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Functions of self-adjoint operators in ideals of compact operators

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Abstract

For self-adjoint operators A, B, a bounded operator J, and a function $f : \mathbb{R} \to \mathbb{C}$, we obtain bounds in quasi-normed ideals of compact operators for the difference $f(A)J - Jf(B)$ in terms of the operator $AJ - JB$. The focus is on functions f that are smooth everywhere except for finitely many points. A typical example is the function $f(t) = |t|^\gamma$ with $\gamma \in (0, 1)$. The obtained results are applied to derive a two-term quasi-classical asymptotic formula for the trace $\operatorname{tr} f(S)$ with S being a Wiener–Hopf operator with a discontinuous symbol.

1. *Introduction*

In this paper, we study a pair of self-adjoint operators A, B on a Hilbert space $\mathfrak{H}$. We are interested in estimates in various quasi-normed ideals of compact operators for the 'quasi-commutators' of the form $f(A)J - Jf(B)$ in terms of the 'perturbation' $AJ - JB$, where $J : \mathfrak{H} \to \mathfrak{H}$ is a bounded operator and $f : \mathbb{R} \to \mathbb{C}$ is a suitable function. There is a vast literature concerned with problems of this type, with a large number of deep results. Our intention is to improve some of the existing estimates for a very specific class of functions f. The focus will be on continuous functions f that are smooth everywhere except possibly for finitely many points. One example of such function is $f(t) = |t|^\gamma$ with $\gamma > 0$. In this introduction, we do not provide a detailed survey of the known results but concentrate on the directly relevant ones only, further references can be found, for example, in [**1**] and [**25**]. By $\mathfrak{S}$ we denote a (quasi)-normed two-sided ideal of compact operators, and by $\mathfrak{S}_p, 0 < p < \infty$, the classical Schatten–von Neumann ideals. By C, c (with or without indices) we denote various positive constants whose precise value is of no importance.

In [**21**, **23**] it was found that if f belongs to the Besov class $B^1_{\infty 1}(\mathbb{R})$ then the function f is $\mathfrak{S}_1$-*operator-Lipschitz*, that is,

$$\|f(A) - f(B)\|_{\mathfrak{S}_1} \leqslant C\|A - B\|_{\mathfrak{S}_1}, C = C(f), \tag{1.1}$$

for arbitrary self-adjoint operators A, B such that $A - B \in \mathfrak{S}_1$. Conversely, as shown in [**21**], the estimate (1.1) implies that $f \in B^1_{11}(\mathbb{R})$ locally. Paper [**8**] identifies a meaningful class of self-adjoint operators, for which the condition $f \in B^1_{11}(\mathbb{R})$ is also sufficient for (1.1).

For the Schatten–von Neumann classes $\mathfrak{S}_p$, $1 < p < \infty$, conditions on the function f look simpler. Precisely, for arbitrary uniformly Lipschitz functions f it was shown in [**25**] that

$$\|f(A)J - Jf(B)\|_{\mathfrak{S}_p} \leqslant c_p\|f\|_{\mathrm{Lip}}\|AJ - JB\|_{\mathfrak{S}_p}, \quad \|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}. \tag{1.2}$$

Note that the main theorem of [**25**] claims the bound (1.2) with $J = I$. However, as the anonymous referee has pointed out to the author, the proof in [**25**] allows an arbitrary bounded

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operator J in (1.2). One can say even more: the bound (1.2) with an arbitrary bounded J can be inferred from (1.2) with $J = I$. This type of result for general operator-Lipschitz functions in arbitrary normed ideals $\mathfrak{S}$, can be deduced from [**15**, Theorem 3.5]. We refer to [**15**], and accompanying papers cited there, for precise statements and further results.

The classes $\mathfrak{S}_p$ with $p \in (0, 1)$, were studied in [**22**] for unitary operators A and B. We discuss this in more detail in Remark 2.5.

The function $f(t) = |t|^\gamma$, $\gamma \in (0, 1)$ was studied in [**2**]. Let $\mathfrak{S}$ be a normed ideal with the *majorization property*, see [**11**] for the definition. This assumption is not too restrictive as any separable ideal (for example, $\mathfrak{S}_p, 0 < p < \infty$) possesses this property. As shown in [**2**] (see also [**4**]), for $\gamma \in (0, 1)$, if $A \geqslant 0$ and $B \geqslant 0$ are such that $|A - B|^\gamma \in \mathfrak{S}$, then

$$\|A^\gamma - B^\gamma\|_{\mathfrak{S}} \leqslant \||A - B|^\gamma\|_{\mathfrak{S}}. \tag{1.3}$$

Observe that the function $|t|^\gamma, \gamma \in (0, 1)$ belongs to the Hölder–Zygmund class $\Lambda_\gamma(\mathbb{R}) = B^\gamma_{\infty,\infty}(\mathbb{R})$ locally. Among other functional spaces, this space was considered in the recent article [**1**]. In fact, [**1**] brings us closer to the objects studied in the current paper as it contains results on the quasi-commutators $f(A)J - Jf(B)$ in quasi-normed ideals. Precisely, for any function $f \in \Lambda_\gamma(\mathbb{R}), \gamma \in (0, 1)$ it was shown in [**1**] that

$$\||f(A)J - Jf(B)|^{1/\gamma}\|_{\mathfrak{S}} \leqslant C(f)\|J\|^{(1-\gamma)/\gamma}\|AJ - JB\|_{\mathfrak{S}}, \tag{1.4}$$

under the assumption that the *Boyd index* $\beta(\mathfrak{S})$ of the quasi-normed ideal $\mathfrak{S}$ is strictly less than 1, see [**1**, Theorem 11.5]. The definition of the Boyd index can be found, for example, in [**1**, Section 3]. For the Schatten–von Neumann ideals $\mathfrak{S}_p, 0 < p < \infty$, the index is found by the simple formula $\beta(\mathfrak{S}_p) = p^{-1}$.

None of the results quoted above generalizes the others but some of them have non-empty intersections. Let us compare, for instance (1.3) and (1.4) for the Schatten–von Neumann classes. Then (1.3) gives

$$\|A^\gamma - B^\gamma\|_{\mathfrak{S}_p} \leqslant \|A - B\|^\gamma_{\mathfrak{S}_{p\gamma}}, \tag{1.5}$$

for any $p \geqslant 1$ and $\gamma \in (0, 1)$, and (1.4) gives (see [**1**, Theorem 11.7])

$$\|f(A)J - Jf(B)\|_{\mathfrak{S}_p} \leqslant C(f)^\gamma\|J\|^{1-\gamma}\|AJ - JB\|^\gamma_{\mathfrak{S}_{p\gamma}}, \tag{1.6}$$

under the condition $p\gamma > 1$. On the one hand, (1.6) is valid for the entire class $\Lambda_\gamma(\mathbb{R})$, and it allows $J \neq I$, but on the other hand, (1.6) holds under the more restrictive assumption $p\gamma > 1$.

One aim of this paper is to derive the following 'hybrid' of (1.3) and (1.4). For the sake of discussion, we state the result in a somewhat simplified form, see Theorem 2.4 for the precise statement. Assume that $f \in \mathsf{C}^\infty(\mathbb{R} \setminus \{z\}), z \in \mathbb{R}$, is a compactly supported function satisfying the condition

$$|f^{(k)}(t)| \leqslant C_k|t - z|^{\gamma-k}, \quad k = 0, 1, \dots, \ t \neq 0, \tag{1.7}$$

with some $\gamma > 0$. Let $\mathfrak{S}$ be a quasi-normed ideal. Then for any $\sigma \in (0, \gamma), \sigma \leqslant 1$, the bound holds

$$\||f(A)J - Jf(B)|^{1/\sigma}\|_{\mathfrak{S}} \leqslant C(f)\|J\|^{(1-\sigma)/\sigma}\|AJ - JB\|_{\mathfrak{S}}. \tag{1.8}$$

Emphasize that in contrast to (1.4), the value $\sigma = \gamma$ is not allowed. On the other hand, there are no restrictions on the ideal $\mathfrak{S}$.

If $\gamma > 1$ then in the formula (1.8) one can take $\sigma = 1$. Thus for $\mathfrak{S} = \mathfrak{S}_p, 1 < p < \infty$, the bound (1.8) is in agreement with (1.2). For $p \in (0, 1)$ and $J = I$ the bound (1.8) is in line with the results of [**22**], see Remark 2.5 for details.

Since our choice of the function f is very specific, the proof of (1.8) does not require sophisticated methods employed in [**1**, **21**–**23**, **25**] where various general functional classes were studied. In particular, we do not make use of the Double Operator Integrals techniques.

Instead, we rely on the representation of $f(A)$ for a self-adjoint operator A in terms of the *quasi-analytic extension* of the function f, which has become known as the Helffer–Sjöstrand formula, see [**5**, **12**]. The convenient quasi-analytic extension is constructed in Lemma 3.3. Note that the recent paper [**9**] also uses quasi-analytic extensions to study differences of the form $f(A) - f(B)$ for self-adjoint operators A, B. Note, however, that [**9**] considers infinitely smooth functions f and normed ideals of compact operators. Recall that the main point of our paper is that the function f may not be even differentiable, see (1.7), and we also allow quasi-normed ideals $\mathfrak{S}$.

In Theorem 2.10, we focus on the following useful special case of the bound (1.8). Let A be a self-adjoint operator and let P be an orthogonal projection. Then, using (1.8) with $J = P$, $B = PAP$ we obtain the bound

$$\|Pf(PAP)P - Pf(A)P\|_{\mathfrak{S}} \leqslant C(f)\||PA(I-P)|^{\sigma}\|_{\mathfrak{S}}. \tag{1.9}$$

A bound of a similar nature was previously derived in [**17**, **18**] for arbitrary $f \in \mathsf{W}^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$:

$$\big|\mathrm{tr}\big(Pf(PAP)P - Pf(A)P\big)\big| \leqslant \tfrac{1}{2}\|f''\|_{\mathsf{L}^\infty}\|PA(I-P)\|^2_{\mathfrak{S}_2}.$$

The above two inequalities are helpful in problems involving Szegő-type estimates and/or asymptotics, see, for example, [**16**–**18**, **27**].

