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PhD Thesis

Teleparallel gravity and its modifications

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Disclaimer

I, Matthew Aaron Wright, confirm that the work presented in this thesis, titled "Teleparallel gravity and its modifications", is my own. Parts of this thesis are based on published work with co-authors Christian Böhmer and Sebastian Bahamonde in the following papers:

- "Modified teleparallel theories of gravity", Sebastian Bahamonde, Christian Böhmer and Matthew Wright, Phys. Rev. D **92** (2015) 10, 104042,
- "Teleparallel quintessence with a nonminimal coupling to a boundary term", Sebastian Bahamonde and Matthew Wright, Phys. Rev. D **92** (2015) 084034,
- "Conformal transformations in modified teleparallel theories of gravity revisited", Matthew Wright, Phys.Rev. D **93** (2016) 10, 103002.

These are cited as [1], [2], [3] respectively in the bibliography, and have been included as appendices. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed:

Date:

Abstract

The teleparallel equivalent of general relativity is an intriguing alternative formulation of general relativity. In this thesis, we examine theories of teleparallel gravity in detail, and explore their relation to a whole spectrum of alternative gravitational models, discussing their position within the hierarchy of Metric Affine Gravity models. Consideration of alternative gravity models is motivated by a discussion of some of the problems of modern day cosmology, with a particular focus on the dark energy problem. Modifications of gravity in the teleparallel framework are examined as potential models to alleviate some of these issues and the relationships between various teleparallel and non-teleparallel modified gravity models are analysed in depth. In particular $f(T, B)$ gravity, where T is a torsion scalar and B is a derivative of a torsion vector, is introduced as a way of analysing both $f(T)$ gravity and $f(R)$ gravity, where R is the Ricci scalar, within the same unified framework. Various theoretical issues of all of these theories are discussed. In a similar way, teleparallel scalar-tensor models are analysed, taking into account coupling between torsion and a scalar field, with dynamical systems techniques utilised to analyse the cosmology of these models. An interesting conformal relationship is found to hold between teleparallel scalar-tensor models and $f(T, B)$ gravity.

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Preface

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Matthew Wright
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List of papers published during my PhD

- "Slowly rotating charged fluid balls in the presence of a cosmological constant", Wright M, General Relativity and Gravitation, 48 (10) 1-15.
- "Buchdahl's inequality in five dimensional Gauss-Bonnet gravity", Wright M, General Relativity and Gravitation, 48 (7), 1-15
- "Conformal transformations in modified teleparallel theories of gravity revisited", Wright M, Phys.Rev. D 93 (2016) 10, 103002
- "Correspondence of $F(R)$ Gravity Singularities in Jordan and Einstein Frames", Bahamonde S, Odintsov S.D., Oikonomou V.K. and Wright M, Annals of Physics 373 (2016) 96-114 .
- "Cosmological dynamics of extended chameleons", Tamanini N and Wright M, JCAP 2016 (2016) 032.
- "The Einstein Static Universe in Scalar-Fluid Theories", Boehmer CG, Tamanini N and Wright M, Phys. Rev. D 92 (2015) 12, 124067
- "Modified teleparallel theories of gravity", Bahamonde S, Boehmer CG and Wright M, Phys. Rev. D 92 (2015) 10, 104042
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- "Teleparallel quintessence with a nonminimal coupling to a boundary term", Bahamonde S and Wright M, Phys. Rev. D 92 (2015) 084034
- "Buchdahl type inequalities in d-dimensions,", Wright M, Class. Quantum Grav. 32 (2015) 215005
- "Interacting quintessence from a variational approach Part II: derivative couplings,", Boehmer CG, Tamanini N and Wright M, Phys. Rev. D 91 (2015) 12, 123003
- "Interacting quintessence from a variational approach Part I: algebraic couplings,", Boehmer CG, Tamanini N and Wright M, Phys. Rev. D 91 (2015) 12, 123002.

Contents

Chapter 1

Introduction

The last 12 months will go down in scientific history as a momentous year for our understanding of the laws of gravity. Just a few months after the general theory of relativity celebrated the one hundredth anniversary of its publication in November 2015, one of its grandest predictions was finally confirmed. Gravitational waves were directly detected by the LIGO collaboration, confirming their existence beyond doubt. This confirmation happened one hundred years after they were first predicted by Albert Einstein in 1916 [4].

Einstein dedicated many years of his life to perfecting arguably his greatest achievement: his general theory of relativity. For the two centuries preceding Einstein, the laws of gravity had been described using Newton's laws of gravitation: an empirical law outlined by Isaac Newton in his seminal Principia in 1687. In Newton's view gravity acted as a conventional force: two objects with mass experience a mutual attraction, in a similar way to two magnets with opposite poles mutually attracting each other. This force law works very well in many circumstances, and to this day Newtonian gravity is an accurate description of reality, provided the objects in question are moving slowly comparative to the speed of light, and that the objects are not excessively massive. Indeed it was Newton's laws which were sufficient for taking man to the moon.

However in 1905, during a seminal year, Einstein published his special theory of relativity, outlining in detail the relationship between space and time, and determining how light and matter travel at high speeds. It revolutionised the way we describe the underlying fabric of the universe. It was based on two postulates: the speed of light is the same for all observers in a vacuum, and that all physical laws are invariant in all inertial (non-accelerating) frames of reference. However, this special theory of relativity was inconsistent with Newton's universal law of gravitation: Newton's gravity acted instantaneously and was not bounded by this upper speed limit.

This set Einstein on a ten year quest to reconcile his theory of special relativity with gravity. In doing so, using powerful ideas from a branch of mathematics known as differential geometry, he completely revolutionised how we think of gravity. His general theory of relativity replaced the gravitational force laws of Newton within an elegant geometric framework. Now matter in a gravitational field travelled along "straight lines" in spacetime, but the gravitational effects of matter caused spacetime itself to deform and curve, so that these straight lines themselves were curved. That is, particles now followed geodesics of a curved manifold. *Gravity was no longer a force, it was a manifestation of the curvature of the underlying spacetime.*

Over the subsequent century, general relativity has passed pretty much every test that has been thrown at it, and continues to agree with observational and experimental evidence to a remarkable degree of precision. The discovery of gravitational waves at the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors in America, provided the first test of general relativity in the strong field regime [5]. Further gravitational wave events have subsequently been detected by the experiment, and the detectors are now at such a sensitivity that many more events are expected to be detected in the near future. The first event detected was that of two black holes coalescing, and the observed signal coincides with what is to be expected from numerical solutions of the field equations of general relativity. On the other scale, in the weak field regime of general relativity, solar system experiments over the last century place very strong bounds, to many decimal places, of any deviations from general relativity.

So it seems like we have a theory of gravity that is theoretically consistent and experimentally supported, at least at the classical level.[1]. In which case, it may appear legitimate to ask: what is the point of this thesis? Why do we need to continue studying gravity theoretically? What is the point of exploring alternative theories of gravity? Well it appears that something is not quite right in our understanding of the universe. There are three unexplained gravitational phenomena, observed by astrophysicists and cosmologists, which are difficult to explain with general relativity alone, which hint at the possibility of new physics. These phenomena appear at length scales beyond those at which we have been able to accurately test general relativity.

Let us outline the phenomena I am referring to. The first of them to be discovered is now commonly referred to as the dark matter problem: essentially this problem is about why there is a huge amount of missing mass in the universe. Experimental evidence for the existence of dark matter comes from two main areas: the behaviour of galactic rotation curves, an issue first observed by Rubin [6] in 1980, and the mass discrepancy problem in galactic clusters. Both of these effects potentially imply there exists additional matter at galactic and extra-galactic scales, that only interacts with normal matter via the weak nuclear force and its main interaction, the gravitational force [7].

Let us take a closer look at the behaviour of galactic rotation curves of spiral galaxies, which give strong evidence for the presence of some additional form of matter. Observations of neutral hydrogen clouds lying at large distances from the

[1]That is ignoring problems with quantum field theory, in which famously there is no known consistent theory unifying general relativity with quantum field theory, in part due to quantum theories of gravity being non-re-normalisable. Alas, this is not a subject for this thesis.

galactic center reveal that these clouds are moving at an approximately constant tangential velocity v_{tg}. However, galactic dynamics are well described by Newton's law of gravity, and this together with the equation for the centrifugal force yield a relation between the tangential velocity and the Newtonian potential,

$$\frac{v_{\mathrm{tg}}^2}{r^2} = \frac{GM}{r^3}, \tag{1.1}$$

with G Newton's gravitational constant. Assuming the validity of this formula, for parts of the outer galaxy to move with approximately constant tangential velocity, it would require the mass of this region to grow linearly with r. This is in stark contrast to observations which show that these regions where the clouds are present contain little luminous matter. Thus, it is proposed an additional matter component is required to explain this behaviour, which has been dubbed dark matter, as we cannot see it. There exists a plethora of dark matter models, for example see [8], which usually have roots derived in theories of particle physics.

Nonetheless, despite many attempts to detect dark matter particles, they have so far evaded all searches. Until they have been conclusively detected, there remains the possibility that dark matter does not exist, and some other phenomena is causing this behaviour. Perhaps equation (1.1) is incorrect, it could be that the right hand side of the equation is modified at large distances to account for the observed discrepancies? Admittedly, this is rather a niche view in the current literature, although it does lend motivation to studying alternative theories of gravity. We will not touch directly on the dark matter problem in this thesis.

Let us move on to the second of these gravitational problems: this is the inflation problem. Inflation was first proposed in 1980 as a way of potentially explain the non-existence of magnetic monopoles in the observable universe. Shortly after, it was also realised that inflation could possibly solve two additional problems of early universe cosmology: the horizon problem and the flatness problem. Broadly speaking, the question is why is the universe as isotropic as it is? Why is it, that when we look into the Cosmic Microwave Background radiation, areas of the sky which have never been in causal contact are in thermal equilibrium? The idea behind inflation is that the universe underwent a period of rapid exponential expansion very shortly after the big bang, increasing the volume of the universe by approximately 60 orders of magnitude. This rapid expansion allowed any inhomogeneities in the early universe to be smoothed out to an extent that gives us the now observed Cosmic Microwave Background.

One can account for a period of early universe inflation in the universe relatively simply with just one additional degree of freedom, assuming the existence of an extra particle in the universe known as the inflaton, equivalently requiring a suitable chosen scalar field to exist. At the moment there are many models of inflation which are consistent with observable data. And thus until we have better observational bounds on the space of possible models, there remain many options open to be explored theoretically.[2]

The final and most recently discovered of these three problems is another problem of cosmology: this is the dark energy problem. This problem was not observed until the late 1990s, with its discovery completely unanticipated by cosmologists. It was found that not only was the universe expanding (as has been known since Hubble in the late 1920s), but that the expansion rate was increasing: *the universe was accelerating*. This Nobel prize winning discovery came about from observations of distant supernovae, and in the subsequent two decades this finding has been confirmed by a variety of increasingly precise cosmological measurements. The reason for this accelerated expansion has been attributed to something now commonly referred to as dark energy. However physicists are still lacking a satisfactory theoretical description as to what this dark energy is.

[2] Additionally there are some physicists, in a minority these days, who do not believe in inflation at all, believing it to raise more philosophical questions and difficulties than it answers.

The standard cosmological model assumes the existence of an object called the cosmological constant to explain dark energy. This constant was first proposed by Einstein himself, when he discovered that the field equations of general relativity could exhibit a steady state or static cosmological universe solution if this constant existed. At the time this was the preferred cosmological model. However, when Hubble discovered the expansion of the universe, there was no longer any need for a static universe solution, and so the constant was disregarded.[3] However, in recent decades, the idea of the constant has been subsequently resurrected as it provides a means of generating an accelerating solution. At the moment, its predictions are consistent with all observations to date, see [9]. However, the cosmological constant model suffers from a variety of theoretical difficulties. Most of these are related to the fact that it has an extremely small observed value, compared to predictions of what its value should be that arise from quantum field theoretical considerations.

Many alternatives have been proposed to this simple cosmological constant model. One such approach is to assume that dark energy is instead a dynamical quantity. One could add an additional form of energy into the universe via an object called a scalar field to account for this: the standard way to do this is called quintessence. However cosmological observations mean that quintessence models are required to be increasingly fine tuned to match the observed behaviour, and so many alternative models have been proposed.

One could solve all three of the dark energy, dark matter and inflation problems by assuming there are additional matter or energy components in the universe. In fact this is the approach taken in the current standard cosmological model, which is known as ΛCDM cosmology, where Λ is the cosmological constant and CDM stands for cold dark matter. But there are alternative models: one could instead consider modifying the laws of gravity i.e. changing the general theory of relativity.

[3]Moreover it was later found that this static solution was in fact not stable to cosmological perturbations.

Despite general relativity's incredible experimental success, these successes have all been at relatively small length scales, that is sub-galactic scales, solar system scales or smaller. Assuming these laws hold at much greater length scales: galactic and greater, is a massive extrapolation of many orders of magnitude. It is therefore imperative that the potential for alternative theories of gravitation to explain these large length scale phenomena should be explored, and indeed it has been the case that over the last couple of decades such alternatives have been increasingly investigated.

A further reason to study modified gravity models is that there is in fact an interesting relationship between certain theories of modified gravity and models of general relativity where additional matter fields are present. In fact if one adds a scalar field into the universe, via a symmetry transformation one can transform this into a modified gravity model. Thus studying modified gravity gives us another way of analysing inflation models or simple scalar field dark energy models.

Modifications of general relativity began being explored almost immediately after Einstein first proposed his theory in 1915. Initial attempts were made by Einstein, Weyl and others to unify general relativity with electromagnetism. Over the past century a plethora of other modifications have been considered, motivated by a variety of different reasons: there have been attempts to quantize gravity, to unify gravity with the strong and weak nuclear forces, to remove singularities from general relativity and to include micro-structural properties such as interactions between spin and the gravitational field.

One could fill many pages with a list of all the different modifications of gravity that have been proposed. To name but a few, there is $f(R)$ gravity, $f(T)$ gravity, scalar tensor gravity, bimetric gravity, modified Gauss-Bonnet gravity, massive gravity, Horndeski gravity, teleparallel dark energy, hybrid metric Palatini gravity and we could go on. With such a plethora of modified theories out there, how do we make sense of them all? One way is to try and unify different theories into broad unified frameworks, that allows one to study the features of the theory needed to agree with observations. Another is to look at if modifications are really different, or are they in fact physically equivalent? One can look at the effects of a conformal transformation, which changes the length and time scales of the theory, and map different modifications into other modifications.

One test any viable modified theory of gravity must pass is that it has a consistent cosmology which is able to replicate the current known cosmological history and observations. This alone does not guarantee that the modification is correct, one must also ensure that the theory of gravitation as a whole is consistent, and that it can pass the many solar system constraints that observations have placed. This means at small scales the theory must well approximate general relativity with any deviations being very small.

One intriguing alternative to general relativity is called teleparallel gravity and it is this theory, along with its modifications, that is the main study of this thesis. Teleparallel gravity was in fact studied by Einstein himself as a way to try and unify general relativity with the laws of electromagnetism. This attempt ultimately failed once it was realised that the gravitational and electromagnetic interactions had to be treated differently due to their different properties, such as their radically different field strengths. But Einstein's work in this area led him to construct a theory which was completely physically equivalent to general relativity, however it has a very different interpretation. This interpretation is arguably less revolutionary than general relativity itself, with the curvature of the universe being replaced by simple force laws. This theory is known as the Teleparallel Equivalent of General Relativity (TEGR).

Teleparallel gravity, as is general relativity, is itself a special case of a more general gravitational theory known as metric affine gravity. Metric affine gravity is a very broad theory which allows one to take into account micro-structural properties of matter, such as spin and hypermomentum. The consideration of this additional structure has shown itself to have some desirable physical features, such as the ability to avoid the singularities that are prevalent in general relativity and is potentially more suitable for quantisation attempts. This theory is very broad, and its structure will in fact encompass all of the gravitational theories we study in this thesis. It has a number of special cases which have been extensively studied in the literature, such as Einstein-Cartan theory, along with the TEGR.

Now with two equivalent theories, general relativity and the TEGR, both leading to the same observational consequences it poses the question: when one wants to modify gravity, which of the two theories do you modify? As soon as one modifies the theories, the equivalence between them breaks down, and so there is not necessarily a reason to favour one approach or the other: it is only studying the physical properties of the theories that can lead us towards the correct approach. Modified teleparallel theories of gravity have shown themselves to be of great potential use in the inflation and dark energy problems. They differ from conventional metric based modifications and possess interesting properties. We will have a look at some of these during this thesis.

Ultimately, it is only observational data that will eventually settle the question of what is the correct theory of gravity, and whether alternatives to general relativity are viable theories. New observational results are appearing all the time, for example the LIGO gravitational wave detections, and such data is continually improving the bounds on any deviations from the general theory of relativity.

Thesis Outline

This thesis will begin in Chapter 2 by introducing the differential geometric framework of a general theory of gravitation, introducing the relevant spaces and geometrical objects required for defining a theory of gravity. In particular we will introduce the metric, tetrad, connection, the curvature tensor, the torsion tensor and the non-metricity tensor. In Chapter 3 these differential geometric concepts will be used to define different theories of gravity. We will start with general relativity, before broadening out to look at some other theories, culminating in the very general metric affine gravity.

Chapter 4 will move on to looking at teleparallel gravity and the TEGR. We will explore various properties of the theory, looking at the equivalent of the geodesic equation and issues of Lorentz invariance. The equivalence with general relativity will be demonstrated explicitly, and we will also see how the theory can be embedded into a Metric Affine framework.

Chapter 5 will look at how we can use the laws of gravity to study the evolution of the universe as a whole: the subject of cosmology. This will include an in-depth look at the dark energy and cosmological constant problems. Dynamical systems techniques in cosmology will be introduced, which will allow us to explore the broad dynamical behaviour of a range of different theories to see if they have potential to exhibit a viable cosmology. A scalar field model is introduced as an illustrative example of these techniques. We will then introduce various metric and teleparallel based modified gravity theories, briefly outlining some of their properties. Two of the theories introduced are known as $f(R)$ gravity, in the general relativistic framework, and $f(T)$ gravity, in the teleparallel framework. We will also briefly mention some other teleparallel modified gravity theories.

Chapter 6 will examine a broader modification of gravity, $f(T, B)$ gravity. This modification will include both of these $f(R)$ and $f(T)$ gravity modifications as particular limiting cases. Writing the theory in this way allows us to explore in detail the structure and advantages/disadvantages of both of these sub-theories, and we will discuss issues of Lorentz invariance and the conservation equation, before taking a glimpse at the cosmology of these models. This chapter is heavily based on the paper [1].

In Chapter 7 we will take a similar approach to considering gravitational modifications within the teleparallel and relativistic frameworks. We will examine scalar-tensor modifications: these are scalar field models, but those which possess a non-minimal coupling between the scalar field and the gravitational sector. We will perform a very general dynamical systems analysis, and see that there are a variety of interesting cosmological phenomena that can be observed, depending on the choice of coupling parameter between the scalar field and gravity. This chapter is based on and extends the results of [2, 10].

Chapter 8 will then look at the topic of conformal transformations within teleparallel theories. It is a well known result that $f(R)$ gravity can be transformed to a simple minimally coupled scalar field model, and vice versa, by simply rescaling the metric appropriately. This result is known not to apply directly to $f(T)$ gravity, but by taking a fresh look at this we will see that there are some very interesting equivalences between certain types of $f(T, B)$ gravity models and nonminimally coupled scalar field models. This forms an interesting link between the models studied in Chapter 6 and Chapter 7. This chapter is heavily based on [3].

Finally, in Chapter 9 we will conclude this thesis. The main ideas will be summarised and some of the broad themes that appear during this thesis will be discussed. Various conventions appear throughout the literature on teleparallel gravity, we will summarise the conventions used in this thesis in Appendix A. The papers [1], [2], [3] are included as Appendix B, C and D respectively.

Chapter 2

General linear spaces: curvature, torsion & non-metricity

One of Einstein's great insights was to realise that the gravitational interaction can be described with a dynamical theory of geometry. And so to analyse gravity, one must first understand geometry. In particular, mathematically, theories of gravity occur in the language of differential geometry and so one needs a firm understanding of differential geometry. In this chapter we will introduce some of the necessary and important geometrical constructs that are required to formulate a theory of gravity.

The definition of a spacetime will be introduced, along with that of a metric, a tetrad and a connection. These will then be used to introduce various geometrical objects that characterise the spacetime, including the curvature, torsion and non-metricity tensors. We will see that the presence or absence of these geometrical objects allow us to define various spaces and subspaces that are of physical and mathematical interest. Altogether, these objects and spaces will form the basis for formulating a very general framework of a theory of gravity that will be introduced fully in the next chapter: *metric affine gravity*.

2.1 Spacetime and geometry

The space in which we will be working in throughout this thesis is called a four dimensional, or a $(3 + 1)$ dimensional, spacetime[1], which has the properties of a continuum. Three of these dimensions are spatial, while the remaining one is temporal. In particular spacetime is an abstract space called a *manifold*, which we denote by $\mathcal{M}$. Broadly speaking a manifold locally has the differential structure of four dimensional Euclidean space $\mathbb{R}^4$. However it does not necessarily share its global properties. This local Euclidean property means that locally at any point we can always choose a particular set of coordinates x^μ that are valid for a suitably small neighbourhood around that point, with μ here denoting these coordinate or spacetime indices, which lie in the range $0, ..., 3$, with 0 denoting the temporal coordinate.

2.1.1 The metric

The manifolds of interest to us will come equipped with some additional structure defined on them, in the form of a *metric*, which we denote by $g_{\mu\nu}$ and a *connection*, $\Gamma^\rho_{\mu\nu}$ which we will discuss later in this chapter. The metric describes how to measure local distances and angles in the spacetime, and determines its causal structure. It is a symmetric rank 2 tensor. In order to describe a spacetime, the metric is required to have a *Lorentzian* signature, that is has signature $(+---)$ or $(-+++)$, depending on convention. This means that the metric, which is required to be non-degenerate, has three eigenvalues of one sign, with the other eigenvalue of the opposing sign. This Lorentzian property means that mathematically the metric is actually only a pseudo-metric, as it is lacking in the property of positive definiteness: there are

[1]One can construct more general theories of gravity in higher $(n + 1)$-dimensions, which, by motivation from theories such as string theory or braneworld models, are frequently studied in the literature.

non-zero vectors whose inner product with respect to the metric is null. It is these null vectors which describe the light-cones and causal structure of the spacetime.

Given a metric and coordinates x^μ, one can define the local line element on the spacetime as follows

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where throughout this thesis Einstein's summation convention is used, with repeated indices being summed over their entire range. The metric's determinant is denoted by g, which due to the metric's non-degeneracy is non-vanishing. Hence the metric possesses an inverse, denoted with upper indices, $g^{\mu\nu}$, so that

$$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu. \tag{2.2}$$

The simplest example of a Lorentzian spacetime manifold is Minkowski space. This is the geometric framework of Einstein's theory of special relativity. Globally Minkowski space is flat and has a constant metric in Cartesian-type coordinates (ct, x, y, z) given by

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \tag{2.3}$$

or alternatively with all the signs switched over depending on signature convention. In the coordinates, c is the speed of light, which from henceforth for the duration of this thesis will be set equal to unity, $c = 1$, using natural units. Minkowski space additionally possesses a trivial connection[2]. Minkowski space describes the geometric setting for the universe without the presence of gravity, and indeed all other physics

[2]Although this does not mean the connection is necessarily identically zero. For example Minkowski space written in spherical polar coordinates has a non-zero connection. This is because the connection is not a tensor, so vanishing in one coordinate system does not imply vanishing in another.

can be formulated on a Minkowski space background, including Electromagnetism and Quantum Field Theory. Minkowski space allows for the formulation of theories that remain the same for all inertial observers. However, if one wants to include gravity, requiring a description of non-inertial observers, one needs to go beyond Minkowski space.

However, for a more general spacetime possessing a non-trivial metric, due to the manifold structure at each point p of the manifold we can define an associated tangent space, $T_p(\mathcal{M})$, which is flat and has the structure of Minkowski space. The union of all of the tangent spaces on the manifold is known as the tangent bundle. This point leads us onto defining the concept of a tetrad.

2.1.2 The tetrad

In this section we introduce the concept of a *tetrad* or *vierbein* (German for four legs). At every point p of our manifold $\mathcal{M}$, the tetrad is defined to be a basis of orthonormal vectors lying in the tangent space $T_p(\mathcal{M})$, and we denote them by $\mathbf{e}_a(x^\mu)$, where the Latin indices are coordinates on the tangent space, and again Greek indices correspond to indices of suitably chosen local spacetime coordinates. Both take values in the range $\mu, a = 0, 1, 2, 3$. The orthonormality condition means that their inner product with respect to the metric, gives the Minkowski metric

$$\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}. \tag{2.4}$$

Now, we can always choose to work in the local coordinate basis ∂_μ of our manifold, and thus we can express the tetrad in component form in terms of this basis

$$\mathbf{e}_a = e^\mu_a \partial_\mu, \tag{2.5}$$

and it is frequently more convenient to explicitly work with e^{μ}_{a}. Written as a matrix, the tetrad is non-singular, and so this means we can define the *inverse tetrad*, alternatively sometimes referred to as the *cotetrad*, e^{a}_{μ} which is defined according to the relations

$$e^{\mu}_{m}e^{n}_{\mu} = \delta^{n}_{m}\,, \tag{2.6}$$

$$e^{\nu}_{m}e^{m}_{\mu} = \delta^{\nu}_{\mu}\,. \tag{2.7}$$

The physical metric $g_{\mu\nu}$ on the manifold can be expressed in terms of the tetrad and the Minkowski metric as

$$g_{\mu\nu} = e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}\,. \tag{2.8}$$

The tetrad uniquely determines the metric, but not vice versa. In general the tetrad possesses 16 individual components, whereas the metric only has 10. These additional 6 components are related to Lorentz invariance: a Lorentz transformation of the form $e'^{a}_{\mu} = \Lambda^{a}{}_{b}e^{b}_{\mu}$ will still result in the same physical metric. Here $\Lambda^{a}{}_{b}$ is an element of the Lorentz group $SO(3,1)$, which is a six dimensional group corresponding to six degrees of freedom (3 boosts and 3 rotations), whose elements satisfy the condition $\eta_{cd}\Lambda^{c}{}_{a}\Lambda^{d}{}_{b} = \eta_{ab}$. Likewise there is a simple relationship between the inverse metric and the cotetrad, with the inverse metric being expressed in terms of the cotetrads as follows

$$g^{\mu\nu} = e^{\mu}_{a}e^{\nu}_{b}\eta^{ab}\,. \tag{2.9}$$

One can also define the quantity e to be the determinant of the tetrad e^{a}_{μ}, which is equivalent to the volume element of the metric, so that $e = \sqrt{-g}$ where g is the determinant of the metric.

The tetrad field allows one to compare the direction of tangent vectors at different points on the manifold, and thus gives rise to the concept of distant parallelism, something we will return to in Chapter 4.

2.1.3 The connection

In this section we will define the concept of a *connection* on our manifold. The connection physically determines the locally inertial observers, and defines the geodesic structure of our spacetime. Mathematically the connection can be completely independent of the metric, in fact a connection can be perfectly well defined on a manifold which possesses no metric. However some compatibility conditions are required when one imposes physical requirements on the spacetime, for example one may want to impose the requirement that photons have to follow null geodesics of the connection.

The connection is an object of fundamental importance if one wants to move beyond the rigid spacetime of Minkowski space. To quote Einstein in [11] (and translated from the original German in [12]):

> *"...the essential achievement of general relativity, namely to overcome rigid space (i.e., the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the displacement field of infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (i.e., the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of rigid space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric ... "* A. Einstein

Let us now formally define this connection. If one parallelly displaces a vector

V^μ from the point x^μ to the point $x^\mu + dx^\mu$, then the vector changes according to

$$dV^\mu = -\Gamma^\mu_{\nu\rho} V^\nu dx^\rho, \tag{2.10}$$

where $\Gamma^\rho_{\mu\nu}$ is the connection on our manifold. In general this connection has $4^3 = 64$ independent components. This connection naturally defines a covariant derivative on our manifold, a tensorial generalisation of the partial derivatives. On a vector V^ν the covariant derivative $\nabla_\mu V^\nu$ is defined as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \tag{2.11}$$

with this relation generalising naturally to higher rank tensors.

An important point to note is that the connection is *not* a tensor. Thus if it vanishes in one frame, that does not mean it vanishes in all frames. Another important remark is that the connection is not a property of the manifold or spacetime. Different connections can be defined upon the same underlying spacetime, the connection itself does not determine the manifold. And hence to specify the geometric setting we are working in, we must define the connection on top of the manifold. This point will be important later on, when we look at equivalent theories of gravity with different connections defined on the same underlying spacetime.

We have now defined some interesting additional structure on our manifold: the metric, telling us how to measure distances and angles, and the connection, telling us how vectors change under parallel transport. Equipped with these two objects, we can now move forward and define some interesting geometrical tensors which will give us information that characterise the underlying spacetime.

2.2 Tensors

2.2.1 The torsion tensor

The first tensor we will define is called the torsion tensor. We have already remarked that the connection does not transform as a tensor under coordinate transformations. However, its antisymmetric part does transform as a tensor, and so we define the *torsion tensor* to be simply (twice) this antisymmetric part of the connection

$$T^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu} \tag{2.12}$$

which has 24 independent components in general. The torsion tensor has an interesting geometrical interpretation: if one builds infinitesimal parallelograms in the manifold (by parallel transport), the failure of the parallelogram to close is proportional to the torsion tensor.

From the definition, it is clear that the torsion tensor is antisymmetric in its last two indices. As a whole, it vanishes identically if and only if the connection is symmetric.

2.2.2 The curvature tensor

Let us now define the curvature tensor. Given a connection, the *Riemann curvature* is defined in terms of the connection as follows

$$R^{\rho}{}_{\mu\lambda\nu} := \partial_{\lambda}\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda}. \tag{2.13}$$

The Riemann curvature tensor provides a measure of the curvature of the manifold. In particular, if one takes a closed loop in the manifold, and transports a vector at a point in the tangent space around the loop, the Riemann tensor measures the deviation of the vector in the tangent space from its original position. We note that the curvature tensor is antisymmetric in its last two indices, with

$$R^{\rho}{}_{\mu\lambda\nu} = -R^{\rho}{}_{\mu\nu\lambda}. \tag{2.14}$$

A priori the curvature tensor possesses no other symmetries.

The curvature tensor we see is defined independently of the metric. And so, without invoking a metric, there are three possible ways to contract the Riemann tensor to define what is known as a *Ricci tensor*. The first of these, the standard Ricci tensor is defined as being the contraction of the first and third indices of the Riemann tensor

$$R_{\mu\nu} := R^{\rho}{}_{\mu\rho\nu}, \tag{2.15}$$

and we emphasise no metric was required to perform this contraction.

Alternatively, one could look at an alternative contraction of the Riemann curvature tensor, instead taking the contraction over the first and second indices. This defines an object known as the *homothetic curvature*

$$\hat{R}_{\mu\nu} = R^{\rho}{}_{\rho\mu\nu}. \tag{2.16}$$

This vanishes if the connection is symmetric, however in general this tensor is non-trivial. This tensor is fully antisymmetric, and so has a vanishing trace. The standard Ricci tensor does not have a direct physical interpretation, however this is not the case for the homothetic curvature. This measures how the length of a vector changes when it is transported along a closed loop. If the homothetic curvature identically vanishes, the connection is found to be volume preserving, so that lengths and volumes are unchanged by parallel transport. Finally, the third possible contraction of the Riemann tensor gives us

$$\bar{R}_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}, \tag{2.17}$$

however, due to the antisymmetry of the Riemann tensor in its last two indices, we have simply that this is negative the standard Ricci tensor

$$\bar{R}_{\mu\nu} = -R_{\mu\nu}, \tag{2.18}$$

and so there are essentially only two independent Ricci tensors.

Now, assuming that there is a metric on our manifold, there is only one non-trivial contraction of these Ricci tensors. The *Ricci scalar*, or scalar curvature, is the contraction of the Ricci tensor with the inverse metric

$$R := R_{\mu\nu} g^{\mu\nu}, \tag{2.19}$$

with the other two Ricci tensors simply contracting to give

$$\hat{R}_{\mu\nu} g^{\mu\nu} = 0 \tag{2.20}$$

$$\bar{R}_{\mu\nu} g^{\mu\nu} = -R. \tag{2.21}$$

The Ricci scalar has the physical interpretation of measuring the deviation of the volume of a unit geodesic ball from the standard unit ball.

A non-trivial torsion means that covariant derivatives of scalar fields do not commute, with their difference being proportional to torsion

$$\nabla_{[\mu} \nabla_{\nu]} \phi = -T_{\mu\nu}{}^{\rho} \nabla_{\rho} \phi. \tag{2.22}$$

On the other hand, the Riemann curvature is relevant when considering the noncommutativity of vectors and other higher rank tensors, for example the following identity holds for an arbitrary vector V^ρ

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = R^\rho{}_{\mu\nu\sigma} V^\sigma - 2T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho, \tag{2.23}$$

a relation which gives rise to the Riemann tensor's physical interpretation: measuring the deviation of a vector after parallel transport over a closed loop.

2.2.3 The non-metricity tensor

We additionally have another tensor which has important physical consequences. The non-metricity tensor is defined on manifolds that are in possession of a metric and a connection, and is simply defined to be the covariant derivative of the metric

$$Q_{\lambda\mu\nu} := -\nabla_\lambda g_{\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\mu\rho}. \tag{2.24}$$

In general this tensor has 40 independent components. By the symmetry of the metric, the non-metricity tensor clearly possesses the symmetry $Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$.

It is often imposed that the non-metricity identically vanishes, that is the metric is covariantly constant. This postulate ensures that both lengths, in particular the unit length, and angles are preserved under a parallel displacement. This ensures that spacetime has a local Minkowski structure, and spacetime can be viewed as a set of Minkowski grains glued together by the affine connection. This postulate is well supported by the many experiments performed to test special relativity. However, it should be clear that this is an a posteriori constraint, which is determined by experimental evidence only, and there is no theoretical justification for this assumption on its own. Thus we will continue to analyse very general manifolds not imposing that the non-metricity vanishes until later in this thesis.

2.2.4 Additional connections and tensors

In this subsection we will briefly discuss some of the other objects and tensors related to the curvature, torsion and non-metricity tensors that will be used throughout this thesis.

But first we will look at a particularly important connection. This is the *Levi-Civita connection* $\bar{\Gamma}$, which is the connection used in general relativity. This is defined as the following combination of first derivatives of the metric

$$\bar{\Gamma}^{\lambda}_{\mu\nu} := \frac{1}{2} g^{\lambda\sigma}(\partial_{\nu} g_{\sigma\mu} + \partial_{\mu} g_{\sigma\nu} - \partial_{\sigma} g_{\mu\nu}). \tag{2.25}$$

This specific connection is torsion free (its torsion tensor vanishes, $\bar{T}^{\lambda}{}_{\mu\nu} = 0$) and metric compatible (its non-metricity tensor vanishes, $\bar{Q}_{\mu\nu\lambda} = 0$). Moreover, it can be shown that it is the unique such connection which satisfies both of these properties. It is readily seen that such a connection is symmetric.

Now let us introduce some important tensors. The *distortion tensor* is defined to be the difference between the connection and the Levi-Civita connection

$$N^{\lambda}{}_{\mu\nu} := \bar{\Gamma}^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\nu}. \tag{2.26}$$

(Again, we see that differences between connections are tensors.) This measures the deviation of our spacetime from being Riemannian, that is it measures the post-Riemannian component. The torsion and non-metricity tensor can be found in terms of the antisymmetric and symmetric parts of the distortion respectively

$$T^{\lambda}{}_{\mu\nu} = 2N^{\lambda}_{[\mu\nu]} \tag{2.27}$$

$$Q_{\mu\nu\lambda} = -2N_{\mu(\nu\lambda)}. \tag{2.28}$$

On the other hand the *contortion tensor* K is defined as the following combinations of the torsion tensor

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\lambda}{}_{\nu} + T_{\nu}{}^{\lambda}{}_{\mu} - T^{\lambda}{}_{\mu\nu}). \tag{2.29}$$

The following relation between the contortion tensor and the distortion tensor can be found to hold

$$N^{\lambda}{}_{\mu\nu} = -K^{\lambda}{}_{\mu\nu} + \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu\nu}{}^{\lambda} - Q_{\nu\mu}{}^{\lambda}), \tag{2.30}$$

so that if the non-metricity tensor vanishes, we have $N^{\lambda}{}_{\mu\nu} = -K^{\lambda}{}_{\mu\nu}$ and so we can choose to work only with one of these tensors. In the teleparallel literature it is the contortion tensor that is most commonly used.

Using everything we have defined so far, we can decompose a general connection into 3 different pieces, a Levi-Civita piece, a piece associated with torsion and a piece associated with non-metricity part, as so

$$\Gamma^{\lambda}_{\mu\nu} = \bar{\Gamma}^{\lambda}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + \frac{1}{2}\left(-Q_{\mu}{}^{\lambda}{}_{\nu} + Q^{\lambda}{}_{\nu\mu} - Q_{\nu\mu}{}^{\lambda}\right). \tag{2.31}$$

Another way of looking at this is we are splitting up the connection into a Riemannian and post-Riemannian part. Using this one can similarly split up the curvature tensor into different components. This sort of decomposition will prove to be useful later on in this thesis.

