# On the well-posedness of SPDEs with singular drift in divergence form

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**Abstract**

We prove existence and uniqueness of strong solutions for a class of second-order stochastic PDEs with multiplicative Wiener noise and drift of the form $\operatorname{div}\gamma(\nabla\cdot)$, where $\gamma$ is a maximal monotone graph in $\mathbb{R}^n \times \mathbb{R}^n$ obtained as the subdifferential of a convex function satisfying very mild assumptions on its behavior at infinity. The well-posedness result complements the corresponding one in our recent work arXiv:1612.08260 where, under the additional assumption that $\gamma$ is single-valued, a solution with better integrability and regularity properties is constructed. The proof given here, however, is self-contained.

## 1 Introduction and main result

Let us consider the stochastic partial differential equation

$$du(t) - \operatorname{div}\gamma(\nabla u(t))\,dt \ni B(t,u(t))\,dW(t), \qquad u(0)=u_0, \tag{1}$$

posed on $L^2(D)$, with $D$ a bounded domain of $\mathbb{R}^n$ with smooth boundary. The following assumptions will be in force: (a) $\gamma$ is the subdifferential of a lower semicontinuous convex function $k:\mathbb{R}^n \to \mathbb{R}_+$ with $k(0)=0$ and such that

$$\lim_{|x|\to\infty} \frac{k(x)}{|x|} = +\infty, \qquad \limsup_{|x|\to\infty} \frac{k(-x)}{k(x)} < +\infty$$

(in particular, $\gamma$ is a maximal monotone graph in $\mathbb{R}^n \times \mathbb{R}^n$ whose domain coincides with $\mathbb{R}^n$); (b) $W$ is a cylindrical Wiener process on a separable Hilbert space $H$, supported by a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},\mathbb{P})$ satisfying the "usual conditions"; (c) $B$ is a map from $\Omega\times[0,T]\times L^2(D)$ to $\mathscr{L}^2(H,L^2(D))$, the space of Hilbert-Schmidt operators from $H$ to $L^2(D)$, that is Lipschitz-continuous and has linear growth with respect to its third argument, uniformly with respect to the other two, and is such that $B(\cdot,\cdot,a)$ is measurable and adapted for all $a\in L^2(D)$.

Under the additional assumption that $\gamma$ is a (single-valued) continuous function, we proved in [7] that (1) admits a strong solution $u$, which is unique within a set of processes satisfying mild integrability conditions. The solution of [7] is constructed pathwise, i.e. for each $\omega \in \Omega$, so that, as is natural to expect, measurability problems arise with respect to the usual $\sigma$-algebras on $\Omega \times [0, T]$ used in the theory of stochastic processes. Precisely because of such an issue we needed to assume $\gamma$ to be single-valued.

The purpose of this note is to provide an alternative approach to establish the well-posedness of (1) that, avoiding pathwise constructions, is simpler than that of [7] and does not need any extra assumption on $\gamma$. The price to pay is that the solution we obtain here is less regular than that of [7]. We also refer to [9] for a related result obtained by analogous methods.

Let us define the concept of solution to (1) we shall be working with.

**Definition 1.1.** *Let $u_0$ be an $L^2(D)$-valued $\mathscr{F}_0$-measurable random variable. A* strong solution *to equation* (1) *is a couple* $(u, \eta)$ *satisfying the following properties:*

*(i) $u$ is a measurable and adapted $L^2(D)$-valued process such that*

$$u \in L^1(0, T; W_0^{1,1}(D)) \qquad \text{and} \qquad B(\cdot, u) \in L^2(0, T; \mathscr{L}^2(U, L^2(D))) \quad \mathbb{P}\text{-a.s.};$$

*(ii) $\eta$ is a measurable and adapted $L^1(D)^n$-valued process such that*

$$\eta \in L^1(0, T; L^1(D)^n) \quad \mathbb{P}\text{-a.s.}, \qquad \eta \in \gamma(\nabla u) \quad \text{a.e. in } \Omega \times (0, T) \times D;$$

*(iii) one has, as an equality in $L^2(D)$,*

$$u(t) - \int_0^t \operatorname{div} \eta(s)\, ds = u_0 + \int_0^t B(s, u(s))\, dW(s) \quad \mathbb{P}\text{-a.s.} \quad \forall\, t \in [0, T]. \quad (2)$$

Note that (2) has to be intended in the sense of distributions. In particular, since $\eta \in L^1(D)^n$, the integrand in the second term of (2) does not, in general, take values in $L^2(D)$. However, the conditions on $B$ imply that the stochastic integral in (2) is an $L^2(D)$-valued local martingale, hence the term involving the divergence of $\eta$ turns out to be $L^2(D)$-valued by comparison.

