|}{|s_0-\rho|}. \tag{3.3}$$

Note that

$$\Big|\frac{s_1-\rho}{s_0-\rho}\Big| = \Big(1-\frac{|s_0-\rho|^2-|s_1-\rho|^2}{|s_0-\rho|^2}\Big)^{\frac12} \le \exp\Big(-2\lambda\frac{\text{Re}(1-\rho)}{|s_0-\rho|^2}\Big)$$
$$= \exp\Big(-2\lambda \text{Re}\Big(\frac{1}{s_0-\rho}\Big)+\frac{2\lambda^2}{|s_0-\rho|^2}\Big).$$

Using this in (3.3), we conclude that

$$\Big|\frac{L(s_1,\chi)}{L(s_0,\chi}\Big| \ll (q(1+|t|))^{\lambda}\exp\Big(2\lambda\sum_{\rho}\Big(-\text{Re}\Big(\frac{1}{s_0-\rho}\Big)+\frac{\lambda}{|s_0-\rho|^2}\Big)\Big). \tag{3.4}$$

On the other hand, taking logarithmic derivatives in (3.1), we see that

$$-\text{Re}\frac{L'}{L}(s_0,\chi) = \frac12\log q(1+|t|)+O(1)-\sum_{\rho}\text{Re}\Big(\frac{1}{s_0-\rho}\Big), \tag{3.5}$$

and the left hand side above is trivially bounded in magnitude by

$$\le \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\lambda}} = \frac{1}{\lambda}+O(1). \tag{3.6}$$

Inserting this bound in (3.4), we conclude that

$$\Big|\frac{L(s_1,\chi)}{L(s_0,\chi)}\Big| \ll \exp\Big(\sum_{\rho}\frac{2\lambda^2}{|s_0-\rho|^2}\Big),$$

and since $|L(s_0,\chi)| \le \zeta(1+\lambda) = \frac{1}{\lambda}+O(1)$, the lemma follows. □

Next, we show how character sums may be related to suitable averages of L-functions.

Lemma 3.2. *Let ϕ be a real number, T a positive real number, and $0\le\lambda\le\frac12$. Let χ_ϕ denote the completely multiplicative function $\chi_\phi(n)=\chi(n)n^{-i\phi}$, and let $S(x,\chi_\phi)=\sum_{n\le x}\chi_\phi(n)$. Then*

$$\sqrt{2\pi T}\int_{-\infty}^{\infty}\frac{S(e^y,\chi_\phi)}{e^y}\exp\Big(\lambda y-\frac{T}{2}y^2\Big)dy = \int_{-\infty}^{\infty}\frac{L(1-\lambda+i\phi+i\xi,\chi)}{1-\lambda+i\xi}\exp\Big(-\frac{\xi^2}{2T}\Big)d\xi.$$

Proof. The Fourier transform of $\frac{S(e^y,\chi_\phi)}{e^y}\exp(y\lambda)$ is

$$\int_{-\infty}^{\infty}\frac{S(e^y,\chi_\phi)}{e^y}\exp(y\lambda)e^{-iy\xi}dy = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{i\phi}}\int_{\log n}^{\infty}\exp(-y(1-\lambda+i\xi))dy$$
$$= \frac{L(1-\lambda+i\phi+i\xi,\chi)}{1-\lambda+i\xi}.$$

The Fourier transform of $\exp(-\frac{T}{2}y^2)$ is

$$\int_{-\infty}^{\infty}\exp\Big(-\frac{T}{2}y^2-iy\xi\Big)dy = \frac{\sqrt{2\pi}}{\sqrt{T}}\exp\Big(-\frac{\xi^2}{2T}\Big).$$

The lemma follows by the Plancherel formula. □

Our last ingredient comes from the theory of mean-values of multiplicative functions.

