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UNIQUENESS OF SIGNATURE FOR SIMPLE CURVES

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Abstract. We propose a topological approach to the problem of determining a curve from its iterated integrals. In particular, we prove that a family of terms in the signature series of a two dimensional closed curve with bounded total variation are in fact moments of its winding number. This relation allows us to prove that the signature series of a class of simple non-smooth curves uniquely determine the curves. This implies that outside a Chordal SLE_κ null set, where $0 < \kappa \leq 4$, the signature series of curves uniquely determine the curves. Our calculations also enable us to express the Fourier transform of the n-point functions of SLE curves in terms of the expected signature of SLE curves. Although the techniques used in this article are deterministic, the results provide a platform for studying SLE curves through the signatures of its sample paths.

Keywords: Rough path theory; Uniqueness of signature problem; SLE curves.

1. Introduction

The signature of a path is a formal series of its iterated integrals. In [6], K.T. Chen observed that the map that sends a path to its signature forms a homomorphism from the concatenation algebra to the tensor algebra and used it to study the cohomology of loop spaces. Recent interests in the study of signature has been sparked by its role in the rough path theory. In particular, it was shown by Hambly and Lyons in [10] that for ODEs driven by paths with bounded total variations, the signature is a fundamental representation of the effect of the driving signal on the solution.

This article has two purposes:

1. To determine the winding number of a curve from its signature.

2. To prove, using a relation obtained from answering 1., that the signature of a class of simple curves uniquely determine the curves.

The first question was first considered as far back as 1936, in a paper by Rado[19], who observed that the second term of the signature series of a smooth path is equal to the integral of its winding number around (x, y), considered as a function of (x, y). In [28], Yam considered the same problem as ours, but used a different approach. He started with the formula

$$\text{Winding number around } z = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\mathrm{d}w.$$

and smoothened the kernel $w \to \frac{1}{w-z}$ around the singularity at $w = z$. He then expanded $\frac{1}{w-z}$ into a power series of w and used the fact that the line integrals along γ of polynomials in w can be expressed in terms of the signature of γ.

Here we take a different approach and obtained a formula for the Fourier transform of the winding number, which appears to be simpler than the formula for the winding number itself. A classical result about iterated integrals, first proved by Chen [7], states that the logarithm of the signature of any path is a Lie series. The first result of this article states that the coefficient of some Hall basis elements in the log signature series are in fact moments of the winding number. Throughout this article, we will use $S(\gamma)_{0,1}$ to denote the signature of a path γ.

Theorem 1. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with bounded total variation. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the standard basis of $\mathbb{R}^2$. Let $N \geq 1$ and let $\mathcal{L}_N(\{\mathbf{e_1}, \mathbf{e_2}\})$ denote the space of Lie polynomials of degree less than or equal to N generated by the alphabets $\mathbf{e_1}$ and $\mathbf{e_2}$ with respect to the concatenation product. Let $\mathcal{B}_N$ denote a Hall basis of $\mathcal{L}_N(\{\mathbf{e_1}, \mathbf{e_2}\})$. Let $\mathcal{W}_{n,k}$ denote the set of all indices $(i_1, \ldots, i_{n+k}) \in \{1,2\}^{n+k}$ which contains n 1s and k 2s and either*

$$\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right] \in \mathcal{B}_N \text{ or } -\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right] \in \mathcal{B}_N.$$

Then

$$\begin{aligned} &\sum_{(i_1,\ldots,i_{n+k}) \in \mathcal{W}_{n,k}} (-1)^k \left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right]^* (\log S(\gamma)) \\ (1.1) \quad = &\int_{\mathbb{R}^2} \frac{x^n y^k}{n!k!} \eta(\gamma - \gamma_0, (x,y))\, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

where $\eta(\gamma - \gamma_0, (x,y))$ is the winding number of the curve $\gamma - \gamma_0$ around the points $x\mathbf{e}_1 + y\mathbf{e}_2$. Here $\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right]^*$ denotes the dual basis corresponding to the basis element

$$\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right]$$

in $\mathcal{B}_N$.

As the winding number of a path does not contain information about the order at which it passes through points, whereas signature does, we cannot expect that the signature of a path can be expressed in terms of just winding numbers. In particular, let a and b be two closed curves in $\mathbb{R}^2$, both starting at 0 and let $\star$ denote the concatenation operation between two paths. Then $a \star b$ and $b \star a$ have the same winding number around any point, but in general do not have the same signature. Nevertheless, it is natural to ask how many terms in the signature series of a path can be represented in terms of its winding numbers. The answer is that the first four terms of a closed curve's signature can be expressed in terms of its winding number.

Corollary 2. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with bounded total variation. The first four terms of $\log(S(\gamma))$ can be expressed in terms of the function $(x,y) \to \eta(\gamma - \gamma_0, (x,y))$ alone.*

At the end of section three, we will prove that the number "four" is sharp. In other words, there are two paths γ, γ' which has the same winding number around every point, but the fifth terms of the signature of γ and γ' differs. The reason that the fifth term of the log signature is not a function of winding number alone is because the Hall basis for $\mathcal{L}_5(\{\mathbf{e_1}, \mathbf{e_2}\})$ will contain a basis element of the form:

$$[[\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_i}, [\mathbf{e_1}, \mathbf{e_2}]]], i = 1, 2.$$

However, we are unable to express $[[\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]]]^* \left(\log S(\gamma)_{0,1}\right)$ in terms of just the winding number of γ. This corresponds to the difficulty in expressing the iterated integral

$$\int_{0<s<t<1} [[\gamma_s, \mathrm{d}\gamma_s], [\gamma_t, [\gamma_t, \mathrm{d}\gamma_t]]]$$

in terms of the moments of the winding number of γ.

Uniqueness of signature

If we consider the signature as a representation of paths, then an interesting question is whether this representation is faithful. This was first considered by Chen himself [8], who proved that irreducible, piecewise regular continuous paths have the same signature if and only if they are equal up to a translation and a reparametrisation.

His result was generalised with a new, quantitative approach by Hambly and Lyons in [10] who showed that two paths γ and $\tilde{\gamma}$ with finite total variations have the same signature if and only if γ can be expressed as the concatenation of $\tilde{\gamma}$ with a tree-like path σ. A continuous function $\sigma : [0,1] \to \mathbb{R}^d$ is tree-like if there exists a continuous function $h : [0,1] \to [0,\infty)$ such that $h(0) = h(1) = 0$ and

$$|\sigma_t - \sigma_s| \leq h(t) + h(s) - 2 \inf_{s \leq u \leq t} h(u).$$

Using the relation between signature of a path and its winding number, we are able to establish uniqueness amongst paths γ satisfying the following conditions:

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$.

(1) γ can be reparametrised as a continuous curve $\tilde{\gamma} : [0,1] \to \overline{\mathbb{D}}$, such that $\tilde{\gamma}(0) = -1$, $\tilde{\gamma}(1) = 1$ and $\tilde{\gamma}(0,1) \subset \mathbb{D}$.
(2) γ has finite p-variation for some $1 \leq p < 2$.
(3) γ is a simple curve.

Let $\mathcal{C}_2(-1, 1, \mathbb{D})$ denote the set of paths γ satisfying these three conditions.

Theorem 3. *Let γ, $\gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$. Then $S(\gamma)_{0,1} = S(\gamma')_{0,1}$ if and only if γ and γ' are equal up to a reparametrisation.*

The $p = 1$ case of Theorem 3 is a direct consequence of the result of Hambly and Lyons. An interesting, but difficult extension is to prove that if the signatures of two curves with finite $p > 1$ -variations are equal, then the paths are equal up to the tree-like path equivalence. The restriction $1 \leq p < 2$ gives us the existence of signature for free, thanks to Young's integration theory.

Theorem 3 only applies to paths with finite p-variations, where $p < 2$. In particular, our results can only be applied to study stochastic processes whose sample paths are almost surely smoother than the Brownian motion sample paths. One example of such processes is the Chordal SLE_κ measure. The SLE measures were born from the study of lattice models which have conformally invariant scaling limit. There are a number of other lattice models whose scaling limit have been proved to be an SLE curve under some boundary conditions, such as the loop erased random walk ($\kappa = 2$, [12]), the Ising model ($\kappa = 3$, [5]), the level lines of Gaussian

Free Field ($\kappa = 4$, [23]), percolation on the triangular lattice ($\kappa = 6$, [4] and [25]), and the Peano curve of the uniform spanning tree ($\kappa = 8$, [12]).

The path regularity and, in particular, the roughness of SLE curves, in relation to the speed κ of the driving Brownian motion, is an extremely interesting topic. It is intuitively clear that the SLE curves becomes rougher as the speed of the driving Brownian motion increases. In [11], the optimal Hölder exponent for SLE curves under the capacity parametrisation was proved to be

$$\min\left(\frac{1}{2}, 1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}\right).$$

In [2], V. Beffara proved that the almost Hausdorff dimension of SLE curves is $\min\left(1 + \frac{\kappa}{8}, 2\right)$. Therefore, the optimal Hölder exponent cannot exceed $\frac{1}{1+\frac{\kappa}{8}}$. B. Werness[27] proved that for $0 < \kappa \leq 4$, almost surely, the SLE curve has finite p variation for any $p > 1 + \frac{\kappa}{8}$. In another words, the roughness of an SLE curve grows linearly with the speed of the driving Brownian motion. It is strongly believed that this remains true for $4 < \kappa < 8$. However, to the best of our knowledge, this problem remains open.

