Benjamin-Ono Kadomtsev-Petviashvili's models in interfacial electro-hydrodynamics

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Abstract

Three-dimensional nonlinear potential free surface flows in the presence of vertical electric fields are considered. Both the effects of gravity and surface tension are included in the dynamic boundary condition. An asymptotic analysis (based on the assumptions of small depth and small free surface displacements) is presented. It is shown that the problem can be modelled by a Benjamin-Ono Kadomtsev-Petviashvili equation. Furthermore a fifth order Benjamin-Ono Kadomtsev-Petviashvili equation is derived to describe the flows in the particular case of values of the Bond number close to 1/3.

Keywords: weakly nonlinear waves, electrohydrodynamics

1. Introduction

Classical nonlinear approaches for water waves leading to model equations such as the Korteweg-de Vries equation (KdV) are useful in establishing an analytical foundation for fully nonlinear studies. Our interest in this paper is in the derivation of such models in the presence of vertical electric fields. The practical importance of interfacial electro-hydrodynamics has been reviewed in [1] and computations of two-dimensional nonlinear waves in the presence of electric fields were presented in [2] and [3] (see also the references cited).

In [4] a Benjamin-Ono Korteweg-de Vries equation was derived to describe two-dimensional waves in the presence of vertical electric fields. Here we consider the complete three-dimensional problem. We show that under appropriate canonical scalings, the problem can be modelled by a Benjamin-Ono Kadomtsev-Petviashvili equation. A special equation valid when the Bond number is close to 1/3 is also derived. This equation can be described as a fifth-order Benjamin-Ono Kadomtsev-Petviashvili equation. The contribution of the electric field to all these equations is in the form of a nonlocal term involving a Cauchy principal value. Such a term appears also in the two-dimensional problems studied in [4].

The fully nonlinear problem is formulated in Section 2. The Benjamin-Ono Kadomtsev-Petviashvili equation is derived asymptotically in Section 3. The fifth-order Benjamin-Ono Kadomtsev-Petviashvili equation is presented in Section 4. Some concluding remarks are given in Section 5.

2. Formulation

Consider a perfectly conducting, inviscid, irrotational and incompressible fluid (region 1) bounded below by a wall electrode at $z = -h$ and bounded above by a free surface $z = \eta(x, y, t)$, here h is the mean depth of the surface. The fluid motion is described by a velocity potential $\varphi(x, y, z, t)$ satisfying Laplace's equation in region 1. Surface tension with coefficient σ and gravity g are included. The region $z > \eta(x, y, t)$, denoted by region 2, is occupied by a hydrodynamically passive dielectric having permittivity ϵ_p (see Figure 1).

Figure 1: Sketch of the flow.

It is assumed that there are no free charges or currents in region 2 and therefore the electric field can be represented as a gradient of a potential function, $\mathbf{E} = -\nabla V$. A vertical electrical field is imposed by requiring that $V \sim -E_0 z$ as $z \to \infty$, where E_0 is constant. The voltage potential satisfies Laplace's equation.

On the free surface the Bernoulli equation holds:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + g\eta + \frac{p}{\rho} = C \tag{1}$$

The pressure p in (1) is obtained through the Young-Laplace equation:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_2^1 = \sigma \nabla_s \cdot \hat{\mathbf{n}} \tag{2}$$

where ∇_s is the surface divergence operator and σ is the surface tension,

$$\mathbf{T} = -p\delta_{ij} + \Sigma_{i,j} \tag{3}$$

is the stress tensor and

$$\hat{\mathbf{n}} = \frac{(\partial_x \eta, \partial_y \eta, -1)}{(1 + (\partial_x \eta)^2 + (\partial_y \eta)^2)^{\frac{1}{2}}} \tag{4}$$

is the unit normal to the interface. The notation $[\dots]_2^1$ denotes the jump across the interface. The quantity $\Sigma_{i,j}$ in (3) is the Maxwell tensor defined by

$$\Sigma_{i,j} = \epsilon_p \left(E_i E_j - \frac{1}{2}\delta_{ij} E_k E_k\right) \tag{5}$$

where E_i, $i = 1, 2, 3$ are the cartesian coordinates of the electric field $\mathbf{E}$.