Lemma 3.3. *Let $y_0 \ge 3$ be a real number, and we assume that $|S(e^{y_0},\chi)| \ge e^{y_0} y_0^{-\frac{1}{100}}$. There exists a real number $\phi = \phi(y_0)$, with $|\phi| \ll e^{y_0}/|S(e^{y_0},\chi)|$, such that for any real number y,*

$$\Big|\frac{S(e^y,\chi_\phi)}{e^y} - \frac{S(e^{y_0},\chi_\phi)}{e^{y_0}}\Big| \ll \Big(\frac{1+|y-y_0|}{y_0}\Big)^{\frac{1}{3}}. \tag{3.7}$$

Moreover

$$S(e^{y_0},\chi_\phi) = (1+i\phi)e^{-i\phi y_0} S(e^{y_0},\chi) + O\Big(\frac{e^{y_0}}{y_0^{\frac{3}{4}}}\Big). \tag{3.8}$$

Finally, if χ has small order k then the following stronger bound for ϕ holds: for some absolute constant $c > 0$,

$$|\phi| \le \frac{1}{y_0}\Big(\frac{ce^{y_0}}{|S(e^{y_0},\chi)|}\Big)^{2k^2}. \tag{3.9}$$

Proof. Take $x = e^{y_0}$ and $f = \chi$ in section 2. By (2.3) and the hypothesis we have $|\phi| \ll e^{y_0}/|S(e^{y_0},\chi)| \le y_0^{\frac{1}{100}}$ as desired, and moreover that $M \le \frac{1}{100}\log y_0 + \log\log y_0 + O(1)$.

To prove (3.7) we may clearly suppose that $|y-y_0| \le y_0/2$, in which case (3.7) follows immediately from (2.5).

Next, by Cauchy-Schwarz (and using $|1-\chi_\phi(p)|^2 \le 2(1-\mathrm{Re}(\chi_\phi(p))))$

$$\begin{aligned}\sum_{p\le e^{y_0}} \frac{|1-\chi_\phi(p)|}{p} &\le \Big(\sum_{p\le e^{y_0}}\frac{1}{p}\Big)^{\frac12}\Big(\sum_{p\le e^{y_0}}\frac{|1-\chi_\phi(p)|^2}{p}\Big)^{\frac12} \le (\log y_0 + O(1))^{\frac12}(2M)^{\frac12}\\ &\le \frac{\log y_0}{7} + O(1).\end{aligned}$$

Using this in (2.4), we obtain (3.8).

Finally, suppose that χ has order k. The triangle inequality (2.1) gives

$$M = \sum_{p\le e^{y_0}} \frac{1-\mathrm{Re}\ \chi_\phi(p)}{p} \ge \frac{1}{k^2}\sum_{p\le e^{y_0}}\frac{1-\mathrm{Re}\ (\chi(p)p^{-i\phi})^k}{p} \ge \frac{1}{k^2}\sum_{p\le e^{y_0}}\frac{1-\cos(k\phi\log p)}{p}.$$

Using the prime number theorem it follows that $\phi \ll \exp(k^2 M)/y_0$ which yields the final assertion (3.9) of the lemma. □

Combining Lemmas 3.1, 3.2, and 3.3, we arrive at the following proposition.

Proposition 3.1. *Let y_0 be large with $|S(e^{y_0},\chi)| =: e^{y_0}/N \ge e^{y_0}y_0^{-\frac{1}{100}}$, and let ϕ be as in Lemma 3.3. If $cN^6/y_0 \le \lambda \le \frac12$ for a suitably large constant c, then there exists $|\xi| \le 2\lambda\sqrt{(\log q)/y_0}$ such that*

$$\sum_\rho \frac{\lambda}{|1+\lambda+i\phi+i\xi-\rho|^2} \ge \frac{y_0}{4}. \tag{3.10}$$

Proof. Put $T = \lambda/y_0$, and note that $T \le 1/(2y_0) < 1$. Using Lemma 3.3 we find that

$$\begin{aligned}&\sqrt{2\pi T}\int_{-\infty}^{\infty}\frac{S(e^y,\chi_\phi)}{e^y}\exp\Big(\lambda y - \frac{T}{2}y^2\Big)dy\\ &\quad= \sqrt{2\pi T}\exp\Big(\frac{\lambda y_0}{2}\Big)\int_{-\infty}^{\infty}\Big(\frac{S(e^{y_0},\chi_\phi)}{e^{y_0}} + O\Big(\frac{1+|y-y_0|^{\frac13}}{y_0^{\frac13}}\Big)\Big)\exp\Big(-\frac{T}{2}(y-y_0)^2\Big)dy.\end{aligned}$$

