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THE SPACE OF HYPERKÄHLER METRICS ON A 4-MANIFOLD WITH BOUNDARY

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Abstract

Let X be a compact 4-manifold with boundary. We study the space of hyperkähler triples $\omega_1, \omega_2, \omega_3$ on X, modulo diffeomorphisms which are the identity on the boundary. We prove that this moduli space is a smooth infinite-dimensional manifold and describe the tangent space in terms of triples of closed anti-self-dual 2-forms. We also explore the corresponding boundary value problem: a hyperkähler triple restricts to a closed framing of the bundle of 2-forms on the boundary; we identify the infinitesimal deformations of this closed framing that can be filled in to hyperkähler deformations of the original triple. Finally we study explicit examples coming from gravitational instantons with isometric actions of SU(2).

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1. Introduction

In this paper we study hyperkähler metrics on a compact 4-manifold X with boundary $\partial X = Y$. Recall that a Riemannian manifold (M, g), of dimension $4n$, is *hyperkähler* if there exists a triple (J_1, J_2, J_3) of orthogonal complex structures which satisfy the quaternionic relations

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -1$$

and such that the corresponding triple $\omega = (\omega_1, \omega_2, \omega_3)$ of 2-forms is closed, where $\omega_i(\cdot,\cdot) = g(J_i\cdot,\cdot)$. Thus (M, g) is Kähler with respect to each of the complex structures J_i.

When $n = 1$, that is in 4 dimensions, the ω_i forms a flat basis of the bundle of self-dual 2-forms Λ^2_+ and the quaternionic relations imply

$$\omega_i \wedge \omega_j = 2\delta_{ij}\,\mu \tag{1.1}$$

where the volume element μ is also determined by the triple:

$$\mu = \tfrac{1}{6}(\omega_1^2 + \omega_2^2 + \omega_3^2). \tag{1.2}$$

Conversely, given a triple of *symplectic forms* $\omega = (\omega_1, \omega_2, \omega_3)$ satisfying (1.1) and (1.2) we may use the span of the ω_i to *define* a rank 3 subbundle of $\Lambda^2 T^*$ on which the natural quadratic form is positive-definite. By the correspondence between such subbundles and conformal structures in 4 dimensions, we see that our triple ω defines first a conformal structure, then a metric by declaring μ in (1.2) to be the metric volume form. The conditions (1.1) and (1.2) together with $d\omega = 0$ then imply that this metric is hyperkähler (see [**9**, **14**] and Lemma 3.1 below).

We propose to study hyperkähler metrics in terms of the corresponding triples. Our work is inspired in part by the following local thickening result of Bryant [**6**]. Suppose that $\iota : Y \subset X$ is a real hypersurface in the hyperkähler manifold (X, ω). The pull-back $\gamma = \iota^*\omega$ is then a *closed framing* of the bundle Λ^2_Y of 2-forms on Y. (For any closed orientable 3-manifold Y, framings of Λ^2_Y exist and a result of Gromov [**13**, page 182] shows that any framing can be deformed to become a closed framing.) Conversely, Bryant showed that if γ is a real-analytic closed framing of Λ^2_Y, then there is a hyperkähler 4-manifold (X, ω) and an embedding $\iota : Y \to X$ such that $\gamma = \iota^*\omega$. This is a consequence of the Cauchy–Kowalevski theorem applied to a certain evolution equation (1.12) and the reader is referred to Theorem 1.4 below for a stronger and more precise statement. It has been pointed out to us by Claude LeBrun that the same result can be proved by twistor-theoretic considerations [**15**, Theorem 3.6.I] using embeddability results for real-analytic CR manifolds.

Suppose now that (X, ω) is a compact hyperkähler manifold with boundary $\partial X = Y$ and inclusion map $\iota : Y \subset X$. As before, $\gamma = \iota^*\omega$ is a closed framing of Λ^2_Y and our boundary value problem is to determine which small deformations of γ as a closed framing of Λ^2_Y arise as boundary values of hyperkähler deformations of ω.

There is a well-known analogue of this problem in the context of self-dual and anti-self-dual (ASD) Einstein metrics of negative scalar curvature. Here one is given a conformally compact ASD Einstein metric g on the interior of X with conformal infinity h, and the problem is to determine which small deformations of h arise as boundary values of conformally compact ASD Einstein metrics near g. If X is the 4-ball, and g is the hyperbolic metric, we are in the realm of LeBrun's positive frequency conjecture [16], proved by Biquard in [5]. We shall see that in the present context of hyperkähler metrics, there is a similar positive frequency condition that needs to be satisfied by deformations of the closed framing γ if they are to arise as boundary values of hyperkähler deformations of ω.

1.1. Overview of results

1.1.1. Formal picture. We begin with a formal, nontechnical discussion of the set-up in order to motivate the statements of our main theorems. Let

$$\mathscr{F} = \{\omega = (\omega_1, \omega_2, \omega_3) \in \Omega^2(X) \otimes \mathbb{R}^3 : \omega_1^2 + \omega_2^2 + \omega_3^2 > 0\}. \tag{1.3}$$

(Unless indicated otherwise, all functions, metrics, forms, and so forth on X are smooth up to (and including) the boundary of X.) The 'gauge group' $\mathscr{G}_0$ of orientation-preserving diffeomorphisms of X which are the identity on ∂X acts on $\mathscr{F}$ by pull-back.

The space of hyperkähler triples $\mathscr{H}$ on X is defined as the subset

$$\mathscr{H} = \{\omega \in \mathscr{F} : \mathrm{d}\omega = 0,\ Q(\omega) = 0\}, \tag{1.4}$$

where Q is the 3×3 matrix whose entries are

$$Q(\omega)_{ij} = \frac{\omega_i \wedge \omega_j}{(1/3)(\omega_1^2 + \omega_2^2 + \omega_3^2)} - \delta_{ij}. \tag{1.5}$$

This follows by combining (1.1) and (1.2).

By linearizing Q at a hyperkähler triple ω, we see that

$$T_\omega\mathscr{H} = \{(\theta_1, \theta_2, \theta_3) \in \Omega^2(X) \otimes \mathbb{R}^3 : \mathrm{d}\theta_i = 0 \text{ and } s_0^2(\theta_i, \omega_j) = 0\}, \tag{1.6}$$

where $(\cdot, \cdot)$ denotes the pointwise inner product on 2-forms and $s_0^2(A)$ denotes the symmetric trace-free part of the matrix A.

The group $\mathscr{G}_0$ of diffeomorphisms acts on $\mathscr{H}$ so we would like to construct the moduli space $\mathscr{M} = \mathscr{H}/\mathscr{G}_0$, and in particular to understand its local structure.

Continuing formally, we note that the tangent space to the $\mathscr{G}_0$-orbit through ω is the infinite-dimensional space

$$\mathscr{B}_\omega = \{\mathscr{L}_v\omega : v \in C^\infty(X, TX),\ v \text{ vanishes on } \partial X\}.$$

If ω is a closed triple, $\mathscr{L}_v\omega = \mathrm{d}(\iota_v\omega)$ by Cartan's formula. Therefore, we define

$$L : C^\infty(X, TX) \to \Omega^2(X) \otimes \mathbb{R}^3, \quad Lv = \mathrm{d}(\iota_v\omega). \tag{1.7}$$

Denote by L^* the formal adjoint of L. This operator

$$L^* : \Omega^2(X) \otimes \mathbb{R}^3 \longrightarrow C^\infty(X, TX) \tag{1.8}$$

is a composite

$$\Omega^2(X) \otimes \mathbb{R}^3 \xrightarrow{\mathrm{d}^*} \Omega^1(X) \otimes \mathbb{R}^3 \xrightarrow{\pi} \Omega^1(X) \tag{1.9}$$

where π is zeroth order, that is given by a linear bundle map.

Continuing formally, we should expect

$$T_{[\omega]}\mathscr{M} = T_\omega\mathscr{H}/\mathscr{B}_\omega = T_\omega\mathscr{H} \cap \ker L^* \tag{1.10}$$

and indeed we shall establish a suitable version of this statement in Sections 2 and 3.

Written this way, the infinitesimal geometry of the moduli space is not so clear. However, there are two subspaces of (1.10) that can be written down by hand, and it turns out that together these give all infinitesimal deformations.

The first family consists of triples $\theta \in \mathscr{Z}^2_-(X) \otimes \mathbb{R}^3$ of closed anti-self-dual (ASD) 2-forms. Such a triple clearly satisfies the conditions (1.6): it is closed by hypothesis, and satisfies $(\theta_i, \omega_j) = 0$ because the self-dual and anti-self-dual 2-forms are pointwise orthogonal. Being closed and ASD also means that θ is coclosed, so it also lies in the kernel of L^* by (1.8) and (1.9).

The second family of deformations comes from diffeomorphisms which 'move the boundary'. More precisely, suppose that $X \subset (X', \omega')$ is a domain in a larger hyperkähler 4-manifold and that ω is the restriction of the hyperkähler triple ω' from X'. Given a smooth map $f : X \to X'$ which is a diffeomorphism onto its image, $f^*(\omega')$ gives a new hyperkähler triple on X. On the infinitesimal level, these give deformations of the form $\theta = \mathscr{L}_v\omega = L(v)$ where v is any vector field on X, not necessarily vanishing along ∂X. By naturality of the equations, θ automatically lies in $T_\omega\mathscr{H}$; if we choose v so that $L^*Lv = 0$, then θ will also lie in the 'gauge-fixed' tangent space (1.10). In other words, if we define

$$\mathscr{W} = \{v \in C^\infty(X, TX) : L^*Lv = 0\},$$

then $L(\mathscr{W}) \subset T_\omega\mathscr{H} \cap \ker L^*$.

Observe that (in contrast to the case where X is a compact manifold without boundary) both of these families of deformations are infinite-dimensional (see Section 5).

1.1.2. Local structure of $\mathscr{M}$. Our first result makes the above statements precise, giving Banach manifold structures on the infinite-dimensional spaces we have discussed. Denote by H^s the Sobolev space of functions with s derivatives in L^2. We shall fix $s > 4$ so that our Sobolev functions have good multiplicative properties. Then we have 'finite-regularity' versions of the above spaces: $\mathscr{F}^s$ is the space of triples of 2-forms with coefficients in H^s, $\mathscr{G}_0^{s+1}$ is the group of orientation-preserving diffeomorphisms which are the identity on ∂X given by mappings in H^{s+1} and so on. In particular, $\mathscr{G}_0^{s+1}$ acts on $\mathscr{H}^s$ and leads to the regularity-s moduli space $\mathscr{M}^s = \mathscr{H}^s/\mathscr{G}_0^{s+1}$. Our first result gives information about the local structure of $\mathscr{M}^s$ at a point $[\omega]$ represented by a *smooth* triple ω.

THEOREM 1.1. *Let ω be a smooth hyperkähler triple on X and fix $s > 4$.*

(i) *(Slice theorem.) A neighbourhood of $[\omega]$ in $\mathscr{F}^s/\mathscr{G}_0^{s+1}$ is homeomorphic to a neighbourhood of 0 in*

$$T_\omega \mathscr{F}^s \cap \ker L^* = H^s(X, \Lambda^2 \otimes \mathbb{R}^3) \cap \ker L^*.$$

(ii) *A neighbourhood of $[\omega]$ in $\mathscr{M}^s = \mathscr{H}^s/\mathscr{G}_0^{s+1}$ is homeomorphic to a neighbourhood of 0 in*

$$[\mathscr{Z}_-^2(X) \otimes \mathbb{R}^3 \cap H^s] + L(\mathscr{W}^{s+1}) \subset H^s(X, \Lambda^2 \otimes \mathbb{R}^3). \tag{1.11}$$

Here $H^s(X, E)$ denotes the space of sections of the bundle E with coefficients in H^s and $\mathscr{Z}_-^2(X)$ is the space of closed anti-self-dual 2-forms on X.

In particular, for each (sufficiently high) degree of regularity s, each smooth point $[\omega]$ of $\mathscr{M}^s$ has a neighbourhood which is homeomorphic to an open subset of a Banach space. Note, however, that triples $\widehat{\omega}$ near ω in the slice $L^*(\widehat{\omega} - \omega) = 0$ are all smooth in the interior of X.

1.1.3. The results of Bryant and Cartan. Bryant's 'thickening' result mentioned above suggests the formulation of a natural boundary value problem for hyperkähler metrics. Namely:

PROBLEM 1.2. Given a closed framing γ of Λ_Y^2, does there exist a hyperkähler triple ω on X with $\iota^*\omega = \gamma$?

We consider, and give an answer to, the following easier question:

PROBLEM 1.3. Given a hyperkähler triple ω on X, with induced boundary framing γ of Λ_Y^2, which nearby closed framings of Λ_Y^2 also arise from hyperkähler triples on X?

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Before describing our results, let us discuss the relation between these boundary value problems and the thickening result of Bryant (which he attributes to Cartan). A statement of this result is as follows:

THEOREM 1.4 (Bryant, [6, Theorem 1]). *Let Y be a closed 3-manifold. Given a closed framing γ of Λ^2_Y which is real-analytic for some analytic structure on Y there is an essentially unique embedding $f : Y \to (X, \omega)$ into a hyperkähler 4-manifold for which $f^*\omega = \gamma$. Here, 'essentially unique' means that if $f' : Y \to (X', \omega')$ is another such embedding then there is an isometry φ from a neighbourhood of $f(Y) \subset X$ to a neighbourhood of $f'(Y) \subset X'$ such that $f' = \varphi \circ f$.*

REMARK 1.5. In [7], Cartan states that such hyperkähler metrics depend, locally and modulo diffeomorphism, on two functions of three variables. This is obvious from the above description. Each closed 2-form is locally determined by 2 functions of 3 variables, making 6 functions of 3 variables in all. Dividing by (ambient) diffeomorphisms reduces this by 4 functions of 3 variables, leaving two functions of 3 variables.

A closed framing γ of Λ^2_Y determines an induced metric on Y, which is fixed by the requirement that γ be an *orthonormal* frame. To see that such a metric exists and is unique, plugging γ into the canonical isomorphism

$$\Lambda^2_Y = TY \otimes \Lambda^3_Y$$

gives us a framing $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of $TY \otimes \Lambda^3_Y$. Hence

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \in \Lambda^3 TY \otimes (\Lambda^3_Y)^3 = (\Lambda^3_Y)^2$$

is nonzero at all points of Y. Since $(\Lambda^3_Y)^2$ is a trivial real line-bundle, there is a volume form $\mathrm{d}\mu_Y$ and a (unique) choice of sign such that $(\mathrm{d}\mu_Y)^2 = \pm\alpha_1 \wedge \alpha_2 \wedge \alpha_3$. Using this volume form to trivialize Λ^3_Y we may now declare the α_i to be orthonormal and we have our induced metric.

However, a closed framing of Λ^2_Y is strictly more information than a metric. We can see this explicitly through the Eguchi–Hanson and Taub–NUT spaces. These provide a pair of nonisometric hyperkähler spaces X and X' in which we can find isometric hypersurfaces Y and Y'. (It is enough to pick Y and Y' to be suitable SU(2)-orbits—further details are provided in Section 6, where we shall see that the induced framings are different.)

In general, a given metric on Y can admit many different closed orthonormal framings of Λ^2_Y. Starting from one such framing γ, the others are of the form

$r(\gamma)$ where $r : Y \to \mathrm{SO}(3)$ is such that $\mathrm{d}(r(\gamma)) = 0$. This is an underdetermined equation with an infinite-dimensional space of solutions. (On the infinitesimal level, it is analogous to prescribing the curl of a vector field on a 3-manifold.) At least when r is nonconstant, Bryant's theorem applied to these different framings gives *nonisometric* hyperkähler extensions of the same metric on Y (assuming the analyticity hypotheses are met).

These remarks are intended to explain that one's first guess, that the boundary value of a hyperkähler metric should be the induced metric on the boundary, leads to an ill-posed boundary problem: the above discussion shows that the same metric can bound nonisometric hyperkähler structures, whereas Theorem 1.4 shows that the formulation in terms of framings does not suffer from this problem.

As an aside, we have a further interpretation of a hyperkähler extension of a closed framing γ of Λ^2_Y as an evolution equation, sometimes called an 'SU(2)-flow'. (Though this is not in any reasonable sense a parabolic equation.) If γ_t is a 1-parameter family of closed framings of Λ^2_Y then $\omega = \mathrm{d}t \wedge *_t\gamma_t + \gamma_t$, where $*_t$ is the Hodge star on Y determined by γ_t, is a hyperkähler triple on $I \times Y$ for some interval $I \subseteq \mathbb{R}$ if and only if

$$\frac{\partial}{\partial t}\gamma_t = \mathrm{d} *_t \gamma_t. \tag{1.12}$$

This is sometimes interpreted as an evolution equation with $\gamma_0 = \gamma$, which evolves amongst closed framings. Thus we can also view our study of the boundary value problem as identifying initial conditions for which the evolution equation has good existence and convergence properties.

1.1.4. Boundary value problem for hyperkähler metrics. We shall now describe what we can say about Problem 1.3.

As first order systems of equations are involved, it is not reasonable to expect that all such nearby closed framings on Y will bound hyperkähler triples on X; instead, only those satisfying a *negative frequency condition* will do so. This is motivated by experience with boundary value problems for Dirac operators, the prototypical example of which is the $\bar\partial$-operator on the disc. In this case, a given function $f : S^1 \to \mathbb{C}$ is only the boundary value of a function u in the disc with $\bar\partial u = 0$ if all negative Fourier coefficients of f vanish.

Our results are cleanest in the case that ω induces *positive mean curvature* on Y. It is worth noting that if ω is a hyperkähler triple on X and $\gamma = \iota^*\omega$, the trace of the matrix of inner products $(\gamma_i, \mathrm{d} * \gamma_j)$ is twice the mean curvature of Y in X (in fact, this matrix encodes the second fundamental form of Y). Denote by $\mathscr{M}^s_+$ the submoduli space of such hyperkähler structures on X. For simplicity, suppose also that $H^1(Y) = 0$.