We can now state our main result. Here and in the following $k^* : \mathbb{R}^n \to \mathbb{R}_+$ is the convex conjugate of $k$, defined as $k^*(y) := \sup_{x \in \mathbb{R}^n} \bigl(x \cdot y - k(x)\bigr)$.

**Theorem 1.2.** *Let $u_0 \in L^2(\Omega, \mathscr{F}_0; L^2(D))$. Then equation* (1) *admits a unique strong solution $(u, \eta)$ such that*

$$\begin{aligned}
&\sup_{t \le T} \mathbb{E}\|u(t)\|^2_{L^2(D)} + \mathbb{E}\int_0^T \|u(t)\|_{W_0^{1,1}(D)}\,dt < \infty,\\
&\mathbb{E}\int_0^T \|\eta(t)\|_{L^1(D)^n}\,dt < \infty,\\
&\mathbb{E}\int_0^T \big(\|k(\nabla u(t))\|_{L^1(D)} + \|k^*(\eta(t))\|_{L^1(D)}\big)\,dt < \infty.
\end{aligned}$$

*Moreover, the solution map $u_0 \mapsto u$ is Lipschitz-continuous from $L^2(\Omega; L^2(D))$ to $L^\infty(0,T; L^2(\Omega; L^2(D)))$, and $u$ is weakly continuous as a function on $[0,T]$ with values in $L^2(\Omega; L^2(D))$.*

Under the extra assumption of $\gamma$ being single-valued, the solution obtained in [7] is more regular in the sense that $\mathbb{E}\sup_{t \le T}\|u(t)\|^2_{L^2(D)}$ is finite, the solution map is Lipschitz-continuous from $L^2(\Omega; L^2(D))$ to $L^2(\Omega; L^\infty(0,T; L^2(D)))$, and $u(\omega, \cdot)$ is weakly continuous as a function on $[0,T]$ with values in $L^2(D)$ for $\mathbb{P}$-a.a. $\omega \in \Omega$.

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## 2 Well-posedness of an auxiliary equation

The goal of this section is to prove well-posedness of a version of (1) with additive noise. Namely, we consider the initial value problem

$$du(t) - \operatorname{div}\gamma(\nabla u(t))\,dt \ni G(t)\,dW(t), \qquad u(0) = u_0, \tag{3}$$

where $G \in L^2(\Omega \times [0,T]; \mathscr{L}^2(H, L^2(D)))$ is a measurable and adapted process.

**Proposition 2.1.** *Equation* (3) *admits a unique strong solution $(u, \eta)$ satisfying the same integrability and weak continuity conditions of Theorem 1.2.*

We introduce the regularized equation

$$du_\lambda(t) - \operatorname{div}\gamma_\lambda(\nabla u_\lambda(t))\,dt - \lambda\Delta u_\lambda(t)\,dt = G(t)\,dW(t), \qquad u_\lambda(0) = u_0,$$

where $\gamma_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\gamma_\lambda := \frac{1}{\lambda}(I - (I + \lambda\gamma)^{-1})$, for any $\lambda > 0$, is the Yosida approximation of $\gamma$, and $\Delta : H_0^1(D) \to H^{-1}(D)$ is the (variational) Dirichlet Laplacian. Since $\gamma_\lambda$ is monotone and Lipschitz-continuous, the classical variational approach (see [4, 8] as well as [5]) yields the existence of a unique predictable process $u_\lambda$ with values in $H^1_0(D)$ such that

$$\mathbb{E}\|u_\lambda\|^2_{C([0,T];L^2(D))} + \mathbb{E}\int_0^T \|u_\lambda(t)\|^2_{H^1_0(D)}\,dt < \infty$$

and

$$u_\lambda(t) - \int_0^t \operatorname{div}\gamma_\lambda(\nabla u_\lambda(s))\,ds - \lambda\int_0^t \Delta u_\lambda(s)\,ds = u_0 + \int_0^t G(s)\,dW(s) \tag{4}$$

$\mathbb{P}$-a.s. in $H^{-1}(D)$ for all $t \in [0,T]$.