The governing equations are then:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad \text{on} \quad -h < z < \eta(x, y, t) \tag{6}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{on} \quad z > \eta(x, y, t) \tag{7}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \varphi}{\partial z} \quad \text{on} \quad z = \eta(x, y, t) \tag{8}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + g\eta - \frac{1}{\rho}\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}} = \frac{\sigma}{\rho}\nabla \cdot \hat{\mathbf{n}} + C \quad \text{on} \quad z = \eta(x, y, t) \tag{9}$$

$$V = 0 \quad \text{on} \quad z = \eta(x, y, t) \tag{10}$$

$$\frac{\partial \varphi}{\partial z} = 0 \quad \text{on} \quad z = -h \tag{11}$$

$$V \sim -E_0 z \quad \text{as} \quad z \to \infty \tag{12}$$

3. Weakly Nonlinear Theory

Given a typical velocity $c_0 = \sqrt{gh}$, typical free surface amplitude a and typical horizontal length scales λ and μ, we define the dimensionless variables:

$$x = \lambda \hat{x}, \quad y = \mu \hat{y}, \quad t = \frac{\lambda}{c_0}\hat{t}, \quad \eta = a\hat{\eta} \quad \varphi = \frac{g\lambda a}{c_0}\hat{\varphi} \tag{13}$$

$$V = \lambda E_0 \hat{V} \quad z^{(1)} = h\hat{z} \quad z^{(2)} = \lambda \hat{Z}, \tag{14}$$

Here, $z^{(1,2)}$ denote the vertical coordinates in regions 1 and 2 respectively. We also introduce the parameters

$$\alpha = \frac{a}{h}, \quad \beta = \frac{h^2}{\lambda^2}, \quad \gamma = \frac{\lambda^2}{\mu^2} \tag{15}$$

In terms of the variables (13) and (14) the governing equations (6)-(12) become

$$\frac{\partial^2 \hat{\varphi}}{\partial \hat{x}^2} + \gamma \frac{\partial^2 \hat{\varphi}}{\partial \hat{y}^2} + \frac{1}{\beta}\frac{\partial^2 \hat{\varphi}}{\partial \hat{z}^2} = 0 \quad -1 < \hat{z} < \alpha \hat{\eta} \tag{16}$$

$$\frac{\partial^2 \hat{V}}{\partial \hat{x}^2} + \gamma \frac{\partial^2 \hat{V}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}}{\partial \hat{Z}^2} = 0 \quad \hat{Z} > \alpha\sqrt{\beta}\hat{\eta} \tag{17}$$

$$\frac{1}{\beta}\frac{\partial \hat{\varphi}}{\partial \hat{z}} = \frac{\partial \hat{\eta}}{\partial \hat{t}} + \alpha \frac{\partial \hat{\varphi}}{\partial \hat{x}}\frac{\partial \hat{\eta}}{\partial \hat{x}} + \alpha\gamma \frac{\partial \hat{\varphi}}{\partial \hat{y}}\frac{\partial \hat{\eta}}{\partial \hat{y}} \quad \text{on} \quad \hat{z} = \alpha\hat{\eta} \tag{18}$$

$$\frac{\partial \hat{\varphi}}{\partial \hat{t}} + \frac{1}{2}\left[\alpha \left(\frac{\partial \hat{\varphi}}{\partial \hat{x}}\right)^2 + \alpha\gamma \left(\frac{\partial \hat{\varphi}}{\partial \hat{y}}\right)^2 + \left(\frac{\partial \hat{\varphi}}{\partial \hat{z}}\right)^2\right] + \hat{\eta} - \frac{1}{2}\frac{E_b}{\alpha}\left[\left(\frac{\partial \hat{V}}{\partial \hat{Z}}\right)^2 + \dots\right]$$

$$-B\left[\beta \frac{\partial^2 \hat{\eta}}{\partial \hat{x}^2}\dots\right] + C = 0 \quad \text{on} \quad \hat{z} = \alpha\hat{\eta} \tag{19}$$

$$\frac{\partial \hat{\varphi}}{\partial \hat{z}} = 0 \quad \text{on} \quad \hat{z} = -1 \tag{20}$$

$$\hat{V} = 0 \quad \text{on} \quad \hat{Z} = \alpha\sqrt{\beta}\hat{\eta} \tag{21}$$

$$\hat{V} = -\hat{Z} \quad \text{as} \quad \hat{Z} \to \infty. \tag{22}$$

Here

$$B = \frac{\sigma}{\rho g h^2}, \quad E_b = \frac{\epsilon_p E_0^2}{\rho g h}. \tag{23}$$

These parameters are an inverse Bond number representing the ratio of capillary to gravitational forces, and an electric Bond number measuring the ratio of electrical to gravitational forces. The dots in (19) correspond to higher order terms in α, β and γ which are not needed in the analysis. The constant C can be calculated by using the solution $\hat{\varphi} = \hat{\eta} = 0$ and $\hat{V} = -\hat{Z}$ and is found to be $C = -E_b/2\alpha$.