The relevant Dirac operator for the hyperkähler problem turns out to be

$$D : \Omega^1(X) \longrightarrow \Omega^0(X) \oplus \Omega^2_+(X), \quad D = \mathrm{d}^* + \mathrm{d}_+. \tag{1.13}$$

On Y, with its induced Riemannian metric, we have the operator

$$D_Y : \Omega^0(Y) \oplus \Omega^1(Y) \longrightarrow \Omega^0(Y) \oplus \Omega^1(Y), \quad D_Y = \begin{bmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & *\mathrm{d} \end{bmatrix}. \tag{1.14}$$

This is a self-adjoint operator of Dirac type, so if we define

$$H_\lambda = \ker(D_Y - \lambda), \tag{1.15}$$

then $\dim H_\lambda < \infty$ and the set of λ for which $H_\lambda \neq 0$ is discrete and unbounded in both directions. Define further

$$G_\lambda = H_\lambda \cap \ker(\mathrm{d}^*). \tag{1.16}$$

Clearly the set of λ with $G_\lambda \neq 0$ is also discrete, $\dim(G_\lambda) < \infty$ and it can also be shown that the set of λ with $G_\lambda \neq 0$ is unbounded in both directions. (We thank Dmitri Vassiliev for useful discussions on this point.)

For any given $s > 1/2$, define $H_+^{s-1/2}(Y)$ to be the completion of $\bigoplus_{\lambda>0} H_\lambda$ with respect to the $H^{s-1/2}$-norm and $G_-^{s-1/2}(Y)$ to be the completion of $\bigoplus_{\lambda<0} G_\lambda$ with respect to the same norm. Then we have the following result.

THEOREM 1.6. *Let ω be a smooth hyperkähler triple on X inducing positive mean curvature on Y. Then the gauge-fixed tangent space*

$$T_{[\omega]}\mathscr{M}^s_+ = T_\omega \mathscr{H}^s \cap \ker L^* \tag{1.17}$$

is naturally isomorphic to the direct sum

$$\mathscr{H}^2_{0,-}(X) \oplus \mathscr{H}^2_-(X) \oplus G_-^{s+1/2}(Y) \otimes \mathbb{R}^3 \oplus H_+^{s+1/2}(Y), \tag{1.18}$$

where $\mathscr{H}^2_{0,-}(X)$ and $\mathscr{H}^2_-(X)$ are certain finite-dimensional spaces of closed anti-self-dual 2-forms whose dimensions depend only on the cohomologies of X, Y and (X, Y).

The spaces $\mathscr{H}^2_-(X)$ and $\mathscr{H}^2_{0,-}(X)$ are defined in (5.30) and (5.32) before Theorem 5.12. Up to these finite-dimensional topological pieces, we have an effective parametrization of the tangent space of $\mathscr{M}^s_+$ in terms of boundary data. In a little more detail, $G_-^{s+1/2}(Y)$ gives a parametrization of the boundary values of the *exact* ASD 2-forms on X: given $\alpha \in G_-^{s+1/2}(Y)$, we find a unique

$u \in H^{s+1}(X, \Lambda^1)$ such that $Du = 0$; then $\mathrm{d}u \in H^s(X, \Lambda^2_-)$ is an exact ASD 2-form. Similarly if $\beta \in H^{s+1/2}_+(Y)$, there is a unique solution $w \in H^{s+1}$ of $L^*Lw = 0$ with boundary value β. This existence and uniqueness of w is true without the frequency condition, but the point is that with this frequency condition imposed, Lw cannot be a triple of ASD 2-forms and so this restriction allows us to replace the sum in Theorem 1.1(ii) by a direct sum.

Theorem 1.6 is in agreement with Cartan's count of degrees of freedom in the local moduli space of hyperkähler metrics (Remark 1.5) in the following sense. Discarding the finite-dimensional spaces in (1.18), the space of negative frequency coclosed 1-forms should be counted as 2 negative frequency functions on Y (that is of 3 variables). We have a triple of such coclosed 1-forms, so 6 negative frequency functions. On the other hand $H^{s+1/2}_+(Y)$ contributes 4 positive frequency functions of 3 variables. Since diffeomorphisms which move the boundary are given by 4 full functions of 3 variables, subtracting this leaves us with just 2 negative frequency functions on Y. This agrees with Cartan's count: for a global boundary value problem, one can only prescribe 'half the data' on the boundary as compared with the thickening problem.

Another way of stating (1.18) is that we have a direct sum decomposition

$$T_{[\omega]}\mathscr{M}^s_+ = [\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s] \oplus L(\mathscr{W}^{s+1}_+),$$

where we introduce the obvious notation $\mathscr{W}^{s+1}_+$ for elements of $\mathscr{W}^{s+1}$ with positive frequency boundary values. This improves the description of this tangent space in (1.11) and allows us to prove the following refinement of Theorem 1.1.

THEOREM 1.7. *The moduli space $\mathscr{M}_+$ of hyperkähler triples on X inducing positive mean curvature on the boundary Y, modulo the action of $\mathscr{G}_0$, is a Fréchet manifold with*

$$T_{[\omega]}\mathscr{M}_+ = [\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3] \oplus L(\mathscr{W}_+).$$

REMARK 1.8. The point here is that while the space on the RHS depends on ω, these spaces are canonically isomorphic for triples ω and ω' in the same path component of $\mathscr{M}_+$.

1.1.5. Examples. An immediate source of examples of hyperkähler 4-manifolds with boundary to which we may apply our theory arises from taking balls (or other bounded domains) in gravitational instantons, by which we mean complete hyperkähler 4-manifolds. The simplest examples are those with an isometric SU(2)-action, and we relate our approach to the standard classification of these spaces in Section 6 and use them to illustrate our main results.

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We start our discussion with the flat metric on $\mathbb{R}^4$. There are two isometric actions of SU(2) on $\mathbb{R}^4$ (corresponding to left and right multiplication by unit quaternions). Consider first the SU(2) action which rotates the hyperkähler triple. This gives rise to a standard framing of S^3 by closed 2-forms. On the other hand, the Taub–NUT metric is a (nonflat) complete hyperkähler metric on $\mathbb{R}^4$ with SU(2)-action which rotates the triple, and depending on a parameter $m > 0$. In Section 6 we shall find a sphere in Taub–NUT such that the induced framing is within $O(m^{-2})$ of the standard round framing of S^3, and we shall be able to verify explicitly that the difference between these framings is negative frequency; this is consistent with Taub–NUT being a small deformation, over this ball, of the flat metric.

Playing a similar game (now choosing the other SU(2) action on $\mathbb{R}^4$), by comparing the flat metric with Eguchi–Hanson, we are also able to construct explicit positive frequency deformations of the induced framing; they cannot be filled by hyperkähler metrics over the ball precisely because the Eguchi–Hanson metric lives on the 4-manifold T^*S^2, and this obstructs the problem of filling this particular positive frequency deformation. Put another way, this provides an initial condition on S^3 for the hyperkähler evolution equation (1.12) which develops a singularity.

1.2. Contents. We begin in Section 2 with the proof of part (i) of Theorem 1.1, showing, roughly speaking, that $\ker(L^*)$ gives a transverse slice to the action of the diffeomorphism group $\mathscr{G}_0$. The rest of Theorem 1.1 is proved in Sections 3 and 4.

Our results on the moduli space $\mathscr{M}_+$, Theorems 1.6 and 1.7 follow in Section 5.5. Finally, in Section 6, we turn to concrete examples, coming from gravitational instantons with isometric SU(2)-actions.

2. Gauge fixing for the action of diffeomorphisms

We begin by recalling some fundamental facts about the group of diffeomorphisms, setting up notation along the way. (Details can be found in [**10**, **11**].) Let X be a compact 4-manifold with boundary Y and let $\omega = (\omega_1, \omega_2, \omega_3)$ be a hyperkähler triple on X. We write $\mathscr{F} = \Omega^2(X) \otimes \mathbb{R}^3$ for the space of triples of 2-forms on X and $\mathscr{G}_0$ for the group of orientation-preserving diffeomorphisms of X which are the identity on the boundary Y. The group $\mathscr{G}_0$ acts on $\mathscr{F}$ by pull-back. The goal in this section is to find a slice for the action.

To do so we work with Hilbert space completions of $\mathscr{F}$ and $\mathscr{G}_0$. Given $s \geqslant 0$, we write $\mathscr{F}^s$ for the Hilbert space of triples of 2-forms whose distributional derivatives are square-integrable up to order s. We can also talk of H^s-maps

$X \to X$. When $s > 3$, H^s embeds into C^1 and so an H^s-diffeomorphism makes sense: we write $\mathscr{G}_0^s$ for the collection of orientation-preserving H^s-maps $X \to X$ which are also C^1-diffeomorphisms and which restrict to the identity on the boundary. The inverse of $\varphi \in \mathscr{G}_0^s$ is again of regularity H^s and the composition of H^s-diffeomorphisms is in H^s. This makes $\mathscr{G}_0^s$ into a topological group.

We next briefly recall the Banach manifold structure on $\mathscr{G}_0^s$. Fix a Riemannian metric on X. (This metric is purely auxiliary and the manifold structure on $\mathscr{G}_0$ turns out not to depend on this choice.) Write $\mathscr{V}_0^s$ for the space of H^s vector fields which vanish on the boundary. The geodesic exponential map is not defined on all tangent vectors, since geodesics can leave through the boundary. However, if $v \in \mathscr{V}_0^s$ is sufficiently small in H^s, with $s > 3$, then it is also bounded by $1/2$ say, in C^1. This means that for any $p \in X$, $|v(p)| \leqslant (1/2)d$ where d is the distance of p from Y. It follows that $\exp(v(p))$ still lies in X and hence that exponentiating vectors to geodesics defines a map from a small neighbourhood of the origin in $\mathscr{V}_0^s$ to $\mathscr{G}_0^s$. One can check that this map is a homeomorphism onto a neighbourhood of the identity giving us a chart there. To define a chart near another point $\varphi \in \mathscr{G}_0^s$, we repeat the same construction, using H^s-sections of $\varphi^* TX$. One then checks that these charts have smooth transition functions making $\mathscr{G}_0^s$ into a Banach manifold. We refer to [10, 11] for details. It is also shown there that right multiplication by a fixed diffeomorphism $\varphi \in \mathscr{G}_0^s$ gives a smooth map $\mathscr{G}_0^s \to \mathscr{G}_0^s$; this will be important in what follows. (On the other hand, left multiplication is merely continuous, so that $\mathscr{G}_0^s$ is a topological group, but not strictly speaking a Banach Lie group.)

The group $\mathscr{G}_0^{s+1}$ acts on $\mathscr{F}^s$ by pull-back. To find a slice for this action we follow the standard approach of taking the orthogonal complement to the infinitesimal group action. Given a vector field v on X which vanishes on Y, the infinitesimal action of v on $\mathscr{F}$ at ω is by Lie derivative, $\mathscr{L}_v\omega$. We write L for the map defined by $Lv = \mathscr{L}_v\omega$ and L^* for its formal adjoint. The main aim of this section is to prove the following, for which it is crucial that ω is smooth.

THEOREM 2.1. *Fix $s > 4$. There exist constants $\epsilon, \delta > 0$ such that for every $\hat{\omega} \in \mathscr{F}^s(X)$ with $\|\hat{\omega} - \omega\|_{H^s} < \epsilon$ there exists a unique diffeomorphism $\varphi \in \mathscr{G}_0^{s+1}$ such that both $L^*(\varphi^*\hat{\omega} - \omega) = 0$ and $\|\varphi^*\hat{\omega} - \omega\|_{H^s} < \delta$. In other words, the slice*

$$S_\delta = \{\omega + \chi : \chi \in \mathscr{F}^s,\ L^*\chi = 0,\ \|\chi\|_{H^s} < \delta\}$$

meets every nearby orbit of $\mathscr{G}_0^{s+1}$ exactly once.

This is the analogue of the Ebin–Palais slice theorem (see for example [19]), which applies to diffeomorphisms acting on metrics (on a compact manifold without boundary) and we follow the outline of their proof closely. Accordingly, we omit the parts of the proof which are identical to Ebin–Palais.

Theorem 2.1 contains two assertions: existence and uniqueness. We begin with existence. The idea is to employ the implicit function theorem, but there is a subtlety because the action

$$\mathscr{G}_0^{s+1} \times \mathscr{F}^s \to \mathscr{F}^s$$

is *not* smooth. To see this note that the derivative at the point $(\mathrm{id}, \hat{\omega})$ should be given by the Lie derivative $\mathscr{L}_v\hat{\omega}$. But if $\hat{\omega} \in H^s$ then $\mathscr{L}_v\hat{\omega} \in H^{s-1}$ which is of insufficient regularity. *The linearized action differentiates $\hat{\omega}$ whereas the full action does not.* This means that the existence part of the slice theorem does not follow from a simple application of the implicit function theorem (in spite of what one sometimes reads!) since this would need the action to be C^1.

Instead, following Ebin, we proceed as follows. Write $\mathscr{E}^s$ for the following subset of $\mathscr{G}_0^{s+1} \times \mathscr{F}^s$:

$$\mathscr{E}^s = \{(\varphi, \chi) : L^*((\varphi^{-1})^*\chi) = 0\}. \tag{2.1}$$

There is a map $\mathscr{E}^s \to \mathscr{F}^s$ given by $F(\varphi, \chi) = \varphi^*\omega + \chi$. The idea is to apply the inverse function theorem to F, to show that it is a diffeomorphism between a neighbourhood of $(\mathrm{id}, 0) \in \mathscr{E}^s$ and a neighbourhood of $\omega \in \mathscr{F}^s$. Once this is done, it will follow that any $\hat{\omega}$ which is H^s-close to ω is of the form $\hat{\omega} = \varphi^*\omega + \chi$ for $(\varphi^{-1})^*\chi \in \ker L^*$. Then $(\varphi^{-1})^*\hat{\omega} - \omega \in \ker L^*$ as required. One should think of $\mathscr{E}^s$ as the normal bundle of the orbit of ω (pulled back to $\mathscr{G}_0^{s+1}$) and F as giving a tubular neighbourhood of the orbit.

To push this argument through, there are three things which must be established:

- $\mathscr{E}^s$ is a Banach manifold. More precisely, we show that $\mathscr{E}^s \to \mathscr{G}_0^{s+1}$ is a Banach vector bundle.
- The map F is smooth.
- $\mathrm{d}F$ at $(\mathrm{id}, 0)$ is an isomorphism.

We now explain how this works. The first step is to note that, whilst the whole action is not smooth, its restriction through a C^∞ triple does give a smooth map. The proof is identical to that in Ebin–Palais, and so we omit it.

LEMMA 2.2. *Let $s > 3$. For a triple of smooth 2-forms ω, the map $A : \mathscr{G}_0^{s+1} \to \mathscr{F}^s$ given by $A(\varphi) = \varphi^*\omega$ is a smooth map of Banach manifolds.*

We next explain why $\mathscr{E}^s$, defined above in (2.1), is a smooth vector bundle over $\mathscr{G}_0^{s+1}$. To do this (and still following Ebin) we define two operators. The first, $L_\varphi : \mathscr{V}_0^{s+1} \to \mathscr{F}^s$, is defined as follows. We use right multiplication to give a smooth trivialization of the tangent bundle $T\mathscr{G}_0^{s+1} \cong \mathscr{G}_0^{s+1} \times \mathscr{V}_0^{s+1}$. Then we set $L_\varphi = \mathrm{d}A_\varphi$ to be the derivative of the action $A : \mathscr{G}_0^{s+1} \to \mathscr{F}^s$ at $\varphi \in \mathscr{G}_0^{s+1}$ with respect to this trivialization.

The second operator, $L_\varphi^* : \mathscr{F}^s \to \mathscr{V}^{s-1}$, is defined to be the formal L^2-adjoint of $L_\varphi : \mathscr{V}_0^{s+1} \to \mathscr{F}^s$, defined with respect to the metric determined by the hyperkähler triple $\varphi^*\omega$. In particular, $L_{\mathrm{id}}^* = L^*$ is precisely the operator appearing in the statement of the Slice Theorem 2.1. (Note that, as is standard with boundary value problems, whilst L_φ is defined on vector fields vanishing on the boundary, its adjoint L_φ^* will in general take values in arbitrary vector fields.)

We need two results concerning these operators. Again the proofs are identical to those in Ebin–Palais and so we omit them.

Lemma 2.3. *We have the following formulae for L_φ and L_φ^*:*

$$L_\varphi = \varphi^* \circ L \circ \varphi_*^{-1}, \quad L_\varphi^* = \varphi_* \circ L^* \circ (\varphi^{-1})^*.$$

In particular, $\ker L_\varphi^* = \varphi^* \ker L_{\mathrm{id}}^*$.

Lemma 2.4. *Let $s > 3$. The operators $L_\varphi \in B(\mathscr{V}_0^{s+1}, \mathscr{F}^s)$ and $L_\varphi^* \in B(\mathscr{F}^s, \mathscr{V}^{s-1})$ depend smoothly on $\varphi \in \mathscr{G}_0^{s+1}$.*

So, by Lemmas 2.3 and 2.4, $\mathscr{E}^s$ is the kernel of the smooth bundle map $\mathscr{G}_0^{s+1} \times \mathscr{F}^s \to \mathscr{V}^{s-1}$, given by $(\varphi, \chi) \mapsto L_\varphi^*(\chi)$. This alone is not enough to ensure that $\mathscr{E}^s$ is a vector bundle (even for finite rank bundles this can fail, since the dimension of the kernel can jump). To prove that $\mathscr{E}^s$ is a Banach vector bundle we use the following result. This is standard and so we do not give the proof.

Lemma 2.5. *Let V be a Banach space and M a Banach manifold. Suppose for each $x \in M$ we have a projection $P_x : V \to V$, so that the map*

$$M \to B(V), \quad x \mapsto P_x$$

is smooth. Then $\operatorname{im} P := \bigcup_{x\in M} \operatorname{im}(P_x)$ *and* $\ker P := \bigcup_{x\in M} \ker(P_x)$ *are smooth subbundles of $M \times V$ and there is a splitting $M \times V = \operatorname{im}(P) \oplus \ker(P)$.*

We ultimately apply Lemma 2.5 to the projection onto $\ker L_\varphi^*$ along $\operatorname{im} L_\varphi$. At this point some parts of the proofs are minor modifications of those in Ebin–Palais and so we give the details. The first step is to show that $L = L_{\mathrm{id}}$ is invertible.