The last section of the paper, Section 4, illustrates the practical use of the bound (1.9) with an example of a multi-dimensional Wiener–Hopf operator with a discontinuous symbol. The operator in question, denoted by S_α, is defined by (4.1), where $\alpha \geqslant 1$ is the 'quasi-classical parameter'. The objective is to obtain a two-term asymptotics of the trace $\mathrm{tr}\, g(S_\alpha)$ with a non-smooth function g, as $\alpha \to \infty$. The function g is allowed to have finitely many singularities of the type described by (1.7). For smooth g the sought two-term asymptotic formula was justified in [**27**] and [**29**]. The main result of Section 4 is contained in Theorem 4.4, and its proof consists in 'closing' the asymptotic formula for smooth g with the help of the bound (1.9). The generalization to non-smooth functions is motivated, in part, by applications in information theory and statistical physics, see, for example, [**10**, **13**, **19**]. Further discussion is deferred until Section 4.

2. *Main results*

2.1. *Quasi-normed ideals of compact operators*

We need some information from the theory of ideals of compact operators. Details can be found in [**3**, **11**, **24**]. Let $\mathfrak{S} \subset \mathfrak{S}_\infty$ be a two-sided ideal. Recall that a functional $\|\cdot\|_{\mathfrak{S}}$ defined for $T \in \mathfrak{S}$ is said to be a quasi-norm if

(1) $\|T\|_{\mathfrak{S}} > 0$ if $T \neq 0$,
(2) $\|zT\|_{\mathfrak{S}} = |z|\|T\|_{\mathfrak{S}}$ for any $z \in \mathbb{C}$,
(3) there exists a number $\varkappa \geqslant 1$ such that

$$\|T_1 + T_2\|_{\mathfrak{S}} \leqslant \varkappa\big(\|T_1\|_{\mathfrak{S}} + \|T_2\|_{\mathfrak{S}}\big).$$

If, in addition, the conditions below are satisfied
(4) $\|XTY\|_{\mathfrak{S}} \leqslant \|X\|\ \|Y\|\ \|T\|_{\mathfrak{S}}$, for any bounded X, Y and $A \in \mathfrak{S}$,
(5) $\|T\|_{\mathfrak{S}} = \|T\|$ for any one-dimensional operator T,

then the quasi-norm $\|\cdot\|_{\mathfrak{S}}$ is said to be symmetric. The ideal $\mathfrak{S}$ is said to be a quasi-normed ideal if it is endowed with a (symmetric) quasi-norm, and is complete. We usually omit the term 'symmetric' for brevity. If $\varkappa = 1$ then the quasi-norm becomes a norm.

Note an important property of quasi-norms. Below by $s_k(T), k = 1, 2, \dots,$ we denote singular numbers of the operator $T \in \mathfrak{S}_\infty$.

LEMMA 2.1. *Let $T \in \mathfrak{S}$ and let $S \in \mathfrak{S}_\infty$ be operators such that $s_k(S) \leqslant M s_k(T)$, $k = 1, 2, \dots$, with some constant $M > 0$. Then $S \in \mathfrak{S}$ and $\|S\|_{\mathfrak{S}} \leqslant M\|T\|_{\mathfrak{S}}$.*

For normed ideals this lemma was proved in [**11**], and the proof for quasi-normed ideals is the same. It shows that the quasi-norm $\|T\|_{\mathfrak{S}}$ depends only on the singular numbers of the operator $T \in \mathfrak{S}$. This means in particular that $\|T\|_{\mathfrak{S}} = \|T^*\|_{\mathfrak{S}} = \||T|\|_{\mathfrak{S}}$, where $|T| = \sqrt{T^*T}$.

We say that a quasi-normed ideal $\mathfrak{S}$ is a *q-normed ideal* if there exists an equivalent quasi-norm $\|\cdot\|_{\mathfrak{S}}$ which satisfies the *q-triangle inequality*:

$$\|T_1 + T_2\|_{\mathfrak{S}}^q \leqslant \|T_1\|_{\mathfrak{S}}^q + \|T_2\|_{\mathfrak{S}}^q, \tag{2.1}$$

for any $T_1, T_2 \in \mathfrak{S}$, see, for example, [**24**]. In fact, any quasi-normed ideal $\mathfrak{S}$ is a q-normed ideal with the $q \in (0, 1]$ found from the equation $\varkappa = 2^{q^{-1}-1}$ ($q = 1$ refers to a normed ideal).

As an example, we can take as $\mathfrak{S}$ any Schatten–von Neumann ideal $\mathfrak{S}_p$, $p \in (0, \infty)$ with the standard (quasi)-norm

$$\|T\|_{\mathfrak{S}_p} = \left[\sum_{k=1}^{\infty} s_k(T)^p\right]^{1/p}.$$

If $p \geqslant 1$, then this functional is a norm, and if $p \in (0, 1)$ then it is a p-norm, see [**26**] and also [**3**].

2.2. *The estimates*

Let A and B be two self-adjoint operators acting on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{G}$, respectively, and let $J : \mathfrak{G} \to \mathfrak{H}$ be a bounded operator. Consider the form

$$V[u, w] = (Ju, Aw) - (JBu, w), u \in D(B), w \in D(A).$$

Suppose that

$$|V[u, w]| \leqslant C\|u\|\ \|w\|,$$

that is, this form defines an operator $V : D(B) \to \mathfrak{H}$ which extends to a bounded operator on the entire space $\mathfrak{G}$. This implies that J maps $D(B)$ into $D(A)$. We use the notation $V = AJ - JB$. Let $R(z; A) = (A - z)^{-1}$, $\mathrm{Im} z \neq 0$. Under the assumption that $V : \mathfrak{G} \to \mathfrak{H}$ is a bounded operator, we can write the resolvent identity

$$R(z; A)J - JR(z; B) = -R(z; A)VR(z; B). \tag{2.2}$$

LEMMA 2.2. *Suppose that the operator $V = AJ - JB$ is such that $|V|^\sigma \in \mathfrak{S}$ with some $\sigma \in (0, 1]$. Then for all $y = Im z \neq 0$ we have*

$$\|R(z; A)VR(z; B)\|_{\mathfrak{S}} \leqslant \||V|^\sigma\|_{\mathfrak{S}}\|J\|^{1-\sigma} \frac{2^{1-\sigma}}{|y|^{1+\sigma}}.$$

Proof. Denote

$$W = R(z; A)VR(z; B).$$

By definition of the quasi-norm,

$$\|W\|_{\mathfrak{S}} \leqslant \||W|^{1-\sigma}\|\ \||W|^\sigma\|_{\mathfrak{S}} = \|W\|^{1-\sigma}\ \||W|^\sigma\|_{\mathfrak{S}} \leqslant \frac{2^{1-\sigma}}{|y|^{1-\sigma}}\|J\|^{1-\sigma}\||W|^\sigma\|_{\mathfrak{S}}, \tag{2.3}$$

where we have used the trivial bound for the left-hand side of (2.2): $\|W\| \leqslant 2|y|^{-1}\|J\|$. In order to estimate the quasi-norm on the right-hand side of (2.3), estimate the singular values $s_k(|W|^\sigma)$:

$$s_k(|W|^\sigma) = s_k(W)^\sigma \leqslant |y|^{-2\sigma} s_k(V)^\sigma.$$

Therefore, by Lemma 2.1

$$\||W|^\sigma\|_{\mathfrak{S}} \leqslant |y|^{-2\sigma} \||V|^\sigma\|_{\mathfrak{S}}.$$

Substituting this bound into (2.3), we get the required estimate. □

We are interested in bounds for the difference

$$f(A)J - Jf(B),$$

where f is a function satisfying the following condition. Below we denote by χ_R the characteristic function of the interval $(-R, R)$, $R > 0$.

Condition 2.3. Assume that for some integer $n \geqslant 1$ the function $f \in \mathsf{C}^n(\mathbb{R} \setminus \{x_0\}) \cap \mathsf{C}(\mathbb{R})$, $x_0 \in \mathbb{R}$, satisfies the bound

$$[\![f]\!]_n = \max_{0\leqslant k\leqslant n} \sup_{x\neq x_0} |f^{(k)}(x)||x - x_0|^{-\gamma+k} < \infty \tag{2.4}$$

with some $\gamma > 0$, and is supported on the interval $[x_0 - R, x_0 + R]$ with some $R > 0$.

For a function f satisfying the above condition the bound holds:

$$|f^{(k)}(x)| \leqslant [\![f]\!]_n |x - x_0|^{\gamma-k} \chi_R(x - x_0), k = 0, 1, \dots, n, \quad x \neq x_0. \tag{2.5}$$

One can immediately deduce from (2.5) that

$$\begin{cases} \|f\|_{\mathsf{L}^\infty} \leqslant [\![f]\!]_0 R^\gamma, \\ |f(t_1) - f(t_2)| \leqslant C_\gamma [\![f]\!]_1 R^{\gamma-\varkappa} |t_1 - t_2|^\varkappa, \ \varkappa = \min\{\gamma, 1\}, \end{cases} \tag{2.6}$$

for any $t_1, t_2 \in \mathbb{R}$, so that $g \in \mathsf{C}^{0,\varkappa}(\mathbb{R})$. Here by $\mathsf{C}^{0,\varkappa}(\mathbb{R}^n), n \geqslant 1$, we denote the standard class of Hölder-continuous functions f with the finite norm

$$\sup_{\mathbf{x}} |f(\mathbf{x})| + \sup_{\mathbf{x}\neq\mathbf{y}} \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\varkappa}.$$

The next theorem constitutes the main result of the paper.

