Testing General Free Functions in Preferred Scale Theories

Ali Mozaffari[1, *]

[1] *Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2BZ, United Kingdom*

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Building on previous work, we explore the parameter space of general free functions in non-relativistic modified gravity theories motivated by k-essence and other scalar-tensor theories. Using a few proposed tests, we aim to update Solar System based constraints on these ideas in line with previous theories and suggest their utility in constraining modification to GR, potentially even being able to test k-essence type theories.

I. INTRODUCTION

Since Einstein's unveiling of his General Theory of Relativity (GR) more than 100 years ago, it has proven to be a *very* resilient idea, surviving tests in the weak, cosmological and strong field regimes with exquisite precision. There remain however both a number of conceptual and experimental questions that GR has yet to answer, such as how can we marry together gravity with quantum theory and what is the nature of the dark sector. In recent years, theories of modified gravity [1] have re-emerged as leading contenders to answer some of these open problems. One such branch of theories involves adding additional *dof* 's such that dynamics around additional (but perhaps arbitrarily inserted) acceleration scale(s) may vary become important. These preferred acceleration scale theories first appeared in the guise of MOdified Newtonian Dynamics (MOND) [2, 3] and have spawned a series of relativistic extensions [4–6]. Such ideas originally appeared as a counterpart to the dark sector, however the two may exist in harmony and perhaps ease the tension on neutrinos as a candidate dark matter particle [7, 8].

In this work, we aim initially to abstract ourselves from the details of any relativistic completion of a preferred scalar theory and instead focus on computing the consequences in the weak field limit of some general effective theory. We will be specialising our computations to two specific tests around gravitational saddle points (SP's)

1. **Tidal Stresses -** The expected (close to linear) tidal stresses of a Newtonian gravitational field may be contrasted with that of some modified gravitational theory that is in some way "switched on" in the low acceleration regime around SP's.

2. **Time Delays -** The experimental gravity favourite with a new twist, the effect on the stress-energy around some region of modified gravity should have an effect on the observed Shapiro time delay [9]

We then consider a particular realisation of this theory, inspired by k-essence models and consider prospects for constraints.

*Electronic address: ali.mozaffari@imperial.ac.uk

The structure of this paper is as follow: Firstly we recap analytical solutions for preferred scale theories around SP's making use of parameterisable free functions to demonstrate effects in different regimes. We move on to the effects possible by generalising our free functions to include both potential ϕ and gradient of potential $|\nabla\phi|$, considering similar analytical regimes. Using first tidal stresses and then time delays as probes, we explore the different effects on observables with these general functions. We conclude with some thoughts on constraints for the future and possibilities for detection.

II. ANALYTICAL SOLUTIONS

A. The U Formalism for Saddle Points

1. *Recap of the basics*

Our starting point is in the non-relativistic limit of these modified gravity theories. A central object here is the (physical) gravitational potential, denoted Φ, we define this simply by saying in the relativistic limit of this theory, particles feel an acceleration $-\nabla\Phi$. We will stick to the Type I,II,III nomenclature (see Appendix A)Φ may be decomposed in Type I theories as

$$\Phi = \Phi_N + \phi \tag{1}$$

such that the Newtonian as usual satisfies

$$\nabla^2\Phi_N = 4\pi G\rho \tag{2}$$

and the fifth force field ϕ has equation of motion

$$\nabla\cdot(a_1\nabla\phi) = C_\rho G\rho \tag{3}$$

where a_1 is a free function, C_ρ is a dimensionless coupling constant (in these models typically $C_\rho \ll 1$) and ρ is the baryonic matter density. An appealing suggestion here is to examine these as effective theories for departures from GR, which we can then be subject to constraints from experiment. Around SP's, we can take the $\rho \to 0$ limit and hence

$$\nabla\cdot(a_1\nabla\phi) = 0 \tag{4}$$

Defining the variable

$$\mathbf{U} = -a_1 \frac{C_\rho}{4\pi}\frac{\nabla\phi}{\alpha} \tag{5}$$

where α is a constant with units of acceleration. Further to this we can define the dimensionless scalar

$$z = \frac{C_\rho}{4\pi}\frac{\nabla\phi}{\alpha} \tag{6}$$

Such that

$$U \;=\; |\mathbf{U}| = a_1\, z \tag{7}$$

This redefinition allows us to rewrite (82) as a system of vector equations

$$\nabla\cdot\mathbf{U} \;=\; 0 \tag{8}$$

$$\nabla\wedge\left(\frac{\mathbf{U}}{a_1}\right) \;=\; 0 \tag{9}$$

Expanding out (9) and collecting terms together,

$$a_1\nabla\wedge\mathbf{U} + \mathbf{U}\wedge\nabla a_1 \;=\; 0 \tag{10}$$

In the case of $a_1 = a_1(z)$, then we can simply rewrite $U = U(z)$ and hence $a_1(z) = a_1(U)$ meaning that

$$\nabla a_1 = \frac{d\,a_1}{d\,U^2}\nabla U^2 \tag{11}$$

and hence Equation (9) becomes

$$M\,U^2\,\nabla\wedge\mathbf{U} + \mathbf{U}\wedge\nabla U^2 \;=\; 0 \tag{12}$$

$$\frac{d\ln U^2}{d\ln a_1} \;=\; M \tag{13}$$

We can illustrate the types of solutions in different regimes using a parameterised free function (previously considered in detail [10]),

$$a_1 = \frac{z^a}{(1+z^b)^{a/b}} \tag{14}$$

but we stress the techniques described here are applicable to any choice of a_1.