LEMMA 2.6. *The map* $L : \mathscr{V}_0^{s+1} \to \mathscr{F}^s$ *is injective.*

Proof. If $Lv = 0$, then v is a Killing field for the metric defined by ω, but v vanishes on Y and so must also vanish on X. □

We next need a concise formula for L^*. We start with the adjoint of the map $TX \to T^*X \otimes \mathbb{R}^3$ given by $v \mapsto \iota_v\omega$. The hyperkähler triple $\omega = (\omega_1, \omega_2, \omega_3)$ determines a triple of complex structures $J = (J_1, J_2, J_3)$ via the requirement that $g(J_j u, v) = \omega_j(u, v)$ for $j = 1, 2, 3$. Using these we define a map $T^*X \otimes \mathbb{R}^3 \to T^*X$ by

$$a = (a_1, a_2, a_3) \mapsto J \cdot a := J_1 a_1 + J_2 a_2 + J_3 a_3. \tag{2.2}$$

The following lemma is the result of a simple calculation.

LEMMA 2.7. *The map* $a \mapsto (J \cdot a)^\sharp$ *is adjoint to* $v \mapsto \iota_v\omega$ *(where* $\alpha^\sharp$ *is the vector metric-dual to the 1-form* α*).*

LEMMA 2.8. *We have* $L^*\eta = (J \cdot \mathrm{d}^*\eta)^\sharp$. *In other words, if* $v \in \mathscr{V}_0$ *and* $\eta \in \mathscr{F}$ *then*

$$\langle Lv, \eta\rangle_{L^2} = \langle v, (J \cdot \mathrm{d}^*\eta)^\sharp\rangle_{L^2}.$$

Proof. This is a direct calculation. Suppose that v is a smooth vector field on X vanishing on the boundary Y. Then

$$\langle Lv, \eta\rangle_{L^2} = \int_X \mathrm{d}(i_v\omega)\wedge{*\eta} = \int_Y i_v\omega\wedge{*\eta} + \int_X i_v\omega\wedge \mathrm{d}{*\eta} = \langle v, (J\cdot \mathrm{d}^*\eta)^\sharp\rangle_{L^2} \tag{2.3}$$

where we have used that $\mathscr{L}_v\omega = \mathrm{d}(i_v\omega)$ (since $\mathrm{d}\omega = 0$) and that the boundary integral vanishes since v is zero on Y. □

We now prove a Hodge decomposition for L and L^*.

PROPOSITION 2.9.

(1) *The map* $L^*L : \mathscr{V}_0^{s+1} \to \mathscr{V}^{s-1}$ *is an isomorphism.*

(2) *There is a splitting* $\mathscr{F}^s = \operatorname{im} L \oplus \ker L^*$.

(3) *The map* $P = L \circ (L^*L)^{-1} \circ L^*$ *is the projection onto* $\operatorname{im} L$ *with respect to this splitting.*

Proof. We start with point 1. The operator L is a first order differential operator with symbol

$$\sigma_L(\xi, v) = \xi \wedge i_v\omega,$$

which is injective. In fact, since $i_v\omega_i = (J_i v)^\flat$ form an orthogonal triple of 1-forms, we have

$$|\sigma_L(\xi, v)|^2 = \sum_{j=1}^{3} |\xi|^2|J_j v|^2 - \langle \xi, i_v\omega_j\rangle^2 = 3|\xi|^2|v|^2 - \sum_{j=1}^{3}\langle \xi, i_v\omega_j\rangle^2 \geqslant 2|\xi|^2|v|^2.$$

Thus $\langle \sigma_{L^*L}(\xi, v), v\rangle \geqslant 2|\xi|^2|v|^2$, meaning L^*L is a strongly elliptic operator in the sense of [18, Ch. 5 (11.79)]. It follows that the Dirichlet problem for L^*L is regular in the sense of [18, page 454] by [18, Ch. 5 Proposition 11.10]. We may thus apply [18, Ch. 5 Proposition 11.16] to conclude that the map

$$\begin{aligned} \mathscr{V}^{s+1}(X) &\to \mathscr{V}^{s-1}(X) \oplus \mathscr{V}^{s+1/2}(Y) \\ v &\mapsto (L^*Lv, v|_Y) \end{aligned} \tag{2.4}$$

is Fredholm, where $\mathscr{V}(Y)$ denotes sections of $TX|_Y$.

Now if $v \in \mathscr{V}_0^{s+1}$ with $L^*Lv = 0$ then

$$0 = \langle L^*Lv, v\rangle_{L^2} = \|Lv\|_{L^2}^2.$$

Since L is injective (Lemma 2.6) it follows that L^*L is injective on $\mathscr{V}_0^{s+1}$. The operator L^*L is formally self-adjoint, and from the formula in the proof of Lemma 2.8 $v|Y = 0$ is a self-adjoint boundary condition. Indeed, if $\eta = Lw$ in (2.3), for another vector field w, we obtain

$$\langle Lv, Lw\rangle_{L^2} - \langle v, L^*Lw\rangle_{L^2} = \int_Y i_v\omega \wedge *Lw$$

and so by skew symmetrizing,

$$\langle L^*Lv, w\rangle_{L^2} - \langle v, L^*Lw\rangle_{L^2} = \int_Y (i_v\omega \wedge *Lw - i_w\omega \wedge *Lv).$$

Since $v|Y = 0$ (as an element of $TX|Y$) implies that $\iota^*(\iota_v\omega) = 0$, we see that $v|Y = 0$ is a self-adjoint boundary condition for the operator L^*L on X. Hence the index of (2.4) is zero, which therefore means it is surjective as well, and so $L^*L : \mathscr{V}_0^{s+1} \to \mathscr{V}^{s-1}$ is an isomorphism.

We next turn to the splitting claimed in point 2. The sum is clearly direct because if $\eta = Lv$ for $v \in \mathscr{V}_0^{s+1}$ and $L^*\eta = 0$ then $L^*Lv = 0$. To show the sum spans, let $\eta \in \mathscr{F}^s$; we must find $v \in \mathscr{V}_0^{s+1}$ such that $\eta - Lv$ is in $\ker L^*$. This amounts to solving $L^*Lv = L^*\eta$ for $v \in \mathscr{V}_0^{s+1}$ which we can do by the surjectivity of (2.4).

Finally, for point 3, note $P^2 = P$, $P(Lv) = Lv$ and that P vanishes on $\ker L^*$. □

COROLLARY 2.10. *Let $s > 3$. For all $\varphi \in \mathscr{G}_0^{s+1}$, the following are true.*

(1) *The map $L_\varphi^* L_\varphi : \mathscr{V}_0^{s+1} \to \mathscr{V}^{s-1}$ is an isomorphism.*

(2) *There is a splitting $\mathscr{F}^s = \operatorname{im} L_\varphi \oplus \ker L_\varphi^*$.*

(3) *The map $P_\varphi = L_\varphi \circ (L_\varphi^* L_\varphi)^{-1} \circ L_\varphi^*$ is the projection onto* $\operatorname{im} L_\varphi$ *with respect to this splitting.*

Proof. Recall that Lemma 2.3 asserts that

$$L_\varphi = \varphi^* \circ L \circ \varphi_*^{-1}, \quad L_\varphi^* = \varphi_* \circ L^* \circ (\varphi^{-1})^*.$$

It follows that $L_\varphi^* L_\varphi = \varphi_* \circ L^* L \circ \varphi_*^{-1}$. Note that when decomposed in this fashion one must be careful to keep track of regularity since, for example, $\varphi_*^{-1} : \mathscr{V}_0^{s+1} \to \mathscr{V}_0^s$. Nonetheless all three maps in this composition are injective and so the same is true of $L_\varphi^* L_\varphi$. Similarly to solve the equation $L_\varphi^* L_\varphi(v) = w$, with $w \in \mathscr{V}^{s-1}$, one first solves $L^* L u = \varphi_*^{-1} w$ for $u \in \mathscr{V}_0^s$ (by invertibility of $L^* L : \mathscr{V}_0^s \to \mathscr{V}^{s-2}$) and then sets $v = \varphi_* u$. Now a priori $v \in \mathscr{V}_0^{s-1}$ but, since $L_\varphi^* L_\varphi(v) = w$, elliptic regularity ensures that $v \in \mathscr{V}_0^{s+1}$ after all, proving surjectivity of $L_\varphi^* L_\varphi$. The other points follow exactly as before in the proof of Proposition 2.9. □

We have now justified the three key points mentioned in the introduction, namely:

- $\mathscr{E}^s \to \mathscr{G}_0^{s+1}$ is a Banach vector bundle. This follows from Lemma 2.5 and the fact that the projection onto $\operatorname{im} L_\varphi$ depends smoothly on φ. This is because $P_\varphi = L_\varphi \circ (L_\varphi^* L_\varphi)^{-1} \circ L_\varphi^*$ and each operator in this composition is smooth in φ (by Lemma 2.4).
- The map $F : \mathscr{E}^s \to \mathscr{F}^s$ given by $F(\varphi, \chi) = \varphi^* \omega + \chi$ is smooth, by Lemma 2.2.
- Its derivative at $(\mathrm{id}, 0)$ is given by

$$\mathrm{d}F : \mathscr{V}_0^{s+1} \oplus \ker L^* \to \mathscr{F}^s, \quad \mathrm{d}F(v, \eta) = Lv + \eta$$

  which is an isomorphism, by Proposition 2.9.

The existence part of Theorem 2.1 now follows, just as in Ebin–Palais, by an application of the implicit function theorem. Accordingly, we state the result without writing the details.

THEOREM 2.11. *Fix $s > 3$. There are constants $C, \epsilon, \delta > 0$ and an open neighbourhood U of $\mathrm{id} \in \mathscr{G}_0^{s+1}$ such that if $\|\hat{\omega} - \omega\|_{H^s} < \epsilon$ then there exists a unique diffeomorphism $\varphi \in U$ such that both $L^*(\varphi^*\hat{\omega} - \omega) = 0$ and $\|\varphi^*\hat{\omega} - \omega\|_{H^s} < \delta$. Moreover, in this case $\|\varphi^*\hat{\omega} - \omega\|_{H^s} < C\|\hat{\omega} - \omega\|_{H^s}$.*

We next turn to uniqueness. The crux is to show that the action of $\mathscr{G}_0^{s+1}$ on $\mathscr{F}^s$ is proper, in a certain sense. Before we state this result, we need a preliminary definition.

DEFINITION 2.12. A triple of 2-forms $(\omega_1, \omega_2, \omega_3)$ on a 4-manifold is called *definite* if there is a nowhere vanishing 4-form Ω such that the 3×3-matrix-valued function $(\omega_i \wedge \omega_j)/\Omega$ is positive-definite.

A hyperkähler triple is an example of a definite triple. Note that definiteness is an open condition in $\mathscr{F}^s$, as long as s is large enough that Sobolev multiplication holds. Given any definite triple $(\omega_1, \omega_2, \omega_3)$ the wedge product is definite on $\langle \omega_1, \omega_2, \omega_3\rangle$ and hence there is a unique conformal class making the ω_j self-dual. One can then specify a metric in this conformal class by taking the volume form to be $\frac{1}{6}(\omega_1^2+\omega_2^2+\omega_3^2)$ (see (1.2)). In this way we canonically associate a Riemannian metric to every definite triple. (Of course, when the triple is hyperkähler, this metric is the obvious one.) For more details on definite triples, see [9].

We are now ready to prove that the action of $\mathscr{G}_0^{s+1}$ on $\mathscr{F}^s$ is proper, at least when restricted to definite triples.

THEOREM 2.13. *Fix $s > 4$. Let $\omega_n \in \mathscr{F}^s$ be a sequence of definite triples and $\varphi_n \in \mathscr{G}_0^{s+1}$ a sequence of diffeomorphisms. Suppose that ω_n converges in H^s to a definite triple ω and that $\varphi_n^*\omega_n$ converges in H^s to a definite triple $\hat{\omega}$. Then there is a subsequence of the φ_n which converges in $\mathscr{G}_0^{s+1}$ to a diffeomorphism φ. Moreover, $\varphi : (X, \hat{g}) \to (X, g)$ is an isometry where g and $\hat{g}$ are the Riemannian metrics associated to ω and $\hat{\omega}$, respectively.*

This is a direct analogue of—and follows immediately from—a theorem of Ebin–Palais for the action of diffeomorphisms on Riemannian metrics (see [19]). (As an aside, the lower bound on s is necessary for the proof of Ebin–Palais which uses Sobolev multiplication at a certain point.)

Proof. Write g_n, g and $\hat{g}$ for the Riemannian metrics corresponding to the definite triples ω_n, ω and $\hat{\omega}$, respectively. We have that $g_n \to g$ and $\varphi_n^* g_n \to \hat{g}$ in H^s. Now the result of Ebin–Palais gives a subsequence of the φ_n which converges in $\mathscr{G}_0^{s+1}$ to a diffeomorphism φ satisfying $\varphi^* g = \hat{g}$. □

The full Slice Theorem 2.1 now follows from Theorems 2.11 and 2.13, in identical fashion to Ebin–Palais's original slice theorem.

3. The hyperkähler equation modulo diffeomorphisms

The goal of this section is to gauge fix the hyperkähler equation in order to be able to apply elliptic theory. The main result is Theorem 3.13 below, which shows that the moduli space of all hyperkähler triples up to diffeomorphism is locally homeomorphic to the zero locus of a nonlinear operator with certain ellipticity properties.

There are complications in arriving at Theorem 3.13 which come from the fact that there are two competing notions of gauge. The first is the differential condition of the previous section, coming from the action of diffeomorphisms on 2-forms. This has the advantage that triples of 2-forms can always be put in 'differential gauge' by the Slice Theorem 2.1. It does not, however, lead to an elliptic equation. The other kind of gauge fixing arises when one parametrizes *cohomologous* triples of 2-forms by triples of 1-forms a via $\omega + \mathrm{d}a$. This leads naturally to an *algebraic* condition on a and with this gauge imposed the hyperkähler equation becomes genuinely elliptic. The problem, however, is that it is not in general possible to put a triple a in 'algebraic gauge' via the action of diffeomorphisms. The proof of Theorem 3.13 involves the interaction between these two notions of gauge.

3.1. A nonlinear Dirac equation. Our starting point is the following formulation of hyperkähler metrics in terms of triples of 2-forms (see [**9**, **14**]). The lemma is standard and accordingly we only sketch the proof.

LEMMA 3.1. *Let X be a 4-manifold and $(\omega_1, \omega_2, \omega_3)$ a triple of closed 2-forms on X. Suppose that*

$$\omega_i \wedge \omega_j = \delta_{ij}\mu$$

for some nowhere vanishing 4-form μ. Then X carries a hyperkähler metric g which is characterized by the fact that the ω_i are all self-dual and the volume form is given by $\mu = 2\mathrm{d}V_g$.

Sketch of proof. Since the wedge product is definite on the subbundle $\langle\omega_i\rangle$ of Λ^2 spanned by the forms ω_i, there is a unique conformal class for which the ω_i are all self-dual. Choosing $\mathrm{d}V_g = \mu/2$ determines a metric in this conformal class. The ω_i now give a metric trivialization $\Lambda^2_+ \cong X \times \mathbb{R}^3$ of the bundle of self-dual 2-forms. Under this identification, the product connection preserves the metric in Λ^2_+ and, since the ω_i are closed, it is also torsion-free. It follows that the product connection is identified with the Levi-Civita connection in Λ^2_+, which is thus flat with trivial holonomy; this is one characterization of a hyperkähler metric. $\square$

As mentioned in the introduction, μ can be recovered from the symplectic forms via $\mu = \frac{1}{3}\sum \omega_i^2$. This means that hyperkähler triples are exactly those triples of symplectic forms, all inducing the same orientation, solving the equation $Q(\omega) = 0$ where $Q(\omega)$ is the symmetric trace-free 3×3-matrix-valued function defined by

$$Q(\omega)_{ij} = \frac{\omega_i \wedge \omega_j}{(1/3)(\omega_1^2 + \omega_2^2 + \omega_3^2)} - \delta_{ij}. \tag{3.1}$$

Linearizing Q at a hyperkähler triple $\omega = (\omega_1, \omega_2, \omega_3)$, we see that infinitesimal hyperkähler deformations of ω are given by triples $\theta = (\theta_1, \theta_2, \theta_3)$ of closed 2-forms which lie in the kernel of the operator $P : \Lambda^2 \otimes \mathbb{R}^3 \to S_0^2\mathbb{R}^3$ (that is, taking values in symmetric trace-free endomorphisms of $\mathbb{R}^3$) defined by

$$P(\theta)_{ij} = \frac{1}{2}(\theta_i, \omega_j) + \frac{1}{2}(\omega_i, \theta_j) - \frac{1}{3}\delta_{ij}\sum_{k=1}^{3}(\theta_k, \omega_k). \tag{3.2}$$

Here (θ_i, ω_j), and so forth denote pointwise inner products. (To obtain this formula, recall that dividing by the volume form converts wedge products with self-dual 2-forms into inner products.) The operator P can be written more succinctly by identifying $\mathbb{R}^3 \cong \Lambda_+^2$ via ω. Then P is the map $\Lambda^2 \otimes \Lambda_+^2 \to S_0^2\Lambda_+^2$ given by $P(\theta) = s_0^2(\theta_+)$, the projection onto the trace-free symmetric part of the self-dual component of θ in $\Lambda_+^2 \otimes \Lambda_+^2$.