2.2.5 Bianchi Identities

The curvature, torsion and non-metricity tensor all obey certain differential geometrical relations that have important physical consequence. There is an identity for each of these tensors, relating their derivatives to algebraic combinations of the three different tensors. These three identities are known collectively as the *Bianchi identities.*

Given a specific connection, the torsion and curvature tensors satisfy two identities known as Bianchi identities which reveal certain symmetries of the tensors. The Bianchi identity for torsion is given by

$$\begin{aligned}\nabla_\nu T^\lambda{}_{\rho\mu} + \nabla_\mu T^\lambda{}_{\nu\rho} + \nabla_\rho T^\lambda{}_{\mu\nu} &= R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} + R^\lambda{}_{\mu\nu\rho} \\ &\quad + T^\lambda{}_{\rho\sigma} T^\sigma{}_{\mu\nu} + T^\lambda{}_{\nu\sigma} T^\sigma{}_{\rho\mu} + T^\lambda{}_{\mu\sigma} T^\sigma{}_{\nu\rho}. \end{aligned} \tag{2.32}$$

The Bianchi identity for curvature is given by

$$\nabla_\nu R^\lambda{}_{\sigma\rho\mu} + \nabla_\mu R^\lambda{}_{\sigma\nu\rho} + \nabla_\rho R^\lambda{}_{\sigma\mu\nu} = R^\lambda{}_{\sigma\mu\delta} T^\delta{}_{\nu\rho} + R^\lambda{}_{\sigma\nu\delta} T^\delta{}_{\rho\mu} + R^\lambda{}_{\sigma\rho\delta} T^\delta{}_{\mu\nu}. \tag{2.33}$$

If the connection does not contain torsion, the latter two identities simplify dramatically and reveal a cyclic symmetric in the Riemann tensor and its first derivatives.

The final Bianchi identity tells us that the derivatives of the non-metricity tensor satisfy

$$\nabla_\mu Q_{\nu\alpha\beta} - \nabla_\nu Q_{\mu\alpha\beta} = R^\lambda{}_{\alpha\mu\nu} g_{\lambda\beta} + R^\lambda{}_{\beta\mu\nu} g_{\lambda\alpha} + T^\lambda{}_{\mu\nu} Q_{\lambda\alpha\beta} \tag{2.34}$$

If the non-metricity vanishes this identity reveals an additional symmetry of the curvature tensor. We note that in the literature these identities are usually displayed using coordinate free notation, but to be consistent with the notation used throughout this thesis we have chosen to display them in coordinate notation.

2.3 Linear spaces

Now we have introduced the various geometrical objects that are needed, and the relationships between them, we can now define and classify different spaces according to the geometrical objects presiding in them.

2.3.1 Linear space L_4

The first space we will look at is known as a general linear space, which we will denote as L_4. This is defined to be a four dimensional manifold with a linear affine connection. If additionally a metric is present the space is given by the pair (L_4, g). This space is the setting for Metric Affine Gravity, which will be introduced in the next chapter.

There are no further restrictions placed on the geometry of this space. By this we mean that in general, in this space the connection is allowed to possess curvature, torsion, and the non-metricity tensor is in general non-vanishing. This means the connection has its full 64 degrees of freedom. The remaining spaces we will define will all be special cases of L_4, with some additional geometric constraints defined upon them.

2.3.2 Riemann-Cartan space: U_4

The first such space we will look at is called Riemann-Cartan space, or U_4 space, and it is a four dimensional manifold in possession of a metric and a linear affine connection, but with the constraint that the non-metricity tensor vanishes identically, so that

$$Q_{\lambda\mu\nu} := -\nabla_\lambda g_{\mu\nu} = 0, \tag{2.35}$$

in other words the metric is covariantly conserved. Therefore in a U_4 space distances and angles are preserved by parallel transport, and this ensures that locally spacetime has the property of Minkowski space, which is experimentally supported to a high precision by many experiments testing special relativity.[3]

Such a Riemann-Cartan spacetime possesses an important property: it is in-

[3]Although it may be noted that experiments have not been at a sufficient sensitivity to detect effects at the level at which we would expect a non-trivial non-metricity to show up.

variant under local Lorentz transformations. This is crucial for building a physical theory that is independent of the observer. In Riemann-Cartan space, the distortion tensor equals the contortion tensor due to the vanishing non-metricity, and thus there is only need to consider one of these objects. In the literature it is the contortion tensor K which is used interchangeably to mean contortion or distortion, a convention which we will adopt from now on. Riemann-Cartan space is the geometric setting for Einstein-Cartan gravity, to be introduced in the next chapter.

2.3.3 V_4 space

The next space we consider is called Riemannian space, or V_4 space, which is a subspace of Riemann-Cartan space U_4. It is a four dimensional manifold with a metric and a linear affine connection, however this time as well as vanishing non-metricity, the additional constraint that the torsion vanishes is imposed. So we have that both

$$Q_{\lambda\mu\nu} = 0, \tag{2.36}$$

$$T^{\lambda}{}_{\mu\nu} = 0 \tag{2.37}$$

must hold. Thus V_4 space is a subspace of U_4 space, with the subspace condition being that the torsion tensor vanishes.

In such a space the connection is symmetric due to the torsion component vanishing. In fact, this condition along with the vanishing non-metricity uniquely determine the connection in terms of the metric and in fact it is the Levi-Civita connection defined in (2.25). This space still allows for interesting geometric structure due to the potential presence of curvature in the space. This space is the setting for General Relativity, to be introduced in the next chapter, and many other curvature based modifications of gravity, which will be examined later in Chapter 5.

2.3.4 R_4 space

At this stage we will now introduce an important subspace of V_4, that is R_4 space. This is defined to be a V_4 space with the additional constraint of vanishing curvature, that is the Riemann tensor vanishes

$$R^{\lambda}{}_{\mu\nu\rho} = 0. \tag{2.38}$$

This ensures that the space is globally flat, and uniquely determines the metric to be isometric to the Minkowski metric, with a trivial connection. Such a spacetime is the setting for Special Relativity, and there is no longer any degrees of freedom for a dynamical geometry in this space.

2.3.5 Weitzenböck space W_4

Finally to conclude this section we note another possible definition of a subspace of L_4. We note that V_4 is a space with curvature but vanishing torsion. However, we could alternatively define a subspace of V_4 by considering this situation reversed, and require the subspace to have torsion but vanishing curvature.

In fact, given a metric compatible torsion free connection on a manifold, it is always possible to define an alternative connection on the manifold with vanishing curvature but non-trivial torsion, this is due to a result going back to Weitzenböck [13]. And as such we will call such a space a Weitzenböck or W_4 space. In such a space there is distant parallelism: lengths and angles are unchanged by parallel transport. Moreover the parallel transport of a vector is path independent. This space forms the basis for teleparallel gravity, which will be introduced in Chapter 4.

2.4 Discussion

In this chapter some of the elementary concepts of differential geometry required to formulate a theory of gravity were explored. We introduced spacetime as a manifold, which possesses a metric for measuring distances and angles, and a connection which defines the concept of parallel transport on the manifold. The most general space that possesses these objects is a general linear space L_4. We have introduced various interesting geometrical objects and tensors that can be defined on L_4, including curvature, torsion, non-metricity, contortion and distortion. We have examined various subspaces of L_4, where constraints are applied on the torsion, curvature and non-metricity tensors, some of which are taken to vanish. In particular we have looked at Riemann-Cartan, Riemann, Weitzenböck and Minkowski space.

Let us briefly mention a few more spaces that have been considered in the gravitational literature. Weyl-Cartan space is a subspace of L_4 such that the only geometrical constraint imposed is that the trace of the non-metricity tensor vanishes identically, that is

$$\mathrm{tr}(Q) = 0. \tag{2.39}$$

If one then considers a vanishing torsion on this Weyl-Cartan space, one gets the space known as Weyl space. Such a space has also been considered as a setting for a gravitational theory. Of course there are a range other subspaces of L_4 with various constraints on the tensors, but we will not discuss any of these further.

Let us end this chapter with a diagram outlining the relationships between the numerous different geometric spaces discussed in this chapter. The Diagram 2.1 outlines the hierarchy of how these different spaces are interrelated. In the next chapter we will look at how one can define theories of gravity within the geometric framework of these different spaces.

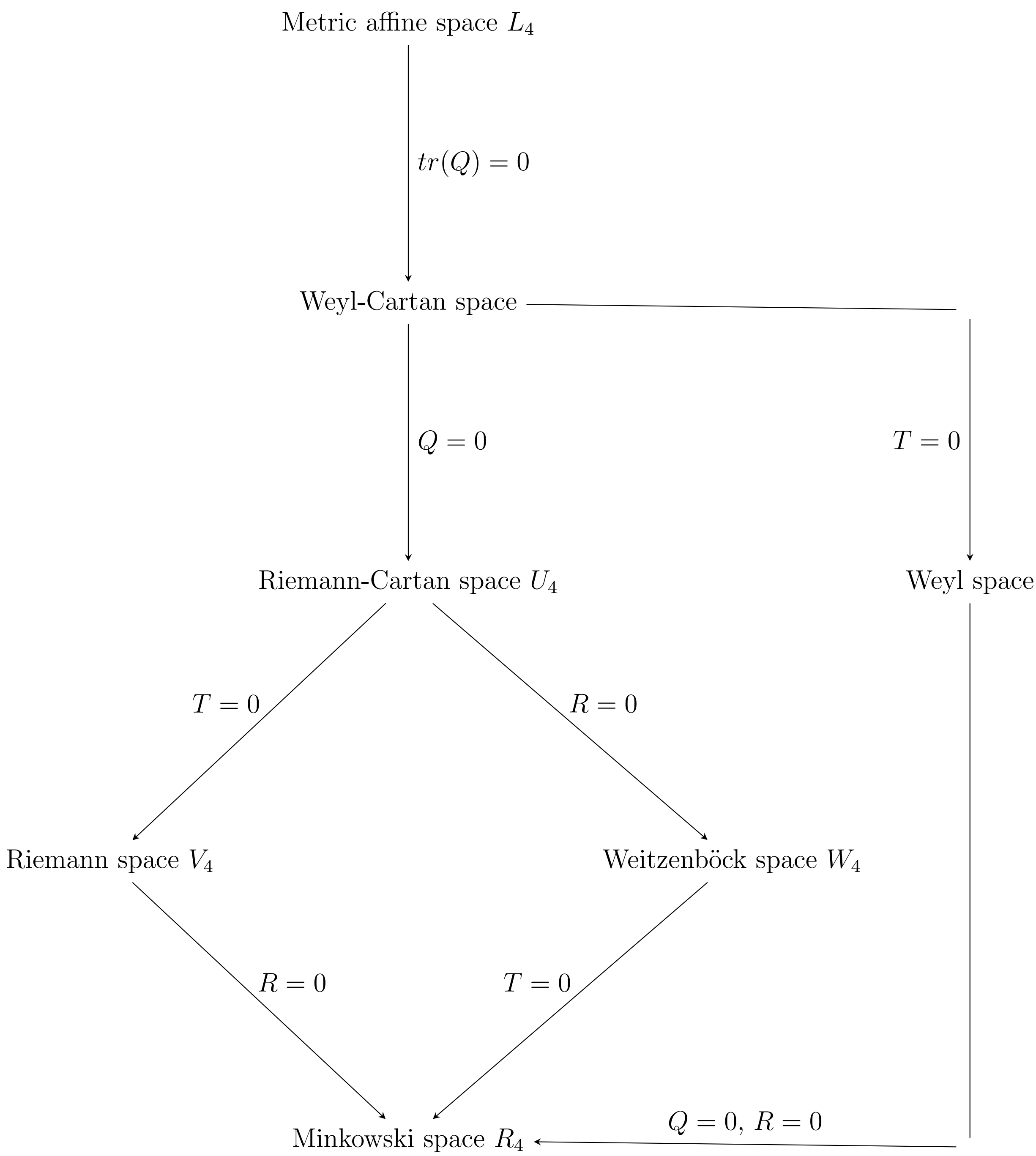

Figure 2.1: Diagram showing the relationships between the various subspaces of a general linear space L_4 discussed in this chapter.

Chapter 3

Gravitational theories

In this chapter we will explore how the differential geometry concepts of the last chapter can be used to construct theories of gravitation. We will begin with a brief review of classical Newtonian gravity. Although a conceptually simple theory, Newtonian gravity is widely applicable and it is still used when performing most solar system scale calculations. It is thus important that any gravitational theory behaves as Newton's in certain appropriate limits.

We will then move on to discussing some aspects of the current accepted theory of gravitation, Einstein's theory of general relativity. Although a very successful and consistent theory, it is based on some assumptions which are not necessarily a priori justified. We will see if we relax some of these restrictions, using the differential geometry concepts from the previous chapter, we can define a more general class of theories of gravity, known as Metric Affine Gravity, of which general relativity is simply a special case. These potentially open the door to interesting physical phenomena, which may be able to alleviate some physical problems associated with general relativity.

We end this chapter by taking a look at a frequently studied particular special case of metric affine gravity, known as Einstein-Cartan theory, that is still more general than general relativity. This will be introduced as a first look at a full gravitational theory in which torsion plays a non-trivial role. In the next chapter we will then move on to the main topic of this thesis, teleparallel gravity, which is a further interesting special case of metric affine gravity.

3.1 Newtonian gravitation

Before Einstein developed his general theory of relativity, the laws of gravity had been described by Isaac Newton in his seminal 1687 Principia (following on from earlier ideas of Kepler and Galileo). Newton's theory is a classical theory, where gravity is an attractive, instantaneous force acting between two massive objects. The theory can successfully account for the planetary motion of every planet in the solar system, except Mercury, to a high degree of accuracy.

In Newtonian gravity, the equations of motion of a particle in a gravitational potential ϕ are given by

$$m\ddot{\mathbf{r}} = -m\nabla\phi(\mathbf{r}) \tag{3.1}$$

where $\mathbf{r}$ is the three dimensional position vector of the particle and m is the mass of the particle. The geometric setting for Newtonian gravity is much simpler than the structures considered in the previous chapter. Newton's gravity takes place in the setting of the product manifold $\mathbb{R} \times \mathbb{R}^3$. Additionally, the metric on the spatial part of the manifold, $\mathbb{R}^3$, is simply the global Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{3.2}$$

with gravity having no effect on the metric.

With the advent of Einstein's special theory of relativity, there was a realisation that the Newtonian theory of gravity could not be the full picture. Special relativity no longer permits instantaneous interactions, with information restricted to travelling at the speed of light or less, and this led to the formulation of the general theory of relativity. However, in many circumstances, Newtonian gravity is still an accurate description of the gravitational interaction, in particular when the gravitational field is weak and sources are slowly moving. This is called the Newtonian limit, and it is crucial that any proposed theory of gravity must coincide with Newtonian gravity in the Newtonian limit in order to be viable.

3.2 General relativity

General relativity is now over one hundred years old and it is the currently accepted, or standard model, for gravitation. Einstein formulated his general theory of relativity in the years leading up to 1915. The force laws of Newtonian gravity were replaced by a more sophisticated geometric framework, where the gravitational interaction is induced via the curvature of the underlying spacetime.

Over the last century general relativity has shown itself to be a phenomenally successful theory, passing countless observational and experimental tests to an extraordinarily high precision. There are three classical experimental tests of general relativity which Newtonian gravity is not able to account for. There is the deflection of light by the sun: Newtonian gravity can only account for half of this deflection, GR is needed to explain the remainder. There is also the perihelion precession of Mercury: Mercury's trajectory is not a closed ellipse, for which GR is needed to fully explain the exact amount of precession. Additionally GR can account for observations of gravitational redshift. Moreover, in 2016 we received observational confirmation of another of general relativity's predictions: the discovery of gravitational waves. The Laser Interferometer Gravitational-Wave Observatory (LIGO) project detected the gravitational waves emitted during the merger of two black holes. This is one of the first tests of GR in the strong field limit, and the observations of the event were consistent with numerical simulations of black hole mergers using the field equations of GR.

One of the most important features of general relativity is the concept of universality of free-fall. All objects, regardless of their internal properties such as inertial mass, will undergo the same acceleration in a gravitational field[1]. No other physical law of nature has this property. It is this concept which naturally suggests that it is the underlying spacetime structure that is changed by gravitation, therefore this suggests that it is the metric that is changed. In general relativity, the change in spacetime is characterised by an underlying curvature of the spacetime, and makes no reference to any force laws. This curvature then means free particles follow the geodesics of the spacetime.

To mathematically describe general relativity, we will need some of the geometrical objects and concepts introduced in the previous chapter. General relativity takes place inside a V_4 space, which we recall means that both the torsion and the non-metricity vanish identically. These two conditions uniquely determine the connection on the manifold in terms of the metric. They imply that the connection is given by the *Levi Civita connection*, which we recall is given by

$$\bar{\Gamma}^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}). \tag{3.3}$$

And so given a metric on the spacetime, we automatically know the connection. Thus the dynamics of the theory only needs to determine the metric.

In general relativity, freely falling particles in the spacetime, with coordinates x^μ, $(\mu = 0, 1, 2, 3)$, will follow geodesics: curves which extremise the length between

[1]In particular this means that the inertial mass m_i of an object is equal to its gravitational mass m_g.

its endpoints. By varying the following action

$$S = \int ds, \tag{3.4}$$

where ds is the line element of the metric, $ds = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$, one finds the geodesic equation. After some algebra, one finds this is given by

$$\ddot{x}^\mu + \bar{\Gamma}^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0. \tag{3.5}$$

This determines how particles move in a curved spacetime in general relativity. It is here we see the fundamental importance of the connection.

The other crucial aspect of general relativity determines how matter and energy induce this curvature in spacetime. This relationship is encoded within the *Einstein equation*, which is given by

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{3.6}$$

Here the tensor $T_{\mu\nu}$ is the energy momentum tensor of matter, determining the type of energy and momentum present at each point of the spacetime. The first Bianchi identity in a V_4 space (that is equation (2.33) in the absence of torsion), readily implies after contraction that the left hand side of Einstein's equation is covariantly conserved

$$\nabla^\mu \left(\bar{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\bar{R}\right) = 0. \tag{3.7}$$

This in turn implies that the energy momentum tensor is conserved, as is required for a sensible physical theory.

One can write the Ricci tensor and scalar explicitly in terms of the metric and its second derivatives, and doing this reveals that the Einstein equation is a set of ten coupled second order partial differential equations. For a general system this is very difficult to solve. However, by invoking symmetry principles, one is able to construct exact solutions for the metric for some physical situations. Perhaps the most important solution is the *Schwarzschild solution*, which describes the vacuum exterior of a static spherical star or black hole of mass M, given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \tag{3.8}$$

Nowadays there are many known solutions to the Einstein equation, and indeed very accurate numerical solutions to many complex physical models that cannot be found analytically.

Much of modern physics is derived from a variational principle, and so we require an action from which the Einstein field equations can be derived. This is given by the Einstein-Hilbert action, first proposed by the mathematician Hilbert in 1915, and is simply given by the volume integral of the Ricci scalar

$$S_{\mathrm{EH}} = \int \left[\frac{1}{16\pi G}\bar{R} + \mathcal{L}_{\mathrm{m}}\right] \sqrt{-g}\, d^4x \,. \tag{3.9}$$

In this action $\mathcal{L}_{\mathrm{m}}$ is called the matter Lagrangian. By varying this action with respect to the metric tensor, one recovers Einstein's field equations, where the energy momentum is defined to be the following variation of the matter Lagrangian

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\mathrm{m}}}{\delta g^{\mu\nu}}. \tag{3.10}$$

As mentioned in the introduction, despite general relativity's observational success, there are a variety of reasons one might want to modify it. The first is that there is no consistent theory of quantum gravity: gravity has not been unified with quantum field theory and the remaining forces of nature. At high energies and small length scales, such as at the start of the universe, this can potentially cause a break down of general relativity. Its inability to be consistently unified with quantum theory might indicate we need to consider a different theory. Another reason we might wish to consider modified theories is classical, and is due to cosmological anomalies which general relativity cannot explain, as outlined in the introduction. These issues will be discussed in detail in Chapter 5.

One way to consider modifications of general relativity is to assume that we are still working in a V_4 space, but modify Einstein's field equations by considering alternatives or extensions of the Einstein-Hilbert action. This maintains a lot of the geometric structure of the spacetime: the Levi-Civita connection remains the connection and free particles will still follow geodesics. We will discuss this approach in Chapter 5. On the other hand, we could instead move beyond the restrictive geometric setting of Riemannian V_4 space and consider theories including torsion and non-metricity. And thus for the remainder of this chapter, we will discuss more broad gravitational theories using the differential geometric concepts of the previous chapter.

3.3 The Palatini variation

The first approach we will look at is in fact equivalent to general relativity, but the theory has a slightly different interpretation and requires less a priori assumptions about the underlying geometry of the spacetime manifold. The Palatini variation[2] of general relativity again is based on the Einstein-Hilbert action. However, we do not begin by making such a restrictive assumption on the form of the connection; that is we are not assuming that the connection is the torsion free Levi-Civita connection, however we still assume it is metric compatible. Thus we begin in a generic U_4 space. We start with an arbitrary affine connection, entering the action through the

[2]The Palatini variation was actually developed by Einstein in 1925, but in the intervening years due to a historical misunderstanding became known by the Palatini variation after the Palatini identity used in the calculation.

Ricci tensor, and treat it as an independent field.

$$S_{\mathrm{EH}} = \int \left[\frac{1}{16\pi G} R(g, \Gamma) + \mathcal{L}_{\mathrm{m}} \right] \sqrt{-g}\, d^4x \,, \tag{3.11}$$

The idea is now to vary this action with respect to both the metric and the connection independently. This leads to two sets of field equations. But in fact, the field equation for the connection is an algebraic one, and its solution simply imposes that the connection is in fact the Levi-Civita one. If one considered a Lagrangian which was not linear in the Ricci scalar, this is no longer necessarily true. But due to this result, it is often thought that there is no particular reason to consider the Palatini variation over the standard metric variational principle if one is simply considering the Einstein-Hilbert action.

However the situation is not necessarily that straightforward. As noted in [14], there is an implicit assumption in the standard Palatini variation: this is that the matter Lagrangian is independent of the connection. This means that any covariant derivatives of the matter fields were implicitly assumed to be defined in terms of the Levi-Civita connection, as opposed to the independent connection. This is certainly not true for certain types of matter. And thus it cannot be consistently argued that the Palatini variation of the Einstein-Hilbert action reduces to general relativity when certain matter fields are present. To work towards a more consistent theory, we will now go beyond this simple Palatini approach into a more general theory incorporating a generic connection.

3.4 Metric Affine Gravity

In this section we will have a brief look at Metric Affine Gravity, which is a very general theory of gravity that takes place in the full linear space L_4. This means torsion, curvature and non-metricity are all present. Metric affine gravity is a gauge theory of gravity: that is interactions of the gravitational field are described by promoting global symmetries to gauge symmetries. It is beyond the scope of this thesis to discuss the gauge theoretical aspects in great detail, however we will mention some of their properties. Frequently in the literature a coordinate free notation is adopted, however for the purposes of this thesis, we will use a coordinate based notation, following the covariant approach to metric affine gravity found in [15].

Metric Affine Gravity takes into account the potential microstructural properties of matter such as spin, dilation current or proper hypercharge as possible physical sources for the gravitational field. It puts them on an equal footing with macroscopic properties like the energy and momentum of matter. This can be motivated by considering extending theories of three dimensional elastic continuum with microstructure to a higher dimensional spacetime, giving interesting physical interpretations to this post-Riemannian geometry. No longer just considering Riemannian spacetime allows the underlying geometry to have interesting microstucture, similar to liquid crystals or dislocated metal. In fact concepts such as torsion and non-metricity are widely used in three dimensional theories of lattice defects, with non-metricity having the interpretation as densities of point defects and torsion having the interpretation of line defects or dislocations.

Considering these post-Riemannian geometries has some other potential upsides. For example, by considering cosmological solutions, it can be deduced that in a Riemannian geometry the universe originated from a singularity: this is problematic as all physics breaks down at singularities. Non-Riemannian geometry potentially alleviates this problem by removing this big bang singularity. There are also motivations from attempts to quantize gravity: the failure to do this to date suggest the need to consider geometries which are not dominated by a classical distance concept.

The first generalisation to the rigid framework of V_4 space is to work within a U_4 space, where a non-trivial torsion is allowed to be present. It was first noted by

Kibble in [16], that one can write gravity in a gauge theoretical approach. Kibble did this by taking gravity to have the local symmetry group $SO(3,1) \rtimes \mathbb{R}^4$, the semidirect product of the Lorentzian special orthogonal group and the four dimensional translation group (with global translations). That is the Lorentz group gets promoted to the Poincaré group. This allows for the presence of torsion and puts gravity into a Riemann-Cartan geometry, or a U_4. The gauge fields are the tetrad which is associated to the global translations, and the (spin) connection, associated to the Lorentz group.

However, this was later generalised to an even more general framework, by Hehl et al [17], called Metric Affine Gravity, taking place in a general linear space L_4. This generalises the local symmetry group from the Lorentz group $SO(3,1)$, promoting it to the full general linear group $GL(4,\mathbb{R})$. Thus the full local symmetry group of the theory is the affine group $A(n,\mathbb{R}) = GL(4,\mathbb{R}) \rtimes \mathbb{R}^4$, which is the semidirect product of the general linear group $GL(4,\mathbb{R})$ and the translation group $\mathbb{R}^4$. Allowing for this broader symmetry group means we go beyond just the Lorentz invariance of $SO(3,1)$, and this means non-metricity becomes present. On a general four dimensional vector x, the affine group acts as follows

$$x \to x' = \Lambda x + \tau, \tag{3.12}$$

where $\Lambda = \Lambda^\alpha{}_\beta \in GL(4,\mathbb{R})$ and $\tau \in \mathbb{R}^4$. The linear connection and the coframe are gauge field potentials, with the curvature and the torsion corresponding to their respective gauge field strengths. Additionally the metric is a further fundamental field with the non-metricity as its corresponding field strength. The matter source of non-metricity is called *hypermomentum*.

Let us take a look at the general field equations of such a gravitational theory, following closely the results found in [15]. There will be two sets of field equations: one for the metric and one for the independent connection. A general Lagrangian for the action, assuming a minimal coupling between matter and gravity, will take the form

$$L = L_{\text{grav}}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}, T_{\mu\nu}{}^{\lambda}, Q_{\mu\nu\lambda}) + L_m(g_{\mu\nu}, \psi^A, \nabla_\mu \psi^A). \tag{3.13}$$

Here L_{grav} is the Lagrangian of the gravitational sector, which in general is an arbitrary function of the curvature, torsion and non-metricity tensor, and the matter Lagrangian L_m depends on the matter fields ψ^A and their covariant derivatives. Thus the connection only enters the matter Lagrangian via these covariant derivatives. In view of the relations between the distortion tensor and torsion (2.27) and non-metricity (2.28), the gravitational Lagrangian can equally be described by

$$L_{\text{grav}} = L_{\text{grav}}(g_{\mu\nu}, R_{\mu\nu\rho}{}^{\lambda}, N_{\mu\nu}{}^{\lambda}), \tag{3.14}$$

which is a function of now only the metric, curvature and distortion, with the degrees of freedom due to torsion and non-metricity contained entirely in the distortion tensor.

The field equations for such a Lagrangian are given by varying with respect to both the metric and the connection independently. The field equations can be displayed in various equivalent forms, the form we choose to show here is the following set of equations

$$(\nabla_\mu + N_{\rho\mu}{}^{\rho}) H^{\lambda\mu}{}_{\nu} + \frac{1}{2} T_{\mu\rho}{}^{\lambda} H^{\mu\rho}{}_{\nu} - E_{\nu}{}^{\lambda} = -\Sigma_{\nu}{}^{\lambda} \tag{3.15}$$

$$(\nabla_\rho + N_{\sigma\rho}{}^{\sigma}) H^{\lambda\rho\mu}{}_{\nu} + \frac{1}{2} T_{\rho\sigma}{}^{\lambda} H^{\rho\sigma\mu}{}_{\nu} - E^{\lambda\mu}{}_{\nu} = \Delta^{\mu}{}_{\nu}{}^{\lambda}. \tag{3.16}$$

where the tensors H are the partial derivatives of the Lagrangian with respect to the torsion and curvature

$$H^{\lambda\mu}{}_{\nu} := -\frac{\partial L_{\rm grav}}{\partial T_{\lambda\mu}{}^{\nu}}, \tag{3.17}$$

$$H^{\mu\nu\rho}{}_{\lambda} := -2\frac{\partial L_{\rm grav}}{\partial R_{\mu\nu\rho}{}^{\lambda}}, \tag{3.18}$$

$$M^{\mu\nu\rho} : = -\frac{\partial L_{\rm grav}}{\partial Q_{\mu\nu\rho}}, \tag{3.19}$$

where in addition we have defined M to be the partial derivative of the Lagrangian with respect to the non-metricity. The gravitational hypermomentum density is defined as

$$E^{\lambda\mu}{}_{\nu} = -H^{\lambda\mu}{}_{\nu} - M^{\lambda\nu}{}_{\mu} = -\frac{\partial L_{\rm grav}}{\partial N_{\lambda\mu}{}^{\nu}}, \tag{3.20}$$

and the generalised energy-momentum tensor of the gravitational field is

$$E_{\nu}{}^{\mu} = \delta^{\mu}_{\nu} L_{\rm grav} + \frac{1}{2} Q_{\nu\rho\lambda} M^{\mu\rho\lambda} + T_{\nu\lambda}{}^{\rho} H^{\mu\lambda}{}_{\rho} + R_{\nu\lambda\rho}{}^{\rho} H^{\mu\lambda\sigma}{}_{\sigma}. \tag{3.21}$$

Finally the sources of the field equations are

$$\Sigma_{\nu}{}^{\mu} := \frac{\partial L_{\rm m}}{\partial \nabla_{\mu}\psi^{A}} \nabla_{\nu}\psi^{A} - \delta^{\mu}_{\nu} L_{\rm m}, \tag{3.22}$$

$$\Delta^{\mu}{}_{\nu}{}^{\lambda} := \frac{\partial L_{\rm m}}{\partial \Gamma_{\lambda\mu}{}^{\nu}}. \tag{3.23}$$

We refer to [18] for an explicit fluid model that possesses microstructure in the form of hypermomentum.

We have thus constructed a very general theory of gravity, with an arbitrary Lagrangian and no prior geometric restrictions on the manifold and we have seen the dynamical structure of the field equations. This theory and field equations incorporate almost *every* single modified gravity model that will be considered within this thesis[3]. Let us now take a look at an explicit special case of this theory.

3.5 Einstein-Cartan Theory

In this section we will look at a particular example of a theory of metric affine gravity, which takes place in the restricted U_4 subspace of L_4, so that the non-metricity tensor vanishes. The particular form we will look at has been much studied in the literature, and it is known as Einstein-Cartan theory, or sometimes as Einstein-Cartan-Sciama-Kibble theory. This is a relatively straightforward way of including a non-trivial torsion into a gravitational theory and the theory shares many features with general relativity. The theory takes into account just one additional piece of microstructure, in the form of the spin of the matter fields.[4]

In this theory, the connection has both curvature and torsion, with the metric Levi-Civita part of the connection being determined by the curvature via the standard matter energy momentum tensor, and the torsion component of the connection determined by the *spin* of the matter. On macroscopic scales this agrees with general relativity as spin is irrelevant at these scales. However on microscopic scales this theory is different as spin becomes important. Although at the time of its formulation, spin was not known to Cartan and its identification as the source of torsion was only made later.

The theory is described by the following action

$$L = \sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_m\right). \tag{3.24}$$

This has the same form as the Einstein-Hilbert action, however one does not impose

[3]The only modification not covered by these equations are those in which there is a nonminimal coupling between gravitational and the matter sector. For the general field equations in this case we refer to [15].

[4]By spin we should emphasise we are referring to intrinsic spin of elementary particles, rather than any macroscopic spin.

a priori that the connection is the Levi-Civita connection, and the matter Lagrangian is not necessarily independent of torsion. The symmetric stress energy tensor is still given by the variation of the matter sources with respect to the metric

$$\mathbf{t}^{\mu\nu} = \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}. \tag{3.25}$$

But additionally we now have a source of torsion, which comes from the variation of the matter Lagrangian with respect to the contortion (or equivalently distortion) tensor

$$\tau_\rho{}^{\nu\mu} = \frac{\delta \mathcal{L}_m}{\delta K_{\mu\nu}{}^\rho}. \tag{3.26}$$

This source of torsion is associated with the spin density of matter.

The variation of the above action yields the following set of field equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G \left(\mathbf{t}^{\mu\nu} + (\nabla_\rho + 2T_{\rho\sigma}{}^\sigma)(\tau^{\mu\nu\rho} - \tau^{\nu\rho\mu} + \tau^{\rho\mu\nu}) \right), \tag{3.27}$$

(where we recall the Ricci scalar is that of an arbitrary metric compatible connection, and not that of the Levi-Civita connection), and

$$T_{\mu\nu}{}^\rho = 8\pi G \tau_{\mu\nu}{}^\rho. \tag{3.28}$$

The second of these equations is purely algebraic and directly relates the spin density to the torsion tensor. This means that torsion does not propagate as a wave, and it will vanish identically in a vacuum.

Due to this algebraic relationship, it is possible to substitute this expression for torsion into the first field equation to retrieve a pure Einstein equation of the form

$$\bar{R}^{\mu\nu} - \frac{1}{2} \bar{R} g^{\mu\nu} = 8\pi G \tilde{\tau}^{\mu\nu} \tag{3.29}$$

where we recall $\bar{R}^{\mu\nu}$ is the Ricci tensor of the Levi-Civita connection and $\tilde{\tau}^{\mu\nu}$ is an effective energy momentum tensor which is formed entirely of terms containing the spin density and the symmetric energy momentum density.

Thus we have an effective Einstein equation. In a vacuum, this equation will coincide with general relativity, it is only the presence of spin in matter that will potentially lead to different solutions. This theory is interesting for a number of reasons. As we discussed earlier, metric affine gravity has the potential to eliminate the big bang singularity prevalent in general relativity. In fact Einstein-Cartan theory can do this alone. The minimal coupling between the torsion tensor and quantum form of matter known as Dirac spinors, generates a spin-spin non-linear self-interaction which becomes significant at high energy densities, such as those occurring in the early universe. This interaction stops the big bang singularity from appearing, replacing it with a big bounce type scenario arising at a non-zero(and hence non-singular) scale factor.

In this chapter we have started having a look at various theories of gravity that can be defined in the geometric spaces introduced in the previous chapter. We have seen an incredibly general theory of gravity, metric affine gravity, whose field equations encompass all gravity models that will be studied through the rest of this thesis. We took a more in depth look at general relativity, along with a generalisation of GR which includes a non-trivial torsion. In the next chapter we will have a look in detail at another special case of Metric Affine Gravity: the *Teleparallel Equivalent of General Relativity.*

Chapter 4

The teleparallel equivalent of general relativity

Almost immediately after Einstein formulated his general theory of relativity, modifications of the theory began to be considered. These early studies were focused on unifying gravity with the other known force of nature at the time, electromagnetism, into a consistent geometric framework. The first attempt at this was made by the mathematician Hermann Weyl in 1918 [19]. His work did not succeed, however Weyl's work introduced for the first time important notions such as gauge transformations and gauge invariance.

In the late 1920s Einstein himself attempted to unify electromagnetism and gravitation, using the mathematical structure of *teleparallelism*, also referred to as absolute parallelism. Teleparallelism means being able to calculate the angle between distant vectors. Einstein's idea was to introduce the tetrad field, which we recall is a field of orthonormal bases of the tangent space, defined at each point of the four dimensional spacetime manifold. This tetrad would have sixteen linearly independent components, six more than the metric of general relativity, which only has ten independent components. Einstein's idea was that these additional six components would be related to the six components of the electromagnetic field. Alas, this attempt again did not work, and it was later realised that these additional six components are eliminated by the Lorentz invariance of the theory: two tetrads described the same physics if one can write one as a Lorentz transformation of the other. However, despite the original unification goal failing, this work led to the development of an alternative description of gravitation, known as *teleparallel gravity* .

Teleparallel gravity has its roots in work done by Einstein and Cartan in the 1920s, where as we saw in the previous chapter, a manifold with both curvature and torsion was introduced. This theory later identified the source of torsion as intrinsic spin. However, teleparallel gravity takes an alternative approach to including torsion, and its theory is based on a geometrical result going back to Weitzenböck [13] who observed that it is always possible to define a specific connection such that the underlying space is globally flat, meaning that the Riemannian curvature of the spacetime is everywhere vanishing. However in order to compensate for this, the Weitzenböck connection possesses a non-trivial torsion. This theory has some appealing properties: it means that the result of parallel transport is independent of the path taken, and lengths and angles are invariant under parallel transport.

Einstein was able to develop a theory of gravity using only this Weitzenböck connection. Moreover, he was able to show that for a particular parameter choice of this teleparallel gravity, the theory had the exact same dynamics as general relativity: they were two physically equivalent theories. This particular choice of parameters is known as the *teleparallel equivalent of general relativity*, or TEGR for short.

During the 1920s there were some other attempts to unify electromagnetism and gravitation. Along with the aforementioned Einstein-Cartan theory, Kaluza and Klein developed a five dimensional theory, Kaluza-Klein theory, with the fifth dimension compactified in an attempt to incorporate electromagnetism. But after this initial period in the 1920s, no new advances were made in the teleparallel theory until the 1960s. The unification idea was dropped, because it was realised that the gravitational and electromagnetic fields must be treated differently due to their vastly different strengths. Moreover they become significant on very different scales. The universe is believed to be overall charge neutral, and so the gravitational force, which is attractive becomes the dominant force on large scales. However on small length scales charge is of far more importance than the gravitational interaction.