We are now going to prove a priori estimates and weak compactness in suitable topologies for $u_\lambda$ and related processes. These will allow us to pass to the limit as $\lambda \to 0$ in (4).

For notational parsimony, we shall often write, for any $p \geq 0$, $L^p_\omega$, $L^p_t$, and $L^p_x$ in place of $L^p(\Omega)$, $L^p(0,T)$, and $L^p(D)$, respectively, and $C_t$ to denote $C([0,T])$. Other similar abbreviations are self-explanatory. The $L^2(D)$-norm will be denoted simply by $\|\cdot\|$. If a function $f : D \to \mathbb{R}^n$ is such that each component $f^j$, $j = 1,\dots,n$, belongs to $L^p(D)$, we shall just write $f \in L^p(D)$ rather than $f \in L^p(D)^n$. The notation $a \lesssim b$ means that $a \leq Nb$ for a positive constant $N$.

**Lemma 2.2.** *There exists a constant $N$ such that*

$$\begin{aligned}\|u_\lambda\|_{L^2_\omega C_t L^2_x} + \lambda^{1/2}\|\nabla u_\lambda\|_{L^2_{t,\omega,x}} + \|\gamma_\lambda(\nabla u_\lambda)\cdot\nabla u_\lambda\|_{L^1_{t,\omega,x},}\\ < N\big(\|u_0\|_{L^2_{\omega,x}} + \|G\|_{L^2_{t,\omega}\mathscr{L}^2(H,L^2_x)}\big).\end{aligned}$$

*Proof.* Itô's formula for the square of the norm in $L^2_x$ yields

$$\begin{aligned}\|u_\lambda(t)\|^2 + 2\int_0^t\int_D \gamma(\nabla u_\lambda(s))\cdot\nabla u_\lambda(s)\,dx\,ds + 2\lambda\int_0^t \|\nabla u_\lambda(s)\|^2\,ds\\ = \|u_0\|^2 + 2\int_0^t u_\lambda(s)G(s)\,dW(s) + \int_0^t \|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds,\end{aligned}$$

hence, taking the supremum in time and expectation,

$$\begin{aligned}\mathbb{E}\|u_\lambda\|^2_{C_tL^2_x} + \mathbb{E}\int_0^T\int_D \gamma_\lambda(\nabla u_\lambda(s))\cdot\nabla u_\lambda(s)\,dx\,ds + \lambda\,\mathbb{E}\|\nabla u_\lambda\|^2_{L^2_{t,x}}\\ \lesssim \mathbb{E}\|u_0\|^2 + \mathbb{E}\|G\|^2_{L^2_t\mathscr{L}^2(H,L^2_x)} + \mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t u_\lambda(s)G(s)\,dW(s)\right|,\end{aligned}$$

where, by Davis' inequality (see, e.g., [6]), the ideal property of Hilbert-Schmidt operators (see, e.g., [1, p. V.52]), and the elementary inequality $ab \leq \varepsilon a^2 + b^2/\varepsilon$

$\forall a, b \geq 0$, $\varepsilon > 0$,

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t u_\lambda(s) G(s)\, dW(s) \right| \lesssim \mathbb{E}\biggl( \int_0^T \|u_\lambda(s)G(s)\|^2_{\mathscr{L}^2(H,\mathbb{R})}\, ds \biggr)^{1/2}$$
$$\leq \varepsilon\, \mathbb{E}\|u_\lambda\|^2_{C_tL^2_x} + N(\varepsilon)\, \mathbb{E}\int_0^T \|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\, ds$$

for any $\varepsilon > 0$. To conclude it suffices to choose $\varepsilon$ small enough. □

**Lemma 2.3.** *The families* $(\nabla u_\lambda)$ *and* $(\gamma_\lambda(\nabla u_\lambda))$ *are relatively weakly compact in* $L^1(\Omega \times (0,T) \times D)$.

