A minimal BV action for Vasiliev's four-dimensional higher spin gravity

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ABSTRACT: The action principle for Vasiliev's four-dimensional higher-spin gravity proposed recently by two of the authors, is converted into a minimal BV master action using the AKSZ procedure, which amounts to replacing the classical differential forms by vectorial superfields of fixed total degree given by the sum of form degree and ghost number. The nilpotency of the BRST operator is achieved by imposing boundary conditions and choosing appropriate gauge transitions between charts leading to a globally-defined formulation based on a principal bundle.

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Contents

1 Introduction

In [1], Vasiliev's four-dimensional higher-spin gravities [2–4], including the minimal bosonic models [5], have been equipped with action principles of generalized-Hamiltonian type. The properties of Vasiliev's theory that underlie the construction of the actions hold true in general models with Lorentzian signature and negative cosmological constant, including models with Yang-Mills sectors and supersymmetries. The off-shell formulation of [1] combine the principle of unfolding [2, 6–9], which lies at the heart of Vasiliev's equations, with a natural extension of the generalized Poisson sigma model from graded-commutative to graded-associative differential algebras.[1]

In the graded-commutative case, the generalized Poisson sigma model was first studied within the two-dimensional context [10–12] whose Batalin-Vilkovisky (BV) formulation [13, 14] was geometrized by Alexandrov-Kontsevish-Schwarz-Zaboronski (AKSZ) in [15], later used for a perturbative path-integral derivation [12, 16, 17] of Kontsevish's star-product [18] on Poisson manifolds. These models are closely related to topological BF models (see the review [19] and refs. therein); interestingly, the BF model without Poisson structure on a non- commutative manifold was studied in [20, 21]. Further developments of the AKSZ formalism can be found in [22–24] and [25–33], and its close ties to unfolded dynamics have been stressed in [34–39]. For related treatments of more general dynamical systems, not necessarily based on differential algebras, see [40, 41] and references therein.

    The two main results of this paper are:

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[1]Preliminary investigations indicate a further natural extension to homotopy-associative differential algebras.

- a set of conditions on the couplings in the generalized Hamiltonian (see eq. (2.39) and (2.61)) and on the boundary values of certain fields and gauge parameters (see eq. (2.60)), that together assure the existence of a globally-defined action principle of fiber-bundle type on a base manifold with non-trivial atlas and boundaries;

- an extension of the AKSZ formalism to unfolded systems on non-commutative base manifolds, in such a way as to construct a minimal BV-AKSZ master action for Vasiliev's four-dimensional higher spin gravities (see eqs. (3.66) and (3.67)).

In all types of generalized Poisson sigma models, whether on commutative or non-commutative base manifolds, the physical degrees of freedom are contained in boundary vertex operators [12, 23]. The boundary lives in a graded target-space manifold equipped with a nilpotent vector field of degree 1, referred to as the Q-structure, and compatible poly-vector fields of suitable degrees depending on the dimension of the base manifold, whose mutual Schouten brackets vanish, thus defining a generalized Poisson structure referred to as a QP-structure in the bi-vector case;[2] see [42] and refs. therein. The bulk lives in a suitably parity-shifted phase space of the boundary target space such that each boundary field becomes paired with a momentum in its turn constrained on the boundary, which thus breaks the group of canonical transformations. Assuming a single boundary, the classical limit is thus determined by the Q-structure and the choice of global formulation used to construct the boundary observables, e.g. formulation on a fiber bundle with structure group corresponding to the manifest gauge symmetries off shell, as we shall discuss more below; for a related analysis, see [43, 44].

In the AKSZ quantization scheme, the free theory consists only of the kinetic bulk terms, which do not depend on the physical vielbein and hence remain well-defined in limits where the metric vanishes. The latter can be gauge-fixed using an auxiliary metric and the physical states can be defined by means of a BRST operator [45–48] whose existence is guaranteed, at least semi-classically, by a vectorial supersymmetry that implies that the AKSZ master action obeys classical as well as quantum BV master equations, as we shall discuss below.

The unfolded framework may thus provide a bridge between deformation quantization and quantum field theories in their metric phases. The idea is that the latter phase may arise within the former in suitable global formulations allowing combinations of nontrivial P structures and boundary vertex operators depending algebraically on the physical vielbein. It may then be possible to draw Feynman diagrams, with propagators only in the bulk and vertices in both bulk and boundary, describing quantum fluctuations for dynamical boundary fields such as scalars, vectors, metrics and higher-spin fields in higher-spin gravities in nontrivial metric backgrounds, unlike the case of bulk actions without P-structures. Another intriguing feature of the AKSZ approach is the cancellation of all vacuum bubbles in flat auxiliary background metrics, which suggests that the Poisson sigma model can be summed over bulk topologies, defining a third-quantization on top

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[2]Which is equivalent to a pure Poisson structure by means of a large graded canonical transformation that exchanges zero-forms and one-forms.

of the second-quantization, that may thus be of importance for addressing the vacuum problem in generally covariant quantum field theory.

1.1 Plan of paper

The plan of the paper is as follows: In section 2 we review off-shell formulations of unfolded systems on commutative base manifold, paying attention to global issues that we have not seen being treated elsewhere to the same level of completeness. In section 3 we extend the AKSZ formalism to unfolded systems on non-commutative base manifolds in such a way as to construct a minimal master action for Vasiliev's theory. We conclude in section 4. The appendix details the usage of vector fields and functional derivatives on non-commutative manifolds.

2 Action principles for unfolded systems on commutative manifolds

2.1 General ideas

Unfolded dynamics. Unfolded dynamics concerns the formulation of field theory in terms of differential algebras. In their basic setting, referred to as graded-commutative free differential algebras, these are sets of differential forms on ordinary commutative (super)manifolds that remain invariant under exterior differentiation and wedging. Their generating elements, denoted by X^α below, are locally-defined forms whose exterior derivatives are completely constrained in a Cartan-integrable fashion, amounting to generalized curvature constraints written $dX^\alpha + Q^\alpha(X) \approx 0$ below.

Various moduli spaces, consisting of gauge orbits subject to boundary conditions, including transitions between charts in the interior of the base manifold, are then described by different types of classical observables as follows. As the observables are dual pairings between elements in the differential algebra and geometric structures on the base manifold, such as points, curves and cycles, they possess two key invariance properties: i) invariance on-shell under small diffeomorphisms, preserving the geometric structures; and ii) invariance off-shell under the generalized structure group containing a subset of all Cartan gauge symmetries.

We wish to stress that as for the off-shell gauge structure, i.e. structure group and the off-shell resolutions of the corresponding set of observables, there exist multiple, physically inequivalent choices. This leads to the notion of a large moduli space of an unfolded system containing physically distinct phases, such as for example unbroken and metric-like phases of a theory of higher-spin gravity. The analysis of phase transitions thus requires a framework for computing partition functions in different ensembles in which the generators of the differential algebra play the role of fundamental fields entering directly into the path integral measure.

BV-AKSZ implementation. Unfolded dynamics, on commutative as well as non-commutative manifolds, admits a natural off-shell formulation of the generalized Hamiltonian type: the bulk action consists of kinetic terms $\sim \int P \wedge dX$, where thus X and P may

have form degrees greater than one, plus a Hamiltonian $\mathscr{H}(X, P)$ containing all interactions; the latter are subject to integrability conditions assuring that the gauge symmetries of the kinetic terms are deformed smoothly[3] into non-abelian gauge symmetries that need not close off-shell. The generalized curvature constraints arise on boundaries of bulk manifolds — on which the momenta variables vanish — upon extremizing the action, and the aforementioned ensembles arise upon perturbing the action by various generalized Poisson structures coupling to the bulk and topological vertex operators inserted at the boundaries, which one may think of as third- quantized analogs of closed- and open-string states, respectively. These key features of the generalized Hamiltonian action can be incorporated into quantization schemes based on the BV field-anti-field formalism or generalizations thereof, which also lends itself to topological summation, master-field descriptions of (topological) vertex operators ensembles and other "third-quantized" concepts, which one may view as being defined in this fashion. Its precise relation to standard "second-quantized" amplitudes remains to be established, however, though proposals for how these may arise — in a suitable metric phase — have been made in the case of higher-spin gravity [51].

As found by AKSZ, the BV formalism can be implemented in the generalized- Hamiltonian case by extending each differential form into a "vectorial" superfield of fixed total degree given by the sum of form degrees, ranging from zero to the top-form degree, and ghost numbers belonging to the integers. This construction manifests the fact that the (canonical) Poisson bracket in target (super)space induces the BV anti-bracket on the space of maps. As a result, substituting each field in the classical action by its corresponding superfield and projecting to zero ghost number yields a master action obeying the classical BV master equation and reducing to the classical action when all anti-fields vanish. Moreover, the corresponding BV Laplacian annihilates any local super-functional, and hence in particular the AKSZ master action, which thus obeys the classical as well as the quantum master equations. The BRST transformations thus remain canonical at the quantum level, and hence, in the absence of anomalies, the quantum field theory will possess a BRST operator acting as a differential within a suitable homotopy associative algebra.

In what follows we shall describe the BV-AKSZ formalism in more detail, after which we shall adapt it in the next section to the case of Vasiliev's 4D higher-spin gravity theory, which is based on a graded-noncommutative and associative free differential algebras.

2.2 Classical sigma model

Classical unfolded dynamics on commutative manifolds. At the classical level, an unfolded system on a commutative base manifold B is a graded-commutative free differential algebra $\mathscr{A}$ on B. Decomposing the base manifold into charts, say $B = \bigcup_\xi B_\xi$, the

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[3]As usual, the term "smooth" refers to constancy of the number of gauge parameters. However, the "number" of physical degrees of freedom, as measured by classical observables, may change as non-abelian gauge interactions change physical-observable conditions abruptly; secondly, the phase-space volume elements themselves depend on strengths of couplings, that may induce critical phenomena. In the case of higher-spin gravities, the free fields are characterized by point-wise defined Weyl tensors (polarization tensors), while for fully non- linear solutions, the physical content in the Weyl tensors is captured by non-local observables [49] such as the eigenvalues of a certain Weyl zero-form operator [50]. In addition, the full solution space exhibits an interesting phase structure with critical "electric" fields [50].

free differential algebras decomposes into sub-algebras, say $\mathscr{A} = \bigoplus_\xi \mathscr{A}_\xi$ that are invariant under the wedge product and the action of the exterior derivative d. The generators $\{X_\xi^\alpha\}$ of $\mathscr{A}_\xi$ are thus differential forms of degrees $p_\alpha := \deg(X_\xi^\alpha) \geqslant 0$, defined locally on B_ξ and valued in some finite-dimensional real spaces Θ_α, referred to as types, and obeying generalized curvature constraints, viz.

$$R_\xi^\alpha := \mathrm{d}X_\xi^\alpha + Q^\alpha(X_\xi^\beta) \approx 0, \tag{2.1}$$

where Q^α are wedge functions obeying the structure equations

$$Q^\beta \partial_\beta Q^\alpha \equiv 0, \tag{2.2}$$

with ∂_α denoting the left-derivative with respect to X^α. These identities imply generalized Bianchi identities (the chart index ξ will be omitted from now on whenever ambiguity cannot arise)

$$\mathrm{d}R^\alpha - R^\beta \partial_\beta Q^\alpha \equiv 0, \tag{2.3}$$

such that the constraints are universally Cartan integrable, i.e. compatible with $\mathrm{d}^2 \equiv 0$ in arbitrary dimensions. It follows that the generalized curvatures transform into each other under Cartan gauge transformations, viz.

$$\delta_\epsilon X^\alpha := \mathrm{d}\epsilon^\alpha - \epsilon^\beta \partial_\beta Q^\alpha \quad \Rightarrow \quad \delta_\epsilon R^\alpha = (-1)^\beta \epsilon^\beta R^\gamma \partial_\gamma \partial_\beta Q^\alpha, \tag{2.4}$$

where ϵ^α are unconstrained gauge parameters of degrees $\deg(\epsilon^\alpha) = p_\alpha - 1$ (hence $\epsilon^\alpha \equiv 0$ if $p_\alpha = 0$) valued in Θ_α and defined on B_ξ. The locally-defined solution spaces consist of gauge orbits

$$X_{C;\lambda}^\alpha = G_\lambda X^\alpha|_{X^\alpha = C^\alpha} \tag{2.5}$$

labeled by zero-form integration constants $C^\alpha = \delta_{p_\alpha,0} C^\alpha$ obeying $dC^\alpha = 0$ and generated by finite Cartan gauge transformations

$$G_\lambda := \exp\left((\mathrm{d}\lambda^\beta - \lambda^\gamma \partial_\gamma Q^\beta(X))\partial_\beta\right), \tag{2.6}$$

where λ^α are gauge functions of degrees $\deg(\lambda^\alpha) = p_\alpha - 1$ (and hence $\lambda^\alpha \equiv 0$ if $p_\alpha = 0$). The locally-defined solution spaces can be glued together into globally-defined solution spaces via gauge transitions, viz.

$$X_\xi^\alpha = G_{t_\xi^{\xi'}} X^\alpha \bigg|_{X^\alpha = X_{\xi'}^\alpha}, \tag{2.7}$$

using suitable locally-defined parameters $t_\xi^{\alpha,\xi'}$ defined on the overlaps $B_\xi \cap B_{\xi'}$. The choice of the structure group leaves room for various physically distinct possibilities depending on the Q-structure (for a discussion, see e.g. [1, 51]). In particular, one may seek global formulations that are direct generalizations of fiber bundles with classical observables that are invariant off-shell under gauge transformations with parameters belonging to the structure algebra, and on-shell under the complete Cartan gauge algebra. For more general geometric formulations, see e.g. [43, 44].

Classical generalized Hamiltonian action.   Classical unfolded systems can be embedded into on-shell configurations of generalized Poisson sigma models, namely as boundary configurations in formulations on open base manifolds of fixed dimension. To this end, one assumes that

$$\dim(B) = \hat{p} + 1\,, \qquad \partial B = \cup_\lambda B'_\lambda\,, \tag{2.8}$$

where each B'_λ is a connected boundary component (which may in itself be covered by an atlas inherited from the bulk), and considers sigma-model maps

$$\phi:\ T[1]B\ \to\ M \tag{2.9}$$

of vanishing intrinsic degree from the parity-shifted tangent bundle $T[1]B$ to a target space M given by an $\mathbb{N}$-graded symplectic Q-manifold. The latter consists of charts,

$$M = \cup_I M_I\,, \tag{2.10}$$

with locally-defined coordinates

$$Z_I^i:\ M_I\ \to\ \theta^i[p_i]\,, \qquad \deg(Z_I^i) = p_i\ \in\ \mathbb{N}\,, \tag{2.11}$$

where $\theta^i[p_i]$ denote p_i-suspended types. It carries two compatible geometric structures: a symplectic two-form $\mathscr{O}$ of degree $\hat{p} + 2$ and a Hamiltonian function $\mathscr{H}$ of degree $\hat{p} + 1$ obeying the *structure equation*

$$\{\mathscr{H}, \mathscr{H}\}_{[-\hat{p}]} \equiv 0\,, \qquad \deg(\mathscr{H}) = \hat{p} + 1\,. \tag{2.12}$$

The canonical Poisson bracket, which has intrinsic degree $-\hat{p}$ and is graded in such a way that $\{\mathscr{H}, \mathscr{H}\}_{[-\hat{p}]}$ does not vanish trivially, is given by

$$\{A, B\}_{[-\hat{p}]} = (-1)^{\hat{p} + (\hat{p}+i+1)A}\ \partial_i A\ \mathscr{P}^{ik}\ \partial_j B \tag{2.13}$$

where we use the conventions

$$\mathscr{O} = \tfrac{1}{2}\, dZ^i dZ^j\, \widetilde{\mathscr{O}}_{ij} \equiv \tfrac{1}{2}\, dZ^i \mathscr{O}_{ij}\, dZ^j\,, \qquad \mathscr{P}^{ik} \mathscr{O}_{kj} = (-1)^{\hat{p}} \delta^i_j\,. \tag{2.14}$$

In particular, the structure equation reads

$$(-1)^{i(\hat{p}+1)}\, \partial_i \mathscr{H}\, \mathscr{P}^{ij} \partial_j \mathscr{H}\ \equiv\ 0\,. \tag{2.15}$$

Locally in target space, one can introduce pre-symplectic forms

$$\mathscr{O}|_{M_I} = \mathrm{d}\vartheta_I\,, \qquad \deg(\vartheta_I) = \hat{p} + 1\,, \tag{2.16}$$

defined modulo $\vartheta \sim \vartheta + \mathrm{d}\mathscr{E}$, and consider the generalized Hamiltonian bulk action

$$S^{\mathrm{cl}}_{\mathrm{bulk}}[\phi|B] = \sum_\xi \int_{B_\xi} \mathscr{L}^{\mathrm{cl}}_\xi = \sum_\xi \int_{B_\xi} \pi\, \phi^*_\xi(\vartheta - \mathscr{H})\,, \tag{2.17}$$

where $\phi_\xi \equiv \phi|_{B_\xi}$ and $\pi : \Omega(T[1]B) \to \Omega(B)$ is a degree-preserving canonical homomorphism that takes k-forms on $T[1]B$ of degree p to p-forms on B, *viz.*

$$\pi : \ \Omega^{[k|p]}(T[1]B) \to \Omega^{[p]}(B) , \tag{2.18}$$

and that intertwines the actions of the exterior derivative d in $\Omega(B)$ and the Lie derivative $\mathscr{L}_q = i_q \circ \mathrm{d} - \mathrm{d} \circ i_q$ in $\Omega(T[1]B)$ along the canonical Q-structure on $T[1]B$ as follows:

$$\mathrm{d} \circ \pi = \pi \circ \mathrm{d} = \pi \circ \mathscr{L}_q , \quad q := \theta^\mu \partial_\mu . \tag{2.19}$$

Equipping $T[1]B$ with coordinates

$$(x^\mu, \theta^\mu) , \qquad \deg(x^\mu, \theta^\mu) = (0, 1) , \tag{2.20}$$

one has

$$\pi(f(x^\mu, \theta^\mu; dx^\mu, d\theta^\mu)) = f(x^\mu, dx^\mu; dx^\mu, 0) . \tag{2.21}$$

Thus the exterior differential d , which has form-degree one, has degree one, i.e.

$$\deg(\mathrm{d}) = \deg(q) = 1 . \tag{2.22}$$

The assumption that the sigma-model maps ϕ have vanishing intrinsic degree implies

$$\Omega^{[k|p]}(M) \xrightarrow{\phi^*} \Omega^{[k|p]}(T[1]B) \xrightarrow{\pi} \Omega^{[p]}(B) , \tag{2.23}$$

that is, the pull-back ϕ^* of a k-form of $\mathbb{N}$-degree p on M is a ditto on $T[1]B$, in its turn sent by π to a p-form on B; the condition that M is $\mathbb{N}$-graded (instead of $\mathbb{Z}$-graded) and $\deg(\mathrm{d}) = 1$ implies that $p \geqslant k$. Thus, since

$$\mathscr{O} = \mathrm{d}\vartheta \in \Omega^{[2|\hat{p}+2]}(M) , \quad \vartheta \in \Omega^{[1|\hat{p}+1]}(M) , \quad \mathscr{H} \in \Omega^{[0|\hat{p}+1]}(M) , \tag{2.24}$$

it follows that

$$\pi\phi^*(\vartheta - \mathscr{H}) \in \Omega^{[\hat{p}+1]}(B) , \tag{2.25}$$

which can then be integrated by decomposing B into charts B_ξ.