Werness's result allowed him to define the signatures of SLE curves using Young's integral and used Green's theorem to compute the first three gradings of the expected signature of SLE curves.

In the study of SLE curves we often do not care about the curves' parametrisations and in some cases, it may be convenient to study the curves' signature instead. In order to do so, one must prove that there is a $1-1$ correspondence between curves and their signatures, outside a null set. Such injectiveness was proved for Brownian motion by Le Jan and Qian in [13] and for general diffusion processes by Geng and Qian. Both results rely on the Strong Markov property. Although the Chordal SLE$_\kappa$ measure is not Markov, the inversion problem can be tackled for $\kappa \leq 4$ since the Chordal SLE$_\kappa$ measure is supported on simple curves from -1 and 1. Therefore, it follows from Theorem 3 that:

Theorem 4. *Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then there exists a $\mathbb{P}$-null set $\mathcal{N}$ such that if $\gamma, \gamma' \in \mathcal{N}^c$ and $S(\gamma)_{0,1} = S(\gamma')_{0,1}$, then γ and γ' are equal up to a reparametrisation.*

The expected signature can be considered as the "Laplace's transform" of a stochastic process and has first been studied in [16]. The sequence of n-point functions of the Chordal SLE measure was first studied by O. Schramm. Using a generalised Green's theorem for non-smooth curves, we may prove the following relationship between the expected signature and the sequence of n-point functions.

Theorem 5. *Let $0 \leq \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points 1 and -1. For each $\gamma \in \mathcal{C}_2(-1, 1, \mathbb{D})$, let $\Phi(\gamma)$ denote the concatenation of γ with the upper semi-circular boundary of $\mathbb{D}$ from 1 to -1. Let Γ_n denotes the n-point function associated with $\mathbb{P}_\kappa$, then*

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} e^{\sum_{i=1}^{N}\lambda_i x_i+\mu_i y_i}\Gamma_N\left(\left(x_1,y_1\right),\ldots,\left(x_N,y_N\right)\right)\mathrm{d}x_1\cdots\mathrm{d}y_N\\ =&\sum_{n_1,\ldots,n_N,k_1\ldots k_N\geq 0}\Pi_{i=1}^{N}\left(\lambda_i\right)^{n_i}\left(-\mu_i\right)^{k_i}\mathbf{e}_1^{*\otimes(n_1+1)}\otimes\mathbf{e}_2^{*\otimes(k_1+1)}\sqcup\ldots\\ &\ldots\sqcup\mathbf{e}_1^{*\otimes(n_N+1)}\otimes\mathbf{e}_2^{*\otimes(k_N+1)}\left(\mathbb{E}\left[S\left(\Phi\left(\gamma\right)\right)_{0,1}\right]\right).\end{aligned}$$

where e_i^* is the dual basis corresponding the standard basis of $\mathbb{R}^2$ (see section 2.1) and $\sqcup$ denotes the shuffle product (see Proposition 7).

The plan for the rest of the article is as follows.
In section 2, we recall the basic results about the signature and winding number.
In section 3, we prove Theorem 1 and Corollary 2.
In section 4, we prove Theorem 3.
In section 5, we prove Theorem 4.
In section 6, we prove Theorem 5.

2. Preliminaries

2.1. **Basic notations.** Let $T^n\left(\mathbb{R}^d\right)$ and $T\left(\mathbb{R}^d\right)$ denote the graded algebras on $\mathbb{R}^d$ defined by

$$T^n\left(\mathbb{R}^d\right):=\oplus_{k=0}^{n}\left(\mathbb{R}^d\right)^{\otimes k}$$

and

$$T\left(\mathbb{R}^d\right):=\oplus_{k=0}^{\infty}\left(\mathbb{R}^d\right)^{\otimes k}$$

where $\left(\mathbb{R}^d\right)^{\otimes 0}:=\mathbb{R}$. We equip $T\left(\mathbb{R}^d\right)$ with the Euclidean metric by identifying $\left(\mathbb{R}^d\right)^{\otimes k}$ with $\mathbb{R}^{d^k}$.

We shall define two projection maps as follow:

(1) $\pi^{(n)}$ will denote the projection map from $T\left(\mathbb{R}^d\right)$ to $\left(\mathbb{R}^d\right)^{\otimes n}$.
(2) π_n will denote the projection map from $T\left(\mathbb{R}^d\right)$ onto $T^n\left(\mathbb{R}^d\right)$.

Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_d\}$ be the standard basis of $\mathbb{R}^d$. Let $\mathbf{e}_1^*,\ldots,\mathbf{e}_d^*$ denote the corresponding dual basis of $\mathbb{R}^{d*}$. Note that $\{\mathbf{e}_{i_1}\otimes\cdots\otimes\mathbf{e}_{i_n}:(i_1,\ldots,i_n)\in\{1,\ldots,d\}^n\}$ forms a basis for $\left(\mathbb{R}^d\right)^{\otimes n}$. We now embed, in the algebraic sense, $T\left(\left(\mathbb{R}^d\right)^*\right)$ into $T\left(\left(\mathbb{R}^d\right)\right)^*$ by extending linearly the map $\mathbf{e}_{i_1}^*\otimes\ldots\otimes\mathbf{e}_{i_n}^*$ in $\left(\mathbb{R}^{d*}\right)^{\otimes n}$ defined by

$$\begin{aligned}\mathbf{e}_{j_1}^*\otimes\ldots\otimes\mathbf{e}_{j_k}^*\left(\mathbf{e}_{i_1}\otimes\ldots\otimes\mathbf{e}_{i_n}\right)&=1\text{ if }n=k\text{ and }j_1=i_1,\ldots,j_n=i_n.\\&=0\text{ otherwise.}\end{aligned}$$

2.2. **Signature.** Let $p>1$ and let $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ denote the set of all continuous functions $\gamma:[0,T]\to\mathbb{R}^d$ such that

$$\|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}^p:=\sup_{\mathcal{P}}\sum_k\left|\gamma_{t_{k+1}}-\gamma_{t_k}\right|^p<\infty. \tag{2.1}$$

where the supremum is taken over all finite partitions $\mathcal{P}:=(t_0,t_1,..,t_{n-1},t_n)$, where $0=t_0<t_1<...<t_{n-1}<t_n=T$.

The elements of $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ will be called curves with finite p-variation.

Note that $\|\cdot\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}$ defines a norm on $\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$.

Definition 6. Let $\gamma \in \mathcal{V}^1\left(\mathbb{R}^d\right)$and let $\triangle_n(s,t) := \{(t_1,\ldots,t_n) : s < t_1 < \cdots < t_n < t\}$. The *lift* of γ is a function $S(\gamma) : \{(s,t) : 0 \le s \le t\} \to T\left(\mathbb{R}^d\right)$ defined by

$$S(\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty} \int_{\triangle_n(s,t)} \mathrm{d}\gamma_{t_1} \otimes \ldots \otimes \mathrm{d}\gamma_{t_n} \tag{2.2}$$

where the sum $+$ is the direct sum operation in $T\left(\mathbb{R}^d\right)$ and the integrals are taken in the Lebesgue-Stieltjes sense.

The signature of a path $\gamma \in \mathcal{V}^1\left(\mathbb{R}^d\right)$ on $[0,T]$ is defined to be $S(\gamma)_{0,T}$.

We shall use the following properties of signature, whose proofs can be found in [14].

1. (Invariance under reparametrisation)For any $t \in [0,\infty)$, $S(\gamma)_{0,t}$ is invariant under any reparametrisation of γ on $[0,t]$.
2. (Inverse) $S(\gamma)_{0,T} \otimes S(\overleftarrow{\gamma})_{0,T} = \mathbf{1}$, where $\overleftarrow{\gamma}(t) := \gamma(T-t)$ is the reversal of γ and $\mathbf{1}$ is the identity element in $T\left(\mathbb{R}^d\right)$.
3. (Chen's Identity)$S(\gamma)_{s,u} \otimes S(\gamma)_{u,t} = S(\gamma)_{0,t}$ for any $0 \le s < u < t \le T$
4. (Scaling and translation)Let $\lambda \in \mathbb{R}^d$, $\mu \in \mathbb{R}$, then

$$S(\lambda + \mu\gamma)_{s,t} = 1 + \sum_{n=1}^{\infty} \mu^n \int_{\triangle_n(s,t)} \mathrm{d}\gamma(t_1) \otimes \ldots \otimes \mathrm{d}\gamma(t_n)$$

5.(Shuffle product formula) We define a *(r,s)-shuffle* to be a permutation of $\{1,2,...,r+s\}$ such that $\sigma(1) < \sigma(2) < .. < \sigma(r)$ and $\sigma(r+1) < ... < \sigma(r+s)$.

Proposition 7. *([14],Theorem 2.15)Let $1 \le p < 2$ and $\gamma \in \mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$, then*

$$\begin{aligned} & \mathbf{e}^*_{k_1} \otimes \ldots \otimes \mathbf{e}^*_{k_r}(S(\gamma))\, \mathbf{e}^*_{k_{r+1}} \otimes \ldots \otimes \mathbf{e}^*_{k_{r+s}}(S(\gamma)) \\ = & \sum_{(r,s)-shuffles\ \sigma} \mathbf{e}^*_{k_{\sigma^{-1}(1)}} \otimes \ldots \otimes \mathbf{e}^*_{k_{\sigma^{-1}(r+s)}}(S(\gamma)). \end{aligned}$$

where $\cdot$ is the multiplication operation in $\mathbb{R}$.