Next we introduce the shallow water scaling:

$$\alpha = \beta = \gamma = \varepsilon \ll 1, \quad T = \varepsilon\hat{t}, \quad X = \hat{x} - \hat{t}. \tag{24}$$

All the derivatives with respect to $\hat{x}$ and $\hat{t}$ are rewritten in terms of derivatives with respect to X and T by using the transformations

$$\frac{\partial}{\partial \hat{x}} = \frac{\partial}{\partial X} \qquad \frac{\partial}{\partial \hat{t}} = -\frac{\partial}{\partial X} + \epsilon\frac{\partial}{\partial T}. \tag{25}$$

We now drop the hats on the various variables and assume the following asymptotic expansions

$$\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + o(\varepsilon^2) \tag{26}$$

$$\eta = \eta_0 + \varepsilon\eta_1 + o(\varepsilon) \tag{27}$$

$$V = -Z + \varepsilon^{\frac{3}{2}}V_1 + o\left(\varepsilon^{\frac{3}{3}}\right) \tag{28}$$

$$p = \varepsilon p_1 + o\left(\varepsilon\right). \tag{29}$$

The asymptotic expansion of V deserves an explanation. Expanding (21) as a Taylor expansion around $Z = 0$ gives

$$V + \epsilon^{3/2}\eta_0\frac{\partial V}{\partial Z} + \cdots = 0 \quad \text{on} \quad Z = 0. \tag{30}$$

Writing the expansion for V as $V = -Z + \delta(\epsilon)V_1$ and inserting it into (30) gives to leading order

$$\delta(\epsilon)V_1 - \epsilon^{3/2}\eta_0 = 0 \tag{31}$$

which shows that $\delta(\epsilon) = \epsilon^{3/2}$. Furthermore (31) implies

$$V_1 = \eta_0 \quad \text{on} \quad Z = 0. \tag{32}$$

The coupling between electrostatics and hydrodynamics enters through the term multiplied by E_b in (19) which is to leading order

$$\frac{E_b}{\varepsilon}\left[-\frac{1}{2} - \varepsilon^{\frac{3}{2}}\frac{\partial V_1}{\partial Z}\right]. \tag{33}$$

The first term in the square bracket of (33) cancels the Bernoulli constant C in (19). This leaves the term of order $\varepsilon^{\frac{3}{2}}$ in (33). To include this term with the other terms of order ε in (19), the electric Bond number needs to be scaled according to

$$E_b = \bar{E}_b\sqrt{\varepsilon}. \tag{34}$$

Equation (17) gives at leading order

$$\frac{\partial^2 V_1}{\partial X^2} + \frac{\partial^2 V_1}{\partial Z^2} = 0 \tag{35}$$

The boundary condition (32) implies

$$\frac{\partial V_1}{\partial x} = \frac{\partial \eta_0}{\partial x} \quad \text{on} \quad Z = 0. \tag{36}$$

It can then be shown that (35) and (36) imply that (see [4] for details)

$$\frac{\partial V_1}{\partial Z} = \mathscr{H}\left[\frac{\partial \eta_0}{\partial X}\right] \tag{37}$$

where $\mathscr{H}$ is the Hilbert Transform operator defined by

$$\mathscr{H}[g] = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{g(\zeta)}{\zeta - X} d\zeta \tag{38}$$

with PV denoting the Cauchy principal value.