A little calculation, together with (3.8), gives that this equals

$$2\pi \exp\Big(\frac{\lambda y_0}{2}\Big)\Big(\frac{(1+i\phi)S(e^{y_0},\chi)}{e^{y_0(1+i\phi)}} + O\Big(\frac{1}{(\lambda y_0)^{\frac16}}\Big)\Big),$$

which, by our assumed lower bound on λ, is in magnitude $\ge \pi\exp(\frac{\lambda y_0}{2})/N$.

Thus, by Lemma 3.2, we see that

$$\pi\exp\Big(\frac{\lambda y_0}{2}\Big)/N \le \int_{-\infty}^{\infty} \frac{|L(1-\lambda+i\phi+i\xi,\chi)|}{|1-\lambda+i\xi|}\exp\Big(-\frac{\xi^2}{2T}\Big)d\xi$$
$$\le \Big(\max_{\xi\in\mathbb{R}} \frac{|L(1-\lambda+i\phi+i\xi,\chi)|}{|1-\lambda+i\xi|}\exp\Big(-\frac{\xi^2}{4T}\Big)\Big)\int_{-\infty}^{\infty}\exp\Big(-\frac{\xi^2}{4T}\Big)d\xi,$$

so that

$$\max_{\xi\in\mathbb{R}} \frac{|L(1-\lambda+i\phi+i\xi,\chi)|}{|1-\lambda+i\xi|}\exp\Big(-\frac{\xi^2}{4T}\Big) \ge \frac12\sqrt{\frac{\pi y_0}{\lambda}}\exp\Big(\frac{\lambda y_0}{2}\Big)/N. \tag{3.11}$$

If $\mathrm{Re}(s)=\sigma>0$ then note that

$$|L(s,\chi)| = \Big|s\int_1^\infty \frac{S(x,\chi)}{x^{s+1}}dx\Big| \le |s|\int_1^\infty \frac{\min(x,q)}{x^{\sigma+1}}dx = |s|\Big(\frac{q^{1-\sigma}-1}{1-\sigma}+\frac{q^{1-\sigma}}{\sigma}\Big).$$

Therefore, if $|\xi| > 2\lambda\sqrt{(\log q)/y_0}$ then

$$\frac{|L(1-\lambda+i\phi+i\xi,\chi)|}{|1-\lambda+i\xi|}\exp\Big(-\frac{\xi^2}{4T}\Big) \le \Big|\frac{1-\lambda+i\phi+i\xi}{1-\lambda+i\xi}\Big|\Big(\frac{q^\lambda-1}{\lambda}+\frac{q^\lambda}{1-\lambda}\Big)q^{-\lambda} \le \frac{2(1+2|\phi|)}{\lambda}.$$

Since $|\phi| \ll N$, and $\lambda y_0 \ge cN^6$ for a suitably large constant c, we may check that the RHS above is smaller than the RHS in (3.11). Therefore the maximum in the LHS of (3.11) is attained for some ξ with $|\xi| \le 2\lambda\sqrt{(\log q)/y_0}$, and at this point we have, by (3.11),

$$|L(1-\lambda+i\phi+i\xi,\chi)| \gg (\lambda y_0)^{1/3}\exp\Big(\frac{\lambda y_0}{2}\Big).$$