Total variation and gauge variation. The total variation of the action can be obtained from the Lie derivative

$$\{\mathrm{d}, i_{\overrightarrow{\delta Z}}\}(\vartheta - \mathscr{H}) = i_{\overrightarrow{\delta Z}}(\mathscr{O} - \mathrm{d}\mathscr{H}) + \mathrm{d}\left(i_{\overrightarrow{\delta Z}}\vartheta\right) , \tag{2.26}$$

where the target space vector field $\overrightarrow{\delta Z} = \delta Z^i \, \vec{\partial}_i$. Writing

$$\vartheta = \mathrm{d}Z^i \vartheta_i , \tag{2.27}$$

one has

$$\delta\mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} = \delta Z^i \mathscr{R}^j \, \widetilde{\mathscr{O}}_{ij} + \mathrm{d}\left(\delta Z^i \vartheta_i\right) , \tag{2.28}$$

with generalized curvatures and Hamiltonian vector field $\overrightarrow{\mathscr{Q}}$ defined by

$$\mathscr{R}^i = \mathrm{d}Z^i + \mathscr{Q}^i \,, \qquad\qquad \mathscr{Q}^i = (-1)^{\hat{p}+1}\,\mathscr{P}^{ij}\partial_j\mathscr{H}\,,$$
$$\overrightarrow{\mathscr{Q}} = \mathscr{Q}^i\,\vec{\partial}_i\,, \qquad\qquad \deg(\overrightarrow{\mathscr{Q}}) = 1\,. \tag{2.29}$$

Demanding the generalized Bianchi identities

$$\mathrm{d}\mathscr{R}^i - \mathscr{R}^j\partial_j\mathscr{Q}^i \;\equiv\; 0\,, \tag{2.30}$$

requires $\overrightarrow{\mathscr{Q}}$ to be a Hamiltonian Q-structure, *viz.*

$$\mathscr{L}_{\overrightarrow{\mathscr{Q}}}\overrightarrow{\mathscr{Q}} = \{\overrightarrow{\mathscr{Q}},\overrightarrow{\mathscr{Q}}\}_{\mathrm{S.B.}} \;\equiv\; 0 \quad\Leftrightarrow\quad \mathscr{Q}^j\partial_j\mathscr{Q}^i \equiv 0 \quad\Leftrightarrow\quad \partial_i\{\mathscr{H},\mathscr{H}\}_{[-\hat{p}]} \equiv 0\,, \tag{2.31}$$

which is equivalent to the structure equation assuming there are no constants of total degree $\hat{p}+2$. The structure equation also implies

$$\mathrm{d}(\vartheta - \mathscr{H}) \equiv \tfrac{1}{2}\mathscr{R}^i\mathscr{R}^j\,\widetilde{\mathscr{O}}_{ij} \equiv \tfrac{1}{2}\mathscr{R}^i\mathscr{O}_{ij}\mathscr{R}^j\,. \tag{2.32}$$

Under the chart-wise defined Cartan gauge transformations

$$\delta_\varepsilon Z^i := \mathrm{d}\varepsilon^i - \varepsilon^j\partial_j\mathscr{Q}^i + \tfrac{1}{2}\,\varepsilon^k\mathscr{R}^l\,\partial_l\widetilde{\mathscr{O}}_{kj}\,\mathscr{P}^{ji}\,, \tag{2.33}$$

the Lagrangian transforms into a total derivative as follows:

$$\delta_\varepsilon\mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}} \equiv \mathrm{d}K_\varepsilon\,, \qquad K_\varepsilon := \varepsilon^i\mathscr{R}^j\,\widetilde{\mathscr{O}}_{ij} + \delta_\varepsilon Z^i\vartheta_i + \mathrm{d}\Upsilon_\epsilon\,, \tag{2.34}$$

where Υ_ϵ is defined on B_ξ and the cancellation of $\mathscr{R}^k$-terms requires that $\mathscr{Q}_i := \mathscr{O}_{ij}\mathscr{Q}^j$ obeys $\partial_i\mathscr{Q}_j \equiv (-1)^{ij}\partial_j\mathscr{Q}_i$ which holds as a consequence of $\mathrm{d}^2\mathscr{H} \equiv 0$. The gauge transformations close as follows [1]:

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]Z^i \equiv \delta_{\varepsilon_{12}}Z^i - \overrightarrow{\mathscr{R}}\varepsilon_{12}^i\,, \tag{2.35}$$

$$\varepsilon_{12}^i \equiv [\varepsilon_1,\varepsilon_2]^i := -\overrightarrow{\varepsilon}_1\,\overrightarrow{\varepsilon}_2\,\mathscr{Q}^i \equiv \overrightarrow{\varepsilon}_2\,\overrightarrow{\varepsilon}_1\,\mathscr{Q}^i\,, \qquad\qquad \overrightarrow{\mathscr{R}} \equiv \mathscr{R}^i\partial_i\,, \tag{2.36}$$

where $\overrightarrow{\mathscr{R}}\varepsilon_{12}^i$ generates a trivial gauge transformation $\delta_{\overrightarrow{\mathscr{R}}\varepsilon_{12}}$ as can be seen from

$$\delta_{\overrightarrow{\mathscr{R}}\varepsilon_{12}}\mathscr{L}^{\mathrm{cl}}_{\mathrm{bulk}}(p') \equiv \int_{p\in B}(\overrightarrow{\mathscr{R}}\varepsilon_{12}^i)(p)\frac{\delta\mathscr{L}(p')}{\delta Z^i(p)} \equiv \mathrm{d}\left[(\overrightarrow{\mathscr{R}}\varepsilon_{12}^i)\vartheta_i\right](p')\,, \tag{2.37}$$

which follows from (2.28).

Global base-manifold formulation of fiber-bundle type. The action is well-defined, i.e.

$$\delta_\varepsilon S^{\mathrm{cl}}_{\mathrm{bulk}} \equiv \sum_\xi \oint_{\partial B_\xi} K_{\epsilon_\xi} = 0\,, \tag{2.38}$$

provided that the locally-defined fields Z^i_ξ and gauge parameters ϵ^i_ξ are subject to suitable conditions at ∂B_ξ — and we note that $\oint_{\partial B_\xi}\mathrm{d}\Upsilon_{\epsilon_\xi} = 0$ since Υ_{ϵ_ξ} is defined on B_ξ and hence

Figure 1. Compatibility condition for the fiber bundle.

globally on ∂B_ξ. Under certain extra assumptions[4] on ϑ and $\mathscr{H}$, the latter amount to conditions at ∂B together with rules for gauge transitions $\hat{\delta}_{t_{\xi'}^\xi}$ across chart boundaries with parameters $t_{\xi'}^{i,\xi}$ defined on overlaps. The assumptions are

$$(i) \quad \hat{\delta}_t K_\varepsilon \equiv 0\,, \quad (ii) \quad \partial_j \partial_k \overrightarrow{t} \mathscr{Q}^i \equiv 0\,, \quad (iii) \quad K_t \equiv 0 \,. \tag{2.39}$$

Assumption (i), which states that K_ε is defined globally, implies the cancellation of contributions to $\delta_\varepsilon S_{\mathrm{bulk}}^{\mathrm{cl}}$ from chart boundaries in the interior of the bulk manifold, leaving

$$K_\varepsilon|_{\partial B} \equiv 0\,, \tag{2.40}$$

as conditions on fields and gauge parameters off shell. Assumptions (ii) and (iii) ensure compatibility between having, on the one hand, gauge transformations $\hat{\delta}_{\varepsilon_\xi}$ on charts acting on fields Z_ξ^i and gauge transition parameters $t_{\xi'}^{i,\xi}$ and, on the other hand, gauge transitions $\hat{\delta}_{t_{\xi'}^\xi}$ between adjacent charts acting acting on fields Z_ξ^i and gauge parameters ε_ξ. As for (ii), the commutativity of the diagram in figure 1 requires

$$Z_\xi^i + \delta_{\varepsilon_\xi} Z_\xi^i + \hat{\delta}_{\tilde{t}_{\xi'}^\xi}(Z_\xi^i + \delta_{\varepsilon_\xi} Z_\xi^i) = Z_\xi^i + \delta_{t_{\xi'}^\xi} Z_\xi^i + \delta_{\varepsilon_{\xi'}}(Z_\xi^i + \delta_{t_{\xi'}^\xi} Z_\xi^i)\,, \tag{2.41}$$

where $\delta_{\varepsilon_{\xi'}}$ only acts on fields and

$$\tilde{t}_{\xi'}^\xi := t_{\xi'}^\xi + \hat{\delta}_{\varepsilon_\xi} t_{\xi'}^\xi \,. \tag{2.42}$$

As $\hat{\delta}_{t_{\xi'}^\xi} \varepsilon_\xi$ drops out, the above condition is equivalent to

$$\delta_{\hat{\delta}_{\varepsilon_\xi} t_{\xi'}^\xi} Z_\xi^i = \left(\delta_{t_{\xi'}^\xi} \delta_{\varepsilon_{\xi'}} - \delta_{\varepsilon_\xi} \delta_{t_{\xi'}^\xi} \right) Z_\xi^i\,, \tag{2.43}$$

whose consistency requires (ii) and one identifies

$$\hat{\delta}_{\varepsilon_\xi} t_{\xi'}^\xi = [t_{\xi'}^\xi, \varepsilon_\xi] \,. \tag{2.44}$$

The transformation $\hat{\delta}_{t_{\xi'}^\xi} \varepsilon_\xi$ is instead fixed by the third assumption *(iii)* which ensures the commutativity between (i) and $\delta_\varepsilon \mathscr{L} \equiv \mathrm{d}K_\varepsilon$; acting with $\hat{\delta}_t$ on the latter identity using $\hat{\delta}_t \delta_\varepsilon \mathscr{L} \equiv \delta_t \delta_\varepsilon \mathscr{L} + \delta_{\hat{\delta}_t \varepsilon} \mathscr{L}$ and (ii) yields

$$\mathrm{d}\left(K_{\hat{\delta}_t \varepsilon + [t,\varepsilon]} + \delta_\varepsilon K_t \right) = 0 \tag{2.45}$$

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[4]For a more general treatment, based on geometrical concepts beyond those of the standard theory of fiber bundles which are used in the present paper, see [43, 44].

from which one deduces that

$$\hat{\delta}_{t_{\xi'}^{\xi}} \varepsilon_\xi = [\varepsilon_\xi, t_{\xi'}^{\xi}] \equiv -\hat{\delta}_{\varepsilon_\xi} t_\xi^{\xi'} \tag{2.46}$$

provided that (iii) holds.

Equations of motion. Applying the variational principle to the action yields the following equations of motion and boundary conditions:

$$\mathscr{R}^i \approx 0, \qquad \delta Z^i \vartheta_i\big|_{\partial B} \approx 0 . \tag{2.47}$$

We recall that $K_\varepsilon|_{\partial B} \equiv 0$ holds off shell as to assure the gauge-invariance of the action and hence the gauge-covariance of the above equations of motion as well as the cancellation of boundary terms in the interior of B in δS, i.e.

$$\delta_t(\delta Z^i \vartheta_i) \equiv 0 . \tag{2.48}$$

Canonical coordinates. We assume[5] that the target manifold has the structure of a $\hat{p}$-suspended cotangent space $M \cong T^*[\hat{p}]N$ with canonical coordinates

$$Z^i = (X^\alpha, P_\alpha), \qquad \deg(X^\alpha) + \deg(P_\alpha) = \hat{p}, \qquad \deg(X^\alpha), \deg(P_\alpha) \in \mathbb{N}. \tag{2.49}$$

Moreover, the pre-symplectic form can be chosen to be given by[6]

$$\vartheta = \mathrm{d}X^\alpha P_\alpha, \qquad \mathscr{O} = (-1)^{\alpha+1}\mathrm{d}X^\alpha \mathrm{d}P_\alpha, \qquad \mathscr{P} = \tfrac{1}{2}\left((-1)^{\hat{p}\alpha}\partial_\alpha\partial^\alpha + (-1)^{\alpha+\hat{p}+1}\partial^\alpha\partial_\alpha\right), \tag{2.50}$$

$$\mathscr{O}_{ij} = \widetilde{\mathscr{O}}_{ij} = \begin{bmatrix} 0 & (-1)^{\alpha+1}\delta_\alpha{}^\beta \\ (-1)^{\hat{p}(\alpha+1)}\delta^\alpha{}_\beta & 0 \end{bmatrix}, \qquad \mathscr{P}^{ij} = \begin{bmatrix} 0 & (-1)^{\hat{p}\alpha}\delta^\alpha{}_\beta \\ (-1)^{\alpha+\hat{p}+1}\delta_\alpha{}^\beta & 0 \end{bmatrix}. \tag{2.51}$$

The equations of motion and structure equation now read

$$\mathscr{R}^\alpha = \mathrm{d}X^\alpha + \mathscr{Q}^\alpha \approx 0, \qquad\qquad \mathscr{R}_\alpha = \mathrm{d}P_\alpha + \mathscr{Q}_\alpha \approx 0, \tag{2.52}$$

$$\mathscr{Q}^\alpha = (-1)^{\hat{p}(\alpha+1)+1}\partial^\alpha\mathscr{H}, \qquad\qquad \mathscr{Q}_\alpha = (-1)^\alpha\partial_\alpha\mathscr{H}, \tag{2.53}$$

$$(-1)^\alpha\partial_\alpha\mathscr{H}\partial^\alpha\mathscr{H} \equiv 0, \qquad\qquad \mathrm{d}(\vartheta - \mathscr{H}) \equiv (-1)^{\alpha+1}\mathscr{R}^\alpha\mathscr{R}_\alpha . \tag{2.54}$$

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[5]This assumption implies no loss of generality provided the starting point is the classical unfolded systems on ∂B (with target space N). It does lead to restrictions, however, starting from systems on B (with target space M) where it excludes models with $\hat{p} = 2(2n+1)$ and coordinates in $Z^{i'}$ of degree $p_{i'} = 2n+1$ contributing $\frac{1}{2}dZ^{i'}dZ^{j'}k_{i'j'}$ to $\mathscr{O}$ where $k_{i'j'}$ is positive definite, such as three-dimensional Chern-Simons theories with compact gauge algebra $\mathfrak{g}_k$. The latter instead admit formulations as four-dimensional BF-models with action $\int_B \mathrm{tr}(TF - \frac{1}{2k}T^2)$ where $F := \mathrm{d}A + A^2$, which is locally on-shell equivalent to $\frac{k}{2}\oint_{\partial B} \mathrm{tr}(A\,\mathrm{d}A + \frac{2}{3}A^3)$. On the other hand, certain non-compact cases admit formulations as three-dimensional BF-models. For example, for the gauge algebra $\mathfrak{g}_k \oplus \mathfrak{g}_{-k}$, which is of relevance for three-dimensional vacuum higher-spin gravities, one has $\frac{k}{2}\int_B \mathrm{tr}(A\,\mathrm{d}A + \frac{2}{3}A^2 - \widetilde{A}\mathrm{d}\widetilde{A} - \widetilde{A}^3 + \mathrm{d}(A\widetilde{A})) \equiv k\int_B \mathrm{tr}(ER + \frac{1}{12}E^3)$ where now $\dim(B) = 3$ and $E := A - \widetilde{A}$, $R := \mathrm{d}\Gamma + \Gamma^2$ and $\Gamma := \frac{1}{2}(A + \widetilde{A})$ — and the total derivative yields manifest invariance under diagonal gauge transformations.