Teleparallelism was later resurrected by Møller in the 1960s, no longer with unification in mind, but instead to find a tensorial complex for the gravitational energy momentum tensor. This work led to Pellegini and Plebanski finding a Lagrangian formulation of teleparallel gravity, which later in 1976 was proven by Cho to be equivalent to the Einstein Hilbert action up to a boundary term.

The theory gained prominence again in the 21st century after the advent of the discovery of dark energy. This will be discussed in detail in the following chapters, but this led to modifications of gravity being increasingly considered. Most of these modifications considered modifying the metric formulation of general relativity. However there are many promising results showing that modifications of the TEGR action could succesfully account for phenomenon such as dark energy and inflation. Moreover these teleparallel modifications had some nice features that were lacking in the GR modifications.

Even though teleparallel gravity is dynamically completely equivalent to general relativity, it has a very different physical interpretation. In this theory torsion, as opposed to curvature, represents the gravitational field. Torsion describes the gravitational interaction as acting as a force, whereas general relativity geometrises the interaction so that no force is in fact present. There are no geodesics in teleparallel gravity, rather there are force equations, similar to Maxwell's theory. Another key difference is that while both formulations are invariant under local Lorentz transformations, in GR all geometrical quantities are naturally Lorentz scalars, while in teleparallel gravity expressions typically depend on the chosen frame. This has important implications, particularly when considering modifications of the theories. For further information on the history and properties of TEGR we refer to the important papers [20–29] and also [30].

In this chapter I will introduce the basics of teleparallel gravity, defining the tetrad and Weitzenböck connections and introducing the general teleparallel Lagrangian. I will then show which Lagrangian is equivalent to general relativity, and prove this equivalence. The issue of local Lorentz invariance will also be discussed, and the theories relation to metric affine gravity will be derived. This will form the basis for later chapters, when we consider modifications of this theory.

4.1 Teleparallelism

General Relativity is a metric theory of gravity, taking place in Riemannian V_4 space where the Riemann curvature tensor is constructed from the Levi-Civita connection, with the metric determined by the Einstein field equations. However, the teleparallel equivalent of general relativity is an equivalent formulation based on a globally flat space where gravity is described by torsion instead of curvature. That this is indeed possible is not trivial and is based on the work of Weitzenböck who noted that by choosing a different connection on the same space[1] in a specific way it is possible to ensure that space is indeed globally flat [13].

Let us take a closer look at this result. Firstly we note that one can decompose a general connection into two pieces: a tetrad part and an object called the *spin connection*

$$\Gamma^{\lambda}_{\mu\nu} = e^{\lambda}_{b}\partial_{\nu}e^{b}_{\mu} + e^{\lambda}_{a}A^{a}_{b\nu}e^{b}_{\mu} \tag{4.1}$$

where $A^{a}_{b\nu}$ is the spin connection. We can solve (4.1) to find the spin connection in

[1]As we discussed in Chapter 2 we can define different connections on the same underlying manifold.

terms of the connection and tetrad

$$A^a_{b\nu} = e^a_\lambda \partial_\nu e^\lambda_b + e^a_\lambda \Gamma^\lambda_{\mu\nu} e^\mu_b. \tag{4.2}$$

Denoting the covariant derivative associated with the connection Γ by ∇_μ, this can alternatively be written as

$$A^a_{b\nu} = e^a_\lambda \nabla_\nu e^\lambda_b. \tag{4.3}$$

In the teleparallel formulation of General Relativity one works with the Weitzenböck connection, that is we work in a W_4 space. This particular connection is given by

$$\Gamma^\lambda_{\mu\nu} = e^\lambda_b \partial_\nu e^b_\mu = -e^b_\mu \partial_\nu e^\lambda_b. \tag{4.4}$$

This is equivalent to have chosen a vanishing spin connection. Using this specific connection, it is found that its Riemann curvature tensor vanishes identically,

$$R^\rho{}_{\mu\lambda\nu} = 0. \tag{4.5}$$

Although imposing that the connection is given by the Weitzenböck connection requires introducing some geometry a priori. As we will see at the end of this chapter, one can impose this by adding a Lagrange multiplier term coupled to the curvature tensor.

The Weitzenböck connection can be decomposed into a sum of the Levi-Civita connection and the contortion tensor, using the relation (2.31) in the absence of non-metricity, as follows

$$\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} + K^\lambda{}_{\mu\nu}. \tag{4.6}$$

This will turn out to be a very important relation for comparing general relativity and teleparallel gravity.

Why is this Weitzenböck connection physically interesting? Well if we consider a vector V^a at a point p of our manifold in the tangent space T_p, and a vector V'^a at another point p' in the tangent space $T_{p'}$, then they will be defined to be parallel if they have the same components expressed in terms of the tetrad basis, that is if $V^\mu = V'^\mu$. That is if we parallel transport the vector V^a, we wish to get the same vector, meaning

$$0 = dV^a = d(V^\mu e^a_\mu) = e^a_\mu dV^\mu + V^\mu \partial_\nu e^a_\mu dx^\nu, \tag{4.7}$$

which after rearrangement and multiplication by the cotetrad gives

$$dV^\mu = -e^\mu_a \partial_\nu e^a_\rho V^\rho dx^\nu = -\Gamma^\mu_{\rho\nu} V^\rho dx^\nu. \tag{4.8}$$

But this is nothing but the definition of the connection (2.10), and we see here that the particular connection here is the Weitzenböck connection. Thus this is the connection that possesses the property of distant parallelism.

4.2 Equivalent of the geodesic equation

In teleparallel gravity, the interpretation of the gravitational field is rather different. Instead of it being a geometric effect, as in general relativity, where particles follow geodesics of a curved manifold, rather free particles follow force law equations. In this section we will show this explicitly, following closely [30, 31].

Teleparallel gravity is a gauge theory of the translation group, with the gauge transformations being local translations of the tangent space coordinates

$$x'^a = x^a + \epsilon^a(x^\mu). \tag{4.9}$$

That is a severely restricted form of the symmetry group of metric affine gravity (3.12). We can decompose the tetrad of teleparallel gravity into a trivial inertial component and a gravitational potential as so

$$e^a_\mu = \partial_\mu x^a + B^a{}_\mu \tag{4.10}$$

where from the gauge theoretical point of view, $B^a{}_\mu$ is the *translational gauge potential.* This is a one-form taking values in the Lie algebra of the translation group

$$B_\mu = B^a{}_\mu \partial_a, \tag{4.11}$$

where ∂_a are the generators of infinitesimal translations.

Let us consider the motion of a spinless particle in this gravitational field $B^a{}_\mu$. The action integral for this motion is given by

$$S = \int_a^b \left[-mc\, d\sigma - mcB^a_\mu u_a\, dx^\mu \right], \tag{4.12}$$

analogously to the action integral for a particle travelling in an electromagnetic field. Here, $d\sigma = (\eta_{ab} dx^a dx^b)^{1/2}$ is the Minkowski space interval in the tangent space. u^a is the particle four velocity in the tetrad frame. The first term in this action represents the motion of a free particle, and the second represents a coupling of the particle to the gravitational field. Variation of the action (4.12) gives the following equation of motion

$$e^a{}_\mu \frac{du_a}{ds} = T^a{}_{\mu\rho} u_a u^\rho \tag{4.13}$$

where for compactness the torsion here is defined as

$$T^a{}_{\mu\nu} = e^a{}_\rho T^\rho{}_{\mu\nu}, \tag{4.14}$$

which also has the interpretation as the translational gauge field strength.

One of the crucial aspects of teleparallel gravity is that this equation is a *force law*. In particular, it is the non-trivial torsion that is playing the role of the force. This is in stark contrast to general relativity, where no forces are present, and free particles simply follow geodesics. The two equations are completely equivalent, but this is a radically different interpretation. Let us indeed show that this equation gives rise to the same dynamics as the geodesic equation. By the antisymmetry of the torsion tensor, and using the definition of the contorsion tensor, it is readily seen that

$$T^{\lambda}{}_{\mu\rho}u^{\rho}u_{\lambda} = -K^{\lambda}{}_{\mu\rho}u^{\rho}u_{\lambda}. \tag{4.15}$$

Now using the relation (4.6), we find that

$$\frac{du^{\lambda}}{ds} + \bar{\Gamma}^{\lambda}_{\mu\nu}u^{\mu}u^{\nu} = 0, \tag{4.16}$$

which is nothing but the geodesic equation of general relativity.

There is another feature of this teleparallel theory of interest. In action (4.12), we implicitly assumed the *weak equivalence principle*, that is that the two masses in the action, the inertial mass m_i and the gravitational mass m_g, are identical with $m_i = m_g = m$. This is a fundamental assumption of general relativity, and without it the theory would be inconsistent, as it would imply that the metric depends on the ratio m_g/m_i of a test particle: this would mean that the gravitational field depends on the properties of test particles. This is not the case in teleparallel gravity, and one does not have to impose the weak equivalence principle from the start. The weak equivalence principle (WEP) is a physical assumption that can only be verified experimentally. To date it has passed all experimental tests, however there are some theoretical difficulties with it [32], mainly from a quantum level [33]. If one does not assume the WEP, the force law is modified to become [31]

$$\left(\partial_\mu x^a + \frac{m_g}{m_i} B^a{}_\mu\right) \frac{du_a}{ds} = \frac{m_g}{m_i} T^a{}_{\mu\rho} u_a u^\rho . \tag{4.17}$$

Because neither $B^a{}_\mu$ nor $T^a{}_{\mu\nu}$ depend on the relation m_g/m_i, the field equations can be solved for the gravitational potential $B^a{}_\mu$, and then (4.17) can be solved to give the motion of the test particle. Thus we can consistently solve the field equations without assuming the WEP, and any future experimental evidence indicating deviations from the WEP may indicate that a teleparallel model should be considered.

4.3 The action

Einstein initially proposed several possibilities for the Lagrangian of teleparallel gravity. He considered all possible contractions of the torsion tensor with itself, which can be parametrised by the following Lagrangian

$$S = \int e(c_1 T^\sigma{}_{\mu\nu} T_\sigma{}^{\mu\nu} + c_2 T^\sigma{}_{\mu\nu} T^{\nu\mu}{}_\sigma + c_3 T_{\sigma\mu}{}^\sigma T^{\nu\mu}{}_\nu)\, d^4x\,, \tag{4.18}$$

$$= \int e\,(c_1 L_1 + c_2 L_2 + c_2 L_3)\; d^4x\,, \tag{4.19}$$

where c_1, c_2 and c_3 are all constants and L_1, L_2 and L_3 are three invariants of the torsion tensor. These invariants were first discussed by Weitzenböck [13]. This action is quadratic in the torsion tensor and has three free parameters: it is a three parameter theory of gravity.

Einstein tried several combinations of the parameters c_1, c_2 and c_3. The first he looked at was $c_2 = 1$ and $c_1 = c_3 = 0$ (which was also found to be equivalent to $c_1 = 1$ and $c_2 = c_3 = 0$). This could be interpreted as a set of vacuum Einstein and Maxwell equations, however this interpretation was viewed as artificial and synthetic and did not in the end provide a consistent theory. However, later he tried the parameter choice $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = -1$, and he found that the field equations of the action (4.19) were completely equivalent to those of general relativity, a result we shall prove shortly. In fact, with this choice of parameters, one can write the action in a more compact form using the *superpotential*, which is defined as the following combination of torsion and contortion

$$S^{\rho\mu\nu} := K^{\mu\nu\rho} - g^{\rho\nu}T^{\sigma\mu}{}_{\sigma} + g^{\rho\mu}T^{\sigma\nu}{}_{\sigma}. \tag{4.20}$$

Associated with this superpotential is the following invariant, sometimes known as the torsion scalar T

$$T = \frac{1}{2}S^{\rho\mu\nu}T_{\rho\mu\nu} = \frac{1}{2}T^{\sigma}{}_{\mu\nu}T_{\sigma}{}^{\mu\nu} + \frac{1}{4}T^{\sigma}{}_{\mu\nu}T^{\nu\mu}{}_{\sigma} - T_{\sigma\mu}{}^{\sigma}T^{\nu\mu}{}_{\nu}. \tag{4.21}$$

This means that the action of the *teleparallel equivalent of general relativity* (TEGR) can be written compactly as

$$S_{\mathrm{TEGR}} = \int \frac{1}{2}S^{\rho\mu\nu}T_{\rho\mu\nu}\, e\, d^4x \; = \int T e\, d^4x\,. \tag{4.22}$$

The field equations of this action are then derived by varying the action (4.22), along with an additional matter Lagrangian with respect to the tetrad field $e^a{}_\mu$. This gives

$$e^{-1}\partial_\mu(eS_a^{\mu\nu}) - T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\nu\mu} - \frac{1}{4}e_a{}^{\nu}T = 4\pi\mathcal{T}_a^{\nu} \tag{4.23}$$

where the tensor $\mathcal{T}_a^{\nu} = e_a{}^{\mu}\mathcal{T}_\mu^{\nu}$ is the standard energy momentum tensor, defined by

$$\mathcal{T}_a^{\nu} = \frac{1}{e}\frac{\delta(e\mathcal{L}_m)}{\delta e^a{}_\nu}\,. \tag{4.24}$$

The following conservation law holds,

$$\partial_\nu \left(e(j_a{}^\nu - \mathcal{T}_a{}^\nu) \right) = 0 \tag{4.25}$$

where $j_a{}^\nu$ is the gauge current, which represents the energy momentum of the gravitational field

$$j_a{}^\nu = \frac{1}{4\pi} \left(e_a{}^\sigma T^\rho{}_{\mu\sigma} S_\rho{}^{\nu\mu} - \frac{1}{4} e_a{}^\nu T \right) . \tag{4.26}$$

4.3.1 Equivalence with general relativity

Let us show that the above TEGR theory is in fact equivalent to general relativity. One could either show this equivalence at the level of the action or at the level of the field equations. Here we choose to show this at the level of the action, as the results derived in this calculation will be very important later on in this thesis.

Let us begin by decomposing the Riemann curvature of a general U_4 connection in terms of the Riemann curvature of the Levi-Civita connection and an additional torsion component. As always, quantities calculated using the Levi-Civita connection will be denoted with a bar. We find

$$R^\rho{}_{\mu\lambda\nu} = \bar{R}^\rho{}_{\mu\lambda\nu} + \bar{\nabla}_\lambda K^\rho{}_{\mu\nu} - \bar{\nabla}_\nu K^\rho{}_{\mu\lambda} + K^\rho{}_{\sigma\lambda} K^\sigma{}_{\mu\nu} - K^\rho{}_{\sigma\nu} K^\sigma{}_{\mu\lambda} \, , \tag{4.27}$$

Now contracting this, we get the Ricci tensor, which is given by

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\rho K^\rho{}_{\mu\nu} - \bar{\nabla}_\nu K^\rho{}_{\mu\rho} + K^\rho{}_{\sigma\rho} K^\sigma{}_{\mu\nu} - K^\rho{}_{\sigma\nu} K^\sigma{}_{\mu\rho} \, . \tag{4.28}$$

And finally we can find the Ricci scalar by contracting this again, to find

$$R = \bar{R} + T - 2\bar{\nabla}^\mu \left(T^\nu{}_{\nu\mu} \right) . \tag{4.29}$$

Now, imposing the teleparallel condition that the Riemann curvature vanishes, $R^{\rho}{}_{\mu\lambda\nu} = 0$, that is choosing the connection to be the Weitzenböck connection, allows us to find the Ricci scalar of the Levi-Civita connection in terms of torsion

$$\bar{R} = -T + 2\bar{\nabla}^{\mu}\left(T^{\nu}{}_{\nu\mu}\right) \tag{4.30}$$

In the derivation of these equations, use has been made of the identities

$$K^{(\mu\nu)\lambda} = T^{\mu(\nu\lambda)} = S^{\mu(\nu\lambda)} = 0, \tag{4.31}$$

$$S^{\mu}{}_{\nu\mu} = 2K^{\mu}{}_{\nu\mu} = -2T^{\mu}{}_{\nu\mu}. \tag{4.32}$$

Now, we can rewrite the covariant derivative in the expression (4.30) using the relation $\nabla_{\mu}V^{\mu} = e^{-1}\partial_{\mu}(eV^{\mu})$, to give

$$\bar{R} = -T + \frac{2}{e}\partial_{\mu}(eT^{\mu})\,, \tag{4.33}$$

where we have also defined $T^{\mu} := T^{\nu}{}_{\nu\mu}$. Now the second term of this relationship is a boundary term: it is just a total derivative, and so in the Lagrangian it will have no effect on the dynamics of the system. Thus the Einstein-Hilbert action is completely equivalent to the teleparallel action, up to a minus sign, and they give rise to the same dynamics.

Later in this thesis we will study this boundary term in greater detail, and so we will introduce the notation

$$B = \frac{2}{e}\partial_{\mu}(eT^{\mu}) = 2\nabla_{\mu}T^{\mu}\,. \tag{4.34}$$

and hence the relationship (4.33) can be written compactly as

$$\bar{R} = -T + B. \tag{4.35}$$

An important issue to note is that although the Ricci scalar contains second derivatives of the metric, the torsion scalar contains only up to first derivatives of the tetrad. The second order quantities are grouped entirely into the boundary term, a fact that has important consequences when considering modifications of the theories.

4.3.2 Lorentz invariance

An important theoretical point which we will address now is the issue of local Lorentz invariance in teleparallel theories of gravity. In any theory of gravity we require two invariance principles to hold. The first is that the theory is invariant under local coordinate transformations, $x^\mu \to x^\mu + \epsilon^\mu$, so that the action is required to be a generally covariant scalar. The second principle is that of local Lorentz invariance, which is required so that special relativity is recovered in local inertial frames. This means that at every point we should be able to redefine our coordinates so that we are in a local inertial coordinate system. To do this we require the action to be invariant under the following infinitesimal Lorentz transformation

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b(x^\mu), \tag{4.36}$$

where $|\omega^a{}_b| \ll 1$ is infinitesimal and antisymmetric due to the Lie Group structure of the Lorentz group.

Let us examine how our action is changed by local Lorentz transformations. Under the transformation (4.36), the tetrad is a Lorentz vector in the latin index, and so transforms as

$$\delta e_a^\mu = \omega^b{}_a e_b^\mu. \tag{4.37}$$

We begin looking at how the matter action transforms under such a transformation.

We find it is modified in the following way

$$\delta S_m = \int \mathcal{T}^a{}_\mu e \delta e^\mu{}_a d^4x = \int \mathcal{T}^a{}_\mu e \omega^b{}_a e^\mu_b d^4x \tag{4.38}$$

Now due to the antisymmetrisation of the function ω_{ab}, it is an easy calculation to show that this vanishes if and only if the energy momentum tensor is symmetric

$$\mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu}. \tag{4.39}$$

Thus the symmetry of the energy momentum tensor is a consequence of local Lorentz invariance of the matter action. In a similar fashion, conservation of the energy momentum tensor is a direct result of the requirement that the matter action is invariant under local coordinate transformations.

Now let us look at the gravitational sector of the Lagrangian. Now we know that even though the connection is not a tensor, the torsion tensor, which is the antisymmetric part of the connection is indeed a tensor under spacetime coordinate transformations due to this antisymmetry. In terms of the Weitzenböck connection explicitly, this is because the partial derivatives can be promoted to covariant derivatives in the torsion tensor. Similarly, as the contortion tensor and superpotential are built algebraically out of the torsion tensor, these too, along with their appropriate contractions, are also invariant under infinitesimal coordinate transformations.

However, the situation is not so straightforward when we check whether they are Lorentz scalars. We know that the Ricci scalar of the Levi-Civita connection is invariant under local Lorentz transformations. This however is not true for the torsion scalar T or the boundary term B: the particular combination of $-T + B$ is invariant, alas the individual terms are not. Applying a Lorentz transformation results in a non-trivial spin connection, which in turn means that the Weitzenböck connection does not transform correctly and the corresponding torsion tensor do not transform as Lorentz scalars. This is not a significant problem for TEGR, insomuch as the term breaking Lorentz invariance is a boundary term, and so the field equations will remain locally Lorentz invariant.

As we will come to see later in this thesis, violation of Lorentz invariance will become an issue when one considers modifications of the TEGR action. It should be emphasised at this stage that local Lorentz invariance is a very well tested assumption experimentally, and sacrificing this principle without evidence contrary to its validity should not be considered lightly. It is a major weakness of such theories that they do not possess this property - losing Lorentz invariance means that special relativity is not valid locally. This means many important physical properties are lost, including the speed of light being constant in all inertial reference frames, the principle of relativity, and many predictions of the standard model of particle physics. Such theories give rise to a frame dependent description of reality.

Let us briefly review the main experimental evidence supporting the validity of local Lorentz invariance. One such way to test Lorentz violations is to assume that electromagnetic interactions suffer a small violation of Lorentz invariance, giving rise to a change in the speed of electromagnetic interactions c compared to the limiting speed of matter. These experiments measure deviations of the speed c by considering the quantity

$$\delta = |1 - c^{-2}|. \tag{4.40}$$

The first such experiment to test for such Lorentz violations were the famous Michelson-Morley experiments, where a beam of light is split and sent down two perpendicular arms. Descendants of the Michelson Morley experiment have bounded δ to below 10^{-9}.

Further experiments measuring local Lorentz variance include measurements of the speed of light independent of the velocity of its source, measurements of the isotropy of the speed of light and time dilation experiments. But those that have obtained the best bounds are the so called clock anisotropy measurements. Lack of local Lorentz invariance would cause shifts in the energy levels of atoms depending on their orienartation relative to the universal reference frame. Looking for these shifts, these experiments have bounded δ to below 10^{-18}. For a review of the various bounds that these different experiments have obtained, we refer to [34] and references within. Thus Lorentz invariance is well supported and any violation to Lorentz invariance would have to be very small, less than 10^{-18}. It should be noted that although some of the teleparallel models alluded to above are Lorentz violating, in recent work methodologies have been developed that restore local Lorentz invariance back into some of these teleparallel modifications [35].

4.4 Embedding teleparallel gravity into MAG

One can also embed teleparallel gravity into a metric affine gravity framework. From a gauge-theoretic point of view, teleparallel gravity is distinguished from other metric affine models by reducing the affine symmetry group to the translation subgroup. This takes place in the previously mentioned Weitzenböck space, where we impose two constraints on the general affine framework by setting the curvature and nonmetricity equal to zero. Thus, geometrically and physically we can treat teleparallelism as a particular case of metric affine gravity.

To impose these conditions on our metric affine framework, we essentially insert two Lagrange multipliers into the action, thus restricting the form of connection in our L_4 space. Following the approach used in [20], we consider the following Lagrangian density

$$S = \int \Big[e\left(c_1 T^{\sigma}{}_{\mu\nu} T_{\sigma}{}^{\mu\nu} + c_2 T^{\sigma}{}_{\mu\nu} T^{\nu\mu}{}_{\sigma} + c_3 T_{\sigma\mu}{}^{\sigma} T^{\nu\mu}{}_{\nu}\right) + \frac{1}{2}\mu^{\alpha\beta} \wedge Q_{\alpha\beta} - \nu^{\alpha}{}_{\beta} \wedge R_{\alpha}{}^{\beta} \Big] d^4x\,, \quad (4.41)$$

Here the quantities $\mu^{\alpha\beta}$ and $\nu^{\alpha}{}_{\beta}$ are Lagrange multipliers, and thus varying the action with respect to these imposes the geometrical constraints $Q_{\alpha\beta} = 0$ and $R_{\alpha}{}^{\beta} = 0$ and so we return to the geometrical framework of a Weitzenböck spacetime W_4.

When the coupling constants are taken again to be $c_1 = 1/2$, $c_2 = 1/4$ and $c_3 = -1$ we again recover the field equations of general relativity. However, the energy momentum tensor on the right hand side of the field equation is not the metrical one of general relativity, rather it is the canonical energy momentum current. This is important, because if the matter field possesses a non-trivial spin, then the energy momentum tensor differs from the metrical one. However, this can cause some consistency problems when spin is included, and thus it has been shown in [20] that teleparallel gravity only makes sense when spinless matter is considered.

An interesting property of the action (4.41) is that the solutions to the field equations only admit black hole solutions such as the Schwarzschild metric or its charged equivalents when the coupling constants c_1, c_2, c_3 are chosen to coincide with the general relativistic case, and thus this naturally picks out TEGR as the sole physically supported theory from this class of theories [20].

4.5 Discussion

In this chapter we have taken an in depth look at some of the properties of the teleparallel equivalent of general relativity (TEGR). We have seen that we can define a theory which leads to the same field equations and physical consequences as general relativity, but taking place in a different geometrical setting. This differing framework dramatically changes the physical viewpoint of the laws of gravity. Curvature is replaced by torsion, meaning the geometric laws of general relativity are replaced by classical force laws and thus the theory sits somewhat more naturally in the spectrum of other physical laws.

However, considering TEGR alone does not help us understand some of the physical phenomena such as dark energy, dark matter and inflation that general relativity fails to explain. Thus there is a potential need to go beyond these theories. In the next chapters we will look at some of the cosmological behaviour a modified theory of gravity should possess, and begin looking at some modifications to this teleparallel action.

Chapter 5

Cosmology, dark energy & modified gravity

In this chapter we will move beyond studying theories of gravity in the abstract sense, and look at one of their most interesting applications: the subject of cosmology. We will discuss some of the challenges facing cosmologists today and introduce a number of models that can potentially deal with these challenges. We shall see that modifications of teleparallel gravity are among those which are excellent candidates to be viable models of our universe.

Cosmology is the study of the dynamics and large scale structure of the universe and its study has fascinated humanity since the dawn of civilisation. Its study aims to understand the origin, evolution and eventual fate of the universe as a whole. It was not until the decades after the publication of the general theory of relativity that cosmology moved from a branch of philosophy to a subject that could be studied using scientific techniques. The pioneering theoretical work of physicists in the 1920s such as Friedman, Lemaître and Einstein, along with Hubble's observations of the expansion of the universe, showed that Einstein's equations were able to model not just local gravitational fields, but the universe as a whole. Gravity was no longer just the force that governed how the planets orbit the sun: it was now able to describe the dynamics of the entire universe. Over the last century the study of cosmology has progressed enormously: from the realisation that the universe began in a big bang in the mid twentieth century, to the current era, where cutting edge satellites are able to record data from the earliest instances of the universe.

As we discussed in the introduction, one of the great challenges for modern day cosmologists is to explain the observed accelerated expansion of the universe. This phenomena is now widely known as dark energy and was discovered in the late 1990s from observations of distant supernovae. General relativity alone does not predict this. The simplest way to account for this, making only a minimal modification to general relativity, is to assume the existence of a small positive cosmological constant. This does not affect the physics at smaller scales, however it modifies the behaviour of the universe at much larger scales. This cosmological constant is currently part of the standard ΛCDM model of cosmology, and it is consistent with all observations to date.

However, the cosmological constant suffers from a number of theoretical difficulties; to be discussed later in this chapter. This has led cosmologists to propose many alternative models to explain the accelerated expansion. One of the simplest ways is to promote this constant to a variable in the form of a scalar field. Adding a scalar field into the universe is one of the simplest modifications we can make, as we simply add one additional degree of freedom and they are usually enough to describe the large-scale effects of high energy gravity theories, at least at an effective level. Scalar field models are also frequently invoked to explain inflation, a period of rapid exponential expansion at the beginning of the universe. The canonical way to do this is adding a kinetic and potential term for a scalar field into the Lagrangian, and in the context of late time cosmology this theory is known as quintessence. Such a model can provide a late time accelerated expansion of the universe, however such simple models are becoming increasingly disfavoured by observations.

This has led to the study of various alternative theories. One way is to consider non-canonical scalar field models, such as a phantom model, where the kinetic term has the incorrect sign, or other various modifications of the scalar field Lagrangian. Another suggestion is to introduce a coupling between the gravitational part of the Lagrangian and the scalar field. This class of models are known as non-minimally coupled scalar field models, and are a generalisation of the Brans-Dicke theories which were first proposed in the 1960s, which were motivated by designing a gravitational theory with a varying gravitational coupling constant.

Instead of considering additional matter components, an alternative approach to the dark energy problem is to consider modified theories of gravity. We examined in the last two chapters different theories of gravity in the metric affine framework, however these alone are not able to account for the accelerated expansion of the universe, and so alternative modifications to these theories are needed. One can consider modifying gravity in the metric framework, taking place in a V_4 space, which leads to theories such as $f(R)$ gravity, where the gravitational Lagrangian is an arbitrary form of the Ricci scalar. Alternatively, one could consider modifications in the teleparallel framework, working in a W_4 space, and this leads to the consideration of theories such as $f(T)$ gravity, where the gravitational Lagrangian is an arbitrary function of the torsion scalar. Both of these approaches have their advantages and disadvantages, which we will discuss later on.

An important criterion for testing whether a theory of gravity is potentially viable is to examine whether it exhibits cosmology consistent with observations. Frequently it is not possible to find the exact cosmological solutions for a theory of gravity, and so to analyse the generic behaviour of the cosmology at hand, one can use the mathematical techniques of dynamical systems. The idea is that one rewrites the cosmological field equations as a system of autonomous ordinary differential equations. By examining the critical points of such systems, one can study the behaviour of the system by looking for the generic asymptotic early time and late time behaviour. This can then be related to physical observables to examine whether the model generically behaves as we expect our universe to.

In this chapter I will introduce the basics of cosmology, looking at the Friedmann-Robertson-Walker (FRW) universe. The necessary dynamical systems techniques relevant for studying cosmology will be introduced: these techniques will be important in later chapters, where they will be applied to various models in the teleparallel framework. I will then discuss the dark energy problem, and problems with the current standard model which assumes the existence of a non-zero cosmological constant. A range of alternative possible models which have been proposed to explain this problem will be discussed, including scalar field models and modified theories of gravity.

5.1 FRW Universe

In order to mathematically describe the universe as a whole, certain physical assumptions need to be made to create a tractable model. Modern day cosmology is based on a postulate known as *the cosmological principle*, or sometimes *the Copernican principle* which says that the universe is *isotropic*, meaning it is invariant under rotations around a point, and *homogeneous*, meaning it is invariant under spatial translations. More broadly speaking, the universe is the same everywhere and in every direction. From the vantage point of the earth, it appears that the universe is highly isotropic around this point. However, it is impossible to verify from one point that the universe is homogeneous: isotropy around another point in space would be required to verify this. It is only by comparing our models to observations that we will be able to verify whether the assumption of homogeneity is valid. However, this principle is assumed so that one can study the universe scientifically, and philosophically the assumption is appealing as it does not put humanity at a special place in the universe.

Isotropy and homogeneity mean that our spacetime is highly symmetric. It implies that each three dimensional spatial hypersurface is a maximally symmetric manifold. This leaves three possible choices for the spatial component of our metric, which in spherical polar type coordinates (r, θ, ϕ) are given by

$$d\Sigma^2_{(k)} = \frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.1}$$

If $k = +1$ the universe is *spatially closed*, or compact and has the topology of a three sphere $\mathbb{S}^3$. If $k = 0$ the universe is said to be *flat*, and the metric is just that of standard Euclidean space. If $k = -1$, the universe is *spatially open*, and the metric describes a hyperboloid $\mathbb{H}^3$.

In cosmology, one frequently chooses to work with a cosmological time t, which is chosen so that an observer at rest will measure this to be the proper time of the universe. This leads to the metric taking the following form, called the Friedmann Robertson Walker metric, displayed here in spherical polar type coordinates

$$ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1-kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{5.2}$$

or alternatively in Cartesian type coordinates

$$ds^2 = -dt^2 + \frac{a(t)^2}{\left(1 + \frac{k}{4}(x^2+y^2+z^2)\right)^2}(dx^2 + dy^2 + dz^2) \tag{5.3}$$

The function $a(t)$ is the *scale factor* of the universe. It determines the relative size and evolution of the universe. If $a(t)$ is increasing then distances between points are increasing, and so the universe is expanding. Likewise, if $a(t)$ were decreasing, then distances between points are decreasing in length, and so the universe would be contracting. These coordinates are known as *comoving coordinates* as an observer at rest in these coordinates will remain at constant (r, θ, ϕ) for all future time.

The following metric can be derived from a tetrad. The metric does not uniquely define a tetrad, and so in some theories of gravity one must be careful in the choice of tetrad. When the spatial curvature is zero, with $k = 0$ the choice of tetrad

$$e^a_\mu = \mathrm{diag}(1, a(t), a(t), a(t)) \tag{5.4}$$

gives the FRW metric and can be consistently used. However when $k = \pm 1$ one needs to be more careful with the choice of tetrad, and this choice depends on what theory of gravity is being worked on. We will discuss this point in a little more detail later on.

Now that we have examined the metric of the FRW universe, we must now consider the right hand side of the field equations, that is the energy content of the universe we are modelling. In order to model the universe as a whole, one assumes that all matter in the universe behaves as a perfect fluid, with energy density ρ and pressure p. This means that the energy momentum tensor takes the following form

$$T_{\mu\nu} = (\rho + p)\, U_\mu U_\nu + p g_{\mu\nu}\,, \tag{5.5}$$

where U_μ is the four velocity of the fluid. Alternatively in comoving coordinates one can write this as the following diagonal matrix

$$T^\mu{}_\nu = \mathrm{diag}(\rho, p, p, p). \tag{5.6}$$

Inserting the FRW metric (5.2) and the energy momentum tensor (5.5), into the Einstein field equations gives the following system of equations, known collectively as the Friedmann equations

$$3\frac{k}{a^2} + 3H^2 = \kappa^2 \rho\,, \tag{5.7}$$

$$\frac{k}{a^2} + 2\dot{H} + 3H^2 = -\kappa^2 p\,. \tag{5.8}$$

Alternatively these equations could equivalently be derived in the TEGR using the FRW tetrad. The conservation of the energy momentum tensor gives the following equation

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{5.9}$$

however this equation is a direct consequence of the two Friedmann equations. We thus have two independent equations for three unknown quantities, H, ρ and p. In order to close the system, one must then posit an equation of state, that is a relationship between the pressure and density of the universe, $p = f(\rho)$. Frequently in cosmology, a simple linear equation of state of the form $p = w\rho$ is considered, with w a constant, physically constrained to lie in the range $0 \leq w \leq 1/3$. When $w = 0$ the fluid is called dust, which on cosmological scales is an accurate model of baryonic matter or cold dark matter. When $w = 1/3$, one has the equation of state of radiation.

For an arbitrary linear equation of state, the Friedmann equations can be solved for the scale factor to give

$$a(t) = \begin{cases} a_0(t - t_0)^{\frac{2}{3(w+1)}}, & w > -1 \\ e^{H_0 t}, & w = -1 \\ a_0(t_0 - t)^{\frac{2}{3(w+1)}}, & w < -1 \end{cases} \tag{5.10}$$

with a_0, t_0 and H_0 constants. Here we have shown that we can analytically solve the Friedmann equations if we have a single perfect fluid matter field with linear equation of state. However, if one looks at more complicated models, with additional or different types of matter, it is frequently too difficult to solve for the scale factor explicitly. And so a way to learn about the phenomenology of such models is to use dynamical systems techniques.

5.2 Dynamical systems in cosmology

Frequently when one encounters Friedmann equations, it is either impossible to solve them analytically, or one has to specify initial conditions which are not always clear in a cosmological context. In order to mathematically study the behaviour of solutions, independently of initial conditions, one can use techniques from a branch of mathematics known as *dynamical systems.* These techniques are widely used in cosmological applications since they allow one to study all possible evolutional paths when there is no possibility of finding an exact solution.

A dynamical system in general is a system of first order ordinary differential equations (ODEs). The dynamical system is said to be autonomous if the ODEs are independent of any explicit dependence of time. The general n-dimensional dynamical system thus takes the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} = (x_1, ..., x_n), \tag{5.11}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a generic (usually differentiable) function, and the dot $\dot{}$ denotes differentiation with respect to the time variable. The system is autonomous as $\mathbf{f}$ is independent of the time t.

One can learn a lot about the mathematical behaviour of the solutions of the system (5.11) by looking at *critical* or *fixed points* of the system. These are defined as follows:

Definition 1 *(Critical point). A point* $\mathbf{x} = \mathbf{x_0}$ *is said to be a critical point of the dynamical system (5.11) if* $\mathbf{f}(\mathbf{x_0}) = 0$.

These critical points are important, because when the system is at one of these points, the dynamical system (5.11) is stationary, and so once the system is at that point it will remain at that point for all time. However it may be that if one applies a small perturbation to the critical point, the system may then move away from the critical point. Therefore it is important to look at the *stability* of the critical point.

There are two related but distinct types of stability that will be introduced. The first is that of a *stable fixed point*:

Definition 2 *(Stable fixed point). Let* $\mathbf{x_0}$ *be a fixed point of system (5.11). It is called stable if* $\forall\epsilon > 0$, $\exists\delta > 0$ *such that if* $\psi(t)$ *is any solution of (5.11) satisfying* $|\psi(t_0) - \mathbf{x_0}| < \delta$*, then the solution* $\psi(t)$ *exists* $\forall t \geq t_0$ *and it will satisfy* $|\psi(t) - \mathbf{x_0}| < \epsilon$ $\forall t \geq t_0$.

But a point being stable is not always the main interest, particularly in a cosmological setting: there is no guarantee that generic solutions will end up attracted to that point. This leads to the further definition of an *asymptotically stable fixed point*:

Definition 3 *(Asymptotically stable fixed point). Let* $\mathbf{x_0}$ *be a stable fixed point of the system (23). It is called asymptotically stable if* $\exists\delta > 0$ *such that if* $\psi(t)$ *is any solution of (5.11) satisfying* $|\psi(t_0) - \mathbf{x_0}| < \delta$*, then* $\lim_{t\to\infty} \psi(t) = \mathbf{x_0}$.