Using now the bound from Lemma 3.1, we conclude that

$$\sum_\rho \frac{\lambda}{|1+\lambda+i\phi+i\xi-\rho|^2} \ge \frac{y_0}{4}.$$

□

4. Proofs of Theorems 1.3 and 1.4 and the Corollaries

Proof of Theorem 1.3. We appeal to Proposition 3.1, taking there $y_0 = \log x$, and let ϕ, λ and ξ be as given there. We then have the lower bound furnished by (3.10). Split the zeros ρ into those with $|1+i\phi-\rho| \ge 40\lambda(\log q)/\log x$ and those zeros lying closer to $1+i\phi$. Note that if $|1+i\phi-\rho| \ge 40\lambda(\log q)/\log x$ then, using the triangle inequality,

$$|1+\lambda+i\phi+i\xi-\rho| \ge \Big|1+20\lambda\frac{\log q}{\log x}+i\phi-\rho\Big| - 20\lambda\frac{\log q}{\log x} - |\xi| \ge \frac{9}{20}\Big|1+20\lambda\frac{\log q}{\log x}+i\phi-\rho\Big|.$$

Therefore the contribution of these zeros to the LHS of (3.10) is

$$\le 5\lambda\sum_\rho \frac{1}{|1+20\lambda(\log q)/\log x+i\phi-\rho|^2} \le \frac{\log x}{4\log q}\sum_\rho \mathrm{Re}\Big(\frac{1}{1+20\lambda(\log q)/\log x+i\phi-\rho}\Big),$$

as $\mathrm{Re}(\rho) \le 1$ for all such ρ. But, arguing as in (3.5) and (3.6), we see that the above is at most

$$\frac{\log x}{4\log q}\Big(\frac{5}{9}\log q\Big) = \frac{5\log x}{36}.$$

We conclude that the contribution of the zeros with $|1+i\phi-\rho| \le 40\lambda(\log q)/\log x$ to the LHS of (3.10) is at least $(\log x)/9$. Since each such zero contributes at most $1/\lambda$, it follows that

$$\Big|\Big\{\rho : |1+i\phi-\rho| \le 40\lambda\frac{\log q}{\log x}\Big\}\Big| \ge \lambda\frac{\log x}{9}.$$

The theorem follows upon setting $L = 40\lambda\log x$. □

Proof of Theorem 1.4. We follow the argument above, now making use of the bound (3.9) which gives $|\phi| \le (cN)^{2k^2}/\log x$. Therefore if now $\lambda \ge (cN)^{2k^2}/\log x$ $(\ge |\phi|)$ and $|1-\rho| \ge 40\lambda(\log q)/\log x$ then $|1+i\phi-\rho| \ge 39\lambda(\log q)/\log x$ and the argument above shows that the contribution of these zeros to the LHS of (3.10) is bounded by $0.15\log x$. Thus we conclude Theorem 1.4. □

Corollary 1.5 follows upon taking $L = c\epsilon^{-6}$ in Theorem 1.3 and $L = (c/\epsilon)^{2k^2}$ in Theorem 1.4. Corollaries 1.1 and 1.2 follow upon taking $L = (\epsilon^2\log q)/4$ in Theorems 1.4 and 1.3, respectively.

5. Locating zeros: Proof of Theorem 1.6

Choosing $L = (c/\eta)^{2k^2}$ for a suitably large constant c, we find by Theorem 1.4 that for each $\chi_j \pmod{q_j}$ there is a zero of $L(s,\chi_j)$ satisfying $s = 1 + w_j/\log q_j$ with $|w_j| \le C_1(\eta)$ for a suitable constant $C_1(\eta)$. Since the region $|w| \le C_1(\eta)$ is compact, we can extract from the sequence w_j a convergent subsequence. Now take z_1 to be the limiting value of w_j from this convergent subsequence.

By restricting to the subsequence above, let us suppose that we now have a sequence of characters χ_j of order k with $L(s,\chi_j)$ having a zero satisfying $1 + (z_1 + o(1))/\log q_j$. Now from the argument of (3.5) and (3.6) we may see that for any L-function there are at most a bounded number of zeros of the form $1 + w/\log q$ with $|w| \le 1 + |z_1|$. Therefore, by appealing to Theorem 1.4 with a suitably large value of L, we may conclude that $L(s,\chi_j)$ has a zero of the form $s = 1 + w_j/\log q_j$ with $|z_1| + 1 \le |w_j| \le C_2(\eta)$ for some suitably large $C_2(\eta)$. Since this region is again compact, we can once again extract a subsequence of characters for which w_j converges, and we call one such limiting value z_2.