We substitute the expansions (26) and (27) into (16) and the boundary conditions (18) and equate powers of ϵ. Since the boundary condition (18) is applied on $z = \alpha\eta$ and since η is itself given as an expansion in powers of ϵ, we need to expand φ and its derivative as Taylor expansions about $z = 0$. This means for example

$$\frac{\partial\varphi}{\partial z}|_{z=\alpha\eta} = \frac{\partial\varphi}{\partial z}|_{z=0} + \alpha\eta\frac{\partial^2\varphi}{\partial z^2}|_{z=0} + \dots \tag{39}$$

This yields

$$\frac{\partial^2\varphi_0}{\partial z^2} = 0, \quad -1 < z < 0 \tag{40}$$

$$\frac{\partial\varphi_0}{\partial z} = 0, \quad \text{on} \quad z = 0 \tag{41}$$

$$\frac{\partial\varphi_0}{\partial z} = 0, \quad \text{on} \quad z = -1 \tag{42}$$

$$\frac{\partial^2\varphi_1}{\partial z^2} = -\frac{\partial^2\varphi_0}{\partial X^2} \quad -1 < z < 0 \tag{43}$$

$$\frac{\partial\varphi_1}{\partial z} = -\frac{\partial\eta_0}{\partial X} \quad \text{on} \quad z = 0 \tag{44}$$

$$\frac{\partial\varphi_1}{\partial z} = 0, \quad \text{on} \quad z = -1 \tag{45}$$

$$\frac{\partial^2\varphi_2}{\partial z^2} = -\frac{\partial^2\varphi_0}{\partial y^2} - \frac{\partial^2\varphi_1}{\partial X^2} \quad -1 < z < 0 \tag{46}$$

$$\frac{\partial\varphi_2}{\partial z} = -\eta_1\frac{\partial^2\varphi_0}{\partial z^2} - \eta_0\frac{\partial^2\varphi_1}{\partial z^2} - \frac{\eta_0^2}{2}\frac{\partial^3\varphi_0}{\partial z^3} - \frac{\partial\eta_1}{\partial X} + \frac{\partial\eta_0}{\partial T} + \frac{\partial\varphi_0}{\partial X}\frac{\partial\eta_0}{\partial X} \quad \text{on} \quad z = 0 \tag{47}$$

$$\frac{\partial\varphi_2}{\partial z} = 0, \quad \text{on} \quad z = -1 \tag{48}$$

Note that equations (41), (44) and (47) represent the first three orders of the kinematic boundary condition, and equations (42), (45) and (48) are the no penetration conditions at the wall. Integrating (40) with the boundary conditions (41) and (42) gives

$$\varphi_0 = \varphi_0(X, y, T) \tag{49}$$

Equation (49) simply says that φ_0 does not depend on z.

Similarly equations (43) -(45) yield

$$\varphi_1 = -\frac{\partial^2 \varphi_0}{\partial X^2}\frac{z^2}{2} - \frac{\partial \eta_0}{\partial X} z + D(X, y, T) \tag{50}$$

where $D(X, y, T)$ is to be determined.

Integrating (46) and using the boundary condition (47) gives

$$\frac{\partial \varphi_2}{\partial z} = -\frac{\partial^2 \varphi_0}{\partial y^2} z + \frac{\partial^4 \varphi_0}{\partial X^4}\frac{z^3}{6} + \frac{\partial^3 \eta_0}{\partial X^3}\frac{z^2}{2} - \frac{\partial^2 D}{\partial X^2} z +$$

$$\eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} - \frac{\partial \eta_1}{\partial X} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X}. \tag{51}$$

Setting $z = -1$ in (51) and using (48) yields

$$\frac{\partial^2 \varphi_0}{\partial y^2} - \frac{1}{6}\frac{\partial^4 \varphi_0}{\partial X^4} + \frac{1}{2}\frac{\partial^3 \eta_0}{\partial X^3} + \frac{\partial^2 D}{\partial X^2} + \eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} - \frac{\partial \eta_1}{\partial X} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} = 0. \tag{52}$$

We now consider the dynamic boundary condition (19) and equating terms of $O(1)$ and of order $O(\epsilon)$ in (19) we obtain

$$\eta_0 = \frac{\partial \varphi_0}{\partial X} \quad \text{on} \quad z = 0 \tag{53}$$

$$\eta_1 = B\frac{\partial^2 \eta_0}{\partial X^2} - \frac{\partial \varphi_0}{\partial T} - \bar{E}_b \frac{\partial V_1}{\partial Z} - \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial X}\right)^2 + \frac{\partial \varphi_1}{\partial X} \quad \text{on} \quad z = 0. \tag{54}$$