The sum

$$\sum_{(r,s)-shuffles\ \sigma} \mathbf{e}^*_{k_{\sigma^{-1}(1)}} \otimes \ldots \otimes \mathbf{e}^*_{k_{\sigma^{-1}(r+s)}}$$

is denoted by $\mathbf{e}^*_{k_1} \otimes \ldots \otimes \mathbf{e}^*_{k_r} \sqcup \mathbf{e}^*_{k_{r+1}} \otimes \ldots \otimes \mathbf{e}^*_{k_{r+s}}$.

2.3. **Winding number.** In this section, we shall recall the definition of winding number and a few key basic facts that we shall use.

Definition 8. A continuous function $\gamma : [0,1] \to \mathbb{R}^2$ is a closed curve if $\gamma_0 = \gamma_1$.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous function. Let $z \in \mathbb{R}^2 \backslash \gamma[0,1]$. Then

$$g^\gamma_z(s) := \frac{\gamma_s - z}{\|\gamma_s - z\|}$$

defines a function $[0,1] \to \mathbb{S}^1$.

Let $p : \mathbb{R} \to \mathbb{S}^1$, $p(x) = e^{ix}$ be a covering map for $\mathbb{S}^1$. Then there exists a lift $\tilde{g}^\gamma_z : [0,1] \to \mathbb{R}$ such that $p \circ \tilde{g}^\gamma_z = g^\gamma_z$. The winding number of γ will be defined in terms of $\tilde{g}_s(z)$ by the following lemma:

Lemma 9. *([18], Chapter 3 Lemma 1 and 2)Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve, and $z \in \gamma[0,1]$. Then the number*

$$\eta(\gamma, z) := \frac{1}{2\pi}\left(\tilde{g}_z^{\gamma}(1) - \tilde{g}_z^{\gamma}(0)\right) \tag{2.3}$$

depends only on γ and z but not on the lift $\tilde{g}_z^{\gamma}$. Moreover, $\eta(\gamma, z)$ is an integer and is called the winding number of γ around the point z.

Remark 10. We may define the winding angle for any $\gamma : [a,b] \to \mathbb{R}^2$ by simply replacing 0 by a, 1 by b in the above definition.

The following theorem, which we shall need, is intuitively clear but is highly non-trivial:

Theorem 11. *([17], p404)Let $\gamma : [0,1] \to \mathbb{R}^2$ be a simple closed curve. Let Int(γ) and Ext(γ) be its interior and exterior respectively. Then $\eta(\gamma, z) = 0$ for all $z \in$Ext(γ). Moreover, either $\eta(\gamma, z) = 1$ for all $z \in$ Int(γ) or $\eta(\gamma, z) = -1$ for all $z \in$ Int (γ). γ is called positively oriented if $\eta(\gamma, z) = 1$ and negatively oriented otherwise.*

A key tool in our proof of Proposition 1 is the following Green's theorem for paths with bounded total variations.

Theorem 12. *([19] and [1])Let $\gamma = \left(\gamma^{(1)}, \gamma^{(2)}\right) : [0,T] \to \mathbb{R}^2$ be a closed curve with bounded total variation. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives in both variables. Then*

$$\int_{\mathbb{C}} \left(\partial_x f(x,y) + \partial_y g(x,y)\right) \eta(\gamma, (x,y))\, \mathrm{d}x\mathrm{d}y = \int_{\gamma} f \mathrm{d}\gamma_s^{(2)} - g \mathrm{d}\gamma_s^{(1)}. \tag{2.4}$$

and

$$\|\eta(\gamma, \cdot)\|_{L^2} \leq \frac{1}{\sqrt{4\pi}} \|\gamma\|_{\mathcal{V}^1([0,T],\mathbb{R}^2)} \tag{2.5}$$

where the equality in (2.5) holds if and only if there exists $(x,y) \in \mathbb{R}^2$, $n \in \mathbb{N}$ and $R > 0$ such that $\gamma_t = (x + R\cos 2\pi nt, x + R\sin 2\pi nt)$.

The $f(x,y) = x$, $g(x,y) = y$ case in (2.4) was proved in [19] and the proof for the general case is essentially the same. New, complete proofs for (2.4) were subsequently given by [26] and [28].

The second inequality is the well-known Banchoff-Pohl isoperimetric inequality[1].

3. Proof of Theorem 1

Before we give a proof of Theorem 1, we would like to first recall some elementary Lie algebra.

3.1. **Hall basis.** Let A be a set. Let K be a field. The Hall basis, introduced in [9], is the simplest way of assigning a basis to a free Lie algebra $\mathcal{L}(A)$ generated by A through the operation of concatenation. Let us describe the construction of the Hall basis.

Let $M(A)$ denote the set of planar rooted complete binary trees with leaves labelled as elements of A. A Hall set H is a subset of $M(A)$ satisfying the following conditions:

- $A \subseteq H$.

- We may assign a total order $\leq$ on H.
- $h = (h_1, h_2)$ in $M(A) \backslash A$ belongs to H if and only if

$$h_1, h_2 \in H \text{ and } h < h_2,\ h_1 < h_2$$

and

$$\text{either } h_1 \in A \text{ or } h_1 = (h_1', h_2') \text{ where } h_2' \geq h_2.$$

There is in general infinitely many Hall sets for each set A, unless $|A| = 1$ or 0.

Let A^* denote the set of all words formed by a finite concatenation of elements of A. Let $f : M(A) \to A^*$ be the map defined by $f(t) = t$ if $t \in A$, and if $t = (t_1, t_2)$, then $f(t) = f(t_1) f(t_2)$.

Let H be a fixed Hall set. For each $h \in f(H)$, define a polynomial P_h by the rule that:

- $P_h = h$ if $h \in A$.
- $P_h = [P_{h_1}, P_{h_2}]$, if $h = h_1 h_2$, where $h_1, h_2 \in f(H)$.

It is a theorem of Hall (for a proof see [20], Theorem 4.9) that for each Hall set H, the set $\{P_h : h \in f(H)\}$ forms a basis for the free Lie algebra generated by A over a field K.

3.2. **Proof of Theorem 1.** A key idea in proving Theorem 1 lies in the fact that the coefficients of some Hall basis elements can be reduced to a single line integral, as illustrated by the following lemma.

Lemma 13. *Let $\gamma : [0,1] \to \mathbb{R}^2$ be a continuous closed curve with bounded total variation. Let $\eta(\gamma, (x,y))$ denote the winding number of γ around $x\mathbf{e}_1 + y\mathbf{e}_2$.*

Then

$$\mathbf{e}_1^{*\otimes(n+1)} \otimes \mathbf{e}_2^{*\otimes(k+1)} \left(S(\gamma)_{0,1} \right) = \frac{(-1)^k}{n!k!} \int_{\mathbb{R}^2} x^n y^k \eta(\gamma - \gamma_0; (x,y))\, \mathrm{d}x \mathrm{d}y. \tag{3.1}$$

Proof. Let $\gamma^{(1)}$and $\gamma^{(2)}$ be the first and second coordinate components of γ respectively.

Recall that

$$\mathbf{e}_1^{*\otimes(n+1)} \otimes \mathbf{e}_2^{*\otimes(k+1)} \left(S(\gamma)_{0,1} \right) = \int_0^1 \mathrm{d}\gamma_{s_1}^{(1)} \ldots \mathrm{d}\gamma_{s_n}^{(1)} \mathrm{d}\gamma_{s_{n+1}}^{(2)} \ldots \mathrm{d}\gamma_{s_{n+k}}^{(2)}.$$

The key idea here is to integrate with respect to $\gamma^{(1)}$s first and then integrate the $\gamma^{(2)}$s.

$$
\begin{aligned}
& \mathbf{e}_1^{*\otimes(n+1)} \otimes \mathbf{e}_2^{*\otimes(k+1)} \left(S\left(\gamma\right)_{0,1} \right) \\
= & \int ... \int_{0<t_1<..<t_{n+1}<s_1<...<s_{k+1}<1} \mathrm{d}\gamma_{t_1}^{(1)} ... \mathrm{d}\gamma_{t_{n+1}}^{(1)} \mathrm{d}\gamma_{s_1}^{(2)} ... \mathrm{d}\gamma_{s_{k+1}}^{(2)} \\
= & \int_{0<s_1<...<s_{k+1}<1} \frac{1}{n!} \left(\gamma_{s_1}^{(1)} - \gamma_0^{(1)} \right)^{n+1} \mathrm{d}\gamma_{s_1}^{(2)} ... \mathrm{d}\gamma_{s_{k+1}}^{(2)} \\
= & \int_0^1 \int_{s_1}^1 ... \int_{s_{k-1}}^1 \int_{s_k}^1 \frac{1}{n!} \left(\gamma_{s_1}^{(1)} - \gamma_0^{(1)} \right)^{n+1} \mathrm{d}\gamma_{s_{k+1}}^{(2)} ... \mathrm{d}\gamma_{s_1}^{(2)} \text{ by Fubini's theorem} \\
= & \frac{1}{(n+1)!} \frac{1}{k!} \int_0^1 \left(\gamma_{s_1}^{(1)} - \gamma_0^{(1)} \right)^{n+1} \left(\gamma_1^{(2)} - \gamma_{s_1}^{(2)} \right)^k \mathrm{d}\gamma_{s_1}^{(2)} \\
= & \frac{1}{n!} \frac{1}{k!} \int_D \left(x - \gamma_0^{(1)} \right)^n \left(\gamma_1^{(2)} - y \right)^k \eta\left(\gamma; (x,y)\right) \mathrm{d}x \mathrm{d}y \text{ by (2.4)} \\
= & \frac{(-1)^k}{n!k!} \int_D x^n y^k \eta\left(\gamma - \gamma_0; (x,y)\right) \mathrm{d}x \mathrm{d}y.
\end{aligned}
$$

□

We will now give a proof of Theorem 1.