In order to perform linear stability analysis, one must consider the $n \times n$ Jacobian matrix formed of the following partial derivatives

$$\mathbf{J} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}. \tag{5.12}$$

One then evaluates $\mathbf{J}$ at each of the critical points, and subsequently find the eigenvalues of the matrix at these points. This matrix will have n eigenvalues, which provide information about the stability of the critical point. A critical point is said to be *hyperbolic* if none of the eigenvalues of the Jacobian matrix have a vanishing real part. Otherwise the point is said to be non-hyperbolic and linear stability analysis does not apply. If all of the eigenvalues have a negative real part, the critical point is stable. If all of the real parts of the eigenvalues are positive, the point is unstable. If there are a mixture of negative and positive real parts, the point is said to be a *saddle*, and along some directions trajectories are attracted towards the point, whereas along other directions trajectories are repelled from the point.

In this thesis I will only consider dynamical systems in which linear stability analysis is sufficient to analyse the stability of critical points. There are some cases in which linear stability analysis is not sufficient, and in this case a more involved stability analysis needs to be performed, for example one can use Lyapunov functions or centre manifold theory[1]. Later in this chapter we will take a first look at an example of a cosmological dynamical system, but for now let us move on to discussing one of the problems of modern day cosmology.

5.3 Dark Energy

One of the great challenges of modern day cosmology is to explain the "dark energy" problem. In the late 1990s it was suggested by type-Ia supernovae surveys [37, 38] that the universe's expansion rate was accelerating. This important observation has subsequently been confirmed by cosmological observations of ever increasing precision, from measurements of the cosmic microwave background (CMB) [9, 39], the Hubble constant [40], baryon acoustic oscillations [41] and again type-Ia supernovae [42]. Unfortunately, despite all this evidence from astronomical observations, on the theoretical ground we still lack a fully satisfactory explanation of this phenomenon.

By accelerated expansion, we mean that the second derivative of the scale factor is positive,

$$\ddot{a} > 0. \tag{5.13}$$

[1]See [36] for an introduction on how to do this.

We can see from (5.10) that this can only happen in a standard matter model with linear equation of state if the equation of state (EoS) parameter satisfies

$$w < -1/3. \tag{5.14}$$

This is problematic: such a matter model is unphysical due to the implication that any such matter would possess a negative pressure.

A related issue to the dark energy problem is the *cosmic coincidence problem*. This is concerned with why we are currently at the point in the history of the universe at which the dark energy and dark matter energy densities are of the same order of magnitude. These two energy densities decay at different rates with respect to the evolution of the scale factor, and so it is only for a very brief cosmic window at which they are of the same order of magnitude. It seems incredibly unlikely that we would be alive at such a special time. It is hoped therefore that an alternative model of dark energy could potentially explain this, perhaps by having a scenario where the two energy densities are of similar orders of magnitude for longer periods of time. The remainder of this chapter will be devoted to examining a range of dark energy models.

5.4 The cosmological constant

Let us first analyse the simplest way one can add a dark energy component into the universe, a cosmological constant. This term was first introduced by Einstein in the 1920s, in an attempt to find static solutions to the cosmological field equations. When it was later observed by Hubble that the universe was in fact expanding, Einstein eliminated this term. However, since the observation of the accelerated expansion of the universe, this constant term has been resurrected as a possible explanation for this dark energy. The constant is added to the Einstein-Hilbert action in the following way

$$S_{\rm EH} = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + L_{\rm m} \right] \sqrt{-g}\, d^4x\,, \tag{5.15}$$

where Λ is the cosmological constant, having units of $(length)^{-2}$. This modifies the Einstein field equations to include an additional cosmological constant term on the left hand side of the equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{5.16}$$

This is a relatively harmless modification to the gravitational action. As the connection is metric compatible, the conservation equation still holds automatically, and provided the constant is small enough it has minimal effect on the dynamics of objects at solar system scales. In fact cosmological observations indicate that the cosmological constant must be approximately

$$\Lambda \approx 10^{-52} m^{-2}. \tag{5.17}$$

With this cosmological term, the Friedmann equations are modified to become

$$3\frac{k}{a^2} + 3H^2 - \Lambda = \kappa^2 \rho\,, \tag{5.18}$$

$$\frac{k}{a^2} + 2\dot{H} + 3H^2 - \Lambda = -\kappa^2 p\,. \tag{5.19}$$

From this we can see that we can interpret the cosmological constant as another perfect fluid, with a constant energy density $\rho_\Lambda = \Lambda/\kappa^2$ and a constant pressure $p_\Lambda = -\Lambda/\kappa^2$, and thus has a linear EoS with $w_\Lambda = -1$. If we are at a scale where the cosmological constant term dominates, so that we neglect the matter energy density and pressure ($\rho = 0$ and $p = 0$), then we can solve these equations exactly to find the form of the scale factor

$$a(t)^2 = e^{Ht}, \tag{5.20}$$

where the Hubble parameter is simply the constant

$$H = \sqrt{\frac{\Lambda}{3}}. \tag{5.21}$$

Such a solution is called the de Sitter universe[2]. This solution can describe the late time asymptotic behaviour of many cosmological models.

The cosmological constant is currently part of the accepted standard model of dark energy, with the model agreeing with cosmological observations to date. However, its presence generates a number of theoretical difficulties. Its extremely small value, as indicated in (5.17) is contrary to theoretical predictions. We will briefly discuss these issues here, however for a more comprehensive discussion we refer to [43].

The cosmological constant can be viewed as a matter fluid with a constant energy density $\rho_\Lambda = \Lambda$ and negative pressure $p_\Lambda = -\rho_\Lambda$. This is analogous to the contribution of the vacuum energy from matter fields, the energy-momentum tensor of a field in its vacuum state

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{\rm vac} g_{\mu\nu} \tag{5.22}$$

with $\rho_{\rm vac}$ being the constant energy density of the vacuum. From a classical perspective, this vacuum energy term can be identified with the value of the matter fields lying at their minimal energy state. Additionally from a quantum perspective, the Heisenberg Uncertainty Principle forbids both the kinetic and potential energies to vanish simultaneously, giving rise to a zero point energy of the quantum fields,

[2]If the cosmological constant were negative one would get an alternative anti-de Sitter solution.

thus giving a source of energy in the form of vacuum energy. Thus two cosmological constant type terms are present in the universe, one from classical considerations and the other from the quantum level.

A non-vanishing value of the vacuum energy becomes present in the Standard Model of particle physics during a symmetry breaking phase transition. It is believed, assuming the Standard Model of particle physics, that the universe has undergone two such phase transitions: the Electro-Weak and the QCD phase transition. These two phase transitions give rise to the following vacuum energy densities

$$\rho_{\text{vac}}^{EW} \approx 10^{8}\,\text{GeV}^{4} \quad \rho_{\text{vac}}^{QCD} \approx 10^{-2}\,\text{GeV}^{4} \tag{5.23}$$

However, the observed value of the cosmological constant, written in GeV is

$$\rho_{\Lambda} = 10^{-47}\text{GeV}^{4}. \tag{5.24}$$

Comparing this value to that of the Electroweak or QCD scale indicates that we have a discrepancy at the level of 50 orders of magnitude.

Moreover, this problem cannot easily be solved if we further consider the quantum vacuum point energy. From quantum mechanical considerations, we would expect the energy density of the quantum vacuum to be approximately

$$\rho_{\text{vac}}^{QM} \approx -10^{8}\,\text{GeV}^{4}. \tag{5.25}$$

This value is still 55 orders of magnitude away from what we would hope for. Now we could suppose there existed a bare cosmological constant in the universe, which just happened to cancel out all of these other quantum and classical contributions. However, this would require an extraordinary degree of fine tuning of over 50 orders of magnitude.

We have already mentioned the cosmic coincidence problem, which the cosmological constant cannot alleviate. In order for galaxy formation to occur, a period of matter domination is required to have taken place. If the cosmological constant were slightly larger, then such a phase would not have occurred, with a direct radiation to dark energy phase transition occurring instead. Moreover, humanity appears to be alive during the very small time frame in the universe's potential history at which the matter to dark energy phase transition is occurring. Again, this is only because of the particular value of the cosmological constant. It was much more likely that we would exist in the late time attracting dark energy phase of the universe. This is a rather philosophical point, but we can see that alternative dark energy models are potentially needed to be able to alleviate this cosmic coincidence problem.

We have thus seen this currently accepted ΛCDM model faces a range of theoretical difficulties. For the remainder of this chapter we will take a look at some alternatives to the standard model, considering dark energy as an additional dynamical matter field or as an artefact of a modified gravity model

5.5 Scalar field dark energy models

The immediate thing one would do if one wants to go beyond the cosmological constant model is to promote the constant to being a dynamical quantity. A scalar field represents the simplest way to add just one dynamical degree of freedom into the cosmological framework, and moreover it is usually enough to describe the large-scale effects of high-energy or modified gravity theories, at least at an effective level. Scalar fields have a prominent role in present cosmology not only since they provide simple inflationary solutions for the early universe, but also for their applications to late-time cosmology. In fact simple scalar field models have been employed to characterize both the inflaton, a hypothetical field introduced to drive the primordial inflationary phase, and dark energy. Moreover, since a scalar field can exhibit an effective negative pressure, it makes it a good candidate for dark energy.

There are different ways one can add a dynamical scalar field into the Lagrangian. Some simple ways are by adding to the matter sector a canonical scalar field [44–57], known as quintessence, a phantom scalar field [58–63], or a combination of both of these fields called quintom [64–73]. Reviews of these models can be found in [74] and [75].

Let us take a look at the canonical quintessence approach. One adds a scalar field to the Lagrangian as follows

$$S = \int \left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) + L_{\mathrm{m}} \right] \sqrt{-g}\, d^4x \,, \tag{5.26}$$

where the scalar field ϕ has a kinetic energy term and a potential $V(\phi)$[3]. In this Lagrangian, the gravitational sector and the scalar field are separated, and so one says the scalar field is *minimally coupled*. Such a model can account for the accelerated expansion of the universe, however to account for the latest cosmological data, these models must be increasingly fine tuned, with the potential required to be very flat.

Current cosmological observations leave open the possibility that the effective equation of state of the universe lies below the phantom barrier at $w_{\mathrm{eff}} = -1$. However quintessence models are always constrained by the condition $w_{\mathrm{eff}} > -1$. In order to account for this possibility, many authors have considered models known as phantom scalar fields. These follow from the same action as above, however the sign in front of the kinetic energy term changes. These models can have interesting cosmological applications, however they suffer from a range of theoretical difficulties. The kinetic energy term having the wrong sign causes problems such as instabilities and negative energies.

There are numerous other ways to consider an additional dark energy field into

[3]One could easily rewrite this action in the teleparallel framework. As there is no coupling between torsion and the scalar field, one could simply change the Ricci scalar to the torsion scalar, and one would recover the exact same dynamics. However we have chosen the curvature formulation as quintessence has mainly been studied in the literature in this form.

the action. One such approach is to consider an interaction of a scalar field with the matter sector. Such models are known as Scalar-Fluid models [76, 77], and are interesting due to their potential to hide the effects of the scalar field at solar system scales, the so called Chameleon mechanism [78–80]. One can also consider higher order fields, such as vector fields or three form fields to potentially describe dark energy, or more complex fluid models such as a Chapylgin gas. A full review of all such models is beyond the scope of this thesis.

5.5.1 Dynamical systems analysis: Quintessence

Let us now use the dynamical systems techniques introduced earlier in the chapter to analyse the quintessence models. This will be particularly relevant later on in this thesis when we discuss a generalisation of this model in the teleparallel framework.

Assuming a FRW universe which is spatially flat[4], one arrives at the following two Friedmann equations

$$3H^2 = \kappa^2 \left(\rho + \rho_\phi\right), \tag{5.27}$$

$$3H^2 + 2\dot{H} = -\kappa^2 \left(p + p_\phi\right). \tag{5.28}$$

In these equations we have defined the energy density and pressure of the scalar field as follows

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{5.29}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5.30}$$

Varying the action (5.26) with respect to the scalar field also gives an equation for

[4]For a dynamical systems analysis of quintessence models in the presence of spatial curvature, see [81]

the scalar field, the Klein-Gordon equation,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{5.31}$$

For simplicity, we will assume that there is only one matter field, which we will assume has the linear equation of state $p = w\rho$. Similarly, one can define the equation of state w_ϕ of dark energy or scalar field as the following ratio of the scalar field pressure and energy density

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{5.32}$$

And also we define the total or effective equation of state as

$$w_{\text{eff}} = \frac{p + p_\phi}{\rho + \rho_\phi}. \tag{5.33}$$

Other important quantities are the matter energy density and the density parameter of the dark energy/scalar field, which are respectively

$$\Omega_m = \frac{\kappa^2 \rho}{3H^2}, \quad \Omega_\phi = \frac{\kappa^2 \rho_\phi}{3H^2}. \tag{5.34}$$

Inspired by these definitions, we now introduce the dimensionless variables

$$\sigma^2 = \frac{\kappa^2 \rho}{3H^2}\,, \quad x^2 = \frac{\kappa^2 \dot{\phi}^2}{6H^2}\,, \quad y^2 = \frac{\kappa^2 V}{3H^2}\,, \tag{5.35}$$

which were first used by Copeland et al. in [82]. The first Friedman equation (5.27) written in these variables becomes the algebraic constraint

$$1 = \sigma^2 + x^2 + y^2, \tag{5.36}$$

which will define the boundary of our phase space. This means we can always write the variable σ in terms of x and y, and in what follows we choose to only work with x and y. We can rewrite physical quantities in terms of only these variables. Of particular importance will be the effective equation of state, which in terms of x and y is given as

$$w_{\text{eff}} = w - (w-1)x^2 - (1+w)y^2. \tag{5.37}$$

We now define the quantity $N = \ln a$ and denote derivatives with respect to N by a prime

$$x' = \frac{dx}{dN} = \frac{1}{H}\frac{dx}{dt}. \tag{5.38}$$

We will use N as a time variable. Using these variables, the equations of motion can be written as the following autonomous system of first order differential equations

$$x' = -\frac{3}{2}\left(2x + (w-1)x^3 + x(w+1)\left(y^2-1\right) - \frac{\sqrt{2}}{\sqrt{3}}\lambda y^2\right), \tag{5.39}$$

$$y' = -\frac{3}{2}y\left((w-1)x^2 + (w+1)\left(y^2-1\right) + \frac{\sqrt{2}}{\sqrt{3}}\lambda x\right). \tag{5.40}$$

Here we have defined the quantity λ by

$$\lambda = -\frac{1}{\kappa V(\phi)}\frac{dV}{d\phi}. \tag{5.41}$$

In order to close the system, we must specify what this quantity λ is, that is we must decide on the form of the potential of the scalar field. The most common choice in the literature is to consider the potential to be of the exponential form

$$V(\phi) = V_0 e^{-\kappa\lambda\phi} \tag{5.42}$$

with V_0 a constant and λ a dimensionless constant. This choice means the parameter λ defined in (5.41) just becomes a constant. A further frequently considered potential is an inverse power law, of the form

$$V(\phi) = \frac{M^{4+\alpha}}{\phi^{\alpha}}. \tag{5.43}$$

However, this potential requires us to consider a third autonomous equation to close the system. In fact, the exponential form for V is the only one which consents to close the autonomous system of equations without introducing another variable, and it is this case we will consider here.

Point	x	y
O	0	0
$A_{\pm}$	± 1	0
B	$\sqrt{\frac{3}{2}}\frac{(1+w)}{\lambda}$	$\sqrt{\frac{3}{2}}\frac{\sqrt{(1+w)(1-w)}}{\lambda}$ 0
C	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$

Table 5.1: Critical points of the autonomous system (5.39)-(5.40).

We are now in a position to find the critical points of the dynamical system, by setting the right hand side of equations (5.39) and (5.40) equal to zero and solving. These are displayed in Table 5.1. We find the following five potential critical points:

- *Point O*. This point occurs at the origin of our phase space. It is the matter dominated era of the universe, with the energy density of matter satisfying $\Omega_m = \sigma^2 = 1$, with the effective EoS $w_{\text{eff}} = 0$.

- *Points $A_{\pm}$*. These two points are dominated by the kinetic energy of the scalar field, with the effective EoS that of a stiff fluid, $w_{\text{eff}} = 1$. No acceleration is present at these point. The points are either unstable or saddle points depending on whether the magnitude of $3 + \sqrt{3/2}\lambda$ is less than $\sqrt{6}$ for A_- or whether $3 - \sqrt{3/2}\lambda$ is less than $\sqrt{6}$ for A_+. Such points are unphysical, yet frequently appear as the early time attractors in quintessence type models; see [82] for example.

- *Point B*. For this point to exist, it is required that the parameter λ of the potential is sufficiently large, requiring $\lambda > 3(1+w)$. This point is a scaling solution, meaning that its effective equation of state mimics that of the matter, with $w_{\rm eff} = w$. These are of great interest from a cosmological perspective, as they are potentially able to solve the cosmic coincidence problem.

- *Point C*. For this point to exist, the potential is required to be sufficiently small, with $\lambda^2 < 6$ for this point to exist. For a suitably flat potential $\lambda^2 < 2$, this point can describe an accelerating universe, however the effective equation of state is bounded below by -1, and thus crossing into the phantom regime is not possible, and λ has to approach zero for $w_{\rm eff}$ to approach -1.

These critical points mean that the dynamical system as a whole exhibits some interesting cosmological behaviour. For certain choices of the parameter values, the system can exhibit late time accelerating attractor solutions, as required to describe dark energy. In this case the system asymptotically approaches a cosmological constant type solution.

Of most interest to us is the point C, which is entirely scalar field dominated, and exists only for $\lambda^2 < 6$. For a suitably flat potential $\lambda^2 < 2$, this point can describe an accelerating universe, however the effective equation of state is bounded below by -1, and thus crossing into the phantom regime is not possible. Moreover λ has to approach zero for $w_{\rm eff}$ to approach -1. For $\lambda^2 < 3$ the late time attractor is given by this point C, whereas for $\lambda^2 > 3$ point B is the global attractor. Hence it is possible in these models to achieve a late time accelerating attracting solution, as is required to explain dark energy.

A typical plot of the two dimensional phase space of these quintessence models is displayed in Fig. 5.1, where the parameter choice $\lambda = 1$ is made. Such a choice means

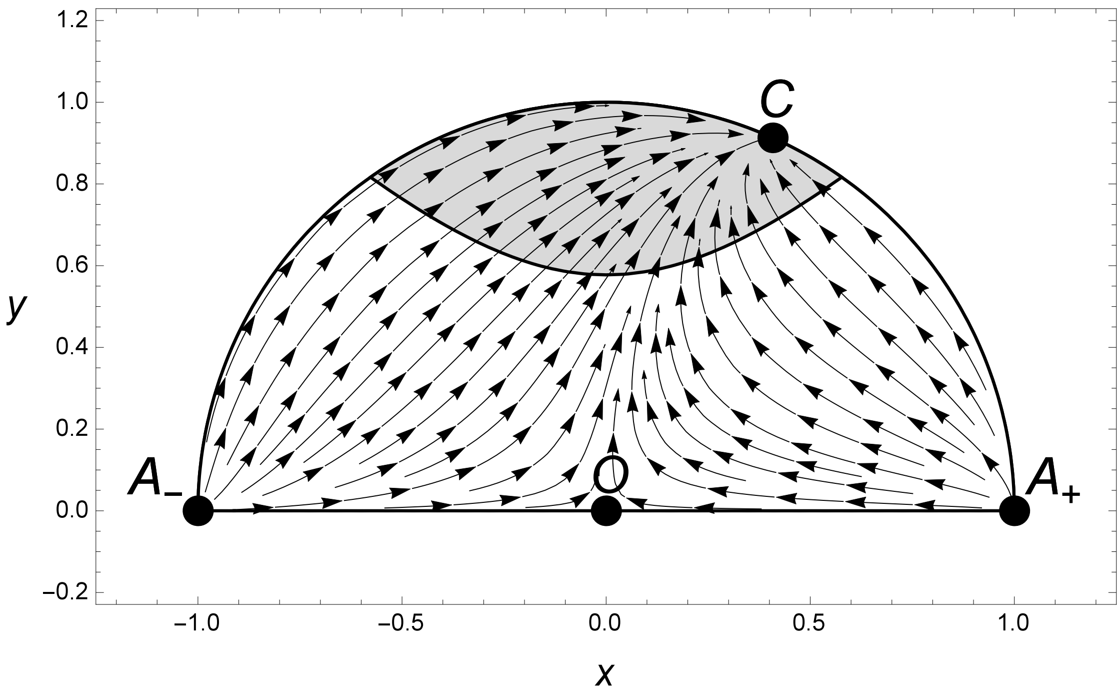

Figure 5.1: Phase space showing trajectories of standard quintessence models, for the particular parameter choice $w = 0$ and $\lambda = 1$. The point C is the late time accelerating attractor, with the shaded region indicating the region of acceleration.

that the scaling solution B does not exist. Here we see that many trajectories pass close to the matter dominated origin before passing through the shaded acceleration region. All trajectories then end at the late time attractor at point C, which in this case is accelerating.

Although initially a promising theory, it was realised that to agree with observations, a large amount of fine tuning of the parameter λ was required in order to get an effective equation of state close to -1. Moreover the effective EoS of the critical points in this model are bounded below by -1, whereas cosmic observations are consistent with an EoS below -1 [9]. Thus models beyond this simple exponential scalar field model are potentially required. In Chapter 7 we will look at generalising this model to include a coupling of the scalar field to the torsion tensor.

5.6 Modifications of gravity

In this section we will discuss an alternative approach to the dark energy problem and consider modified theories of gravity. It is not just the dark energy problem that leads to considering alternative models of gravity: as discussed in the introduction there are also the dark matter and inflation problems.

There are essentially two approaches one could take when attempting to solve the dark matter, dark energy and inflation problems. We could modify the matter content of the universe, by inserting an additional dark matter and dark energy component into the right hand side of the field equations of general relativity or TEGR. We have already seen this approach earlier in the chapter where dark energy was inserted into the right hand side of Einstein's equations via either a cosmological constant, or a dynamic quintessence scalar field. This approach can similarly be considered for dark matter, where simply one adds additional matter into the right hand side of the equations, and similarly for inflation one can consider a scalar field known as an *inflaton*.

An alternative approach is to instead consider modifying the gravitational sector (i.e. the left hand side of Einstein's equations). Doing so means examining modified theories of gravity. This is a completely sensible and consistent approach: the phenomena we need to explain is entirely of a gravitational nature (dark matter or dark energy has not been detected via any of the other three fundamental forces). Or it may be the case that modified gravity is required to describe dark energy, but dark matter is indeed a particle such as a weakly interacting massive particle (WIMP). Whatever the case, further degrees of freedom, beyond the ones of general relativity and Standard Model particles, are needed in order to account for the observations at both early and late times and all such approaches should continue to be explored.

There are many modifications of gravity that have been proposed in the literature with the aim of describing these phenomena, and I will discuss some of them in this chapter. I will take a particular focus on the models that have been studied in both the teleparallel and curvature frameworks. This list is certainly not comprehensive, and for a recent review of modified theories of gravity, see [83].

5.6.1 Brans-Dicke and Scalar-Tensor gravity

One of the first well studied modifications of gravity was formulated in the early 1960s, called Brans-Dicke theory (sometimes referred to as Jordan-Brans-Dicke theory). Originally the theory was motivated by generalising general relativity to include a gravitational constant G that was allowed to vary dynamically in space and time. To do this they replaced the inverse of the gravitational constant, $1/G$ by a scalar field. This leads to the consideration of the following action

$$S = \int \frac{1}{16\pi}\left(\phi R - \frac{\omega \partial_\mu \phi \partial^\mu \phi}{\phi}\right)\sqrt{-g}d^4x + S_m \tag{5.44}$$

where ω is the dimensionless Dicke coupling constant. Note the $1/G$ factor in front of the Ricci scalar is no longer present and has been replaced by the scalar field. Brans-Dicke theory predicts different behaviour for the perihelion precession of Mercury and gravitational light deflection of the Sun, with both of these being dependent on the value of ω. The Cassini-Huygens experiment from 2003 puts a bound of $\omega > 40,000$ to 2σ. Brans-Dicke theory today is currently still a viable theory, however there is little experimental evidence for a varying gravitational constant.

However, predecessors of Brans-Dicke theory are actively studied today. In modern day language, Brans-Dicke theory is a particular example of a wider class of theories known as *scalar-tensor theories*. They have the following general action

$$S = \int \left(\frac{1}{2\kappa}A(\phi)R - \omega(\phi)\partial_\mu \phi \partial^\mu \phi - V(\phi)\right)\sqrt{-g}d^4x + S_m, \tag{5.45}$$

where both A and ω are generic functions of the scalar field, with V the potential of the scalar field. In these models, there is an interaction term between the scalar field and the gravitational sector, and this is referred to as a *nonminimal coupling*. The coupling becomes minimal again if the function $A(\phi)$ = constant. Such a class of theories is very general, and contains many interesting models as subclasses. They contain Brans-Dicke theories, along with the quintessence models and phantom quintessence models, and many more. Alternatively, one could consider coupling a scalar field to suitable torsion scalars in the TEGR. We will discuss these theories more extensively, along with their teleparallel equivalents in Chapter 8.

5.6.2 $f(R)$ gravity

So far we have considered adding a scalar field into the Einstein-Hilbert action, with a potential for a nonminimal coupling between the scalar field and the gravitational sector. In this section, we will discuss a different approach to modifying gravity: rather than adding in an extra ingredient into the Einstein-Hilbert action, such as a scalar field, we will directly modify the form of the Einstein-Hilbert action.

One way to do this is to promote the Ricci scalar in the Einstein-Hilbert action to rather be a general function of the Ricci scalar, which we denote by $f(R)$. Such a modification is naturally known as $f(R)$ gravity, and it is one of the most well studied modified forms of gravity [84–86]. The function f is taken to be an arbitrary (sufficiently smooth) function of the Ricci scalar, and one recovers general relativity when one sets $f(R) = R - 2\Lambda$. Thus the gravitational sector of the action takes the following form[5]

$$S_{f(R)} = \frac{1}{2\kappa} \int f(R)\sqrt{-g}\, d^4x\,. \tag{5.46}$$

[5]Note that we are still working in a V_4 space, however one can also consider $f(R)$ gravity in a U_4 or an L_4 space.

Recall that with the Einstein-Hilbert action, one could vary the action with respect to just the metric, or alternatively with respect to both the metric and connection, treating the two fields as a priori independent (the Palatini variation). In that case, both types of variations led to the same field equations, with the connection constraint imposing that the connection was just of the Levi-Civita form. However, this is no longer the case when one considers $f(R)$ gravity, and depending on the variation used one gets different field equations, and so two different theories: metric $f(R)$ gravity and Palatini-$f(R)$ gravity. The Palatini version of $f(R)$ gravity suffers some theoretical difficulties, for example it appears to be in conflict with the standard model [87, 88] and suffers from the existence of singularities appearing in stars [89]. We will thus not discuss this case further, and consider only metric $f(R)$ gravity.

Now assuming a Levi-Civita connection, the Ricci scalar depends on second derivatives of the metric tensor. Varying the action with respect to the metric will thus result in a double integration by parts, which will give rise to terms taking the following form $\nabla_\mu \nabla_\nu F$ where, as is common in the $f(R)$ literature, $F = f'(R)$. Thus generically the theory will be fourth order (unless f' is constant, in which case we recover general relativity). Performing this variation gives the following field equations

$$FR_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + g_{\mu\nu} \Box F - \nabla_\mu \nabla_\nu F = 8\pi G T_{\mu\nu}, \tag{5.47}$$

where $\Box = \nabla^\mu \nabla_\mu$. Such theories have been extensively studied in a variety of applications. The Starobinsky model of $f(R)$ gravity, where f takes the form $f(R) = R + \alpha R^2$, is a strong candidate to describe inflation, whereas functions of the form $f(R) = R^n$ have been shown to have interesting properties when analysing galaxy rotation curves [90]. They also are of great interest for applications in late time acceleration. For a dynamical systems analysis of the cosmology of general $f(R)$

models, see [91].

These $f(R)$ gravity theories have interesting relations to scalar tensor theories. By applying a Legendre transformation to the metric, one can show that the two theories are essentially physically equivalent: one can conformally transform an $f(R)$ gravity to get a scalar tensor theory, and vice versa. This will be discussed in detail in Chapter 8.

5.6.3 $f(T)$ gravity

Considering $f(R)$ gravity makes sense when we are working in the metric formulation of gravity. If instead we took the TEGR to be the fundamental theory, it is no longer natural to consider working with a function of the Ricci scalar. In this case it is natural instead to consider modifying the teleparallel Lagrangian to promote the torsion scalar T to be a general function of the torsion scalar: such a theory is called $f(T)$ gravity, where again f is a sufficiently smooth function of its argument. The Lagrangian takes the following form

$$S_{f(T)} = \frac{1}{2\kappa} \int f(T)\, e\, d^4x \,. \tag{5.48}$$

Despite the fact that teleparallel gravity and general relativity are equivalent, due to the relation $R = -T + B$ where B is a boundary term, this is no longer the case when one considers non-linear functions of the torsion scalar. One can no longer write that $f(R) = f(T) +$ boundary term. Thus $f(T)$ gravity is a distinct modification of gravity to $f(R)$ gravity, and this distinction leads to the consideration of $f(T, B)$ gravity, which will be discussed in the next section.

This class of $f(T)$ theories were first considered in [92], motivated by potential early universe inflationary applications. However, they have also been shown to potentially exhibit interesting late time phenomenology, with applications in the dark energy problem [93]. If we vary action (5.48) with respect to the tetrad, we arrive at the following set of field equations

$$4e\Big[f_{TT}(\partial_\mu T)\Big]S_\nu{}^{\mu\lambda} + 4e^a_\nu\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta^\lambda_\nu = 16\pi e\Theta^\lambda_\nu\,, \quad (5.49)$$

where we have used Θ to represent the energy momentum tensor in the teleparallel context. At first appearance this theory is somewhat nicer than $f(R)$ gravity because these field equations are second order, in contrast to $f(R)$ gravity which we recall has fourth order terms in its field equations. However a deeper analysis reveals there are a few theoretical difficulties with $f(T)$ gravity. The theory in general is not locally-Lorentz invariant, which $f(R)$ gravity manifestly is.

For a time there was much confusion in the literature about the correct tetrads that must be used when working in $f(T)$ gravity. For example, consider the case of a FRW metric with spatial curvature. What tetrad should one use? In spherical polar type coordinates, naively one could pick the following tetrad

$$e = \ \mathrm{diag}(1, \frac{a(t)}{\sqrt{1-kr^2}}, a(t)\,r, a(t)\,r\,\sin\theta), \quad (5.50)$$

which recovers the correct FRW metric (5.2). However, in $f(T)$ gravity this imposes a constraint on the field equations which means that they can only be solved for a particular choice of functional form of $f(T)$. However, the choice of functional form of f should certainly be independent of the tetrad, and this would appear as if certain types of solutions of the theory were inconsistent with $f(T)$ gravity.

To remedy this issue, one must instead choose between "Good and bad tetrads", see [94]. A good tetrad is one that does not impose a functional form on $f(T)$ from the field equations. This issue has been clarified further in [35,95], where it is noted that by not setting the spin connection equal to zero, that is not specifying the gauge, one can recover a consistent set of field equations without resorting to 'good tetrads', which turn out to be necessary only when the gauge is restricted. We will take a more in-depth look at $f(T)$ gravity in the next chapter.

5.6.4 Other teleparallel modifications

There are many more modifications of gravity which have been considered by various authors, and it is beyond the scope of this thesis to discuss them all in detail here. However we will take a look at a few more here, focusing on those which have been considered within the teleparallel framework.

Modified Gauss-Bonnet Gravity

Gauss-Bonnet gravity is a theory of gravity based on the Gauss-Bonnet theorem of geometry. Gravity in higher dimensions can be extended in directions beyond simply considering higher dimensional general relativity, while still preserving many of the features that make general relativity appealing. Gauss-Bonnet gravity is a particular natural theory to consider, and appears in the low energy effective action of string theory. This theory is a generalisation of Einstein gravity that adds an extra term to the standard Einstein-Hilbert action

$$L_{\mathrm{GB}} = R^2 - 4R^{AB}R_{AB} + R_{ABCD}R^{ABCD} =: G, \tag{5.51}$$

which is quadratic in the Riemann tensor. When varying this extra term with respect to the metric only second order derivatives remain in the field equations, with the higher derivative terms cancelling out exactly, and thus the theory shares many of the nice properties of general relativity. In four dimensions Gauss-Bonnet gravity and general relativity are equivalent, since by the Gauss-Bonnet theorem, the Gauss-Bonnet term in the action reduces to a total dervative, giving a surface integral and thus does not add a contribution to Einstein's equation. But when analysing gravity in higher dimensions this extra term is non-trivial and it is thus natural to consider this extra Gauss-Bonnet contribution when considering higher dimensional theories.

However if one is only interested in four dimensional theories one can still consider the Gauss-Bonnet term in a non trivial way. This led to a formulation of gravity known as modified Gauss-Bonnet gravity, with the following action

$$S = \int f(R, G)\sqrt{-g}d^4x, \tag{5.52}$$

where f is a sufficiently smooth function of both of its arguments: the Ricci scalar and the Gauss-Bonnet term. If f has non-linear terms in G, then this theory departs from that of standard $f(R)$ gravity. However, the cost of adding a non-linear term in G is that the field equations will no longer be second order.

One can follow a similar approach in teleparallel gravity. There is an equivalent to the Gauss-Bonnet term in teleparallel gravity [96], given by $\mathcal{T}_G$, leading to the modified gravity

$$S = \int f(T, \mathcal{T}_G)ed^4x \tag{5.53}$$

which differs from the metric equivalent. Additionally one can extend this to include additional terms beyond the Gauss-Bonnet term, Lovelock polynomials, that also satisfy the Gauss-Bonnets term's nice properties. This is known as Lovelock gravity, and a teleparallel equivalent of this model has been discussed in [97].

Nonminimal matter coupling

Another modification that has been studied in both the metric and teleparallel frameworks has been to consider coupling the matter sector to the gravitational sector. In the context of torsion, the following action has been considered [98], where a coupling between the torsion scalar and the matter Lagrangian is present

$$S = \int \frac{1}{2\kappa}\left\{T + f_1(T) + \left[1 + f_2(T)\right]L_m\right\} ed^4x. \tag{5.54}$$

Here the f_i are arbitrary functions of the torsion scalar, and λ is a factor with dimensions $(\text{mass})^{-2}$ which controls the strength of the coupling. Such a model can exhibit very interesting cosmological phenomenology. Late time de Sitter type solutions are possible, along with a crossing of the phantom barrier, and early time inflationary solutions. Thus the model is able to obtain a unified cosmic history.

A further type of nonminimal matter coupling in the teleparallel framework has been to consider coupling the torsion scalar to the trace of the energy momentum tensor. The following action was considered in [99]

$$S = \int \frac{1}{2\kappa} \left(T + f(T, \mathcal{T})\right) e \, d^4x + S_m \,, \tag{5.55}$$

where $\mathcal{T}$ is the trace of the energy momentum tensor, $\mathcal{T} = \Theta^{\mu}{}_{\mu}$. Similar to the torsion matter coupling, this model can also provide a unified cosmic history, with early time inflationary solutions, late time quintessence, phantom or de Sitter type solutions along with a dynamical crossing of the phantom barrier. Such couplings have previously been considered in the curvature formulation, with their counterparts defined in the natural way, making the replacements: $T \rightarrow R$, $e \rightarrow \sqrt{-g}$.

Born-Infield teleparallel gravity

A further interesting modification, a particular type of $f(T)$ gravity, is the Born-Infield action [100]. This has the following action

$$S = \int \frac{\lambda}{2\kappa} \left(\sqrt{1 + \frac{2T}{\lambda}} - 1 \right) e d^4x. \tag{5.56}$$

The parameter λ controls the scale at which deviations from general relativity occur. In the limit $\lambda \rightarrow 0$, this action simple recovers the standard Lagrangian of the TEGR. Such a theory is motivated from the Born-Infield approach to electrodynamics, which is able to regularise the singularities which appear, for example, around a point charge. In terms of the teleparallel Born-Infield action, it has been shown that it is able to regularise the early time Big Bang singularity which appears in general relativity, with the scale factor converging to a non-zero constant. Alternatively a potential bounce universe scenario is possible [101].

The above is certainly not a comprehensive list of the possible types of teleparallel gravity, but gives a flavour of the type of modifications being studied today. One could easily consider a plethora other types of modification, for example one could consider a teleparallel version of the unimodular $f(R)$ gravity, where one constrains the determinant of the tetrad to be equal to one. Now that we have given an overview of the state of research and the challenges facing cosmology and modified gravity today, we will devote the rest of this thesis to exploring some of these modified teleparallel models in detail.

Chapter 6

$f(T, B)$ gravity

In this chapter we will explore the modification of gravity, $f(T, B)$ gravity. This model was first considered in [1] and this chapter follows that work closely.