Using (50) we see that the last term in (54) is equal to $\frac{\partial D}{\partial X}$. Eliminating η_1 between (52) and (54) yield

$$\frac{\partial^2 \varphi_0}{\partial y^2} - \frac{1}{6}\frac{\partial^4 \varphi_0}{\partial X^4} + \frac{1}{2}\frac{\partial^3 \eta_0}{\partial X^3} + \eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} - B\frac{\partial^3 \eta_0}{\partial X^3} + \frac{\partial^2 \varphi_0}{\partial T \partial X} +$$

$$\bar{E}_b \frac{\partial^2 V_1}{\partial X \partial Z} + \frac{\partial \varphi_0}{\partial X}\frac{\partial^2 \varphi_0}{\partial X^2} = 0. \tag{55}$$

The final equation is obtained by differentiating (55) with respect to X and using (37) and (53). This gives after some algebra

$$\frac{\partial}{\partial X}\left[\frac{\partial \eta_0}{\partial T} + \frac{1}{2}\left(\frac{1}{3} - B\right)\frac{\partial^3 \eta_0}{\partial X^3} + \frac{3}{2}\eta_0 \frac{\partial \eta_0}{\partial X} + \frac{\bar{E}_b}{2}\mathscr{H}\left(\frac{\partial^2 \eta_0}{\partial X^2}\right)\right] + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial y^2} = 0. \tag{56}$$

In terms of the dimensional variables, (56) becomes

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t}+c_0\frac{\partial \eta}{\partial x}+\frac{1}{2}\left(\frac{1}{3}-B\right)c_0h^2\frac{\partial^3\eta}{\partial x^3}+\frac{3c_0}{2h}\eta\frac{\partial \eta}{\partial x}+\frac{E_bc_0h}{2}\mathscr{H}\left(\frac{\partial^2\eta}{\partial x^2}\right)\right]$$
$$+\frac{c_0}{2}\frac{\partial^2\eta}{\partial y^2}=0 \tag{57}$$

where we have dropped the subscript 0 on η.

4. Analysis around $B = 1/3$

A new equation can be derived by assuming that B close to 1/3. We replace the scaling (24) by

$$\alpha=\varepsilon^2,\quad \beta=\varepsilon,\quad \gamma=\varepsilon^2,\quad T=\varepsilon^2\hat{t},\quad X=\hat{x}-\hat{t}. \tag{58}$$

The expansions are now given by

$$\varphi=\varphi_0+\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3+o(\varepsilon^3) \tag{59}$$

$$\eta=\eta_0+\varepsilon\eta_1+\varepsilon^2\eta_2+o(\varepsilon^2) \tag{60}$$

$$V=-Z+\varepsilon^{\frac{5}{2}}V_1+o\left(\varepsilon^{\frac{5}{2}}\right) \tag{61}$$

$$B=\frac{1}{3}+\varepsilon B_1+o(\varepsilon). \tag{62}$$

The asymptotic expansion (61) of V can be justified by an argument similar to that used in Section 3 . Expanding (21) as a Taylor expansion around $Z=0$ gives

$$V+\epsilon^{5/2}\eta_0\frac{\partial V}{\partial Z}+\cdots=0\quad \text{on}\quad Z=0. \tag{63}$$

Writing the expansion for V as $V=-Z+\delta(\epsilon)V_1$ and inserting it into (63) gives to leading order

$$\delta(\epsilon)V_1-\epsilon^{5/2}\eta_0=0 \tag{64}$$

which shows that $\delta(\epsilon)=\epsilon^{5/2}$. Furthermore (64) implies (32)

The coupling between electrostatics and hydrodynamics enters through the term multiplied by E_b in (19), which is, to leading order

$$\frac{E_b}{\varepsilon^2}\left[-\frac{1}{2}-\varepsilon^{5/2}\frac{\partial V_1}{\partial Z}\right] \tag{65}$$

The first term in the square bracket of (33) cancels the Bernoulli constant C in (19). This leaves the term of order $\varepsilon^{\frac{5}{2}}$ in (65). The size of E_b is chosen to compete with the third- and fifth-order dispersion terms. This happens at the order ϵ^2. Therefore we need to scale E_b according to

$$E_b = E_b^* \epsilon^{3/2} \tag{66}$$

The potential V_1 still satisfies (37).