Proof. (of Theorem 1)By Proposition 13, it suffices to prove that

$$
\begin{aligned}
& \sum_{(i_1,\dots,i_{n+k}) \in \mathcal{W}_{n,k}} (-1)^k \left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \left[\dots, \left[\mathbf{e}_{i_{n+k}}, \left[\mathbf{e}_1, \mathbf{e}_2 \right] \right] \right] \right] \right]^* \left(\log S\left(\gamma\right) \right) \\
= & \left(-1\right)^k \mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S\left(\gamma\right) \right).
\end{aligned}
$$

Let $\mathcal{L}_N \{\mathbf{e_1}, \mathbf{e_2}\}$ denote the set of Lie polynomials of degree lass than or equal N generated by $\{\mathbf{e_1}, \mathbf{e_2}\}$. Let $\mathcal{B}_N$ be a Hall basis for $\mathcal{L}_N$. Any basis element in $\mathcal{B}_N$ of degree at least two can be written as

$$
\pm \left[\mathcal{P}_n, \left[\dots \mathcal{P}_1, \left[\mathbf{e_1}, \mathbf{e_2} \right] \right] \right]
$$

where $\mathcal{P}_1, \dots, \mathcal{P}_n$ are homogeneous Lie polynomials.

As γ is closed,

$$
\mathbf{e}_1^* \left(\log S\left(\gamma\right) \right) = 0 = \mathbf{e}_2^* \left(\log S\left(\gamma\right) \right)
$$

and therefore we may write $\pi_N \left(S\left(\gamma\right) \right)$ as

$$
\exp \left(\sum_{n,k \geq 0} \sum_{(i_1,\dots,i_{n+k}) \in \mathcal{W}_{n,k}} a_{i_1,\dots,i_{n+k}} \left[\mathbf{e}_{i_1}, \left[\dots, \left[\mathbf{e}_{i_{n+k}}, \left[\mathbf{e}_1, \mathbf{e}_2 \right] \right] \right] \right] + \mathbf{B} \right) \tag{3.2}
$$

where $\mathbf{B}$ denote a linear combination of terms of the form

$$
\left[\mathcal{P}_j, \left[\dots \mathcal{P}_1, \left[\mathbf{e_1}, \mathbf{e_2} \right] \right] \right] \tag{3.3}
$$

where $\mathcal{P}_1, \dots, \mathcal{P}_j$ are homogeneous Lie polynomials and at least one of $\mathcal{P}_1, \dots, \mathcal{P}_j$ has degree at least 2.

Note that the the word $\mathbf{e_1}^{\otimes n+1} \otimes \mathbf{e}_2^{\otimes k+1}$ has only one change of alphabet, from the $n+1$ th to the $n+2$th alphabet.

On the other hand, in the series expansion of exp, the words appearing in the expansion of the tensor powers of $\mathbf{B}$ or the cross terms between

$$\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]$$

and $\mathbf{B}$ or the quadratice or higher tensor powers of $\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]$ will have at least two change of alphabet. This means that the only contribution to the coefficient of $\mathbf{e_1}^{\otimes n+1} \otimes \mathbf{e}_2^{\otimes k+1}$ in the series expansion in (3.2) comes from the term

$$\sum_{(i_1,\ldots,i_{n+k})\in\mathcal{W}_{n,k}} a_{i_1,\ldots,i_{n+k}} \left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right],$$

with n 1s and k 2s in front of $[\mathbf{e}_1, \mathbf{e}_2]$. In another words, we have

$$\begin{aligned}&\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S\left(\gamma\right)\right)\\ =&\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(\sum_{(i_1,\ldots,i_{n+k})\in\mathcal{W}_{n,k}} a_{i_1,\ldots,i_{n+k}} \left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right).\end{aligned}$$

Note that

$$\begin{aligned}&\underbrace{\mathbf{e}_1^* \otimes \ldots \otimes \mathbf{e}_1^*}_{n} \otimes \underbrace{\mathbf{e}_2^* \otimes \ldots \otimes \mathbf{e}_2^*}_{k} \left(\left[\mathbf{e}_{i_1}, \left[\mathbf{e}_{i_2}, \ldots, \left[\mathbf{e}_{i_{n+k}}, [\mathbf{e}_1, \mathbf{e}_2]\right]\right]\right]\right)\\ =&\left(-1\right)^k.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S\left(\gamma\right)\right)\\ =&\sum_{(i_1,\ldots,i_{n+k})\in\mathcal{W}_{n,k}} a_{i_1,\ldots,i_{n+k}} \left(-1\right)^k\end{aligned}$$

for $1 \leq n + k \leq N$. □

We now prove Corollary 2.

Proof. (of Corollary 2)
According to [20] (Example 4.8), a Hall basis for the Lie polynomials up to degree 4 generated by the set $\{\mathbf{e_1}, \mathbf{e_2}\}$ is

$$\mathbf{e_1}, \mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}], [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]], [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]], \tag{3.4}$$

$$[\mathbf{e_1}, [\mathbf{e_1}, [\mathbf{e_1}, \mathbf{e_2}]]], [\mathbf{e_1}, [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]]], [\mathbf{e_2}, [\mathbf{e_2}, [\mathbf{e_1}, \mathbf{e_2}]]]. \tag{3.5}$$

Let $\mathcal{B}_4^*$ be the dual basis corresponding to the basis (3.4). To prove Corollary 2, it is sufficient to express, for each $f \in \mathcal{B}_4^*$, the value $f\left(\log S\left(\gamma\right)\right)$ in terms of the winding number of γ.

As γ is a closed curve, $\mathbf{e}_i^*\left(\log\left(S\left(\gamma\right)_{0,1}\right)\right) = 0$ for $i = 1, 2$.

By Theorem 1,

$$\begin{aligned}
[\mathbf{e_1},\mathbf{e_2}]^*(\log(S(\gamma))) &= \int \eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1,[\mathbf{e_1},\mathbf{e_2}]]^*(\log(S(\gamma))) &= \int x\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_2,[\mathbf{e_1},\mathbf{e_2}]]^*(\log(S(\gamma))) &= \int y\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1,[\mathbf{e}_1,[\mathbf{e_1},\mathbf{e_2}]]]^*(\log(S(\gamma))) &= \frac{1}{2}\int x^2\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_1,[\mathbf{e}_2,[\mathbf{e_1},\mathbf{e_2}]]]^*(\log(S(\gamma))) &= \int xy\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y \\
[\mathbf{e}_2,[\mathbf{e}_2,[\mathbf{e_1},\mathbf{e_2}]]]^*(\log(S(\gamma))) &= \frac{1}{2}\int y^2\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

□

3.3. **Sharpness of Corollary 2.** The purpose of this section is to prove the following sharpness compliment to Corollary 2.

Proposition 14. *There exists two paths γ,γ' such that the winding number of γ and γ' around every point is equal, but the fifth term of their signature differs.*

Proof. Let $\mathbf{e}_i$ denote the path $t\to t\mathbf{e}_i$, $t\in[0,1]$ and let

$$\gamma = \mathbf{e_1}\star\mathbf{e_2}\star-\mathbf{e_1}\star-\mathbf{e_2}\star-\mathbf{e_1}\star-\mathbf{e_2}\star\mathbf{e_1}\star\mathbf{e_2}$$

and

$$\gamma' = -\mathbf{e_1}\star-\mathbf{e_2}\star\mathbf{e_1}\star\mathbf{e_2}\star\mathbf{e}_1\star\mathbf{e}_2\star-\mathbf{e}_1\star-\mathbf{e}_2.$$

where $\star$ denote the concatenation operation on paths.

By Theorem 11 and the additivity of the winding number with respect to the concatenation product,

$$\eta(\gamma,(x,y)) = 1_{[0,1]\times[0,1]\cup[-1,0]\times[-1,0]}(x,y) = \eta\left(\gamma'(x,y)\right).$$

By a directly calculation, we see that the signature of $\mathbf{e}_i$ is

$$e^{\mathbf{e}_i}.$$

Therefore, by Chen's identity,

$$S(\gamma) = e^{\mathbf{e}_1}e^{\mathbf{e}_2}e^{-\mathbf{e}_1}e^{-\mathbf{e}_2}e^{-\mathbf{e}_1}e^{-\mathbf{e}_2}e^{\mathbf{e}_1}e^{\mathbf{e}_2} \tag{3.6}$$

and

$$S(\gamma') = e^{-\mathbf{e}_1}e^{-\mathbf{e}_2}e^{\mathbf{e}_1}e^{\mathbf{e}_2}e^{\mathbf{e}_1}e^{\mathbf{e}_2}e^{-\mathbf{e}_1}e^{-\mathbf{e}_2}. \tag{3.7}$$

We claim that

$$\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma)\rangle = 1$$

and

$$\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma)\rangle = -1.$$

Note that the word $\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1\otimes\mathbf{e}_2\otimes\mathbf{e}_1$ is "square-free", i.e. none of the alphabet in the word is identical to the alphabet on its immediate left or right. This means the contribution to the value of both

$$\left\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma)\right\rangle$$

and

$$\left\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma)\right\rangle$$

only comes from the first order term in exponentials in (3.6) and (3.7). For both, the contribution can only comes in one of the following five combinations:

Combination 1. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}$ exponentials.