*Proof.* Recall that, for any $y$, $r \in \mathbb{R}^n$, ones has $k(y) + k^*(r) = r \cdot y$ if and only if $r \in \partial k(y) = \gamma(y)$. Therefore, since

$$\gamma_\lambda(x) \in \partial k\bigl((I + \lambda\gamma)^{-1}x\bigr) = \gamma\bigl((I + \lambda\gamma)^{-1}x\bigr) \qquad \forall x \in \mathbb{R}^n,$$

we deduce by the definition of $\gamma_\lambda$ that

$$\begin{aligned} k\bigl((I + \lambda\gamma)^{-1}x\bigr) + k^*\bigl(\gamma_\lambda(x)\bigr) &= \gamma_\lambda(x) \cdot (I + \lambda\gamma)^{-1}x \\ &= \gamma_\lambda(x) \cdot x - \lambda\,|\gamma_\lambda(x)|^2 \leq \gamma_\lambda(x) \cdot x \qquad \forall x \in \mathbb{R}^n\,. \end{aligned} \tag{5}$$

(See, e.g., [3] for all necessary facts from convex analysis used in this note.) Hence, taking Lemma 2.2 into account, there exists a constant $N > 0$, independent of $\lambda$, such that

$$\mathbb{E}\int_0^T\!\!\int_D k^*\bigl(\gamma_\lambda(\nabla u_\lambda)\bigr) \leq \mathbb{E}\int_0^T\!\!\int_D \gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda < N.$$

The assumptions on $k$ imply that its convex conjugate $k^*$ is also convex, lower semicontinuous and such that $\lim_{|y|\to\infty} k^*(y)/|y| = +\infty$. Therefore a simple modification of the criterion by de la Vallée Poussin implies that $(\gamma_\lambda(\nabla u_\lambda))$ is uniformly integrable on $\Omega \times (0,T) \times D$, hence that it is relatively weakly compact in $L^1_{t,\omega,x}$ by the Dunford-Pettis theorem. A completely analogous argument shows that

$$\mathbb{E}\int_0^T\!\!\int_D k\bigl((I + \lambda\gamma)^{-1}\nabla u_\lambda\bigr) \leq \mathbb{E}\int_0^T\!\!\int_D \gamma_\lambda(\nabla u_\lambda) \cdot \nabla u_\lambda < N,$$

hence that $(I + \lambda\gamma)^{-1}\nabla u_\lambda$ is relatively weakly compact in $L^1_{t,\omega,x}$. Moreover, since $\nabla u_\lambda = (I + \lambda\gamma)^{-1}\nabla u_\lambda + \lambda\gamma_\lambda(\nabla u_\lambda)$, it also follows that $(\nabla u_\lambda)$ is relatively weakly compact in $L^1_{t,\omega,x}$. □

Thanks to Lemmata 2.2 and 2.3, there exists a subsequence of $\lambda$, denoted by the same symbol, and processes $u \in L^\infty_t L^2_{\omega,x} \cap L^1_{t,\omega} W^{1,1}_0$ and $\eta \in L^1_{t,\omega,x}$ such that

$$\begin{aligned} u_\lambda &\longrightarrow u && \text{weakly* in } L^\infty_t L^2_{\omega,x},\\ u_\lambda &\longrightarrow u && \text{weakly in } L^1_{t,\omega} W^{1,1}_0,\\ \gamma_\lambda(\nabla u_\lambda) &\longrightarrow \eta && \text{weakly in } L^1_{t,\omega,x},\\ \lambda u_\lambda &\longrightarrow 0 && \text{weakly in } L^2_{t,\omega} H^1_0. \end{aligned}$$

as $\lambda \to 0$. Let $t \in [0,T]$ be arbitrary but fixed. The fourth convergence above implies

$$\lambda \int_0^t \Delta u_\lambda(s)\, ds \longrightarrow 0 \qquad \text{in } L^2_\omega H^{-1},$$

while the third yields, for any $\varphi \in L^\infty_\omega W^{1,\infty}$,

$$\mathbb{E}\int_0^t\!\!\int_D \gamma_\lambda(\nabla u_\lambda(s)) \cdot \nabla\varphi\, dx\, ds \longrightarrow \mathbb{E}\int_0^t\!\!\int_D \eta(s) \cdot \nabla\varphi\, dx\, ds,$$

hence $\mathbb{E}\int_0^t \langle \operatorname{div} \gamma_\lambda(\nabla u_\lambda(s)), \varphi\rangle\, ds \longrightarrow \mathbb{E}\int_0^t \langle \operatorname{div} \eta(s), \varphi\rangle\, ds$. Therefore, recalling (4), by difference we deduce that

$$\mathbb{E}\langle u_\lambda(t), \varphi\rangle \longrightarrow \mathbb{E}\langle u(t), \varphi\rangle.$$

