

## A NOTE ON CHARACTERISTIC FUNCTIONS

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1. Let  $F(t)$  be a probability distribution function. Let  $\phi(x)$  be its characteristic function, given by

$$\phi(x) = \int_{-\infty}^{\infty} e^{ixt} dF(t),$$

and define

$$(1.1) \quad \psi(x) = 1 - \phi(x).$$

We obtain some elementary inequalities for  $\psi(X)$  from which we deduce a number of facts about characteristic functions. To provide an application of these results, we prove the following theorem. For  $\alpha = 1$ , this theorem is contained in a theorem of Pitman [6] and the proof of Boas' Theorem 1 of [1] yields the case  $0 < \alpha < 1$ . See also Pitman [7].

**THEOREM 1.** *If  $0 < \alpha < 2$ , a necessary and sufficient condition that*

$$(1.2) \quad v^\alpha \int_{|t|>v} dF(t) = o(1) \quad (v \rightarrow \infty)$$

*is that*

$$(1.3) \quad (1 - \Re\phi(u))/u^\alpha = o(1) \quad (u \rightarrow 0+).$$

Boas' method, in fact, establishes that, for  $0 < \alpha < 1$ , (1.2) is equivalent to

$$(1.4) \quad (1 - \phi(u))/u^\alpha = o(1) \quad (u \rightarrow 0+)$$

and so conditions (1.3) and (1.4) are equivalent for  $0 < \alpha < 1$ .

**COROLLARY 1.** *Let  $S_n$  denote the sum of  $n$  independent random variables each with distribution  $F(t)$ . If  $\alpha > 0$ , then (1.4) is a necessary and sufficient condition that*

$$(1.5) \quad n^{-1/\alpha} S_n \xrightarrow{p} 0 \quad (n \rightarrow \infty).$$

**REMARKS.** (1) Theorem 1 remains true if the  $o$  in both (1.2) and (1.3) is replaced by  $O$ .

(2) If  $\alpha = 2n + \beta$ , where  $n$  is a positive integer and  $0 < \beta < 2$ , a necessary and sufficient condition for (1.2) is that

$$\Re\phi^{(2n)}(0) - \Re\phi^{(2n)}(u) = o(u^\beta) \quad (u \rightarrow 0+).$$

(3) If  $0 < \alpha < 1$  in Corollary 1, then condition (1.4) may be replaced by (1.3). If  $\alpha = 1$ , this is not true. If  $\alpha > 1$ , (1.3) implies the existence of the

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mean  $\mu$  of the distribution  $F(t)$  and (1.4) in Corollary 1 may be replaced by (1.3) as long as  $S_n$  is replaced by  $S_n - n\mu$ . If  $\alpha = 1$ , (1.3) does not imply the existence of  $\mu$ . (See e.g. Lukacs [5], p. 29).

2. An elementary property of characteristic functions is that they are positive definite, i.e.

$$\sum_{j,k=1}^n \phi(t_j - t_k) z_j \bar{z}_k \geq 0,$$

for any real  $t_1, t_2, \dots, t_n$  and complex  $z_1, z_2, \dots, z_n$ . Since  $\phi(0) = 1$ , it follows that

$$|\sum_{j=1}^n z_j|^2 \geq \sum_{j,k=1}^n \psi(t_j - t_k) z_j \bar{z}_k,$$

where  $\psi(x)$  is defined by (1.1). But  $\psi(-x) = \bar{\psi}(x)$ , and so

$$(2.1) \quad \frac{1}{2} |\sum_{j=1}^n z_j|^2 \geq \Re \{ \sum_{j < k} \psi(t_j - t_k) z_j \bar{z}_k \}.$$

We take  $n = 3, x = t_1 - t_2, y = t_2 - t_3$ , and  $x + y = t_1 - t_3$ . If  $x > 0$  and  $y > 0$ , the choice  $z_1 = y, z_2 = -(x + y)$  and  $z_3 = x$  yields

$$(2.2) \quad \Re \psi(x + y) / (x + y) \leq \Re \psi(x) / x + \Re \psi(y) / y,$$

i.e. the function  $x^{-1} \Re \psi(x)$  is subadditive on  $(0, \infty)$ . For any non-negative function  $h(x)$  which is subadditive on  $(0, \infty)$  we have that

$$(2.3) \quad y h(y) \leq 2 \int_0^y h(x) dx \quad (y > 0).$$

In particular, therefore,

$$(2.4) \quad \Re \psi(y) \leq 2 \int_0^y x^{-1} \Re \psi(x) dx.$$

The choice  $z_1 = 1, z_2 = -(1 + re^{i\theta})$  and  $z_3 = re^{i\theta}$ , where

$$r = \{ \Re \psi(x) / \Re \psi(y) \}^{\frac{1}{2}} \quad \text{and} \quad \theta = \arg \{ \psi(x + y) - \psi(x) - \psi(y) \},$$

yields that

$$(2.5) \quad |\psi(x + y) - \psi(x) - \psi(y)| \leq 2 \{ \Re \psi(x) \cdot \Re \psi(y) \}^{\frac{1}{2}}.$$