Next we substitute the expansions (59)-(62) into (16) and (18) and proceed as in Section 3. This yields the equations (40)-(45)

$$\frac{\partial^2 \varphi_2}{\partial z^2} = -\frac{\partial^2 \varphi_1}{\partial X^2} \quad -1 < z < 0 \tag{67}$$

$$\frac{\partial \varphi_2}{\partial z} = -\frac{\partial \eta_1}{\partial X} \quad \text{on} \quad z = 0 \tag{68}$$

and (48).

Solving (40)-(45), (67), (68) and (48) (we are omitting the details since the calculations are very similar to those of Section 3) gives (49), (50) and

$$\varphi_2 = \frac{\partial^3 \eta_0}{\partial X^3}\left(\frac{z^4}{24} + \frac{z^3}{6}\right) - \frac{\partial^2 D}{\partial X^2}\frac{z^2}{2} - \frac{\partial \eta_1}{\partial X} z + H(X, y, T) \tag{69}$$

where the function H is to be determined.

Proceeding to order ϵ^3, equations (16), (18) and (20) give

$$\frac{\partial^2 \varphi_3}{\partial z^2} = -\frac{\partial^2 \varphi_0}{\partial y^2} - \frac{\partial^2 \varphi_2}{\partial X^2} \tag{70}$$

$$\frac{\partial \varphi_3}{\partial z} = -\frac{\partial \eta_2}{\partial X} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} + \eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} \quad \text{on} \quad z = 0 \tag{71}$$

$$\frac{\partial \varphi_3}{\partial z} = 0 \quad \text{on} \quad z = -1. \tag{72}$$

We now expand (19) up to order ϵ^2. This yields the second equation (49),

$$\eta_1 - \frac{1}{3}\frac{\partial^2 \eta_0}{\partial X^2} - \frac{\partial \varphi_1}{\partial X} = 0 \quad \text{on} \quad z = 0 \tag{73}$$

and

$$B_1 \frac{\partial^2 \eta_0}{\partial X^2} + \frac{1}{3}\frac{\partial^2 \eta_1}{\partial X^2} = \frac{\partial \varphi_0}{\partial T} - \frac{\partial \varphi_2}{\partial X} + \eta_2 + \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial X}\right)^2 - E_b^* \frac{\partial V_1}{\partial Z}. \tag{74}$$

Using (50) and (69) we can rewrite (73) and (74) as

$$\eta_1 - \frac{1}{3}\frac{\partial^2 \eta_0}{\partial X^2} - \frac{\partial D}{\partial X} = 0 \quad \text{on} \quad z = 0 \tag{75}$$

and

$$B_1 \frac{\partial^2 \eta_0}{\partial X^2} + \frac{1}{3}\frac{\partial^2 \eta_1}{\partial X^2} = \frac{\partial \varphi_0}{\partial T} - \frac{\partial H}{\partial X} + \eta_2 + \frac{1}{2}\left(\frac{\partial \varphi_0}{\partial X}\right)^2 + E_b^* \frac{\partial V_1}{\partial Z}. \tag{76}$$

Next we substitute (69) into (70) and integrate with respect to z with the boundary condition (71). This gives

$$\frac{\partial \varphi_3}{\partial z} = -\frac{\partial^2 \varphi_0}{\partial y^2} z - \frac{\partial^5 \eta_0}{\partial X^5}\left(\frac{z^5}{120} + \frac{z^4}{24}\right) + \frac{\partial^4 D}{\partial X^4}\frac{z^3}{6} + \frac{\partial^3 \eta_1}{\partial X^3}\frac{z^2}{2} - \frac{\partial^2 H}{\partial X^2} z$$
$$- \frac{\partial \eta_2}{\partial X} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} + \eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} \tag{77}$$

Setting $z = -1$ in (77) and using (72) gives

$$\frac{\partial^2 \varphi_0}{\partial y^2} - \frac{1}{30}\frac{\partial^5 \eta_0}{\partial X^5} - \frac{1}{6}\frac{\partial^4 D}{\partial X^4} + \frac{1}{2}\frac{\partial^3 \eta_1}{\partial X^3} + \frac{\partial^2 H}{\partial X^2} - \frac{\partial \eta_2}{\partial X} + \frac{\partial \eta_0}{\partial T} + \frac{\partial \varphi_0}{\partial X}\frac{\partial \eta_0}{\partial X} + \eta_0 \frac{\partial^2 \varphi_0}{\partial X^2} = 0. \tag{78}$$