Consequently, since $u_\lambda(t)$ is bounded in $L^2_\omega L^2_x$, we also have that $u_\lambda(t) \to u(t)$ weakly in $L^2_\omega L^2_x$. Taking the limit as $\lambda \to 0$ in (4) thus yields

$$u(t) - \int_0^t \operatorname{div} \eta(s)\, ds = u_0 + \int_0^t G(s)\, dW(s) \qquad \text{in } L^1_\omega V_0',$$

where $V_0'$ is the (topological) dual of a separable Hilbert space $V_0$ embedded continuously and densely in $H^1_0$, and continuously in $W^{1,\infty}$. The identity immediately implies that $u \in C_t L^1_\omega V_0'$. Since $u \in L^\infty_t L^2_\omega L^2_x$, it follows by a result of Strauss (see [10, Theorem 2.1]) that $u$ is a weakly continuous function on $[0,T]$ with values in $L^2_\omega L^2_x$.

By Mazur's lemma there exist sequences of convex combinations of $\gamma_\lambda(\nabla u_\lambda)$ that converge $\eta$ in (the norm topology of) $L^1_{t,\omega,x}$, thus also, passing to a subsequence, $\mathbb{P} \otimes dt$-almost everywhere in $L^1_x$. Similarly, since $u_\lambda \to u$ weakly* in $L^\infty_t L^2_{\omega,x}$ implies that $u_\lambda \to u$ weakly in $L^2_{t,\omega,x}$, there exist sequences of convex combinations of $u_\lambda$ that converge to $u$ $\mathbb{P} \otimes dt$-almost everywhere in $L^2_x$. Since convex combinations of $(u_\lambda)$ and of $(\gamma_\lambda(\nabla u_\lambda))$ are (at least) predictable and adapted, respectively, it follows that $u$ is predictable and $\eta$ is measurable and adapted. Moreover, thanks to the weak lower semicontinuity of convex integrals, one has

$$\mathbb{E}\int_0^T\!\!\int_D \bigl(k(\nabla u) + k^*(\eta)\bigr) < \infty.$$

In order to show that $\eta \in \gamma(\nabla u)$ for a.a. $(\omega, t, x)$, we need the following "energy identity".

**Lemma 2.4.** *Assume that*

$$y(t) + \alpha \int_0^t y(s)\,ds - \int_0^t \operatorname{div}\zeta(s)\,ds = y_0 + \int_0^t C(s)\,dW(s)$$

*in* $L^2_x$ $\mathbb{P}$*-a.s. for all* $t \in [0,T]$*, where* $\alpha \in \mathbb{R}$*,* $y_0 \in L^2_{\omega,x}$ *is* $\mathscr{F}_0$*-measurable, and*

$$y \in L^\infty_t L^2_{\omega,x} \cap L^1_{t,\omega} W^{1,1}_0, \qquad \zeta \in L^1_{t,\omega,x}, \qquad C \in L^2_{t,\omega}\mathscr{L}^2(H, L^2_x)$$

*are measurable and adapted processes such that* $k(c\nabla y) + k^*(c\zeta) \in L^1_{t,\omega,x}$ *for a constant* $c > 0$*. Then*

$$\begin{aligned}\mathbb{E}\|y(t)\|^2 &+ 2\alpha\,\mathbb{E}\int_0^t \|y(s)\|^2\,ds + 2\,\mathbb{E}\int_0^t\int_D \zeta\cdot\nabla y\,dx\,ds\\ &= \mathbb{E}\|y_0\|^2 + \mathbb{E}\int_0^t \|C(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds \qquad \forall t \in [0,T].\end{aligned}$$

*Proof.* Let $m \in \mathbb{N}$ be such that such that $(I - \delta\Delta)^{-m}$ maps $L^1_x$ into $H^1_0 \cap W^{1,\infty}$, and use the notation $h^\delta := (I - \delta\Delta)^{-m}h$ for any $h$ taking values in $L^1_x$. One has

$$y^\delta(t) + \alpha \int_0^t y^\delta(s)\,ds - \int_0^t \operatorname{div}\zeta^\delta(s)\,ds = y^\delta_0 + \int_0^t C^\delta(s)\,dW(s) \tag{6}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, as an equality in $L^2_x$, for which Itô's formula yields

$$\begin{aligned}\|y^\delta(t)\|^2 &+ 2\alpha \int_0^t \|y^\delta(s)\|^2\,ds + 2\int_0^t\int_D \zeta^\delta\cdot\nabla y^\delta\,dx\,ds\\ &= \|y^\delta_0\|^2 + \int_0^t \|C^\delta(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds + \int_0^t y^\delta(s)C^\delta(s)dW(s).\end{aligned}$$