We next consider infinitesimal deformations of ω which fix the cohomology class. These correspond to $\theta = \mathrm{d}a$ for $a \in \Omega^1 \otimes \mathbb{R}^3$ a triple of 1-forms which solve $P(\mathrm{d}a) = s_0^2(\mathrm{d}_+a) = 0$. There is ambiguity in the choice of a with $\mathrm{d}a = \theta$ fixed which we can reduce by requiring that a is in 'Coulomb gauge', $\mathrm{d}^*a = 0$. Such an a can always be found (see Lemma 3.9 below) but there is still redundancy in this parametrization; there are many different solutions a to $\mathrm{d}a = \theta$ with $\mathrm{d}^*a = 0$. Indeed on a manifold with boundary they form an infinite-dimensional space. Lemma 3.9 shows how to cut this down to a space of dimension $b^1(X)$ by imposing appropriate boundary conditions. Before discussing this, we look at the 'Coulomb gauge-fixed' operator $D(a) = (P(\mathrm{d}a), \mathrm{d}^*a)$ whose kernel parametrizes infinitesimal cohomologous hyperkähler deformations of ω.

As written, D is a differential operator $D : \Omega^1(\mathbb{R}^3) \to C^\infty(S_0^2\mathbb{R}^3 \oplus \mathbb{R}^3)$ between sections of bundles of different ranks and so cannot be elliptic. This is to be expected because of the action of vector fields: given a vector field v, the triple $\mathscr{L}_v\omega$ gives an infinitesimal hyperkähler deformation of ω and so must lie in the kernel of P. Since $\mathscr{L}_v\omega = \mathrm{d}(i_v\omega)$, this suggests that on the level of 1-forms we should work orthogonal to triples of the form $i_v\omega$, that is, consider a with

$$J \cdot a = 0 \tag{3.3}$$

(where $J \cdot a$ is defined in (2.2)). Notice that this is an *algebraic* condition, and is not the same as the *differential* gauge fixing condition $L^*(\mathrm{d}a) = 0$ of Section 2.

The advantage of (3.3) is that it leads directly to an elliptic operator; as we show shortly, when suitably interpreted in this way D is a Dirac operator. The disadvantage of (3.3) is that it cannot be imposed by acting via diffeomorphisms. The problem occurs at the boundary. On the infinitesimal level, to put a in 'algebraic gauge', one must solve $J \cdot (i_v \omega + a) = 0$, which amounts to $v = \frac{1}{3}(J \cdot a)^\flat$. For arbitrary a this vector field will not vanish on the boundary and so the action of $\mathscr{G}_0$ is not sufficient to ensure a given triple satisfies (3.3).

Nonetheless, understanding the restriction of D to those a with $J \cdot a = 0$ will be crucial in the sequel. The most efficient way to proceed is via spinors. Write S_+, $S_- \to X$ for the positive and negative spin bundles of X and $S_\pm^m$ for the mth tensor product of $S_\pm$. In what follows we only ever encounter tensor products $S_+^m \otimes S_-^n$ with an even number of factors, $m+n = 2k$, and so the question of whether or not X is spin can safely be ignored. Moreover, when $m+n = 2k$ this tensor product carries a real structure and we write $S_+^m \otimes S_-^n$ to mean the *real* locus of this bundle, a real vector bundle of rank $(m+1)(n+1)$.

We begin by recalling, without proof, some spinorial isomorphisms (see [2]).

LEMMA 3.2. *There are the following natural isomorphisms of vector bundles:*

- $S_+ \otimes S_- \cong TX \cong \Lambda^1$;
- $S_+^2 \cong \Lambda_+^2$;
- $S_+ \otimes S_+^m \cong S_+^{m+1} \oplus S_+^{m-1}$;
- $S_0^2(S_+^2) \cong S_+^4$;

where in the last isomorphism, $S_0^2(S_+^2)$ denotes trace-free symmetric endomorphisms of S_+^2.

COROLLARY 3.3. *Let (X, ω) be a hyperkähler 4-manifold. Using the hyperkähler triple to identify $\Lambda_+^2 \cong \mathbb{R}^3$, there are isomorphisms*

$$\Lambda^1 \otimes \mathbb{R}^3 \cong (S_- \otimes S_+) \oplus (S_- \otimes S_+^3), \tag{3.4}$$

$$S_0^2(\mathbb{R}^3) \oplus \mathbb{R}^3 \cong S_+ \otimes S_+^3. \tag{3.5}$$

Moreover, the first summand in (3.4) is identified with triples of the form $\iota_v \omega$ where v is a vector field whilst the second summand is identified with triples a such that $J \cdot a = 0$.

Proof. The isomorphisms follow from Lemma 3.2. To prove the last claim, note that the map $v \mapsto \iota_v\omega$ from $TX \to \Lambda^1 \otimes \mathbb{R}^3$ is SU(2)-equivariant under the natural action of SU(2) on TX, Λ^1 and $\mathbb{R}^3 \cong \Lambda^2_+$. It follows that the image of this map agrees with the first summand in (3.4) by Schur's Lemma. Finally, since $a \mapsto (J \cdot a)^\sharp$ is the adjoint of $v \mapsto i_v\omega$, the second summand in (3.4) is identified with solutions to $J \cdot a = 0$. □

PROPOSITION 3.4. *Let (X, ω) be a hyperkähler 4-manifold. On restriction to sections of the subbundle $S_- \otimes S^3_+ \subset \Lambda^1 \otimes \mathbb{R}^3$, and under the isomorphisms of Corollary 3.3, the operator $D(a) = (s_0^2(\mathrm{d}_+ a), \mathrm{d}^* a)$ is identified with the negative Dirac operator coupled to the Levi-Civita connection on S^3_+:*

$$\mathscr{D} : C^\infty(S_- \otimes S^3_+) \to C^\infty(S_+ \otimes S^3_+).$$

Proof. We start with the standard fact that the operator $\mathrm{d}^* + \mathrm{d}_+ : \Omega^1 \to \Omega^0 \oplus \Omega^2_+$ is a Dirac operator. Namely, under the isomorphisms $\Lambda^1 \cong S_- \otimes S_+$ and $\mathbb{R} \oplus \Lambda^2_+ \cong S_+ \otimes S_+$, $\mathrm{d}^* + \mathrm{d}_+$ is identified with the negative Dirac operator coupled to the Levi-Civita connection on S_+:

$$\mathscr{D}_1 : C^\infty(S_- \otimes S_+) \to C^\infty(S_+ \otimes S_+).$$

(For a proof of this, see, for example, [2] where they consider Λ^2_- rather than Λ^2_+ but the idea is the same.) Next, we couple this Dirac operator to the bundle $S^2_+ \cong \mathbb{R}^3$, which is flat since X is hyperkähler. This means that on *triples* of 1-forms, the operator $\mathrm{d}^* + \mathrm{d}_+$ is again identified with a negative Dirac operator, this time coupled to the Levi-Civita connection on $S_+ \otimes S^2_+$:

$$\mathscr{D}_2 : C^\infty(S_- \otimes S_+ \otimes S^2_+) \to C^\infty(S_+ \otimes S_+ \otimes S^2_+).$$

Finally, the following decompositions are parallel with respect to the Levi-Civita connection:

$$S_- \otimes S_+ \otimes S^2_+ \cong (S_- \otimes S^3_+) \oplus (S_- \otimes S_+);$$
$$S_+ \otimes S_+ \otimes S^2_+ \cong (S_+ \otimes S^3_+) \oplus (S_+ \otimes S_+).$$

It follows that the restriction of $\mathscr{D}_2$ to $C^\infty(S_- \otimes S^3_+)$ maps into $C^\infty(S_+ \otimes S^3_+)$ where it agrees with the negative Dirac operator coupled to the Levi-Civita connection on S^3_+ as claimed. □

COROLLARY 3.5. *The map $F : C^\infty(S_- \otimes S^3_+) \to C^\infty(S_+ \otimes S^3_+)$ given by*

$$F(a) = Q(\omega + \mathrm{d}a) + \mathrm{d}^* a$$

is a nonlinear Dirac operator, whose zeros define hyperkähler triples. (Here, we use Corollary 3.3 to identify the domain of F with the subspace of triples in $\Omega^1(X) \otimes \mathbb{R}^3$ satisfying (3.3) and the range of F with $C^\infty(X, S^2_0\mathbb{R}^3 \oplus \mathbb{R}^3)$.)

Another elementary calculation that will be used later concerns the operators d_+ and d^* on triples in the summand $S_- \otimes S_+$ in (3.4).

Let α be a 1-form and let $\tau_i = J_i\alpha$ be the corresponding triple of 1-forms. Then

$$\mathrm{d}^*\tau_i \in C^\infty(X, \mathbb{R}^3) \tag{3.6}$$

and

$$\mathrm{d}_+\tau_i \in \Omega^2_+(X) \otimes \mathbb{R}^3 \cong C^\infty(X, \mathbb{R}) \oplus C^\infty(X, \mathbb{R}^3) \oplus C^\infty(X, S^2_0\mathbb{R}^3). \tag{3.7}$$

PROPOSITION 3.6. *Under the identification $\mathbb{R}^3 = \Lambda^2_+$ by ω, $\mathrm{d}^*\tau$ in (3.6) is identified with $\mathrm{d}_+\alpha$.*

For $\mathrm{d}_+\tau$, we have

$$s^2_0(\mathrm{d}_+\tau) = 0 \tag{3.8}$$

while the $\mathbb{R} \oplus \mathbb{R}^3$ components are identified respectively with $\mathrm{d}^*\alpha$ and $\mathrm{d}_+\alpha$.

Proof. We first consider the s^2_0-projection of the matrix

$$(\omega_i, \mathrm{d}\tau_j) = \frac{\omega_i \wedge \mathrm{d}(J_j\alpha)}{\mathrm{d}V_g} = \frac{\mathrm{d}(\omega_i \wedge J_j\alpha)}{\mathrm{d}V_g}.$$

The complex structures J_i and Kähler forms ω_i are related by $J_i\alpha = *(\alpha \wedge \omega_i)$ and so

$$(\omega_i, \mathrm{d}\tau_j) = \frac{\mathrm{d} * (J_iJ_j\alpha)}{\mathrm{d}V_g}. \tag{3.9}$$

Now the quaternion relations for the J_i imply that the s^2_0-projection of this matrix vanishes and the $\mathbb{R}$-component of $\mathrm{d}_+\tau$ is $\mathrm{d}^*\alpha$ as claimed.

Meanwhile $\mathrm{d}^*\tau_i = - * \mathrm{d} * (J_i\alpha) = *\mathrm{d}(\omega_i \wedge \alpha) = *(\omega_i \wedge \mathrm{d}\alpha) = (\omega_i, \mathrm{d}\alpha)$. □

REMARK 3.7. Another way of stating the second part of Proposition 3.6 is in terms of the matrix $(\omega_i, \mathrm{d}\tau_j) = \frac{1}{2}(\omega_i, \mathrm{d}_+\tau_j)$: specifically, it says that the trace part is equal to $\mathrm{d}^*\alpha$, the skew part is equal to $\mathrm{d}_+\alpha$ and the s^2_0 part is zero.

3.2. Proof of Theorem 1.1. Having considered cohomologous triples $\omega + \mathrm{d}a$, with the additional conditions $\mathrm{d}^*a = 0 = J \cdot a$, we now turn to the general case. We shall write down a smooth map $\mathcal{Q}$ with domain essentially triples a of 1-forms on X with coefficients in the Sobolev space H^{s+1} such that: $\mathscr{Q}$ is a submersion at any given hyperkähler triple; and $\mathscr{Q}^{-1}(0)$ is precisely the set of all hyperkähler triples $\widehat{\omega}_i$ with $L^*(\widehat{\omega} - \omega) = 0$. Theorem 1.1 will follow from this.

3.2.1. Hodge theory on X. We begin by recalling some Hodge theory for manifolds with boundary. The standard reference for this material is [17]. As before, denote by ι the boundary inclusion $\iota : Y \to X$. Given $\alpha \in \Omega^p(X)$, define forms on Y as follows:

$$\alpha_\top = \iota^*(\alpha) \quad \text{and} \quad \alpha_\perp = \iota^*(*\alpha). \tag{3.10}$$

Use these to define boundary conditions for two spaces of harmonic forms:

$$\mathscr{H}^p_\top = \{\alpha \in \Omega^p(X) : \mathrm{d}\alpha = 0, \mathrm{d}^*\alpha = 0, \alpha_\top = 0\}, \tag{3.11}$$
$$\mathscr{H}^p_\perp = \{\alpha \in \Omega^p(X) : \mathrm{d}\alpha = 0, \mathrm{d}^*\alpha = 0, \alpha_\perp = 0\}. \tag{3.12}$$

Elements of $\mathscr{H}^p_\top$ are called *Hodge forms satisfying Dirichlet boundary conditions* and elements of $\mathscr{H}^p_\perp$ are called *Hodge forms satisfying Neumann boundary conditions* (even though the traditional Neumann condition involves a normal derivative, unlike here). The Hodge theorem for manifolds with boundary, due to Morrey–Friedrichs, is as follows.

THEOREM 3.8. *The inclusions $\mathscr{H}^p_\top \to \Omega^p(X, Y)$ and $\mathscr{H}^p_\perp \to \Omega^p(X)$ induce isomorphisms*

$$\mathscr{H}^p_\top \cong H^p(X, Y), \qquad \mathscr{H}^p_\perp \cong H^p(X).$$

(Here $\Omega^p(X, Y)$ is the space of p-forms on X which restrict to 0 on Y; these vector spaces give a complex under exterior derivative and $H^p(X, Y)$ is the resulting cohomology group.)

With this in hand, we can give a convenient parametrization of closed 2-forms as follows.

LEMMA 3.9. *Let θ be a closed triple of 2-forms on X. There exist triples $\chi \in \mathscr{H}^2_\perp \otimes \mathbb{R}^3$ and $a \in \Omega^1 \otimes \mathbb{R}^3$ such that $\theta = \chi + \mathrm{d}a$ with $\mathrm{d}^*a = 0$ and $a_\perp = 0$. Moreover, χ is unique and a is unique up to addition of a triple $b \in \mathscr{H}^1_\perp \otimes \mathbb{R}^3$.*

Proof. By the Hodge theorem for manifolds with boundary there is a unique $\chi \in \mathscr{H}^2_\perp \otimes \mathbb{R}^3$ such that $\theta - \chi$ is exact. Write $\theta - \chi = \mathrm{d}\hat{a}$ for some triple $\hat{a}$ of 1-forms. Now let $f : X \to \mathbb{R}^3$ solve $\Delta f = -\mathrm{d}^*\hat{a}$, with the Neumann boundary condition $(\mathrm{d}f)_\perp = -\hat{a}_\perp$ and write $a = \hat{a} + \mathrm{d}f$. By choice of f, $\mathrm{d}^*a = 0$ and $a_\perp = 0$. □

3.2.2. A nonlinear operator of mixed order. We next impose the slice condition $L^*(\theta) = 0$ which, by Theorem 2.1, is equivalent to dividing out by the action of diffeomorphisms which are the identity on the boundary (at least for small θ).

LEMMA 3.10. *Let $\theta = \chi + \mathrm{d}a$ where $\chi \in \mathscr{H}^2_\perp \otimes \mathbb{R}^3$ and $a \in \Omega^1 \otimes \mathbb{R}^3$ with $\mathrm{d}^*a = 0$. Then $L^*(\theta) = 0$ if and only if $\Delta(J \cdot a) = 0$.*

Proof. Note that $L^*(\theta) = (J \cdot \mathrm{d}^*\theta)^\sharp$ so $L^*(\theta) = 0$ if and only if $J \cdot (\mathrm{d}^*\mathrm{d}a) = 0$. Since $\mathrm{d}^*a = 0$ this is equivalent to $J \cdot \Delta a = 0$, where $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ is the Hodge Laplacian. A hyperkähler metric is Ricci-flat, so on 1-forms the Hodge Laplacian is equal to the rough Laplacian. Moreover, J is covariant constant (since each of J_1, J_2, J_3 are) and thus J commutes with the rough Laplacian, and hence the Hodge Laplacian on 1-forms. The result follows. □

Thus given $\theta = \chi + \mathrm{d}a$ as above, the conditions

$$Q(\omega + \chi + \mathrm{d}a) = 0, \quad \mathrm{d}^*a = 0, \quad \Delta(J \cdot a) = 0 \tag{3.13}$$

(where Q is defined in equation (3.1)) are equivalent to

$$Q(\omega + \theta) = 0, \quad \mathrm{d}^*a = 0, \quad L^*(\theta) = 0, \tag{3.14}$$

and we know by the slice theorem that these conditions define a neighbourhood in $\mathscr{M}$ of ω if the norm of θ is sufficiently small. We shall combine the three conditions in (3.13) to define our smooth map $\mathscr{Q}$, but before doing so, we must take care of the fact that θ does not determine a uniquely, even if $\mathrm{d}^*a = 0$. However, Lemma 3.9 shows us how to fix this problem. Thus we make the following definition.

DEFINITION 3.11 (The gauge-fixed hyperkähler equation). For (χ, a) as above, define

$$\mathscr{Q}(\chi, a) = (Q(\omega + \chi + \mathrm{d}a), \mathrm{d}^*a, \Delta(J \cdot a), a_\perp). \tag{3.15}$$

The domain of $\mathscr{Q}$ is defined to be the open set of

$$\mathscr{U}^{s+1} \subset ((\mathscr{H}^2_\perp \otimes \mathbb{R}^3) \oplus H^{s+1}(X, T^*X \otimes \mathbb{R}^3))/(\mathscr{H}^1_\perp \otimes \mathbb{R}^3) \tag{3.16}$$

satisfying the condition

$$\sum (\omega_i + \chi_i + \mathrm{d}a_i)^2 > 0 \tag{3.17}$$

and the degree of regularity s is taken to be > 4.

REMARK 3.12. The map $\mathscr{Q}$ is well defined on this domain because $\chi + \mathrm{d}a$ does not change if a triple of harmonic 1-forms is added on to a.

Note further that $\mathscr{Q}$ maps into

$$H^s(X, S_0^2\mathbb{R}^3) \oplus H^s(X, \mathbb{R}^3) \oplus H^{s-1}(X, T^*X) \oplus H^{s+1/2}(Y, \Lambda^3 T^*Y \otimes \mathbb{R}^3), \quad (3.18)$$

the last term being essentially the restriction to Y of the triple of normal components of a.