[6]This choice is equivalent to using $\vartheta_{\mathrm{alt}} = \frac{1}{2}Z^i dZ^j \mathscr{O}_{ij} = \frac{1}{2}\left(dX^\alpha P_\alpha - (-1)^{\alpha(\hat{p}+1)}dP_\alpha X^\alpha\right)$ and adding Gibbons-Hawking-type boundary terms of the form $\frac{1}{2}\sum_\xi \oint_{\partial B_\xi} X^\alpha P_\alpha$.

The power-series expansion of $\mathscr{H}$ in P_α yields rank-n poly-vector fields $\Pi(n)$ on N of degrees $1 + (1 - n)\hat{p}$ whose mutual Schouten brackets vanish, viz.

$$\{\Pi_{(n_1)}, \Pi_{(n_2)}\}_{\text{S.B.}} \equiv 0 \quad \text{for all } n_1, n_2 \geqslant 0 \,. \tag{2.55}$$

Using the notation $\varepsilon^i = (\epsilon^\alpha, \eta_\alpha)$ and choosing $\Upsilon_\varepsilon = -\epsilon^\alpha P_\alpha$, the gauge variation of $S_{\text{bulk}}^{\text{cl}}[X, P|B]$ reads

$$\delta_\varepsilon \mathscr{L}_{\text{bulk}}^{\text{cl}} = \mathrm{d}K_\varepsilon \,, \qquad K_\varepsilon = (-1)^{\hat{p}(\alpha+1)} \eta_\alpha \mathscr{R}^\alpha + \left((\overrightarrow{P} - 1)\overrightarrow{\epsilon} + \overrightarrow{P}\,\overrightarrow{\eta} \right) \mathscr{H} \,, \tag{2.56}$$

where we have defined

$$\overrightarrow{P} := P_\alpha \frac{\partial}{\partial P_\alpha} \,, \qquad \overrightarrow{\epsilon} = \epsilon^\alpha \frac{\partial}{\partial X^\alpha} \,, \qquad \overrightarrow{\eta} = \eta_\alpha \frac{\partial}{\partial P_\alpha} \,. \tag{2.57}$$

Globally-defined formulations of fiber-bundle type, as discussed above, thus arise by using transition functions with parameters $t_\xi^{\xi'} = (t^\alpha, 0)_{\xi'}^\xi$ obeying[7]

$$(\overrightarrow{P} - 1)\overrightarrow{t}\,\mathscr{H} = 0 \quad \Leftrightarrow \quad \overrightarrow{t}\,\Pi_{(n)} = 0 \quad \text{for } n \neq 1 \,, \tag{2.58}$$

and imposing the boundary conditions

$$K_\varepsilon|_{\partial B} \equiv 0 \,. \tag{2.59}$$

The latter can be implemented by the following Dirichlet conditions:

$$\eta_\alpha|_{\partial B} \equiv 0 \,, \qquad P_\alpha|_{\partial B} \equiv 0 \,, \tag{2.60}$$

provided that the function

$$\Pi_{(0)} \equiv \mathscr{H}|_{P_\alpha=0} \equiv 0 \,. \tag{2.61}$$

For these globally-defined models, the resulting integrable structures in the target space encompass

(i) a vector field $Q := \Pi_{(1)}$ of degree 1 that is nilpotent in the sense that $\mathscr{L}_Q Q = 2\{Q, Q\} \equiv 0$, referred to as the Q-structure;

(ii) a tower of generalized Poisson structures $\Pi_{(n)}$ with $n \geqslant 2$ that are compatible with Q in the sense that $\mathscr{L}_Q \Pi_{(n)} \equiv 0$;

(iii) if in addition $\Pi_{(n)} = 0$ for $n \geqslant 3$ then $\Pi_{(2)}$ is a Poisson structure equipping N with a Poisson bracket of intrinsic degree $-\hat{p} + 1$, referred to together with Q as a QP-structure.

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[7]As for (2.48), it can be checked that $\delta_t(\delta X^\alpha P_\alpha) = \overrightarrow{\delta X}(\overrightarrow{P} - 1)\overrightarrow{t}\,\mathscr{H} = 0$.

Transition amplitudes. Proceeding by assuming that $\partial B = \cup_\lambda B'_\lambda$, where B'_λ are connected boundary components, the space $\mathscr{M}$ of saddle points consists of gauge-equivalence classes of maps $\phi : T[1]B \to T^*[\hat{p}]N$ obeying $\mathscr{R}^\alpha \approx 0 \approx \mathscr{R}_\alpha$ on B and $P_\alpha|_{\partial B} \equiv 0$. Conversely, a set $\{\phi_\lambda : T[1]B'_\lambda \to N\}$ of boundary configurations obeying

$$R^\alpha|_{B'_\lambda} \approx 0, \qquad R^\alpha := \mathrm{d}X^\alpha + Q^\alpha, \tag{2.62}$$

may be referred to as being (classically) compatible with $(\vartheta, \mathscr{H})$ provided there exists an extrapolating bulk manifold B with $\partial B = \cup_\lambda B'_\lambda$ and a map $\phi : T[1]B \to T^*[\hat{p}]N$ obeying $\mathscr{R}^\alpha \approx 0 \approx \mathscr{R}_\alpha$ on B and $\phi|_{B'_\lambda} = \phi_\lambda$, which requires generalized Poisson structures in the non-trivial case. Semi-classically, the corresponding "third-quantized" transition amplitude

$$\mathscr{A}[\phi_\lambda] \sim \sum_B J(B) \exp\left(\frac{i}{\hbar} S^{\mathrm{cl}}_{\mathrm{bulk}}[\phi|B]\right), \quad \text{where} \quad \phi|_{B'_\lambda} = \phi_\lambda, \tag{2.63}$$

where $J(B)$ comprises functional determinants — combining into finite topological invariants once contributions from gauge-fixing sectors are included.

Generalized action-angles. Modified amplitudes arise upon perturbing $S^{\mathrm{cl}}_{\mathrm{bulk}}$ by topological vertex operators which are functionals $\oint_C \mathscr{V}(X, dX)$ obeying

$$\delta\mathscr{V}(X, dX) = \delta X^\alpha M_{\alpha\beta}(X, dX)R^\beta + \mathrm{d}(\delta X^\alpha \mathscr{P}_\alpha(X, dX)), \tag{2.64}$$

for some matrices $M_{\alpha\beta}$. Adding such perturbations with $C \subseteq \partial B$ to $S^{\mathrm{cl}}_{\mathrm{bulk}}$ yields a modified action

$$S^{\mathrm{cl}}_{\mathrm{tot}}[X, P; \mu_i|B; C_i] := S^{\mathrm{cl}}_{\mathrm{bulk}}[X, P|B] + \sum_r \mu_r \int_{C_r} \mathscr{V}^r, \qquad C_r \subseteq \partial B, \tag{2.65}$$

where μ_r are parameters. The total variation of the action now consists of bulk terms, which impose $\mathscr{R}^\alpha \approx 0 \approx \mathscr{R}_\alpha$, plus boundary terms that all vanish on-shell due to the boundary condition $P_\alpha|_{\partial B} \equiv 0$ (which holds off-shell and that implies $R^\alpha|_{\partial B} \approx 0$). Hence

$$\delta \int_{C_r} \mathscr{V}^r \approx 0, \tag{2.66}$$

that is, the on-shell values of the perturbations are classical observables

$$\mathscr{O}^r[X|C_r] := \int_{C_r} \mathscr{J}^r(X), \qquad \mathscr{J}^r := \mathscr{V}^r(X^\alpha, -Q^\alpha), \tag{2.67}$$

that are defined intrinsically in the sense that if δ_{C_r} denotes a small variation of C_r then

$$\mathrm{d}\mathscr{J}^r \approx 0 \qquad \Rightarrow \qquad \delta_{C_r}\mathscr{O}^r \approx 0. \tag{2.68}$$

On general grounds, such functionals are locally-defined functions on $\mathscr{M}$ as their finiteness requires further boundary conditions on $X^\alpha|_{B'_\lambda}$. Perturbatively, in weak-field expansions, the latter amount to taking linearized boundary zero-form integration constants and gauge-functions in suitable representations R_Σ of the underlying Cartan gauge algebra; for related

analyses in the case of higher-spin gravity, see [50]. In other words, finiteness of $\mathscr{O}^r$ holds in a super-selection sector given by a region of $\mathscr{M}$ labelled by a set $\{R_\Sigma\}$ of representations of the gauge algebra. One may refer to a set $\{\int_{C_r} \mathscr{V}^r\}$ of topological vertex operators as being complete if $\{\mathscr{O}^r\}$ is a set of (locally-defined) coordinates on (a super-selection sector of) $\mathscr{M}$.

Treating μ_r as generalized chemical potentials leads to the notion of a grand canonical ensemble with partition function

$$Z\{\mu_r; w\} = \left\langle \prod_r e^{\frac{i\mu_r}{\hbar} \int_{C_r} \mathscr{V}^r} \right\rangle_B , \tag{2.69}$$

where w denotes the moduli hidden in the transition functions and

$$\langle (\cdot) \rangle_B \sim \int \mathscr{D}X \mathscr{D}P (\cdot) e^{\frac{i}{\hbar} S_{\text{bulk}}^{\text{cl}}[X,P|B]} , \tag{2.70}$$

denotes a suitably regularized path integral measure (to be out-lined below). Micro-canonical ensembles with fixed $\int_{C_r} \mathscr{V}^r = q^r$ are then described by partition functions

$$\widetilde{Z}\{q^r; w\} = \prod_r \int \frac{d\mu_r}{2\pi} e^{-\frac{iq^r \mu_r}{\hbar}} Z\{\mu_r; w\} , \tag{2.71}$$

given by path integrals with fixed boundary conditions, *viz.*

$$\widetilde{Z}\{q^r\} \sim \left\langle \prod_r \delta \left(\int_{C_r} \mathscr{V}^r - q^r \right) \right\rangle_B . \tag{2.72}$$

The open Poisson sigma model can be made closed by filling in the boundary components B'_λ with open bulk manifolds B_λ obeying $\partial B_\lambda = -B'_\lambda$, which may require additional transition functions introducing further moduli that we denote by w', and considering the partition function

$$\overline{Z}\{\mu_r; w, w'\} := \left\langle \prod_r e^{\frac{i\mu_r}{\hbar} \int_{C_r} \mathscr{V}^r} \right\rangle_{\overline{B}} , \qquad \overline{B} := B \cup \bigcup_\lambda B_\lambda . \tag{2.73}$$

In the semi-classical limit, the filled-in bulk actions $S_{\text{bulk}}^{\text{cl}}[X, P|\overline{B}]$ become total derivatives (depending on w') which may play the role of counter-terms possibly along the lines of the recent work in [52].

2.3 BV master action

AKSZ quantization. The path integral measure (2.70) can be defined using the BV field-anti-field formalism following the AKSZ approach — see e.g. [31] for a review and references. To this end, the first step is to extend the classical sigma model by introducing layers of ghosts in correspondence with the classical gauge structure. The first layer of ghosts, which have the same form degree as the gauge parameters, have their own gauge symmetries, corresponding to gauge-for-gauge symmetries, which induce a second layer of ghosts, or ghost-for-ghosts, with one unit less of form degree, and so on. Proceeding this

way, via a canonical procedure to be reviewed below, yields a minimal quantum sigma model in which all fields have non-negative ghost numbers and which exhibits the complete gauge structure. As for the second step, which is the actual gauge-fixing procedure, involving the pairing of ghosts with suitable ghost-momenta and the introduction of Lagrange multipliers, it need not be unique, as various gauge-fixing schemes may refer to different additional special structures in target space over and above the generalized Poisson structures going into the (unique) minimal model. We shall not enter any further into these details but simply note the existence of a canonical (maximal) gauge-fixing scheme, that does not refer to any special target-space structures, with the salient features of a (classically) conserved BRST current and vacuum-bubble cancellation [33].

In order to arrive at the minimal quantum model, the classical map $\phi : T[1]B \to M$ is extended into

$$\phi : \ T[1]B \to \boldsymbol{M} \,, \tag{2.74}$$

where $\boldsymbol{M}$ is a bi-graded symplectic manifold containing M as a sub-manifold. As observed by AKSZ, the symplectic structure on M induces dittos on $\boldsymbol{M}$ and Maps $[T[1]B, \boldsymbol{M}]$ with the graded Poisson bracket of the latter being the BV bracket $(\cdot, \cdot) \equiv (\cdot, \cdot)_{\mathrm{BV}}$, the basic geometric structure underlying the BV field-anti-field formalism. Thus, in a certain space of local and ultra-local superfunctionals, based on a suitable extension of $\Omega(M)$ into $\Omega(\boldsymbol{M})$, the BV bracket is equivalent to the graded Poisson bracket on M. Moreover, the BV bracket-adjoint action of the integral of the pre-symplectic form on $\boldsymbol{M}$ over $T[1]B$ generates the exterior derivative. Taken together, these two lemmas imply that the classical BV master equation $(S, S) = 0$, subject to the functional boundary condition that S reduces to S^{cl} as all anti-fields are set to zero, has a simple and elegant solution given by the AKSZ action $\boldsymbol{S}$, which then also solves the quantum master equation, as we shall review next.

Vectorial superfields. Each classical coordinate $Z^i \equiv Z^{i\,\langle 0 \rangle}_{[p_i]}$ on M is extended into a tower of coordinates and conjugated anti-coordinates on $\boldsymbol{M}$ as follows:

$$\left\{ Z^{i\,\langle g \rangle}_{[p_i-g]} \,, \qquad Z^{\langle -1-g \rangle}_{i[\hat{p}+1-p_i+g]} := \left(Z^{i\,\langle g \rangle}_{[p_i-g]} \right)^{+} \right\}, \qquad g = 0, \ldots, p_i \,, \tag{2.75}$$

$$|Z^{i\,\langle g \rangle}_{[p_i-g]}| = p_i \,, \quad |Z^{\langle -1-g \rangle}_{i[\hat{p}+1-p_i+g]}| = \hat{p} - p_i \,, \tag{2.76}$$

where $O^{\langle g \rangle}_{[p]}$ denotes a component with distinct ghost number g and form degree p. The total degree and Grassmann parity (for classical theories consisting of only bosonic fields) are defined, respectively, by

$$|\cdot| := \deg(\cdot) + \mathrm{gh}(\cdot) \,, \quad \mathrm{Gr}(\cdot) = |\cdot| \mod 2 \,. \tag{2.77}$$

Given a differential form $L \in \Omega(\boldsymbol{M})$ of fixed total degree $|L|$, described locally on $\boldsymbol{M}$ by a function $L(Z, Z^+, dZ, dZ^+)$, with pull-back $\pi\phi^*(L) \equiv \sum_{p=0}^{\hat{p}+1} [\pi\phi^*(L)]^{\langle |L|-p \rangle}_{[p]} \in \Omega(B)$ and a p-cycle $C \subseteq B$, the integral

$$I(L|C) \equiv \sum_{\xi} \int_{B_\xi \cap C} \pi\,\phi^*_\xi(L) := \sum_{\xi} \int_{B_\xi \cap C} [\pi\phi^*L]^{\langle |L|-p \rangle}_{[p]} \ \text{ i.e. } \ \mathrm{gh}(I(L|C)) = |L| - p \,. \tag{2.78}$$

The canonical coordinates $Z^i = (X^\alpha, P_\alpha)$ of M induce supercoordinates $\boldsymbol{Z}^i = (\boldsymbol{X}^\alpha, \boldsymbol{P}_\alpha)$ of $\boldsymbol{M}$ of fixed total degree as follows:

$$\boldsymbol{X}^\alpha = \underbrace{X^{\alpha\,\langle p_\alpha\rangle}_{[0]} + X^{\alpha\,\langle p_\alpha - 1\rangle}_{[1]} + \ldots + X^{\alpha\,\langle 0\rangle}_{[p_\alpha]}}_{\text{fields}} + \underbrace{P^{\alpha\,\langle -1\rangle}_{[p_\alpha+1]} + P^{\alpha\,\langle -2\rangle}_{[p_\alpha+2]} + \ldots + P^{\alpha\,\langle p_\alpha - \hat{p}-1\rangle}_{[\hat{p}+1]}}_{\text{anti-fields}}, \qquad (2.79)$$

$$\boldsymbol{P}_\alpha = \underbrace{P^{\langle \hat{p}-p_\alpha\rangle}_{\alpha\,[0]} + P^{\langle \hat{p}-p_\alpha-1\rangle}_{\alpha\,[1]} + \ldots + P^{\langle 0\rangle}_{\alpha\,[\hat{p}-p_\alpha]}}_{\text{fields}} + \underbrace{X^{\langle -1\rangle}_{\alpha\,[\hat{p}-p_\alpha+1]} + X^{\langle -2\rangle}_{\alpha\,[\hat{p}-p_\alpha+2]} + \ldots + X^{\langle -p_\alpha-1\rangle}_{\alpha\,[\hat{p}+1]}}_{\text{anti-fields}}.$$
$$(2.80)$$