It is evident from (6) that $y^\delta$ is a continuous $L^2_x$-valued process, hence the stochastic integral $(y^\delta C^\delta)\cdot W$ on the right-hand side of the above identity is a continuous local martingale. Let $(T_n)$ be a localizing sequence, and multiply the previous identity by $1_{[0,T_n]}$, to obtain, thanks to $\mathbb{E}(y^\delta C^\delta)\cdot W(\cdot\wedge T_n) = 0$,

$$\begin{aligned}\mathbb{E}\|y^\delta(t\wedge T_n)\|^2 &+ 2\alpha\,\mathbb{E}\int_0^{t\wedge T_n} \|y^\delta(s)\|^2\,ds + 2\,\mathbb{E}\int_0^{t\wedge T_n}\int_D \zeta^\delta\cdot\nabla y^\delta\,dx\,ds\\ &= \mathbb{E}\|y^\delta_0\|^2 + \mathbb{E}\int_0^{t\wedge T_n} \|C^\delta(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds.\end{aligned}$$

Letting $n$ tend to $\infty$, the dominated convergence theorem yields

$$\begin{aligned}\mathbb{E}\|y^\delta(t)\|^2 + 2\alpha\,\mathbb{E}\int_0^t \|y^\delta(s)\|^2\,ds + 2\,\mathbb{E}\int_0^t\!\!\int_D \zeta^\delta\cdot\nabla y^\delta\,dx\,ds\\ = \mathbb{E}\|y_0^\delta\|^2 + \mathbb{E}\int_0^t \|C^\delta(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds\end{aligned}$$

for all $t\in[0,T]$. We are now going to pass to the limit as $\delta\to 0$: the first and second terms on the left-hand side and the first on the right-hand side clearly converge to $\mathbb{E}\|y(t)\|^2$, $2\alpha\,\mathbb{E}\int_0^t\|y(s)\|^2\,ds$ and $\mathbb{E}\|y_0\|^2$, respectively. Properties of Hilbert-Schmidt operators and the dominated convergence theorem also yield

$$\lim_{\delta\to 0}\mathbb{E}\int_0^t \|C^\delta(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds = \mathbb{E}\int_0^t \|C(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds$$

for all $t\in[0,T]$. To conclude it then suffices to show that $\nabla y^\delta\cdot\zeta^\delta\to\nabla y\cdot\zeta$ in $L^1_{t,\omega,x}$. Since $\nabla y^\delta\to\nabla y$ and $\zeta^\delta\to\zeta$ in measure in $\Omega\times(0,t)\times D$, Vitali's theorem implies strong convergence in $L^1_{t,\omega,x}$ if the sequence $(\nabla y^\delta\cdot\zeta^\delta)$ is uniformly integrable in $\Omega\times(0,t)\times D$. In turn, the latter is certainly true if $(|\nabla y^\delta\cdot\zeta^\delta|)$ is dominated by a sequence that converges strongly in $L^1_{t,\omega,x}$. Indeed, using the assumptions on the behavior of $k$ at infinity as well as the generalized Jensen inequality for sub-Markovian operators (see [2]), one has

$$\pm c^2\zeta^\delta\cdot\nabla y^\delta \lesssim 1 + k(c\nabla y^\delta) + k^*(c\zeta^\delta) \le 1 + (I-\delta\Delta)^{-m}\left(k(c\nabla y) + k^*(c\zeta)\right),$$

where the sequence on the right-hand side converges in $L^1_{t,\omega,x}$ as $\delta\to 0$ because, by assumption, $k(c\nabla y)+k^*(c\zeta)\in L^1_{t,\omega,x}$. □