From (3.2), if

$$\theta = \chi + \mathrm{d}a, \quad a = \sigma + \tau, \quad (3.19)$$

where

$$J \cdot \sigma = 0 \quad (3.20)$$

and τ is the component of a in the subbundle isomorphic to S_-S_+ (recall (3.4) again), we have

$$Q(\omega+\chi+\mathrm{d}a) = Q(\omega)+P(\omega, \chi+\mathrm{d}a)+Q(\chi+\mathrm{d}a) = P(\omega, \chi+\mathrm{d}a)+Q(\chi+\mathrm{d}a). \quad (3.21)$$

Thus the linearization of $\mathscr{Q}$ at 0 is

$$\mathrm{d}\mathscr{Q}_0(\chi, a) = (s_0^2(\omega, \chi + \mathrm{d}\sigma), \mathrm{d}^*\sigma + \mathrm{d}^*\tau, \Delta(J \cdot \tau), a_\perp) \quad (3.22)$$

using (3.20). Combining the first two summands to make $S_+ \otimes S_+^3$ as in (3.5), and using Proposition 3.4, we obtain

$$\mathrm{d}\mathscr{Q}_0(\chi, a) = (\mathscr{D}\sigma + \mathrm{d}^*\tau + s_0^2(\omega, \chi), \Delta(J \cdot \tau), a_\perp). \quad (3.23)$$

3.3. Regularity of $\mathscr{Q}^{-1}(0)$. Whilst not strictly speaking an elliptic operator, $\mathscr{Q}$ is built from elliptic parts. In particular, it enjoys the following regularity property.

THEOREM 3.13. *Fix $s > 4$. There exists $\epsilon > 0$ such that if $\mathscr{Q}(\chi, a) = 0$ with $\chi \in \mathscr{H}_\perp^2 \otimes \mathbb{R}^3$, $a \in H^{s+1}(X, T^*X \otimes \mathbb{R}^3)$ and $\|(\chi, a)\|_{H^{s+1}} < \epsilon$, then in fact a is smooth in the interior of X.*

It follows that a neighbourhood of $[\omega]$ in the moduli space $\mathscr{M}^s$ of hyperkähler triples that are smooth in the interior of X and of regularity $H^{s+1/2}$ on Y is homeomorphic to a neighbourhood of $\mathscr{Q}^{-1}(0)$ in $\mathscr{U}^{s+1}$.

Proof. We begin with the proof that a is smooth in the interior of X. Note first that χ is automatically smooth, since it solves the linear elliptic system $\mathrm{d}\chi = 0 = \mathrm{d}^*\chi$.

The component τ of a in (3.20) is smooth, for the map $a \mapsto J \cdot a$ identifies the subbundle of $\Lambda^1 \otimes \mathbb{R}^3$ that τ lives in with Λ^1, and $\Delta(J \cdot \tau) = 0$.

To show that σ is smooth, note first from (3.21) and (3.22) that $\mathscr{Q}(\chi, a) = 0$ is equivalent to

$$\mathscr{D}\sigma = -\mathrm{d}^*\tau - s_0^2(\omega, \chi) - Q(\chi + \mathrm{d}(\sigma + \tau)), \tag{3.24}$$

which we write in the schematic form

$$\mathscr{D}\sigma = -q(\mathrm{d}\sigma, \mathrm{d}\sigma) - l(\mathrm{d}\sigma) - r, \tag{3.25}$$

where q is quadratic and l is linear in $\mathrm{d}\sigma$, the coefficients of q, l and r all depending real-analytically on the smooth data τ and χ.

Equation (3.25) is a first order fully nonlinear equation for σ. To prove regularity one can work directly with the first order equation, but it is more straightforward to use a standard device and take another derivative to turn (3.25) into a second order quasilinear equation. To do this, we apply the adjoint Dirac operator $\mathscr{D}^*$. Schematically, we obtain

$$\mathscr{D}^*\mathscr{D}\sigma = -\nabla q \cdot \mathrm{d}\sigma \cdot \mathrm{d}\sigma - 2q \cdot \nabla(\mathrm{d}\sigma) \cdot \mathrm{d}\sigma - \nabla l \cdot \mathrm{d}\sigma - l \cdot \nabla(\mathrm{d}\sigma) - \nabla \cdot r, \tag{3.26}$$

where the dots denote various algebraic contractions whose precise form is not important. Write $\mathscr{P} : C^\infty(S^3_+ \otimes S_-) \to C^\infty(S^3_+ \otimes S_-)$ for the second order linear operator

$$\mathscr{P}(\rho) = \mathscr{D}^*\mathscr{D}\rho + 2q \cdot \nabla(\mathrm{d}\rho) \cdot \mathrm{d}\sigma + l \cdot \nabla(\mathrm{d}\rho).$$

We have absorbed all the second order behaviour from (3.26) into $\mathscr{P}$, making it linear by letting $\mathrm{d}\sigma$ appear in its coefficients.

The coefficients of $\mathscr{P}$ depend on those of $\mathscr{D}^*\mathscr{D}$ and on $\mathrm{d}\sigma$, χ and τ. Since χ and τ are smooth and $\mathscr{D}^*\mathscr{D}$ has smooth coefficients, the coefficients of $\mathscr{P}$ are in the same Holder space as $\mathrm{d}\sigma$. Since $\sigma \in H^{s+1}$, Sobolev embedding gives that the coefficients are in $C^{k,\alpha}$ for some $k \geqslant 0$ and $0 < \alpha < 1$. (At this stage $k = s - 3$ is the best we can arrange.)

Next notice that the C^0 norm of the coefficients of $\mathscr{P}$ depends continuously on $\mathrm{d}\sigma$, χ, τ (in the C^0-topology). Moreover, $\mathscr{P} = \mathscr{D}^*\mathscr{D}$ when $\chi = 0 = \tau$ (since l vanishes in this case). Hence, for $\mathrm{d}\sigma$, χ, τ sufficiently small in C^0, and so in particular in H^{s+1}, $\mathscr{P}$ is an elliptic operator.

Now rearranging (3.26) gives

$$\mathscr{P}(\sigma) = -\nabla q \cdot \mathrm{d}\sigma \cdot \mathrm{d}\sigma - \nabla l \cdot \mathrm{d}\sigma - \nabla \cdot r. \tag{3.27}$$

Since $\mathscr{P}$ is elliptic with coefficients in $C^{k,\alpha}$ and the right-hand side of (3.27) is in $C^{k,\alpha}$ as well, Schauder estimates apply, giving $\sigma \in C^{k+2,\alpha}$. It follows in turn that the coefficients of $\mathscr{P}$ and the right-hand side of (3.27) are actually in $C^{k+1,\alpha}$ and so $\sigma \in C^{k+3,\alpha}$. Bootstrapping this argument then gives that σ is smooth in the interior of X. $\square$

REMARK 3.14. We stress that while this result gives that all gauge-fixed hyperkähler perturbations of ω are smooth in the interior of X, though there is no reason to believe that they will extend smoothly up to or through the boundary Y.

3.4. $\mathscr{Q}$ is a submersion. We show next that for any sufficiently large s, $\mathscr{Q}$ is a submersion. For this we need to know that $\mathscr{D}$ is surjective (with suitable domain and range). We gather the results we need first, before proceeding to the proof in Section 3.4.2.

3.4.1. On Dirac operators. Since $\mathscr{D}$ is an operator of Dirac type, we have

$$\mathscr{D}^*\mathscr{D} = \nabla_1^*\nabla_1 + R_1, \quad \mathscr{D}\mathscr{D}^* = \nabla_2^*\nabla_2 + R_2 \tag{3.28}$$

where ∇_1 is the metric connection on $S_-S_+^3$, ∇_2 is the metric connection on $S_+S_+^3$, R_1 is an endomorphism of $S_-S_+^3$ and R_2 is an endomorphism of $S_+S_+^3$. The endomorphisms R_1 and R_2 depend only upon the curvature of the bundles in question. Because the only nonvanishing piece of curvature on a hyperkähler 4-manifold is the anti-self-dual part of the Weyl curvature and this is a section of S_-^4 (the symmetric fourth power of S_-) it follows that R_1 and R_2 both vanish identically.

PROPOSITION 3.15. *The operator*

$$\mathscr{D} : H^s(X, S_-S_+^3) \to H^{s-1}(X, S_+S_+^3) \tag{3.29}$$

is surjective.

Proof. From the formula $\mathscr{D}\mathscr{D}^* = \nabla_2^*\nabla_2$, we have that the spectrum of $\mathscr{D}\mathscr{D}^*$ on sections satisfying Dirichlet boundary conditions is strictly positive. Hence, there exists

$$G : L^2(X, S_+S_+^3) \longrightarrow H^2(X, S_+S_+^3)$$

with $\mathscr{D}\mathscr{D}^* \circ G = 1$. So if $f \in L^2$, $u = \mathscr{D}^*Gf \in H^1$ and $\mathscr{D}u = f$. If we know that f is also in H^{s-1}, then we still have $\mathscr{D}u = f$ so elliptic regularity gives $u \in H^s$. $\square$

3.4.2. Proof that $\mathscr{Q}$ is a submersion. We now show that the linearization of $\mathscr{Q}$ is surjective at every smooth hyperkähler triple ω.

PROPOSITION 3.16. *Let ω be a smooth hyperkähler triple on X. Then the operator* $\mathrm{d}\mathscr{Q}_0$ *in* (3.23) *is surjective onto* (3.18).

Proof. Let (ψ, υ, b) lie in (3.18). To prove surjectivity of $\mathrm{d}\mathscr{Q}_0$, it suffices to find $a = \sigma + \tau$ with

$$\mathscr{D}\sigma + \mathrm{d}^*\tau = \psi, \tag{3.30}$$

$$\Delta(J \cdot \tau) = \upsilon, \tag{3.31}$$

$$a_\perp = b. \tag{3.32}$$

First, let τ' solve $\Delta(J \cdot \tau') = \upsilon$ with Dirichlet boundary conditions $\tau'|_Y = 0$. Next we use the surjectivity of $\mathscr{D}$ on X to solve $\mathscr{D}\sigma' = \psi - \mathrm{d}^*\tau'$. With these choices we have satisfied (3.30) and (3.31). Let $a' = \sigma' + \tau'$. We adjust a' so as to satisfy (3.32) without spoiling (3.30) and (3.31).

To do this we consider $a = a' + \mathrm{d}f$ where f is a triple of harmonic functions, $\Delta f = 0$. We have that

$$\mathrm{d}\mathscr{Q}_0(\mathrm{d}f) = (s_0^2(\mathrm{d}_+(\mathrm{d}f)) + \mathrm{d}^*(\mathrm{d}f), \Delta(J \cdot \mathrm{d}f), (\mathrm{d}f)_\perp).$$

Now $\mathrm{d}_+(\mathrm{d}f) = (\mathrm{d} + *\mathrm{d})(\mathrm{d}f) = 0$ and $\mathrm{d}^*\mathrm{d}f = \Delta f = 0$. Moreover, $\Delta(J \cdot \mathrm{d}f) = J \cdot \Delta \mathrm{d}f$ (since the metric is hyperkähler) and this also vanishes, since $\Delta \mathrm{d}f = \mathrm{d}\Delta f$. The conclusion is that when f is a triple of harmonic functions,

$$\mathrm{d}\mathscr{Q}_0(a' + \mathrm{d}f) = (\psi, \upsilon, a'_\perp + (\mathrm{d}f)_\perp).$$

To prove that $\mathrm{d}\mathscr{Q}_0$ is surjective, we choose f to be harmonic functions with the Neumann boundary condition $(\mathrm{d}f)_\perp = b - a'_\perp$; then $a = a' + \mathrm{d}f$ solves $\mathrm{d}\mathscr{Q}_0(a) = (\psi, \upsilon, b)$. □

We have now shown that $\mathscr{Q}$ is a submersion. Since we have already seen that for any smooth triple, a small neighbourhood of 0 in $\mathscr{Q}^{-1}(0)$ is homeomorphic to a small neighbourhood of $[\omega]$ in $\mathscr{M}^s$, we have now proved part (ii) of Theorem 1.1, apart from the identification of the tangent space in (1.11). This will be done in the next section.

4. The tangent space to $\mathscr{M}^s$

We have now seen that for any smooth hyperkähler triple ω, a neighbourhood of $[\omega]$ in $\mathscr{M}^s$ is homeomorphic to a ball containing the origin in $\ker(\mathrm{d}\mathscr{Q}_0)$. We shall now prove (1.11), thereby giving a more satisfactory interpretation of this tangent space. The assertion to be proved is the following.

CLAIM 4.1. Let X be a compact 4-manifold with boundary Y and ω a hyperkähler triple on X. Then

$$T_\omega \mathscr{H}^s \cap \ker(L^*) = [\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s] + L(\mathscr{W}^{s+1}) \subset H^s(X, \Lambda^2 \otimes \mathbb{R}^3). \quad (4.1)$$

REMARK 4.2. The space $\mathscr{Z}^2_-(X)$ of closed anti-self-dual (ASD) 2-forms consists of elements that are smooth in the interior—any element is harmonic—but they can be arbitrarily bad at the boundary. The notation $\cap H^s$ means that we consider those closed ASD 2-forms which are in $H^s(X)$, so having boundary values in $H^{s-1/2}(Y)$.

This result depends on two facts. The first is proved exactly as for the surjectivity of $\mathscr{D}$.

LEMMA 4.3. *On the hyperkähler manifold X with boundary Y, the operator $D = \mathrm{d}^* + \mathrm{d}_+$ is surjective.*

Proof. The only thing to check in copying the proof of Proposition 3.15 is that $DD^* = \nabla^*\nabla$. In fact,

$$D^*D = \nabla_1^*\nabla_1 \quad \text{and} \quad DD^* = \nabla_2^*\nabla_2 \quad (4.2)$$

for the same reason that $R_1 = 0$ and $R_2 = 0$ in (3.28): the anti-self-dual part of the Weyl curvature cannot act as a nonzero endomorphism of S_-S_+ or S_+S_+. □

The second observation we need is contained in the following.

LEMMA 4.4. *Let $L_+v = \mathrm{d}_+(\iota_v\omega)$ be the self-dual part of the operator L. Then*

$$L^*L = L^*L_+. \quad (4.3)$$

Furthermore, if $\theta = \theta_+ + \theta_-$ is a triple of closed 2-forms decomposed into self-dual and anti-self-dual parts which satisfies $L^*\theta = 0$, then we also have

$$L^*\theta_+ = 0 = L^*\theta_-.$$

Proof. Since $L^*\theta = J\cdot \mathrm{d}^*\theta = -J\cdot *\mathrm{d} * \theta$,

$$L^*L_+v = -J\cdot *\mathrm{d} * (1 + *)\mathrm{d}(\iota_v\omega) = -J\cdot *\mathrm{d} * \mathrm{d}(\iota_v\omega) = L^*Lv.$$

For the second part, since θ is closed, we trivially have

$$\mathrm{d}\theta_+ + \mathrm{d}\theta_- = 0 \quad \text{and so} \quad J\cdot *\mathrm{d}\theta_+ + J\cdot *\mathrm{d}\theta_- = 0. \quad (4.4)$$

Writing $L^* = -J \cdot (*\mathrm{d}*)$, we see that $L^*(\theta) = 0$ becomes

$$0 = J \cdot *\mathrm{d}(*\theta_+ + *\theta_-) = J \cdot *\mathrm{d}\theta_+ - J \cdot *\mathrm{d}\theta_-. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$J \cdot *\mathrm{d}\theta_+ = 0 = J \cdot *\mathrm{d}\theta_- \tag{4.6}$$

from which the result follows because $J \cdot *\mathrm{d}$ equals $-L^*$ on triples of self-dual forms and equals L^* on triples of anti-self-dual forms. □

4.1. Proof of Claim. In one direction, it is clear that the right-hand side of (4.1) is contained in the left-hand side. For the converse, suppose that for any given $\theta \in T_\omega \mathscr{H}^s \cap \ker(L^*)$, we can solve $L_+ v = \theta_+$ for some $v \in C^\infty(X, TX)$. Write

$$\theta = (\theta - Lv) + Lv.$$

Then by construction, $\theta - Lv$ lies in $\Omega^2_-(X) \otimes \mathbb{R}^3$. It is also closed, because θ is closed by hypothesis and Lv is exact. So we just have to show

$$L^*\theta = 0 \Rightarrow L^*Lv = 0.$$

However, the second part of Lemma 4.4 applies to give $L^*\theta_+ = 0$ and $L^*\theta_- = 0$. Hence if $L_+ v = \theta_+$, we have

$$L^*Lv = L^*L_+(v) = L^*(\theta_+) = 0,$$

using the first part of Lemma 4.4 as well.

It remains only to discuss the solvability of $L_+(v) = \theta_+$.

Recall from Proposition 3.6 that $L_+(v) = \mathrm{d}_+ J\alpha$, if α is the 1-form dual to v, and that this map is the composite of $D = \mathrm{d}^* + \mathrm{d}_+$ with the algebraic inclusion

$$\Omega^0 \oplus \Omega^2_+ \hookrightarrow \Omega^2_+ \otimes \Omega^2_+.$$

The definition (1.6) of $T_\omega \mathscr{H}^s$ includes the condition $s_0^2(\omega, \theta) = 0$ which says precisely that θ_+ lies in the image of this inclusion. Since D is surjective (Lemma 4.3), it follows that the equation $L_+(v) = \theta_+$ can be solved for any $\theta \in T_\omega \mathscr{H}^s$. The proof of the claim is complete, as is the proof of Theorem 1.1.

5. The moduli space $\mathscr{M}_+$

In this section, we make the following assumption.

ASSUMPTION 5.1. The mean curvature H of Y is everywhere nonnegative, and positive at least one point. (A definition of H appears in (5.3).)

For the avoidance of doubt, our convention is that the mean curvature of the boundary of the ball in $\mathbb{R}^4$ is positive.