In the proximity of a gravitational saddle point (SP), it is well know that the Newtonian field is linearised. To a good approximation, we may introduce a truncated multi-pole expansion of the Newtonian field (in spherical polar coordinates, r, ψ, θ) around the SP,

$$\nabla\Phi_N \;=\; -Ar\mathbf{N} \tag{15}$$

$$\mathbf{N} \;=\; N_r\mathbf{e}_r + N_\psi\mathbf{e}_\psi \tag{16}$$

$$N_r \;=\; \frac{1}{4}\left(1+3\cos 2\psi\right) \tag{17}$$

$$N_\psi \;=\; -\frac{3}{4}\sin 2\psi \tag{18}$$

where A is the expected Newtonian tidal stress and the additional θ coordinate is absent due to the spherical symmetry present. We assign the Newtonian contribution to the field $\mathbf{U}$, found for $a_1 \to 1$ as

$$\mathbf{U}_0 \;=\; -\left(\frac{C_\rho}{4\pi}\right)^2\frac{\nabla\Phi_N}{\alpha} \tag{19}$$

The utility of this linear approximation is great, particularly as it sets up a separable ansatz for solutions in the large and small limits of U.

The boundary between the two regimes in U is found at $|\mathbf{U}|^2 \simeq 1$. The intuition here (as justified in [11]) is that departures from spherical symmetry are subdominant, such that

$$\mathbf{U}\simeq\mathbf{U}_0 \;\Rightarrow\; r^2\left(\cos^2\psi + \frac{1}{4}\sin^2\psi\right) = \left(\frac{4\pi}{C_\rho}\right)^4\left(\frac{\alpha}{A}\right)^2 = r_0^2 \tag{20}$$

$$\Rightarrow \mathbf{U}_0 \;=\; A\left(\frac{r}{r_0}\right)\mathbf{N} \tag{21}$$

This suggests an ellipsoidal boundary around the saddle point, with semi-major axes r_0 between which there exists two different regimes.

It is found in general [10] that solutions in the $U \gg 1$ limit take the form

$$\mathbf{U} \;=\; \mathbf{U}_0 + \mathbf{U}_2 \tag{22}$$

$$\mathbf{U}_2 \;=\; \left(\frac{r}{r_0}\right)^{1-b}\left(F(\psi)\,\mathbf{e}_r + G(\psi)\,\mathbf{e}_\psi\right) \tag{23}$$

where b refers to the fall off power of z in the expansion of a_1 in (14). We note that this sets up a perturbative expansion in $\mathbf{U}$ with

$$\frac{|\mathbf{U}_2|}{|\mathbf{U}_0|} \sim \left(\frac{r}{r_0}\right)^{-b} \tag{24}$$

and suggests that in the large r limit, the fifth force becomes a rescaled Newtonian contribution (effectively renormalising G_N). In the $U \ll 1$ limit, the behaviour follows

$$\mathbf{U} = C_M\left(\frac{r}{r_0}\right)^{\gamma_a}\left(F_a(\psi)\,\mathbf{e}_r + G_a(\psi)\,\mathbf{e}_\psi\right) \tag{25}$$

where γ_a and C_M and are constants that can be calculated depending on the model picked (as detailed in [10]) and a is the leading order exponent of z in the expansion of a_1 in (14). Finally we may attempt to compute the behaviour of ϕ field by solving (12) to find $\mathbf{U}$ (in the appropriate limit) and then inverting through

$$-\nabla\phi = \frac{4\pi\alpha}{C_\rho}\frac{\mathbf{U}}{a_1(U)} \tag{26}$$

and this we may compute both tidal stresses

$$S_{ij} = -\frac{\partial^2\phi}{\partial x_i\partial x_j} \tag{27}$$

2. More General $a_1(z,\phi)$

In the more general case of $a_1(z,\phi)$, obviously then $U = U(z,\phi)$. But *if* we can find $z(U,\phi)$ (which is often possible in particular limits e.g. $z \ll 1$) then we may write $a_1 = a_1(U,\phi)$. In this way,

$$\nabla a_1 = \frac{\partial a_1}{\partial U^2}\nabla U^2 + \frac{\partial a_1}{\partial \phi}\nabla\phi \tag{28}$$

Also since $\mathbf{U} \propto \nabla\phi$ then $\mathbf{U}\wedge\nabla\phi = 0$ and so the final system of equations are now mimic the $U(z)$ case of Equation (12), except that

$$M \;=\; 1 \Bigg/ \frac{\partial \ln a_1}{\partial \ln U^2} = \frac{2}{1 - a_1\, z_{,U}} \tag{29}$$

where $_{,U} \equiv \frac{\partial}{\partial U}$ Thus for a particular model $P(z,\phi)$, we must first proceed to identify the free function a_1, then depending on the limit in consideration expand $a_1(z,\phi)$ as some power series in z. Next compute the relation $U = U(z,\phi)$ and so $z_{,U}$ to find M.