Theorem 2.4. *Suppose that f satisfies Condition 2.3 with some $\gamma > 0$, $n \geqslant 2$ and $R > 0$. Let $\mathfrak{S}$ be a q-normed ideal where $(n - \sigma)^{-1} < q \leqslant 1$ with some number $\sigma \in (0, 1]$, $\sigma < \gamma$.*

Let A, B be two self-adjoint operators as described above such that $V = AJ - JB$ is a bounded operator. Suppose that $|V|^\sigma \in \mathfrak{S}$. Then

$$\|f(A)J - Jf(B)\|_{\mathfrak{S}} \leqslant C_n R^{\gamma-\sigma} [\![f]\!]_n \|J\|^{1-\sigma} \||V|^\sigma\|_{\mathfrak{S}}, \tag{2.7}$$

with a positive constant C_n independent of the operators A, B, J, function f and parameter R.

One should observe that the parameters n and σ in Theorem 2.4 are not entirely independent of each other. Indeed, the condition $(n - \sigma)^{-1} < q \leqslant 1$ does not allow $n = 2$ and $\sigma = 1$ at the same time. We'll need to remember this fact in the proof of Theorem 4.4 later on.

REMARK 2.5. It is appropriate to compare Theorem 2.4 with the results of the paper [22] mentioned in the introduction. In [22] it was shown for a pair of unitary operators U_1 and U_2 that

$$f(U_1) - f(U_2) \in \mathfrak{S}_p \text{ under the assumption that } U_1 - U_2 \in \mathfrak{S}_p, p \in (0,1), \tag{2.8}$$

if $f \in B^{1/p}_{\infty p}(\mathbb{T}_1)$, where $\mathbb{T}_1$ is the unit circle. Conversely, (2.8) implies that $f \in B^{1/p}_{pp}(\mathbb{T}_1)$. These conditions can certainly be appropriately rephrased for self-adjoint operators with the help of the Cayley transform.

For the sake of comparison, in Theorem 2.4 assume for simplicity that $f \in \mathsf{C}^\infty(\mathbb{R} \setminus \{0\})$ is a function such that $f(t) = |t|^\gamma, \gamma > 0$, for all $|t| \leqslant 1$ and $f(t) = 0$ for $|t| \geqslant 2$. Then using (2.7) with $\mathfrak{S} = \mathfrak{S}_p$, $J = I$, $R = 2$ and $\gamma > 1$ we get that

$$\|f(A) - f(B)\|_{\mathfrak{S}_p} \leqslant C_p \|A - B\|_{\mathfrak{S}_p},$$

for arbitrary $p \in (0, 1]$ with a constant C_p independent of A, B. The chosen function f belongs to $B^\nu_{rq}(\mathbb{R})$, $r \in (0, \infty], q \in (0, \infty)$, if and only if $\nu < \gamma + r^{-1}$. Thus f does not satisfy the sufficient condition $f \in B^{1/p}_{\infty p}(\mathbb{R})$ from [22], if $\gamma \leqslant p^{-1}$. On the other hand, the necessary condition $f \in B^{1/p}_{pp}(\mathbb{R})$ is satisfied for any $\gamma > 0$.

Note the following scaling property.

REMARK 2.6. Theorem 2.4 for arbitrary $R > 0$ follows from Theorem 2.4 for $R = 1$. Indeed, without loss of generality one may assume that $x_0 = 0$. Note that the function $g(t) = R^{-\gamma} f(Rt)$ satisfies (2.5) with $R = 1$ and that $\pmb{|} g \pmb{|}_n = \pmb{|} f \pmb{|}_n$. Now use bound (2.7) for the function g and the operators $A' = R^{-1}A, B' = R^{-1}B$.

It is also convenient to have a separately stated result for smooth functions f.

COROLLARY 2.7. *Suppose that $g \in \mathsf{C}^n_0(-\rho, \rho)$, with some $\rho > 0$ and $n \geqslant 2$. Let $\mathfrak{S}$ be a q-normed ideal where $(n - \sigma)^{-1} < q \leqslant 1$ with some number $\sigma \in (0, 1]$. Let A, B be two self-adjoint operators as in Theorem 2.4. Then*

$$\|g(A)J - Jg(B)\|_{\mathfrak{S}} \leqslant C \max_{0 \leqslant k \leqslant n} \big(\rho^k \|g^{(k)}\|_{\mathsf{L}^\infty}\big) \rho^{-\sigma} \|J\|^{1-\sigma} \||V|^\sigma\|_{\mathfrak{S}}, \tag{2.9}$$

with a constant C independent of the operators A, B, J, function g and parameter R.

Proof. Suppose first that $\rho = 1$ and without loss of generality set

$$\max_{0 \leqslant k \leqslant n} \|g^{(k)}\|_{\mathsf{L}^\infty} = 1.$$

Then the function

$$f(t) = (t-2)^2 \big(g(t)(t-2)^{-2}\big)$$

clearly satisfies (2.5) with $\gamma = 2$, $x_0 = 2$, $R = 3$ and $\pmb{|} f \pmb{|}_n \leqslant C$. Therefore by Theorem 2.10,

$$\|g(A)J - Jg(B)\|_{\mathfrak{S}} \leqslant C \|J\|^{1-\sigma} \||V|^\sigma\|_{\mathfrak{S}},$$

which proves (2.9) for $\rho = 1$.

If $\rho > 0$ is arbitrary, then use the first part of the proof for the function $f(t) = g(\rho t)$ and operators $A' = A\rho^{-1}$, $B' = B\rho^{-1}$. □

Now we use Corollary 2.7 to obtain bounds similar to (2.7) for functions with unbounded supports. We concentrate on smooth functions g satisfying the bound

$$|g^{(k)}(x)| \leqslant (1 + |x|)^{-\beta}, \beta > 0, \quad k = 1, 2, \dots, n, x \in \mathbb{R}. \tag{2.10}$$

Corollary 2.8. *Suppose that g satisfies (2.10) with some $n \geqslant 2$ and $\beta > 0$. Let the ideal $\mathfrak{S}$ and operators A, B be as in Corollary 2.7. If $q\beta > 1$, then*

$$\|g(A)J - Jg(B)\|_{\mathfrak{S}} \leqslant C_n \|J\|^{1-\sigma} \||V|^\sigma\|_{\mathfrak{S}}, \tag{2.11}$$

with a positive constant C_n independent of the operators A, B, J and function g.

Proof. Let $\Upsilon \in \mathsf{C}_0^\infty(\mathbb{R})$ be a function such that $\Upsilon(t) = 0$ for all $|t| \geqslant 1$. We pick Υ in such a way that $\sum_{m\in\mathbb{Z}} \Upsilon(x-m) = 1$, $x \in \mathbb{R}$. Let $g_m(x) = \Upsilon(x-m)g(x)$, $m \in \mathbb{Z}$. Since $\|g_m^{(k)}\|_{\mathsf{L}^\infty} \leqslant C(1+|m|)^{-\beta}$, $k = 0, 1, \dots, n$, it follows from Corollary 2.7 that

$$\|g_m(A)J - Jg_m(B)\|_{\mathfrak{S}} \leqslant C(1+|m|)^{-\beta} \|J\|^{1-\sigma} \||V|^\sigma\|_{\mathfrak{S}}, \quad m \in \mathbb{Z}, \tag{2.12}$$

with a constant C independent of m and g. Now use the q-triangle inequality (2.1):

$$\begin{aligned}\|g(A)J - Jg(B)\|_{\mathfrak{S}}^q &\leqslant \sum_{m\in\mathbb{Z}} \|g_m(A)J - Jg_m(B)\|_{\mathfrak{S}}^q \\ &\leqslant C\|J\|^{q(1-\sigma)} \||V|^\sigma\|_{\mathfrak{S}}^q \sum_{m\in\mathbb{Z}} (1+|m|)^{-q\beta}.\end{aligned}$$

Since $q\beta > 1$, the above bound leads to (2.11). □

2.3. *An important special case*

As explained in the Introduction, it is of particular interest for us to consider the case when the operator J is an orthogonal projection.

Condition 2.9. Let A be a self-adjoint operator on $\mathfrak{H}$, and let P be an orthogonal projection such that $PD(A) \subset D(A)$, and $PA(I-P)$ extends to $\mathfrak{H}$ as a bounded operator.

The condition $PD(A) \subset D(A)$ guarantees that PAP is self-adjoint on the domain $PD(A) \oplus (I-P)\mathfrak{H}$.

Theorem 2.10. *Suppose that f satisfies Condition 2.3 with some $\gamma > 0$, $n \geqslant 2$ and $R > 0$. Let $\mathfrak{S}$ be a q-normed ideal where $(n-\sigma)^{-1} < q \leqslant 1$ with some number $\sigma \in (0,1]$, $\sigma < \gamma$. Let A, P be a self-adjoint operator and an orthogonal projection satisfying Condition 2.9. Suppose that $|PA(I-P)|^\sigma \in \mathfrak{S}$. Then*

$$\|f(PAP)P - Pf(A)\|_{\mathfrak{S}} \leqslant C \,{\vert\!\vert\!\vert} f {\vert\!\vert\!\vert}_n R^{\gamma-\sigma} \||PA(I-P)|^\sigma\|_{\mathfrak{S}}$$

with a positive constant C independent of the operators A, P, function f and parameter R.

Proof. Denote

$$B_1 = PAP, \quad B_2 = A, \quad J = P.$$

Then

$$f(PAP)P - Pf(A) = f(B_1)J - Jf(B_2).$$

Since $V = B_1J - JB_2 = -PA(I-P)$, Theorem 2.4 leads to the required estimate. □

We also state the following consequence of Corollary 2.7.

Corollary 2.11. *Suppose that* $g \in \mathsf{C}_0^n(-\rho, \rho)$, *with some* $\rho > 0$ *and* $n \geqslant 2$. *Let* $\mathfrak{S}$ *be a* q*-normed ideal with* $(n-\sigma)^{-1} < q \leqslant 1$, *where* $\sigma \in (0, 1]$. *Let the operator* A *and orthogonal projection* P *be as in Theorem 2.10. Then*

$$\|g(PAP)P - Pg(A)\|_{\mathfrak{S}} \leqslant C \max_{0 \leqslant k \leqslant n} \big(\rho^k \|g^{(k)}\|_{\mathsf{L}^\infty}\big)\rho^{-\sigma} \||PA(I-P)|^\sigma\|_{\mathfrak{S}}, \tag{2.13}$$

with a constant C *independent of the operator* A *and projection* P.