So far we have only used the surjectivity of Dirac operators on manifolds with boundary. We now bring in APS type boundary conditions for D giving surjectivity of this operator. We shall refer to the literature for most of the proofs; we have found [3] and [4] to be good references for this material.

5.1. Geometry near ∂X. We first introduce some notation. Let $\rho : X \to \mathbb{R}$ measure the distance of a point to the boundary Y. This function is smooth near Y. Using geodesics which are orthogonal to Y we can identify a neighbourhood U of Y with the product $[0, \epsilon) \times Y$, with the function ρ corresponding to projection onto the first factor. If $\rho \in [0, \varepsilon)$, $y \in Y$, use parallel transport along orthogonal geodesics to identify $T_{\rho,y}X$ with $\mathbb{R} \oplus T_y Y$, and similarly for 1-forms, and so forth. The $\mathbb{R}$ summand here corresponds to the coefficient of ν, the outward unit vector field tangent to the orthogonal geodesics. A consequence of this identification is that the normal component ∇_ν of the metric connection acts simply as $-\partial/\partial\rho$ on the $\mathbb{R}$ and TY components of vector fields. We can write any 1-form a in the form

$$a = f\,\mathrm{d}\rho + b \tag{5.1}$$

where f is a path of functions on Y and b is a path of 1-forms on Y, and

$$\nabla_\nu a = -\partial_\rho f\,\mathrm{d}\rho - \partial_\rho b. \tag{5.2}$$

Similarly the metric takes the form $g = \mathrm{d}\rho^2 + h(\rho)$ where $h(\rho)$ is a path of Riemannian metrics on Y. Recall that the metric volume element $\mathrm{d}\mu_Y$ of the path of metrics h on TY is not closed: instead we have

$$\mathrm{d}[\mathrm{d}\mu_Y] = H\mathrm{d}\mu_X = -H\mathrm{d}\rho \wedge \mathrm{d}\mu_Y, \tag{5.3}$$

where H is the mean curvature of the family of level sets of ρ. (We think of H as a path of functions on Y.)

Similarly, any self-dual 2-form θ has the form

$$\theta = -\mathrm{d}\rho \wedge c + *_Y c \tag{5.4}$$

in U, where $c \in T^*Y$ and $*_Y : T^*Y \to \Lambda^2 T^*Y$ is the boundary $*$ operator. Then mapping θ to $c = \iota_\nu\theta$ identifies $\Lambda^2_+ X|U$ with $T^*Y|U$.

LEMMA 5.2. *In the collar neighbourhood U of Y, we have that $D = \mathrm{d}^* + \mathrm{d}_+$ is given by*

$$D : \begin{bmatrix} f \\ b \end{bmatrix} \longmapsto \begin{bmatrix} \nu + H & 0 \\ 0 & \nu \end{bmatrix} \begin{bmatrix} f \\ b \end{bmatrix} + D_Y \begin{bmatrix} f \\ b \end{bmatrix} \tag{5.5}$$

where

$$D_Y = \begin{bmatrix} 0 & \mathrm{d}_Y^* \\ \mathrm{d}_Y & *_Y \mathrm{d}_Y \end{bmatrix}. \tag{5.6}$$

Proof. We start by computing $\mathrm{d}^* a$ for a given in (5.1):

$$\begin{aligned} \mathrm{d}^* a &= - * \mathrm{d} * (f \,\mathrm{d}\rho + b) \\ &= - * \mathrm{d}(-f \mathrm{d}\mu_Y - \mathrm{d}\rho \wedge *_Y b) \\ &= - * (\nu(f) \mathrm{d}\rho \wedge \mathrm{d}\mu_Y - H f \mathrm{d}\mu_X + \mathrm{d}\rho \wedge \mathrm{d}_Y *_Y b) \\ &= \nu(f) + H f + \mathrm{d}_Y^* b. \end{aligned}$$

Similarly,

$$\mathrm{d}a = \mathrm{d}(f \mathrm{d}\rho + b) = -\mathrm{d}\rho \wedge \mathrm{d}_Y f - \mathrm{d}\rho \wedge \nu(b) + \mathrm{d}_Y b.$$

Hence

$$(1 + *)\mathrm{d}a = -\mathrm{d}\rho \wedge (\nu(b) + \mathrm{d}_Y f + *_Y \mathrm{d}_Y b) + *_Y (\nu(b) + \mathrm{d}_Y f + *_Y \mathrm{d}_Y b)$$

which gets identified with

$$\nu(b) + \mathrm{d}_Y f + *_Y \mathrm{d}_Y b.$$

These computations complete the proof. □

We now turn to the formal adjoint of $D = \mathrm{d}^* + \mathrm{d}_+$.

PROPOSITION 5.3. *The formal adjoint D^* of D, with the same identifications, is given by*

$$D^* \begin{bmatrix} f \\ b \end{bmatrix} \longmapsto \begin{bmatrix} -\nu & 0 \\ 0 & -\nu - H \end{bmatrix} \begin{bmatrix} f \\ b \end{bmatrix} + D_Y \begin{bmatrix} f \\ b \end{bmatrix}.$$

Moreover,

$$(Du, v) - (u, D^* v) = \int_Y \langle u, v \rangle \mathrm{d}\mu_Y. \tag{5.7}$$

Proof. This follows from our formula (5.3) which shows that $\nu + H$ and $-\nu$ are formal adjoints to each other. The second equation also follows from this formula. □

5.2. Green's formulae. By combining (5.7) with the formulae (4.2), we obtain the following useful result, which will be used to obtain sharp statements about the injectivity and surjectivity of D with suitable boundary conditions.

PROPOSITION 5.4. *Let $D = \mathrm{d}^* + \mathrm{d}_+$ on the hyperkähler manifold X with boundary Y. Then for $u \in \Omega^1(X)$ we have*

$$\|Du\|^2 = \|\nabla u\|^2 + \int_Y (H|\iota_\nu u|^2 + (u, D_Y u)_Y)\mathrm{d}\mu_Y. \tag{5.8}$$

Furthermore,

$$\|D^* v\|^2 = \|\nabla v\|^2 + \int_Y (H|\iota_\nu c|^2 - (v, D_Y v)_Y)\mathrm{d}\mu_Y \tag{5.9}$$

for $v \in \Omega^0(X) \oplus \Omega^2_+(X)$, c being the component of v in $\Omega^2_+(X)$.

Proof. For (5.9), put $u = D^* v$ into (5.7), to get

$$(v, DD^* v) - \|D^* v\|^2 = \int_Y (v, D^* v)\, \mathrm{d}\mu_Y.$$

We have an analogous formula for $\nabla^*\nabla$:

$$(v, \nabla^*\nabla v) - \|\nabla v\|^2 = -\int_Y (v, \nabla_\nu v)\, \mathrm{d}\mu_Y.$$

Subtracting and recalling that $DD^* = \nabla^*\nabla$ gives

$$\|D^* v\|^2 - \|\nabla v\|^2 = -\int_Y (v, \nu(v) + D^* v)\, \mathrm{d}\mu_Y.$$

Now substitute the formula for D^* from Proposition 5.3 into the right-hand side to obtain (5.9). The formula (5.8) follows in precisely the same way. □

Another useful result analogous to those in Proposition 5.4 relates the L^2-norms of Du and $\widetilde{D}u$, where

$$\widetilde{D} = \mathrm{d}^* + \mathrm{d}_-. \tag{5.10}$$

PROPOSITION 5.5. *Let the notation be as above. Then, for $u \in \Omega^1(X)$ and $b = \iota^* u$, we have*

$$\|\widetilde{D}u\|^2 - \|Du\|^2 = -2\int_Y (b, *_Y \mathrm{d}_Y b)\, \mathrm{d}\mu_Y. \tag{5.11}$$

Proof. We note that

$$\widetilde{D}^*\widetilde{D} = \nabla^*\nabla \tag{5.12}$$

by the same argument that gives (4.2). Computations similar to those in the proofs of Lemma 5.2 and Proposition 5.3 give

$$\widetilde{D} = \begin{bmatrix} \nu + H & 0 \\ 0 & \nu \end{bmatrix} + \begin{bmatrix} 0 & \mathrm{d}_Y^* \\ \mathrm{d}_Y & - *_Y \mathrm{d}_Y \end{bmatrix} \tag{5.13}$$

with formal adjoint

$$\widetilde{D}^* = \begin{bmatrix} -\nu & 0 \\ 0 & -\nu - H \end{bmatrix} + \begin{bmatrix} 0 & \mathrm{d}_Y^* \\ \mathrm{d}_Y & - *_Y \mathrm{d}_Y \end{bmatrix}$$

in the collar neighbourhood U of Y. Arguing now as in the proof of Proposition 5.4, we obtain the formulae

$$(u, D^* D u) - \|Du\|^2 = -\int_Y (u, Du)\, \mathrm{d}\mu_Y$$

and

$$(u, \widetilde{D}^* \widetilde{D} u) - \|\widetilde{D} u\|^2 = -\int_Y (u, \widetilde{D} u)\, \mathrm{d}\mu_Y.$$

The first term on the left-hand side of each of these two equations is $(u, \nabla^* \nabla u)$, so subtracting we obtain

$$\|\widetilde{D} u\|^2 - \|Du\|^2 = \int_Y (u, (\widetilde{D} - D)u)\, \mathrm{d}\mu_Y. \tag{5.14}$$

The result now follows from our formulae for $\widetilde{D}$ and D, (5.5) and (5.13). □

5.3. The kernel of D in terms of boundary data. We shall now combine the formulae just obtained with standard Fredholm results for operators of Dirac type on a manifold with boundary to parametrize the null space of D in terms of boundary data.

The operator D_Y is (formally) self-adjoint and of first order, so it has a discrete real spectrum which is unbounded above and below, with no (finite) accumulation points. Denote by H_λ the eigenspace of D_Y corresponding to the eigenvalue λ. Fix a real number $s > 1/2$.

DEFINITION 5.6. Denote by $H_+^{s-1/2}(Y)$ the completion in the Sobolev $(s - 1/2)$-norm of $\bigoplus_{\lambda>0} H_\lambda$. Similarly, denote by $H_-^{s-1/2}(Y)$ the completion in the $(s-1/2)$-norm of $\bigoplus_{\lambda<0} H_\lambda$.

REMARK 5.7. We shall refer to the elements of $H_+^{s-1/2}(Y)$ as positive frequency boundary data, and similarly to the elements of $H_-^{s-1/2}(Y)$ as negative frequency boundary data.

Then we have

$$H^{s-1/2}(Y) = H_-^{s-1/2}(Y) \oplus H_0(Y) \oplus H_+^{s-1/2}(Y), \tag{5.15}$$

with $H_0(Y)$ being the (finite-dimensional) kernel of D_Y. Similarly define:

$$H_+^s(X) = \{u \in H^s(X, \Lambda^1) : u|Y \in H_+^{s-1/2}(Y)\};$$
$$H_-^s(X) = \{u \in H^s(X, \Lambda^1) : u|Y \in H_-^{s-1/2}(Y)\};$$
$$H_0^s(X) = \{u \in H^s(X, \Lambda^1) : u|Y \in H_0^{s-1/2}(Y)\}.$$

The basic results we need are as follows.

THEOREM 5.8. *Let X be a hyperkähler manifold with smooth boundary and mean curvature $H \geqslant 0$, and strictly positive at at least one point. Then for $s > 1/2$, the operator*

$$D = \mathrm{d}^* + \mathrm{d}_+ : H_{\geqslant 0}^s(X, \Lambda^1) \to H^{s-1}(X, \mathbb{R} \oplus \Lambda_+^2) \tag{5.16}$$

is surjective, with finite-dimensional kernel isomorphic to $H^1(X)$.

Further, there is a Poisson operator

$$\mathscr{P} : H_-^{s-1/2}(Y) \to \ker(D) \cap H^s(X, T^*X),$$

that is the projection to $H_-^{s-1/2}(Y)$ of the restriction $\mathscr{P}(f)|Y$ is equal to f.

REMARK 5.9. Here we have written $H_{\geqslant 0}$ for the direct sum of H_+ and H_0.

Proof. Without any restriction on the mean curvature, that (5.16) is Fredholm is standard in the theory of Dirac operators on manifolds with boundary [3, 4]. This theory also identifies the cokernel of (5.16) with the null space of the adjoint operator D^* with domain $H_-^s(X)$.

Consider (5.9) applied to v with

$$D^* v = 0, \quad v \in H_-^s(X). \tag{5.17}$$

The first term on the right-hand side of (5.9) is manifestly $\geqslant 0$, the second term is $\geqslant 0$ by Assumption 5.1 and the third is strictly positive if $0 \neq v|Y \in H_-^{s-1/2}(Y)$ by (5.17). But the left-hand side of (5.9) is 0 by (5.17), which means that $v|Y = 0$ and $\nabla v = 0$. Hence v is identically zero and D^* is injective on $H_-^s(X)$.

To identify the kernel of D we use the formula (5.8). We see that if $Du = 0$ then $\nabla u = 0$ and $D_Y(u|Y) = 0$. Looking at the formula for D_Y, it follows that if we write $u = f\,\mathrm{d}\rho + b$ on Y then

$$\mathrm{d}_Y f = 0, \quad \mathrm{d}_Y b = 0 = \mathrm{d}_Y^* b.$$

Hence f is constant and $\int_Y Hf^2 = 0$ implies $f = 0$ if $H \geqslant 0$ and is positive at a point.

Now the standard Weitzenböck formula for 1-forms on a Ricci-flat 4-manifold shows that every harmonic 1-form a with $\iota_\nu(a) = 0$ is parallel. So the null space of (5.16) is isomorphic to this space of forms, which is in turn identifiable with $H^1(X)$ by Hodge theory.

The construction of the Poisson operator is standard, but we recall the details. For any given s, we can define a bounded extension operator

$$E : H_-^{s-1/2}(Y) \longrightarrow H^s(X, \Lambda^1)$$

so that $Ef|Y = f$. Let $G : H^{s-1}(X, \mathbb{R} \oplus \Lambda_+^2) \to H_{\geqslant 0}^s(X, \Lambda^1)$ be a right inverse of (5.16). Set

$$\mathscr{P} f = Ef - GD(Ef).$$

By definition $\mathscr{P}$ maps into $\ker(D) \cap H^s$. Since $(G\sigma|Y)_- = 0$ for any σ in $H^{s-1}(X, \mathbb{R} \oplus \Lambda_+^2)$, it follows that $(\mathscr{P} f|Y)_- = (Ef|Y)_- = f$ as required. □

5.4. The kernel of D. We now wish to give a precise description of $\ker(D)$ in terms of boundary data. Recall the decomposition (5.15) of boundary data

$$H^{s-1/2}(Y) = H_-^{s-1/2}(Y) \oplus H_0(Y) \oplus H_+^{s-1/2}(Y), \tag{5.18}$$

in terms of the spectrum of D_Y, and that the coefficient bundle here is $T^*X|Y = \mathbb{R} \oplus T^*Y$.

The finite-dimensional space $H_0(Y)$ consists of pairs (f, b) where f is a constant function and b is a harmonic 1-form on Y. Split

$$H_0(Y) = H_{0,-}(Y) \oplus H_{0,+}(Y) \tag{5.19}$$

where

$$H_{0,-}(Y) = \operatorname{im}(H^1(X) \to H^1(Y))$$

and $H_{0,+}(Y)$ is the orthogonal complement of this space in $H_0(Y)$.

LEMMA 5.10. *Suppose that $u \in \ker(D) \cap H^s$ and $u|Y \in H_0(Y) \oplus H_+^{s-1/2}(Y)$. Then if $H \geqslant 0$ and is strictly positive at at least one point, it follows that the component u_0 of $u|Y$ in $H_0(Y)$ must lie in $H_{0,-}(Y)$ and the positive frequency part u_+ of $u|Y$ is zero.*

Proof. For u as given, we have, from (5.8),

$$0 = \|\nabla u\|^2 + \int_Y (H|\iota_\nu u|^2 + (u, D_Y u)_Y) \mathrm{d}\mu_Y, \tag{5.20}$$

and all terms on the RHS are separately $\geqslant 0$. Hence they are all zero. It follows in particular that $\int_Y (u, D_Y u)_Y d\mu_Y = 0$, so $u_+ = 0$. Thus, $u|Y = u_0 = (f, b)$ where f is constant and b is harmonic. Since $H \geqslant 0$ with strict inequality at some point, $f = \iota_\nu u = 0$. Moreover, $\nabla u = 0$ in X which, since X is Ricci-flat, is equivalent to $du = 0 = d^* u$. Thus $u \in \mathscr{H}^1_\perp(X)$, given in (3.12), which is isomorphic to $H^1(X)$, and b is its restriction to the boundary. Since $b = 0$ implies $u = 0$ (since $\nabla u = 0$ in X), we have that u is uniquely determined by its boundary value b, which defines a unique element in $H_{0,-}(Y)$. □

Combining Lemma 5.10 with Theorem 5.8, we obtain the following.

PROPOSITION 5.11. *Under the positive mean curvature assumption 5.1, we have a natural isomorphism*

$$\ker(D) \cap H^s \cong H_-^{s-1/2}(Y) \oplus H_{0,-}(Y). \tag{5.21}$$

Proof. The map is given by restriction to the boundary followed by projection onto $H_-^{s-1/2}(Y) \oplus H_{0,-}(Y)$.

Given $v = (v_-, v_0) \in H_-^{s-1/2}(Y) \oplus H_{0,-}(Y)$, by definition there exists u_0 with $Du_0 = 0$ and $u_0|Y = v_0$. Then

$$\mathscr{P} v_- + u_0 \in \ker D \tag{5.22}$$

and the projection to $H_-^{s-1/2}(Y) \oplus H_{0,-}(Y)$ of this element is (v_-, v_0). Hence (5.21) is surjective.