Itô's formula yields

$$\begin{aligned}\mathbb{E}\|u_\lambda(t)\|^2 + 2\,\mathbb{E}\int_0^t\!\!\int_D \gamma_\lambda(\nabla u_\lambda)\cdot\nabla u_\lambda + 2\lambda\,\mathbb{E}\int_0^t\|\nabla u_\lambda\|^2\\ = \mathbb{E}\|u_0\|^2 + \mathbb{E}\int_0^t \|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds\end{aligned}$$

and, by Lemma 2.4,

$$\mathbb{E}\|u(t)\|^2 + 2\,\mathbb{E}\int_0^t\!\!\int_D \eta\cdot\nabla u = \mathbb{E}\|u_0\|^2 + \mathbb{E}\int_0^t \|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds.$$

One then has

$$
\begin{aligned}
2\limsup_{\lambda\to 0}\mathbb{E}\int_0^T\int_D \gamma_\lambda(\nabla u_\lambda(s))\cdot\nabla u_\lambda(s)\,dx\,ds\\
&\leq \mathbb{E}\|u_0\|^2+\mathbb{E}\int_0^T\|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds-\liminf_{\lambda\to 0}\mathbb{E}\|u_\lambda(T)\|^2\\
&\leq \mathbb{E}\|u_0\|^2+\mathbb{E}\int_0^T\|G(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds-\mathbb{E}\|u(T)\|^2\\
&= \mathbb{E}\int_0^T\int_D \eta(s)\cdot\nabla u(s)\,dx\,ds.
\end{aligned}
$$

Since $\nabla u_\lambda \to \nabla u$ and $\gamma_\lambda(\nabla u_\lambda)\to\eta$ weakly in $L^1_{t,\omega,x}$, this implies that $\eta\in\gamma(\nabla u)$ a.e. in $\Omega\times(0,T)\times D$. We have thus proved the existence and weak continuity statements of Proposition 2.1.

In order to show that the solution is unique, we are going to prove that *any* solution depends continuously on $(u_0,G)$. Let $(u_i,\eta_i)$, $i=1,2$, satisfy

$$
u_i(t)-\int_0^t \operatorname{div}\eta_i(s)\,ds = u_0+\int_0^t G_i(s)\,ds
$$

in the sense of Definition 1.1, as well as the integrability conditions (on $u$ and $\eta$) of Theorem 1.2. Setting $y:=u_1-u_2$, $y_0:=u_{01}-u_{02}$, $\zeta:=\eta_1-\eta_2$, and $F:=G_1-G_2$, one has

$$
y(t)-\int_0^t \operatorname{div}\zeta(s)\,ds = y_0+\int_0^t F(s)\,dW(s)
$$

$\mathbb{P}$-a.s. in $L^2(D)$ for all $t\in[0,T]$. For any process $h$, let us use the notation $h^\alpha(t):=e^{-\alpha t}h(t)$. For any $\alpha>0$, the integration-by-parts formula yields

$$
y^\alpha(t)+\int_0^t\bigl(-\operatorname{div}\zeta^\alpha(s)+\alpha y^\alpha(s)\bigr)\,ds = y_0+\int_0^t F^\alpha(s)\,dW(s),
$$

hence also, thanks to Lemma 2.4,

$$
\begin{aligned}
\mathbb{E}\|y^\alpha(t)\|^2+2\alpha\,\mathbb{E}\int_0^t\|y^\alpha(s)\|^2\,ds+2\,\mathbb{E}\int_0^t\int_D\zeta^\alpha(s)\cdot\nabla y^\alpha(s)\,dx\,ds\\
&\leq \mathbb{E}\|y_0\|^2+\mathbb{E}\int_0^t\|F^\alpha(s)\|^2_{\mathscr{L}^2(H,L^2_x)}\,ds,
\end{aligned}
$$

where $\zeta^\alpha\cdot\nabla y^\alpha\geq 0$ by monotonicity. Therefore, taking the $L^\infty_t$ norm,

$$
\|y^\alpha\|_{L^\infty_t L^2_{\omega,x}}+\sqrt{\alpha}\|y^\alpha\|_{L^2_{t,\omega,x}}\lesssim \|y_0\|_{L^2_{\omega,x}}+\|F^\alpha\|_{L^2_{t,\omega}\mathscr{L}^2(H,L^2_x)},
$$

that is, using the notation $L^p_t(\alpha):=L^p([0,T],e^{-\alpha t}dt)$ for any $p\geq 0$,

$$
\begin{aligned}
\|u_1-u_2\|_{L^\infty_t(\alpha)L^2_{\omega,x}}+\sqrt{\alpha}\|u_1-u_2\|_{L^2_t(\alpha)L^2_{\omega,x}}\\
&\lesssim \|u_{01}-u_{02}\|_{L^2_{\omega,x}}+\|G_1-G_2\|_{L^2_t(\alpha)L^2_\omega\mathscr{L}^2(H,L^2_x)}.
\end{aligned}
\tag{7}
$$

Taking $\alpha = 0$ and $G_1 = G_2$ immediately yields the uniqueness of solutions (as well as Lipschitz-continuous dependence on the initial datum). The proof of Proposition 2.1 is thus complete.