3. *Proof of Theorem* 2.4

3.1. *A quasi-analytic extension*

In order to study functions of self-adjoint operators we use the formula known as the Helffer–Sjöstrand formula, see [**5**, **12**]. It requires the notion of a quasi-analytic extension of f. We use a somewhat more complicated definition than that in [**12**] since we are working with non-smooth functions. For the sake of simplicity we concentrate on compactly supported functions, although all the definitions with appropriate modifications can be given for more general functions. Let

$$\Pi = \Pi_+ \cup \Pi_-, \ \Pi_\pm = \{z = (x, y) : \pm y > 0\}.$$

Definition 3.1. Let $f \in \mathsf{C}_0^{0,\varkappa}(\mathbb{R}), 0 < \varkappa \leqslant 1$, and let $\tilde{f} \in \mathsf{C}_0^{0,\varkappa}(\mathbb{R}^2) \cap \mathsf{C}^1(\Pi)$ be a function such that

(i) $\tilde{f}(x, 0) = f(x)$, for all $x \in \mathbb{R}$, and
(ii) $|y|^{-1}\omega \in \mathsf{L}^1(\mathbb{R}^2)$, where

$$\omega(x, y) = \omega(x, y; \tilde{f}) = \frac{\partial}{\partial \overline{z}} \tilde{f}(x, y) = \frac{1}{2}\left[\frac{\partial}{\partial x}\tilde{f}(x, y) + i\frac{\partial}{\partial y}\tilde{f}(x, y)\right].$$

Then $\tilde{f}$ is said to be a quasi-analytic extension of f.

Proposition 3.2. *Let* A *be a self-adjoint operator on a Hilbert space* $\mathfrak{H}$. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function as in Definition 3.1, and let* $\tilde{f}$ *be its quasi-analytic extension. Then*

$$f(A) = \frac{1}{\pi}\iint \frac{\partial}{\partial \overline{z}}\tilde{f}(x, y) R(x + iy; A)\, dx\, dy. \tag{3.1}$$

Proof. It suffices to show that

$$f(t) = \frac{1}{\pi}\iint \frac{\partial}{\partial \overline{z}}\tilde{f}(x, y)(t - x - iy)^{-1}\, dx\, dy, \quad \forall t \in \mathbb{R}.$$

For a $\delta > 0$ split the plane into two regions:

$$\mathcal{D}_1 = \mathcal{D}_1(\delta) = \{z : |y| \geqslant \delta\}, \quad \mathcal{D}_2 = \mathcal{D}_2(\delta) = \{z : |y| < \delta\}.$$

First estimate the contribution from $\mathcal{D}_2$:

$$\left|\iint_{\mathcal{D}_2} \omega(x, y)(t - x - iy)^{-1}\, dx\, dy\right| \leqslant \iint_{|y|<\delta} |y|^{-1}|\omega(x, y)|\, dx\, dy.$$

By Definition 3.1, the integral tends to zero as $\delta \to 0$. Using the property

$$\frac{\partial}{\partial \overline{z}}\tilde{f}(x, y)(t - x - iy)^{-1} = \frac{\partial}{\partial \overline{z}}\Big(\tilde{f}(x, y)(t - x - iy)^{-1}\Big),$$

we simplify the remaining integral:

$$\iint_{\mathcal{D}_1} \omega(x,y)(t-x-iy)^{-1}\,dx\,dy = \int_{\mathbb{R}} F_\delta(x;t)\,dx,$$

where

$$F_\delta(x;t) = \frac{1}{2\pi i}\big(\tilde{f}(x,\delta)(t-x-i\delta)^{-1} - \tilde{f}(x,-\delta)(t-x+i\delta)^{-1}\big).$$

Rewrite:

$$\begin{aligned} 2\pi i F_\delta(x;t) &= f(x)\big((t-x-i\delta)^{-1} - (t-x+i\delta)^{-1}\big) \\ &\quad + \big(\tilde{f}(x,\delta) - \tilde{f}(x,0)\big)(t-x-i\delta)^{-1} \\ &\quad + \big(\tilde{f}(x,-\delta) - \tilde{f}(x,0)\big)(t-x+i\delta)^{-1}. \end{aligned}$$

The last two terms converge to zero for all $x \neq t$ as $\delta \to 0$. Moreover, since $f \in \mathsf{C}^{0,\varkappa}$, we have

$$|\tilde{f}(x,\pm\delta) - \tilde{f}(x,0)| \leqslant C\delta^{\varkappa},$$

and hence the last two terms on the right-hand side do not exceed $C|t-x|^{\varkappa-1}$. Thus their integral over x converges to zero as $\delta \to 0$. Consequently,

$$\int F_\delta(x;t)\,dx = \frac{1}{2\pi i}\int f(x)\big((t-x-i\delta)^{-1} - (t-x+i\delta)^{-1}\big)\,dx + o(1), \quad \delta \to 0.$$

Since f is continuous, the integral converges to $f(t)$, as claimed. □

Versions of the formula (3.1) have been known well before the paper [**12**]. In [**6**] Dyn'kin developed functional calculus for operators in Banach spaces, based on a formula in the spirit of (3.1). Similar functional constructions can be found in Hörmander's book [**14**, Section 3.1], so (3.1) must have been known to him earlier. In [**7**] Dyn'kin found a characterization of the classical Besov and Sobolev classes in terms of quasi-analytic extensions. These results were used in [**8**].

Let us describe a convenient quasi-analytic extension of the function f satisfying Condition 2.3. For convenience assume that $x_0 = 0$. For $b > 0$ introduce the domain

$$F_b = \{(x,y) \in \mathbb{R}^2 : |y| < b|x|\}. \tag{3.2}$$

By $U_b = U_b(x,y)$ we denote the characteristic function of F_b, that is,

$$U_b(x,y) = \begin{cases} 1, & |y| < b|x|, \\ 0, & |y| \geqslant b|x|. \end{cases} \tag{3.3}$$

Let $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ be a function such that

$$\zeta(t) = 1 \quad \text{for } |t| \leqslant 1/2, \quad \text{and} \quad \zeta(t) = 0, |t| \geqslant 1. \tag{3.4}$$

Lemma 3.3. *Let f satisfy Condition (2.3) with $x_0 = 0$ and some $n \geqslant 2$ and $R = 1$. Then f has a quasi-analytic extension $\tilde{f} \in \mathsf{C}^{0,\varkappa}(\mathbb{R}^2) \cap \mathsf{C}^1(\Pi)$, with the $\varkappa$ defined in (2.6), such that $\tilde{f}(x,y) = 0$ if $|y| > |x|$. Moreover, the derivative*

$$\omega(x,y) = \frac{\partial}{\partial \bar{z}}\tilde{f}(x,y),$$

satisfies the bound

$$|\omega(x,y)| \leqslant C_n \,|\!|\!| f |\!|\!|_n |x|^{\gamma-n}|y|^{n-1}U_1(x,y)\chi_1(x), \tag{3.5}$$

for $x \neq 0, y \in \mathbb{R}$. The constant C_n does not depend on f.

Proof. We use a slight modification of the 'standard' construction of a quasi-analytic extension which can be found, for example, in [**14**, Section 3.1], or [**5**, Chapter 2]. Let the function ζ be defined as in (3.4). Without loss of generality assume $[\![f]\!]_n = 1$. Define for all $x \neq 0$ and $y \in \mathbb{R}$:

$$\tilde{f}(x,y) = \left[\sum_{l=0}^{n-1} f^{(l)}(x)\frac{(iy)^l}{l!}\right]\sigma(x,y),\ \sigma(x,y) = \zeta\left(\frac{y}{x}\right).$$

For $x = 0$ we set $\tilde{f}(0,y) = 0, y \in \mathbb{R}$. Clearly, $\tilde{f}(x,0) = f(x)$, for all $x \in \mathbb{R}$. Moreover, $\tilde{f}$ is trivially continuous for all $x \neq 0$. At $x = 0$ it is continuous because of the bound $|\tilde{f}(x,y)| \leqslant C|x|^\gamma$, for all $x \neq 0, y \in \mathbb{R}$, which follows from (2.5). Furthermore, $\tilde{f} \in \mathsf{C}^1(\mathbb{C} \setminus \{0\})$ and one checks directly for $x \neq 0$ that

$$\frac{\partial \tilde{f}}{\partial x}(x,y) = \left[\sum_{l=0}^{n-1} f^{(l+1)}(x)\frac{(iy)^l}{l!}\right]\sigma(x,y) + \left[\sum_{l=0}^{n-1} f^{(l)}(x)\frac{(iy)^l}{l!}\right]\frac{\partial\sigma}{\partial x}(x,y),$$

and

$$\frac{\partial \tilde{f}}{\partial y}(x,y) = i\left[\sum_{l=1}^{n-1} f^{(l)}(x)\frac{(iy)^{l-1}}{(l-1)!}\right]\sigma(x,y) + \left[\sum_{l=0}^{n-1} f^{(l)}(x)\frac{(iy)^l}{l!}\right]\frac{\partial\sigma}{\partial y}(x,y).$$

Thus

$$\begin{aligned}\omega(x,y) = \frac{\partial}{\partial \bar{z}}\tilde{f}(x,y) &= \frac{1}{2}\left(\frac{\partial \tilde{f}}{\partial x} + i\frac{\partial \tilde{f}}{\partial y}\right)(x,y)\\ &= \frac{1}{2}f^{(n)}(x)\frac{(iy)^{n-1}}{(n-1)!}\sigma(x,y) + \left[\sum_{l=0}^{n-1} f^{(l)}(x)\frac{(iy)^l}{l!}\right]\frac{\partial\sigma}{\partial\bar{z}}(x,y),\end{aligned} \tag{3.6}$$

for all $x \neq 0$. Since we have assumed $[\![f]\!]_n = 1$, (2.5) implies that for $x \neq 0$ we have

$$|\tilde{f}_y(x,y)| \leqslant C_n\left[\sum_{l=1}^{n-1}|x|^{\gamma-l}|y|^{l-1}|\zeta(yx^{-1})| + |x|^{-1}\left[\sum_{l=0}^{n-1}|x|^{\gamma-l}|y|^l\right]|\zeta'(yx^{-1})|\right]\chi_1(x). \tag{3.7}$$

Recall that $|y| < |x| \leqslant 1$ on the support of $\zeta(yx^{-1})$, so that

$$|\tilde{f}_y(x,y)| \leqslant C|x|^{\gamma-1}U_1(x,y)\chi_1(x), \quad x \neq 0.$$

Using the bound $|y| < |x| \leqslant 1$ again, we deduce that

$$|\tilde{f}_y(x,y)| \leqslant C|y|^{\varkappa-1}U_1(x,y)\chi_1(x), \quad y \neq 0,$$

where $\varkappa = \min\{\gamma, 1\}$. One easily proves the same bounds for the derivative $\tilde{f}_x$. These bounds imply that $\tilde{f} \in \mathsf{C}^{0,\varkappa}(\mathbb{R}^2)$.