In these coordinates, the symplectic and pre-symplectic forms $\boldsymbol{O}$ and $\boldsymbol{\vartheta}$, respectively, on $\boldsymbol{M}$ read

$$\boldsymbol{O} = \left[(-1)^{\alpha+1}\mathrm{d}\boldsymbol{X}^\alpha \mathrm{d}\boldsymbol{P}_\alpha\right]^{\langle 0\rangle}_{[\hat{p}+2]} = \mathrm{d}\boldsymbol{\vartheta}, \qquad \boldsymbol{\vartheta} = [\mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha]^{\langle 0\rangle}_{[\hat{p}+1]}, \qquad (2.81)$$

and we denote the corresponding graded Poisson bracket on $\boldsymbol{M}$ by

$$\{\cdot,\cdot\} \equiv \{\cdot,\cdot\}^{\langle 0\rangle}_{[-\hat{p}]}, \qquad (2.82)$$

which thus has intrinsic quantum numbers $\mathrm{gh}\left(\{\cdot,\cdot\}\right) = 0$ and $\deg\left(\{\cdot,\cdot\}\right) = -\hat{p}$. The evaluation maps $\boldsymbol{Z}^i(p) : \phi \in \mathrm{Maps}\left[T[1]B, \boldsymbol{M}\right] \mapsto (\phi^*\boldsymbol{Z}^i)(p)$ for fixed $p \in T[1]B$ (see appendix A) define canonical coordinates on $\mathrm{Maps}\left[T[1]B, \boldsymbol{M}\right]$ in which its symplectic form

$$\boldsymbol{\Omega}(\delta\boldsymbol{Z}_1, \delta\boldsymbol{Z}_2) = I\left((-1)^{\alpha+1}\delta\boldsymbol{X}^\alpha_1 \delta\boldsymbol{P}_{2\alpha}|B\right) - (1 \leftrightarrow 2), \qquad \mathrm{gh}\left(\boldsymbol{\Omega}\right) = -1, \qquad (2.83)$$

where $\delta\boldsymbol{Z}$ denotes a vector field on $\mathrm{Maps}\left[T[1]B, \boldsymbol{M}\right]$ of total degree zero with component expansion

$$\delta\boldsymbol{Z}\big|_\phi = \int_{p\in T[1]B} \sum_{k=0}^{\hat{p}+1}\left[\pi\left(\phi^*\left(\delta Z^{i\langle p_i - k\rangle}_{[k]}\right)(p)\right) \frac{\delta}{\delta Z^{i\langle p_i-k\rangle}_{[k]}(p)}\bigg|_\phi\right.$$
$$\left. + \pi\left(\phi^*\left(\delta Z^{+\langle \hat{p}-p_i-k\rangle}_{i[k]}\right)(p)\right) \frac{\delta}{\delta Z^{+\langle \hat{p}-p_i-k\rangle}_{i[k]}(p)}\bigg|_\phi\right]. \qquad (2.84)$$

The corresponding graded Poisson bracket on $\mathrm{Maps}\left[T[1]B, \boldsymbol{M}\right]$, referred to as the BV bracket, is denoted by

$$(\cdot,\cdot) \equiv (\cdot,\cdot)^{\langle 1\rangle}_{[0]}, \qquad (2.85)$$

which thus has intrinsic quantum numbers $\mathrm{gh}\left((\cdot,\cdot)\right) = 1$ and $\deg\left((\cdot,\cdot)\right) = 0$.

BV bracket induced from Poisson bracket. As observed by AKSZ, the BV bracket $(\cdot,\cdot)$ on $\mathrm{Maps}\left[T[1]B, \boldsymbol{M}\right]$ is induced from the graded Poisson bracket $\{\cdot,\cdot\}$ on $\Omega^{[0]}(\boldsymbol{M})$ via the formula

$$\left(I(F|B),\,\phi^*(F')\right) \equiv \phi^*(\{F,F'\}), \qquad (2.86)$$

for $F, F' \in \Omega^{[0]}(\boldsymbol{M})$. It follows that the BV-adjoint action of the pre-symplectic form is related to the exterior derivative as follows:

$$(I(\mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha | B), \, \boldsymbol{\phi}^*(L)) \equiv \mathrm{d}\, \boldsymbol{\phi}^*(L) \equiv \boldsymbol{\phi}^*(\mathrm{d}\, L), \qquad (2.87)$$

for $L \in \Omega(\boldsymbol{M})$. We note that $\boldsymbol{\phi}^*(L)$ is an ultra-local functional, i.e. a function on $\mathrm{Maps}\,[T[1]B, \boldsymbol{M}]$, *idem* $F, F' \in \Omega^{[0]}(\boldsymbol{M})$, and that since $\deg(\mathrm{d}) = 1$ and $\mathrm{gh}(\mathrm{d}) = 0$ one has $\left(I(\mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha | B), \boldsymbol{\phi}^*(L_{[p]}^{\langle g \rangle}) \right) \equiv \boldsymbol{\phi}^*(\mathrm{d}L_{[p-1]}^{\langle g+1 \rangle})$.

Superfunctionals: are functionals built from ultra-local superfunctionals $\boldsymbol{\phi}^*(\boldsymbol{G})$ where $\boldsymbol{G} \in \Omega(\boldsymbol{M})$ have local representatives of the form $\boldsymbol{G} = G(\boldsymbol{Z}^i, d\boldsymbol{Z}^i)$ where $G \in \Omega(M)$. In particular, if $\boldsymbol{F}, \boldsymbol{F}'$ are superfunctions it follows that

$$\{\boldsymbol{F}, \boldsymbol{F}'\} = \left(\{F, F'\}_{[-\hat{p}]}(Z^i) \right)\big|_{Z^i \to \boldsymbol{Z}^i}, \qquad (2.88)$$

where $\{F, F'\}_{[-\hat{p}]}$ denotes the Poisson bracket evaluated in the classical target space M.

The AKSZ action: is given by the superfunctional

$$\boldsymbol{S}_{\mathrm{bulk}}[\boldsymbol{\phi}|B] := I(\boldsymbol{L}|B) = \sum_\xi \int_{B_\xi} \pi \phi_\xi^*(\boldsymbol{L}), \qquad \boldsymbol{L} := \mathrm{d}\boldsymbol{X}^\alpha \boldsymbol{P}_\alpha - \mathscr{H}(\boldsymbol{X}, \boldsymbol{P}), \qquad (2.89)$$

with $\mathscr{H}$ being a solution to the classical structure equation (2.15) obeying $\mathscr{H}|_{P_\alpha = 0} = 0$. Defining

$$\mathrm{s}(\cdot) := (\boldsymbol{S}_{\mathrm{bulk}}, (\cdot)), \qquad (2.90)$$

one has

$$\mathrm{s}\boldsymbol{Z}^i = \boldsymbol{R}^i, \qquad (2.91)$$

where the generalized supercurvatures

$$\boldsymbol{R}^i := \mathrm{d}\boldsymbol{Z}^i + \boldsymbol{Q}^i, \qquad \boldsymbol{Q}^i := \mathscr{Q}^i(\boldsymbol{Z}^j) = (-1)^{\hat{p}+1} \mathscr{P}^{ij} \partial_j \mathscr{H}(\boldsymbol{Z}^i), \qquad (2.92)$$

with $\boldsymbol{Q}^i$ being the superfield extension of the classical Hamiltonian Q-structure in (2.29). The locally-defined field configurations form equivalence classes modulo gauge transformations

$$\delta_{\boldsymbol{\varepsilon}} \boldsymbol{Z}^i := \mathrm{d}\varepsilon^i - \varepsilon^j \partial_j \boldsymbol{Q}^i, \qquad (2.93)$$

where the parameters have total degree $|\varepsilon^i| = |\boldsymbol{Z}^i| - 1$ and expansions into components with fixed ghost numbers and form degrees given by the suspension of eqs. (2.79) and (2.80) with one unit of form degree, and zero units of ghost number. As in the classical case, it follows from

$$\delta_{\boldsymbol{\varepsilon}} \boldsymbol{S}_{\mathrm{bulk}} \equiv \sum_\xi \oint_{\partial B_\xi} \boldsymbol{K}_{\boldsymbol{\varepsilon}}, \quad \boldsymbol{K}_{\boldsymbol{\varepsilon}} = (-1)^{\hat{p}(\alpha+1)} \boldsymbol{\eta}_\alpha \boldsymbol{R}^\alpha + \left((\overrightarrow{\boldsymbol{P}} - 1)\overrightarrow{\boldsymbol{\epsilon}} + \overrightarrow{\boldsymbol{P}}\,\overrightarrow{\boldsymbol{\eta}} \right) \mathscr{H}, \quad (2.94)$$

that the AKSZ action can be defined globally using fiber-bundle type geometries in which

(i) the local representatives $\boldsymbol{Z}_\xi^i$ are glued together using transition functions with parameters $\boldsymbol{t}_{\xi'}^{i,\xi} = (\boldsymbol{t}^\alpha, 0)_{\xi'}^\xi$, obeying

$$(\overrightarrow{\boldsymbol{P}} - 1)\overrightarrow{\boldsymbol{t}}\,\mathscr{H} \equiv 0 \quad \text{i.e.} \quad \overrightarrow{\boldsymbol{t}}\,\Pi_{(n)} \equiv 0 \ \text{ for } \ n \neq 1\,, \tag{2.95}$$

and

(ii) the following Dirichlet conditions are imposed:

$$\boldsymbol{\eta}_\alpha|_{\partial B} = 0\,, \qquad \boldsymbol{P}_\alpha|_{\partial B} = 0\,. \tag{2.96}$$

The AKSZ relation between the BV bracket and the Poisson bracket given above implies that

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) = (-1)^{\hat{p}} \sum_\xi \oint_{\partial B_\xi} \pi \phi_\xi^* \left(\boldsymbol{R}^\alpha \boldsymbol{P}_\alpha - 2\boldsymbol{L}\right) = 0\,, \tag{2.97}$$

where the latter equality follows from (2.96) and the facts that $\delta_{\boldsymbol{t}}\boldsymbol{L} \equiv \boldsymbol{K_t} \equiv 0$ and that

$$\delta_{\boldsymbol{t}}\boldsymbol{P}_\alpha = -(-1)^\alpha \overrightarrow{\boldsymbol{t}}\,\partial_\alpha\mathscr{H}\,, \qquad \delta_{\boldsymbol{t}}\boldsymbol{R}^\alpha = (-1)^{\hat{p}(\alpha+1)}\overrightarrow{\boldsymbol{R}}_X \overrightarrow{\boldsymbol{t}}\,\partial^\alpha\mathscr{H}\,, \tag{2.98}$$

where we have defined $\overrightarrow{\boldsymbol{R}}_X := \boldsymbol{R}^\alpha \partial_\alpha$, which implies

$$\delta_{\boldsymbol{t}}(\boldsymbol{R}^\alpha \boldsymbol{P}_\alpha) \equiv \overrightarrow{\boldsymbol{R}}_X \overrightarrow{\boldsymbol{t}}(\overrightarrow{\boldsymbol{P}} - 1)\mathscr{H} \equiv 0\,. \tag{2.99}$$

In other words, the AKSZ action $\boldsymbol{S}_{\text{bulk}}$ solves the classical BV master equation

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) = 0 \quad \Leftrightarrow \quad \text{s}^2 = 0\,, \tag{2.100}$$

subject to the functional boundary condition

$$\boldsymbol{S}_{\text{bulk}}[\phi|B]\big|_{\phi=\phi} = S_{\text{bulk}}^{\text{cl}}[\phi|B]\,. \tag{2.101}$$

Quantum master equation. A remarkable property of the AKSZ formalism is that any local super-functional $\boldsymbol{L}$ obeys

$$\Delta\boldsymbol{L} = 0\,, \tag{2.102}$$

where Δ is the BV-Laplacian. It follows that $\boldsymbol{S}_{\text{bulk}}$ obeys both classical and quantum master equations (see e.g. [33] and refs. therein), *viz.*

$$(\boldsymbol{S}_{\text{bulk}}, \boldsymbol{S}_{\text{bulk}}) = 0 = \Delta\boldsymbol{S}_{\text{bulk}}\,. \tag{2.103}$$

Hence $D\boldsymbol{Z}\exp(\frac{i}{\hbar}\boldsymbol{S}_{\text{bulk}})$ defines a BRST-invariant path-integral measure (on suitable Lagrangian submanifolds): The classical BRST transformation $\delta_{\text{BRST}}\mathscr{O} := \epsilon\,s(\mathscr{O})$, with constant fermionic parameter ϵ with $\text{gh}(\epsilon) = -1$, leaves both gauge-fixed action and $D\boldsymbol{Z}$ invariant; the former invariance requires the classical master equation while the latter invariance requires[8] $\Delta\boldsymbol{S}_{\text{bulk}} = 0$. The quantization thus deforms the classical differential

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[8]More generally, it follows from (2.102) that any canonical transformation, *viz.* $\delta_{\boldsymbol{E}}\mathscr{O} := (\boldsymbol{E}, \mathscr{O})$ with $\text{gh}(\boldsymbol{E}) = -1$, leaves $D\boldsymbol{Z}$ invariant.

algebra with differential d and Q-structure $\boldsymbol{Q}$, which one may view as a first-quantized algebra, into a second-quantized operator algebra with BRST current $\boldsymbol{j}_{\mathrm{BRST}}$ (which is conserved on shell barring anomalies) and differential $\mathrm{ad}_{\boldsymbol{q}}$ where $\boldsymbol{q} := \oint \boldsymbol{j}_{\mathrm{BRST}}$. Thus, acting on second-quantized ultra-local superfunctionals $\boldsymbol{F}$, one has $\mathrm{ad}_{\boldsymbol{q}}\boldsymbol{F} = \mathrm{d}\boldsymbol{F} + \rho(\boldsymbol{Q})\boldsymbol{F}$ where $\rho(\boldsymbol{Q})$ denotes the realization of $\boldsymbol{Q}$ in the second-quantized algebra, that, on general grounds, carries the structure of a graded homotopy-associative differential algebra.

Deformed master action. The BRST cohomology at ghost number zero consists of on-shell gauge-invariant observables [53].[9] Although the latter can be extended into off-shell functionals in various ways, the super-field framework leads to a unique extension: Given a set of classical observables, $\{\mathscr{O}^r\}$ say, with super-field extensions $\boldsymbol{O}^r = \mathscr{O}^r[\boldsymbol{Z}]$ obeying $\mathrm{s}\boldsymbol{O}^r = 0$, one seeks further off-shell extensions

$$\widehat{\boldsymbol{O}}^r := \boldsymbol{O}^r + \int_{C_r} \boldsymbol{R}^r \boldsymbol{L}_r \,, \qquad \mathrm{s}\boldsymbol{L}_r = 0 \,, \qquad (\widehat{\boldsymbol{O}}^r, \widehat{\boldsymbol{O}}^r) = 0 \,, \tag{2.104}$$

i.e. $\widehat{\boldsymbol{O}}^r = \boldsymbol{O}^r + \mathrm{s}\left(\int_{C_r} \boldsymbol{Z}^r \boldsymbol{L}_r\right)$. The total master action

$$\boldsymbol{S}_{\mathrm{tot}} := \boldsymbol{S}_{\mathrm{bulk}} + \sum_r \mu_r \widehat{\boldsymbol{O}}^r \tag{2.105}$$

then obeys the classical master equation. As for boundary conditions [24], the undeformed ones (2.96) (imposed off shell in order to have a globally-defined bulk action) are compatible with those following the variational principle provided that the off-shell extensions are super-field extensions $\boldsymbol{V}^r = \mathscr{V}^r(\boldsymbol{X}, \mathrm{d}\boldsymbol{X})$ of topological vertex operators as defined in (2.64), i.e.

$$\boldsymbol{S}_{\mathrm{tot}}[\boldsymbol{X}, \boldsymbol{P}; \mu_i | B; C_i] := \boldsymbol{S}_{\mathrm{bulk}}[\boldsymbol{X}, \boldsymbol{P} | B] + \sum_r \mu_r \int_{C_r} \mathscr{V}^r(\boldsymbol{X}, \mathrm{d}\boldsymbol{X}) \,, \tag{2.106}$$

where $C_r \subseteq \partial B$.

3 Vasiliev's theory: a graded-associative non-commutative case

In this section we begin by reviewing selected features of the action principle for Vasiliev's theory in four dimensions given in [1]. We then construct a minimal classical BV master action using a natural generalization of the AZSZ formalism to the case of graded associative differential algebras. In addition, we shall refine the analysis of [1] concerning compatibility conditions for globally-defined formulations of fiber-bundle type at the level of classical action as well as classical BV master actions.