# 3 Proof of Theorem 1.2

For any $v \in L^2_{t,\omega,x}$ measurable and adapted, and any $\mathscr{F}_0$-measurable random variable $u_0 \in L^2_{\omega,x}$, the process $B(\cdot, v)$ is measurable, adapted, and belongs to $L^2_{t,\omega}\mathscr{L}^2(H, L^2_x)$, hence the equation

$$du(t) - \operatorname{div}\gamma(\nabla u(t))\,dt \ni B(t, v(t))\,dW(t), \qquad u(0) = u_0,$$

is well-posed in the sense of Proposition 2.1. Moreover, for any $v_1$, $v_2$ and $u_{01}$, $u_{02}$ satisfying the same hypotheses on $v$ and $u_0$, respectively, (7) yields

$$\begin{aligned}\|u_1 - u_2\|_{L^\infty_t(\alpha)L^2_{\omega,x}} &+ \sqrt{\alpha}\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \\ &\lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|B(\cdot, v_1) - B(\cdot, v_2)\|_{L^2_t(\alpha)L^2_\omega\mathscr{L}^2(H,L^2_x)}.\end{aligned}$$

It hence follows by the Lipschitz-continuity of $B$ that

$$\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \lesssim \frac{1}{\sqrt{\alpha}}\Big(\|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|v_1 - v_2\|_{L^2_t(\alpha)L^2_{\omega,x}}\Big), \tag{8}$$

where the implicit constant does not depend on $\alpha$. In particular, denoting by $\Gamma$ the map $(u_0, v) \mapsto u$, one has that $v \mapsto \Gamma(u_0, v)$ is a strict contraction of $L^2_t(\alpha)L^2_{\omega,x}$ for $\alpha$ large enough. Therefore, by equivalence of norms, $v \mapsto \Gamma(u_0, v)$ admits a unique fixed point in $L^2_{t,\omega,x}$, which solves (1) and satisfies all integrability conditions. Such solution is unique as any solution is a fixed point of $v \mapsto \Gamma(u_0, v)$.

Let us show that the solution map $u_0 \mapsto u$ is Lipschitz-continuous: (8) yields, choosing $\alpha$ large enough,

$$\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}} \le N_1\|u_{01} - u_{02}\|_{L^2_{\omega,x}} + N_2\|u_1 - u_2\|_{L^2_t(\alpha)L^2_{\omega,x}}$$

with constants $N_1 > 0$ and $0 < N_2 < 1$, hence, by equivalence of norms,

$$\|u_1 - u_2\|_{L^2_tL^2_{\omega,x}} \lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}}.$$

This in turn implies, in view of (7) (with $\alpha = 0$) and the Lipschitz-continuity of $B$,

$$\begin{aligned}\|u_1 - u_2\|_{L^\infty_tL^2_{\omega,x}} &\lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|B(\cdot, u_1) - B(\cdot, u_2)\|_{L^2_{t,\omega}\mathscr{L}^2(H,L^2_x)} \\ &\lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}} + \|u_1 - u_2\|_{L^2_tL^2_{\omega,x}} \lesssim \|u_{01} - u_{02}\|_{L^2_{\omega,x}},\end{aligned}$$

which completes the proof.

*Remark.* A priori estimates entirely analogous to those of Lemma 2.2, as well as weak compactness results exactly as in Lemma 2.3, can be proved for the regularized equation obtained by replacing $\gamma$ with $\gamma_\lambda + \lambda\nabla$ directly in (1). It is however not immediately clear how to pass to the limit as $\lambda \to 0$ in the stochastic integrals appearing in such regularized equations with multiplicative noise, i.e. to show that $B(u_\lambda) \cdot W$ converges to $B(u) \cdot W$ in a suitable sense.

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