In order to establish (3.5) we use (3.6), so that for all $x \neq 0$ we have

$$|\omega(x,y)| \leqslant C_n\left[|x|^{\gamma-n}|y|^{n-1}|\zeta(yx^{-1})| + |x|^{-1}\left[\sum_{l=0}^{n-1}|x|^{\gamma-l}|y|^l\right]|\zeta'(yx^{-1})|\right]\chi_1(x). \tag{3.8}$$

The first term on the right-hand side already satisfies (3.5). Using the formula

$$\zeta'(yx^{-1}) = \zeta'(yx^{-1})U_1(x,y)\big(1 - U_{\frac{1}{2}}(x,y)\big),$$

we conclude that in the second term on the right-hand side of (3.8) we have $|x|/2 \leqslant |y| \leqslant |x|$. Hence this term satisfies (3.5) as well.

Since $\gamma > 0$, the bound (3.5) ensures that $|y|^{-1}\omega \in \mathsf{L}^1(\mathbb{R}^2)$. This completes the proof. $\square$

3.2. *Proof of Theorem 2.4*

Without loss of generality we may assume that $x_0 = 0$, and that $\|f\|_n = 1$. Also, in view of Remark 2.6, it suffices to obtain (2.7) for $R = 1$ only.

Let $\tilde f$ be the quasi-analytic extension constructed in Lemma 3.3. By the formula (3.1) and resolvent identity (2.2) we can write

$$T = f(A)J - Jf(B) = -\frac{1}{\pi}\iint \omega(x,y)R(z;A)VR(z;B)\,dx\,dy, \quad z = x+iy.$$

Let $\rho(x,y) = 8^{-1}|y|$, and let $\{\mathcal{D}_j\}, j = 1,2,\dots$, be a family of open discs with finite intersection property centred at some points $z_j = (x_j, y_j) \in \Pi$, of radius $\rho(x_j, y_j)$ such that

$$\cup_j \mathcal{D}_j = \Pi.$$

Let $\phi_j \in \mathsf{C}_0^\infty(\Pi)$ be an associated partition of unity such that

$$|\partial_x^l \partial_y^k \phi_j(x,y)| \leqslant C_{l,k}\rho(x,y)^{-l-k}.$$

By the 'finite intersection property' we mean that the number of discs having non-empty common intersection is uniformly bounded. The existence of such a covering and such a partition of unity follows from [**14**, Theorem 1.4.10]. Estimate the quasi-norm of

$$T_j = \iint \phi_j(x,y)\omega(x,y)R(z;A)VR(z;B)\,dx\,dy.$$

For $z \in \mathcal{D}_j$ expand $R(z;A)$ and $R(z;B)$ in the uniformly norm-convergent series

$$R(z;K) = \sum_{k=0}^\infty (z-z_j)^k R(z_j;K)^{k+1},$$

where $K = A$ or B. The uniformity of convergence is guaranteed by the bound $|z - z_j|\|R(z_j;K)\| \leqslant 1/8$. Denote $\mathbf{k} = (k_1, k_2),\ k_1 \geqslant 0, k_2 \geqslant 0$. Therefore, we can now expand T_j in the norm-convergent series:

$$T_j = \sum_{\mathbf{k}} T_{j\mathbf{k}},\ T_{j\mathbf{k}} = \left[\iint \omega(x,y)\phi_j(x,y)(z-z_j)^{k_1+k_2}\,dx\,dy\right] R(z_j;A)^{k_1} W_j R(z_j;B)^{k_2},$$

$$W_j = R(z_j;A)VR(z_j;B).$$

By Lemma 2.2,

$$\|W_j\|_{\mathfrak{S}} \leqslant 2^{1-\sigma}\|J\|^{1-\sigma}\||V|^\sigma\|_{\mathfrak{S}}|y_j|^{-1-\sigma}.$$

Estimate the $\mathfrak{S}$-quasi-norm of each $T_{j\mathbf{k}}$. It follows from (3.5) that

$$\|T_{j\mathbf{k}}\|_{\mathfrak{S}} \leqslant C|y_j|^{-1-\sigma}8^{-k_1+k_2}\|J\|^{1-\sigma}\ \||V|^\sigma\|_{\mathfrak{S}} \iint_{\mathcal{D}_j} |x|^{\gamma-n}|y|^{n-1}U_1(x,y)\chi_1(x)\,dx\,dy.$$

A straightforward calculation shows that if $\mathcal{D}_j \cap F_1 \neq \varnothing$ then $(x_j, y_j) \in F_2$ and $\mathcal{D}_j \subset F_4$, see (3.2) for the definition of $F_b, b > 0$. Thus for all $(x,y) \in \mathcal{D}_j$ we have

$$|x - x_j| < 8^{-1}|y_j| < 4^{-1}|x_j|, \quad |y - y_j| < 8^{-1}|y_j|,$$

so

$$\tfrac{3}{4}|x_j| < |x| < \tfrac{5}{4}|x_j|, \quad \tfrac{7}{8}|y_j| < |y| < \tfrac{9}{8}|y_j|.$$

Consequently,

$$\|T_{j\mathbf{k}}\|_{\mathfrak{S}} \leqslant C8^{-k_1+k_2}\|J\|^{1-\sigma}\ \||V|^\sigma\|_{\mathfrak{S}}|x_j|^{\gamma-n}|y_j|^{n-\sigma}U_2(x_j,y_j)\chi_2(x_j).$$

Since $\mathcal{D}_j \subset F_4$, by the q-triangle inequality, we have

$$\begin{aligned}\|T_j\|_{\mathfrak{S}}^q &\leqslant \sum_{\mathbf{k}} \|T_{j\mathbf{k}}\|_{\mathfrak{S}}^q \\ &\leqslant C\|J\|^{q(1-\sigma)}\ \||V|^\sigma\|_{\mathfrak{S}}^q\ |x_j|^{q(\gamma-n)}|y_j|^{q(n-\sigma)}U_2(x_j,y_j)\chi_2(x_j)\sum_{\mathbf{k}} 8^{-(k_1+k_2)q} \\ &\leqslant C\|J\|^{q(1-\sigma)}\ \||V|^\sigma\|_{\mathfrak{S}}^q \iint\limits_{\mathcal{D}_j} |x|^{q(\gamma-n)}|y|^{q(n-\sigma)-2}U_4(x,y)\chi_4(x)\,dx\,dy.\end{aligned}$$

Use the q-triangle inequality again to sum over j:

$$\begin{aligned}\|T\|_{\mathfrak{S}}^q &\leqslant \sum_j \|T_j\|_{\mathfrak{S}}^q \\ &\leqslant C\|J\|^{q(1-\sigma)}\||V|^\sigma\|_{\mathfrak{S}}^q \sum_j \iint\limits_{\mathcal{D}_j} |x|^{q(\gamma-n)}|y|^{q(n-\sigma)-2}U_4(x,y)\chi_4(x)\,dx\,dy \\ &\leqslant C\|J\|^{q(1-\sigma)}\||V|^\sigma\|_{\mathfrak{S}}^q \iint |x|^{q(\gamma-n)}|y|^{q(n-\sigma)-2}U_4(x,y)\chi_4(x)\,dx\,dy,\end{aligned}$$

where we have used the finite intersection property. By assumption, we have $q(n-\sigma)-2 > -1$, and hence the integral on the right-hand side is bounded by

$$C\int_{|x|<4} |x|^{q(\gamma-n)}\left[\int_{|y|\leqslant 4|x|} |y|^{q(n-\sigma)-2}\,dy\right]dx \leqslant \tilde{C}\int_0^4 x^{q(\gamma-\sigma)-1}\,dx \leqslant C.$$

This proves (2.7) for $R=1$. As explained in Remark 2.6, this immediately leads to (2.7) for general $R>0$.

4. *Trace asymptotics for multidimensional Wiener–Hopf operators with discontinuous symbols*

4.1. *Definitions*

Now we derive from the theorems established above estimates for some Wiener–Hopf operators on $\mathsf{L}^2(\mathbb{R}^d)$. Let $\Lambda, \Omega \subset \mathbb{R}^d, d \geqslant 2$, be two domains, and let χ_Λ, χ_Ω be their characteristic functions. For a bounded complex-valued function $a = a(\mathbf{x}, \boldsymbol{\xi})$, called *symbol*, define the pseudo-differential operator

$$\big(\mathrm{Op}_\alpha(a)u\big)(\mathbf{x}) = \frac{\alpha^d}{(2\pi)^{d/2}}\iint e^{i\alpha\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{y})}a(\mathbf{x},\boldsymbol{\xi})u(\mathbf{y})\,d\mathbf{y}\,d\boldsymbol{\xi}, u \in \mathsf{S}(\mathbb{R}^d).$$

It is a standard fact that under the condition $a \in \mathsf{W}^{d+1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ the norm of the operator $\mathrm{Op}_\alpha(a)$ is bounded uniformly in $\alpha \geqslant 1$, see, for example, [**27**, Lemma 3.9].