Proceeding in this manner, we obtain Theorem 1.6.

6. Relations among characters with large partial sums

We begin by showing that if a multiplicative function f is at a small distance from the function $n^{i\phi}$ then the partial sums of f get large in suitable ranges.

Proposition 6.1. *Let f be a multiplicative function with $|f(n)| \le 1$ for all n. Let x be large, and put $\lambda = M + \log(1+|\phi|) + c$ where $M = M_f(x)$ and $\phi = \phi_f(x)$ are as in Section 2, and c is a large constant. Then there exists y in the range $x^{1/(\lambda e^{\lambda})} \le y \le x$ such that*

$$\Big|\sum_{n\le y} f(n)\Big| \gg \frac{e^{-M}}{|1+i\phi|}\; y.$$

Proof. By (2.2) we know that

$$\Big|F\Big(1+\frac{\lambda}{\log x}+i\phi\Big)\Big| \asymp \frac{\log x}{\lambda}\exp(-\mathbb{D}(f,n^{i\phi};x^{1/\lambda})^2) \gg \frac{\log x}{\lambda}e^{-M}.$$

On the other hand, with $\delta=\lambda/\log x$,

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^{1+\delta+i\phi}} = (1+\delta+i\phi)\int_1^\infty \frac{1}{y^{2+\delta+i\phi}}\sum_{n\le y} f(n)dy.$$

Put $\eta=1/(\lambda e^\lambda)$. Assuming that $|\sum_{n\le y} f(n)|\le e^{-\lambda}y$ for all $x^\eta\le y\le x$, and using the trivial bound $|\sum_{n\le y} f(n)|\le y$ otherwise, we find that the right hand side above is bounded in size by

$$\begin{aligned}
&\le |1+\delta+i\phi|\Big(\int_1^{x^\eta}\frac{dy}{y^{1+\delta}}+e^{-\lambda}\int_{x^\eta}^{x}\frac{dy}{y^{1+\delta}}+\int_x^\infty\frac{dy}{y^{1+\delta}}\Big)\\
&= |1+\delta+i\phi|\frac{\log x}{\lambda}\left((1-e^{-\eta\lambda})+e^{-\lambda}(e^{-\eta\lambda}-e^{-\lambda})+e^{-\lambda}\right)\\
&\le 6(1+|\phi|)e^{-\lambda}\ \frac{\log x}{\lambda}.
\end{aligned}$$

This yields a contradiction, provided c is sufficiently large. □

Our next result shows that if the partial sums of two completely multiplicative functions get large, then the product of these functions also has large partial sums. Corollary 1.7 follows immediately from this result.

Theorem 6.1. *Let f_1 and f_2 be completely multiplicative functions with $|f_1(n)|$ and $|f_2(n)|$ bounded by 1 for all n. Suppose that η is a positive real number and x_1, and x_2 are such such that (for $j=1,2$)*

$$\Big|\sum_{n\le x_j} f_j(n)\Big| \ge \eta x_j. \tag{6.1}$$

Then, with $\xi=c\eta^6$ for a suitable absolute constant $c>0$, there exists $x\ge(\min(x_1,x_2))^\xi$ such that, for some absolute constant $c>0$,

$$\Big|\sum_{n\le x} f_1(n)f_2(n)\Big| \ge \xi x.$$

Proof. By (2.3) there exists ϕ_1,ϕ_2 with $|\phi_j|\ll 1/\eta$ such that

$$\mathbb{D}(f_j,n^{i\phi_j};x_j)^2 \le \log(1/\eta)+\log\log(1/\eta)+O(1).$$

Let $X=\min\{x_1,x_2\}$, $f=f_1f_2$ and $\phi=\phi_1+\phi_2$. Since $\mathbb{D}(f_j,n^{i\phi_j};X)\le\mathbb{D}(f_j,n^{i\phi_j};x_j)$, the triangle inequality (2.1) gives

$$\mathbb{D}(f,n^{i\phi};X)\le\mathbb{D}(f_1,n^{i\phi_1};X)+\mathbb{D}(f_2,n^{i\phi_2};X),$$

so that $\mathbb{D}(f,n^{i\phi};X)^2 \le 4\log(1/\eta)+4\log\log(1/\eta)+O(1)$. The result now follows from Proposition 6.1. □