Combination 2. $1^{st}, 2^{nd}, 3^{rd}, 4^{th}, 7^{th}$ exponentials.

Combination 3. $1^{st}, 2^{nd}, 3^{rd}, 6^{th}, 7^{th}$ exponentials.

Combination 4. $1^{st}, 2^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.

Combination 5. $1^{st}, 4^{nd}, 5^{rd}, 6^{th}, 7^{th}$ exponentials.

For $S(\gamma)$, the contributions from Combination 1 and Combination 5 is -1, while the contribution from Combination $2-4$ is 1. Therefore,

$$\begin{aligned} & \left\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma)\right\rangle \\ = \; & -1+1+1+1-1 \\ = \; & 1. \end{aligned}$$

For $S(\gamma')$, the contributions from Combination 1 and Combination 5 is 1, while the contribution from Combination $2-4$ is -1. Therefore,

$$\begin{aligned} & \left\langle \mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*\otimes\mathbf{e}_2^*\otimes\mathbf{e}_1^*, S(\gamma')\right\rangle \\ = \; & 1-1-1-1+1 \\ = \; & -1. \end{aligned}$$

□

3.4. **Tree-like path and winding number.**

Lemma 15. *A two dimensional tree-like path γ with bounded total variation is closed and has winding number zero around all points (x,y) in $\mathbb{R}^2\backslash\gamma[0,1]$.*

Proof. A path γ with bounded total variation is tree-like if and only if it has signature $\mathbf{1} := (1,0,0,0\ldots)$. As the first term of the signature of γ is zero, we have

$$\int_0^1 \mathrm{d}\gamma = \gamma_1-\gamma_0 = 0.$$

By Theorem 1,

$$\int_{\mathbb{R}^2} \frac{x^n y^k}{n!k!}\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y = 0$$

for all $n,k\geq 0$. Therefore,

$$\int_{\mathbb{R}^2} e^{\lambda_1 x+\lambda_2 y}\eta(\gamma-\gamma_0,(x,y))\,\mathrm{d}x\mathrm{d}y = 0$$

for all $\lambda_1,\lambda_2\in\mathbb{R}$. As the function $(x,y)\to\eta(\gamma-\gamma_0,(x,y))$ lies in L^2 by (2.5), we have by the injectiveness of Fourier transform on L^2 that

$$\eta(\gamma,(x,y)+\gamma_0) = \eta(\gamma-\gamma_0,(x,y)) = 0$$

for all $(x,y)\in\mathbb{R}^2$ except a Lebesgue null set. As the function $(x,y)\to\eta(\gamma,(x,y)+\gamma_0)$ is locally constant on $\mathbb{R}^2\backslash\gamma[0,1]$, we have

$$\eta(\gamma-\gamma_0,(x,y))=0$$

for all $(x,y)\in\mathbb{R}^2\backslash\gamma[0,1]$. □

Remark 16. The converse of Lemma 15 is not true. Let γ and γ' be the paths defined in the proof of Proposition 14 and η be the concatenation of γ and the reversal of γ'. Then by the additivity of winding number with respect to the concatenation product, η has zero winding number around every point. As the signature of γ and γ' are different, we have by Chen's identity that the signature of η is not $\mathbf{1}$. Therefore, η is not tree-like.

4. Uniqueness of signature

4.1. **Young's integrals and approximation theorems.** While the Lebesgue-Stieltjes integration theory is sufficient to define integrals against functions with finite 1-variations, we need Young's integration theory to define integrals against functions with finite p variations, where $1\leq p<2$. A proof of the following criterion for the existence of Young's integral can be found in [14]:

Theorem 17. *(L. Young[29]) Let $T>0$ and $t\in[0,T]$ and let $p,q\geq1$ be such that $\frac{1}{p}+\frac{1}{q}>1$. Then for $\gamma_\cdot\in\mathcal{V}^p([0,t],\mathbb{R})$ and $\gamma'_\cdot\in\mathcal{V}^q([0,t],\mathbb{R})$, the following limit exists:*

$$\int_0^t\gamma\mathrm{d}\gamma':=\lim_{\|\mathcal{P}\|\to0}\sum\gamma_{t_i}\left(\gamma'_{t_{i+1}}-\gamma'_{t_i}\right)\tag{4.1}$$

where the sum is over partition points $0=t_0<t_1<..<t_n=t$ in $\mathcal{P}$, and $\|\mathcal{P}\|:=\sup_i|t_{i+1}-t_i|$.

Furthermore, as a function of t, $t\to\int_0^t\gamma\mathrm{d}\gamma'\in\mathcal{V}^q([0,T],\mathbb{R})$, and there is a constant $C_{p,q}>0$ depending only on p and q such that

$$\left\|\int_0^\cdot(\gamma_s-\gamma_0)\,\mathrm{d}\gamma'_s\right\|_{\mathcal{V}^q([0,T],\mathbb{R})}\leq C_{p,q}\|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R})}\|\gamma'\|_{\mathcal{V}^q([0,T],\mathbb{R})}.\tag{4.2}$$

The limit in (4.1) is called the *Young's integral of γ with respect to γ'*. In the case when γ' has finite 1-variation, the integral coincides with the Lebesgue-Stieltjes integral.

Theorem 17 allows us to define the signature of $\gamma\in\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$ for $1\leq p<2$. We will use $\triangle_n(s,t)$ to denote the set $\{(t_1,..,t_n):s<t_1<...<t_n<t\}$ and $\triangle$ to denote the set $\{(s,t):0\leq s\leq t\leq T\}$.

Definition 18. Let $1\leq p<2$ and $\gamma\in\mathcal{V}^p\left([0,T],\mathbb{R}^d\right)$. The *lift* of γ is a function $S(\cdot):\triangle\to T\left(\mathbb{R}^d\right)$, defined by

$$S(\gamma)_{s,t}:=1+\sum_{n=1}^{\infty}\int_{\triangle_n(s,t)}\mathrm{d}\gamma_{t_1}\otimes...\otimes\mathrm{d}\gamma_{t_n}\tag{4.3}$$

where the sum $+$ is the direct sum operation in $T\left(\mathbb{R}^d\right)$.

The signature of γ on $[0,T]$ is defined as $S(\gamma)_{0,T}$.

Remark 19. $S(\gamma)_{\cdot,\cdot}$ is well-defined for $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$, with $p < 2$ because, for instance, $\int \mathrm{d}\gamma \otimes \mathrm{d}\gamma$ exists because $\frac{1}{p} + \frac{1}{p} > 1$. Moreover, the resulting integral remains in $\mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$, so the resulting integral can be integrated again with respect to γ.

Remark 20. The signatures of rougher paths, if exist, will have to be defined using the rough path theory. We shall not need it in this article.

For a continuous function γ and a partition $\mathcal{P} := t_0 = 0 < t_1 < .. < t_n = T$, the piecewise linear interpolation of γ with respect to $\mathcal{P}$ is defined as the following function on $[0,T]$:

$$\gamma_t^{\mathcal{P}} := \gamma_{t_i} + \left(\frac{\gamma_{t_{i+1}} - \gamma_{t_i}}{t_{i+1} - t_i}\right)(t - t_i) \text{ for } t \in [t_i, t_{i+1}]$$

Then the following approximation theorem holds:

Lemma 21. *(Lemma 1.12 and Proposition 1.14,* [14]*)Let p and q be such that $1 \le p < q$. Let $\gamma \in \mathcal{V}^p\left([0,T], \mathbb{R}^d\right)$. Then for all finite partitions $\mathcal{P}$,*

$$\left\|\gamma^{\mathcal{P}}\right\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)} \le \|\gamma\|_{\mathcal{V}^p([0,T],\mathbb{R}^d)}$$

Furthermore for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all partitions $\mathcal{P}$ of $[0,T]$ satisfying $\|\mathcal{P}\| < \delta$ we have

$$\begin{aligned} \left\|\gamma - \gamma^{\mathcal{P}}\right\|_{\mathcal{V}^q([0,T],\mathbb{R}^d)} &< \varepsilon, \text{ and} \\ \sup_{t \in [0,T]} \left\|\gamma_t - \gamma_t^{\mathcal{P}}\right\| &< \varepsilon. \end{aligned}$$

We shall need to use Green's theorem to evaluate the signature. In order to ensure that the domain we are integrating over is Jordan, we will approximate our curves by simple curves. This is made possible by the following lemma:

Lemma 22. *(*[27]*, Lemma 4.3) Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous simple curve. Then for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[0,1]$ such that $\|\mathcal{P}\| < \varepsilon$ and $\gamma^{\mathcal{P}}$ is simple.*

The following corollary, which is the only place we have used the convexity of $\mathbb{D}$, follows immediately.