Under (truncated) Wiener–Hopf operators with discontinuous symbols here, we understand operators of the form

$$S_\alpha(a) = S_\alpha(a;\Lambda,\Omega) = \chi_\Lambda P_{\Omega,\alpha}\mathrm{Re}\,\mathrm{Op}_\alpha(a)P_{\Omega,\alpha}\chi_\Lambda, \quad P_{\Omega,\alpha} = \mathrm{Op}_\alpha(\chi_\Omega). \tag{4.1}$$

The function a is assumed to be smooth, and it is the presence of the projection $P_{\Omega,\alpha}$ that suggests the term 'discontinuous symbol'. In Section 5, we consider somewhat more general discontinuous symbols.

We impose the following conditions on the domains Λ and Ω.

CONDITION 4.1. The domains $\Lambda, \Omega \subset \mathbb{R}^d, d \geqslant 2$, are both Lipschitz domains; Ω is bounded, and either Λ or $\mathbb{R}^d \setminus \Lambda$ is bounded.

CONDITION 4.2. The domain Λ is piece-wise C^1, and Ω is piece-wise C^3.

By the Lipschitz domain we understand a domain which locally looks like a set of points above the graph of a suitable Lipschitz function. Precise definitions of this property as well as of piece-wise smoothness are given in [**29**, Definition 2.1].

We are interested in the large α asymptotics of the trace of the operator

$$D_\alpha(a, \Lambda, \Omega; g) = \chi_\Lambda g\big(S_\alpha(a, \Lambda, \Omega)\big)\chi_\Lambda - \chi_\Lambda g\big(S_\alpha(a, \mathbb{R}^d, \Omega)\big)\chi_\Lambda \tag{4.2}$$

with a function $g : \mathbb{R} \to \mathbb{C}$ which is smooth except for finitely many points.

Let us define the asymptotic coefficients entering the main asymptotic formulas. For a symbol $b = b(\mathbf{x}, \boldsymbol{\xi})$ let

$$\mathfrak{W}_0(b) = \mathfrak{W}_0(b; \Lambda, \Omega) = \frac{1}{(2\pi)^d} \int_\Omega \int_\Lambda b(\mathbf{x}, \boldsymbol{\xi})\, d\mathbf{x}\, d\boldsymbol{\xi}. \tag{4.3}$$

For any $(d-1)$-dimensional Lipschitz surfaces L, P denote

$$\mathfrak{W}_1(b) = \mathfrak{W}_1(b; L, P) = \frac{1}{(2\pi)^{d-1}} \int_L \int_P b(\mathbf{x}, \boldsymbol{\xi}) |\mathbf{n}_L(\mathbf{x}) \cdot \mathbf{n}_P(\boldsymbol{\xi})|\, dS_{\boldsymbol{\xi}}\, dS_{\mathbf{x}}, \tag{4.4}$$

where $\mathbf{n}_L(\mathbf{x})$ and $\mathbf{n}_P(\boldsymbol{\xi})$ denote the exterior unit normals to L and P defined for a.e. $\mathbf{x}$ and $\boldsymbol{\xi}$, respectively. For any function $g \in \mathsf{C}^{0,\varkappa}(\mathbb{C})$, $\varkappa > 0$, and any number $s \in \mathbb{C}$, we also define

$$\mathfrak{A}(g; s) = \frac{1}{(2\pi)^2} \int_0^1 \frac{g(st) - (1-t)g(0) - tg(s)}{t(1-t)}\, dt. \tag{4.5}$$

The next result is found in [**29**, Theorem 2.5]:

PROPOSITION 4.3. *Let $\Lambda, \Omega \subset \mathbb{R}^d, d \geqslant 2$, be two domains satisfying Conditions 4.1 and 4.2. Let $a \in \mathsf{W}^{d+2,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ be a complex-valued function. Let $g_p(t) = t^p$ with some $p = 0, 1, \dots$. Then*

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \operatorname{tr} D_\alpha(a, \Lambda, \Omega; g_p) = \mathfrak{W}_1(\mathfrak{A}(g_p; \operatorname{Re} a), \partial\Lambda, \partial\Omega). \tag{4.6}$$

Observe that for the polynomials $g_0(t) \equiv 1$ and $g_1(t) = t$ both sides of the above formula equal zero, so the asymptotics hold trivially.

The main focus of [**29**] was on non-smooth domains Λ and Ω. As a result the above proposition was formally proved in [**29**] for the case $d \geqslant 2$ only, although a similar, somewhat simplified argument should give (4.6) for the case $d = 1$ as well, with an appropriately modified definition of the coefficient $\mathfrak{W}_1$, see, for example, [**30**]. However, we do not pursue this objective in the current paper.

Our aim here is to extend Proposition 4.3 to functions g that have just C^2 local smoothness, and may lose differentiability at finitely many points.

THEOREM 4.4. *Let $d \geqslant 2$, and let the domains Λ, Ω be two domains satisfying Conditions 4.1 and 4.2. Let the symbol $a = a(\mathbf{x}, \boldsymbol{\xi})$ be a globally bounded C^∞-function. Let $X = \{z_1, z_2, \dots, z_N\} \subset \mathbb{R}$, $N < \infty$, be a collection of points on the real line. Suppose that*

$g \in \mathsf{C}^2(\mathbb{R} \setminus X)$ is a function such that in a neighbourhood of each point $z \in X$ it satisfies the bound

$$|g^{(k)}(t)| \leqslant C_k |t-z|^{\gamma-k}, \quad k = 0, 1, 2,$$

with some $\gamma > 0$. Then

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \operatorname{tr} D_\alpha(a, \Lambda, \Omega; g) = \mathfrak{W}_1\big(\mathfrak{A}(g; \operatorname{Re} a); \partial\Lambda, \partial\Omega\big). \tag{4.7}$$

Let us make some comments on Theorem 4.4.

REMARK 4.5. (i) The assumption $a \in \mathsf{C}^\infty$ is made for simplicity. In fact, some finite smoothness, depending on the value of the parameter γ, would suffice, but we have chosen to avoid ensuing technicalities.

(ii) Suppose that Λ is bounded and that $g \in \mathsf{C}^\infty(\mathbb{R})$ is a function such that $g(0) = 0$. Then both operators on the right-hand side of (4.2) are trace class, and formula (4.7) is just an indirect way to write the asymptotics

$$\begin{aligned} \operatorname{tr} g(S_\alpha) = \alpha^d\ & \mathfrak{W}_0(g(\operatorname{Re} a); \Lambda, \Omega) \\ & + \alpha^{d-1}\log\alpha\ \mathfrak{W}_1(\mathfrak{A}(g; \operatorname{Re} a); \partial\Lambda, \partial\Omega) + o(\alpha^{d-1}\log\alpha), \end{aligned} \tag{4.8}$$

as $\alpha \to \infty$, established in [**29**, Theorem 2.3]. Indeed, one can show that

$$\operatorname{tr}\chi_\Lambda g(S_\alpha(a; \mathbb{R}^d, \Omega))\chi_\Lambda = \alpha^d \mathfrak{W}_0(g(\operatorname{Re} a); \Lambda, \Omega) + O(\alpha^{d-1}), \quad \alpha \to \infty,$$

see, for example, [**27**, Section 12.3], where a similar calculation was done. Substituting this in (4.7) one obtains (4.8).

If the symbol a depends only on the variable $\boldsymbol{\xi}$, that is, $a = a(\boldsymbol{\xi})$, then the reduction of (4.7) to (4.8) for bounded Λ is more straightforward. Indeed, in this case the second operator on the right-hand side of (4.2) is given by

$$\chi_\Lambda \operatorname{Op}_\alpha\big(g(\operatorname{Re} a\chi_\Omega)\big)\chi_\Lambda. \tag{4.9}$$

This operator is clearly trace class for any continuous g such that $g(0) = 0$. Integrating its kernel along the diagonal, one easily finds the exact value of its trace: $\alpha^d \mathfrak{W}_0(g(\operatorname{Re} a); \Lambda, \Omega)$.

(iii) Suppose that $a = a(\boldsymbol{\xi})$, and that $g = 0$ on the range of the function $\operatorname{Re} a$. Then the operator (4.9) equals zero, so that Theorem 4.4 implies that $g(S_\alpha(a, \Lambda, \Omega))$ is trace class and its trace satisfies the asymptotic formula

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \operatorname{tr} g\big(S_\alpha(a, \Lambda, \Omega)\big) = \mathfrak{W}_1\big(\mathfrak{A}(g; \operatorname{Re} a); \partial\Lambda, \partial\Omega\big). \tag{4.10}$$

It is interesting to point out that this formula holds for both bounded or unbounded domains Λ.

Another proof of formula (4.10) in the special case $a \equiv 1$ and $g(0) = g(1) = 0$ was given in [**19**]. It was motivated by the study of the *entanglement entropy* for free Fermions at zero temperature, see also [**10**] and [**13**]. Mathematically speaking, the entropy is found as trace of the operator $\eta_\beta(S_\alpha), \beta > 0$, with the operator $S_\alpha = S_\alpha(1, \Lambda, \Omega)$ for bounded Λ and Ω, and with the function η_β defined by

$$\eta_\beta(t) = \begin{cases} \dfrac{1}{1-\beta}\log(t^\beta + (1-t)^\beta), \ \beta > 0, \beta \neq 1 \text{ (Rényi entropy)}, \\ -t\log t - (1-t)\log(1-t), \ \beta = 1 \text{ (von Neumann entropy)}, \end{cases} \tag{4.11}$$

if $t \in [0, 1]$, end extended by 0 to the rest of the real line. Clearly, for $\beta \neq 1$ the function η_β satisfies the conditions of Theorem 4.4 with $\gamma = \beta$, and for $\beta = 1$ – with arbitrary $\gamma < 1$.