We discussed in the previous chapter how in the curvature formulation of gravity, when working in a V_4 space, one can consider modifying the Einstein-Hilbert action to promote the curvature scalar to a general function of the curvature scalar, i.e. considering $f(R)$ type gravities. Alternatively, if one works in a Weitzenböck space, W_4, one can generalise the TEGR action to a general function of the torsion scalar, giving the $f(T)$ class of gravities. Despite the equivalence of the Einstein-Hilbert action and the TEGR action, this no longer holds true between $f(T)$ and $f(R)$ gravity: one can no longer write that $f(T) = f(R) + \text{boundary term}$, and so the actions lead to different dynamics.

These two different modifications have different properties, some of which are desirable, others not so. $f(R)$ gravity has fourth order field equations, which are very uncommon for physical laws[1], where usually one hopes to understand the system by just knowing position and velocity. Moreover fourth order equations are very difficult to solve mathematically, and can give rise to instabilities. On the other

[1]One such example of a physical law which has a fourth order equation is the Euler - Bernoulli equation, which describes how a beam is deflected given a certain load.

hand, the field equations of $f(T)$ gravity remain second order, and so are easier to deal with, and physically more desirable. The second order parts of the Ricci scalar are not present in the torsion scalar, they are entirely contained within the boundary term B. However, $f(T)$ gravity suffers from a different theoretical difficulty: it is not locally Lorentz invariant[2] [102, 103]. $f(R)$ gravity on the other hand is completely locally Lorentz invariant.

The difference between these two actions is simply the effect of the boundary term. Thus if we also consider a function depending on both the torsion scalar and the boundary term, we can construct a theory which contains both $f(R)$ and $f(T)$ gravity as limiting cases. In order to better understand the structure of these modifications in the teleparallel framework, the following action is proposed

$$S_{\mathrm{TB}} = \int \left[\frac{1}{16\pi G} f(T,B) + L_{\mathrm{m}}\right] e\, d^4x\,, \tag{6.1}$$

where f is a suitably smooth function of both of its arguments, the torsion T and boundary term B. The Lagrangian density L_{m} is the standard matter Lagrangian. We are considering this action in the Weitzenböck space, that is assuming our manifold has a Weitzenböck connection.

A similar model was proposed shortly after we proposed our model, in [104], where the action contained the function $f(T-a_1B)$, where a_1 is some constant. This is less general than the $f(T,B)$ model considered here, with $f(T,B)$ containing this as a particular special case. However this restricted model does still include both $f(T)$ (when $a_1=0$) and $f(R)$ gravity (when $a_1=1$) as limiting cases.

In this chapter, we will derive the field equations of $f(T,B)$ gravity, explore its limiting cases and take a look at the theoretical issues of Lorentz invariance and the conservation equation. We will end by taking a glimpse at the cosmology of these

[2]However recently the issue of local Lorentz invariance in $f(T)$ gravity has been subject to some debate [35, 95]. The authors claim that by choosing an appropriate non-vanishing spin connection, one can restore local Lorentz invariance. This is tied up with the issue of "Good and Bad tetrads" [94].

models.

6.1 Field equations

Now that we have described the action of the $f(T,B)$ gravity, we now must derive the field equations of the theory. The derivation of the field equations for this Lagrangian is quite an involved process, so we will present the lengthy derivation explicitly. Variations of the action with respect to the tetrad gives

$$\delta S_{\rm TB} = \int \left[\frac{1}{\kappa}\Big(f(T,B)\delta e + e f_B(T,B)\delta B + e f_T(T,B)\delta T\Big) + \delta(eL_{\rm m})\right] d^4x\,, \quad (6.2)$$

where the three variations explicitly are

$$e f_B(T,B)\delta B = \Big[2ee_a^{\nu}\nabla^{\lambda}\nabla_{\mu}f_B - 2ee_a^{\lambda}\Box f_B - Bef_B e_a^{\lambda} - 4e(\partial_\mu f_B)S_a{}^{\mu\lambda}\Big]\delta e^a_{\lambda}\,, \quad (6.3)$$

$$e f_T(T,B)\delta T = \Big[-4e(\partial_\mu f_T)S_a{}^{\mu\lambda} - 4\partial_\mu(eS_a{}^{\mu\lambda})f_T + 4ef_T T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu}\Big]\delta e^a_{\lambda}\,, \quad (6.4)$$

$$f(T,B)\delta e = ef(T,B)e_a^{\lambda}\delta e^a_{\lambda}\,. \quad (6.5)$$

The two variations (6.4) and (6.5) are both standard variations that are required to derive the field equations for $f(T)$ gravity, and the presence of the boundary term does not change this. For that reason we will not present this derivation explicitly.

On the other hand, the variation involving the δB term (6.3) is not standard, so we will explicitly derive this below. Let us first derive some useful variations and relations that will be needed during our derivation. Using that $\delta e = \frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}$ and $g^{\mu\nu} = \eta^{ab}e_a^{\mu}e_b^{\nu}$, we can find the variation of the inverse metric and the volume element

$$\delta g^{\mu\nu} = -\Big(g^{\nu\lambda}e_a^{\mu} + g^{\mu\lambda}e_a^{\nu}\Big)\delta e^a_{\lambda}\,, \quad (6.6)$$

$$\delta e = ee_a^{\lambda}\delta e^a_{\lambda}\,. \quad (6.7)$$

Now varying the relationship between the tetrad and the cotetrad (2.7) we find the relation of the variation of the inverse tetrad is given by

$$\delta e^{\sigma}_{m} = -e^{\sigma}_{n} e^{\mu}_{m} \delta e^{n}_{\mu} \,. \tag{6.8}$$

And by taking partial derivatives of (2.7), one can also find a similar relation for the partial derivatives of the inverse tetrad

$$\partial_\nu e^{\sigma}_{m} = -e^{\sigma}_{n} e^{\mu}_{m} \partial_\nu e^{n}_{\mu} \,. \tag{6.9}$$

Using (6.6) and (6.9), the variation of the torsion vector δT^μ can be written as

$$\delta T^\mu = -\Big(e^{\mu}_{a} T^\lambda + g^{\mu\lambda} T_a - T^{\lambda}{}_{a}{}^{\mu}\Big)\delta e^{a}_{\lambda} + g^{\mu\nu} e^{\lambda}_{a}\Big(\partial_\lambda \delta e^{a}_{\nu} - \partial_\nu \delta e^{a}_{\lambda}\Big) \,. \tag{6.10}$$

Using $\partial_\lambda e = e g^{\mu\nu} \partial_\lambda g_{\mu\nu}$ and the compatibility equation for the metric $\nabla_\lambda (g^{\mu\nu}) = 0$ we find

$$\partial_\lambda e = e \Gamma_\lambda{}^{\rho}{}_{\rho} \,, \tag{6.11}$$

$$\partial_\lambda g^{\mu\nu} = -\Big(\Gamma_\lambda{}^{\nu\mu} + \Gamma_\lambda{}^{\mu\nu}\Big) \,. \tag{6.12}$$

Finally we note for reference that the affine connection in terms of the Weitzenböck connection and contorsion tensor is given by

$$\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma_\mu{}^{\lambda}{}_{\nu} - K_\mu{}^{\lambda}{}_{\nu} = \Gamma_\nu{}^{\lambda}{}_{\mu} - K_\nu{}^{\lambda}{}_{\mu} \,. \tag{6.13}$$

Initially performing the boundary term variation of our $f(T, B)$ action, using the explicit form of the boundary term, it is found that

$$e f_B(T, B) \delta B = -\Big(f_B B + 2(\partial_\mu f_B) T^\mu\Big)\delta e - 2e(\partial_\mu f_B)\delta T^\mu \,, \tag{6.14}$$

where we used that the torsion vector is given by the contraction of the torsion tensor, which explicitly in terms of the tetrad yields

$$T^{\mu} = g^{\mu\nu}T^{\sigma}{}_{\sigma\nu} = g^{\mu\nu}e^{\sigma}_{a}\Big(\partial_{\sigma}e^{a}_{\nu} - \partial_{\nu}e^{a}_{\sigma}\Big)\,. \tag{6.15}$$

Now we can start working out the individual terms in (6.14). If we integrate by parts and disregard the boundary term, the final term on the right hand side of (6.14) becomes

$$\begin{aligned} e(\partial_{\mu}f_B)\delta T^{\mu} = \Big[&\partial_{\nu}\Big(e^{\lambda}_{a}(eg^{\mu\nu})(\partial_{\mu}f_B)\Big) - \partial_{\nu}\Big(e^{\nu}_{a}(eg^{\mu\lambda})(\partial_{\mu}f_B)\Big) \\ &- e(\partial_{\mu}f_B)\Big(e^{\mu}_{a}T^{\lambda} + g^{\mu\lambda}T_a + T^{\lambda}{}_{a}{}^{\mu}\Big)\Big]\delta e^{a}_{\lambda}\,. \end{aligned} \tag{6.16}$$

Using (6.11), (6.12) and the above equation (6.15), the first term of (6.16) can be written in terms of covariant derivatives as

$$\partial_{\nu}\Big(e^{\lambda}_{a}(eg^{\mu\nu})(\partial_{\mu}f_B)\Big) = ee^{\lambda}_{a}\Box f_B - e(\partial_{\mu}f_B)\Big(e^{\lambda}_{a}\Gamma_{\nu}{}^{\mu\nu} - e^{\lambda}_{a}\Gamma^{\nu\mu}{}_{\nu} + \Gamma^{\mu\lambda}{}_{a}\Big)\,. \tag{6.17}$$

Using the same idea, the second term of (6.16) becomes

$$\begin{aligned} \partial_{\nu}\Big(e^{\nu}_{a}(eg^{\mu\lambda})(\partial_{\mu}f_B)\Big) = ee^{\nu}_{a}\nabla^{\lambda}\nabla_{\mu}f_B + e(\partial_{\mu}f_B)\Big(&g^{\mu\lambda}(\Gamma_{a}{}^{\nu}{}_{\nu} - \Gamma_{\nu}{}^{\nu}{}_{a}) \\ &- \Gamma_{a}{}^{\lambda\mu} - \Gamma_{a}{}^{\mu\lambda} + \Gamma^{\lambda\mu}{}_{a} - K^{\lambda\mu}{}_{a}\Big)\,. \end{aligned} \tag{6.18}$$

By replacing these last two equations (6.17) and (6.18) into (6.16) we find

$$\begin{aligned} e(\partial_{\mu}f_B)\delta T^{\mu} = -\Big[&e(\partial_{\mu}f_B)\Big(e^{\mu}_{a}T^{\lambda} + g^{\mu\lambda}T_a + T^{\lambda}{}_{a}{}^{\mu} + g^{\mu\lambda}(\Gamma_{a}{}^{\nu}{}_{\nu} - \Gamma_{\nu}{}^{\nu}{}_{a}) \\ &- \Gamma_{a}{}^{\lambda\mu} - \Gamma_{a}{}^{\mu\lambda} + \Gamma^{\lambda\mu}{}_{a} - K^{\lambda\mu}{}_{a} - \Gamma_{a}{}^{\mu\lambda} + \Gamma^{\mu\lambda}{}_{a} + \Gamma_{\nu}{}^{\mu\nu} - \Gamma^{\nu\mu}{}_{\nu}\Big) \\ &- ee^{\lambda}_{a}\Box f_B + ee^{\nu}_{a}\nabla^{\lambda}\nabla_{\mu}f_B\Big]\delta e^{a}_{\lambda}\,. \end{aligned} \tag{6.19}$$

And finally, if we use the symmetry of the affine connection, that is equation (6.13), we can simplify the term as

$$e(\partial_\mu f_B)\delta T^\mu = -\Big[e(\partial_\mu f_B)\Big(e_a^\mu T^\lambda + \Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big) \\ - ee_a^\lambda \Box f_B + ee_a^\nu \nabla^\lambda \nabla_\mu f_B\Big]\delta e^a_\lambda\,. \quad (6.20)$$

Now, by replacing this expression into (6.14) and using (6.7) we find

$$ef_B(T,B)\delta B = \Big[2ee_a^\nu \nabla^\lambda \nabla_\mu f_B - 2ee_a^\lambda \Box f_B - Bef_B e_a^\lambda \\ + 2e(\partial_\mu f_B)\Big(e_a^\mu T^\lambda - e_a^\lambda T^\mu + \Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big)\Big]\delta e^a_\lambda\,. \quad (6.21)$$

Into this we will now introduce the superpotential, which we recall is given in terms of the torsion and contorsion as $2S_a{}^{\lambda\mu} = K_a{}^{\lambda\mu} + e_a^\mu T^\lambda - e_a^\lambda T^\mu$, to obtain

$$ef_B(T,B)\delta B = \Big[2ee_a^\nu \nabla^\lambda \nabla_\mu f_B - 2ee_a^\lambda \Box f_B - Bef_B e_a^\lambda \\ + 2e(\partial_\mu f_B)\Big(2S_a{}^{\lambda\mu} - K_a{}^{\lambda\mu} + \Gamma^{\lambda\mu}{}_a - \Gamma_a{}^{\mu\lambda} - K^{\lambda\mu}{}_a\Big)\Big]\delta e^a_\lambda\,. \quad (6.22)$$

The last four terms on the right hand side are identically zero due to the relationship (6.13). Thus, cancelling these we obtain the final result which is

$$ef_B(T,B)\delta B = \Big[2ee_a^\nu \nabla^\lambda \nabla_\mu f_B - 2ee_a^\lambda \Box f_B - Bef_B e_a^\lambda - 4e(\partial_\mu f_B)S_a{}^{\mu\lambda}\Big]\delta e^a_\lambda\,. \quad (6.23)$$

Now we have obtained the variations, we can now formulate the field equations. The energy momentum tensor is defined as follows

$$\Theta_a^\lambda = \frac{1}{e}\frac{\delta(eL_m)}{\delta e^a_\lambda}\,. \quad (6.24)$$

Note we have used Θ to denote the energy momentum tensor, as opposed to T to avoid confusion with the torsion tensor. Putting everything together, we find that the field equations for $f(T,B)$ gravity are given by

$$2ee_a^{\lambda}\Box f_B - 2ee_a^{\sigma}\nabla^{\lambda}\nabla_{\sigma}f_B + eBf_Be_a^{\lambda} + 4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_a{}^{\mu\lambda} + 4\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^{\sigma}{}_{\mu a}S_{\sigma}{}^{\lambda\mu} - efe_a^{\lambda} = 16\pi e\Theta_a^{\lambda}. \quad (6.25)$$

And contracting this with e_ν^a we arrive at the field equations in spacetime indices only

$$2e\delta_\nu^{\lambda}\Box f_B - 2e\nabla^{\lambda}\nabla_{\nu}f_B + eBf_B\delta_\nu^{\lambda} + 4e\Big[(\partial_\mu f_B) + (\partial_\mu f_T)\Big]S_\nu{}^{\mu\lambda} + 4e_\nu^a\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^{\sigma}{}_{\mu\nu}S_{\sigma}{}^{\lambda\mu} - ef\delta_\nu^{\lambda} = 16\pi e\Theta_\nu^{\lambda}\,. \quad (6.26)$$

where $\Theta_\nu^{\lambda} = e_\nu^a\Theta_a^{\lambda}$ is the standard energy momentum tensor. In the following sections we will explore the limiting cases of this theory, that result in $f(T,B)$ gravity coinciding with $f(T)$ gravity and $f(R)$ gravity. In particular we will derive the teleparallel equivalent of $f(R)$ gravity.

6.2 $f(T)$ gravity limit

Let us begin with examining the field equations (6.26) when we choose the function f to be independent of the boundary term. In order to match the sign convention employed, we simply set

$$f(T,B) = f(T)\,, \quad (6.27)$$

so that $f_B = 0$. Doing this, we find

$$4e\Big[f_{TT}(\partial_\mu T)\Big]S_\nu{}^{\mu\lambda} + 4e_\nu^a\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^{\sigma}{}_{\mu\nu}S_{\sigma}{}^{\lambda\mu} - ef\delta_\nu^{\lambda} = 16\pi e\Theta_\nu^{\lambda}\,, \quad (6.28)$$

which, as expected, are the standard $f(T)$ field equations. Let us make an important remark about this limit. One verifies immediately that this is the unique form of the function f which will give second order field equations. Recall that linear terms in the boundary term B do not effect the field equations. Therefore, the generic field equations contain terms of the form $\partial_\mu\partial_\nu f_B$ which are always of fourth order and can vanish if and only if f_B is a constant, so that f is linear in the boundary term.

Therefore, for a nonlinear function f, $f(T)$ gravity is the only possible second order modified theory of gravity constructed out of R, T and B. As mentioned before, the price to pay is the violation of local Lorentz invariance.

6.3 $f(R)$ gravity limit

Here we will show carefully how we recover standard $f(R)$ gravity in this model, and also find the teleparallel equivalent of $f(R)$ gravity. The teleparallel equivalent of $f(R)$ gravity was found in the Einstein frame, where one conformally transforms the theory to a scalar tensor theory, in [102], however here we will derive the equivalent in the Jordan frame. Recall the relationship (4.33)

$$R = -T + B\,, \tag{6.29}$$

which suggests to consider our function f to be of the particular form

$$f(T, B) = f(-T + B)\,. \tag{6.30}$$

We also introduce the standard notation for the derivative of f from $f(R)$ gravity

$$F(R) = f'(-T + B) = -f_T = f_B\,. \tag{6.31}$$

Inserting this form of function into our general $f(T,B)$ field equation (6.26) leads to the following field equations

$$2e\delta_\nu^\lambda \Box F - 2e\nabla^\lambda \nabla_\nu F + eBF\delta_\nu^\lambda - 4e_\nu^a \partial_\mu (eS_a{}^{\mu\lambda})F + 4eFT^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 16\pi e\Theta_\nu^\lambda\,. \quad (6.32)$$

This equation is the field equation for the teleparallel equivalent of $f(R)$ gravity[3]. As this is not an obvious observation, let us prove this statement by rewriting the field equation in the usual form, as expressed in the $f(R)$ literature (5.47).

We can rewrite the fourth term in (6.32) as

$$4e_\nu^a \partial_\mu (eS_a{}^{\mu\lambda}) = 2\partial_\mu (eK_\nu{}^{\mu\lambda}) - 2\partial_\nu (eT^\lambda) + eB\delta_\nu^\lambda + 4eS_\sigma{}^{\lambda\mu}W_\mu{}^\sigma{}_\nu\,. \quad (6.33)$$

Inserting this back into (6.32) gives

$$2e\delta_\nu^\lambda \Box F - 2e\nabla^\lambda \nabla_\nu F - 2F\partial_\mu (eK_\nu{}^{\mu\lambda}) + 2F\partial_\nu (eT^\lambda) - 4eFS_\sigma{}^{\lambda\mu}\Gamma_\nu{}^\sigma{}_\mu - ef\delta_\nu^\lambda = 16\pi e\Theta_\nu^\lambda\,. \quad (6.34)$$

Now, we need to replace torsion with curvature. The Ricci tensor of the Levi-Civita connection satisfies the identity

$$\bar{R}_{\mu\nu} = \nabla_\nu K_\lambda{}^\lambda{}_\mu - \nabla_\lambda K_\nu{}^\lambda{}_\mu + K_\lambda{}^\rho{}_\mu K_\nu{}^\lambda{}_\rho - K_\lambda{}^\lambda{}_\rho K_\nu{}^\rho{}_\mu\,. \quad (6.35)$$

We can rewrite this to derive the following identity

$$\bar{R}_\nu^\lambda = \frac{1}{e}\Big(\partial_\sigma (eK_\nu{}^{\lambda\sigma}) + \partial_\nu (eT^\lambda)\Big) - 2S_\sigma{}^{\lambda\mu}\Gamma_\nu{}^\sigma{}_\mu\,. \quad (6.36)$$

[3] Although for simplicity we have expressed this equation using covariant derivatives ∇ of the Levi-Civita connection, these can easily be rewritten in the teleparallel framework using the relation $\nabla_\mu V^\mu = \frac{1}{e}\partial_\mu (eV^\mu)$.

Using this final identity (6.36), it is then easy to see that the field equations reduce to the $f(R)$ field equations in standard form

$$F\bar{R}_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} + g_{\mu\nu}\Box F - \nabla_\mu\nabla_\nu F = 8\pi\Theta_{\mu\nu}\,, \tag{6.37}$$

where $\Theta_{\mu\nu}$ is the energy-momentum tensor. Thus we conclude that equation (6.32) is indeed the correct field equation of the teleparallel equivalent of $f(R)$ gravity.

6.4 $f(B)$ gravity

In this section we will consider a possible new modification of gravity. A naturally intriguing possibility would be to consider a functional form of $f(T, B)$ that is independent of the torsion scalar T, that is an $f(B)$ gravity. However, such a theory could not reduce to general relativity, or TEGR, in a suitable limit and thus is not of great theoretical interest. However, if we considered instead a functional form

$$f(T, B) = -T + f(B) \tag{6.38}$$

then we have a genuinely new modification of gravity not covered by either of the $f(T)$ or $f(R)$ subcases. This now has the property of reducing to GR or TEGR when the function $f(B)$ is simply linear (as linear boundary terms in the Lagrangian do not effect the field equations). Moreover, we will see in a later chapter, when we consider conformal issues, that this subcase has some interesting relationships to some particular scalar field models.

In this case, let us derive the field equations. We now have $f_T = -1$ and so

$$\begin{aligned} 2e\delta_\nu^\lambda\Box f_B - 2e\nabla^\lambda\nabla_\nu f_B + eBf_B\delta_\nu^\lambda + 4e\partial_\mu f_B S_\nu{}^{\mu\lambda} \\ - 4e_\nu^a\partial_\mu(eS_a{}^{\mu\lambda}) + 4eT^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 16\pi e\Theta_\nu^\lambda\,. \end{aligned} \tag{6.39}$$

Of course, these field equations will still remain fourth order and suffer issues of lack of Lorentz invariance. We will come back to this model in Chapter 8.

6.5 Lorentz invariance

In this section we will investigate the issue of local Lorentz invariance in these modified torsion models. As we did earlier in this chapter, when we considered the $f(R)$ limit, let us rewrite our general field equation in a covariant form in terms of the Einstein tensor and the metric. If we insert the expression for the Ricci tensor (6.36) into the field equation (6.26) we find

$$\begin{aligned} 2e\delta_\nu^\lambda \Box f_B - 2e\nabla^\lambda\nabla_\nu f_B + eBf_B\delta_\nu^\lambda + 4e\Big[(f_{BB}+f_{BT})(\partial_\mu B) + (f_{TT}+f_{BT})(\partial_\mu T)\Big]S_\nu{}^{\mu\lambda} \\ + 4e_\nu^a\partial_\mu(eS_a{}^{\mu\lambda})f_T - 4ef_T T^\sigma{}_{\mu\nu}S_\sigma{}^{\lambda\mu} - ef\delta_\nu^\lambda = 16\pi e\Theta_\nu^\lambda\,. \quad (6.40) \end{aligned}$$

Using the two relations $R = -T + B = -T + 2\partial_\mu T^\mu$ and $R_\nu^\lambda = G_\nu^\lambda + \frac{1}{2}(B-T)\delta_\nu^\lambda$, along with some algebra, we find that we can write the field equation in the following covariant form

$$\begin{aligned} H_{\mu\nu} := -f_T G_{\mu\nu} + g_{\mu\nu}\Box f_B - \nabla_\mu\nabla_\nu f_B + \frac{1}{2}(Bf_B + Tf_T - f)g_{\mu\nu} \\ + 2\Big[(f_{BB}+f_{BT})(\nabla_\lambda B) + (f_{TT}+f_{BT})(\nabla_\lambda T)\Big]S_\nu{}^\lambda{}_\mu = 8\pi\Theta_{\mu\nu}\,. \quad (6.41) \end{aligned}$$

It is readily seen that if one considers the $f(T)$ limit, then this equation coincides with the covariant form of the $f(T)$ field equations, as presented in [103].

We note that this equation is manifestly covariant. However, it is not in general invariant under local Lorentz transformations. As we saw in Chapter 4, a necessary condition for the field equation to be locally Lorentz invariant is for the antisymmetric part of the equation to be identically zero, as the energy-momentum tensor is required to be symmetric. In particular, this implies that the coefficient of $S_\nu{}^\lambda{}_\mu$

must vanish identically, see for example [103]. Requiring this straight away gives two conditions that must be satisfied

$$f_{BB} + f_{BT} = 0\,, \quad \text{and} \quad f_{TT} + f_{BT} = 0\,. \tag{6.42}$$

This coupled system of second order partial differential equations can be solved to give the following first order condition

$$f_T + f_B = c_1\,, \tag{6.43}$$

where c_1 is a constant of integration. Solving this first order equation gives us a general f of the form

$$f(T,B) = \tilde{f}(-T+B) + c_1 B = \tilde{f}(R) + c_1 B\,. \tag{6.44}$$

Since B is a total derivative term, the resulting field equations are unchanged by terms linear in B. Hence, we can set $c_1 = 0$ without loss of generality. We already showed that an f of this form simply reduces to the $f(R)$ field equations, which are manifestly Lorentz invariant. Hence we can conclude that the above field equations are Lorentz invariant if and only if they are equivalent to $f(R)$ gravity. Therefore, the teleparallel equivalent of $f(R)$ gravity is the only possible Lorentz invariant theory of gravity constructed out of R, T and B. Conversely to the above, the price to pay is the presence of higher order derivative terms.

6.6 Conservation equations

We have just investigated the issue of local Lorentz invariance. Now let us move onto another potential problem: requiring that the matter action is invariant under infinitesimal coordinate transformations. As we saw in Chapter 4, this gives thee condition that the energy momentum tensor $\Theta_{\mu\nu}$ is divergence free

$$\nabla^\mu \Theta_{\mu\nu} = 0\,. \tag{6.45}$$

(also shown in [103]). Hence we require the left-hand side of our field equations to also have this property. Let us show that this is indeed the case and that there is no need for this to be imposed as an extra (independent) condition.

For compactness, let us define the vector

$$X_\lambda = \Big[(f_{BB} + f_{BT})(\nabla_\lambda B) + (f_{TT} + f_{BT})(\nabla_\lambda T)\Big]. \tag{6.46}$$

Taking the covariant derivative of $H^{\mu\nu}$, we find after some simplification

$$\nabla^\mu H_{\mu\nu} = -\left[R_{\mu\nu} - \frac{1}{2}B g_{\mu\nu} + 2\nabla^\rho S_{\nu\rho\mu}\right] X^\mu. \tag{6.47}$$

Now using

$$R_{\mu\nu} = -2\nabla^\rho S_{\nu\rho\mu} + \frac{1}{2}B g_{\mu\nu} - 2S^{\rho\sigma}{}_\mu K_{\nu\sigma\rho}, \tag{6.48}$$

this simplifies to

$$\nabla^\mu H_{\mu\nu} = 2S^{\rho\sigma}{}_\mu K_{\nu\sigma\rho} X^\mu. \tag{6.49}$$

However, we know that the energy momentum tensor is symmetric, and hence

$$H_{[\mu\nu]} = -S_{[\nu\mu]}{}^\lambda X_\lambda = 0. \tag{6.50}$$

This implies

$$\nabla^\mu H_{\mu\nu} = 2H^{[\rho\sigma]} K_{\nu\rho\sigma} = 0, \tag{6.51}$$

which follows from K being antisymmetric in its last two indices. This means that on shell the left-hand side of the field equations are conserved.

6.7 Cosmology

In this section we will take a first look at the cosmology of a general $f(T,B)$ model. Both $f(R)$ and $f(T)$ gravity both can exhibit interesting late time and early time phenomenology, so it is in some sense trivial that $f(T,B)$ gravity will do so too. Nonetheless we will briefly analyse the cosmology of such models looking at the field equations and possible implications.

Let us work in a spatially flat FRW universe, so that we choose the diagonal FRW tetrad, that gives rise to the standard FRW metric. The torsion scalar and boundary term of such a tetrad are given by

$$T = -6H^2, \quad B = -18H^2 - 6\dot{H}, \tag{6.52}$$

respectively. Plugging the tetrad ansatz into the field equations, assuming a perfect fluid energy momentum tensor, gives the following cosmological field equations, as can be readily seen from [105]

$$f + \frac{6\dot{a}\dot{f}_B}{a} - \frac{12 f_T \dot{a}^2}{a^2} - \frac{6 f_B\left(a\ddot{a} + 2\dot{a}^2\right)}{a^2} = \kappa^2 \rho \tag{6.53}$$

$$f - \frac{4\dot{a}\dot{f}_T}{a} - \frac{6 f_B\left(a\ddot{a} + 2\dot{a}^2\right)}{a^2} - \frac{4 f_T(a\ddot{a} + 2\dot{a}^2)}{a^2} + 2\ddot{f}_B = -2\kappa^2 p. \tag{6.54}$$

Traditionally, one would now specify the form of the function $f(T,B)$ and attempt to find the scale factor. However, one can also invert this process, these field equations can also be used to find functional forms of $f(T,B)$ that are able to produce particular cosmological solutions. This process is known as reconstruction. For example, in a recent paper [104], using a restricted form of the type of $f(T,B)$, it was found that a de-Sitter type solution can be produced by the following $f(T,B)$ gravity

$$f(T,B) = -T + a_1 B + [(2-3a_1)(-T+a_1 B)]^{\frac{1-3a_1}{2-3a_1}}. \tag{6.55}$$

Other approaches one could take to analysing the above field equations would be to rewrite them as a dynamical system, in line with what we did in the previous chapter. This has been analysed in depth for $f(R)$ gravity, however in order to close the system one is generally required to perform function inversions, a process which is typically not very easy to perform (although recently a new approach to dealing with this issue has been considered [106]). Such an analysis is beyond the scope of this thesis, and later in this thesis we will see that with the help of a conformal transformation we can analyse the cosmology in a simpler way.

6.8 Discussion

In this chapter we have taken a look at a very general teleparallel modification which incorporates a number of teleparallel modifications into one framework. The results of this chapter can be visualised using Fig. 6.1. The starting point is a gravitational action based on an arbitrary function $f(T,B)$ which depends on the torsion scalar and a torsion boundary term. If this function is assumed to be independent of the boundary term, one arrives at $f(T)$ gravity which we identified as the unique second order gravitational theory in this approach. Likewise, if the function takes the special form $f(-T+B)$, we find the teleparallel equivalent of $f(R)$ gravity. This theory is identified as the unique locally Lorentz invariant theory. Any other form of $f(T,B)$ will result in gravitational theories which are neither of second order nor locally Lorentz invariant.

This analysis has subsequently been expanded on further, where the effects of

$$
\begin{tikzcd}
f(T,B) \arrow[r, "f = f(T)"] \arrow[d, "f = f(-T+B)"'] & f(T) \arrow[d, "f = T"] \\
f(R) \arrow[r, "f = R"'] & \mathrm{GR}
\end{tikzcd}
$$

Figure 6.1: The relationship between different modified gravity models. $f(R)$ gravity is the unique Lorentz invariant theory. $f(T)$ gravity is the unique theory with second order field equations.

a Gauss-Bonnet term, a boundary Gauss-Bonnet term and the trace of the energy momentum tensor have been incorporated into a large very general theory, $f(T, B, T_G, B_G, \mathcal{T})$ which incorporates a very large class of theories [105].

Only the beginning of a study of $f(T, B)$ cosmology has been considered here, but further investigations of these models would be of great interest to study in the future. Also of interest would be to see if there are potential non-trivial $f(T, B)$ models which undergo interesting cosmic phenomenology, such as bounce universes which have the potential to eliminate the big bang singularity.

Whether $f(T, B)$ gravity is a viable description of the universe remains to be seen. However, even if that were not to be the case, the framework of studying these models is very useful from a theoretical point of view, allowing us to understand the differences and structure of the underlying sub-models $f(R)$ and $f(T)$ gravity. We should mention at this point that there have been several studies in the literature of $f(R, T)$ models, being functions of the Ricci scalar and the torsion scalar[4]. Of course, such models are completely equivalent to $f(T, B)$ gravity, using the relationship $R = -T + B$. However, it appears that considering the boundary term directly is a more natural way of framing the theory from a teleparallel point of view.

[4] It is important not to get these papers confused with the many more papers on $f(R, T)$ models where T is defined to be the trace of the energy momentum tensor!

Chapter 7

Nonminimally coupled teleparallel models

In the last chapter we explored how one can consider a broader class of theories in the teleparallel framework by considering a function depending on the boundary term B as well as the usual teleparallel scalar. In this chapter we will perform a similar analysis, where we will look at the equivalence between scalar-tensor theories in the teleparallel and curvature frameworks.

In Chapter 5 we briefly discussed scalar-tensor theories as a generalisation of Brans-Dicke theories in the curvature framework. These are a generalisation of quintessence theories where one considers a coupling between the scalar field and the gravitational sector. The standard approach is to consider a coupling between the scalar field and the Ricci scalar, of the form $\xi R\phi^2$ [107–109]. Such a nonminimal coupling has motivations from different contexts. It appears as a result of quantum corrections to the scalar field in curved spacetimes [110, 111] and it is also required by renormalisation considerations [109]. It also appears in the context of superstring theories [112]. Such models have attempted to explain the early time inflationary epoch, however the simple model with a quadratic scalar potential is now disfavoured by the current Planck data [113–115].

In recent years an alternative formulation has been considered where the coupling occurs between the scalar field and torsion of the form $\xi T\phi^2$ [116–127]. This gives rise to different dynamics and interesting phenomenology, for example phantom behaviour and dynamical crossing of the phantom barrier: the universe is able to pass from an effective EoS $w_{\rm eff} > -1$ to an effective EoS $w_{\rm eff} < -1$. A dynamical systems analysis of these models were considered in [117], and the observational constraints on such models were found in [121]. Other types of nonminimal coupling to the torsion sector have been considered, for example in [128] a coupling between torsion and derivatives of the scalar field are considered.

In this chapter, we consider an approach where we examine a nonminimal coupling between the scalar field to both the torsion scalar T and the boundary term B. We note that coupling a scalar field to a boundary term is not a new idea, for example a nonminimal coupling between a scalar field boundary term such as the Gauss-Bonnet term and higher Lovelock polynomials have previously been considered [129, 130]. This theory encompasses both nonminimally coupled teleparallel gravity and nonminimally coupled general relativity in suitable limits.

This chapter is based on and extends the results derived in the joint work [2], and the erratum [10].

7.1 Scalar field - torsion coupling

Let us first briefly review previous studies of nonminimally coupled scalar fields. The first approach to nonminimally coupling a scalar field to the gravitational sector is to consider a coupling to the Ricci tensor as follows

$$S = \int \left[\frac{R}{2\kappa^2} + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi + \xi R\phi^2) - V(\phi) + L_{\rm m} \right] \sqrt{-g}\, d^4x \,. \tag{7.1}$$

As we discussed in Chapter 5, this approach was originally considered in the context of Brans-Dicke theories motivated by introducing a variable gravitational constant.

Minimally coupled quintessence corresponds to taking $\xi = 0$ in the above Lagrangian. As we have already seen, quintessence alone can give rise to many interesting features from late time accelerated expansion of the universe to inflation [44–46]. However simple models of scalar field inflation are becoming disfavoured by the latest Planck data. A further issue with this simple quintessence approach is that the effective equation of state must always satisfy $w_{\rm eff} > -1$ and require a very flat fine tuned potential in order to explain current cosmological observations.

When $\xi \neq 0$, the system is said to have a nonminimal coupling and is referred to as being in the Jordan frame. This can be transformed to a minimal coupling via a conformal transformation, with the resulting frame known as the Einstein frame. Such a transformation reduces the system to a quintessence model with a coupling between the scalar field and dark matter. Physical quantities in this frame can then be transformed back into physical quantities in the Jordan frame. We will look at this conformal relationship in the next chapter. For a review of the dynamics of these models, see chapter 9 of [74] and references therein. Alternatively, one can work directly in the Jordan frame: a dynamical systems analysis for various potentials have been considered by various authors, see [131–134] and references within.

In recent years, an alternative approach to nonminimally coupled scalar fields has been considered in a teleparallel setting. Coupling a scalar field to torsion has been considered, giving rise to what are known as teleparallel dark energy theories [116]. The following action is considered

$$S = \int \left[-\frac{T}{2\kappa^2} + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \xi T\phi^2) - V(\phi) + L_{\rm m} \right] e\, d^4x \,. \tag{7.2}$$

This gives rise to different dynamics to the case of the nonminimal coupling to the

Ricci scalar. Of course with a minimal coupling, setting $\xi = 0$, the two theories again become equivalent due to the teleparallel equivalence. This theory again has a richer structure than simple standard quintessence behaviour, with both phantom and quintessence type dynamics possible, along with dynamical crossing of the phantom barrier.

The equivalence between general relativity and teleparallel gravity breaks down as soon as one nonminimally couples a scalar field: the field equations result in different dynamics. In this chapter, we will consider a more general action, as considered in [2, 135], with the aim of unifying both of the previously considered approaches. We examine

$$S = \int \left[-\frac{T}{2\kappa^2} + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \xi T\phi^2 - \chi B\phi^2) - V(\phi) + L_{\mathrm{m}} \right] e\, d^4x. \tag{7.3}$$

When one sets $\chi = -\xi$ one will recover an action which is equivalent to (7.1), and when one sets $\chi = 0$ the action (7.2) is recovered.

One can also recast this action in a different form via an integration by parts of the boundary term.

$$S = \int \left[-\frac{T}{2\kappa^2} + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \xi T\phi^2) + \chi\phi T^\mu\partial_\mu\phi - V(\phi) + L_{\mathrm{m}} \right] e\, d^4x. \tag{7.4}$$

where now we have a coupling between a derivative of the scalar field and the torsion tensor contraction T^μ. Such a term has been considered several times in the literature, and first goes back to [136] in the 1980's, who considered this in a Brans-Dicke type model in the context of Møller's tetradic theory [137]. More recently such a coupling has also been considered by others in [127, 138].