Corollary 23. *Let $\gamma : [0,1] \to \overline{\mathbb{D}}$ be a continuous simple curve such that $\gamma_0 = -1$, $\gamma_1 = 1$ and $\gamma_t \subset \mathbb{D}$ for all $t \in (0,1)$. Then for all $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[0,1]$ such that $\|\mathcal{P}\| < \varepsilon$ and $\gamma^{\mathcal{P}}$ is simple. Furthermore, $\gamma_t^{\mathcal{P}} \subset \mathbb{D}$ for $t \in (0,1)$ and $\gamma_0^{\mathcal{P}} = -1$, $\gamma_1^{\mathcal{P}} = 1$.*

Proof. The only thing to prove is $\gamma^{\mathcal{P}}(0,1) \subset \mathbb{D}$. Since $\gamma_{t_i}, \gamma_{t_{i+1}} \subset \mathbb{D}$ for $i \neq 0$ or $n-1$, thus by the convexity of $\mathbb{D}$, the line segment between γ_{t_i} and $\gamma_{t_{i+1}}$ lies in $\mathbb{D}$. Note also that the line segment strictly between γ_0 and γ_1 and the segment between γ_{n-1} and γ_n also lies in $\mathbb{D}$ by convexity. $\square$

The following lemma is extremely useful in proving the properties of Young's integral.

Lemma 24. *Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuous curve with finite p-variation, where $p < 2$. Let $\mathcal{P}_m$ be a sequence of partitions such that $\mathcal{P}_m$ contains both 0 and 1 for all m and $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$. For any $(i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$,*

$$\mathbf{e}_{i_1}^* \otimes \ldots \otimes \mathbf{e}_{i_n}^* \left[S(\gamma)_{0,1} \right] = \lim_{m\to\infty} \mathbf{e}_{i_1}^* \otimes \ldots \otimes \mathbf{e}_{i_n}^* \left[S\left(\gamma_s^{\mathcal{P}_m}\right)_{0,1} \right]. \tag{4.4}$$

Proof. See Corollary 2.11 in [14]. □

4.2. **Proof of Theorem 3.** The following lemma is a direct consequence of Lemma 13.

Lemma 25. *Let $\eta : [0,1] \to \mathbb{R}^2$ be a positively oriented, simple closed curve with bounded total variation and D be its interior, then*

$$\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S(\eta)_{0,1} \right) = \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \int_D \left(x - \eta_0^{(1)} \right)^{n-1} \left(\eta_0^{(2)} - y \right)^{k-1} \mathrm{d}x\mathrm{d}y \tag{4.5}$$

Proof. This follows from Lemma 13 and Theorem 11. □

Note that Lemma 25 cannot be applied directly to elements of $\mathcal{C}_2(-1,1,\mathbb{D})$ as they are not closed curves and are "too rough". For the first problem, we need to "closed off" the curves in $\mathcal{C}_2(-1,1,\mathbb{D})$ by concatenating it with the upper semi-circular boundary of $\mathbb{D}$ from 1 to -1. Let us describe this more precisely.

Recall that $\mathcal{C}_2(-1,1,\mathbb{D})$ is defined just before the statement of Theorem 3.

Let ϕ denote the anti-clockwise semi-circular boundary of $\mathbb{D}$, or more precisely, $\phi(t) := (\cos t, \sin t), 0 \le t \le \pi$.

Let $p \ge 1$. For elements γ and $\tilde{\gamma}$ in $\mathcal{V}^p\left([0,T_2], \mathbb{R}^d\right)$ and $\mathcal{V}^p\left([0,T_1], \mathbb{R}^d\right)$, define a concatenation product $\star : \mathcal{V}^p\left([0,T_2], \mathbb{R}^d\right) \times \mathcal{V}^p\left([0,T_1], \mathbb{R}^d\right) \to \mathcal{V}^p\left([0,T_1+T_2], \mathbb{R}^d\right)$ by

$$\begin{aligned} \gamma \star \tilde{\gamma}(u) &:= \gamma(u), \quad u \in [0,T_1], \\ \gamma \star \tilde{\gamma}(u) &:= \tilde{\gamma}(u - T_1) + \gamma(T_1) - \tilde{\gamma}(0), \quad u \in [T_1, T_1+T_2] \end{aligned}$$

Then for $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$, $\eta = \gamma \star \phi$ is a simple closed curve.

As η does not in general has bounded total variation, we will prove a version of Lemma 25 that works for η.

Lemma 26. *Let $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$. If $\eta = \gamma \star \phi$ and D is the interior of η, then*

$$\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S(\eta)_{0,1} \right) = \frac{1}{(n-1)!} \frac{(-1)^{k-1}}{(k-1)!} \int_D (x+1)^{n-1} y^{k-1} \mathrm{d}x\mathrm{d}y. \tag{4.6}$$

If $(k_1, l_1), ..., (k_n, l_n) \in \mathbb{N}^2$, we have

$$\Pi_{i=1}^n \mathbf{e}_1^{*\otimes k_i} \otimes \mathbf{e}_2^{*\otimes l_i} \left(S(\eta)_{0,1} \right) \tag{4.7}$$

$$= C_{\mathbf{k},\mathbf{l}} \int_{\mathbb{R}^{2n}} \left[\Pi_{j=1}^n (x_j+1)^{k_j-1} y_j^{l_j-1} \right] 1_{D^n} \mathrm{d}x_1 \mathrm{d}y_1 ... \mathrm{d}x_n \mathrm{d}y_n \tag{4.8}$$

where $C_{\mathbf{k},\mathbf{l}} = \Pi_{j=1}^n \left[\frac{(-1)^{l_j-1}}{(k_j-1)!(l_j-1)!} \right]$.

Proof. Let $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$. By Corollary 23, there exists a sequence of partitions $(\mathcal{P}_m)_{m=1}^\infty$ such that $\|\mathcal{P}_m\| \to 0$ as $m \to \infty$, $\gamma_0^{\mathcal{P}_m} = -1$, $\gamma_1^{\mathcal{P}_m} = 1$ and $\gamma^{\mathcal{P}_m} \star \phi$ is a simple closed curve. Let D_m be the interior of $\gamma^{\mathcal{P}_m} \star \phi$, which we shall denote as $\eta^{(m)}$.

Since $\gamma^{\mathcal{P}_m} \to \gamma$ in $\|\cdot\|_\infty$, we have for each $(x,y) \in \mathbb{R}^2 \backslash \eta \cup_{m=1}^{\infty} \eta^{(m)}$, $1_{D_m}(x,y) \to 1_D(x,y)$, where D is the interior of η. As γ has finite p-variation, where $p < 2$, γ can be reparametrised to be a $\frac{1}{p}$-Hölder continuous path ([14], Section 1.2.2) and hence $\gamma[0,1]$ has Hausdorff dimension strictly less than 2. Therefore, the set $\eta \cup_{m=1}^{\infty} \eta^{(m)}$ has two dimensional Lebesgue measure is zero. Thus $1_{D_m} \to 1_D$ almost everywhere on $\mathbb{R}^2$ in Lebesgue measure.

By the bounded convergence theorem,

$$\begin{aligned} & \lim_{m\to\infty} \int_{\mathbb{R}^2} (x+1)^{n-1} (-y)^{k-1} 1_{D_m} \mathrm{d}x\mathrm{d}y \\ = & \int_{\mathbb{R}^2} (x+1)^{n-1} (-y)^{k-1} 1_D \mathrm{d}x\mathrm{d}y \end{aligned}$$

By Lemma 24,

$$\lim_{m\to\infty} \mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S\left(\gamma^{\mathcal{P}_m}\right)_{0,1} \right) = \mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k} \left(S\left(\gamma\right)_{0,1} \right).$$

(4.6) then follows from Lemma 25.

(4.7) follows by multiplying n terms of the form (4.6) together. □

Remark 27. B. Werness [27] is the first to realise that the Green's theorem can be used to compute some terms in the signature of a curve. He used it to prove the $n = 2, k = 1$ case of Lemma 26 and to compute the first three gradings of the expected signature of SLE curve. The main idea in generalising Werness's result to Lemma 26 is an interchange of integration, see the proof of Lemma 13.

Before proving our main result, we need just one more technical lemma.

Lemma 28. *Let $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$. If $\gamma[0,1] = \gamma'[0,1]$, then there exists a continuous strictly increasing function $r(t)$ such that*

$$\gamma_{r(t)} = \gamma'_t$$

for all $t \in [0,1]$.

Proof. Let γ^{-1} denote the inverse of the function $t \to \gamma_t$, which exists as γ is a simple curve.

Define a function $r : [0,1] \to [0,1]$ by $r(t) = \gamma^{-1} \circ \gamma'(t)$.

As both γ and γ' are injective continuous functions and $\gamma[0,1] = \gamma'[0,1]$, thus r is a bijective continuous function from $[0,1]$ to $[0,1]$. Hence it is monotone.

But $\gamma_0 = \gamma'_0 = -1$, $\gamma_1 = \gamma'_1 = 1$, so $r(0) = 0$ and $r(1) = 1$. Hence r is an increasing function and the result follows. □

We now prove Theorem 3.

Proof. (of Theorem 3)The only if direction follows from the invariance of signature under reparametrisation.