Before proving Theorem 4.4, we list some useful facts.

Lemma 4.6. (i) *If* $g \in \mathsf{W}^{1,\infty}(\mathbb{R})$, *then*

$$|\mathfrak{A}(g;s)| \leqslant \frac{1}{\pi^2}|s|\|g'\|_{\mathsf{L}^\infty}. \tag{4.12}$$

(ii) *Suppose that* g *satisfies* (2.5) *with some* $0 < R \leqslant 1$. *Then*

$$|\mathfrak{A}(g;s)| \leqslant C|s|^{\varkappa/2}R^{\gamma/2}{\mathopen{|\!|\!|}} g {\mathclose{|\!|\!|}}_1, \quad \varkappa = \min\{1,\gamma\}. \tag{4.13}$$

Proof. Using the formula $t^{-1}(1-t)^{-1} = t^{-1} + (1-t)^{-1}$, we rewrite $\mathfrak{A}$ in the form

$$(2\pi)^2\mathfrak{A}(g;s) = \int_0^1 \frac{g(st)-g(0)}{t}\,dt + \int_0^1 \frac{g(st)-g(s)}{1-t}\,dt. \tag{4.14}$$

Now (4.12) follows from the bounds

$$|g(st)-g(0)| \leqslant \|g'\|_{\mathsf{L}^\infty}|s||t|, \quad |g(st)-g(s)| \leqslant \|g'\|_{\mathsf{L}^\infty}|s||t-1|.$$

Proof of (4.13). Assume without loss of generality that ${\mathopen{|\!|\!|}} g {\mathclose{|\!|\!|}}_1 = 1$. By (2.6) the first integral in (4.14) is estimated by

$$\big(2\|g\|_{\mathsf{L}^\infty}\big)^{1/2}\int_0^1 \frac{|g(st)-g(0)|^{1/2}}{t}\,dt \leqslant CR^{\gamma/2}|s|^{\varkappa/2}\int_0^1 t^{(\varkappa/2)-1}dt \leqslant \tilde{C}R^{\gamma/2}|s|^{\varkappa/2}.$$

Similarly, the second integral is bounded by

$$\big(2\|g\|_{\mathsf{L}^\infty}\big)^{1/2}\int_0^1 \frac{|g(st)-g(s)|^{1/2}}{1-t}\,dt \leqslant CR^{\gamma/2}|s|^{\varkappa/2}\int_0^1 (1-t)^{(\varkappa/2)-1}dt \leqslant \tilde{C}R^{\gamma/2}|s|^{\varkappa/2}.$$

This proves (4.13). □

A crucial ingredient in the proof of Theorem 4.4 is the following lemma.

Lemma 4.7. *Let* Λ *and* Ω *satisfy Condition* 4.1, *and let the symbol* a *be as in Theorem* 4.4. *Then for any* $q \in (0,1]$, *and all* $\alpha \geqslant 2$ *we have*

$$\|\chi_\Lambda P_{\Omega,\alpha}Op_\alpha(a)P_{\Omega,\alpha}(I-\chi_\Lambda)\|_{\mathfrak{S}_q}^q \leqslant C_q\alpha^{d-1}\log\alpha,$$

with a constant C_q *independent of* α.

The above bound can be derived from [**28**, Theorem 4.6] in the same way as [**28**, Corollary 4.7].

4.2. *Proof of Theorem* 4.4

Throughout the proof, we denote for brevity $D_\alpha(g) = D_\alpha(a,\Lambda,\Omega;g)$, and $\mathfrak{W}_1(g) = \mathfrak{W}_1(\mathfrak{A}(g;\operatorname{Re}a);\partial\Lambda,\partial\Omega)$.

The proof splits into two parts.

Step 1: Proof of formula (4.7) for $g \in \mathsf{C}^2(\mathbb{R})$. Recall that the norm of the operator $\mathrm{Op}_\alpha(a)$ is bounded uniformly in $\alpha \geqslant 1$, so without loss of generality we may assume that $\|\mathrm{Op}_\alpha(a)\| \leqslant 1/2$ and that g is supported on the interval $[-1,1]$, and it is real-valued.

By the Weierstrass Theorem, for any $\varepsilon > 0$ one can find a real polynomial g_ε such that the function $f_\varepsilon = g - g_\varepsilon$ satisfies the bound

$$\max_{0\leqslant k\leqslant 2}\max_{|t|\leqslant 1}|f_\varepsilon^{(k)}(t)| < \varepsilon. \tag{4.15}$$

Now we use Corollary 2.11 with $\mathfrak{S} = \mathfrak{S}_1$, $n = 2$, $R = 1$, arbitrary $\sigma \in (0,1)$, and the operators

$$A = P_{\Omega,\alpha} \mathrm{Re}\, \mathrm{Op}_\alpha(a) P_{\Omega,\alpha}, \quad P = \chi_\Lambda. \tag{4.16}$$

Corollary 2.11, Lemma 4.7 and bound (4.15) give that

$$\|D_\alpha(f_\varepsilon)\|_{\mathfrak{S}_1} \leqslant C\varepsilon \|\chi_\Lambda P_{\Omega,\alpha} \mathrm{Op}_\alpha(a) P_{\Omega,\alpha}(I - \chi_\Lambda)\|_{\mathfrak{S}_\sigma}^\sigma$$

$$\leqslant C\varepsilon \alpha^{d-1} \log \alpha, \quad \alpha \geqslant 2,$$

and as a consequence,

$$\mathrm{tr} D_\alpha(g) \leqslant \mathrm{tr} D_\alpha(g_\varepsilon) + \|D_\alpha(f_\varepsilon)\|_{\mathfrak{S}_1} \leqslant \mathrm{tr} D_\alpha(g_\varepsilon) + C\varepsilon \alpha^{d-1} \log \alpha.$$

Now, using Proposition 4.3 for the polynomial g_ε we get

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} D_\alpha(g) \leqslant \mathfrak{W}_1(g_\varepsilon) + C\varepsilon.$$

Due to (4.12) and (4.15), the asymptotic coefficient $\mathfrak{W}_1(f_\varepsilon)$ tends to zero as $\varepsilon \to 0$, so that

$$\mathfrak{W}_1(g_\varepsilon) = \mathfrak{W}_1(g) - \mathfrak{W}_1(f_\varepsilon) \to \mathfrak{W}_1(g), \quad \varepsilon \to 0.$$

This implies that

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} D_\alpha(g) \leqslant \mathfrak{W}_1(g).$$

In the same way one obtains the appropriate lower bound for the lim inf. This completes the proof of (4.7) for $g \in \mathsf{C}^2(\mathbb{R})$.

Step 2: Completion of the proof. Let g be a function as specified in the theorem. As before we assume that g is real-valued. By choosing an appropriate partition of unity, we may assume that the set X consists of one point only, which, without loss of generality, we set to be zero. Now, as in the first part of the proof, we assume that $\|\mathrm{Op}_\alpha(a)\| \leqslant 1/2$, and that g is supported on the interval $[-1, 1]$.

Let $\zeta \in \mathsf{C}_0^\infty(\mathbb{R})$ be a real-valued function, satisfying (3.4). Represent $g = g_R^{(1)} + g_R^{(2)}, 0 < R \leqslant 1$, where $g_R^{(1)}(t) = g(t)\zeta(tR^{-1})$, $g_R^{(2)}(t) = g(t) - g_R^{(1)}(t)$. It is clear that $g_R^{(2)} \in \mathsf{C}^2(\mathbb{R})$, so one can use the formula (4.7) established in Part 1 of the proof:

$$\lim_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} D_\alpha(g_R^{(2)}) = \mathfrak{W}_1(g_R^{(2)}). \tag{4.17}$$

For $g_R^{(1)}$ we use Theorem 2.10 with $\mathfrak{S} = \mathfrak{S}_1$, $n = 2$, an arbitrary $\sigma \in (0,1], \sigma < \gamma$, and with the operators A, P defined in (4.16). Noticing that $\boldsymbol{|} g_R^{(1)} \boldsymbol{|}_2 \leqslant C \boldsymbol{|} g \boldsymbol{|}_2$, we get from Theorem 2.10 and Lemma 4.7 that

$$\left\| D_\alpha(g_R^{(1)}) \right\|_{\mathfrak{S}_1} \leqslant C_\sigma R^{\gamma-\sigma} \|(I - \chi_\Lambda) A \chi_\Lambda\|_{\mathfrak{S}_\sigma}^\sigma \leqslant C_\sigma R^{\gamma-\sigma} \alpha^{d-1} \log \alpha, \quad C_\sigma = C_\sigma(g),$$

for all $\alpha \geqslant 2$. Therefore,

$$\mathrm{tr} D_\alpha(g) \leqslant \mathrm{tr} D_\alpha(g_R^{(2)}) + C_\sigma R^{\gamma-\sigma} \alpha^{d-1} \log \alpha.$$

Using (4.17), we obtain the bound

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1} \log \alpha} D_\alpha(g) \leqslant \mathfrak{W}_1(g_R^{(2)}) + CR^{\gamma-\sigma}.$$

Due to (4.13), the asymptotic coefficient $\mathfrak{W}_1(g_R^{(1)})$ converges to zero as $R \to 0$. Thus

$$\mathfrak{W}_1(g_R^{(2)}) = \mathfrak{W}_1(g) - \mathfrak{W}_1(g_R^{(1)}) \to \mathfrak{W}_1(g), \quad R \to 0.$$

This implies that

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} D_\alpha(g) \leqslant \mathfrak{W}_1(g).$$

In the same way, one obtains the appropriate lower bound for the lim inf. This completes the proof.