We now eliminate η_1 and η_2 between (75), (76) and (78). The constants D and H cancel exactly. The final equation is obtained by using (37) and (53) to eliminate φ_0 and V_1. It gives

$$\frac{\partial}{\partial X}\left[\frac{\partial \eta_0}{\partial T} + \frac{1}{90}\frac{\partial^5 \eta_0}{\partial X^5} + \frac{3}{2}\eta_0 \frac{\partial \eta_0}{\partial X} - \frac{B_1}{2}\frac{\partial^3 \eta_0}{\partial X^3} + \frac{E_b^*}{2}\mathscr{H}\left(\frac{\partial^2 \eta_0}{\partial X^2}\right)\right] + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial y^2} = 0. \tag{79}$$

In terms of the original dimensional variables, (79) can be rewritten as

$$\frac{\partial}{\partial x}\left[\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{1}{90}h^4 c_0 \frac{\partial^5 \eta}{\partial x^5} + \frac{3c_0}{2h}\eta \frac{\partial \eta}{\partial x} + \frac{1}{2}(\frac{1}{3} - B)h^2 c_0 \frac{\partial^3 \eta}{\partial x^3} + \frac{E_b h c_0}{2}\mathscr{H}\left(\frac{\partial^2 \eta}{\partial x^2}\right)\right]$$
$$+ \frac{c_0}{2}\frac{\partial^2 \eta}{\partial y^2} = 0 \tag{80}$$

where we have dropped the subscript 0 on η.

5. Conclusions

We have studied nonlinear three-dimensional free surface flows in the presence of vertical electric fields. We have shown that these problems can be reduced to two model equations, namely the equations (57) and (80). These equations are useful because the computation of fully nonlinear three dimensional free surface flows require a very large number of mesh points even in the absence of electric fields (see [5], [6], [7] and [8]).

Equation (57) is a Benjamin-Ono Kadomtsev-Petviashvili equation while equation (80) is a fifth order Benjamin-Ono Kadomtsev-Petviashvili equation. It can be shown that the linear terms in (57) and (80) can be obtained by taking the long wave limit of the dispersion relation of linear waves (see [9]).

These equations have a rich structure in the sense that they include many canonical equations as particular cases. For solutions which do not depend on y, (57) and (80) reduce (after integration with respect to x) to the two-dimensional equations derived in [4]. If we assume that $E_b = 0$ (i.e. that there are no electrical fields), (57) and (80) are the Kadomtsev-Petviashvili equation and the fifth order Kadomtsev-Petviashvili equation. The reader can find a review of the properties of the Kadomtsev-Petviashvili in [10] and an application of the fifth order Kadomtsev-Petviashvili equation in [11]. For other related works on similar model equations without electric fields see [12], [13], [14] and [15].

We now comment on the relevance of equations of (57) and (80) to experimental situations. We use here the discussion presented in [16]. Experiments in the cases when the region 1 is water and mercury were performed in [17] and [18] respectively. In both papers, the region 2 is air which can be considered as a dielectric. Mercury and (impure) water can be approximated as perfect conductors. The critical value $B = 1/3$ corresponds to $h_W \approx 4.7mm$ and $h_M \approx 3.3mm$ (see [16]). Here the subscript W and M refer to water and mercury. The experiments in water (see [17]) were performed for values of $h = 5cm$ and correspond therefore to $B < 1/3$. The experiments in mercury (see [18]) were performed for values of h between $2.12mm$ and $8.5mm$ and include therefore values of $B \approx 1/3$. All the results in [17] and [18] were obtained in the absence of electric fields. There (57) with $E_b = 0$ is relevant to the experiments in [17] whereas (80) with $E_b = 0$ is relevant to the experiments in [18]. Relations (34) and (66) give some guidance on the magnitude of the electrical fields to be used in experiments which can be modelled by

(57) and (80).

Finally let us mention that the results presented can be generalised to include some forcing (such as a prescribed distribution of pressure). Details can be found in [9].

Acknowledgements. The work of DTP was partly supported by the EPSRC grants EP/K041134 and EP/L020564. The work of J.-M. VDB was partly supported by the EPSRC grants EP/J019569 and EP/N018559..

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