(If  $\Re \psi(x) = 0$  or  $\Re \psi(y) = 0$ , the inequality follows trivially from (2.1) with  $n = 3$ ). From (2.5) we deduce that

$$(2.6) \quad |(\psi(x + y))^{\frac{1}{2}}| \leq |\psi(x)|^{\frac{1}{2}} + |\psi(y)|^{\frac{1}{2}},$$

i.e. the function  $|\psi(x)|^{\frac{1}{2}}$  is subadditive on  $(-\infty, \infty)$ . A similar argument establishes that  $\{ \Re \psi(x) \}^{\frac{1}{2}}$  is subadditive on  $(-\infty, \infty)$ . As an application of (2.6), we have the following.

$$(2.7) \quad \lim_{t \rightarrow 0} |\psi(t)| / t^2 = \sup_{-\infty < t < \infty} |\psi(t)| / t^2.$$

(See Theorem 7.1.11 of Hille and Phillips [3], p. 250). The following well-known result is a corollary of (2.7). If  $\phi(x) = 1 + o(x^2)(x \rightarrow 0)$ , then  $\phi(x) \equiv 1$ .

In view of the inequality of the arithmetic and geometric means, we obtain

further from (2.5) that

$$|\psi(x + y) - \psi(x) - \psi(y)| \leq \Re\psi(x) + \Re\psi(y).$$

This, in turn, implies that  $\Re\psi(x) + \Re\psi(y) \geq \frac{1}{2}\Re\psi(x + y)$ , an inequality which is well-known for the case  $x = y$  (see e.g. Lukacs [5], p. 60).

3. PROOF OF THEOREM 1. Using the notation

$$Q_\alpha(v) = v^\alpha p(|x| > v) = v^\alpha \int_{|t|>v} dF(t),$$

we have the following. Suppose  $0 < \alpha < 2$ , and

$$(3.1) \quad \limsup_{v \rightarrow \infty} Q_\alpha(v) = l.$$

Then

$$(3.2) \quad \limsup_{u \rightarrow 0+} u^{-\alpha} \Re\psi(u) \leq 2^{2-\alpha} l / \alpha (2 - \alpha).$$

From (2.4) we have that

$$\begin{aligned} \Re\psi(u)/u^\alpha &\leq 2u^{-\alpha} \int_0^u \Re\psi(x)/x \, dx \\ &= 2u^{-\alpha} \int_{-\infty}^{\infty} dF(t) \int_0^u (1 - \cos xt)/x \, dx \\ &= 2u^{-\alpha} \int_{-\infty}^{\infty} dF(t) \int_0^{ut} (1 - \cos y)/y \, dy \\ &= 2 \int_0^\infty (1 - \cos y)y^{-(1+\alpha)} Q_\alpha(y/u) \, dy. \end{aligned}$$

Since  $0 < \alpha < 2$ ,  $(1 - \cos y)/y^{1+\alpha} \in \mathcal{L}(0, \infty)$ . Hence

$$\begin{aligned} \limsup_{u \rightarrow 0+} u^{-\alpha} \Re\psi(u) &\leq 2l \int_0^\infty (1 - \cos y)/y^{1+\alpha} \, dy \\ &\leq l \int_0^2 y^{1-\alpha} \, dy + 4l \int_2^\infty y^{-1-\alpha} \, dy \\ &= 2^{2-\alpha} l / \alpha (2 - \alpha). \end{aligned}$$

(For an exact formula for  $\int_0^\infty y^{-\alpha-1}(1 - \cos y) \, dy$  see Feller's "An introduction to probability theory" vol. II p. 542).

From the Truncation Inequality (see e.g. Loève [4], p. 196), we have that

$$\int_{|t|>v} dF(t) \leq 14v \int_0^{1/v} \Re\psi(u) \, du \quad (v > 0).$$

It follows immediately that, if  $\alpha > 0$ , and

$$\limsup_{u \rightarrow 0+} u^{-\alpha} \Re\psi(u) = m,$$

then

$$\limsup_{v \rightarrow \infty} Q_\alpha(v) \leq 14m.$$

Theorem 1 is now an immediate consequence of the above results.

Before proving Corollary 1, we require a lemma.