Another variant of the argument of Theorem 6.1 is the following. Suppose f is completely multiplicative with $|f(n)| \leq 1$ and $|\sum_{n\leq x} f(n)| \geq \eta x$. Then for any natural number k, there exists $y \geq x^{c\eta^{2k^2}}$ (for a suitable absolute constant $c > 0$) with

$$\Big|\sum_{n\leq y} f(n)^k\Big| \geq c\eta^{2k^2} y.$$

To see this, note that our hypothesis on f implies (as in Theorem 6.1) that

$$\mathbb{D}(f, n^{i\phi}; x)^2 \leq \log(1/\eta) + \log\log(1/\eta) + O(1)$$

for some $|\phi| \ll 1/\eta$. By the triangle inequality it follows that $\mathbb{D}(f^k, n^{ik\phi}; x) \leq k\mathbb{D}(f, n^{i\phi}; x)$. Now we invoke Proposition 6.1, and obtain the stated conclusion. One application of this variant is that (stated informally) if a small power of a character χ equals a non-principal character of small conductor, then one can obtain cancelations in the character sums for χ.

7. Producing many quadratic residues below $p^{\frac{1}{4}}$

Corollary 1.8 follows immediately from the Burgess bound together with the following general result on completely multiplicative functions taking values in $[-1, 1]$, which largely follows from the work in [9].

Proposition 7.1. *Let x be large, and let f be a completely multiplicative function with $-1 \leq f(n) \leq 1$ for all n. Suppose that*

$$\sum_{n\leq x} f(n) = o(x).$$

Then for $1/\sqrt{e} \leq \alpha \leq 1$ we have

$$\frac{1}{x^\alpha}\Big|\sum_{n\leq x^\alpha} f(n)\Big| \leq \max\Big(|\delta_1|, \frac{1}{2} + 2(\log\alpha)^2\Big) + o(1),$$

where

$$\delta_1 = 1 - 2\log(1+\sqrt{e}) + 4\int_1^{\sqrt{e}} \frac{\log t}{t+1} dt = -0.656999\ldots.$$

Proof. We make free use of the work in [9]. Put $y = \exp((\log x)^{\frac{2}{3}})$ and let g be the completely multiplicative function defined by $g(p) = 1$ for $p \leq y$ and $g(p) = f(p)$ for $p > y$. Then (see page 439 of [9]) for $x^{1/\sqrt{e}} \leq z \leq x$ we have

$$\sum_{n\leq z} f(n) = \Big(\prod_{p\leq y}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1}\Big)\sum_{n\leq z} g(n) + o(z).$$

If now

$$\prod_{p\leq y}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1} \leq \frac{1}{10},$$

then the result follows at once. So let us assume that the product above is at least $\frac{1}{10}$, so that we have

$$\sum_{n\leq x} g(n) = o(x).$$

Now we may pass from mean values of multiplicative functions to solutions of integral equations as in [9], and use the results established there. Put

$$\tau(\alpha) = \sum_{p \le x^\alpha} \frac{1-g(p)}{p}.$$

By inclusion-exclusion (see Proposition 3.6 of [9]) we have

$$o(x) = \sum_{n \le x} g(n) \ge x(1 - \tau(1) + o(1)),$$

so that $\tau(1) \ge 1 + o(1)$ and more generally

$$\tau(\alpha) \ge 1 - 2 \sum_{x^\alpha \le p \le x} \frac{1}{p} = 1 + 2\log\alpha + o(1).$$