A particularly interesting subcase of these models will be when $\xi = 0$, corresponding to a pure coupling between the boundary term and ϕ, as this is a potentially unexplored model and is not covered by either of the previous two limiting theories. Such a coupling was studied in detail in [2]. One could in principle choose a more general coupling $\eta(\phi)B$ between the potential and the boundary, however for this work we will restrict ourselves to analysing the choice $\eta(\phi) = \phi^2$, which ensures that the constant χ is dimensionless. A more general function might have some interesting inflationary applications.

We now derive the field equations of the action (7.3). Varying the action with respect to the tetrad field yields the following field equations

$$-\left(\frac{2}{\kappa^2} + 2\xi\phi^2\right)\left[e^{-1}\partial_\mu(eS_a{}^{\mu\nu}) - e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu} - \frac{1}{4}e_a^\nu T\right] - e_a^\nu\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$
$$+e_a^\mu\partial^\nu\phi\partial_\mu\phi - 4(\xi+\chi)e_a^\rho S_\rho{}^{\mu\nu}\phi\partial_\mu\phi - \chi\left[e_a^\nu\Box(\phi^2) - e_a^\mu\nabla^\nu\nabla_\mu(\phi^2)\right] = \mathcal{T}_a^\nu, \tag{7.5}$$

where $\Box = \nabla^\mu\nabla_\mu$ and ∇ is the covariant derivative with respect to the Levi-Civita connection. Here $\mathcal{T}_a^\nu$ is the standard energy momentum tensor derived from varying the matter sector, and is not to be confused with torsion.

As we saw in the last chapter, the term in the field equations containing $S_\rho{}^{\mu\nu}$ is Lorentz violating [103]: it has a non-trivial antisymmetric component. This term vanishes only in the case when we have $\xi = -\chi$, which corresponds to the case where the nonminimal coupling reduces to that of a coupling to only the Ricci scalar. This result is very similar to the one obtained in the last chapter and in [1], where the only Lorentz invariant $f(T,B)$ modification was $f(T,B) = f(-T+B) = f(R)$.

Now it can be shown that the Einstein tensor of the Levi-Civita connection can be related to the torsion sector via the relation

$$G^\sigma{}_\nu = -\left[2e^{-1}\partial_\mu(eS_a{}^{\mu\nu}) - 2e_a^\lambda T^\rho{}_{\mu\lambda}S_\rho{}^{\nu\mu} - \frac{1}{2}e_a^\nu T\right]e_\sigma^a. \tag{7.6}$$

This means we can write the field equations in a covariant form as follows

$$\left(\frac{2}{\kappa^2}+2\xi\phi^2\right)G_{\mu\nu}-g_{\mu\nu}\left[\frac{1}{2}\nabla_\lambda\phi\nabla^\lambda\phi-V(\phi)\right]+\nabla_\mu\phi\nabla_\nu\phi$$
$$-4(\xi+\chi)S_\mu{}^\lambda{}_\nu\phi\partial_\lambda\phi-\chi\Big[g_{\mu\nu}\Box(\phi^2)-\nabla_\mu\nabla_\nu(\phi^2)\Big]=\mathcal{T}_{\mu\nu}. \tag{7.7}$$

It is readily seen that this equation reduces to the correct field equation for a non-minimal coupling of a scalar field to the Ricci scalar when one takes $\chi=-\xi$.

Finally we have the modified scalar field equation. This is obtained by varying the action with respect to the scalar field, yielding the following Klein-Gordon equation

$$\Box\phi+V'(\phi)=-(\xi T+\chi B)\phi. \tag{7.8}$$

Now we have examined the field equations of the system, we will devote the rest of this chapter to examining the cosmological applications of this model.

7.2 Cosmology

In this section we will derive the background equations for the cosmology of the above models. We will consider the standard spatially flat FLRW tetrad, and assume the energy momentum tensor of the matter sector is standard barotropic matter given by an isotropic perfect fluid, with a linear equation of state. We will also assume all dynamical quantities, including the scalar field ϕ, are homogeneous, depending only the time t.

Inserting the FLRW tetrad into the field equations (7.5) gives us the following

Friedmann equations

$$3H^2 = \kappa^2 \left(\rho + \rho_\phi\right), \tag{7.9}$$

$$3H^2 + 2\dot{H} = -\kappa^2 \left(p + p_\phi\right). \tag{7.10}$$

Here we have defined the energy density and pressure of the scalar field as follows

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 3\xi H^2\phi^2 + 6\chi H\phi\dot{\phi}, \tag{7.11}$$

$$\begin{aligned} p_\phi =& \frac{1}{2}(1-4\chi)\dot{\phi}^2 - V(\phi) + 2H\phi\dot{\phi}(2\xi + 3\chi) \\ &+ 2\phi^2\dot{H}(\xi + 6\chi^2) + 2\chi\phi V'(\phi) + 3H^2\phi^2(\xi + 4\chi\xi + 12\chi^2). \end{aligned} \tag{7.12}$$

Using the FLRW metric, the Klein-Gordon equation (7.8) reduces to

$$\ddot{\phi} + 3H\dot{\phi} + 6(\xi H^2 + \chi(3H^2 + \dot{H}))\phi + V'(\phi) = 0. \tag{7.13}$$

In the above derivations we have used the relations (6.52) for the FLRW torsion scalar and boundary term. In this model matter obey the standard conservation equation,

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{7.14}$$

as can be seen by a lengthy but straightforward calculation. Similarly, the scalar field energy density and pressure also satisfy

$$\dot{\rho_\phi} + 3H(\rho_\phi + p_\phi) = 0, \tag{7.15}$$

so even in the presence of a nonminimal coupling there is no transfer of energy between the matter sector and the dark energy sector.

One can define the equation of state of the dark energy/scalar field as the following ratio of the scalar field pressure and energy density, and the total or effective equation of state respectively as

$$w_\phi = \frac{p_\phi}{\rho_\phi}, \quad w_{\rm eff} = \frac{p + p_\phi}{\rho + \rho_\phi}. \tag{7.16}$$

We can also define the standard matter energy density and the energy density of the scalar field, respectively, as

$$\Omega_m = \frac{\kappa^2 \rho}{3H^2}, \quad \Omega_\phi = \frac{\kappa^2 \rho_\phi}{3H^2}, \tag{7.17}$$

so that the relation $1 = \Omega_m + \Omega_\phi$ holds.

7.3 Dynamical systems analysis

Now let us use dynamical systems techniques introduced in Chapter 5 to analyse the behaviour of the cosmology of this system. This system has been analysed for various special cases in the literature. When $\chi = 0$, the model reduces to a "teleparallel dark energy model", where there is a nonminimal coupling only between the torsion scalar and the scalar field: the dynamical systems analysis in this case has been studied in [117, 123]. Likewise, when $\chi = -\xi$, the system becomes a model in which there is a nonminimal coupling between the scalar field and the Ricci scalar, which has been studied extensively, see [131–134]. Finally, a further limiting case of this framework is when $\xi = 0$, when there is a pure coupling between only the boundary term: the dynamical systems analysis of this was performed in [2]. In this thesis, we will present, for the first time, an analysis of the dynamics of the full action (7.3), without restricting ourselves to these limiting cases.

The first thing we must do is define appropriate dimensionless variables. We choose the following definitions

$$\sigma^2 = \frac{\kappa^2 \rho}{3H^2}\,, \quad x^2 = \frac{\kappa^2 \dot{\phi}^2}{6H^2}\,, \quad y^2 = \frac{\kappa^2 V}{3H^2}\,, \quad z = \kappa\phi\,, \tag{7.18}$$

which straightforwardly generalise the normalised variables used to analyse standard quintessence, introducing just one extra variable z, see Chapter 5 or [82]. The first Friedmann equation (7.9) written in these variables is simply the constraint

$$1 = \sigma^2 + x^2 + y^2 + 2\sqrt{6}\chi xz - \xi z^2, \tag{7.19}$$

which will define the boundary of our phase space. The shape of this boundary will of course depend on the values of the coupling constants χ and ξ.

Due to this algebraic relation, the phase space will be three dimensional, and we choose to work with the variables x, y, and z. Since the energy density of matter is non-negative, the relation

$$x^2 + y^2 + 2\sqrt{6}\chi xz - \xi z^2 \leq 1 \tag{7.20}$$

must be satisfied. As in standard quintessence, due to symmetries of the system, we can assume without loss of generality that our potential is positive and so we only need to consider $y > 0$. There is no restriction on the sign of x or z, since $\dot{\phi}$ can be positive or negative, as can the scalar field. This means that the phase space is potentially not compact, it will depend on the values and signs of the coupling constants.

To that end, if we further introduce the variables

$$u = x + \sqrt{6}\chi z, \quad v = \sqrt{|6\chi^2 + \xi|}z \tag{7.21}$$

then the phase space boundary can be rewritten as

$$u^2 \pm v^2 + y^2 \leq 1, \tag{7.22}$$

where the minus sign corresponds to the case $6\chi^2 + \xi > 0$, and the plus sign is the case when $6\chi^2 + \xi < 0$. In the former case, we see that the boundary is simply hyperbolic space, $\mathbb{H}^2$. In the latter, the space is that of a unit sphere, which is indeed compact. In the limiting case when $\xi = -6\chi^2$, the above $u - v$ variables break down, and the phase space instead becomes an infinitely long cylinder.

As before, we define the quantity $N = \ln a$ and denote derivatives with respect to N by a prime. The equations of motion can be written as the following slightly lengthy autonomous system of first order differential equations

$$\begin{aligned} x' =& \frac{1}{2\left(\left(\xi+6\chi^2\right)z^2+1\right)}\Big(- 3x^3(4\chi+w-1) - 3x\big(y^2(2\lambda\chi z+w+1) \\ & - z^2(\xi(12\chi+w-2)+\chi(1-12\chi(w-1))) - w + 1\big) \\ & + \sqrt{6}x^2 z(4\xi - 3\chi(4\chi+3w-3)) + \sqrt{6}\big(y^2\left(\lambda+\lambda\xi z^2 - 3\chi(w+1)z\right) \\ & + 3\chi z\left(z^2(\xi(4\chi+w-2)+\chi)+w-1\right)\big)\Big), \end{aligned} \tag{7.23}$$

$$\begin{aligned} y' =& -\frac{y}{2\left(\left(\xi+6\chi^2\right)z^2+1\right)}\Big(3x^2(4\chi+w-1) \\ & + \sqrt{6}x\left(\lambda+\lambda\left(\xi+6\chi^2\right)z^2 + z(6\chi(w-1)-4\xi)\right) \\ & + 3\left(y^2(2\lambda\chi z+w+1) - z^2\left(4\xi\chi+\xi w+12\chi^2+\chi\right) - w - 1\right)\Big), \end{aligned} \tag{7.24}$$

$$z' = \sqrt{6}x. \tag{7.25}$$

The above dynamical system is rather lengthy, so let us write (7.23)-(7.25) in the more compact form

$$x_i' = f_i(x,y,z), \quad (x_1,x_2,x_3) = (x,y,z). \tag{7.26}$$

In the system we have defined the quantity λ by

$$\lambda = -\frac{V'(\phi)}{\kappa V(\phi)}. \tag{7.27}$$

In order to close the system we will have to specify a form of λ. For the autonomous system to remain three dimensional, one needs to choose a form of the potential such that λ can be written in terms of the variables x, y and z. As in Chapter 5, we will assume the potential V to have an exponential form of the kind

$$V(\phi) = V_0\, e^{-\lambda\kappa\phi}\,. \tag{7.28}$$

This ensures that λ is simply a constant.

However this is not the only choice that will give a closed three dimensional system. One could also consider a power law potential of the form

$$V(\phi) = \frac{M^{\alpha+4}}{\phi^{\alpha}}, \tag{7.29}$$

where α is a constant and M is a positive constant with the units of mass. This would allow one to write λ in terms of z as

$$\lambda = \frac{2\sqrt{6}\alpha\chi}{z}, \tag{7.30}$$

however we will leave the analysis of such a potential for future work.

We can rewrite the important physical quantities in terms of the new variables x, y and z. We find that the effective equation of state can be written as

$$\begin{aligned} w_{\mathrm{eff}} = \frac{1}{3\left(\left(\xi+6\chi^2\right)z^2+1\right)}\Big(& -3x^2(w+4\chi-1)+2\sqrt{6}xz(2\xi-3(w-1)\chi) \\ & +3\left(w-y^2(2\chi\lambda z+w+1)+z^2\left(4\xi\chi+6\chi^2+\xi w\right)\right)\Big). \end{aligned} \tag{7.31}$$

Now at a fixed point, one can integrate the second Friedmann equation (7.10) explicitly to find a, and it is found that

$$a \propto (t - t_0)^{2/[3(1+w_{\rm eff})]}. \tag{7.32}$$

This means that the universe's expansion will be accelerating if the effective equation of state satisfies $w_{\rm eff} < -1/3$. Another useful physically important quantity is the deceleration parameter q, defined to be

$$q = -1 - \frac{\dot{H}}{H^2} = \frac{1}{2} + \frac{3}{2}w_{\rm eff}, \tag{7.33}$$

which will tell us if the universe is accelerating or not, depending on whether it is negative or positive respectively. Similarly, we can also express the energy density of the matter and scalar field in terms of x, y and z

$$\Omega_m = \frac{\kappa^2\rho_m}{3H^2} = \sigma^2 = 1 - x^2 - y^2 - 2\sqrt{6}\chi xz + \xi z^2, \tag{7.34}$$

$$\Omega_\phi = \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 + 2\sqrt{6}\chi xz - \xi z^2. \tag{7.35}$$

7.3.1 Finite critical points

We are now able to find the finite critical points of the dynamical system, corresponding to the solutions of

$$f_i(x, y, z) = 0, \quad i = 1, 2, 3, \tag{7.36}$$

that lie within the phase space. We emphasise that here we are looking for finite critical points, however in some cases the phase space is not compact, so there may be critical points lying at infinity. These will have to be considered separately and we will investigate these shortly.

Point	x	y	z	Existence
O	0	0	0	$\forall\lambda,\ \xi,\ \chi$
$A_{\pm}$	± 1	0	0	$\xi=\chi=0$
B	$\sqrt{\frac{3}{2}}\frac{(1+w)}{\lambda}$	$\sqrt{\frac{3}{2}}\frac{\sqrt{(1+w)(1-w)}}{\lambda}$	0	$\xi=\chi=0,\ \lambda^2>3(1+w)$
C	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	$\xi=\chi=0\ \&\ \lambda^2<6$
D	0	1	$\frac{\lambda}{6\chi}$	$\xi=0\ \&\ \forall\lambda,\ \chi$
E	0	1	0	$\lambda=0\ \&\ \chi\neq 0/\xi\neq 0$
F	0	$\frac{\sqrt{2}(\xi+6\chi)}{\sqrt{\lambda}}\sqrt{z}$	$\frac{\xi+3\chi-\sqrt{(\xi+3\chi)^2-\lambda^2\xi}}{\lambda\xi}$	$\lambda^2\xi<(\xi+3\chi)^2\ \&\ \xi>0$
G	0	$\frac{\sqrt{2}(\xi+6\chi)}{\sqrt{\lambda}}\sqrt{z}$	$\frac{\xi+3\chi+\sqrt{(\xi+3\chi)^2-\lambda^2\xi}}{\lambda\xi}$	$\lambda^2\xi<(\xi+3\chi)^2\ \&\ \xi>0$
$H_{\pm}$	0	0	$\pm\frac{1}{\sqrt{-\xi}}$	$\xi<0\ \&\ \chi\neq 0\ \&\ \forall\lambda$

Table 7.1: Potential finite critical points of the autonomous system (7.23)-(7.25).

A list of all of the potential critical points is displayed in Tab. 7.1, along with the points' conditions for existence. We have listed all potential critical points, however not all of them exist for all values of the coupling constants χ and ξ. We can essentially break down the analysis into five distinct cases that we need to consider:

- $\chi=\xi=0$. In this case the action is minimally coupled and becomes simply standard quintessence. The points O, $A_{\pm}$, B and C are the only points which potentially exist, with the existence of B and C depending on the size of the parameter λ. This model was discussed in Chapter 5. It should be emphasised that the points $A_{\pm}$, B and C can only exist in the minimally coupled case, as soon as χ or ξ are non-zero they are no longer critical points, but they are displayed here for completeness.

- $\chi=0,\ \xi\neq 0$. This case corresponds to a nonminimal coupling between the scalar field and the torsion scalar. This dynamical system was extensively analysed in [117]. In this case, the critical points O, E, F and G are the only points that can potentially exist.

- $\xi = 0$, $\chi \neq 0$. In this case, there is a nonminimal coupling to the boundary term only. This dynamical system was analysed in detail in [2]. In this case, only the finite critical points O and D exist.

- $\lambda = 0$. This case, when λ vanishes, has to be treated separately, and corresponds to the potential simply being a constant. In this case, the points O, E and $H_{\pm}$ can potentially exist.

- $\chi, \xi, \lambda \neq 0$. Finally, in the case when the constants λ, χ and ξ are all non-zero, the system can possess the critical points O, F, G and $H_{\pm}$. Such a case has not been considered previously in the literature, except for the limiting case when there is a coupling to the Ricci scalar, that is when $\chi = -\xi$.

As we saw in Chapter 5, in order to determine the linear stability of these points, one must analyse the Jacobian matrix of partial derivatives

$$J = \frac{\partial f_i(x, y, z)}{\partial x_j}, \quad i, j = 1, 2, 3 \tag{7.37}$$

evaluated at each of the critical points and examine the sign of the eigenvalues. The eigenvalues and stability properties of the critical points are displayed in Tab. 7.2. The points $A_{\pm}$, B and C exist only for $\chi = \xi = 0$, in which case the variable z is superfluous and the system reduces to a two dimensional system. Hence only the eigenvalues of the reduced two dimensional system are displayed. The eigenvalues for the points F and G are too lengthy to be displayed, and it is not possible to analytically derive conditions for stability. Instead we will have to resort to analysing the stability by numerically investigating phase space plots.

We are interested in the behaviour of the universe at each of these points, that is: how is the scale factor evolving? We have displayed the effective equation of

Point	Eigenvalues	Stability
O	$\frac{3}{2}, \frac{1}{4}\left(-3 \pm \sqrt{-96\xi - 144\chi + 9}\right)$	Saddle node
$A_\pm$	$3,\ 3 \mp \sqrt{\frac{3}{2}}\lambda$	Unstable node: A_+ $\lambda < \sqrt{6}$ Unstable node: A_- $\lambda > -\sqrt{6}$ Saddle node: otherwise
B	$\frac{3}{4} + \frac{3\sqrt{24-7\lambda^2}}{4\lambda}, -\frac{3}{4} + \frac{3\sqrt{24-7\lambda^2}}{4\lambda}$	Stable node: $3 < \lambda^2 < 24/7$ Stable spiral: $\lambda^2 > 24/7$
C	$\lambda^2 - 3,\ \frac{1}{2}(\lambda^2 - 6)$	Stable node: $\lambda^2 < 3$ & $\lambda\beta > (\lambda^2 - 3)$ Saddle node: $\beta < (\lambda^2 - 3)/\lambda$
D	$-3, \frac{3}{2}\left(-1 + \frac{\sqrt{\lambda^2 - 48\chi + 6}}{\sqrt{\lambda^2+6}}\right),$ $\frac{3}{2}\left(-1 - \frac{\sqrt{\lambda^2 - 48\chi + 6}}{\sqrt{\lambda^2+6}}\right)$	Stable spiral: $48\chi > \lambda^2 + 6$ Stable point: $0 < 48\chi < \lambda^2 + 6$ Saddle point: $\chi < 0$
E	$-3,\ \frac{1}{2}\left(-3 - \sqrt{-24\xi - 72\chi + 9}\right),$ $\frac{1}{2}\left(-3 + \sqrt{-24\xi - 72\chi + 9}\right)$	Stable spiral: $8\xi + 24\chi > 3$ Stable node: $0 < 8\xi + 24\chi < 3$ Saddle point: $8\xi + 24\chi < 0$
F	$\Delta_1, \Delta_2, \Delta_3$	See discussion.
G	$\Delta_4, \Delta_5, \Delta_6$	See discussion.
$H_\pm$	$\frac{\xi}{\chi}, \frac{\xi}{\chi} + 3, \frac{2\xi}{\chi} + 3$	Unstable node: $\chi < 0$ Saddle point: $-3\chi < 2\xi < 0$ Stable node: $2\xi < -3\chi < 0$

Table 7.2: Stability and eigenvalues of the critical points of the dynamical system (7.23)-(7.25).

Point	$w_{\rm eff}$	q	Acceleration
O	w	$\frac{1+3w}{2}$	No
$A_\pm$	1	2	No
B	w	$\frac{1+3w}{2}$	No
C	$\frac{\lambda^2-3}{3}$	$-1+\frac{\lambda^2}{2}$	$\lambda^2<2$
D	-1	-1	Yes
E	-1	-1	Yes
F	-1	-1	Yes
G	-1	-1	Yes
$H_\pm$	$2+\frac{\xi}{\chi}$	$1+\frac{2\xi}{3\chi}$	$\frac{\xi}{\chi}<-2$

Table 7.3: The effective equation of state, deceleration parameter and acceleration properties of the critical points of the dynamical system (7.23)-(7.25).

state, the decceleration parameter, and whether acceleration is present in Tab. 7.3.

Let us discuss the behaviour of each of the finite critical points in turn:

- *Point O.* This is the only point that exists for all values of the parameters. It is a matter dominated point, with the energy density of matter, $\sigma^2 = 1$.
- *Points $A_\pm$, B and C.* These four points only exist in the minimally coupled quintessence model, with $\xi = \chi = 0$. We refer the reader to Chapter 5 for a discussion of the properties of these points. However we note that they can also appear as quasi-critical points if the parameters χ and ξ are small.
- *Point D.* This point only exists when the coupling between the torsion scalar and the scalar field vanishes, that is $\xi = 0$, but the coupling between the boundary term remains $\chi \neq 0$. In fact when these conditions are satisfied, the Point D is the only finite non-zero critical point. The point is dominated by

the energy density of the potential of the scalar field. It describes an accelerating cosmological constant type universe, with $w_{\text{eff}} = -1$, and is generically attracting, either being a stable spiral or stable node if $\chi > 0$. If $\chi < 0$ the point is a saddle.

- *Point E*. This point only exists in the limiting case when the parameter λ vanishes, that is when the potential is just that of a cosmological constant. This point is a dark energy dominated, de Sitter type universe, and is stable if $\xi + 3\chi > 0$.

- *Point F*. This point exists generically for non-zero ξ. The eigenvalues of this point have been denoted by Δ_1, Δ_2 and Δ_3, this is as the full expressions are extremely long. These eigenvalues are also complicated to analyse their stability properties analytically, but via numeric investigations, in Figure 7.1 we have plotted the stability region in χ−ξ parameter space when the potential parameter satisfies $\lambda = 1$.

- *Point G*. Similar to Point F, this point exists generically for non-zero ξ. The eigenvalues of this point have been denoted by Δ_4, Δ_5 and Δ_6, again as the full expressions are too long. This point has similar difficulties as Point F, the regions of stability are too complex to analyse analytically, however numerical plots for the stability region in $\chi - \xi$ parameter space indicate that this point is generically not stable.

- *Points $H_\pm$*. These two points only exist when $\xi < 0$ and χ is non-vanishing. Thus they can exists in the general relativistic case when $\chi = -\xi$. They have an effective EoS depending on the value of χ and ξ, with $w_{\text{eff}} = 2 + \frac{\xi}{\chi}$. The total energy density of this point comes from the scalar field, with $\Omega_\phi = 1$. These points are stable only when they describe a phantom universe, with $w_{\text{eff}} < -1$.

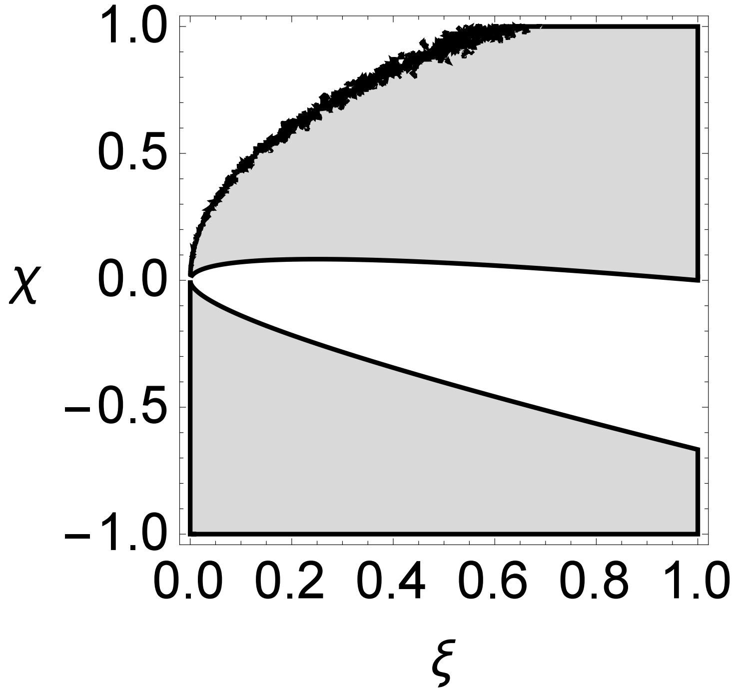

Figure 7.1: Parameter space indicating the regions of stability in $\chi - \xi$ space of the critical point F when the potential parameter $\lambda = 1$. Note that these points only exist when $\xi > 0$. The non-smoothness of the boundary indicates numerical difficulties due to the complicated form of the eigenvalues, with the true boundary expected to be smooth.

7.3.2 Critical points at infinity

We have so far discussed the critical points of the dynamical system lying at finite points in the $x - y - z$ phase space. However, if the condition $\xi + 6\chi^2 > 0$ holds, then the phase space is non-compact, and thus potentially there are critical points lying at infinity, which trajectories of the system may asymptote towards, and so to understand the global stability of the system we must carefully check whether there are critical points at infinity. There are a number of approaches one can take to doing this. One approach is to use projective coordinates, such as the technique used in [131]. Another approach would be to compactify the variables by using a function such as arctan or tanh, which was an approach utilised in [76].

For the purposes of this thesis, we will apply the method used in [2,117], and use Poincaré variables to compactify the phase space. For full details of this method we refer to [139]. The main idea is to introduce compactified coordinates x_r, y_r and z_r which are defined as

$$x_r = \frac{x}{\sqrt{1+r^2}}, \quad y_r = \frac{y}{\sqrt{1+r^2}}, \quad z_r = \frac{z}{\sqrt{1+r^2}}, \tag{7.38}$$

where r is the standard spherical polar coordinate defined as $r^2 = x^2 + y^2 + z^2$. This projects the dynamics into the Poincaré sphere. These variables can thus only take finite values, as they lie in the restricted range $-1 < x_r, y_r, z_r < 1$.

Now let us define the compactified radius measure ρ such that

$$\rho = \frac{r}{\sqrt{1+r^2}}, \tag{7.39}$$

with this implying that $x_r^2 + y_r^2 + z_r^2 = \rho^2$. This definition is useful, since any points lying at infinity will have $\rho = 1$, and thus we can study the dynamics at infinity by considering the limit $\rho \to 1$.

So that we can work with this new radius measure, we make a further coordinate transformation, transforming our Poincaré variables into spherical polar Poincaré coordinates as so

$$x_r = \rho\cos\theta\sin\varphi, \quad z_r = \rho\sin\theta\sin\varphi, \quad y_r = \rho\cos\varphi. \tag{7.40}$$

Before we begin to consider the Friedmann constraint, these variables are constrained to lie in the compact range $\rho \in [0,1]$, $\theta \in [0, 2\pi]$ and since we are restricting ourselves to $y \geq 0$ the angle φ lies in the restricted range $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The Friedmann constraint (7.20) can be written in Poincaré variables as

$$2x_r^2 + 2y_r^2 + (1-\xi)z_r^2 + 2\sqrt{6}\chi x_r z_r \leq 1. \tag{7.41}$$

The physical phase space will therefore be the intersection of this ellipsoid region with the Poincaré sphere

$$x_r^2 + y_r^2 + z_r^2 \leq 1. \tag{7.42}$$

Now, to understand the dynamics at infinity, we must transform the system (7.23)-(7.25) into Poincaré spherical polar coordinates, and take the limit $\rho \to 1$. In this limit, the following equations are obtained

$$\rho' = 0, \tag{7.43}$$

$$\sqrt{1-\rho^2}\theta' = -\frac{\lambda \cos\varphi \cot\varphi \left(\sqrt{6}\xi \sin\theta - 6\chi \cos\theta\right)}{2\left(\xi + 6\chi^2\right)}, \tag{7.44}$$

$$\sqrt{1-\rho^2}\varphi' = \frac{\lambda \cos\varphi \left(\sqrt{6}\cos\theta \left(\xi - 3\chi^2 \cos 2\varphi + 3\chi^2\right) + 6\chi \sin\theta \cos^2\varphi\right)}{2\left(\xi + 6\chi^2\right)}, \tag{7.45}$$

and hence the angular part of the equation decouples. Setting the right hand side of these equations equal to zero, we find there are two classes of solutions. There are the critical points at infinity which obey the solution

$$\cos\theta = \pm\frac{\xi}{\sqrt{\xi + 6\chi^2}}, \quad \cos\varphi = \pm\frac{\sqrt{\xi}}{\sqrt{1-\xi}}, \tag{7.46}$$

which exist only for certain values of the parameters ξ and χ, namely only when $0 < \xi < 1/2$. Alternatively the other class of solution is when

$$\cos\varphi = 0. \tag{7.47}$$

In this latter case we still need to solve for θ. At infinity we can thus parametrise the critical points as

$$\begin{aligned} x_r &= \pm\cos\theta, \\ z_r &= \pm\sin\theta, \\ y_r &= 0. \end{aligned} \tag{7.48}$$

Now we can use this ansatz to find an equation for θ'. We go back to the equations derived for x_r', y_r' and z_r' and inserting (7.48), and using the chain rule to find an expression for θ', we obtain

$$\begin{aligned} \theta' = &\frac{\csc\theta}{4\left(\xi+6\chi^2\right)}\big(2\sqrt{6}\sin\theta\big(-2\xi^2+\xi(2-3\chi) \\ &+\big(2\xi^2+\xi(3\chi+2)+3\chi(3-8\chi)\big)\cos 2\theta+3\chi(3-8\chi)\big) \\ &-3\left(\xi(12\chi-1)+12\chi^2+4\chi-1\right)\cos 3\theta+\left(\xi(36\chi-3)+9(1-2\chi)^2\right)\cos\theta\big). \end{aligned} \tag{7.49}$$

Now setting the right hand side of this equation equal to zero allows us to find the value of θ for these critical points at infinity. In general this equation has six solutions, however the obtained solutions are too complex to display here analytically. Moreover not all of these points at infinity obey the Friedmann constraint (7.41), however determining how many of the solutions do is difficult analytically for a general ξ and χ. This is not a problem for the analysis, it is just important we are aware of their existence when numerically investigating the phase space of the dynamical system. However, let us mention the form of these points in a couple of the limiting cases. When $\chi = 0$, the critical points at infinity, in Poicaré coordinates (x_r, y_r, z_r) are given by [117]

$$K_\pm : \left(\mp\frac{\sqrt{2}}{2}, 0, \pm\frac{\sqrt{2}}{2}\right), \quad L_\pm : \left(\pm\frac{\sqrt{2}}{2}, 0, \pm\frac{\sqrt{2}}{2}\right), \tag{7.50}$$

with the additional two points being unphysical. On the other hand, if $\xi = 0$, all of the six critical points are found to be unphysical.

In the Poincaré variables we have that the dark energy density parameter is given by

$$\Omega_\phi = \frac{x_r^2 + y_r^2 + 2\sqrt{6}\chi x_r z_r - \xi z_r^2}{1 - x_r^2 - y_r^2 - z_r^2}. \tag{7.51}$$

At the critical points of (7.46), this dark energy density parameter is divergent. Similarly, since the relation $1 = \Omega_m + \Omega_\phi$ the matter energy density will also be divergent at this point, and so these points are not of physical interest [117], they are only of mathematical interest. The remainder of the critical points may or may not be unphysical, it will depend on the values of the parameters χ or ξ, but to be physical, we see that they must lie on the boundary of the Friedmann constraint, otherwise the energy density parameter will be divergent (7.51). This is quite a restrictive condition, it requires two ellipsoids to agree, that is the denominator and numerator of (7.51) must both be zero, which in general will have at most four solutions. Now that we have an understanding of some of the issues of the system at infinity, we can now look at the system as a whole.

7.4 Cosmological implications

In this section we will discuss the cosmological interpretations of the dynamics of the above system. The dynamics change extensively based on the parameter choices for the coupling constants χ and ξ, and so the various subcases of parameter choices will be analysed and discussed separately. We will not discuss the minimally coupled quintessence case, which was analysed in Chapter 5.

7.4.1 Teleparallel dark energy: $\chi = 0$

Teleparallel dark energy is the subcase of parameter choices where there is a non-minimal coupling to the torsion scalar only, so we must take $\chi = 0$ but keep a generic non-zero ξ into the action (7.3). The phase space analysis of this system was first explored in [123], and a more detailed analysis, where the dynamics of the system at infinity are taken into account, was performed in [117]. It is this latter analysis that we review here. For generic ξ and λ, the system has three finite critical points, the origin O, along with two further points F and G (and note these point labels agree with those used in [117]). Along with these finite points, there are the further critical points at infinity, $K_\pm$ and $L_\pm$ (7.50).

The critical points at infinity are found to either be saddle points or unstable. The points $K_\pm$ can give rise to accelerating universe solutions, with suitably negative effective equation of state, and thus these models are able to undergo transient inflationary periods if they are drawn towards the accelerating saddle point at infinity. The points F and G both describe dark energy dominated points, with effective equation of state $w_{\text{eff}} = -1$. The point G is only a saddle node, however the point F is stable for $\lambda^2 < \xi$, and thus for a large range of non-zero paramters λ and ξ these models possess a late time accelerating attractor. Thus these models require little fine tuning to get desirable cosmological behaviour. The critical points have effective EoS $w_{\text{eff}} = -1$ for an arbitrary value of the potential constant λ, which is a significant advantage over standard quintessence, which requires a very small λ to achieve such an acceleration.

Another interesting consequence of this model is that it allows the effective equation of state to crossover into the phantom region with $w_{\text{eff}} < -1$, a property which standard quintessence models cannot do alone, and is a region which is perhaps slightly favoured by current cosmological observations.

7.4.2 Boundary term coupling $\xi = 0$

Now let us consider another limiting case, where there is a coupling between the scalar field and the boundary term only, which means we restore a non-zero χ but now consider a vanishing ξ. Such a model was studied in the work [2].

The points $A_{\pm}$, B and C only exist in the limit $\chi \to 0$, and of course the points exhibit the same behaviour as the above discussion about standard quintessence. However, we mention these here as for small choices of our paramter χ they will appear in the system as quasi-critical points. The point O exists also for $\chi \neq 0$, and corresponds to a matter dominated universe with no scalar field contribution. This point remains a saddle point for all χ and λ.

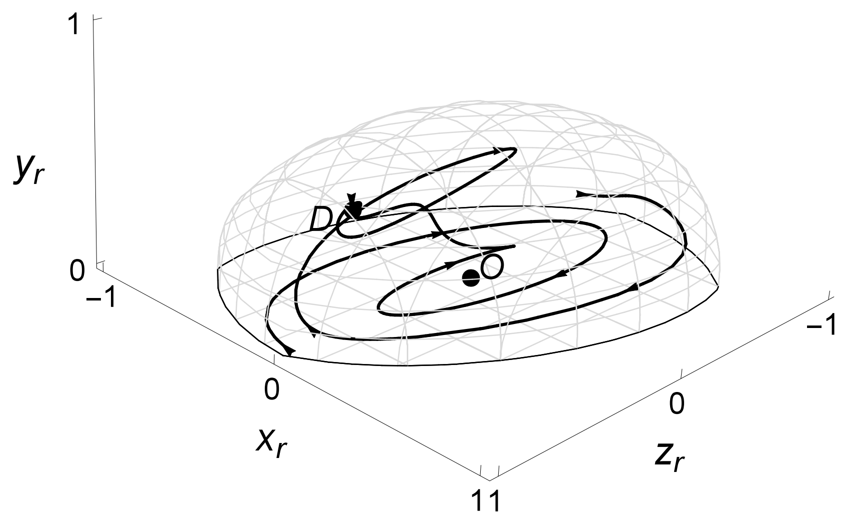

Figure 7.2: Phase space showing trajectories of the dynamical system (7.23)-(7.25) in Poincaré variables when $\chi = 1, \lambda = 2$ and $w = 0$. Point D is the global attractor. The phase space is an ellipsoid intersecting the Poincaré sphere.