Let $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$ be such that $S(\gamma) = S(\gamma')$.

Let $\eta := \gamma \star \phi$ and$\eta' := \gamma' \star \phi$. By Chen's identity, $S(\gamma) = S(\gamma')$ implies

$$S(\eta)_{0,1} = S(\eta')_{0,1}.$$

Let D and D' be the interior of η and η' respectively.

Since $\gamma, \gamma' \in \mathcal{C}_2(-1,1,\mathbb{D})$, we have by Lemma 26 that for $\phi = \eta, \eta'$, $A = D, D'$,

$$\mathbf{e}_1^{*\otimes n} \otimes \mathbf{e}_2^{*\otimes k}\left(S(\phi)_{0,1}\right) = \frac{1}{(n-1)!}\frac{(-1)^{k-1}}{(k-1)!}\int_A (x+1)^{n-1} y^{k-1} \mathrm{d}x\mathrm{d}y \tag{4.9}$$

Then $S(\eta)_{0,1} = S(\eta')_{0,1}$ implies that

$$\int_{\mathbb{R}^2} (x+1)^{n-1} y^{k-1} 1_D(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^2} (x+1)^{n-1} y^{k-1} 1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all n and k.

Thus

$$e^{i\lambda_1}\int_{\mathbb{R}^2} e^{i\lambda_1 x + i\lambda_2 y} 1_D(x,y)\,\mathrm{d}x\mathrm{d}y = e^{i\lambda_1}\int_{\mathbb{R}^2} e^{i\lambda_1 x + i\lambda_2 y} 1_{D'}(x,y)\,\mathrm{d}x\mathrm{d}y$$

for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

By the fact that Fourier transform is injective on $L^1(\mathbb{R}^2)$,

$$1_D(x,y) = 1_{D'}(x,y) \tag{4.10}$$

for almost all $(x,y) \in \mathbb{R}^2$.

Therefore, both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are null sets with respect to the Lebesgue measure.

As both $D\backslash\overline{D'}$ and $D'\backslash\overline{D}$ are open, so they must both be empty. This means $D \subset \overline{D'}$ and $D' \subset \overline{D}$. Thus $\overline{D} = \overline{D'}$.

Note that as γ, γ' are simple curves and $\gamma, \gamma' \subset \mathbb{D}$ except at the endpoints, the domains D and D' are Jordan domains. Using the Jordan curve theorem, we can prove that $\overline{\mathbb{R}^2\backslash\overline{D}} = \mathbb{R}^2\backslash D$ for any Jordan domain D.

As $\overline{D} = \overline{D'}$, we have $\mathbb{R}^2\backslash D = \mathbb{R}^2\backslash D'$, implying that $\gamma[0,1] = \gamma'[0,1]$. The result follows from Lemma 28. □

5. Uniqueness of signature for Schramm-Loewner Evolution

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space. Let $(B_t : t \geq 0)$ be a one-dimensional standard Brownian Motion. Let $0 < \kappa$. Let $z \in \overline{\mathbb{H}}\backslash\{0\}$. For each $\omega \in \Omega$, consider the initial value problem:

$$\frac{\mathrm{d}g_t(z,\omega)}{\mathrm{d}t} = \frac{2}{g_t(z,\omega) - \sqrt{\kappa}B_t(\omega)} \quad g_0(z) = z \tag{5.1}$$

We shall recall the following facts about g_t from [21].

(1) For each ω, a unique solution to this equation exists up to time $T_z > 0$, where T_z is the first time such that $g_t - \sqrt{\kappa}B_t \to 0$ as $t \to T_z$.
(2) Define
$$H_t = \{z \in \mathbb{H} : t < T_z\} \text{ and } K_t = \mathbb{H}\backslash H_t$$
Then H_t is open and simply connected.
(3) For each time $t > 0$, g_t defines a conformal map from H_t onto $\mathbb{H}$. In particular, g_t is invertible.
(4) Let $\hat{f}_t(z) := g_t^{-1}(z + \sqrt{\kappa}B_t)$. There exists a $\mathbb{P}$-null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$, the limit
$$\hat{\gamma}(t,\omega) := \lim_{z\to 0, z\in\mathbb{H}} \hat{f}_t(z)$$

exists and $t \to \hat{\gamma}(t)$ is continuous. The two dimensional stochastic process $(\hat{\gamma}_t : t \geq 0)$ is called the *Chordal SLE$_\kappa$ curve.*

The Loewner correspondence from a continuous path $t \to B_t(\omega)$ to $t \to \hat{\gamma}(\cdot, \omega)$ is in fact deterministic and one-to-one. Therefore, the measure on the Brownian paths induces, through this correspondence, a measure on paths in $\overline{\mathbb{H}}$ from 0 to ∞, which we shall call the Chordal SLE$_\kappa$ measure in $\mathbb{H}$.

Proposition 29. *Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$. Then with probability one, the following holds:*

1. ([27], Section 4.1)If $0 < \kappa \leq 4$, then for any $p > 1+\frac{\kappa}{8}$, γ has finite p-variation.

2.([21],Theorem 7.1 and Theorem 6.1)$\gamma : [0,\infty) \to \overline{\mathbb{H}}$ satisfies $\gamma_0 = 0$ and $\liminf_{t\to\infty} |\hat{\gamma}_t| = \infty$.

3.([21], Theorem 6.1)For $0 \leq \kappa \leq 4$, $t \to \hat{\gamma}_t$ is a simple curve and $\hat{\gamma}(0,\infty) \subset \mathbb{H}$.

The fact that $\lim_{t\to\infty} \hat{\gamma}_t = \infty$ *a.s.* means that the signature $S(\hat{\gamma})_{0,\infty}$ will not be defined. Therefore, we shall follow [27] and opt to study the Chordal SLE$_\kappa$ curve in the unit disc $\mathbb{D}$, from -1 to 1. The Chordal SLE_κ measure in domain $\mathbb{D}$ with marked points -1 and 1 is defined as follows:

Definition 30. For $\kappa > 0$. Let $\mathbb{P}$ be the Chordal SLE$_\kappa$ measure in $\mathbb{H}$, D be a simply connected subdomain of $\mathbb{C}$, $a, b \in \partial D$ and f be a conformal map from $\mathbb{H}$ to D, with $f(0) = a$ and $f(\infty) = b$. Then the Chordal SLE$_\kappa$ measure in D with marked points a and b is defined as the measure $\mathbb{P} \circ f^{-1}$.

Remark 31. Although there is a one dimensional family of conformal maps f such that f maps $\mathbb{H}$ to D, 0 to a and ∞ to b, the scale invariance of the Chordal SLE measure in $\mathbb{H}$ means that the measure $\mathbb{P} \circ f^{-1}$ is the same no matter which member f in this one dimensional family we use.

We now prove our almost sure uniqueness theorem concerning the signature of SLE curves.

Proof. (of Theorem 4)Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1. Then by Proposition 29, there exists a $\mathbb{P}_\kappa$-null set $\mathcal{N}$, such that for all $\gamma \in \mathcal{N}^c$,

1. $\gamma(0) = -1$, $\gamma(1) = 1$ and $\gamma(0,1) \subset \mathbb{D}$.
2. γ has with finite $1 \leq p < 2$ variations.
3. γ is simple.

Therefore, in particular, $\mathcal{N}^c \subset \mathcal{C}_2(-1,1,\mathbb{D})$.

Let $\gamma, \gamma' \in \mathcal{N}^c$ be such that $S(\gamma) = S(\gamma')$, then by Theorem 3, γ and γ' are reparametrisations of each other. $\square$

6. Expected signature and n-point functions

6.1. **n-point functions from expected signature.** We will need the following immediate consequence of the shuffle product formula.

Lemma 32. *Let $(k_1, l_1), ..., (k_n, l_n) \in \mathbb{N}^2$. Then*

$$\Pi_{i=1}^{n} \mathbf{e}_1^{*\otimes k_i} \otimes \mathbf{e}_2^{*\otimes l_i} \left(S(\gamma)_{0,1}\right) = \mathbf{e}_1^{*\otimes k_1} \otimes \mathbf{e}_2^{*\otimes l_1} \sqcup \ldots \sqcup \mathbf{e}_1^{*\otimes k_n} \otimes \mathbf{e}_2^{*\otimes l_n} \left(S(\gamma)_{0,1}\right)$$

where the operation $\sqcup$ is the shuffle product operation defined in Proposition 7.

Proof. This follows from an iterated use of Proposition 7. $\square$

A well-known observable in the theory of SLE is the following sequence of n-point functions:

Definition 33. Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ denote the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked point -1 and 1. For each $\gamma \in \mathcal{C}_2(-1,1,\mathbb{D})$, let $\Phi(\gamma)$ denote the concatenation of γ and the upper semi-circular boundary of $\mathbb{D}$, oriented in the anti-clockwise direction. We shall define the n-point function associated with the probability measure $\mathbb{P}_\kappa$ to be:

$$\Gamma_n(x_1, y_1, .., x_n, y_n) = \mathbb{P}_\kappa\left[(x_1, y_1), \ldots (x_n, y_n) \in \mathrm{Int}\Phi(\cdot)\right].$$

The $n-$point functions for SLE$_\kappa$ curves were first studied by O. Schramm who calculated the 1-point function explicitly in terms of hypergeometric functions (see [22]). Although PDEs can be written down for the n-point functions, the analytic expressions for general n and κ are not known. The only exception is $n = 2$ and $\kappa = \frac{8}{3}$, which was predicted in [24] and computed rigorously in [3].