Conversely, suppose $u \in \ker(D)$ has $u|Y \in H_0(Y) \oplus H_+^{s-1/2}(Y)$. By Lemma 5.10, $u|Y \in H_{0,-}(Y)$ and this proves that (5.21) is also injective. □

5.4.1. More Hodge theory. On our compact manifold X with boundary inclusion $\iota : Y \to X$, the intersection pairing is well defined on the space

$$\ker(H^2(X) \to H^2(Y)) = \operatorname{im}(H^2(X, Y) \to H^2(X)) \tag{5.23}$$

by the usual formula

$$[\alpha] \cup [\beta] = \int_X \alpha \wedge \beta, \tag{5.24}$$

where we need $\iota^*(\alpha) = \iota^*(\beta) = 0$ for this to be well defined in cohomology. Thus we may choose a decomposition

$$\ker(H^2(X) \to H^2(Y)) = H^2_+(X) \oplus H^2_-(X) \tag{5.25}$$

such that (5.24) is positive-definite on $H^2_+(X)$, negative-definite on $H^2_-(X)$. By choosing a complement $H^2_0(X)$ of (5.23) in $H^2(X)$, we complete (5.25) to a decomposition

$$H^2(X) = H^2_+(X) \oplus H^2_-(X) \oplus H^2_0(X). \tag{5.26}$$

and the dimensions of these spaces depend only on the topology of the pair (X, Y).

By the Hodge theory in Theorem 3.8,

$$H^2(X) \cong \mathscr{H}^2_\perp = \{\alpha \in \Omega^2(X) : \mathrm{d}\alpha = \mathrm{d}^*\alpha = 0,\ \iota^*(*\alpha) = 0\} \tag{5.27}$$

and so we have an isomorphism

$$\ker(H^2(X) \to H^2(Y)) \cong \{\alpha \in \mathscr{H}^2_\perp : [\iota^*\alpha] = 0 \in H^2(Y)\}. \tag{5.28}$$

Notice that the projections from Λ^2 to $\Lambda^2_\pm$ given by

$$P_\pm(\alpha) = \tfrac{1}{2}(\alpha \pm *\alpha) \tag{5.29}$$

map closed and coclosed 2-forms to $\mathscr{Z}^2_\pm(X)$ and if $\iota^*(*\alpha) = 0$ then $\iota^*(2P_\pm(\alpha)) = \iota^*(\alpha)$. Thus, if we define finite-dimensional spaces

$$\mathscr{H}^2_\pm(X) = \{P_\pm(\alpha) : \alpha \in \mathscr{H}^2_\perp,\quad [\iota^*\alpha] = 0 \in H^2(Y)\} \subset \mathscr{Z}^2_\pm(X), \tag{5.30}$$

$$\mathscr{H}^2_0(X) = \{\alpha \in \mathscr{H}^2_\perp : [*\alpha] = 0 \in H^2(X)\}, \tag{5.31}$$

$$\mathscr{H}^2_{0,\pm}(X) = P_\pm \mathscr{H}^2_0(X) \subset \mathscr{Z}^2_\pm(X), \tag{5.32}$$

where $\mathscr{Z}^2_\pm(X)$ are the closed self-dual/anti-self-dual 2-forms on X, then we have the following.

THEOREM 5.12. *In the notation above, $\mathscr{H}^2_\pm(X) \cong H^2_\pm(X)$ and $\mathscr{H}^2_{0,\pm}(X) \cong H^2_0(X)$.*

Proof. Given the isomorphism (5.28) and the fact that $P_+ + P_- = \mathrm{id}$, we see that

$$\mathscr{H}^2_+ \oplus \mathscr{H}^2_- \cong \ker(H^2(X) \to H^2(Y)).$$

Moreover, the intersection form is positive-definite on $\mathscr{H}^2_+$ and negative-definite on $\mathscr{H}^2_-$. The decomposition (5.25) then implies that $\mathscr{H}^2_\pm \cong H^2_\pm(X)$.

Now consider $H^2_0(X)$, which is isomorphic to the cokernel of the map

$$H^2(X, Y) \to H^2(X). \tag{5.33}$$

Since the $*$-operator interchanges the spaces in (5.33), it also interchanges the kernel and cokernel of this map. In particular, (5.25) is complemented in $H^2(X)$

by the classes represented by $\alpha \in \mathscr{H}^2_\perp(X)$ such that $*\alpha$ is in the kernel of (5.33); that is, by

$$\{[\alpha] \in H^2(X) : \alpha \in \mathscr{H}^2_0(X)\}. \tag{5.34}$$

In fact, (5.34) is the L^2 orthogonal complement of (5.25). (It is easy to see that these two spaces are orthogonal inside $H^2(X)$, and the argument just given shows that they span $H^2(X)$.) Thus we may set $H^2_0(X)$ equal to (5.34) so that (5.26) holds.

For $\alpha \in \mathscr{H}^2_0(X)$, we see that $[2P_\pm(\alpha)] = [\alpha] \in H^2(X)$. It follows that $\mathscr{H}^2_{0,\pm}(X) = P_\pm \mathscr{H}^2_0(X)$ are isomorphic to $H^2_0(X)$ as claimed. $\square$

REMARK 5.13. The fact that we can choose a complement $H^2_0(X)$ of (5.25) in $H^2(X)$ which can be represented equally well by self-dual or anti-self-dual forms shows clearly that the cup product is not well defined on this space!

The next result gives a 'standard form' for any element of $\mathscr{Z}^2_-(X)$.

PROPOSITION 5.14. *We have the following direct sum decomposition:*

$$\mathscr{Z}^2_-(X) = \mathscr{H}^2_{0,-}(X) \oplus \mathscr{H}^2_-(X) \oplus \{\mathrm{d}a \in \mathrm{d}\Omega^1(X) : (\mathrm{d}^* + \mathrm{d}_+)a = 0\}. \tag{5.35}$$

Moreover, with respect to the decomposition of 1*-forms in a collar neighbourhood*

$$a = f\,\mathrm{d}\rho + b$$

(see (5.1)*) we may assume* $f|Y = 0$.

Proof. It is clear that the right-hand side of (5.35) is contained in $\mathscr{Z}^2_-(X)$ since if $\mathrm{d}_+ a = 0$ then $\mathrm{d}a$ is anti-self-dual and exact. Let $\alpha \in \mathscr{Z}^2_-(X)$. By Theorem 5.12, the corresponding cohomology class $[\alpha]$ has components only in $H^2_-(X) \oplus H^2_0(X)$, and these have unique representatives $(\alpha_-, \alpha_0) \in \mathscr{H}^2_-(X) \oplus \mathscr{H}^2_{0,-}(X)$. Then $\alpha - \alpha_- - \alpha_0$ is exact, so we may write

$$\mathrm{d}a' = \alpha - \alpha_- - \alpha_0,$$

and automatically

$$\mathrm{d}_+ a' = 0.$$

Suppose further that

$$a' = f'\,\mathrm{d}\rho + b' \text{ near } Y.$$

We have not yet arranged $f'|Y = 0$ or $\mathrm{d}^* a' = 0$. For this, define $a = a' + \mathrm{d}u$, so $\mathrm{d}a = \mathrm{d}a'$,

$$\mathrm{d}^* a = \mathrm{d}^*\mathrm{d}u + \mathrm{d}^* a'$$

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and if $a = f\,\mathrm{d}\rho + b$ then

$$f = f' + \partial_\rho u \text{ on } Y.$$

Solving Poisson's equation $\mathrm{d}^*\mathrm{d}u = -\mathrm{d}^*a'$ with the Neumann condition $\partial_\rho u|Y = -f'|Y$ yields a satisfying $\mathrm{d}^*a = 0$ and $f|Y = 0$ as required. □

Let

$$K^s = \{a \in \ker(D) \cap H^s(X, \Lambda^1) : a_\perp = 0\}.$$

Proposition 5.14 shows that $\mathrm{d}K^s$ is isomorphic to the space of exact ASD 2-forms. The next result shows that, up to $H^1(X)$, d gives an isomorphism of K^s onto $\mathrm{d}K^s$.

PROPOSITION 5.15. *With the above definitions, the following sequence is exact:*

$$0 \to H^1(X) \to K^s \to \mathrm{d}K^s \to 0. \tag{5.36}$$

Proof. Proposition 5.14 shows that the sequence is exact at $\mathrm{d}K^s$. It is also clear that it is exact at $H^1(X)$ and that it is a complex. It remains to show that the kernel of d is precisely $H^1(X)$, identified as $\mathscr{H}^1_\perp(X)$, the harmonic 1-forms a with $a_\perp = 0$. Suppose $\mathrm{d}a = 0$, with $a \in K^s$. Since $\mathrm{d}^*a = 0$ and $a_\perp = 0$ as part of the definition of K^s, $a \in \mathscr{H}^1_\perp(X)$ as required. □

REMARK 5.16. Recall that H_λ is the λ-eigenspace of D_Y. For real λ, put

$$G_\lambda = \{u \in H_\lambda : \mathrm{d}^*u = 0\}. \tag{5.37}$$

Clearly G_λ is finite-dimensional for every λ and the set of λ with $G_\lambda \neq 0$ is discrete. It can also be shown that the set of λ with $G_\lambda \neq 0$ is unbounded above and below, just as for the H_λ. Denote by $G^{s-1/2}_-(Y)$ the completion of the direct sum $\bigoplus_{\lambda<0} G_\lambda$. Then K^s is isomorphic to $H_{0,-}(Y) \oplus G^{s-1/2}_-(Y)$ (and is infinite-dimensional), and $\mathrm{d}K^s$ is isomorphic to $G^{s-1/2}_-(Y)$. This follows at once from (5.11).

We may now prove Theorem 1.6, for which we need a definition of $\mathscr{W}_+$. Recall the splitting

$$H^{s-1/2}(Y) = H^{s-1/2}_-(Y) \oplus H_{0,-}(Y) \oplus H_{0,+}(Y) \oplus H^{s-1/2}_+(Y)$$

where the suppressed bundle is $T^*X|Y = \mathbb{R} \oplus T^*Y$. The space $\mathscr{W}^s$ is by definition

$$\mathscr{W}^s = \ker(L^*L) \cap H^s$$

and restriction to the boundary gives an isomorphism

$$\mathscr{W}^s \cong H^{s-1/2}(Y)$$

(where we identify vector fields with 1-forms using the metric). Define

$$\mathscr{W}^s_+ = \{w \in \mathscr{W}^s : w|Y \in H_{0,+}(Y) \oplus H^{s-1/2}_+(Y)\}.$$

We shall prove the following sharpened version of Theorem 1.6.

THEOREM 5.17. *Let ω be a smooth hyperkähler triple on X, inducing positive mean curvature on the boundary Y. Then the gauge-fixed tangent space*

$$T_{[\omega]}\mathscr{M}^s_+ = T_\omega \mathscr{H}^s \cap \ker(L^*) \tag{5.38}$$

is isomorphic to the direct sum

$$[\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s] \oplus L(\mathscr{W}^{s+1}_+). \tag{5.39}$$

Moreover, the summands are naturally isomorphic to the spaces of boundary values

$$\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s \cong \mathscr{H}^2_{0,-}(X) \oplus \mathscr{H}^2_-(X) \oplus G^{s+1/2}_-(Y) \tag{5.40}$$

and

$$L(\mathscr{W}^{s+1}_+) \cong H_{0,+}(Y) \oplus H^{s+1/2}_+(Y). \tag{5.41}$$

Proof. Note first of all that L is injective on $\mathscr{W}^{s+1}_+$. Indeed, if $w \in \mathscr{W}^{s+1}_+$ and $Lw = 0$, then in particular $L_+w = 0$. But Proposition 3.6 shows that L_+w can be identified with Dw. By Lemma 5.10 $Dw = 0$ and $w \in \mathscr{W}^{s+1}_+$ implies that $w = 0$.

The same argument shows

$$[\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3] \cap L(\mathscr{W}^{s+1}_+) = 0.$$

Indeed, if $w \in \mathscr{W}^{s+1}_+$ is such that Lw lies in the intersection, then $L_+w = 0$, so $w = 0$ as before.

Since (5.40) follows from our earlier discussion and the isomorphism (5.41) follows from the injectivity of L on $\mathscr{W}^{s+1}_+$, it remains only to prove that the direct sum (5.39) is equal to the tangent space as given in (4.1):

$$[\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s] + L(\mathscr{W}^{s+1}) \subset H^s(X, \Lambda^2 \otimes \mathbb{R}^3).$$

For this, let $w \in \mathscr{W}^{s+1}$ and

$$Lw = L_+w + L_-w \tag{5.42}$$

be the self-dual/anti-self-dual decomposition of the triple Lw. Since $L_-w = \mathrm{d}_-(\iota_w\omega) \in \mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s$, we just need to show that we can find $w' \in \mathscr{W}^{s+1}_+$ with

$$L_+w = L_+w'.$$

Let the boundary value of w be written $w_- + w_+$ where

$$w_- \in H^{s-1/2}_-(Y) \oplus H_{0,-}(Y), \quad w_+ \in H^{s-1/2}_+(Y) \oplus H_{0,+}(Y). \tag{5.43}$$

Using Proposition 5.11, we find u with $Du = 0$ and $u|Y = w_- + u_+$, where $u_+ \in H^{s-1/2}_+(Y) \oplus H_{0,+}(Y)$. Recalling again that $D = L_+$ and that $L^*L = L^*L_+$ by Lemma 4.4, if we define

$$w' = w - u,$$

then we have

$$L_+w' = L_+w, \quad L^*Lw' = L^*L_+w' = L^*L_+w = 0,$$

and $w'|Y = w_+ - u_+$ is positive frequency. Hence $w' \in \mathscr{W}^{s+1}_+$ with $L_+w = L_+w'$ as required. □

5.5. Proof of Theorem 1.7. We now show that the moduli space $\mathscr{M}_+$ of smooth (up to the boundary) hyperkähler triples inducing positive mean curvature on the boundary is a manifold.

First we note that $\mathscr{M}_+$ is well defined: every smooth hyperkähler triple ω (or rather its $\mathscr{G}^{s+1}_0$-equivalence class) has a neighbourhood in $\mathscr{M}^s_+$ homeomorphic to a ball in

$$[\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3 \cap H^s] \oplus L(\mathscr{W}^{s+1}_+).$$

The elements of this ball are smooth in the interior and of finite regularity at the boundary. However, the parametrization in terms of boundary values shows that there is a nonzero subspace of smooth elements of this space: simply choose boundary values in H^s on Y for every s (and also satisfying the relevant frequency conditions).

The issue is that the gauged-fixed tangent spaces

$$T_{[\omega]}\mathscr{M}_+ \cong \mathscr{T} = [\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3] \oplus L(\mathscr{W}_+) \tag{5.44}$$

depend on ω: the notion of anti-self-duality depends on the metric, as does L, and the operator D_Y, which defines the frequency decomposition that defines $\mathscr{W}_+$.

Although these spaces move, the claim is that they are all naturally isomorphic on the path components of $\mathscr{M}_+$.

PROPOSITION 5.18. *Let ω_0 and ω_1 be two smooth hyperkähler triples in the same path component of $\mathscr{M}_+$. Let $\mathscr{T}_0$ and $\mathscr{T}_1$ be the corresponding gauge-fixed tangent spaces as given by (5.44). Then the restriction to $\mathscr{T}_0$ of the L^2-orthogonal projection on $\mathscr{T}_1$ is an isomorphism.*

Proof. Denote by $\mathscr{T}_i^{\perp}$ the L^2-orthogonal complement of $\mathscr{T}_i$ in $L^2(X, \Lambda^2 \otimes \mathbb{R}^3)$. Note first the standard fact that the restricted L^2-orthogonal projection maps are isomorphisms if and only if

$$\mathscr{T}_0 \cap \mathscr{T}_1^{\perp} = 0 = \mathscr{T}_1 \cap \mathscr{T}_0^{\perp}. \tag{5.45}$$

To see this, let $\pi : \mathscr{T}_0 \to \mathscr{T}_1$ be the restricted projection map. Then π is injective if and only if $\mathscr{T}_0 \cap \mathscr{T}_1^{\perp} = 0$. If π is not surjective, there is $\xi \in \mathscr{T}_1$, orthogonal to the image $\pi(\mathscr{T}_0)$ of $\mathscr{T}_0$ in $\mathscr{T}_1$. If $\eta \in \mathscr{T}_0$ and we write $\eta = \eta_1 + \eta_1^{\perp} \in \mathscr{T}_1 \oplus \mathscr{T}_1^{\perp}$, then

$$\langle \xi, \eta \rangle = \langle \xi, \eta_1 \rangle = 0$$

because $\pi(\eta) = \eta_1$. This is true for all $\eta \in \mathscr{T}_0$ so $\xi \in \mathscr{T}_1 \cap \mathscr{T}_0^{\perp}$. So the assumption (5.45) implies that $\xi = 0$, and π is surjective.

It therefore suffices to prove (5.45). By hypothesis, there is a path of hyperkähler triples $\omega(t)$, $0 \leqslant t \leqslant 1$, connecting ω_0 to ω_1 in $\mathscr{M}_+$, and a corresponding continuous path $\mathscr{T}_t$ of gauge-fixed tangent spaces. If one of (5.45) fails, then we may suppose by symmetry that $\mathscr{T}_1 \cap \mathscr{T}_0^{\perp} \neq 0$.

We shall use the boundary value description:

$$\begin{aligned} &\mathscr{Z}_-^2(X) \otimes \mathbb{R}^3 \cong \mathscr{H}_{0,-}^2(X) \oplus \mathscr{H}_-^2(X) \oplus G_-(Y) \quad \text{and} \\ &L(\mathscr{W}_+) \cong H_{0,+}(Y) \oplus H_+(Y). \end{aligned}$$

Note that G_λ given in (5.37) can also be characterized as the subspace of H_λ, the λ-eigenspace of D_Y, with the function component zero. So we have a decomposition

$$\Omega^1(Y) = G_-(t) \oplus G_0(t) \oplus G_+(t)$$

for all t. Notice that $G_0(t)$ consists of the harmonic 1-forms on Y, so $G_0(t) \cong H^1(Y)$. Let $F_+(t)$ denote the space $H_{0,+}(Y) \oplus H_+(Y)$ as defined by ω_t and $F_-(t)$ be its orthogonal complement. Thus the boundary values of $\mathscr{W}_+(t)$ lie in $F_+(t)$. Moreover, recall that $H_{0,+}(Y) \cong H^1(Y)/\operatorname{im}(H^1(X) \to H^1(Y))$, so $H_{0,+}(Y)$ has topologically determined dimension.