The model has a further critical point at D. The point D exists only for $\chi \neq 0$, and so it is unique to this model, although its coordinates are independent of χ. At this point the energy density from the kinetic energy of the scalar field vanishes, but it has both contributions from both the matter sector and the potential energy of

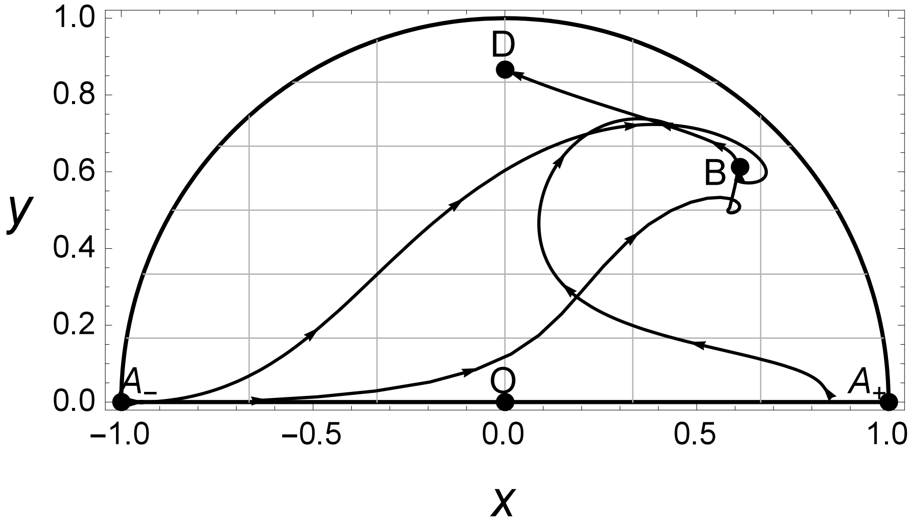

Figure 7.3: Phase space showing trajectories of the dynamical system (7.23)-(7.25) projected onto the $x - y$ plane when $\chi = 10^{-3}, \lambda = 2$ and $w = 0$. The points $A_\pm$ and B are quasi-stationary. Point D is again the global attractor.

the scalar field. It has an effective equation of state $w_{\text{eff}} = -1$ independent of the values of χ and λ. For positive χ this point is always a stable spiral, independent of λ and hence it will always describe a late time accelerating attractor solution without requiring any fine tuning.

In Fig. 7.2 we display some typical trajectories in the three dimensional Poincaré phase space for the particular parameter choice $\lambda = 2$ and $\chi = 1$. The boundary of the phase space is given in Poincaré coordinates by the intersection of (7.41) and (7.42). Trajectories can pass close to the matter dominate origin O, before all trajectories end at the late time accelerating point D.

In Fig. 7.3 we display a two dimensional projection onto the $x - y$ plane for the phase space when the parameter values are $\lambda = 2$ and χ is chosen so that it is close to zero, $\chi = 10^{-3}$. In this case the critical points of standard quintessence, points $A_\pm$, B and C behave as quasi-stationary points. Trajectories are still attracted close to these points. In the plot shown, trajectories start near the early time unstable points $A_\pm$ and are drawn towards the quasi scaling solution B. However, B is no longer a true critical point and so trajectories then travel to the stable global attractor D, which is not present in standard quintessence.

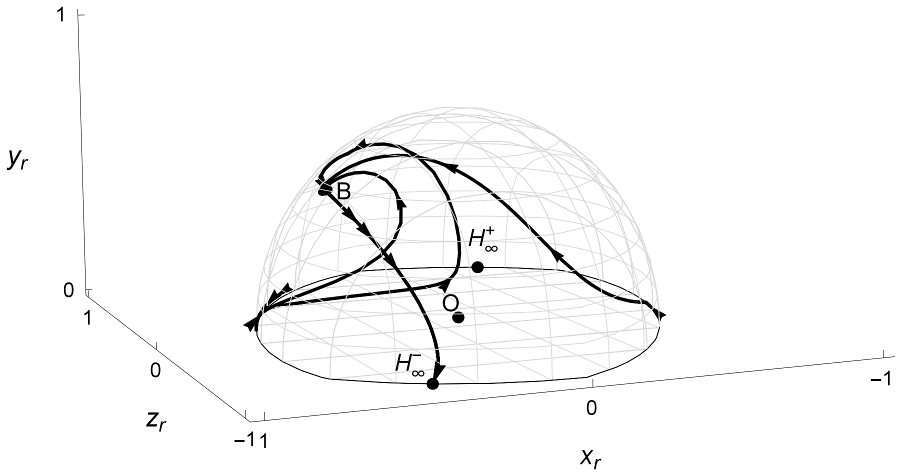

Figure 7.4: Phase space showing trajectories of the dynamical system (7.23)-(7.25) when $\chi = -10^{-3}, \lambda = 2$ and $w = 0$. Trajectories end at unphysical critical points lying at infinity.

The dynamics of the system are less interesting from a cosmological point of view when one considers a negative coupling χ. In this case the point D is no longer a global attractor, and trajectories are instead drawn towards the unphysical critical points at infinity. Such a scenario is displayed in Fig 7.4. The points $H_\infty^\pm$ lie on the boundary of the Poincaré sphere, and these are the mathematical critical points at infinity corresponding to the fixed point of equation (7.49). Trajectories move towards the quasi-critical point B before ending at one of these points at infinity, but as already noted above they are unphysical since both Ω_m and Ω_ϕ are divergent at these points. Such models are therefore not physically viable.

To summarise this model, it is found that for a positive coupling, the system generically evolves to a late time dark energy dominated attractor, whose effective equation of state is exactly -1. This is independent of the potential, and thus requires absolutely no tuning of the potential to achieve this. Moreover while the system is evolving close to this late time attractor, the phantom barrier can indeed be crossed, a scenario impossible without the presence of the coupling. We display a plot typical of such behaviour in Fig. 7.5. It is seen that the effective equation of state

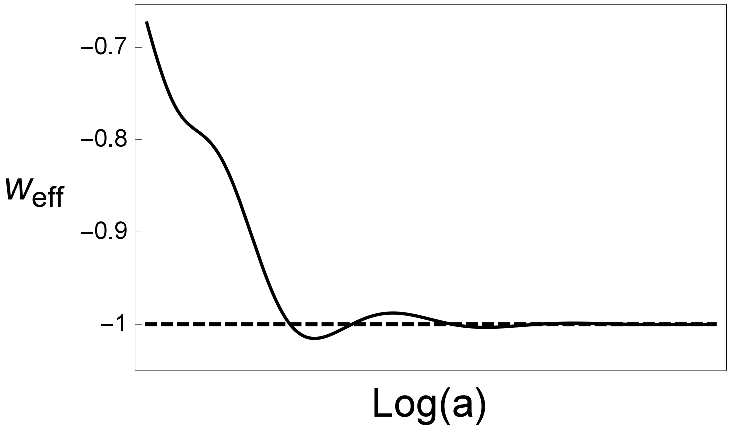

Figure 7.5: Plot of $w_{\rm eff}$ against N for a typical trajectory when the parameters $\lambda = 2$, $\chi = 1$ are chosen. The dashed line indicates the phantom barrier.

can cross the phantom barrier, and indeed cross from both directions, oscillating around the barrier before settling at its final late time de Sitter type expansion.

The global dynamics of these models are simpler than the case of the nonminimally coupled torsion scenario. There are fewer critical points, and there are no longer any physical critical points at infinity. Teleparallel dark energy also possesses saddle points describing an accelerating universe and hence can exhibit transient periods of inflation. Such a scenario is not possible in this model as we have no accelerating saddle points.

7.4.3 Cosmological constant potential $\lambda = 0$

The case when the parameter $\lambda = 0$ is relatively simple to analyse, and corresponds to the potential just taking the form of a cosmological constant.

The only critical points existing in this scenario are the points O, E and $H_\pm$, with $H_\pm$ existing only if $\xi < 0$. In fact the point E exists *only* when the parameter $\lambda = 0$, meaning that the potential is simply a constant. This point is entirely dominated by the potential term, and the dynamics at this point simply corresponds to standard de Sitter type expansion, with the potential behaving exactly as a cosmological constant.

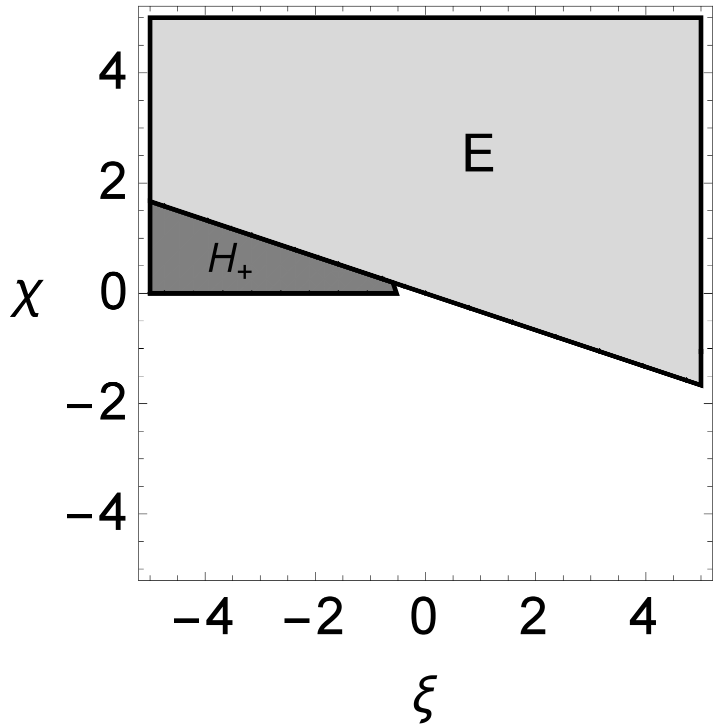

Figure 7.6: Region plot of $\chi - \xi$ parameter space, showing when the critical points E and $H_\pm$ are the late time accelerating attractor solutions when $\lambda = 0$, that is when the potential takes the form of a cosmological constant.

The point E is the late time attractor of the system, with all its eigenvalues entirely negative, provided $\xi + 3\chi > 0$. Otherwise the point is a saddle. Alternatively, if $\xi < 0$, $\chi > 0$ and $2\xi + 3\chi > 0$, then the point $H_\pm$ is a stable late time attractor. We can see in Figure 7.6 the different regions of parameter space where E and $H_\pm$ are the stable solutions. We observe that they can never be stable simultaneously. It is interesting to note that when $H_\pm$ is the stable attractor, this coincides with the region in which it describes an accelerating universe: in fact it generically describes a phantom acceleration, with $w_{\text{eff}} < -1$ exactly when it is stable.

In Figure 7.7 we see plots of a few trajectories in phase space of this $\lambda = 0$ model for the particular choice of parameters $\xi = -5/4$ and $\chi = -1/\sqrt{6}$. With this choice of parameters, the effective EoS of the point H_+ lies just inside the phantom zone, with $w_{\text{eff}} = -1.06$. We see that the phase space is described by an upper

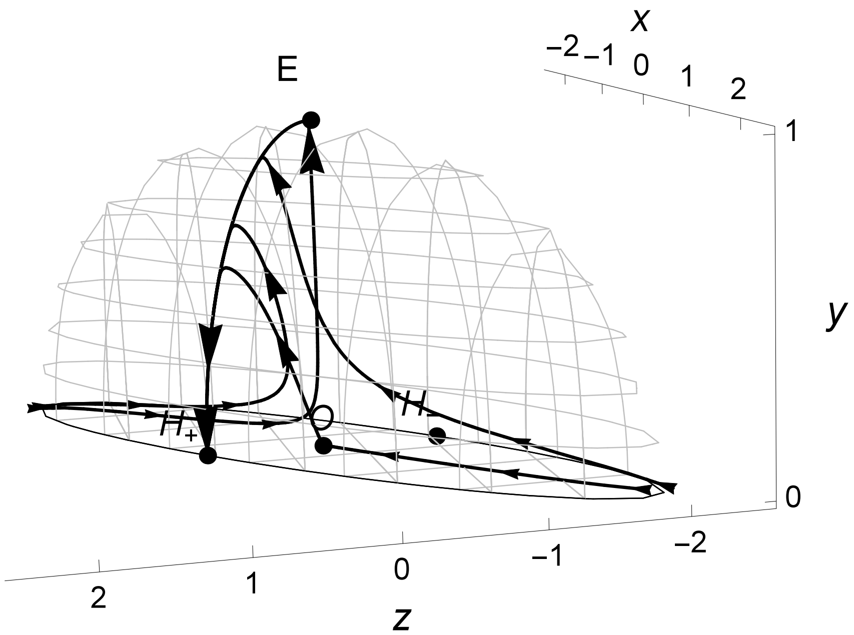

Figure 7.7: Phase space plot of trajectories in the $\lambda = 0$ model, for a particular choice of parameters $\xi = -5/4$ and $\chi = 1/\sqrt{6}$.

half ellipsoid. Many trajectories experience an era of matter domination, before undergoing an era of cosmological constant type acceleration, before settling down at the late time attractor, which is slightly in the phantom region. This behaviour is potentially very interesting from a phenomenological point of view.

7.4.4 Full nonminimal coupling: $\xi \neq 0$ and $\chi \neq 0$

Now we analyse the case where both of the coupling constant ξ and χ are non-zero. This generic case has not been analysed previously in the literature.

Let us first analyse the case when the phase space of the system is compact, which we recall is when the coupling parameters satisfy the condition $6\chi^2 + \xi < 0$ (which immediately implies that the constant $\xi < 0$). This means we can display the phase space of the system directly, without the need to transform to Poincaré variables. In this scenario, the points O, G and $H_\pm$ are the critical points of the system. In this case, we find that generically only the points $H_\pm$ are the late time

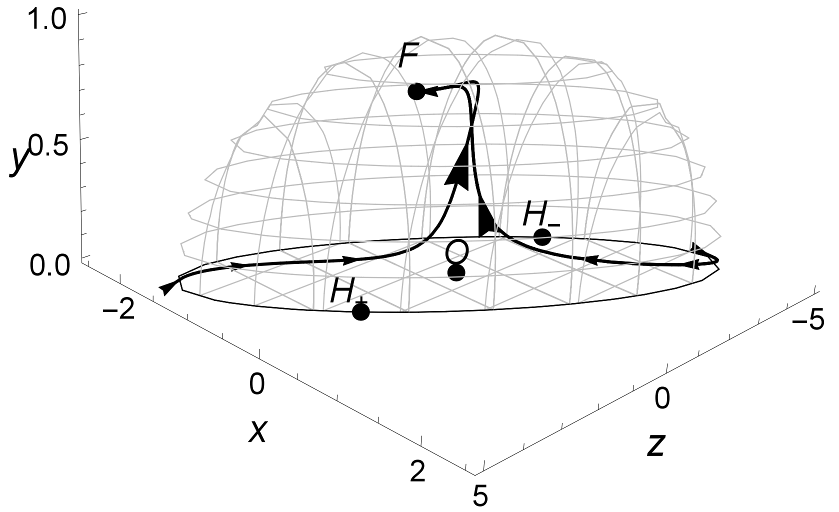

Figure 7.8: Phase space plot of trajectories, for a particular choice of parameters $\lambda = 1$, $\xi = -9/64$ and $\chi = 1/8$.

attractor solutions. In this case, phase diagrams are very similar to the diagram 7.7, where we had a cosmological constant potential, with the only difference being the point E is replace by the point F, which in both cases lie on the boundary near the top of the phase space, acting like saddle points. In fact the point E agrees with the point F in the limit $\lambda \to 0$, so this behaviour makes sense.

However, the point F can be fine tuned so that it becomes stable: this occurs in the range of parameter values when $0 < \chi < 1/4$ and $-3\chi/2 < \xi < -6\chi^2$. We have displayed the phase space and some trajectories for this particular scenario in Figure 7.8 when the parameter choice $\lambda = 1$, $\xi = -9/64$ and $\chi = 1/8$ was made. We see again the phase space is an upper half ellipsoid, this time it is slightly more stretched. We see that the trajectories undergo a period of matter domination before ending at the late time attractor F, which is an accelerating solution. However, we emphasise that to get this behaviour requires a fair degree of fine tuning, and this behaviour is not typical.

Now let us take a look at what happens when the phase space is non-compact,

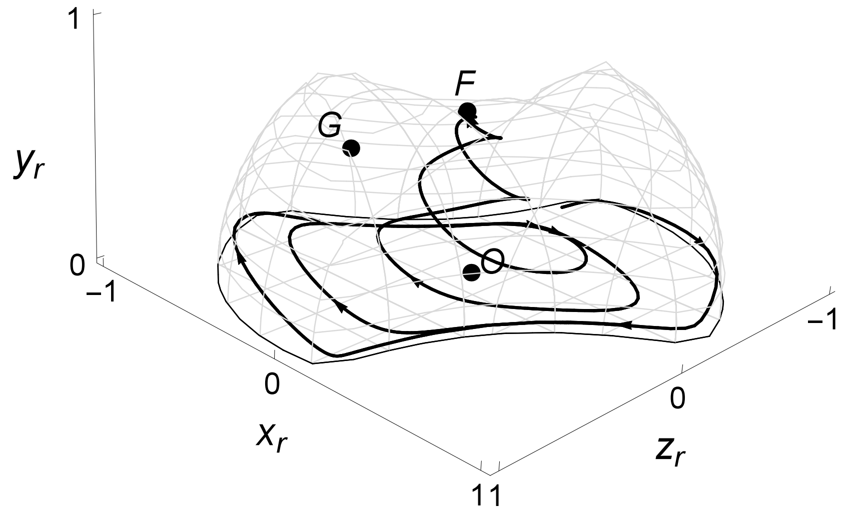

Figure 7.9: Phase space plot in Poincaré coordinates, for a particular choice of parameters $\lambda = 1$, $\xi = 1$ and $\chi = 1$.

that is when $\xi + 6\chi^2 > 0$. We can essentially break this down into two scenarios: when $\xi < 0$ we have the critical points O, G and $H_\pm$ present in the system, whereas when $\xi > 0$ we have the points O, F and G. On top of this we potentially have critical points lying at infinity.

The complicated analytic nature of the points F and G means that unfortunately it is hard to say anything too detailed about exactly what is going on in this model for any particular choice of parameters χ and ξ. Let us take a look at an example when $\xi = 1$ and $\chi = 1$. We have displayed a phase space plot in Poincaré coordinates in Figure 7.9. We see that trajectories start on the boundary of the phase space, before undergoing a period of matter domination, ending at the accelerating critical point F on the boundary of the phase space. This behaviour is very similar to the boundary term only coupling model $\xi = 0$, and it seems like the addition of the torsion coupling term does not change the phenomenology of what can happen in these models dramatically. However, despite this let us next discuss one of the parameter choices in a little more detail.

7.4.5 Ricci scalar coupling $\chi = -\xi$

Having a non-zero χ and ξ contains one particularly physically interesting model, that is when the parameters are chosen so that there is a nonminimal coupling present between the Ricci scalar and the scalar field. Because of the relation $R = -T + B$ this occurs when $\chi = -\xi$. The dynamical analysis of our previous discussion is completely applicable to this case, however due to this models physical importance, we will discuss a few of the key features of the dynamics of this model.

The compactness condition becomes $\xi + 6\chi^2 = \chi(6\chi - 1) < 0$. And thus if the coupling constant $0 < \chi < 1/6$ the phase space is compact, whereas if not the phase space is not compact. An interesting observation is that if $\chi = 1/6$, the phase space becomes an infinite cylinder. But it is this exact choice of coupling which renders the scalar field equation conformally invariant, see for example [127]. In this model, the points O, F, G and $H_{\pm}$ are the potential critical points of the system. The points $H_{\pm}$ are always stiff matter states, with $w_{\text{eff}} = 1$, in this scenario, and so are not of immediate physical interest to cosmological applications. Additionally they only exist if $\xi < 0$ and are always found to be saddle points. The point F is generically found to be the late time accelerating attractor solution for a positive ξ.

We note that the dynamics of this model are easier to study after a conformal transformation has been applied. With the use of the conformal transformation, as we shall see in the subsequent chapter, one can transform this theory into a standard nonminimally coupled quintessence model, but one with an interaction term between dark energy and matter, see for example [74]. The dynamics of that model are easily understood, and one can then transform these dynamics back to the nonminimally coupled frame (the Jordan frame) to get a full understanding of the system[1].

[1]However, a dynamical analysis of this model has been completed in the Jordan frame as well. This can be found in [140].

7.5 Discussion

In this chapter we introduced a model where a nonminimal coupling of a scalar field to both the torsion scalar T and the boundary term B. This model incorporates both nonminimal coupling to the Ricci scalar and a nonminimal coupling to the torsion scalar in suitable limits, unifying both approaches into one general framework. The cosmological dynamics of these models were investigated, with a detailed breakdown into a range of different sub-cases and we found a variety of interesting cosmological behaviour. We found that we could have a generic evolution to a late time accelerating attractor solution without the need for a great deal of fine tuning: an addition of a coupling to a boundary term generically led to late time accelerating attractors. Additionally, we saw that for many of the models considered a dynamical crossing of the phantom barrier was shown to be possible, a scenario not possible with generic quintessence models.

In this chapter we focused on exploring the background cosmology of these models. In future work the cosmological perturbations should also be investigated, along the lines of [122], which examined perturbation theory in the $\chi = 0$ case. Also of interest would be to analyse the Einstein static universe and stability issues of these general models, in line with [141]. We should also note that all the models analysed in this chapter used an exponential potential. It would be interesting to go beyond this potential, analysing the model with either a different form such as a power law, or attempting to analyse the dynamical system in general without restricting ourselves to any potential in particular. Also of interest would be to look at the existence of scaling solution in these models: these are of great interest in attempts to solve the cosmic coincidence problem.

In the next chapter, we will see how these scalar field models are related to the modified gravity models discussed in Chapter 6. Thus an understanding of the cosmological dynamics in this chapter will allow us to understand the cosmological dynamics of $f(T, B)$ gravity.

Chapter 8

Conformally equivalent theories of gravity

In the previous two chapters we have introduced two different general modifications in the teleparallel framework: first we looked at $f(T,B)$ gravity and then we looked at a teleparallel "scalar-tensor" type gravity, where a nonminimal coupling between a scalar field and the torsion scalars T and B was present. In this chapter, after an introduction to conformal transformations in a variety of different theories, we will show that $f(T,B)$ gravity and nonminimally coupled torsion models are in fact physically related to each other in an interesting way. This chapter is largely based on the work done in [3].

Theories of gravitation are dynamically equivalent if one can map one theory to another by applying a conformal transformation to the metric. Conformal symmetry is a fundamental symmetry of spacetime and generalises the concept of scale invariance. Within the curvature formulation of gravity, this has been used to show that $f(R)$ gravity, introduced in Chapter 5, is dynamically equivalent to just general relativity with an additional matter field present, taking the form of a canonical scalar field. This conformal transformation can be chosen so that this scalar field is minimally coupled to the gravitational sector, meaning there are no coupling $A(\phi)R$ terms present within the action (where $A(\phi)$ is some generic function of the scalar field ϕ). This transformed theory is known as the *Einstein frame* of $f(R)$ gravity, whereas the original action is referred to as the *Jordan frame*. There has been much debate over the years about '*Which frame is the true physical frame?*' [142–144]. In fact, this choice of physical frame depends solely on the definition of clocks and rulers.

This physical equivalence between the Jordan and the Einstein frame is often exploited to great power. It allows one to choose which frame to perform calculations in, or derive physical predictions, based on which frame it is simpler to do so. The Jordan frame $f(R)$ gravity has field equations which are fourth order, whereas the Einstein frame has only second order field equations, and so in $f(R)$ gravity it is often simpler to work in the Einstein frame. There is also an interesting conformal relationship between phantom scalar field theories and $f(R)$ gravity. Recall phantom theories are those in which the kinetic energy term of the scalar field has the incorrect sign. In such a case, the phantom theory is generically conformally equivalent to a complex $f(R)$ gravity, that is the function $f(R)$ has a non-trivial imaginary part [145].

The purpose of this chapter is to consider the conformal equivalence between different modifications in the teleparallel framework. $f(T)$ gravity, unlike $f(R)$ gravity, is known not to be equivalent to Einstein gravity with a minimally coupled scalar field [146], a result we will review in this chapter. The transformation results in an additional torsion scalar field coupling being present. In fact we will show that this additional coupling can take the interpretation of a nonminimal coupling between the scalar field and the boundary term B, just like the model studied in the previous chapter. However, as observed in [103], the kinetic energy of this nonminimally coupled scalar field has the incorrect sign, that is it is of the phantom form, which can lead to instabilities at the level of perturbations.

Furthermore, we look at the conformal behaviour of coupled $A(\phi)T$ theories and show that they can be mapped to a particular class of $f(T,B)$ theories of gravity. We then start with an $f(T,B)$ gravity, and examine conditions on the functional form of f required to map to particular types of minimial and nonminimally coupled scalar field theories. We also look at a particular subset of these models, which we call $f(B)$ gravity, where $f(T,B) = -T + f(B)$, and are able to show that these are conformally equivalent to a teleparallel dark energy theory, where only a nonminimal coupling between a scalar field and the torsion scalar T is present.

8.1 Conformal transformations

In this section we will introduce the concept of a conformal transformation. A conformal transformation $g \to \hat{g}$ is simply a rescaling of the metric of our space. We multiply the metric by a scalar field Ω which is dependent on the spacetime coordinates x^μ, called the conformal factor, as so

$$\hat{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad \hat{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}. \tag{8.1}$$

Ω is required to be real, meaning that the conformal transformation is positive definite. Such a transformation preserves the causal structure of the spacetime: null vectors in one frame will remain null in another. Hence theories related to each other via a conformal transformation are dynamically equivalent, with the only difference between the two theories being how one measures time and distance.

Equivalently we can define the conformal transformation acting on the tetrad. Under the conformal transformation (8.1), it is easy to see that the tetrad and the inverse tetrad must transform as [127, 146, 147]

$$\hat{e}^a_\mu = \Omega(x) e^a_\mu, \quad \hat{e}_a^\mu = \Omega^{-1}(x) e_a^\mu. \tag{8.2}$$

We also have that the volume element e, or equivalently $\sqrt{-g}$ of our gravitational actions transforms under this rescaling of the metric as

$$\hat{e} = \Omega^4 e. \tag{8.3}$$

This concept of a conformal transformation is valid in a general metric affine linear space L_4. In such a space, where there are no geometric relations between the metric and connection, the transformation of the connection under a conformal rescaling must be specified independently. Only if we impose some additional structure does an explicit transformation law for the connection become a priori specified. In a metric affine space, it is usually proposed that the connection remains unchanged under a conformal transformation [147, 148], that is it is conformally invariant, with

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma}. \tag{8.4}$$

This is required in for example, Poincare gravity, taking place in the Riemann-Cartan spacetime U_4 where the connection's canonical dimension coincides with the electromagnetic and Yang-Mills potentials, which are conformally invariant.

However, as this transformation law for the connection is independent of the metric transformation, one could consider other types of transformation being possible. For example, in [149] the authors considered both additive and multiplicative transformation laws for metric affine gravity of the form

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = f(\Omega)\Gamma^{\alpha}_{\beta\gamma} + \Delta(\Omega)^{\alpha}_{\beta\gamma}. \tag{8.5}$$

The authors of this work were able to show that considering such a transformation they were able to eliminate some of the issues present in conformal general relativity, such as the existence of ghost modes. However, if we work in either V_4 or W_4 space, we will see that the connection transformation law becomes automatically specified.

8.1.1 Conformal transformations in V_4 space

Let us consider now conformal transformations in the restricted setting of V_4 space, that is the geometrical setting of general relativity. In this case, we are working with a pure metric theory, with the connection determined solely by the metric in terms of the Levi-Civita connection.

Thus we can use the transformation law of the metric into the definition of the Levi-Civita connection, to obtain the following transformation law for the connection. The Levi-Civita connection, $\Gamma \to \hat{\Gamma}$ becomes

$$\hat{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \Omega^{-1}\left(\delta^{\alpha}_{\beta}\nabla_{\gamma}\Omega + \delta^{\alpha}_{\gamma}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega\right). \tag{8.6}$$

Writing out the definition of the curvature tensor in terms of this connection then induces the Riemann tensor transformation law [150]

$$\begin{aligned}\hat{R}_{\alpha\beta\gamma}{}^{\delta} = R_{\alpha\beta\gamma}{}^{\delta} + 2\delta^{\delta}_{[\alpha}\nabla_{\beta]}\nabla_{\gamma}(\ln\Omega) - 2g^{\delta\sigma}g_{\gamma[\alpha}\nabla_{\beta]}\nabla_{\sigma}(\ln\Omega) + 2\nabla_{[\alpha}(\ln\Omega)\delta^{\delta}_{\beta]}\nabla_{\gamma}(\ln\Omega)\\ -2\nabla_{[\alpha}(\ln\Omega)g_{\beta]\gamma}g^{\delta\sigma}\nabla_{\sigma}(\ln\Omega) - 2g_{\gamma[\alpha}\delta^{\delta}_{\beta]}g^{\sigma\rho}\nabla_{\sigma}(\ln\Omega)\nabla_{\rho}(\ln\Omega),\end{aligned} \tag{8.7}$$

while the Ricci tensor transforms according to

$$\begin{aligned}\hat{R}_{\alpha\beta} = R_{\alpha\beta} - 2\nabla_{\alpha}\nabla_{\beta}(\ln\Omega) - g_{\alpha\beta}g^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma}(\ln\Omega) + 2\nabla_{\alpha}(\ln\Omega)\nabla_{\beta}(\ln\Omega)\\ -2g_{\alpha\beta}\,g^{\rho\sigma}\nabla_{\rho}(\ln\Omega)\nabla_{\sigma}(\ln\Omega).\end{aligned} \tag{8.8}$$

Finally, contracting this, simplifying and inverting, it is seen that the conformal transformation modifies the Ricci scalar $R \to \hat{R}$, giving

$$R = \Omega^2 \hat{R} - 12\hat{\partial}^\mu \Omega \hat{\partial}_\mu \Omega + 6\Omega \Box \Omega. \tag{8.9}$$

Thus we can see that the conformal transformation determines uniquely the transformation laws of the other tensors defined on our spacetime.

8.1.2 Conformal transformations in Weitzenböck space

Now let us consider what happens in the situation when torsion is present. In a Riemann-Cartan spacetime an additional geometric structure is imposed which determines the transformation law of the connection and torsion tensor under a conformal rescaling [147].

Teleparallel gravity is a special case of the Riemann-Cartan space, thus under such a conformal transformation, it is easily calculated that the torsion tensor transforms as [146, 147]

$$\hat{T}^\rho{}_{\mu\nu} = T^\rho{}_{\mu\nu} + \Omega^{-1}(\delta^\rho_\nu \partial_\mu \Omega - \delta^\rho_\mu \partial_\nu \Omega), \tag{8.10}$$

which can be derived by writing out the torsion tensor in terms of the tetrad. A similar calculation shows that the Weitzenböck connection transforms as

$$\hat{\Gamma^\mu_{\nu\rho}} = \Gamma^\mu_{\nu\rho} - \Omega^{-1}\delta^\mu_\rho \partial_\nu \Omega. \tag{8.11}$$

Using the relation between contortion and the torsion tensor (2.29), we can derive that the contortion tensor transforms as

$$\hat{K}^{\mu\nu}{}_\rho = \Omega^{-2} K^{\mu\nu}{}_\rho + \Omega^{-3}(\delta^\nu_\rho \partial^\mu \Omega - \delta^\mu_\rho \partial^\nu \Omega). \tag{8.12}$$

Together these imply that the superpotential transforms as

$$\hat{S}_{\rho}{}^{\mu\nu} = \Omega^{-2} S_{\rho}{}^{\mu\nu} + \Omega^{-3}(\delta^{\mu}_{\rho}\partial^{\nu}\Omega - \delta^{\nu}_{\rho}\partial^{\mu}\Omega). \tag{8.13}$$

Contracting these transformations, this allows one to calculate how the torsion scalar transforms

$$\hat{T} = \Omega^{-2} T + 4\Omega^{-3} g^{\mu\nu}\partial_{\nu}\Omega T^{\rho}{}_{\rho\mu} - 6\Omega^{-4} g^{\mu\nu}\partial_{\mu}\Omega\partial_{\nu}\Omega, \tag{8.14}$$

from which the inverse transformation can be derived

$$T = \Omega^2 \hat{T} - 4\Omega\hat{g}^{\mu\nu}\partial_{\mu}\Omega\hat{T}_{\nu} - 6\hat{g}^{\mu\nu}\partial_{\mu}\Omega\partial_{\nu}\Omega. \tag{8.15}$$

where we note that partial derivatives remain unchanged under conformal transformations, so that $\partial_{\mu} = \hat{\partial}_{\mu}$. Finally for the purposes of this chapter, we need to investigate how the boundary term B changes under a conformal transformation. We find

$$B = \frac{2}{e}\partial_{\mu}(eT^{\mu}) = \frac{2\Omega^4}{\hat{e}}\partial_{\mu}(\hat{e}(\Omega^{-2}\hat{T}^{\mu} + 3\Omega^{-3}\hat{\partial}^{\mu}\Omega)), \tag{8.16}$$

where, by contracting (8.10) we have used that the vector T^{μ} transforms as

$$T^{\mu} = \Omega^2\hat{T}^{\mu} + 3\Omega\hat{\partial}^{\mu}\Omega. \tag{8.17}$$

Expanding out the partial derivative, this means that B transforms as follows

$$\begin{aligned} B &= \Omega^2\hat{B} - 4\Omega\hat{T}^{\mu}\hat{\partial}_{\mu}\Omega - 18\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}\Omega + \frac{6}{\hat{e}}\Omega\hat{\partial}_{\mu}(\hat{e}\hat{g}^{\mu\nu}\hat{\partial}_{\nu}\Omega) \\ &= \Omega^2\hat{B} - 4\Omega\hat{T}^{\mu}\hat{\partial}_{\mu}\Omega - 18\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}\Omega + 6\Omega\hat{\Box}\Omega. \end{aligned} \tag{8.18}$$

As a consistency check, we also note that the combination $-T+B$ transforms as

$$-T+B=\Omega^2(-\hat{T}+\hat{B})-12\hat{\partial}^\mu\Omega\hat{\partial}_\mu\Omega+6\Omega\Box\Omega \tag{8.19}$$

which using the relation $R=-T+B$, gives the correct transformation law for the Ricci scalar (8.9).

As an aside, we note that in [127] it was observed that the tensor defined as

$$C^\rho{}_{\mu\nu}=T^\rho{}_{\mu\nu}+S^\rho{}_{\mu\nu} \tag{8.20}$$

is conformally invariant, that is under a conformal transformation it remains the same

$$\hat{C}^\rho{}_{\mu\nu}=C^\rho{}_{\mu\nu}. \tag{8.21}$$

This can henceforth be thought of as the teleparallel equivalent of the Weyl tensor of general relativity. There is a modified theory of gravity known as conformal gravity, where the action of the theory is constructed out of the square of the Weyl tensor. This tensor $C^\rho{}_{\mu\nu}$ can be used to construct a teleparallel version of this conformal gravity, by considering an action based on the square of this tensor, and it would be interesting to study the properties of this theory - this has yet to be done.

8.2 f(R) gravity transformed

In this section we will analyse conformal relations between modified theories of gravity in the geometric setting of a V_4 space, reviewing the relationship between the Einstein and the Jordan frames of $f(R)$ gravity.

Let us start with the following generic real $f(R)$ theory of gravity, which has the following action

$$S_{f(R)} = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\,d^4x\,. \tag{8.22}$$

The first step is to now introduce two new auxiliary fields, denoted by ϕ and χ. This allows us to consider the equivalent action

$$S = \frac{1}{16\pi G}\int \left[\chi(R-\phi) + f(\phi)\right]\sqrt{-g}\,d^4x\,. \tag{8.23}$$

To see that this action is indeed equivalent to (8.22), we observe that varying this action with respect to the auxiliary field χ gives the equation of motion $\phi = R$, and inserting this solution back into the action (8.23) (so that the action is on shell), we recover the $f(R)$ gravity action (8.22).

Instead however, if we first vary the action (8.23) with respect to the other auxiliary field ϕ, we find that $\chi = f'(\phi)$ and so we can eliminate χ. We therefore arrive at the following equivalent action

$$S = \frac{1}{16\pi G}\int \left[f'(\phi)(R-\phi) + f(\phi)\right]\sqrt{-g}\,d^4x\,. \tag{8.24}$$

This action is a particular type of scalar tensor gravity, see Chapter 5, with a nonminimal coupling between the Ricci tensor and the scalar field. However, at the moment the scalar field is non-dynamical, since it possesses no kinetic term.

Now we will apply a conformal transformation to the metric to transform to a minimally coupled scalar frame. In this case we will choose the particular conformal factor

$$\hat{g}_{\mu\nu} = f'(\phi)g_{\mu\nu}. \tag{8.25}$$

Now using the conformal transformation laws in the action (8.24), and defining a new scalar field σ as the following function of ϕ

$$\sigma = \sqrt{3}\ln f'(\phi), \tag{8.26}$$

the action (8.24) takes the form

$$S = \int \frac{1}{16\pi G}\left[\hat{R} - \frac{1}{2}\hat{g}^{\rho\nu}\partial_\rho\sigma\partial_\nu\sigma - V(\sigma)\right]\sqrt{-\hat{g}}\, d^4x\,. \tag{8.27}$$

The potential of the scalar field $V(\sigma)$ has been defined as

$$V(\sigma) = \frac{\phi}{f'(\phi)} - \frac{f(\phi)}{f'(\phi)^2}, \tag{8.28}$$

which can be rewritten in terms of σ by inverting the relation (8.26). This frame is referred to as the *Einstein frame*, where the theory takes the form of a scalar tensor theory in which the gravitational sector and the scalar field are minimally coupled and the action of the scalar field is of the canonical form.

Let us give an explicit example of how we can transform a particular $f(R)$ gravity to a scalar tensor theory, and observe how physical quantities become modified. We also observe how cosmological singluarities can change their structure, which serves as a warning that care must be taken in some circumstances when conformally transforming between frames. We use an example which was considered in detail in both [145, 151], when $f(R) = R^{-n}$ in the Jordan frame.