LEMMA 1. If  $\alpha < 3$ , and  $Q_\alpha(v) \rightarrow 0 (v \rightarrow \infty)$ , then

$$u^{1-\alpha} \int_{-1/u}^{1/u} t \, dF(t) - u^{-\alpha} g_m \phi(u) \rightarrow 0 \quad (u \rightarrow 0+).$$

PROOF. We show that, if (3.1) holds and  $\alpha < 3$ , then

$$(3.3) \quad \limsup_{u \rightarrow 0+} |u^{1-\alpha} \int_{-1/u}^{1/u} t dF(t) - u^{-\alpha} g_m \phi(u)| \leq ((7 - 2\alpha)/(6 - 2\alpha))l.$$

We have that

$$u^{-\alpha} g_m \phi(u) = u^{-\alpha} \int_{-1/u}^{1/u} \sin ut dF(t) + u^{-\alpha} \int_{|t|>1/u} \sin ut dF(f),$$

and so

$$(3.4) \quad \limsup_{u \rightarrow 0+} |u^{-\alpha} g_m \phi(u) - u^{-\alpha} \int_{-1/u}^{1/u} \sin ut dF(t)| \leq l.$$

But, writing  $F(t) + F(-t) = f(t)$ , we have

$$\begin{aligned} |u^{1-\alpha} \int_{-1/u}^{1/u} t dF(t) - u^{-\alpha} \int_{-1/u}^{1/u} \sin ut dF(t)| &= |u^{-\alpha} \int_0^{1/u} (ut - \sin ut) dF(t)| \\ &= |u^{-\alpha} \int_0^{1/u} dF(t) \int_0^{ut} (1 - \cos y) dy| \\ &= |u^{-\alpha} \int_0^1 (1 - \cos y) dy \int_{y/u}^{1/u} dF(t)| \\ &\leq \int_0^1 (1 - \cos y) y^{-\alpha} Q_\alpha(y/u) dy. \end{aligned}$$

Since  $\alpha < 3$ ,  $(1 - \cos y)/y^\alpha \in \mathcal{L}(0, 1)$ . Hence

$$(3.5) \quad \limsup_{u \rightarrow 0+} |u^{1-\alpha} \int_{-1/u}^{1/u} t dF(t) - u^{-\alpha} \int_{-1/u}^{1/u} \sin ut dF(t)| \leq l \int_0^1 (1 - \cos y)/y^\alpha dy \leq l/2(3 - \alpha).$$

Inequality (3.3) now follows from (3.4) and (3.5).

PROOF OF COROLLARY 1. Heyde and Rohatgi [2] have shown that, for  $0 < \alpha < 2$ , (1.5) is equivalent to the following pair of conditions:

$$(3.6) \quad u^{1-\alpha} \int_{-1/u}^{1/u} t dF(t) \rightarrow 0 \quad (u \rightarrow 0+);$$

$$(3.7) \quad Q_\alpha(v) \rightarrow 0 \quad (v \rightarrow \infty).$$

For  $0 < \alpha < 2$ , the corollary is therefore obtained from Theorem 1, Lemma 1 and the above result. If  $\alpha \geq 2$ , we noted in section 2 that (1.4) cannot hold unless  $\phi(x) \equiv 1$ , and this is also true of (1.5) by the Central Limit Theorem.

REMARKS. (1) Heyde and Rohatgi note that, for  $\alpha < 1$ , (3.6) is implied by (3.7). We note that, if  $\alpha > 1$ , then (3.7) implies the existence of  $\mu$ . An easy argument, similar to that employed in the proof of Lemma 1, shows that, if  $\alpha > 1$  and (3.7) holds, then

$$(3.8) \quad u^{1-\alpha} \mu - u^{-\alpha} g_m \phi(u) \rightarrow 0 \quad (u \rightarrow 0+).$$

This result, together with Lemma 1, imply that (3.6) may be replaced by the condition  $\mu = 0$ , as long as  $\alpha > 1$ .

(2) From Boas' proof of his Theorem 3 in [1], we have that, for  $0 < \alpha < 2$ , a necessary and sufficient condition that  $\int_{-\infty}^{\infty} |t|^\alpha dF(t)$  be finite is that

$$(3.9) \quad \Re\psi(x)/x^{\alpha+1} \in \mathcal{L}(0, 1).$$

Taking  $h(x) = x^{-\alpha-1}\mathcal{R}\psi(x)$  in (2.3), we see that (3.9) implies that  $x^{-\alpha}\mathcal{R}\psi(x) \rightarrow 0(x \rightarrow 0+)$ . It is also true that, for  $0 < \alpha < 2$ ,

$$(3.10) \quad \psi(x)/x^{\alpha+1} \varepsilon \mathcal{L}(0, 1)$$

implies that  $x^{-\alpha}\psi(x) \rightarrow 0(x \rightarrow 0+)$ . When  $0 < \alpha < 1$ , there is nothing to prove (see comment after Theorem 1). When  $\alpha \geq 1$ , then  $\mu$  exists and  $x^{-1}\psi(x) \rightarrow -i\mu(x \rightarrow 0+)$ . Therefore, from (3.10),  $\mu = 0$ . This deals with the case  $\alpha = 1$ . For  $\alpha > 1$ , we need to show that  $x^{-\alpha} \mathcal{I}_m \psi(x) \rightarrow 0(x \rightarrow 0+)$ . This follows from (3.8).

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