Proof. (of Theorem 5)
Let $0 < \kappa \leq 4$. Let $\mathbb{P}_\kappa$ be the Chordal SLE$_\kappa$ measure in $\mathbb{D}$ with marked points -1 and 1.

As in the proof of Theorem 4, there exists a $\mathbb{P}_\kappa$ null set $\mathcal{N}$ such that $\mathcal{N}^c \subset \mathcal{C}_2(-1,1,\mathbb{D})$.

By Lemma 26, we have for each $\gamma \in \mathcal{N}^c$,

$$\begin{aligned} & C_{\mathbf{n}} \int_{\mathbb{C}^N} \Pi_{i=1}^N (1+x_i)^{n_i} y_i^{k_i} 1_{(\mathrm{Int}\Phi(\gamma))^N} \mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_N \mathrm{d}y_N \\ = \quad & \Pi_{i=1}^N \mathbf{e}_1^{*\otimes(n_i+1)} \otimes \mathbf{e}_2^{*\otimes(k_i+1)} \left(S(\Phi(\gamma))_{0,1}\right). \end{aligned}$$

where $(\mathrm{Int}\Phi(\gamma))^n$ is the set

$$\cap_{k=1}^N \{(x_1, y_1, \ldots, x_N, y_N) : (x_k, y_k) \in \mathrm{Int}\Phi(\gamma)\}$$

and

$$C_{\mathbf{n},\mathbf{k}} := \Pi_{i=1}^N \frac{(-1)^{k_i}}{n_i! k_i!}.$$

By Lemma 32,

$$\Pi_{i=1}^N \mathbf{e}_1^{*\otimes n_i} \otimes \mathbf{e}_2^{*\otimes k_i} \left(S(\Phi(\gamma))_{0,1}\right) = \mathbf{e}_1^{*\otimes n_1} \otimes \mathbf{e}_2^{*\otimes k_1} \sqcup \ldots \sqcup \mathbf{e}_1^{*\otimes n_N} \otimes \mathbf{e}_2^{*\otimes k_N} \left(S(\Phi(\gamma))_{0,1}\right).$$

By taking linear combinations, we have

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} e^{\sum_{i=1}^N \lambda_i (x_i+1)+\mu_i y_i} \mathbb{E}\left[1_{D^N}\right] \mathrm{d}x_1 \cdots \mathrm{d}y_N \\ = \quad & \sum_{n_1,\ldots,n_N,k_1\ldots k_N \geq 0} \Pi_{i=1}^N (\lambda_i)^{n_i} (-\mu_i)^{k_i} \mathbf{e}_1^{*\otimes(n_1+1)} \otimes \mathbf{e}_2^{*\otimes(k_1+1)} \sqcup \ldots \\ & \ldots \sqcup \mathbf{e}_1^{*\otimes(n_N+1)} \otimes \mathbf{e}_2^{*\otimes(k_N+1)} \left(\mathbb{E}\left[S(\Phi(\gamma))_{0,1}\right]\right) \end{aligned}$$

The result then follows by noting $\mathbb{E}\left[1_{D^N}(\cdot)\right] = \Gamma_N(\cdot)$. $\square$

As we may determine the signature of $\Phi(\gamma)$ from the signature of γ using Chen's identity, this formula gives a relationship between the expected signature of the Chordal SLE measure and the n-point functions.

6.2. **Expected signature from n-point functions.** We may ask whether it is possible to obtain the expected signature from the n-point functions. Unfortunately, here we can do no better than the deterministic case and are only able to obtain an explicit formula only up to the fourth term. To obtain a simpler formula, we choose to study the Chordal SLE$_\kappa$ measure on $1+\mathbb{D}$ so that all paths start from 0. The n-point functions can be defined for the domain $1+\mathbb{D}$ in the obvious way.

Proposition 34. *Let $0<\kappa\leq 4$. Let γ denote the Chordal SLE$_\kappa$ curve from 0 to 2 in $1+\mathbb{D}$. Let $\Phi(\gamma)$ denote the concatenation of γ with the upper semi-circle of the unit disc $1+\mathbb{D}$, oriented in the anti-clockwise direction. Then the first four terms of the tensor element $\Phi(\gamma)$ is*

$$1+\int_{\mathbb{D}}\left([\mathbf{e}_1,\mathbf{e}_2]+[\mathbf{x}_1,[\mathbf{e}_1,\mathbf{e}_2]]+\frac{1}{2}[\mathbf{x}_1,[\mathbf{x}_1,[\mathbf{e}_1,\mathbf{e}_2]]]\right)\Gamma_1((x_1,y_1))\,\mathrm{d}x_1\mathrm{d}y_1 \tag{6.1}$$

$$+\frac{1}{2}\int_{\mathbb{R}^4}[\mathbf{e}_1,\mathbf{e}_2]\otimes[\mathbf{e}_1,\mathbf{e}_2]\,\Gamma_2((x_1,y_1),(x_2,y_2))\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_2\mathrm{d}x_4 \tag{6.2}$$

where $\mathbf{x}_1=x_1\mathbf{e}_1+y_1\mathbf{e}_2$ and $\mathbf{x}_2=x_2\mathbf{e}_1+y_2\mathbf{e}_2$, and Γ_n is the n-point function for the Chordal SLE$_\kappa$ measure.

Proof. Let $\mathcal{N}$ be a Chordal SLE$_\kappa$ null set such that for all $\gamma\in\mathcal{N}^c$, γ is simple, $\gamma(0,1)\subset\mathbb{D}$, starts from 0 and ends at 2 and has finite p variation for some $p<2$.

Let $\gamma\in\mathcal{N}^c$. Using the exactly same computation as in the proof of Corollary 2 and replacing the use of Green's theorem with Lemma 26, we have the following

$$\begin{aligned}&[\mathbf{e_1},[\mathbf{e_1},\ldots[\mathbf{e_1}[\mathbf{e_2},\ldots[\mathbf{e_2},[\mathbf{e_1},\mathbf{e_2}]]]]]]^*(\log S(\Phi(\cdot)))\\ =&\int_{\mathbb{R}^2}\frac{x^n y^k}{n!k!}1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

where $[\mathbf{e_1},[\mathbf{e_1},\ldots[\mathbf{e_1}[\mathbf{e_2},\ldots[\mathbf{e_2},[\mathbf{e_1},\mathbf{e_2}]]]]]]$ contains n 1s and k 2s in front of $[\mathbf{e}_1,\mathbf{e}_2]$.

As $\Phi(\gamma)$ is closed, $\mathbf{e}_1^*(\log S(\Phi(\gamma)))=\mathbf{e}_2^*(\log S(\Phi(\gamma)))=0$. Hence

$$\begin{aligned}\pi_4(\log S(\Phi(\gamma)))=&\int_{\mathbb{R}^2}[\mathbf{e}_1,\mathbf{e}_2]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^2}x[\mathbf{e}_1,[\mathbf{e}_1,\mathbf{e}_2]]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^2}y[\mathbf{e}_2,[\mathbf{e}_1,\mathbf{e}_2]]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^2}\frac{x^2}{2}[\mathbf{e}_1,[\mathbf{e}_1,[\mathbf{e}_1,\mathbf{e}_2]]]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^2}xy[\mathbf{e}_1,[\mathbf{e}_2,[\mathbf{e}_1,\mathbf{e}_2]]]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ &+\int_{\mathbb{R}^2}\frac{y^2}{2}[\mathbf{e}_2,[\mathbf{e}_2,[\mathbf{e}_1,\mathbf{e}_2]]]\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\\ =&\int_{\mathbb{R}^2}\Big([\mathbf{e}_1,\mathbf{e}_2]+[x\mathbf{e}_1+y\mathbf{e}_2,[\mathbf{e}_1,\mathbf{e}_2]]\\ &+\frac{1}{2}[x\mathbf{e}_1+y\mathbf{e}_2,[x\mathbf{e}_1+y\mathbf{e}_2,[\mathbf{e}_1,\mathbf{e}_2]]]\Big)\,1_{\mathrm{Int}\Phi(\gamma)}(x,y)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

By taking the exponential and writing $x\mathbf{e}_1 + y\mathbf{e}_2$ as $\mathbf{x}$,

$$\begin{aligned}\pi_4\left(S\left(\Phi\left(\gamma\right)\right)\right) = 1 &+ \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^2} \left[\mathbf{x}, \left[\mathbf{e}_1, \mathbf{e}_2\right]\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^2} \left[\mathbf{x}, \left[\mathbf{x}, \left[\mathbf{e}_1, \mathbf{e}_2\right]\right]\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\ &+ \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y\end{aligned}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \otimes \int_{\mathbb{R}^2} \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x, y\right) \mathrm{d}x\mathrm{d}y \\ = &\int_{\mathbb{R}^4} \left[\mathbf{e}_1, \mathbf{e}_2\right] \otimes \left[\mathbf{e}_1, \mathbf{e}_2\right] 1_{\mathrm{Int}\Phi(\gamma)}\left(x_1, y_1\right) 1_{\mathrm{Int}\Phi(\gamma)}\left(x_2, y_2\right) \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\end{aligned}$$

The proof is completed by taking expectation. □

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