Let us consider the vacuum cosmology of this model, assuming a flat FRW metric. With such a form of $f(R)$, it can be seen from the cosmological field equations of the theory that the corresponding scale factor solution behaves as a power law, taking the form

$$a \sim (t_0 - t)^{\frac{(n+1)(2n+1)}{n+2}}\,. \tag{8.29}$$

Therefore, we can see that if either $n < -2$ or $-1 < n < -1/2$, a Big Rip singularity appears at the time instance $t = t_0$ in the Jordan frame with the scale factor diverging, whereas if these conditions do not hold a Big Crunch singularity, with the scale factor vanishing, is present at this point.

Let us see how the cosmology changes if one transforms to the Einstein frame. The Einstein frame canonical scalar field reads,

$$\sigma \sim (n+1)\ln R \sim -2(n+1)\ln(t_0 - t) \ , \tag{8.30}$$

where we used the fact that the Ricci scalar of the power law FRW scale factor (8.29) reads, up to a proportionality constant,

$$R \sim \frac{6(n+1)(2n+1)(4n+5)n}{(n+2)^2(t_0-t)^2} \ . \tag{8.31}$$

In the corresponding scalar-tensor theory, the time coordinate $\hat{t}$ is given by,

$$d\hat{t} = \pm e^{\frac{1}{2}\sigma} dt \sim \pm (t_0 - t)^{-(n+1)} dt \, , \tag{8.32}$$

and consequently, we have $\hat{t} = \pm (t_0 - t)^{-n}$. Therefore, in the case that $n > 0$, when t approaches $t \to t_0$ in the Jordan frame, this corresponds to $\hat{t} \to \pm\infty$ in the Einstein frame. As a consequence, the singularity changes its structure, since it does not appear in finite time for the scalar-tensor theory. However a new additional singularity may be present, as when t approaches infinity in the Einstein frame, it corresponds to the new time coordinate $\hat{t} \to 0$, and thus any singularities at infinity can be brought towards a finite time. On the other hand, when $n < 0$, the limit $t \to t_0$ in the Jordan frame corresponds to $t \to 0$ in the Einstein frame. We also find that the metric in the scalar-tensor theory behaves as

$$ds_{ST}^2 = e^{\sigma}\left(-dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2\right)$$
$$\sim -d\hat{t}^2 + \hat{a}(\hat{t})^2 \sum_{i=1,2,3} (dx^i)^2 \,, \quad \hat{a}(\hat{t})^2 \sim a_0^2 \hat{t}^{\frac{2(n^2-1)}{n(n+2)}} \,, \tag{8.33}$$

where the constant a_0 is an arbitrary parameter. In this case the power of the scale factor is negative only when $-2 < n < -1$ or $0 < n < 1$, and thus a Big Rip singularity becomes present then. Thus for the Big Rip singularity in the Jordan frame, the scale factor now behaves as $\hat{a}(\hat{t})^2 \to 0$ when $\hat{t} \to 0$, in the Jordan frame it becomes a Big Crunch singularity.

8.3 f(T) gravity transformed

Let us now consider the equivalent situation in a W_4 space, examining what happens when one attempts to conformally transform $f(T)$ gravity to a scalar frame. To do this we follow the approach used in [103, 146]. Let us start with the following $f(T)$ action

$$S_{f(T)} = -\frac{1}{16\pi G}\int f(T) e \, d^4x \,. \tag{8.34}$$

Following the same procedure as the case of $f(R)$ gravity, we can introduce two auxiliary fields χ and ϕ such that the $f(T)$ action takes the following form

$$S = -\frac{1}{16\pi G}\int [\chi(T-\phi) + f(\phi)] \, e \, d^4x \,. \tag{8.35}$$

Varying this action with respect to χ yields $\phi = T$ showing that the action is indeed equivalent to the $f(T)$ action, unless $f''(T) \equiv 0$, in which case we are already working in Einstein gravity. Instead varying with respect to ϕ then yields $\chi = f'(\phi)$.

Now introducing the notation $F(\phi) = f'(\phi)$, we can recast the theory into the following scalar-tensor type theory

$$S = \frac{1}{16\pi G} \int \left[-F(\phi)T - \omega(\phi) g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] e \, d^4x \, . \tag{8.36}$$

In this particular case the kinetic term coefficient $\omega(\phi)$ is identically zero, $\omega(\phi) = 0$, and the scalar field potential V is given by

$$V(\phi) = f(\phi) - \phi f'(\phi). \tag{8.37}$$

Now let us apply a general conformal transformation as outlined above (8.1). Then the action (8.36) transforms to the following

$$\begin{aligned} S = \frac{1}{16\pi G} \int \Big[&- F(\phi)(\Omega^{-2}\hat{T} - 4\Omega^{-3}\hat{g}^{\mu\nu}\partial_\mu\Omega\hat{T}_\nu - 6\Omega^{-4}\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega) \\ &- \Omega^{-4}V(\phi)\Big] \hat{e} \, d^4x \, . \end{aligned} \tag{8.38}$$

In order for the coupling between the gravitational sector and the scalar field to be minimal, we need to choose the conformal factor to be

$$\Omega^2 = F(\phi). \tag{8.39}$$

This means the action becomes

$$\begin{aligned} S = \frac{1}{16\pi G} \int \Big[&(-\hat{T} + 2F(\phi)^{-1}\hat{g}^{\mu\nu}\partial_\mu F(\phi)\hat{T}_\nu \\ &+ \frac{3F'(\phi)^2}{2F(\phi)^2}\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi) - F^{-2}(\phi)V(\phi)\Big] \hat{e} \, d^4x \, . \end{aligned} \tag{8.40}$$

Now in order to get the kinetic term to be of the correct form, we define a new scalar field implicitly by

$$\frac{d\psi}{d\phi} = \sqrt{3}\frac{F'(\phi)}{F(\phi)}, \tag{8.41}$$

which can be solved for ψ to give

$$\psi = \sqrt{3}\ln F(\phi). \tag{8.42}$$

This results in the action taking the following form

$$S = \frac{1}{16\pi G}\int\left[-\hat{T} + 2F^{-1}\hat{\partial}_\mu F\hat{T}^\mu + \frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - U(\psi)\right]\hat{e}\,d^4x\,, \tag{8.43}$$

where the new potential U is given by $U(\psi) = V(\phi)/F^2(\phi)$. We note that we have corrected a few algebraic errors which were presented in the original work of [146]. Let us examine the second term in the action (8.38). Using that

$$F^{-1}\hat{\partial}_\mu F = \hat{\partial}_\mu(\ln F), \tag{8.44}$$

we can integrate this term by parts to find that the action takes the following form

$$S = \frac{1}{16\pi G}\int\left[-\hat{T} - \ln(F)\hat{B} + \frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - U(\psi)\right]\hat{e}\,d^4x\,. \tag{8.45}$$

Now the factor in front of $\hat{B}$ in the action can be expressed in terms of ψ, and so finally the action becomes

$$S = \frac{1}{16\pi G}\int\left[-\hat{T} - \frac{\psi}{\sqrt{3}}\hat{B} + \frac{1}{2}g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi - U(\psi)\right]\hat{e}\,d^4x\,. \tag{8.46}$$

this action represents a scalar field with a linear nonminimal coupling to a boundary term. However, as has been noted in [103] the kinetic energy term has the incorrect sign, thus the action is that of a phantom field, which generically leads to instabilities at the level of perturbations. This is very similar to the model discussed in the previous chapter, where a quadratic coupling between B and the scalar field, $\psi^2 B$, was present. However, there the kinetic term was of the canonical form and thus was not subject to the instabilities that are present in $f(T)$ gravity.

8.4 Teleparallel dark energy and f(T,B) gravity

In this section we will work the other way around, that is we will start with a teleparallel scalar tensor model, and transform it into a modified gravity theory. Let us begin with the following action, where a nonminimal coupling between a scalar field and the torsion scalar is present, sometimes referred to as *teleparallel dark energy*

$$S = \frac{1}{16\pi G}\int \left[-A(\phi)T - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \mathcal{L}_m\right] e d^4x. \tag{8.47}$$

This action was first introduced in [116] for the particular choice $A(\phi) = 1 + \xi\phi^2$. This was later generalised to include a more general coupling between ϕ and T in [119, 120, 152]. Immediately from the results of the previous section, we know that such a theory cannot be conformally transformed to an $f(T)$ gravity theory as there is no coupling between ϕ and B present. However we will show in this section that it can be conformally transformed to the broader class of theories known as $f(T, B)$ gravity.

Let us apply a conformal transformation to this theory in an attempt to remove the kinetic term from this action. A general conformal transformation changes the action to the following

$$S = \frac{1}{16\pi G}\int \Omega^{-4}\Big[- A(\phi)(\Omega^2\hat{T} - 4\Omega\partial_\mu\Omega\hat{T}^\mu - 6\hat{g}^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega) \\ - \frac{1}{2}\Omega^2\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\Big]\hat{e}\, d^4x. \tag{8.48}$$

Now requiring that the kinetic term of the scalar field vanishes gives the following condition

$$A(\phi)\left(\frac{d\Omega}{d\phi}\right)^2 = \frac{1}{12}\Omega^2. \tag{8.49}$$

Solving this will enable us to choose the conformal factor in terms of ϕ

$$\Omega = \exp\left(\int \frac{1}{2\sqrt{3A(\phi)}}d\phi\right). \tag{8.50}$$

We can also formally invert this relation meaning we can write ϕ as a function of Ω, $\phi = \phi(\Omega)$. So now the action becomes

$$S = \frac{1}{16\pi G}\int \left[-A(\Omega)\Omega^{-2}\hat{T} + 4\Omega^{-3}A(\Omega)\partial_\mu\Omega\hat{T}^\mu - U(\Omega)\right]\hat{e}\, d^4x, \tag{8.51}$$

where again the new potential $U(\Omega)$ is given simply by

$$U(\Omega) = \frac{V(\phi)}{\Omega^4}. \tag{8.52}$$

It appears at this stage that the presence of $A(\Omega)$ in the second term of (8.51) ruins the possibility of this being equivalent to an $f(T, B)$ gravity. However if we introduce the function

$$G(\Omega) = \int \frac{A(\Omega)}{\Omega^3}d\Omega, \tag{8.53}$$

we can write the second term of (8.51) as

$$\Omega^{-3}A(\Omega)\partial_\mu\Omega = \partial_\mu G(\Omega). \tag{8.54}$$

Now we can integrate this term by parts so that the action takes the form

$$S = \frac{1}{16\pi G}\int \left[-A(\Omega)\Omega^{-2}\hat{T} - 2G(\Omega)\hat{B} - U(\Omega)\right] \hat{e}\, d^4x. \tag{8.55}$$

Now the scalar field Ω has no kinetic term and is just an auxiliary field. Varying the action with respect to Ω and finding its equation of motion gives

$$\frac{2A(\Omega) - \Omega A'(\Omega)}{\Omega^3}\hat{T} - \frac{2A(\Omega)}{\Omega^3}\hat{B} - U'(\Omega) = 0. \tag{8.56}$$

Now this can be formally solved to find Ω in terms of $\hat{T}$ and $\hat{B}$, $\Omega = \Omega(\hat{T}, \hat{B})$ and so the action can be written as an $f(T, B)$ theory, with the function f given by

$$f(T, B) = -A(\Omega)\Omega^{-2}\hat{T} - 2G(\Omega)\hat{B} - U(\Omega). \tag{8.57}$$

Thus we have established that a teleparallel dark energy theory with an arbitrary coupling between T and the scalar field can be written as a particular instance of $f(T, B)$ gravity. In the next section we will derive conditions on the functional form of f for such a teleparallel dark energy model.

We note there is one particular class of models for which the functional form of the $f(T, B)$ gravity will take the form

$$f(T, B) = -\alpha T + f(B), \tag{8.58}$$

so that we have an Einstein gravity plus some additional $f(B)$ contribution. This is when the coefficient of $\hat{T}$ vanishes in (8.56), so that one can invert $U'(\Omega)$ to find Ω as a function of $\hat{B}$ only. This is when the coupling function A takes the form

$$A(\Omega) = \alpha\Omega^2, \tag{8.59}$$

where α is some positive constant. Finding this originally in terms of ϕ, we derive that the coupling function is given by

$$A(\phi) = \beta^2(1 + \frac{\phi}{2\beta\sqrt{3\alpha}})^2, \tag{8.60}$$

where β is some arbitrary constant.

To conclude this section let us give an explicit toy example of a nonminimally coupled teleparallel dark energy theory and transform it to an $f(T,B)$ gravity. Let us suppose the coupling function $A(\phi)$ is given by the simple form

$$A(\phi) = \frac{\phi^2}{\sqrt{3}}. \tag{8.61}$$

This means that we can write Ω in terms of ϕ, using (8.50), simply as $\Omega = \sqrt{\phi}$. Now let us choose a simple quadratic potential so that it can easily be inverted

$$U(\phi) = m^2\phi^2 = m^2\Omega^4, \tag{8.62}$$

where m is a constant. Then solving (8.56) for Ω gives

$$\Omega = \left(\frac{T+B}{2\sqrt{3}m^2}\right)^{1/2}. \tag{8.63}$$

Inserting this back into (8.57) will give us the following functional form of $f(T,B)$

$$f(T,B) = \frac{1}{12m^2}(T+B)^2. \tag{8.64}$$

We can thus see that this teleparallel dark energy theory is conformally equivalent to a particular type of $f(T,B)$ gravity. And thus the study of $f(T,B)$ gravity is important; it is not simply a theory of purely theoretical interest.

8.5 f(T,B) gravity transformed

In this section we will explore the consequences when one conformally transforms a general $f(T,B)$ gravity to a scalar frame. We start with the gravitational sector of the $f(T,B)$ action

$$S_{\mathrm{f(T,B)}} = \frac{1}{16\pi G}\int f(T,B)e\, d^4x\,, \tag{8.65}$$

and we introduce the four auxiliary fields χ_1, χ_2, ϕ_1 and ϕ_2, writing the above action in the equivalent form

$$S = \frac{1}{16\pi G}\int \left[f(\phi_1,\phi_2) + \chi_1(T-\phi_1) + \chi_2(B-\phi_2)\right] e\, d^4x\,. \tag{8.66}$$

Varying with respect to χ_1 yields $T = \phi_1$ and with respect to χ_1 gives $B = \phi_2$. Now varying with respect to ϕ_1 and ϕ_2 gives $\chi_1 = f^{(1,0)}(\phi_1,\phi_2)$ and $\chi_2 = f^{(0,1)}$ respectively. This leaves the following action

$$S = \frac{1}{16\pi G}\int \left[f(\phi_1,\phi_2) + (T-\phi_1)f^{(1,0)}(\phi_1,\phi_2) + (B-\phi_2)f^{(0,1)}(\phi_1,\phi_2)\right] e\, d^4x\,, \tag{8.67}$$

assuming that neither of the second derivatives $f^{(2,0)}$ or $f^{(0,2)}$ vanishes (the case of $f^{(0,2)} = 0$ is equivalent to $f(T)$ gravity and was examined earlier in this chapter. We will cover the remaining case in the next section). We can rewrite this slightly differently as the following scalar tensor type action with two scalar fields

$$S = \frac{1}{16\pi G}\int \left[-F(\phi_1,\phi_2)T + G(\phi_1,\phi_2)B - V(\phi_1,\phi_2)\right] e\, d^4x\,, \tag{8.68}$$

where we introduce the notation $F(\phi_1,\phi_2) = -f^{(1,0)}(\phi_1,\phi_2)$ and $G(\phi_1,\phi_2) = f^{(0,1)}(\phi_1,\phi_2)$. The double potential $V(\phi_1,\phi_2)$ is given by

$$V(\phi_1,\phi_2) = \phi_1 f^{(1,0)}(\phi_1,\phi_2) + \phi_2 f^{(0,1)}(\phi_1,\phi_2) - f(\phi_1,\phi_2). \tag{8.69}$$

This is a particular instance of a teleparallel scalar-tensor theory with two scalar fields. Conformal transformations with multiple scalar fields have been discussed in [153,154].

Now let us apply a conformal transformation to the action (8.68). We find

$$\begin{aligned} S = \frac{1}{16\pi G}\int \Big[\, &(-\Omega^{-2}\hat{T} + 4\Omega^{-3}\hat{\partial}_\mu\Omega\hat{T}^\mu + 6\Omega^{-4}\hat{\partial}_\mu\Omega\hat{\partial}^\nu\Omega)F(\phi_1,\phi_2) \\ &+ (\Omega^{-2}\hat{B} - 4\Omega^{-3}\hat{T}^\mu\hat{\partial}_\mu\Omega + 6\Omega^{-3}\hat{\partial}^\mu\hat{\partial}_\mu\Omega - 18\Omega^{-4}\hat{\partial}^\mu\Omega\hat{\partial}_\mu\Omega \\ &+ \frac{6}{\hat{e}}\Omega^{-3}\hat{\partial}^\mu\Omega\hat{\partial}_\mu\hat{e})G(\phi_1,\phi_2) - \Omega^{-4}V(\phi_1,\phi_2) \, \Big] \, \hat{e}\, d^4x \,. \end{aligned} \tag{8.70}$$

We can integrate the term with $\partial_\mu\hat{e}$ by parts to obtain

$$\begin{aligned} S = \frac{1}{16\pi G}\int &((-\Omega^{-2}\hat{T} + 4\Omega^{-3}\hat{\partial}_\mu\Omega\hat{T}^\mu + 6\Omega^{-4}\hat{\partial}_\mu\Omega\hat{\partial}^\nu\Omega)F(\phi_1,\phi_2) \\ &+ (\Omega^{-2}\hat{B} - 4\Omega^{-3}\hat{T}^\mu\hat{\partial}_\mu\Omega)G(\phi_1,\phi_2) - 6\Omega^{-3}\hat{\partial}^\mu\Omega\partial_\mu G(\phi_1,\phi_2) - U(\phi_1,\phi_2))\hat{e}\, d^4x \,, \end{aligned} \tag{8.71}$$

where we have disregarded a boundary term and we also introduce the new potential $U(\phi_1,\phi_2) = \Omega^{-4}V(\phi_1,\phi_2)$.

We want to explore under what conditions one can choose a suitable conformal factor to eliminate the coupling between either T or the boundary term B. It is straightforward to observe that it is always possible to eliminate the coupling between the scalar field and the torsion scalar T, one simply chooses the conformal factor to be $\Omega^2 = F(\phi_1,\phi_2)$. However, to eliminate couplings between the scalar fields and B, or equivalently the vector T^μ, requires a longer calculation. Integrating the boundary term by parts, we get left with the following coefficient of the vector T^μ in the above action (8.71)

$$\begin{aligned}\Big(4\Omega^{-3}\hat{\partial}_\mu\Omega(F(\phi_1,\phi_2)-G(\phi_1,\phi_2))-2\partial_\mu(\Omega^{-2}G(\phi_1,\phi_2))\Big)\,T^\mu\\ =\Big(4\Omega^{-3}\hat{\partial}_\mu\Omega F(\phi_1,\phi_2)-2\Omega^{-2}\partial_\mu(G(\phi_1,\phi_2))\Big)\,T^\mu.\end{aligned}\tag{8.72}$$

And thus a sufficient condition for the coupling between T^μ to vanish is that

$$2\Omega^{-1}\hat{\partial}_\mu\Omega F(\phi_1,\phi_2)-\partial_\mu(G(\phi_1,\phi_2))=0.\tag{8.73}$$

Now can we choose a sufficient $\Omega=\Omega(\phi_1,\phi_2)$ such that this will vanish? Expanding (8.73) in terms of ϕ_1 and ϕ_2 partial derivatives gives the following two first order partial differential equations

$$2\Omega^{-1}\Omega^{(1,0)}F(\phi_1,\phi_2)-G^{(1,0)}(\phi_1,\phi_2)=0,\tag{8.74}$$

$$2\Omega^{-1}\Omega^{(0,1)}F(\phi_1,\phi_2)-G^{(0,1)}(\phi_1,\phi_2)=0,\tag{8.75}$$

which we can rewrite as

$$\Omega^{(1,0)}=\frac{\Omega}{2F}G^{(1,0)},\tag{8.76}$$

$$\Omega^{(0,1)}=\frac{\Omega}{2F}G^{(0,1)}.\tag{8.77}$$

Now for such a solution Ω to exist for these partial differential equations, we simply require that the second mixed derivatives agree, that is if we differentiate the first of these equations with respect to ϕ_2 it must equal the second equation differentiated with respect to ϕ_1. After doing this calculation, we find the following condition on our original function f, which must be satisfied in order for such an Ω to exist

$$f^{(2,0)}f^{(0,2)} = (f^{(1,1)})^2. \tag{8.78}$$

One such solution to this equation is $f(R)$ gravity, when $f^{(1,0)} = -f^{(0,1)}$, but the equation has other solutions too, including separable solutions. For example, there is the solution

$$f(T,B) = \alpha T^{\frac{1}{1-k}} B^{\frac{k}{k-1}}, \tag{8.79}$$

for some constants k and α.

Finally we mention, for the couplings between both T and T^μ to simultaneously vanish, we require $\Omega = F^{1/2}$ and the system (8.76)-(8.77) to hold. But then solving for these two conditions requires that $F = -G$, which is simply the case of the teleparallel equivalent of $f(R)$ gravity, when the functional form of $f(T,B)$ is $f(T,B) = f(-T+B) = f(R)$. And so the unique class of $f(T,B)$ gravity which has an Einstein frame is $f(R)$ gravity, as to be expected.

8.6 f(B) gravity

For completeness, in this section we will examine the final case of $f(T,B)$ gravity we have yet to explore. This is the case when we have the following action

$$S_{\mathrm{f(T,B)}} = \frac{1}{16\pi G}\int [\alpha T + f(B)]\, e\, d^4x\,, \tag{8.80}$$

and without loss of generality we will set $\alpha = -1$ so that the action takes the form of Einstein gravity plus a boundary term modification. This action was not covered by the analysis in the previous section since for this particular action $f_{TT} = 0$. Performing the standard transformation using auxiliary variables, the action can be recast into the following form

$$S_{\mathrm{f(T,B)}} = \frac{1}{16\pi G}\int \left[-T + F(\phi)B - V(\phi)\right] e\, d^4x\,, \tag{8.81}$$

where $F(\phi) = f'(\phi)$ and $V(\phi) = \phi f'(\phi) - f(\phi)$.

Let us attempt to remove the coupling between ϕ and B in this action. Now applying a conformal transformation and performing some integration by parts recasts this into the following form

$$S = \frac{1}{16\pi G}\int \Big[- \Omega^{-2}\hat{T} + \Omega^{-2}(1+G(\phi))\hat{B} - 4\Omega^{-3}\hat{T}^{\mu}\hat{\partial}_{\mu}\Omega F(\phi) + 6\Omega^{-4}\hat{\partial}_{\mu}\Omega\hat{\partial}^{\nu}\Omega - 6\Omega^{-3}\hat{\partial}^{\mu}\Omega\hat{\partial}_{\mu}F(\phi) - U(\phi)\Big]\hat{e}\, d^4x\,, \tag{8.82}$$

where the new potential $U(\phi) = \Omega^{-4}V(\phi)$. We now explore whether or not we can choose a suitable $\Omega = \Omega(\phi)$ to remove the couplings between ϕ and both $\hat{B}$ and $\hat{T}^{\mu}$. Rewriting the partial derivatives in terms of ϕ and integrating by parts the term with $\hat{T}^{\mu}$ gives

$$S = \frac{1}{16\pi G}\int \Big[- \Omega^{-2}\hat{T} + \Omega^{-2}(1+F(\phi)+\Omega^2 H(\phi))\hat{B} + \left(6\Omega^{-4}\left(\frac{d\Omega}{d\phi}\right)^2 - 6\Omega^{-3}F'(\phi)\frac{d\Omega}{d\phi}\right)\hat{\partial}^{\mu}\phi\hat{\partial}_{\mu}\phi - U(\phi)\Big]\hat{e}\, d^4x\,, \tag{8.83}$$

where $H(\phi)$ is given by the following integral

$$H(\phi) = 2\int \frac{F(\phi)}{\Omega(\phi)^3}\frac{d\Omega}{d\phi}d\phi. \tag{8.84}$$

In order for the boundary term to have no effect on the action, we require that the coefficient of $\hat{B}$ is simply a constant β, so that

$$\Omega^{-2}(1+F(\phi)+\Omega^2 H(\phi)) = \beta. \tag{8.85}$$

This will allow us to find Ω in terms of ϕ. Differentiating this with respect to ϕ gives

$$-\frac{2}{\Omega^3}\frac{d\Omega}{d\phi}+\frac{F'(\phi)}{\Omega^2}=0. \tag{8.86}$$

And so solving this for Ω gives

$$\Omega=e^{F(\phi)/2}. \tag{8.87}$$

Inserting this solution for Ω into our action (8.83) gives us

$$S=\frac{1}{16\pi G}\int\left[-e^{-F(\phi)}\hat{T}-\frac{3}{2}\left(e^{-F(\phi)}F'(\phi)^2\right)\hat{\partial}^\mu\phi\hat{\partial}_\mu\phi-U(\phi)\right]\hat{e}\,d^4x\,. \tag{8.88}$$

Finally, introducing the new scalar field $\varphi=2\sqrt{3}(e^{-F(\phi)/2}-\beta)$ recasts the action into the following form

$$S=\frac{1}{16\pi G}\int\left[-A(\varphi)\hat{T}-\frac{1}{2}\hat{\partial}^\mu\varphi\hat{\partial}_\mu\varphi-U(\varphi)\right]\hat{e}\,d^4x\,, \tag{8.89}$$

where the coupling function $A(\varphi)$ is given by

$$A(\varphi)=\beta^2(1+\frac{\varphi}{2\sqrt{3}\beta})^2, \tag{8.90}$$

in agreement with the result (8.60).. Thus we have found that $f(B)$ gravity, where we have an additional $f(B)$ term added to the Einstein Hilbert action, is conformally equivalent to a particular instance of teleparallel dark energy. Moreover, as opposed to $f(T)$ gravity, the kinetic term has the correct sign and so will not suffer from the same potential instabilities to perturbations.

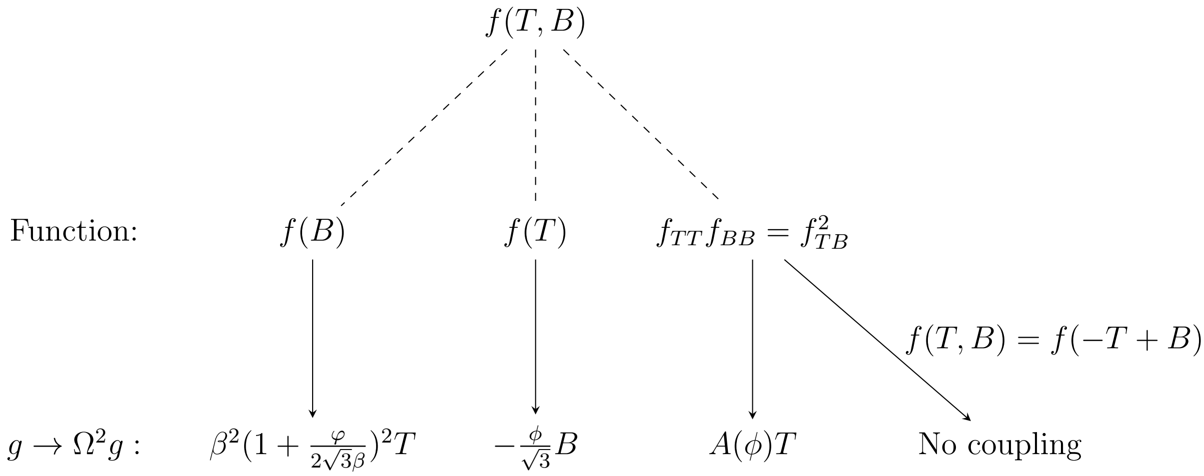

Figure 8.1: The conformal equivalence of different $f(T,B)$ gravity models. The top line shows the particular functional form of the $f(T,B)$ gravity considered, and the bottom line shows the type of nonminimal coupling present in the action after a particular conformal transformation. A minimal coupling is only possible in the case when $f(T,B) = f(R)$. The kinetic and potential energy of the scalar field are also present in the conformally transformed action, with the kinetic term being either the canonical or phantom type.

8.7 Discussion

In this chapter we have explored the conformal relationships between various modified teleparallel gravity theories. Conformally transforming these theories indicates the need to take into account first derivatives of torsion. In particular the scalar B given by the divergence of a contraction of the torsion tensor arises naturally when one considers transformations of these teleparallel theories. We first reviewed $f(T)$ gravity, showing that it is conformally equivalent to a phantom teleparallel scalar tensor theory with a linear nonminimal coupling between the scalar field and the boundary term only, similar to a model discussed in Chapter 7.

Furthermore we looked at a teleparallel dark energy theory, where a generic nonminimal coupling between the scalar field and the torsion scalar T was present. We showed that in general this is conformally equivalent to a particular $f(T,B)$

gravity, moreover if this $f(T,B)$ gravity is nonlinear in both T and B then it must satisfy the condition

$$f_{TT}f_{BB} = (f_{TB})^2. \tag{8.91}$$

We gave an explicit toy example of such a coupling and transformed it into a simple $f(T,B)$ theory, which did indeed satisfy condition (8.91). The other possibility is that the coupling between ϕ and T takes the particular form

$$A(\phi) = \beta^2(1 + \frac{\phi}{2\beta\sqrt{3\alpha}})^2, \tag{8.92}$$

in which case the model is conformally equivalent to a particular $f(T,B)$ theory of the form

$$f(T,B) = -\alpha T + f(B), \tag{8.93}$$

with only a linear dependence on the torsion scalar, and the particular $f(B)$ is dependent on the structure of the potential of the scalar field.

Moreover, we also looked at the case of $f(B)$ gravity, where the functional form of $f(T,B)$ gravity is given by $f(T,B) = -T + f(B)$. It was shown that this can always be conformally transformed to a frame where there is a particular type of nonminimal coupling between the scalar field and T. Thus we have an interesting duality relation, $f(T)$ can always be transformed to a $A(\phi)B$ phantom scalar field theory, whereas $f(B)$ can always be transformed to a canonical $A(\phi)T$ theory.

We have derived various relationships between modified teleparallel theories of gravity and teleparallel scalar-tensor theories. The unique form of these different theories which has an Einstein frame is given by $f(T,B)$ gravity which takes the form $f(-T+B)$, which is equivalent to $f(R)$. In all other cases a form of nonminimal coupling between the scalar field and the gravitational sector remains present. The various conformal relationships between the different theories considered have been summarised in Figure 8.1.

We can see that a full consideration of the boundary term B is necessary for a true understanding of the equivalence between scalar-tensor representations and modified gravity representations of torsion based theories.

Chapter 9

General Conclusions

In this thesis we have looked at a variety of different models in the teleparallel framework of gravitational theories. We have discussed some of the theoretical issues surrounding them, along with a focus on their cosmological applications, exploring the challenges and issues that contemporary cosmologists and astrophysicists must address in the near future. The dark energy, dark matter and inflation problems are unlikely to disappear any time soon, and a wide range of approaches to solving them should continue to be explored, until cosmological and experimental data provides us with some more definitive answers.

The teleparallel equivalent of general relativity is an interesting alternative theory of gravity. Despite possessing the same dynamics as general relativity, it offers a radically different physical interpretation. Einstein's revolutionary view, viewing gravity as a merely fictional force as a product of the curvature of space and time, is turned on its head. We return to the viewpoint of thinking of gravity as a force, acting through the medium of torsion. It would be interesting to speculate how physics would have differed over the last century had Einstein developed TEGR first. Would general relativity have come to be as popular today, or would it be seen as some esoteric reworking of an ordinary force law?

Modifying gravity continues to be an interesting avenue of study for a number of reasons: one might want to attempt to solve the problems of modern cosmology, or consider the effect of microstructure on gravity, or remove singularities present in the early universe, or develop theories which are potentially able to be unified with quantum field theory. In that regard, we introduced metric affine gravity, an incredibly general theory of gravity whose various sub-models can potentially address many of the above motivations for studying modified gravity. In this thesis, we focused in particular on the dark energy problem, and to that end we considered modifications of the TEGR.

The equivalence of general relativity and the TEGR poses the question: if we want to modify the theory, which one of the two theories do we modify? Historically it has been general relativity which is the most frequently and first to be modified. TEGR has in some ways always been viewed as a curiosity, and its study has been rather neglected by the mainstream physics community, due in part to the dynamics of the theory being identical to general relativity. But modifying the two theories gives rise to different dynamics, and thus this question is of great importance.

This thesis in some way attempts to unify the two approaches to modification, using a larger framework enabling the study of both curvature and torsion based modifications at the same time. We first looked at how we could consider a theory more general than $f(R)$ and $f(T)$ gravity, which we dubbed $f(T,B)$ gravity, which reduces to both of the other models in suitable limits. Moreover it also gives rise to a greater potential class of modifications, which may have potentially interesting cosmological applications in the future.

The second class we looked at is scalar-tensor type theories, again examining a general model that included both the teleparallel and metric based theories as limiting cases of the theory. A variety of interesting cosmological phenomenology was observed that is potentially able to match observations well, and moreover on the whole very little fine tuning is required to generate this interesting behaviour.

These latter two theories, $f(T,B)$ gravity and the teleparallel scalar-tensor models, were then shown to be in fact physically equivalent, up to a local rescaling of time and space. It is much easier to work with the scalar field models from a computational point of view: dynamical systems techniques do not require complicated function inversions, and so any study of $f(T)$ or $f(T,B)$ gravity models in future should perhaps take place in their Einstein frame equivalents.

An underlying theme to this thesis is that rather than focusing on very specific types of modified gravity model, our goal is to study very broad classes of theories and examine their general features and properties. The mathematics of dynamical systems has shown itself to be a vital tool in this regard, allowing us to understand the broad behaviour of the phenomenology of different models, without having to find exact solutions to the equations. On the other hand we have also reduced the space of possible models available to study, by showing some types of model are in fact just a physically equivalent way of writing another model. Broadening our space of possible models brought new insights, showing that previously studied theories that were thought to be different were in fact just different sides of the same coin.

In this thesis we have talked a lot about the dark energy problem, and the ability of modified teleparallel theories of gravity to potentially alleviate this issue. However, modified teleparallel theories have other potential uses. In fact $f(T)$ gravity was originally constructed as an inflationary type model, see [92], and subsequently many further teleparallel inflation models have been considered, for example [155, 156]. In fact in [155] reconstruction techniques were used to generate an $f(T)$ gravity that underwent a period of inflation and dark energy expansion in one consistent cosmology. Modified gravity models have also been applied in the literature to the dark matter problem, for example $f(R) = R^n$ models have been applied to the rotation curve problem [157]. The author is unaware of any similar studies in modified teleparallel theories, but they would be potentially interesting to investigate in the future.

It is hoped that this thesis will provide the reader with a useful introduction to the subject of teleparallel gravity and its history and its place within the hierarchy of different theories of gravity. We also hope to have left the reader with an understanding of the current topics that are being researched in the fields of cosmology and modified teleparallel gravity today. It is an exciting time to be working within the field of gravity, with the advent of gravitational wave astronomy and new increasingly precise cosmological data arriving every year.

To that end, it may be the case that in a few years new observational data comes to light that rules out the modified teleparallel models considered in this thesis. Despite this meaning the contents of this thesis was out of date, I would celebrate that day. Science only progresses by formulating new hypotheses and continuously and rigorously testing them until we can disprove them. If our hypotheses fail, it means we have the opportunity to move on and develop even more novel and exciting ideas.

Appendix A

Conventions

There are many different conventions in the literature on teleparallel and modified gravity. This thesis has attempted to use uniform conventions throughout, which we list below in this appendix. Largely we have stuck to the conventions used in [158].

Throughout this thesis we have used natural units, setting the speed of light and Plank's constant equal to unity, $c = \hbar = 1$. Four dimensional spacetime coordinates are depicted with Greek letters, $\mu, \nu, \rho, ...$ whereas four dimensional tangent space coordinates are depicted with Latin letters, $a, b, c, ...$ where both sets of indices run over the range $0, 1, 2, 3$. We ave adopted the Einstein summation convention, where repeated indices are summed over their entire range. We have used both signatures $(+, -, -, -)$ and $(-, +, +, +)$ throughout this thesis, in an attempt to remain consistent with the literature: the former is used when studying teleparallel gravity, whereas the latter is used when studying general relativity and its modifications.

The Levi-Civita connection is denoted by $\bar{\Gamma}^{\mu}_{\lambda\rho}$, whereas a general connection is denoted without the bar. The Weitzenböck connection additionally has been denoted as $\Gamma^{\mu}_{\lambda\rho}$, but we have been explicit when we are using it to denote this particular connection. The signs of tensors such as torsion, non-metricity, contorsion have all been taken to be the same as when they were first introduced.

Appendix B

Modified teleparallel theories of gravity

Sebastian Bahamonde, Christian Böhmer and Matthew Wright,

Modified teleparallel theories of gravity,

Phys. Rev. D **92** (2015) 10, 104042

Appendix C

Teleparallel quintessence with a nonminimal coupling to a boundary term

Sebastian Bahamonde and Matthew Wright,

Teleparallel quintessence with a nonminimal coupling to a boundary term,

Phys. Rev. D **92** (2015) 084034

Appendix D

Conformal transformations in modified teleparallel theories of gravity revisited

Matthew Wright,

Conformal transformations in modified teleparallel theories of gravity revisited,

Phys. Rev. D **93** (2016) 10, 103002

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