Before turning to the details, we wish to emphasize that while the BV anti-bracket generalizes straightforwardly to the non-commutative case, the corresponding generalization

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[9]At the level of locally-defined densities, one has that $H^{\langle g \rangle}(s) \cong H^{\langle g \rangle}(\gamma, H(\delta))$ where γ generates the classical gauge symmetries and δ is the Koszul-Tate differential implementing the equations of motion [54]. The construction of $H^{\langle 0 \rangle}(s)$ then passes via globally-defined formulations distinguishing between manifest off-shell gauge symmetries and non-manifest Cartan gauge symmetries on shell [1, 51].

of the BV Laplacian requires the introduction of distributive two-point functions (delta functions) on non-commutative manifold, that we defer to a future work together with the analysis of the quantum BV master equation. It is natural, however, to expect that the classical BV master action principle presented here also solves the quantum master equation.

3.1 Classical theory

Correspondence space. Vasiliev's formulation of higher-spin gravities is in terms of associative differential algebras on non-commutative *correspondence spaces* $\mathfrak{C} \cong T^*\mathfrak{M}$ introducing the following basic notions:

- the differential algebra $\Omega(\mathfrak{C})$ with differential d and compatible graded-associative product $\star$, i.e. if $f, g \in \Omega(\mathfrak{C})$ then $\mathrm{d}(f \star g) = (\mathrm{d}f) \star g + (-1)^{\deg(f)} f \star (\mathrm{d}g)$. These two operations are assumed to be real in the sense that there exists an anti-linear anti-automorphism $\dagger$ obeying $(f \star g)^{\dagger} = (g^{\dagger}) \star (f^{\dagger})$ and $(\mathrm{d}f)^{\dagger} = \mathrm{d}(f^{\dagger})$ for all $f, g \in \Omega(\mathfrak{C})$;

- a graded cyclic trace operation $\mathrm{Tr}: \Omega(\mathfrak{C}) \to \mathbb{C}$ obeying $\mathrm{Tr}[\mathrm{d}(\cdot)] \equiv 0$ (modulo boundary terms), given essentially by the integral over $\mathfrak{C}$, that defines a non-degenerate bi-linear form compatible with d and $\star$;

- a subalgebra consisting of d-closed central elements J^r, i.e. $\mathrm{d}J^r = 0$ and $J^r \star f = f \star J^r$ for all $f \in \Omega(\mathfrak{C})$;

In the case of four-dimensional bosonic higher-spin gravities, including the minimal bosonic models, the differential forms take their values in the algebra $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by two outer Kleinians $(k, \bar{k})$ obeying

$$k \star k = 1, \qquad [k, \mathrm{d}]_{\star} = 0, \qquad k^{\dagger} = \bar{k}. \tag{3.1}$$

The subalgebra of d-closed central elements is generated by various projections of the symplectic form on $\mathfrak{C}$ together with the elements

$$(J^I_{[2]})_{I=1,2} = -\tfrac{i}{4}\left(1, k\kappa\right) \star P_+ \star d^2z,$$
$$(J^{\bar{I}}_{[2]})_{\bar{I}=\bar{1},\bar{2}} = -\tfrac{i}{4}\left(1, \bar{k}\bar{\kappa}\right) \star P_+ \star d^2\bar{z},$$
$$P_+ = \tfrac{1}{2}\left(1 + k\bar{k}\right), \tag{3.2}$$

where the two inner Kleinians

$$\kappa := (2\pi)^2 \delta^2(y) \star \delta^2(z), \qquad \bar{\kappa} := (\kappa)^{\dagger} = (2\pi)^2 \delta^2(\bar{y}) \star \delta^2(\bar{z}), \tag{3.3}$$

using Weyl-ordered symbols, and $(y^{\alpha}, \bar{y}^{\dot{\alpha}}; z^{\alpha}, \bar{z}^{\dot{\alpha}})$ (with $\alpha, \dot{\alpha} = 1, 2$) are local coordinates on the doubled twistor space $\mathfrak{Z}_{\xi} \times \mathfrak{Y}_{\xi} \subset \mathfrak{C}$ obeying

$$\{k, y^{\alpha}\}_{\star} = \{k, z^{\alpha}\}_{\star} = 0 = [k, \bar{y}^{\dot{\alpha}}]_{\star} = [k, \bar{z}^{\dot{\alpha}}]_{\star}, \tag{3.4}$$

$$[y^{\alpha}, y^{\beta}]_{\star} = 2\mathrm{i}\epsilon^{\alpha\beta}, \qquad [z^{\alpha}, z^{\beta}]_{\star} = -2\mathrm{i}\epsilon^{\alpha\beta}, \tag{3.5}$$

and the reality conditions

$$(y^{\alpha})^{\dagger} = \bar{y}^{\dot{\alpha}}, \qquad (z^{\alpha})^{\dagger} = \bar{z}^{\dot{\alpha}}. \tag{3.6}$$

The inner Kleinian obey $\kappa \star \kappa = 1$ and

$$\kappa \star f \star \kappa = (-1)^{\deg_{\mathfrak{y}\times\mathfrak{z}}(f)}\pi(f)\,, \qquad \pi(f) := k \star f \star k\,, \tag{3.7}$$

where $\deg_{\mathfrak{y}\times\mathfrak{z}}(f)$ denotes the holomorphic form degree of f, *idem* $\bar{\kappa}$ and $\bar{\pi}(f) := \bar{k} \star f \star \bar{k}$. The full correspondence space is thus of the form

$$\mathfrak{C} = \bigcup_{\xi} \mathfrak{C}_{\xi}\,, \qquad \mathfrak{C}_{\xi} = T^*\mathfrak{M}_{\xi} \times \mathfrak{Z}_{\xi} \times \mathfrak{Y}_{\xi}\,, \tag{3.8}$$

where $T^*\mathfrak{M}_{\xi}$ has real canonical coordinates (X^M, P_M) obeying

$$\left[X^M, P_M\right]_{\star} = i\delta^M_N\,. \tag{3.9}$$

Requiring $(X^M, P_M; y^\alpha, \bar{y}^{\dot\alpha}; z^\alpha, \bar{z}^{\dot\alpha})$ to commute with the line-elements $(dX^M, dP_M; dy^\alpha, d\bar{y}^{\dot\alpha}; dz^\alpha, d\bar{z}^{\dot\alpha})$ it follows that the latter generate a graded commutative algebra.

Chiral trace operation. The basic chiral trace operation is defined by

$$\mathrm{Tr}[f] := \sum_{\xi} \int_{T^*\mathfrak{M}_{\xi}\times\mathfrak{Y}\times\mathfrak{Z}} f|_{k=0=\bar{k}}\,, \tag{3.10}$$

where the integral projects onto the top form degree; the integrand should be understood as the symbol of f in a suitable order;[10] and the twistor variables are integrated along independent real contours. This trace operation is graded cyclic, i.e.

$$\mathrm{Tr}[f \star g] = (-1)^{\deg(f)\deg(g)}\mathrm{Tr}[g \star f]\,. \tag{3.11}$$

Various other graded-cyclic trace operations can be obtained by projecting Tr. Inserting

$$P_\pm = \frac{1}{2}(1 \pm k\bar{k})\,, \tag{3.12}$$

yields a trace that is graded cyclic and non-degenerate on the bosonic subalgebras

$$\mathscr{A}_\pm := \{f \in \Omega(\mathfrak{C}) \times \mathbb{Z}_2 \times \mathbb{Z}_2 : \ f = \pi\bar{\pi}(f) = P_+ \star f\}\,. \tag{3.13}$$

Inserting $\Omega_{\mathfrak{Y}} := \frac{d^2y\, d^2\bar{y}}{(2\pi)^2}$ yields a trace that is non-degenerate on $\Omega(T^*\mathfrak{M} \times \mathfrak{Z}) \otimes \Omega^{[0]}(\mathfrak{Y})$. Inserting also $\Pi_{\mathfrak{M}} := \frac{1}{n!}\epsilon^{M_1\cdots M_n}dP_{M_1}\cdots dP_{M_n}\delta^n(P_N)$, defined using Weyl order, we obtain a reduced trace operation that remains non-degenerate on $\Omega(\mathfrak{M} \times \mathfrak{Z}) \otimes \Omega^{[0]}(\mathfrak{Y})$, *viz.*

$$\check{\mathrm{Tr}}[f] := \mathrm{Tr}[\Pi_{\mathfrak{M}} \star \Omega_{\mathfrak{Y}} \star f] \equiv \sum_{\xi} \int_{\mathfrak{M}_{\xi}} \mathrm{Tr}'[f]\,, \tag{3.14}$$

where the twistor-space trace operation

$$\mathrm{Tr}'[f] := \int_{\mathfrak{Y}\times\mathfrak{Z}} \left[\Omega_{\mathfrak{Y}} \star f|_{k=\bar{k}=0\,;\,dP_M=0\,;\,P_M=0}\right]\,. \tag{3.15}$$

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[10]If the trace is well-defined, it does not depend on the choice of order.

The reduced trace remains graded cyclic, i.e.

$$\check{\mathrm{Tr}}[f \star g] = (-1)^{\deg(f)\deg(g)}\check{\mathrm{Tr}}[g \star f] \, . \tag{3.16}$$

In order to make contact with the previous section, one thus treats

$$\mathfrak{M} \times \mathfrak{Z} \equiv B \, , \tag{3.17}$$

as the bulk manifold, hence

$$\hat{p} = \dim(\mathfrak{M}) + 3 \, , \tag{3.18}$$

and $\mathfrak{Y}$ as a fiber manifold, i.e. all quantities are expanded in sets $\{T_\lambda(y^\alpha, \bar{y}^{\dot\alpha})\}$ of functions on $\mathfrak{Y}$ treated as types forming a basis for an associative $\star$- product algebra with coefficients in $\Omega(B)$ that remains closed under $\star$-product composition with κ and $\bar{\kappa}$; for a concrete example of this separation of variables, see [50]. The choice of types is adapted to the boundary conditions on B and may hence manifest various symmetry algebras, such as generalized Lorentz algebras or compact algebras, leading to the notion of (inverse) harmonic expansions [55, 56]. In what follows, for the purpose of setting up the AKSZ formalism, it suffices, however, to treat the $\mathfrak{Y}$-dependence formally.

Classical action: odd-dimensional bulk. If $\dim(\mathfrak{M}) = 2n + 1$ with $n \geqslant 0$, that is $\hat{p} = 2n + 4$, a duality extension of Vasiliev's equations of motion for four-dimensional higher-spin gravities, which is locally equivalent to Vasiliev's original equations, follows from the variational principle based on the generalized Hamiltonian bulk action

$$S_{\mathrm{bulk}}^{\mathrm{cl}}[\{A, B, U, V\}_\xi] = \sum_\xi \int_{\mathfrak{M}_\xi} \mathrm{Tr}' \left[U \star DB + V \star \left(F + \mathscr{G}(B, U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \right) \right]_\xi \, , \tag{3.19}$$

where the locally-defined master fields have decompositions under total form degree into

$$A = A_{[1]} + A_{[3]} + \cdots + A_{[2n+3]} \, , \qquad B = B_{[0]} + B_{[2]} + \cdots + B_{[2n+2]} \, , \tag{3.20}$$

$$U = U_{[2]} + U_{[4]} + \cdots + U_{[2n+4]} \, , \qquad V = V_{[1]} + V_{[3]} + \cdots + V_{[2n+3]} \, . \tag{3.21}$$

The interaction freedom in $\mathscr{G}$ needs to be constrained in order for the action to be gauge invariant. Making the ansatz[11]

$$\mathscr{G} = \mathscr{F}(B; J^I, J^{\bar{I}}, J^{I\bar{I}}) + \widetilde{\mathscr{F}}(U; J^I, J^{\bar{I}}, J^{I\bar{I}}) \, , \tag{3.22}$$

$$\mathscr{F} = \mathscr{F}_0(B) + \mathscr{F}_I(B) \star J_{[2]}^I + \mathscr{F}_{\bar{I}}(B) \star J_{[2]}^{\bar{I}} + \mathscr{F}_{I\bar{I}}(B) \star J_{[4]}^{I\bar{I}} \, , \tag{3.23}$$

$$\widetilde{\mathscr{F}} = \widetilde{\mathscr{F}}_0(U) + \widetilde{\mathscr{F}}_I(U) \star J_{[2]}^I + \widetilde{\mathscr{F}}_{\bar{I}}(U) \star J_{[2]}^{\bar{I}} + \widetilde{\mathscr{F}}_{I\bar{I}}(U) \star J_{[4]}^{I\bar{I}} \, , \tag{3.24}$$

the following two cases yields integrable equations of motion:

$$\text{bilinear } Q\text{-structure} : \mathscr{F} = B \star J \, , \qquad J = J_{[2]} + J_{[4]} \, , \tag{3.25}$$

$$\text{bilinear } P\text{-structure} : \widetilde{\mathscr{F}} = U \star J' \, , \qquad J' = J'_{[2]} + J'_{[4]} \, , \tag{3.26}$$

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[11]The coupling $\tilde{f} := \partial_U \widetilde{\mathscr{F}}_0|_{U=0}$ determines whether the target space is a symplectic manifold ($\tilde{f} \neq 0$) or a proper Poisson manifold ($\tilde{f} = 0$). In the symplectic case the U and V variables can be integrated out after which the action becomes a boundary term while this is no longer possible in the proper Poisson case.

where the central elements are defined via

$$B \star J_{[2]} \equiv \mathscr{F}_I \star J_{[2]}^I + \mathscr{F}_{\bar{I}} \star J_{[2]}^{\bar{I}} \,, \qquad B \star J_{[4]} \equiv \mathscr{F}_{I\bar{I}} \star J_{[4]}^{I\bar{I}} \,, \tag{3.27}$$

$$J_{[2]} = -\tfrac{i}{4} \left[\mathrm{d}z^2 (b_1 + b_2 \, k \, \kappa) + \mathrm{d}\bar{z}^2 (b_{\bar{1}} + b_{\bar{2}} \, \bar{k} \, \bar{\kappa}) \right] \star P_+ \,, \tag{3.28}$$

$$J_{[4]} = -\tfrac{i}{4} \, \mathrm{d}z^2 \mathrm{d}\bar{z}^2 \left[c_{1\bar{1}} + c_{2\bar{1}} \, k \, \kappa + c_{1\bar{2}} \, \bar{k} \, \bar{\kappa} + c_{2\bar{2}} \, \kappa \, \bar{\kappa} \right] \star P_+ \,. \tag{3.29}$$

Indeed, letting $Z^i = (A, B, U, V)$, the general variation of the action reads

$$\delta S_{\mathrm{bulk}}^{\mathrm{cl}} = \sum_{\xi} \int_{\mathfrak{M}_\xi} \mathrm{Tr}' \left[\delta Z^i \star \mathscr{R}^j \tilde{\mathscr{O}}_{ij} \right] + \sum_{\xi} \int_{\partial \mathfrak{M}_\xi} \mathrm{Tr}' \left[U \star \delta B - V \star \delta A \right] \,, \tag{3.30}$$

where $\mathscr{O}_{ij}$ is a constant non-degenerate matrix defining the symplectic form of degree $\hat{p}+2$ on the target space and the generalized curvatures

$$\mathscr{R}^A = F + \mathscr{F} + \widetilde{\mathscr{F}} \,, \qquad\qquad \mathscr{R}^B = DB + (V \partial_U) \star \widetilde{\mathscr{F}} \,, \tag{3.31}$$

$$\mathscr{R}^U = DU - (V \partial_B) \star \mathscr{F} \,, \qquad\qquad \mathscr{R}^V = DV + [B, U]_\star \,, \tag{3.32}$$

and the bulk equations of motion $\mathscr{R}^i \approx 0$ are Cartan integrable for the above choices of $\mathscr{F}$ and $\widetilde{\mathscr{F}}$. As shown in [1], the on-shell Cartan gauge transformations

$$\delta_{\epsilon,\eta} A = D\epsilon^A - (\epsilon^B \partial_B) \star \mathscr{F} - (\eta^U \partial_U) \star \widetilde{\mathscr{F}} \,, \tag{3.33}$$

$$\delta_{\epsilon,\eta} B = D\epsilon^B - [\epsilon^A, B]_\star - (\eta^V \partial_U) \star \widetilde{\mathscr{F}} - (\eta^U \partial_U) \star (V \partial_U) \star \widetilde{\mathscr{F}} \,, \tag{3.34}$$

$$\delta_{\epsilon,\eta} U = D\eta^U - [\epsilon^A, U]_\star + (\eta^V \partial_B) \star \mathscr{F} + (\epsilon^B \partial_B) \star (V \partial_B) \star \mathscr{F} \,, \tag{3.35}$$

$$\delta_{\epsilon,\eta} V = D\eta^V - [\epsilon^A, V]_\star - [\epsilon^B, U]_\star + [\eta^U, B]_\star \,, \tag{3.36}$$

remain symmetries off shell modulo boundary terms, *viz.*

$$\delta_{\epsilon,\eta} S_{\mathrm{bulk}}^{\mathrm{cl}}[A, B, U, V] = \sum_{\xi} \int_{\partial \mathfrak{M}_\xi} K_\eta \,, \tag{3.37}$$

where

$$K_\eta = \mathrm{Tr}' \left[\eta^U \star DB + \eta^V \star (F + \mathscr{F} + (1 - U \partial_U) \star \widetilde{\mathscr{F}}) \right] \,. \tag{3.38}$$

As found in [1], the closure formula for Cartan gauge transformations generalized straightforwardly from the commutative to the non-commutative case, i.e.

$$[\delta_\varepsilon, \delta_\varepsilon] Z^i = \delta_{\varepsilon_{12}} Z^i - \overrightarrow{\mathscr{R}} \star \varepsilon_{12}^i \,, \tag{3.39}$$

with composite parameters

$$\varepsilon_{12}^i = -\overrightarrow{\varepsilon}_1 \star \overrightarrow{\varepsilon}_2 \star \mathscr{Q}^i \,, \tag{3.40}$$

which can be used to construct globally-defined bulk actions within the context of fiber bundles. Thus, the contributions to $\delta_{\epsilon,\eta} S_{\mathrm{bulk}}^{\mathrm{cl}}$ from the chart boundaries in the interior of

$\mathfrak{M}$ can be made to cancel by gluing together the locally-defined field configurations and broken η-gauge parameters using gauge transitions $\hat{\delta}_t Z^i = \delta_t Z^i$ and

$$\hat{\delta}_t \eta^U = -[t^A, \eta^U] - (t^B \partial_B) \star (\eta^V \partial_B) \mathscr{F} , \tag{3.41}$$

$$\hat{\delta}_t \eta^V = -[t^A, \eta^V] + \{\eta^U, t^B\} , \tag{3.42}$$

i.e.

$$\hat{\delta}_t K_\eta = 0 , \tag{3.43}$$

where, moreover, the compatibility conditions on $\{t^A, t^B\}$ read as follows:

$$\overrightarrow{\mathscr{R}} \star [\overrightarrow{t}, \overrightarrow{\epsilon}]_\star \star \mathscr{Q}^i = 0 \qquad \text{for all } i, \overrightarrow{\mathscr{R}} \text{ and } \overrightarrow{\epsilon} . \tag{3.44}$$

The conditions on t^A hold for all $\mathscr{F}$ while those for t^B hold only if $\mathscr{F}$ is at most bi-linear. Thus, if $\mathscr{F}$ is at least tri-linear then t^B-transitions must be discarded.