Applying Theorem 5.1 of [9], if $\tau(\alpha) \ge 1$ then

$$\Big| \sum_{n \le x^\alpha} g(n) \Big| \le (|\delta_1| + o(1)) x^\alpha.$$

If $1 + 2\log\alpha \le \tau(\alpha) \le 1$, then an inclusion-exclusion argument (see again Proposition 3.6 of [9]) gives

$$\Big| \sum_{n \le x^\alpha} g(n) \Big| \le \Big(1 - \tau(\alpha) + \frac{\tau(\alpha)^2}{2} + o(1)\Big) x^\alpha \le \Big(\frac{1}{2} + 2(\log\alpha)^2 + o(1)\Big) x^\alpha.$$

The proposition follows. □

There is some scope to improve the bound in Proposition 7.1, especially when α is close to 1. Here Lipschitz estimates like (2.5) show that

$$x^{-\alpha} \sum_{n \le x^\alpha} f(n) \ll (1-\alpha)^{1-\frac{2}{\pi}+o(1)},$$

which is plainly better than the bound in Proposition 7.1 for α sufficiently close to 1.

References

[1] W. D. Banks, M. Z. Garaev, D. R. Heath-Brown, and I. E. Shparlinski. Density of non-residues in Burgess-type intervals and applications. *Bull. Lond. Math. Soc.*, 40(1):88–96, 2008.

[2] William D. Banks and Konstantin A. Makarov. Convolutions with probability distributions, zeros of l-functions, and the least quadratic nonresidue. *Funct. Approx. Comment. Math.*, 55: 243–280, 2016.

[3] D. A. Burgess. On character sums and L-series. *Proc. London Math. Soc. (3)*, 12:193–206, 1962.

[4] D. A. Burgess. On character sums and L-series. II. *Proc. London Math. Soc. (3)*, 13:524–536, 1963.

[5] Harold Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, second edition, 1980. Revised by Hugh L. Montgomery.

[6] Harold G. Diamond, Hugh L. Montgomery, and Ulrike M. A. Vorhauer. Beurling primes with large oscillation. *Math. Ann.*, 334(1):1–36, 2006.

[7] P. D. T. A. Elliott. Extrapolating the mean-values of multiplicative functions. *Nederl. Akad. Wetensch. Indag. Math.*, 51(4):409–420, 1989.

[8] Andrew Granville and K. Soundararajan. Large character sums. *J. Amer. Math. Soc.*, 14(2):365–397, 2001.

[9] Andrew Granville and K. Soundararajan. The spectrum of multiplicative functions. *Ann. of Math. (2)*, 153(2):407–470, 2001.

[10] Andrew Granville and K. Soundararajan. Decay of mean values of multiplicative functions. *Canad. J. Math.*, 55(6):1191–1230, 2003.

[11] Andrew Granville and K. Soundararajan. Large character sums: pretentious characters and the Pólya-Vinogradov theorem. *J. Amer. Math. Soc.*, 20(2):357–384, 2007.

[12] Hugh L. Montgomery. *Ten lectures on the interface between analytic number theory and harmonic analysis*, volume 84 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

[13] K. A. Rodosskiĭ. On non-residues and zeros of L-functions. *Izv. Akad. Nauk SSSR. Ser. Mat.*, 20:303–306, 1956.

[14] Terence Tao. The Elliott-Halberstam conjecture implies the Vinogradov least quadratic nonresidue conjecture. *Algebra Number Theory* 9:1005–1034, 2015.

[15] E. C. Titchmarsh The theory of the Riemann zeta-function. Second edition, revised by D. R. Heath-Brown. OUP, 1986.

AG: Département de mathématiques et de statistique, Université de Montréal, CP 6128 succ. Centre-Ville, Montréal, QC H3C 3J7, Canada.; and Department of Mathematics, University College London, Gower Street, London WC1E 6BT, England.

E-mail address: andrew@dms.umontreal.ca

KS: Department of Mathematics, Stanford University, Stanford, CA 94305, USA.

E-mail address: ksound@stanford.edu