Suppose $\mathscr{T}_1 \cap \mathscr{T}_0^{\perp} = 0$ fails. Then we have

$$G_-(1) \cap [G_0(0) \oplus G_+(0)] \neq 0 \text{ or } F_+(1) \cap F_-(0) \neq 0.$$

Suppose the first possibility occurs. Then for some t, $G_-(t)$ contains an element of $G_0(0)$. However, as we observed, $G_0(t)$ is of fixed dimension equal to $\dim H^1(Y)$, giving a contradiction. The second possibility is ruled out for a similar reason, since $H_{0,+}(Y)$ has a fixed dimension. $\square$

We now prove Theorem 1.7 which we restate for convenience.

THEOREM. *The moduli space $\mathscr{M}_+$ of hyperkähler triples on X inducing positive mean curvature on the boundary Y, modulo the action of $\mathscr{G}_0$, is a Fréchet manifold with*

$$T_{[\omega]}\mathscr{M}_+ = [\mathscr{Z}^2_-(X) \otimes \mathbb{R}^3] \oplus L(\mathscr{W}_+).$$

It should be noted that the spaces on the right-hand side depend on ω.

Proof. We have seen that on each connected component all tangent spaces are canonically identifiable with each other. It follows from this that the transition maps between different coordinate patches are smooth as follows.

On any component of an overlap between two charts, which are necessarily determined by $[\omega_0]$ and $[\omega_1]$ which are path-connected, $\mathscr{M}_+$ can be written as a smooth graph over the tangent spaces $\mathscr{T}_0$ and $\mathscr{T}_1$. Since $\mathscr{T}_1$ is a graph over $\mathscr{T}_0$ by Proposition 5.18, the transition map on the component will be a composition of projections from and to smooth graphs over open sets in Fréchet spaces, and thus is smooth. □

6. SU(2)-invariant examples

Complete SU(2)-invariant hyperkähler metrics in 4 dimensions have been well understood for many years [1, 12]. We give a brief description of the classification from our present point of view as a further illustration of the formalism of triples and to justify explicitly the claim of the Introduction that a given metric on a 3-manifold can arise by restriction of two nonisometric hyperkähler metrics.

The SU(2)-invariant hyperkähler metrics fall into two classes, according as the corresponding hyperkähler triple is fixed or rotated under the SU(2)-action. In both cases one seeks hyperkähler triples of the form $\omega = \mathrm{d}t \wedge \eta_t + *_t\eta_t$ where η_t is a family of left-invariant coclosed coframes on SU(2) (or quotients thereof) and $*_t$ is the induced Hodge star on each hypersurface in the 4-manifold given by fixing t. We briefly review the analysis of these gravitational instantons.

For the case where the triple is fixed one chooses the standard left-invariant coframing $\eta = (\eta_1, \eta_2, \eta_3)$ of SU(2) such that $\mathrm{d}\eta_i = \epsilon_{ijk}\eta_j \wedge \eta_k$ and considers $\omega = (\omega_1, \omega_2, \omega_3)$ where

$$\begin{gathered}\omega_1 = f_1\mathrm{d}t \wedge \eta_1 + f_2 f_3 \eta_2 \wedge \eta_3, \quad \omega_2 = f_2\mathrm{d}t \wedge \eta_2 + f_3 f_1 \eta_3 \wedge \eta_1, \\ \omega_3 = f_3\mathrm{d}t \wedge \eta_3 + f_1 f_2 \eta_1 \wedge \eta_2,\end{gathered} \tag{6.1}$$

for a triple of t-valued functions $f = (f_1, f_2, f_3)$. This triple automatically satisfies the orthogonality conditions (1.1) provided that $f_1 f_2 f_3 \neq 0$, and so will define a hyperkähler structure if $\mathrm{d}\omega_i = 0$. This is equivalent to the following system of ODEs:

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} = \frac{f_2^2 + f_3^2 - f_1^2}{f_2 f_3}, \quad \frac{\mathrm{d}f_2}{\mathrm{d}t} = \frac{f_3^2 + f_1^2 - f_2^2}{f_3 f_1}, \quad \frac{\mathrm{d}f_3}{\mathrm{d}t} = \frac{f_1^2 + f_2^2 - f_3^2}{f_1 f_2}. \tag{6.2}$$

There are then three possibilities.

- When $f_1 = f_2 = f_3$, one quickly obtains the standard flat metric on $\mathbb{R}^4$ and the standard triple where $f_1 = f_2 = f_3 = t = r$, the radial distance from the origin. The closed framings of the 2-forms on the S^3 orbits of the action are in this case simply
$$r^2(\eta_2 \wedge \eta_3, \eta_3 \wedge \eta_1, \eta_1 \wedge \eta_2).$$
- When $f_1 \neq f_2 = f_3 = r$, then one finds that $f_1 = r(1 - c^4/r^4)^{1/2}$ for a constant $c > 0$ where $r \geqslant c$, which gives the Eguchi–Hanson metric on T^*S^2, given for $r > c$ by:
$$\left(1 - \frac{c^4}{r^4}\right)^{-1} \mathrm{d}r^2 + r^2\left(1 - \frac{c^4}{r^4}\right)\eta_1^2 + r^2\eta_2^2 + r^2\eta_3^2.$$
The orbits where $r > c$ is constant are $\mathbb{RP}^3$s, whereas the exceptional orbit where $r = c$ gives the S^2 'bolt' which is the zero section. The closed framings of $\Lambda^2 T^*\mathbb{RP}^3$ are:
$$(r^2\eta_2 \wedge \eta_3, r^2(1 - c^4/r^4)^{1/2}\eta_3 \wedge \eta_1, r^2(1 - c^4/r^4)^{1/2}\eta_1 \wedge \eta_2).$$
As $r \to \infty$ these framings approach the standard closed framing on $\mathbb{RP}^3$. The induced metric on each $\mathbb{RP}^3$ is a Berger metric $r^2(1 - c^4/r^4)\eta_1^2 + r^2\eta_2^2 + r^2\eta_3^2$, where the relative 'squashing' of the circle corresponding to η_1 can take any value in $(0, 1)$. Taking the same closed framings on S^3 will not lead to a complete invariant hyperkähler metric, but instead to a double cover of the Eguchi–Hanson space.
- When all of the f_i are distinct, one does not obtain a complete metric.

If one now wants to study invariant hyperkähler metrics where the action rotates the frame one views the standard left-invariant coframe on SU(2) as a 1-form taking values in the imaginary quaternions (rather than $\mathbb{R}^3$). If we also identify points $q \in \mathrm{SU}(2) \cong S^3$ with unit quaternions, we may define a triple $\hat{\omega}$ of 2-forms by $\hat{\omega}|_q = q\omega q^{-1}$, where ω is as in (6.1) (now viewed as taking values in the imaginary quaternions). This time, in place of (6.2), we obtain

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} = \frac{f_2^2 + f_3^2 - f_1^2 - 2f_2 f_3}{f_2 f_3}, \quad \frac{\mathrm{d}f_2}{\mathrm{d}t} = \frac{f_3^2 + f_1^2 - f_2^2 - 2f_3 f_1}{f_3 f_1}, \quad \frac{\mathrm{d}f_3}{\mathrm{d}t} = \frac{f_1^2 + f_2^2 - f_3^2 - 2f_1 f_2}{f_1 f_2}. \tag{6.3}$$

Again, there are three possibilities.

- When $f_1 = f_2 = f_3$, one unsurprisingly again obtains the flat hyperkähler metric on $\mathbb{R}^4$ since $f_1 = f_2 = f_3 = -t$.

- When $f_1 \neq f_2 = f_3$, one can solve (6.3) with $f_1 = 2m(r-m)^{1/2}(r+m)^{-1/2}$ and $f_2 = f_3 = (r^2 - m^2)^{1/2}$ for a constant $m > 0$ where $r \geqslant m$. This leads again to a metric on $\mathbb{R}^4$ which now has cubic volume growth at infinity, known as the Taub–NUT metric (with 'mass' m):

$$\frac{1}{4}\frac{r+m}{r-m}\mathrm{d}r^2 + 4m^2\frac{r-m}{r+m}\eta_1^2 + (r^2-m^2)\eta_2^2 + (r^2-m^2)\eta_3^2.$$

  So we see on the hyperspheres where r is constant the induced metric, as for the Eguchi–Hanson metric, is a Berger metric where the relative 'squashing' of the circle factor on S^3 corresponding to η_1 can again take any value in $(0, 1)$. This shows that although the metrics on S^3 here and in the double cover of the Eguchi–Hanson metric are the same, the closed framings of the bundle of 2-forms are different so that one finds different hyperkähler triples (and hence metrics) extending them, as we must have by Theorem 1.4. One can describe the closed framings of $\Lambda^2 T^* S^3$ as $q\gamma q^{-1}$ where γ is the triple

$$((r^2-m^2)\eta_2 \wedge \eta_3, 2m(r-m)\eta_3 \wedge \eta_1, 2m(r-m)\eta_1 \wedge \eta_2).$$

- When the f_i are all distinct, one can solve explicitly (6.3) using elliptic functions and obtain the Atiyah–Hitchin metric, defined on $S^4 \setminus \mathbb{RP}^2$, which arises in the study of moduli spaces of monopoles on $\mathbb{R}^3$. The metric near the Veronese $\mathbb{RP}^2$ at infinity is asymptotic to the Taub–NUT metric with mass $m < 0$. Here the orbits are $\mathrm{SO}(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, except for an exceptional $\mathbb{RP}^2$ orbit, and one can write the induced closed framings of the 2-forms in terms of elliptic functions, and also observe that the induced metrics are no longer Berger metrics. (One may also consider the double cover of the Atiyah–Hitchin metric on $\mathbb{CP}^2 \setminus S^2$, that has $\mathrm{SU}(2)/\mathbb{Z}_4$ as the orbits of the action except for a special S^2 orbit, and which now can be deformed in a 1-parameter family of gravitational instantons that are not $\mathrm{SU}(2)$-invariant [8].)

We focus on the simplest example of a hyperkähler 4-manifold with boundary arising from this analysis, namely the unit 4-ball with the flat metric. We let η be the standard left-invariant coframe on S^3, let ω be the standard hyperkähler triple on B^4 and let $\gamma = \omega|S^3$.

As we have seen, a key point is to study the closed anti-self-dual 2-forms. We see that in this case they are simple to describe explicitly using negative frequency data on the boundary.

LEMMA 6.1. *Let* $E_k = \{\alpha \in \Omega^2(S^3) : \mathrm{d} * \alpha = -k\alpha\}$ *for* $k \in \mathbb{N} \setminus \{1\}$. *Closed anti-self-dual 2-forms on* B^4 *are given by* $\sum_{k=2}^{\infty} \mathrm{d}(r^k * \alpha_k)$ *where* $\alpha_k \in E_k$. *Hence,* $\mathscr{Z}^2_-(B^4)$ *is isomorphic to*

$$\mathscr{Z}^2_-(S^3) = \left\{\alpha \in \Omega^2(S^3) : \mathrm{d}\alpha = 0,\ \alpha \in \bigoplus_{k=2}^{\infty} E_k\right\}.$$

Proof. The eigenvalues of $\mathrm{d}*$ on closed 2-forms on S^3 are well known to be $k \in \mathbb{Z} \setminus \{0, \pm 1\}$ with multiplicity $k^2 - 1$. The result then follows from the one-to-one correspondence between closed anti-self-dual 2-forms on B^4 and eigenforms for $\mathrm{d}*$ on S^3. □

This lemma together with our main results allow us to explicitly describe the moduli space of hyperkähler triples on B^4 in terms of boundary values on S^3 as follows. Notice that, by Theorem 1.1, the true space of hyperkähler deformations of the flat metric on B^4, working up to the action of diffeomorphisms which can move the boundary S^3, is described by the quotient $\mathscr{T}/L(\mathscr{W})$ where $\mathscr{T} = [\mathscr{Z}^2_-(B^4) \otimes \mathbb{R}^3] + L(\mathscr{W})$.

PROPOSITION 6.2. *On* B^4, $\mathscr{T}/L(\mathscr{W}) \cong \{\alpha \in \mathscr{Z}^2_-(S^3) \otimes \mathbb{R}^3 : (\alpha_i, \gamma_j) \in C^\infty(S^3, S^2_0\mathbb{R}^3)\}$.

Proof. Elements $Lv \in L(\mathscr{W})_- = L(\mathscr{W}) \cap [\mathscr{Z}^2_-(B^4) \otimes \mathbb{R}^3]$ satisfy $L_+v = 0$, which is equivalent to the Dirac equation $Dv = 0$ by Proposition 3.6, and thus are determined by the boundary values of v. Moreover, we know that Lv is given as a sum of forms which are homogeneous in r by Lemma 6.1. We thus restrict to the case where $v = r^k f\partial_r + r^{k-1}w$ where f is a function on S^3, w is a vector field on S^3 and $k \in \mathbb{N}$. We calculate from the equation $L_+v = 0$ that we have (viewing w as a 1-form)

$$\mathrm{d}^* w = (k+1)f \quad \text{and} \quad \mathrm{d}f - *\mathrm{d}w = (k+3)w.$$

We deduce that, recalling that η is the standard coframe on S^3,

$$Lv|S^3 = L_-v|S^3 = *\mathrm{d}(i_w\eta) - (k+1)f\gamma + (k+3)w \wedge \eta.$$

Hence, $\alpha \in \mathscr{Z}^2_-(Y)$ is L^2-orthogonal to $Lv|S^3$ if and only if

$$\langle \alpha, *\mathrm{d}(i_w\eta) - (k+1)f\gamma + (k+3)w\wedge\eta\rangle = -(k+1)f\langle\alpha, \gamma\rangle + (k+3)\langle\alpha, w\wedge\eta\rangle = 0,$$

since $\mathrm{d}\alpha = 0$. Hence, by imposing this condition for all $Lv \in L(\mathscr{W})_-$, which amounts to varying f and w (and hence k) so that $Lv|S^3 \in \mathscr{Z}^2_-(S^3)$, we deduce that we must have, in summation convention: $\alpha_i \wedge \eta_i = 0$ and $\epsilon_{ijk}\alpha_j \wedge \eta_k = 0$. This is equivalent to the vanishing of the trace and skew parts of the matrix (α_i, γ_j) of inner products. □

It is natural to ask what happens when one takes positive frequency data on S^3 instead. One knows that this cannot fill in to a hyperkähler triple on B^4, but in general one cannot say more than that. However, in a special case we can explicitly demonstrate that we can take arbitrarily small positive frequency data which has no hyperkähler filling, by relating the deformation of the boundary data to the Eguchi–Hanson metric.

PROPOSITION 6.3. *For $c \in (0, 1)$ let*

$$\hat{\gamma} = (\eta_2 \wedge \eta_3, (1 - c^4)^{1/2}\eta_3 \wedge \eta_1, (1 - c^4)^{1/2}\eta_1 \wedge \eta_2).$$

Then $\hat{\gamma} - \gamma$ has positive frequency with respect to $\mathrm{d}*$ *on S^3 and there does not exist a hyperkähler triple $\hat{\omega}$ on B^4 such that $\hat{\omega}|S^3 = \hat{\gamma}$.*

Proof. We see that $\hat{\gamma} - \gamma$ consists of constant multiples of $\eta_i \wedge \eta_j$ which have eigenvalue 2 with respect to $\mathrm{d}*$ and so are positive frequency. We know that $\hat{\gamma}$ has a unique hyperkähler extension $\hat{\omega}$ given by the ODEs (6.2) derived above, which lead to the Eguchi–Hanson triple

$$\hat{\omega} = (r\mathrm{d}r \wedge \eta_1 + r^2\eta_2 \wedge \eta_3, rf^{-1}\mathrm{d}r \wedge \eta_2 + r^2 f\eta_3 \wedge \eta_1, rf^{-1}\mathrm{d}r \wedge \eta_3 + r^2 f\eta_1 \wedge \eta_2)$$

where $f(r) = (1 - c^4/r^4)^{1/2}$ for $r > c$. The issue is whether this can be extended smoothly to $r = c$ to give a hyperkähler metric on B^4, but this is not possible by the classification of SU(2)-invariant hyperkähler 4-manifolds. □

As we saw above, there are two SU(2)-invariant hyperkähler metrics on B^4: the flat metric and the Taub–NUT metric. We have an induced closed framing of the 2-forms on S^3 in Taub–NUT when $r = m + 1/2m$ given by $q\hat{\gamma}q^{-1}$ where

$$\hat{\gamma} = \left(\left(1 + \frac{1}{4m^2}\right)\eta_2 \wedge \eta_3, \eta_3 \wedge \eta_1, \eta_1 \wedge \eta_2\right).$$

Hence, if we consider the second standard framing of the 2-forms on S^3 given by $q\gamma q^{-1}$, then $\hat{\gamma} - \gamma = (4m^2)^{-1}(\eta_2 \wedge \eta_3, 0, 0)$, which can clearly be made arbitrarily small by making the mass m sufficiently large. Notice that in Proposition 6.3 this difference was seen to be positive frequency with respect to γ. However, we observe that in the analysis of the SU(2)-invariant hyperkähler 4-manifolds above that the induced orientation on S^3 is such that $-\eta_1 \wedge \eta_2 \wedge \eta_3 > 0$ (that is reversed). Hence, $q(\hat{\gamma} - \gamma)q^{-1}$ is now negative frequency as we would expect.

We summarize this discussion in a final proposition.

PROPOSITION 6.4. *Let S^3 be endowed with the closed framing of the 2-forms given at $q \in S^3$ by $q\gamma q^{-1}$. For any $m > 0$ there exist closed framings $q\hat{\gamma}q^{-1}$ of the 2-forms on S^3 such that $q(\hat{\gamma} - \gamma)q^{-1}$ is negative frequency with respect to* $\mathrm{d}*$ *on S^3 and the hyperkähler filling of $q\hat{\gamma}q^{-1}$ to B^4 is given by Taub–NUT with mass m.*

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