Classical action: even-dimensional bulk. If $\dim(\mathfrak{M}) = 2n$ with $n \geqslant 0$, that is $\hat{p} = 2n - 1$, the duality-extended equations of motion follow from the variational principle based on the generalized Hamiltonian bulk action

$$S_{\text{bulk}}^{\text{cl}}[A, B; S, T] = \sum_\xi \int_{\mathfrak{M}_\xi} \text{Tr}' \left[S \star DB + T \star (F + \mathscr{F} + \widetilde{\mathscr{F}}(S; J^I, J^{\bar{I}}, J^{I\bar{J}})) \right]_\xi , \tag{3.45}$$

where the interaction function obeys

$$\widetilde{\mathscr{F}}(-S) = \widetilde{\mathscr{F}}(S) , \qquad \widetilde{\mathscr{F}}|_{S=0} = 0 , \tag{3.46}$$

and the fields are assigned form degrees as follows:

$$A = A_{[1]} + A_{[3]} + \cdots + A_{[2n-1]} , \qquad B = B_{[0]} + B_{[2]} + \cdots + B_{[2n-2]} , \tag{3.47}$$

$$S = S_{[1]} + S_{[3]} + \cdots + S_{[2n-1]} , \qquad V = T_{[1]} + T_{[2]} + \cdots + T_{[2n-2]} . \tag{3.48}$$

From the variation one obtains

$$\mathscr{R}^A = F + \mathscr{F} + \widetilde{\mathscr{F}}(S) , \qquad \mathscr{R}^B = DB - (T\partial_S) \star \widetilde{\mathscr{F}}(S) , \tag{3.49}$$

$$\mathscr{R}^S = DS + (T\partial_B) \star \mathscr{F} , \qquad \mathscr{R}^T = DT + [S, B]_\star . \tag{3.50}$$

and the integrability of the equation of motion $D\mathscr{R}^I - (\mathscr{R}^J \partial_J) \star Z^I \equiv 0$ requires

$$D\mathscr{R}^A - (\mathscr{R}^B \partial_B) \star \mathscr{F} - (\mathscr{R}^S \partial_S) \star \widetilde{\mathscr{F}}$$
$$= ((T\partial_S) \star \widetilde{\mathscr{F}} \partial_B) \star \mathscr{F} - ((T\partial_B) \star \mathscr{F} \partial_S) \star \widetilde{\mathscr{F}} \equiv 0 , \tag{3.51}$$

$$D\mathscr{R}^B - [\mathscr{R}^A, B] + (\mathscr{R}^T \partial_T) \star \widetilde{\mathscr{F}} + (\mathscr{R}^S \partial_S) \star ((T\partial_S) \star \widetilde{\mathscr{F}})$$
$$= ((T\partial_B) \star \mathscr{F} \partial_S) \star ((T\partial_S) \star \widetilde{\mathscr{F}}) \equiv 0 , \tag{3.52}$$

$$D\mathscr{R}^S - [\mathscr{R}^A, S] - (\mathscr{R}^T \partial_B) \star \mathscr{F} - (\mathscr{R}^B \partial_B) \star ((T\partial_B) \star \mathscr{F})$$
$$= ((T\partial_S) \star \widetilde{\mathscr{F}} \partial_B) \star ((T\partial_B) \star \mathscr{F}) \equiv 0 , \tag{3.53}$$

whereas

$$DR^T - [\mathcal{R}^A, T] - [\mathcal{R}^S, B] + \{\mathcal{R}^B, S\} \equiv 0 \,, \tag{3.54}$$

as follows from the even functions $\widetilde{\mathcal{F}}$ obey

$$\{S, (T\partial_S) \star \widetilde{\mathcal{F}}\}_\star \equiv [T, \widetilde{\mathcal{F}}]_\star \,. \tag{3.55}$$

The remaining conditions are satisfied in two cases:

$$\mathcal{F} = B \star f(J^I, J^{\bar{I}}, J^{I\bar{J}}) \,, \quad \widetilde{\mathcal{F}} = \sum_n S^{\star 2n} \star w_n(J^I, J^{\bar{I}}, J^{I\bar{J}}) \tag{3.56}$$

or

$$\mathcal{F} = \sum_n B^{\star n} \star f_n(J^I, J^{\bar{I}}, J^{I\bar{J}}) \,, \quad \widetilde{\mathcal{F}} = 0. \tag{3.57}$$

where f_n, w_n are arbitrary functions of the central terms $J^I, J^{\bar{I}}, J^{I\bar{J}}$. This choice makes the action invariant under the gauge transformations

$$\delta_{\epsilon,\eta} A = D\epsilon^A - (\epsilon^B \partial_B) \star \mathcal{F} - (\eta^S \partial_S) \star \widetilde{\mathcal{F}} \,, \tag{3.58}$$

$$\delta_{\epsilon,\eta} B = D\epsilon^B - [\epsilon^A, B]_\star + (\eta^T \partial_S) \star \widetilde{\mathcal{F}} + (\eta^S \partial_S) \star (T\partial_S) \star \widetilde{\mathcal{F}} \,, \tag{3.59}$$

$$\delta_{\epsilon,\eta} S = D\eta^S - [\epsilon^A, S]_\star + (\eta^T \partial_B) \star \mathcal{F} + (\epsilon^B \partial_B) \star (T\partial_B) \star \mathcal{F} \,, \tag{3.60}$$

$$\delta_{\epsilon,\eta} T = D\eta^T - [\epsilon^A, T]_\star + \{\epsilon^B, S\}_\star - [\eta^S, B]_\star \,, \tag{3.61}$$

up to the boundary terms

$$\delta_{\epsilon,\eta} S^{\mathrm{cl}}_{\mathrm{bulk}} = \sum_\xi \int_{\partial\mathfrak{M}_\xi} \mathrm{Tr} \left[\eta^S \star DB + \eta^T \star (F + \mathcal{F} + (1 - S\partial_S) \star \widetilde{\mathcal{F}}) \right] \,. \tag{3.62}$$

The construction of a globally-defined action and the required compatibility conditions are analogous to the case of even $\hat{p}$ using

$$\hat{\delta}_t \eta^S = -[t^A, \eta^S]_\star + (t^B \partial_B) \star (\eta^T \partial_B) \star \mathcal{F} \,, \tag{3.63}$$

$$\hat{\delta}_t \eta^T = -[t^A, \eta^T]_\star - [t^B, \eta^S]_\star \,. \tag{3.64}$$

3.2 AKSZ master action

The bulk action. In this section we give the minimal solutions $\boldsymbol{S}$ of the classical master equation corresponding to the classical action principles given in the previous sections.

The classical fields Z^i become coordinates $\boldsymbol{Z}^i$ of a supermanifold and contain all the ghosts and antifields of the BRST-BV spectrum similarly to what is explained below (2.79) and (2.80). The AKSZ master actions are obtained by taking the classical bulk actions (3.19) and (3.45); replacing Z^i by $\boldsymbol{Z}^i$ therein; integrating as in (2.78) so as to select only the top form component of the resulting Lagrangian density; and projecting onto ghost number zero, *viz.*

$$\boldsymbol{S} = S^{\mathrm{cl}}_{\mathrm{bulk}}[\boldsymbol{Z}^i]\Big|^{\langle 0 \rangle} \equiv \sum_\xi \int_{B_\xi} \mathrm{Tr}' \boldsymbol{L}_\xi \Big|^{\langle 0 \rangle} \,, \tag{3.65}$$

that is

$$\text{Even } \hat{p}: \quad \boldsymbol{L} = \boldsymbol{U} \star \boldsymbol{DB} + \boldsymbol{V} \star \left(\boldsymbol{F} + \mathscr{F}(\boldsymbol{B}; J^r) + \widetilde{\mathscr{F}}(\boldsymbol{U}; J^r) \right), \tag{3.66}$$

$$\text{Odd } \hat{p}: \quad \boldsymbol{L} = \boldsymbol{S} \star \boldsymbol{DB} + \boldsymbol{T} \star \left(\boldsymbol{F} + \mathscr{F}(\boldsymbol{B}; J^r) + \widetilde{\mathscr{F}}(\boldsymbol{S}; J^r) \right). \tag{3.67}$$

As for the BV bracket $(\cdot, \cdot)$ in non-commutative space, it is defined analogously to (2.83) and defines a derivation in the sense that for any local star-functional F and ultra local star-functionals $A(p)$ and $B(p)$, evaluated on $p \in \mathfrak{C}$, it satisfies

$$(F, A(p) \star B(p)) = (F, A(p)) \star B(p) + (-1)^{A(F+1)} A(p) \star (F, B(p)). \tag{3.68}$$

Thus, similarly to the commutative case, we have the following basic BV brackets:

$$(\boldsymbol{S}, \boldsymbol{Z}^i) = \boldsymbol{R}^i, \qquad \boldsymbol{R}^i = \mathrm{d}\boldsymbol{Z}^i + \boldsymbol{Q}^i, \tag{3.69}$$

and where $\boldsymbol{Q}^i = \mathscr{Q}^i(\boldsymbol{Z}^i)$.

It is then a direct computation to verify that the master equation $(\boldsymbol{S}, \boldsymbol{S}) = 0$ is satisfied up to boundary terms as follows:

$$\underline{\text{Even } \hat{p}}: (\boldsymbol{S}, \boldsymbol{S}) = -\oint_{\partial B} \mathrm{Tr}' \left[\boldsymbol{U} \star \boldsymbol{DB} + \boldsymbol{V} \star (\boldsymbol{F} + \mathscr{F}(\boldsymbol{B}; J)) + \boldsymbol{V} \star (1 - \boldsymbol{U}\partial_{\boldsymbol{U}}) \star \widetilde{\mathscr{F}}(\boldsymbol{U}; J) \right], \tag{3.70}$$

$$\underline{\text{Odd } \hat{p}}: (\boldsymbol{S}, \boldsymbol{S}) = \oint_{\partial B} \mathrm{Tr}' \left[\boldsymbol{S} \star \boldsymbol{DB} + \boldsymbol{T} \star (\boldsymbol{F} + \mathscr{F}(\boldsymbol{B}; J)) + \boldsymbol{T} \star (1 - \boldsymbol{S}\partial_{\boldsymbol{S}}) \star \widetilde{\mathscr{F}}(\boldsymbol{S}; J) \right] \tag{3.71}$$

which one indeed identifies as the non-commutative generalization of (2.97), i.e.

$$(\boldsymbol{S}, \boldsymbol{S}) = (-1)^{\hat{p}} \sum_{\xi} \oint_{\partial B_\xi} \mathrm{Tr}' \left[(\boldsymbol{R}^\alpha \star \boldsymbol{P}_\alpha - 2\boldsymbol{L} \right]_\xi, \tag{3.72}$$

which vanishes upon imposing

$$\boldsymbol{P_\alpha}|_{\partial B} = 0, \tag{3.73}$$

and using gauge transitions between charts, acting as follows:

$$\delta_{\boldsymbol{t}} \boldsymbol{A} = \boldsymbol{D}\boldsymbol{t}^A - (\boldsymbol{t}^B \partial_B) \star \mathscr{F}, \tag{3.74}$$

$$\delta_{\boldsymbol{t}} \boldsymbol{B} = \boldsymbol{D}\boldsymbol{t}^B - [\boldsymbol{t}^A, B]_\star, \tag{3.75}$$

$$\delta_{\boldsymbol{t}} \boldsymbol{U} = -[\boldsymbol{t}^A, \boldsymbol{U}]_\star + (\boldsymbol{t}^B \partial_B) \star (\boldsymbol{V}\partial_B) \star \mathscr{F}, \tag{3.76}$$

$$\delta_{\boldsymbol{t}} \boldsymbol{V} = -[\boldsymbol{t}^A, \boldsymbol{V}]_\star - [\boldsymbol{t}^B, \boldsymbol{U}]_\star, \tag{3.77}$$

$$\delta_{\boldsymbol{t}} \boldsymbol{S} = -[\boldsymbol{t}^A, \boldsymbol{S}]_\star + (\boldsymbol{t}^B \partial_B) \star (\boldsymbol{T}\partial_B) \star \mathscr{F}, \tag{3.78}$$

$$\delta_{\boldsymbol{t}} \boldsymbol{T} = -[\boldsymbol{t}^A, \boldsymbol{T}]_\star + \{\boldsymbol{t}^B, \boldsymbol{S}\}_\star, \tag{3.79}$$

with parameters $(\boldsymbol{t}^A, \boldsymbol{t}^B)$ obeying the super-field extension of (3.44).

Some boundary deformations. An example of a set of boundary deformations of minimal bosonic models [51] (for the corresponding projection of the off-shell system, see [1]) is given by topological vertex operators of the form [51][12]

$$
\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]} = \mathrm{Tr}' \left[d^4 Z \kappa \star \left(\prod_{i=1}^{n} \left(R \star E^{2m_i} \right) - \frac{(-1)^n m}{(m+n)} E^{\star 2(m+n)} \right) \right] , \tag{3.80}
$$

where $\overrightarrow{m} = (m_1, \ldots, m_n) \equiv (m_2, \ldots, m_n, m_1)$ with $m_i \geqslant 0$ and $\sum_{i=1}^{n} m_i = m$ denotes a cyclic order, and

$$
E := \tfrac{1}{2}(1-\pi)A_{[1]}|_{\mathfrak{M}}, \qquad R := d\Gamma + \Gamma \star \Gamma, \qquad \Gamma = \tfrac{1}{2}(1+\pi)A_{[1]}|_{\mathfrak{M}}, \tag{3.81}
$$

obeying

$$
\nabla E \approx 0, \qquad R + E \star E \approx 0, \tag{3.82}
$$

with $\nabla = (\mathrm{d} + \mathrm{ad}_\Gamma)|_{\mathfrak{M}}$. It follows that $\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}$ obeys (2.64) (with total derivatives on $\mathfrak{M}$) and that

$$
\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]} \approx \mathscr{J}^{\overrightarrow{m}}_{[2(m+n)]} := \frac{(-1)^n n}{(m+n)} \mathrm{Tr}' \left[d^4 Z \kappa \star E^{\star 2(m+n)} \right] , \tag{3.83}
$$

which is indeed a non-trivial element of the on-shell de Rham cohomology on $\mathfrak{M}$ (and hence $\partial \mathfrak{M}$) assuming a globally-defined formulation of fiber-bundle type with structure group containing π-even but not π-odd gauge parameters in form degree zero. In other words, the insertion of $\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}$ at $\partial \mathfrak{M}$ deforms the unbroken phase into a broken phase with smaller structure group and hence additional observables; the broken gauge symmetries instead resurface on shell with π-odd parameters $\xi := \tfrac{1}{2}(1-\pi)\epsilon^A_{[0]}$ forming a section together with the soldering one-form E on a fiber bundle with π-even structure group in degree zero. Indeed, under the gauge transformations δ_ξ, the variation $\delta_\xi \mathscr{J}^{\overrightarrow{m}}_{[2(m+n)]}$ consists of total derivatives that cancel across chart boundaries provided (ξ, E) forms a section. Clearly, the on-shell values of $\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}$ are non-trivial only on submanifolds of $\partial \mathfrak{M}$ where E is non-degenerate, which is also where the parameter ξ can be converted into diffeomorphisms. In other words, perturbing the action by $\int_C \mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}$ on $2(m+2)$-cycles $C \subseteq \partial \mathfrak{M}$, and imposing non-trivial on-shell values for $\int_C \mathscr{J}^{\overrightarrow{m}}_{[2(m+n)]}$ leads to a metric phase on C.