5. *More on symbols with jump discontinuities*

In this section, we give a variant of Theorem 4.4 for more general discontinuous symbols: instead of the symbols $a(\mathbf{x},\boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi})$ we study $a(\mathbf{x},\boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi}) + a_1(\mathbf{x},\boldsymbol{\xi})\chi_{\Omega_1}(\boldsymbol{\xi})$, $\Omega_1 = \mathbb{R}^d \setminus \Omega$, that is, symbols allowed to have jump discontinuities on the surface $\partial\Omega$.

Along with the operator (4.1) introduce the notation for its non-symmetric variant:

$$T_\alpha(a) = T_\alpha(a;\Lambda,\Omega) = \chi_\Lambda P_{\Omega,\alpha} \mathrm{Op}_\alpha(a) P_{\Omega,\alpha}\chi_\Lambda,$$

so that $S_\alpha(a) = \operatorname{Re} T_\alpha(a)$. Let $\Omega_1 = \mathbb{R}^d \setminus \Omega$, and let a, a_1 be two smooth symbols. Define

$$V_\alpha(a,a_1) = V_\alpha(a,a_1;\Lambda,\Omega) = T_\alpha(a;\Lambda,\Omega) + T_\alpha(a_1;\Lambda,\Omega_1),$$
$$H_\alpha(a,a_1) = H_\alpha(a,a_1;\Lambda,\Omega) = S_\alpha(a;\Lambda,\Omega) + S_\alpha(a_1;\Lambda,\Omega_1).$$

Both symbols a and a_1 are assumed to have compact supports in the variable $\boldsymbol{\xi}$.

To state the result, we need to define instead of (4.5) the coefficient

$$\mathfrak{D}(g;s,s_1) = \frac{1}{(2\pi)^2}\int_0^1 \frac{g(st+s_1(1-t)) - tg(s) - (1-t)g(s_1)}{t(1-t)}\,dt, \tag{5.1}$$

$s, s_1 \in \mathbb{C}$.

Theorem 5.1. *Let the domains Λ, Ω and function g be in Theorem 4.4. Suppose that the symbols $a, a_1 \in \mathsf{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ are globally bounded functions compactly supported in the variable $\boldsymbol{\xi}$. Then*

$$\begin{aligned}\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} &\operatorname{tr}\big[\chi_\Lambda g\big(H_\alpha(a,a_1;\Lambda,\Omega)\big)\chi_\Lambda - \chi_\Lambda g\big(H_\alpha(a,a_1;\mathbb{R}^d,\Omega)\big)\chi_\Lambda\big] \\ &= \mathfrak{W}_1\big(\mathfrak{D}(g;\operatorname{Re} a,\operatorname{Re} a_1);\partial\Lambda,\partial\Omega\big).\end{aligned} \tag{5.2}$$

It is appropriate to make a comment in the spirit of Remark 4.5(ii).

If the domain Λ is bounded and $g \in \mathsf{C}^\infty(\mathbb{R})$ is such that $g(0)=0$, then both operators on the left-hand side of (5.2) are trace class, and formula (5.2) is just another way to write the asymptotics

$$\begin{aligned}\operatorname{tr} g\big(H_\alpha(a,a_1;\Lambda,\Omega)\big) &= \alpha^d\big(\mathfrak{W}_0(g(\operatorname{Re} a);\Lambda,\Omega) + \mathfrak{W}_0(g(\operatorname{Re} a_1);\Lambda,\Omega_1)\big) \\ &\quad + \alpha^{d-1}\log\alpha\ \mathfrak{W}_1\big(\mathfrak{D}(g;\operatorname{Re} a,\operatorname{Re} a_1);\partial\Lambda,\partial\Omega\big) + o(\alpha^{d-1}\log\alpha), \quad \alpha\to\infty.\end{aligned}$$

The derivation of this fact from (5.2) repeats almost word for word the proof of formula (4.8).

Similarly to Theorem 4.4, Theorem 5.1 is derived from formula (5.2) for polynomials $g_p(t) = t^p$, $p = 1, 2, \dots$:

THEOREM 5.2. *Let the domains Λ, Ω, and the symbols a, a_1 satisfy the conditions of Theorem 5.1. Then for any $p = 1, 2, \dots$ we have*

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \operatorname{tr}\big[g_p\big(V_\alpha(a, a_1; \Lambda, \Omega)\big) - \chi_\Lambda g_p\big(V_\alpha(a, a_1; \mathbb{R}^d, \Omega)\big)\chi_\Lambda\big]$$
$$= \mathfrak{W}_1\big(\mathfrak{D}(g_p; a, a_1); \partial\Lambda, \partial\Omega\big). \tag{5.3}$$

If V_α is replaced by H_α, then the same formula (5.3) holds with a, a_1 replaced with $\operatorname{Re} a, \operatorname{Re} a_1$.

The derivation of Theorem 5.1 from Theorem 5.2 follows the plan of the proof of Theorem 4.4, and is omitted.

As far as Theorem 5.2 itself is concerned, its proof essentially repeats that of Proposition 4.3 (given in [**29**]) with some obvious modifications. Below we provide only a sketch of this proof, leaving out some details that can be easily reconstructed.

We'll need the following estimates.

PROPOSITION 5.3. *Let the domain Λ be as in Theorem 5.1, and let the symbol $a \in \mathsf{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be compactly supported in both variables. Then for any $q \in (0, 1]$, and all $\alpha \geqslant 1$ we have*

$$\|\chi_\Lambda Op_\alpha(a)(I - \chi_\Lambda)\|_{\mathfrak{S}_q}^q \leqslant C_q \alpha^{d-1},$$

and

$$\|P_{\Omega,\alpha} Op_\alpha(a)(I - P_{\Omega,\alpha})\|_{\mathfrak{S}_q}^q \leqslant C_q \alpha^{d-1},$$

with a constant independent of α.

See [**28**, Corollary 4.4].

Proof of Theorem 5.2 (*sketch*). We give the proof only for the operator V_α. The version of (5.3) for the self-adjoint operator H_α can be obtained following the elementary argument detailed in [**27**, p. 77]. Furthermore, for simplicity we assume that Λ is a bounded domain, so that (5.3) amounts to

$$\lim_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \big[\operatorname{tr}\big(g_p(V_\alpha(a, a_1; \Lambda, \Omega))\big) - \alpha^d\big(\mathfrak{W}_0(g_p(a); \Lambda, \Omega) + \mathfrak{W}_0(g_p(a_1); \Lambda, \Omega_1)\big)\big]$$
$$= \mathfrak{W}_1(\mathfrak{D}(g_p; a, a_1); \partial\Lambda, \partial\Omega), \tag{5.4}$$

see the remark after Theorem 5.1. Since both Λ and Ω are bounded, we may assume that the symbols a, a_1 are compactly supported in both variables. In what follows we use the following convention: for any two operators A, B depending on the parameter $\alpha \geqslant 1$ we write $A \sim B$ if $\|A - B\|_{\mathfrak{S}_1} \leqslant C\alpha^{d-1}$ with a constant C independent of α.

By Proposition 5.3,

$$P_{\Omega,\alpha} a P_{\Omega,\alpha} + P_{\Omega_1,\alpha} a_1 P_{\Omega_1,\alpha} \sim a P_{\Omega,\alpha} + a_1 P_{\Omega_1,\alpha} = b P_{\Omega,\alpha} + a_1, \quad b = a - a_1.$$

Expanding $g_p(V(a, a_1; \Lambda, \Omega))$ and repeatedly using Proposition 5.3 again, we obtain that

$$g_p\big(V(a, a_1; \Lambda, \Omega)\big) \sim \sum_{l=0}^{p} \binom{p}{l} \operatorname{Op}_\alpha(a_1^{p-l}) g_l\big(T_\alpha(b; \Lambda, \Omega)\big).$$

Traces of operators similar to the ones in the sum above have been studied in [**29**]. By [**29**, Lemma 3.3 and Theorem 4.1],

$$\operatorname{tr} g_p\big(V(a, a_1; \Lambda, \Omega)\big) = \alpha^d \mathfrak{W}_0(a_1^p; \Lambda, \mathbb{R}^d) + \alpha^d \sum_{l=1}^{p} \binom{p}{l} \mathfrak{W}_0(a_1^{p-l} g_l(b); \Lambda, \Omega)$$
$$+ \alpha^{d-1} \log \alpha \sum_{l=1}^{p} \binom{p}{l} \mathfrak{W}_1\big(a_1^{p-l} \mathfrak{A}(g_l; b);\ \partial\Lambda, \partial\Omega\big) + o(\alpha^{d-1} \log \alpha). \tag{5.5}$$

The second sum starts with $l = 1$ since $\mathfrak{A}(g_0; b) = 0$. By definition (4.3), the first two terms on the right-hand side amount to

$$\frac{\alpha^d}{(2\pi)^d} \int_\Lambda \int_{\mathbb{R}}^{d} \big(b(\mathbf{x}, \boldsymbol{\xi})\chi_\Omega(\boldsymbol{\xi}) + a_1(\mathbf{x}, \boldsymbol{\xi})\big)^p \, d\boldsymbol{\xi} \, d\mathbf{x}$$
$$= \alpha^d \big(\mathfrak{W}_0(g_p(a); \Lambda, \Omega) + \mathfrak{W}_0(g_p(a_1); \Lambda, \Omega_1)\big).$$

To evaluate the second sum in (5.5) note that by definition (4.4),

$$\sum_{l=1}^{p} \binom{p}{l} z^{p-l} \mathfrak{A}(g_l; s) = \frac{1}{(2\pi)^2} \int_0^1 \frac{g_p(z + ts) - g_p(z) - t g_p(z + s) + t g_p(z)}{t(1-t)} \, dt$$
$$= \mathfrak{D}(g_p; s + z, z),$$

for any $z, s \in \mathbb{C}$. Therefore, the second sum on the right-hand side of (5.5) equals

$$\alpha^{d-1} \log \alpha \ \mathfrak{W}_1\big(\mathfrak{D}(g_p; a, a_1); \partial\Lambda, \partial\Omega\big).$$

This completes the proof of (5.4). As explained earlier, this leads to (5.3), as required. □

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