Turning to the total AKSZ master action, it is straightforward to check using the BRST transformations

$$
s\boldsymbol{E} = \boldsymbol{D}\boldsymbol{E}, \qquad s\boldsymbol{\Gamma} = \boldsymbol{R} + \boldsymbol{E} \star \boldsymbol{E} \tag{3.84}
$$

that the BRST transformations of each one of the two terms making up $\boldsymbol{V}^{\overrightarrow{m}}_{[2(m+n)]} := \mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}(\boldsymbol{Z}, \mathrm{d}\boldsymbol{Z})$ transforms into a total derivative such that

$$
s\boldsymbol{V}^{\overrightarrow{m}}_{[2(m+n)]} = \mathrm{d}\boldsymbol{W}^{\overrightarrow{m}}_{[2(m+n)]}, \tag{3.85}
$$

independently of the relative coefficient in $\mathscr{V}^{\overrightarrow{m}}_{[2(m+n)]}$, which is instead fixed by demanding the super-field analog of (2.64).

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[12]For manifestly Lorentz-covariant vertex operators, see [51].

4 Conclusions

In this paper we have taken the first steps of the BV-AKSZ quantization of four-dimensional higher-spin gravity based on the classical action proposed in [1] by constructing the corresponding minimal AKSZ master action obeying the classical BV master equation. We have also given the details of the global formulation within the framework of fiber bundles, which was described only briefly given in [1].

Besides the gauge-fixing procedure, which may require non-minimal sectors containing ghost-momenta and Nakanishi-Laudrup auxiliary fields, there are several lines of developments that present themselves at the present stage of which some are:

(i) the classification of the bulk Hamiltonians consistent with Vasiliev's theory on the boundary and corresponding globally-defined formulations, that may extend beyond the realm of fiber bundles;

(ii) the classification of possible deformations of the bulk action, which in general depend on the choice of global formulation in (i);

(iii) to forgo the associativity of the star-product on the correspondence by considering more general homotopy-associative differential algebras.

Finally, it remains an open problem whether contact can be made with the perturbative Fronsdal program. The natural procedure is to add a deformation four-form within a suitable global formulation to be identified as the generating function of holographic correlation functions, possibly in accordance with the various observations and conjectures made in [57–60]. In the latter respect, the four-form proposed by [51], that is, the quantity $\mathscr{V}^{(2)}_{[4]}$ given in eq. (3.80) (for $m = n = 1$), which depends only on zero-forms and one-forms on $\partial\mathfrak{M}$, is an interesting candidate: Assuming that $\hat{p} = 8$ so that $\dim(\mathfrak{M}) = 5$, and that $\partial\mathfrak{M}$ is non-compact with non-trivial external states on $\partial^2\mathfrak{M}$, it follows that $\mathscr{V}^{(2)}_{[4]}$ is non-trivial on-shell (constructed from boundary-to-bulk propagators) and hence a candidate for an on-shell action. Its vertices, on the other hand, cannot be used to close any loops as follows from conservation of form degree on $\mathfrak{M}$ (bulk vertices of the form $\mathrm{Tr}'[J^r \star U^{\star n} \star V]_{\deg_{\mathfrak{M}}=5}$ cannot yield correlation functions on $\partial\mathfrak{M}$ between forms $X^\alpha|_{\partial\mathfrak{M}}$ if all degrees $p_\alpha \leqslant 1$). Hence, it appears treating $\mathscr{V}^{(2)}_{[4]}$ as a deformation four-form may give rise to non-trivial tree diagrams and trivial loop corrections, in accordance with the general pattern expected from free conformal field theories.

On-shell equivalence to Fronsdal approach. Concerning the correspondence with the free $O(N)$ vector model [57] and Gross-Neveu model [61], we make the following observations:

- for any $\mathscr{H}(U, V; B)$ and applying perturbation theory in which $\int_{\mathfrak{M}} \mathrm{Tr}'[dX^\alpha \star P_\alpha]$ is treated as the kinetic term, it follows from the fact that the vertices in $\mathscr{H}(U, V; B)$ are built from exterior (star-) products that boundary correlation functions that involve only zero-forms and one-forms are given by their semi-classical limits (as vacuum

bubbles cancel), *viz.*

$$\langle B_{[0]}(p_1) \cdots B_{[0]}(p_n) A_{[1]}(p_{n+1}) \cdots A_{[1]}(p_{n+m}) \rangle|_{p_i \in \partial \mathfrak{M}}$$
$$= \langle B_{[0]}(p_1) \rangle \cdots \langle B_{[0]}(p_n) \rangle \langle A_{[1]}(p_{n+1}) \rangle \cdots \langle A_{[1]}(p_{n+m}) \rangle ; \qquad (4.1)$$

- assuming the existence of a perturbative completion $\int_{\partial \mathfrak{M}} \mathcal{V}_{\mathrm{FV}}(B_{[0]}, dB_{[0]}; A_{[1]}, dA_{[1]})$ of the Fradkin-Vasiliev action,[13] it can be added as a topological vertex operator and treated as an interaction, including its kinetic terms;

- it follows that the expectation value of the Fradkin-Vasiliev action is tree-level exact, i.e.

$$Z(\mu) := \left\langle \exp\left(\frac{i\mu}{\hbar} \int_{\partial \mathfrak{M}} \mathcal{V}_{\mathrm{FV}}\right) \right\rangle = \exp\left(\frac{i\mu}{\hbar} \int_{\partial \mathfrak{M}} \mathcal{V}_{\mathrm{FV}}\right)\bigg|_{B_{[0]}=\langle B_{[0]} \rangle; A_{[1]}=\langle A_{[1]} \rangle} , \qquad (4.2)$$

with expectation values $\langle B_{[0]} \rangle$ and $\langle A_{[1]} \rangle$ obeying the Vasiliev equations of motion subject to boundary conditions at the three-dimensional conformal boundary $\bar{\partial}\partial\mathfrak{M}$ of $\partial\mathfrak{M}$;

- thus, assuming a suitable topology for $\partial\mathfrak{M}$ and that $\langle B_{[0]} \rangle$ and $\langle A_{[1]} \rangle$ are asymptotic to AdS_4, hence built from the boundary data using boundary-to-bulk propagators, we expect that $Z(\mu)$ with $\mu N = \hbar$ is equal to the generating functional of the free $O(N)$ model in the case of the Type A model with scalar field obeying $\Delta = 1$ boundary conditions, and to the generating functional of the free Gross-Neveu model (with N free fermions) in the case of the Type B model with scalar field obeying $\Delta = 2$ boundary conditions.

We wish to stress the fact that both of the latter higher-spin gravity models are manifestly tree-level unitary: by the very nature of the perturbative treatment of the Poisson sigma models (with kinetic PdX-terms on $\mathfrak{M}$), the partition function $Z(\mu)$ is completely free from loop corrections in the Fradkin-Vasiliev sector, in perfect agreement with free three-dimensional CFTs. In other words, $Z(\mu)$ is given by the sum of tree Witten-diagrams in AdS_4 with external boundary-to-bulk and internal bulk-to-bulk Green's functions arising as the result of solving classical equations of motion subject to boundary sources (and not of performing any Gaussian integrals starting from the Fronsdal kinetic terms in the Fradkin-Vasiliev action).

In the case of the strongly-coupled fixed points of the $O(N)$ vector model [58] and the Gross-Neveu model [61], reached by suitable double-trace deformations, the Fradkin-Vasiliev action needs to be modified with a Gibbons-Hawking term

$$\int_{\bar{\partial}\partial\mathfrak{M}} \mathcal{V}_{\mathrm{GH}} = \int_{\bar{\partial}\partial\mathfrak{M}} \phi \, \partial_n \phi + \cdots , \qquad (4.3)$$

where the $\cdots$ contain a non-linear completion achieving higher-spin gauge invariance.

---

[13]Whether the completion is given in the standard Fronsdal formulation or in the frame-like formulation is immaterial as in both cases the dynamical field content can be obtained by applying projections to the Vasiliev master fields.

In the standard perturbative approach, in which the kinetic terms are taken from $\int_{\partial \mathfrak{m}} \mathscr{V}_{\mathrm{FV}}$, this modification induces a shift in the scalar two-point function $G_{\Delta=1}$ as follows (for a recent treatment, see [62]):

$$G_{\Delta=1}(p; r, r') + |p| K_{\Delta=1}(p; r) K_{\Delta=1}(p; r') \equiv G_{\Delta=2}(p; r, r') . \tag{4.4}$$

In the Poisson sigma model, on the other hand, the Gibbons-Hawking modification is instead treated as an additional vertex. As a result, pairs of external scalar legs of the tree diagrams are sewn together leading to additional scalar loops that are restricted in the configuration space as to touch the boundary. Likewise, the non-linear completion of $\int_{\bar{\partial}\partial \mathfrak{m}} \mathscr{V}_{\mathrm{GH}}$ may induce loop-corrections involving higher-spin fields running in similar boundary loops.

Acknowledgments

A Vector fields and functional derivatives on non-commutative manifold

Star-vector fields. A graded-associative quasi-free differential algebra on a non-commutative base manifold $\mathfrak{B}$ consists of local representatives $\mathfrak{R}_\xi$ (ξ labels charts $B_\xi \subset B$) generated by sets $\{Z^i_\xi\}_{i \in \mathscr{S}}$ of locally-defined differential forms subject to generalized curvature constraints

$$\mathscr{R}^i_\xi := \mathrm{d}Z^i_\xi + \mathscr{Q}^i(Z_\xi, J) \approx 0 , \tag{A.1}$$

where $\overrightarrow{\mathscr{Q}} := \mathscr{Q}^i \partial_i$ (with $\partial_i \equiv \overrightarrow{\partial}_i$) is a composite $\star$-vector field of total degree one subject to the Cartan integrability condition

$$\overrightarrow{\mathscr{Q}} \star \mathscr{Q}^i \equiv 0 . \tag{A.2}$$

A composite $\star$-vector field $\overrightarrow{\mathscr{X}}$ (see appendix B of [1] for more details) is a graded inner derivation of the graded associative $\star$-product algebra $\mathfrak{R} := \mathrm{Env}[Z^i] \otimes \mathfrak{J}$ where $\mathfrak{J}$ is a space of central and d-closed elements (including the identity), i.e. if $\mathscr{F}, \mathscr{F}' \in \mathfrak{R}$ then

$$\overrightarrow{\mathscr{X}} \star (\mathscr{F} \star \mathscr{F}') = (\overrightarrow{\mathscr{X}} \star \mathscr{F}) \star \mathscr{F}' + (-1)^{\deg(\overrightarrow{\mathscr{X}}) \deg(\mathscr{F})} \mathscr{F} \star (\overrightarrow{\mathscr{X}} \star \mathscr{F}') , \tag{A.3}$$

provided that $\overrightarrow{\mathscr{X}}$ and $\mathscr{F}$ have fixed degrees. In components, one writes $\overrightarrow{\mathscr{X}} := \mathscr{X}^i(Z^j) \partial_i$ where $\mathscr{X}^i := \overrightarrow{\mathscr{X}} \star Z^i$. The graded bracket between two composite $\star$-vector fields is defined by

$$[\overrightarrow{\mathscr{X}}, \overrightarrow{\mathscr{X}}']_\star \star \mathscr{F} := \overrightarrow{\mathscr{X}} \star (\overrightarrow{\mathscr{X}}' \star \mathscr{F}) - (-1)^{\deg(\overrightarrow{\mathscr{X}}) \deg(\overrightarrow{\mathscr{X}}')} \overrightarrow{\mathscr{X}}' \star (\overrightarrow{\mathscr{X}} \star \mathscr{F}) , \tag{A.4}$$

is a degree-preserving graded Lie bracket, i.e. $[\overrightarrow{\mathscr{X}}, \overrightarrow{\mathscr{X}}']_\star$ is a graded inner derivation obeying the graded Jacobi identity $[[\overrightarrow{\mathscr{X}}, \overrightarrow{\mathscr{X}}']_\star, \overrightarrow{\mathscr{X}}'']_\star + \text{graded cyclic} \equiv 0$. In components, one has

$$[\overrightarrow{\mathscr{X}}, \overrightarrow{\mathscr{X}}']_\star = \left(\overrightarrow{\mathscr{X}} \star \mathscr{X}'^i - (-1)^{\overrightarrow{\deg}(\mathscr{X})\deg(\overrightarrow{\mathscr{X}}')} \overrightarrow{\mathscr{X}}' \star \mathscr{X}^i \right) \partial_i \,. \qquad (A.5)$$

The Cartan integrability condition (A.2), that can be rewritten $[\overrightarrow{\mathscr{Q}}, \overrightarrow{\mathscr{Q}}]_\star \equiv 0$, amounts to that $\overrightarrow{\mathscr{Q}}$ is a nilpotent composite $\star$-vector field of degree one. This condition ensures that the generalized curvature constraints $\mathscr{R}_\xi^i \approx 0$ are compatible with $\mathrm{d}^2 \equiv 0$ without further algebraic constraints on the generating elements Z_ξ^i. One can also show [1] that the nilpotency of $\overrightarrow{\mathscr{Q}}$ is separately equivalent to that the generalized curvatures $\mathscr{R}^i$ obey the generalized Bianchi identities

$$\mathrm{d}\mathscr{R}^i - \overrightarrow{\mathscr{R}} \star \mathscr{Q}^i \equiv 0\,, \qquad \text{where} \qquad \overrightarrow{\mathscr{R}} := \mathscr{R}^i \partial_i\,, \qquad (A.6)$$

and transform into each other under the following Cartan gauge transformations

$$\delta_\varepsilon Z^i \equiv \mathscr{T}_\varepsilon^i := d\varepsilon^i - \overrightarrow{\varepsilon} \star \mathscr{Q}^i\,, \qquad \text{where} \qquad \overrightarrow{\varepsilon} := \varepsilon^i \partial_i \qquad (A.7)$$

and where ε^i is considered infinitesimal and independent of Z^i, *viz.*

$$\delta_\varepsilon \mathscr{R}^i = -\overrightarrow{\mathscr{R}} \star \left((\overrightarrow{\varepsilon} \star \mathscr{Q}^i) \right)\,. \qquad (A.8)$$

Functional derivative on commutative manifold. We define the variational functional left derivative $\delta_{f(p)} F[f] \equiv \frac{\delta^L}{\delta f(p)} F[f]$ at $p \in B$ of a functional $F[f]$ with respect to a differential form f via the relation

$$\int_{p \in B} \delta f(p)\, \delta_{f(p)} F[f] = F[f + \delta f] - F[f] + O((\delta f)^2)\,. \qquad (A.9)$$

We assign a total degree and a Grassmann parity, respectively, to variables, operations and maps as follows:

$$|\cdot| := \deg(\cdot) + \mathrm{gh}(\cdot)\,, \qquad \mathrm{Gr}(\cdot) = |\cdot| \mod 2\,, \qquad (A.10)$$

which implies that the total exterior derivative d anti-commutes with the BRST operator. We refer to a functional $F[f]$ as being *ultra-local* if $F[f] = L(f, \mathrm{d}f)$ where L is an algebraic function of f and $\mathrm{d}f$, and as being *local* if $F[f] = \int_B \mathscr{L}(f, \mathrm{d}f)$ where $\mathscr{L}$ is ultra-local. We refer to a functional as being *intrinsically defined* on B if it does not refer to any auxiliary frame on B. The functional derivatives of local functionals are intrinsically defined and ultra-local, *viz.*

$$\delta_{f(p)} \int_B \mathscr{L}(f, \mathrm{d}f) = \left(\partial_f \mathscr{L} - (-1)^{|f|} \mathrm{d}(\partial_{\mathrm{d}f} \mathscr{L}) \right)(p) \overset{\text{def.}}{=} \frac{\delta \mathscr{L}(f, \mathrm{d}f)}{\delta f}(p)\,, \qquad (A.11)$$

where throughout the paper all the derivatives are left-derivatives, so that $\partial_f \mathscr{L} = \frac{\partial^L}{\partial f} \mathscr{L}$ and $\partial_{\mathrm{d}f} \mathscr{L} = \frac{\partial^L}{\partial \mathrm{d}f} \mathscr{L}$. The functional derivatives of ultra-local functionals are given by

$$\delta_{f(p)} \big(L(f, \mathrm{d}f)(p') \big) = [\delta_{f(p)} f(p')] \, (\partial_f L) \, (p') + (-1)^{\hat{p} + 1 + |f|} \, (\mathrm{d}_{p'}[\delta_{f(p)} f(p')])(\partial_{\mathrm{d}f} L)(p')\,, (A.12)$$

and refers to an auxiliary frame h^A via the distribution (taking f to be a q-form):

$$\frac{\delta f(p)}{\delta f(p')} \equiv \delta_{f(p)} f(p') = (-1)^{\hat{p}|f|+\text{gh}(f)} \, h^{A[\hat{p}+1-q]}(p) \, h^{B[q]}(p') \, \epsilon_{A[\hat{p}+1-q]B[q]} \, \delta(p,p') \,, \quad \text{(A.13)}$$

where the Dirac function is the zero-form defined by

$$\int_{p\in B} h(p)\varphi(p)\delta(p,p') = \varphi(p') \,, \qquad \varphi \in \Omega^{[0]}(B) \,, \quad \text{(A.14)}$$

where we use the definitions and conventions

$$h^{A[n]} = \frac{1}{n!} h^{A_1} \cdots h^{A_n} \,, \qquad h = h^{A[\hat{p}+1]} \epsilon_{A[\hat{p}+1]} \,, \quad \text{(A.15)}$$

$$\epsilon^{A[n]C[\hat{p}+1-n]} \epsilon_{B[n]C[\hat{p}+1-n]} = (-1)^{\eta_{AB}} n! (\hat{p}+1-n)! \delta^{A[n]}_{B[n]} \,. \quad \text{(A.16)}$$

Then, the functional derivative of an ultra-local functional $F(f, \mathrm{d}f)$ is such that one has

$$\int_{p\in B} \delta f(p) \left[\delta_{f(p)} \left(F(f, \mathrm{d}f)(p') \right) \right] = \delta f(p') \frac{\delta F}{\delta f}(p') + \mathrm{d}_{p'} \left[\delta f(p') \, (\partial_{\mathrm{d}f} F)(p') \right] \quad \text{(A.17)}$$

using the notation and definition of (A.11). Therefore, expanding the total derivative on the right-hand side of the above equation, one has

$$F(f + \delta f, \mathrm{d}(f + \delta f))(p') - F(f, \mathrm{d}f)(p') = (\mathrm{d}_{p'}\delta f(p')) \, \partial_{\mathrm{d}f} F(p') + \delta f(p') \, \partial_f F(p') \,. \quad \text{(A.18)}$$

Functional variations in the non-commutative case. In the case of a non-commutative graded manifold one defines the functional variation $\frac{\delta F}{\delta Z^i}$ of a functional $F[Z]$ by

$$F[Z + \delta F] - F[Z] = \delta F = \int_{p\in B} \left(\delta Z^i(p) \star \frac{\delta F[Z]}{\delta Z^i(p)} \right) + \mathcal{O}((\delta Z)^2) \,. \quad \text{(A.19)}$$

Starting from the functional $F[Z] = \int_B L_\star(Z, \mathrm{d}Z)$ where $L_\star(Z, \mathrm{d}Z)$ is a star-function of $(Z, \mathrm{d}Z)$, one has

$$\frac{\delta F[Z]}{\delta Z^i(p)} = \partial_i^{cycl} L_\star(p) - (-1)^i \, \mathrm{d}(\partial_{\mathrm{d}Z^i}^{cycl} L_\star)(p) =: \frac{\delta L_\star(Z, \mathrm{d}Z)}{\delta Z^i}(p) \quad \text{(A.20)}$$

where, for $P_\star(Z) = f_{i_1,\dots,i_n} \, Z^{i_1} \star \dots \star Z^{i_n} \equiv (-1)^{i_1(i_2+\dots+i_n)} f_{i_2,\dots,i_n,i_1} \, Z^{i_1} \star \dots \star Z^{i_1}$, the *cyclic derivative*

$$\partial_i^{cycl} P_\star(Z) = n \, f_{i,i_2,\dots,i_n} \, Z^{i_2} \star \dots \star Z^{i_n} \,. \quad \text{(A.21)}$$

One then defines

$$\frac{\delta}{\delta Z^i(p)} \left[L_\star(Z, \mathrm{d}Z)(p') \right] = \frac{\delta Z^j(p')}{\delta Z^i(p)} \star \frac{\partial^{cycl} L_\star}{\partial Z^j}(p') + (-1)^{\hat{p}+i+1} \left(\mathrm{d}_{p'} \frac{\delta Z^j(p')}{\delta Z^i(p)} \right) \star \frac{\partial^{cycl} L_\star}{\partial \mathrm{d}Z^j}(p') \quad \text{(A.22)}$$

where $\frac{\delta Z^j(p')}{\delta Z^i(p)}$ has total degree $j - i - \hat{p} - 1$ and is such that

$$\int_{p \in \mathfrak{M}} \mathrm{Tr}\left[\delta Z^i(p) \star \frac{\delta Z^j(p')}{\delta Z^i(p)} \star \frac{\partial^{cycl} L_\star}{\partial Z^j}(p')\right] = \delta Z^i(p') \star \frac{\partial^{cycl} L_\star}{\partial Z^i}(p') . \qquad (A.23)$$

As a result, the action of $\delta = \int_{p \in B} \delta Z^i(p) \star \frac{\delta}{\delta Z^i(p)}$ on the ultra-local functional $L_\star(Z, \mathrm{d}Z)(p')$ yields

$$\begin{aligned}
\delta L_\star(Z, \mathrm{d}Z)(p') &= \delta Z^i(p') \star \frac{\delta L_\star}{\delta Z^i}(p') + \mathrm{d}_{p'}\left[\delta Z^i(p') \star \frac{\partial^{cycl} L_\star}{\partial \mathrm{d}Z^i}(p')\right] \\
&= \delta Z^i(p') \star \frac{\partial^{cycl} L_\star}{\partial Z^i}(p') + \delta(\mathrm{d}Z^i)(p') \star \frac{\partial^{cycl} L_\star}{\partial \mathrm{d}Z^i}(p') , \qquad (A.24)
\end{aligned}$$

as it should.

References

[1] N. Boulanger and P. Sundell, *An action principle for Vasiliev's four-dimensional higher-spin gravity*, *J. Phys.* **A 44** (2011) 495402 [arXiv:1102.2219] [INSPIRE].

[2] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in* $(3 + 1)$-*dimensions*, *Phys. Lett.* **B 243** (1990) 378 [INSPIRE].

[3] M.A. Vasiliev, *Properties of equations of motion of interacting gauge fields of all spins in* $(3 + 1)$-*dimensions*, *Class. Quant. Grav.* **8** (1991) 1387 [INSPIRE].

[4] M.A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in* $(3 + 1)$-*dimensions*, *Phys. Lett.* **B 285** (1992) 225 [INSPIRE].

[5] E. Sezgin and P. Sundell, *Analysis of higher spin field equations in four-dimensions*, *JHEP* **07** (2002) 055 [hep-th/0205132] [INSPIRE].

[6] M.A. Vasiliev, *Equations of motion of interacting massless fields of all spins as a free differential algebra*, *Phys. Lett.* **B 209** (1988) 491 [INSPIRE].

[7] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Annals Phys.* **190** (1989) 59 [INSPIRE].

[8] M.A. Vasiliev, *Unfolded representation for relativistic equations in* $(2 + 1)$ *anti-de Sitter space*, *Class. Quant. Grav.* **11** (1994) 649 [INSPIRE].

[9] M. Vasiliev, *Actions, charges and off-shell fields in the unfolded dynamics approach*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 37 [hep-th/0504090] [INSPIRE].

[10] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, *Annals Phys.* **235** (1994) 435 [hep-th/9312059] [INSPIRE].

[11] P. Schaller and T. Strobl, *Poisson structure induced (topological) field theories*, *Mod. Phys. Lett.* **A 9** (1994) 3129 [hep-th/9405110] [INSPIRE].

[12] A.S. Cattaneo and G. Felder, *On the AKSZ formulation of the Poisson σ-model*, *Lett. Math. Phys.* **56** (2001) 163 [math/0102108] [INSPIRE].

[13] I. Batalin and G. Vilkovisky, *Gauge algebra and quantization*, *Phys. Lett.* **B 102** (1981) 27 [INSPIRE].

[14] I. Batalin and G. Vilkovisky, *Quantization of gauge theories with linearly dependent generators*, *Phys. Rev.* **D 28** (1983) 2567 [*Erratum ibid.* **D 30** (1984) 508] [ɪɴSPIRE].

[15] M. Alexandrov, M. Kontsevich, A. Schwartz and O. Zaboronsky, *The geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys.* **A 12** (1997) 1405 [hep-th/9502010] [ɪɴSPIRE].

[16] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, *Commun. Math. Phys.* **212** (2000) 591 [math/9902090] [ɪɴSPIRE].

[17] A.S. Cattaneo and G. Felder, *Poisson σ-models and deformation quantization*, *Mod. Phys. Lett.* **A 16** (2001) 179 [hep-th/0102208] [ɪɴSPIRE].

[18] M. Kontsevich, *Deformation quantization of Poisson manifolds. 1*, *Lett. Math. Phys.* **66** (2003) 157 [q-alg/9709040].

[19] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Topological field theory*, *Phys. Rept.* **209** (1991) 129 [ɪɴSPIRE].

[20] A. Blasi, N. Maggiore and M. Montobbio, *Noncommutative two dimensional BF model*, *Nucl. Phys.* **B 740** (2006) 281 [hep-th/0512006] [ɪɴSPIRE].

[21] L. Vilar, O. Ventura, R. Amaral, V. Lemes and L. Buffon, *Seiberg-Witten map for the 4D noncommutative BF theory*, *J. Phys.* **A 41** (2008) 425203 [arXiv:0710.3954] [ɪɴSPIRE].

[22] M. Grigoriev and P. Damgaard, *Superfield BRST charge and the master action*, *Phys. Lett.* **B 474** (2000) 323 [hep-th/9911092] [ɪɴSPIRE].

[23] J.-S. Park, *Topological open p-branes*, hep-th/0012141 [ɪɴSPIRE].

[24] C. Hofman and J.-S. Park, *Topological open membranes*, hep-th/0209148 [ɪɴSPIRE].

[25] N. Ikeda, *A deformation of three-dimensional BF theory*, *JHEP* **11** (2000) 009 [hep-th/0010096] [ɪɴSPIRE].

[26] N. Ikeda, *Deformation of BF theories, topological open membrane and a generalization of the star deformation*, *JHEP* **07** (2001) 037 [hep-th/0105286] [ɪɴSPIRE].

[27] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, in *Contemp. Math. Vol. 3315: Quantization, Poisson brackets and beyond*, AMS Publishing, Providence U.S.A. (2002) [math/0203110] [ɪɴSPIRE].

[28] C. Hofman and J.-S. Park, *BV quantization of topological open membranes*, *Commun. Math. Phys.* **249** (2004) 249 [hep-th/0209214] [ɪɴSPIRE].

[29] N. Ikeda, *Chern-Simons gauge theory coupled with BF theory*, *Int. J. Mod. Phys.* **A 18** (2003) 2689 [hep-th/0203043] [ɪɴSPIRE].

[30] N. Ikeda, *Deformation of Batalin-Vilkovisky structures*, math/0604157 [ɪɴSPIRE].

[31] D. Roytenberg, *AKSZ-BV formalism and courant algebroid-induced topological field theories*, *Lett. Math. Phys.* **79** (2007) 143 [hep-th/0608150] [ɪɴSPIRE].

[32] G. Barnich and M. Grigoriev, *A Poincaré lemma for σ-models of AKSZ type*, *J. Geom. Phys.* **61** (2011) 663 [arXiv:0905.0547] [ɪɴSPIRE].

[33] N. Ikeda, *Lectures on AKSZ topological field theories for physicists*, arXiv:1204.3714 [ɪɴSPIRE].

[34] G. Barnich, M. Grigoriev, A. Semikhatov and I. Tipunin, *Parent field theory and unfolding in BRST first-quantized terms*, *Commun. Math. Phys.* **260** (2005) 147 [hep-th/0406192] [InSPIRE].

[35] G. Barnich and M. Grigoriev, *BRST extension of the non-linear unfolded formalism*, hep-th/0504119 [InSPIRE].

[36] M. Grigoriev, *Off-shell gauge fields from BRST quantization*, hep-th/0605089 [InSPIRE].

[37] G. Barnich and M. Grigoriev, *First order parent formulation for generic gauge field theories*, *JHEP* **01** (2011) 122 [arXiv:1009.0190] [InSPIRE].

[38] M. Grigoriev, *Parent formulation at the Lagrangian level*, *JHEP* **07** (2011) 061 [arXiv:1012.1903] [InSPIRE].

[39] D. Kaparulin, S. Lyakhovich and A. Sharapov, *On Lagrange structure of unfolded field theory*, *Int. J. Mod. Phys.* **A 26** (2011) 1347 [arXiv:1012.2567] [InSPIRE].

[40] P. Kazinski, S. Lyakhovich and A. Sharapov, *Lagrange structure and quantization*, *JHEP* **07** (2005) 076 [hep-th/0506093] [InSPIRE].

[41] S. Lyakhovich and A. Sharapov, *Quantizing non-Lagrangian gauge theories: an augmentation method*, *JHEP* **01** (2007) 047 [hep-th/0612086] [InSPIRE].

[42] R. Zucchini, *The Lie algebroid Poisson σ-model*, *JHEP* **12** (2008) 062 [arXiv:0810.3300] [InSPIRE].

[43] A. Kotov and T. Strobl, *Characteristic classes associated to Q-bundles*, arXiv:0711.4106 [InSPIRE].

[44] A. Kotov and T. Strobl, *Generalizing geometry — algebroids and σ-models*, arXiv:1004.0632 [InSPIRE].

[45] C. Becchi, A. Rouet and R. Stora, *The Abelian Higgs-Kibble Model. Unitarity of the S Operator*, *Phys. Lett.* **B 52** (1974) 344 [InSPIRE].

[46] C. Becchi, A. Rouet and R. Stora, *Renormalization of the Abelian Higgs-Kibble Model*, *Commun. Math. Phys.* **42** (1975) 127 [InSPIRE].

[47] C. Becchi, A. Rouet and R. Stora, *Renormalization of gauge theories*, *Annals Phys.* **98** (1976) 287 [InSPIRE].

[48] I. Tyutin, *Gauge invariance in field theory and statistical physics in operator formalism*, arXiv:0812.0580 [InSPIRE].

[49] N. Colombo and P. Sundell, *Twistor space observables and quasi-amplitudes in 4D higher spin gravity*, *JHEP* **11** (2011) 042 [arXiv:1012.0813] [InSPIRE].

[50] C. Iazeolla and P. Sundell, *Families of exact solutions to Vasiliev's 4D equations with spherical, cylindrical and biaxial symmetry*, *JHEP* **12** (2011) 084 [arXiv:1107.1217] [InSPIRE].

[51] E. Sezgin and P. Sundell, *Geometry and observables in Vasiliev's higher spin gravity*, *JHEP* **07** (2012) 121 [arXiv:1103.2360] [InSPIRE].

[52] A. Perez, D. Tempo and R. Troncoso, *Higher spin gravity in 3D: black holes, global charges and thermodynamics*, arXiv:1207.2844 [InSPIRE].

[53] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in gauge theories*, *Phys. Rept.* **338** (2000) 439 [hep-th/0002245] [InSPIRE].

[54] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton University Press, Princeton U.S.A. (1992), pg. 520.

[55] C. Iazeolla and P. Sundell, *A Fiber approach to harmonic analysis of unfolded higher-spin field equations*, *JHEP* **10** (2008) 022 [`arXiv:0806.1942`] [ɪɴSPIRE].

[56] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding mixed-symmetry fields in AdS and the BMV conjecture. II. Oscillator realization*, *JHEP* **07** (2009) 014 [`arXiv:0812.4438`] [ɪɴSPIRE].

[57] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys.* **B 644** (2002) 303 [*Erratum ibid.* **B 660** (2003) 403] [`hep-th/0205131`] [ɪɴSPIRE].

[58] I. Klebanov and A. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys. Lett.* **B 550** (2002) 213 [`hep-th/0210114`] [ɪɴSPIRE].

[59] S. Giombi and X. Yin, *Higher spins in AdS and twistorial holography*, *JHEP* **04** (2011) 086 [`arXiv:1004.3736`] [ɪɴSPIRE].

[60] O. Aharony, G. Gur-Ari and R. Yacoby, *D = 3 bosonic vector models coupled to Chern-Simons gauge theories*, *JHEP* **03** (2012) 037 [`arXiv:1110.4382`] [ɪɴSPIRE].

[61] E. Sezgin and P. Sundell, *Holography in 4D (super) higher spin theories and a test via cubic scalar couplings*, *JHEP* **07** (2005) 044 [`hep-th/0305040`] [ɪɴSPIRE].

[62] S. Giombi and X. Yin, *On higher spin gauge theory and the critical O(N) model*, *Phys. Rev.* **D 85** (2012) 086005 [`arXiv:1105.4011`] [ɪɴSPIRE].