B. Examples

We can illustrate our techniques using an adaptation of our parameterised free function,

$$a_1 \;=\; \frac{z^a\, u^c}{(1 + z^b\, u^{bc/a})^{a/b}} \tag{30}$$

$$u \;=\; \left(\frac{\phi}{v^2}\right)^c \tag{31}$$

$$M \;=\; \frac{2}{a}\left(1 + a + z^b u^{bc/a}\right) \tag{32}$$

where v is some constant with dimension of velocity. We argue that the value of v should be taken from

$$v^2 \sim A\, r_0^2 \tag{33}$$

since it is of the correct dimension and we see from the case of the Newtonian

$$\frac{\Phi_N}{v^2} = -\left(\frac{r}{r_0}\right)^2 N_r \tag{34}$$

that it sets up a useful cutoff scale. This model has the virtue of relatively independently controlled behaviour in each of the $z \ll 1$ and $z \gg 1$ regimes. We add to this the effect of scaling the modified potential ϕ and seek to explore its behaviour in different regimes around SP's.

1. $z \gg 1$

In this regime

$$a_1 \;\simeq\; \left(1 - \frac{a}{b}\frac{1}{z^b u^{bc/a}} + \dots\right) \tag{35}$$

$$M \;\simeq\; \frac{2}{a} U^b u^{bc/a} \tag{36}$$

Subsequently (12) takes the form

$$\frac{2\, u^{bc/a}}{a}\, U^{b+2}\nabla\wedge\mathbf{U} + \mathbf{U}\wedge\nabla U^2 = 0 \tag{37}$$

which may be solved by assuming the ansatz for $\mathbf{U}$

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_2 \tag{38}$$

where $\mathbf{U}_0$ is again the curl free contribution to the solution and $\mathbf{U}_2$ is a perturbative contribution with non-zero curl, sourced by $\mathbf{U}_0$. If at lowest order $a_1 \to$ constant, then we are assured to split up $\mathbf{U}$ in this way, however in general a curl free contribution $\mathbf{U}_0$ may not be present (c.f. the $z \ll 1$ regime). In the case of (30) however we can be satisfied it will be present since $\phi^c\,\nabla\phi \propto \nabla\left(\phi^{c+1}\right)$ which obviously remains curl free.

In this regime, the presence of the ϕ becomes problematic however our expectation is that we can expand the field as a power series in the large r limit,

$$\phi \simeq \phi_\infty + \epsilon\phi_0 + \epsilon^2\phi_1 + \dots \tag{39}$$

where ϕ_∞ will be some constant contribution to ϕ at $r \to \infty$. This means that *a priori* we can find ϕ_0,

$$\nabla\cdot\left(\left(\frac{\phi_0}{v^2}\right)^c\nabla\phi_0\right) \;=\; \frac{v^{-2c}}{c+1}\nabla^2(\phi_0)^{c+1} = \frac{C_\rho}{4\pi}\nabla^2\Phi_N \tag{40}$$

$$\phi_0 \;=\; \left(\frac{C_\rho}{4\pi}\frac{|c+1|}{|\Phi_N|^c}v^{2c}\right)^{\frac{1}{c+1}}\Phi_N \tag{41}$$

where the choice of notation is picked to ensure the correct sign of the force $\mathbf{F}_\phi$. It is required that we know *both* of $\mathbf{U}_0$ and ϕ_0 in order to solve for $\mathbf{U}_2$

$$\nabla\wedge\mathbf{U}_2 \;=\; -\frac{a}{2\,b\,v^{2bc}}\frac{\mathbf{U}_0\wedge\nabla(U_0)^2}{(U_0)^{b+2}}(\phi_0)^{bc} \tag{42}$$

What becomes obvious here is that

$$\frac{|\mathbf{U}_2|}{|\mathbf{U}_0|} \sim \left(\frac{r}{r_0}\right)^{b(c-1)/(c+1)} \tag{43}$$

which suggests only values of c satisfying $-1 < c \leq 1$ will permit perturbative solutions with $\mathbf{U}_2$ subdominant to $\mathbf{U}_0$ as before. For values of c outside of this, the dominant contribution in expanding $\mathbf{U}$ will now be $\mathbf{U}_2$. We stress that to avoid violations of Solar System constraints, the fifth force can only follow certain limiting behaviour:

- Φ mimics the Newtonian potential with an appropriately small scaling, such that G_N is effectively renormalised (within limits, such as BBN and the CMB)

- $\nabla\phi$ becomes subdominant in the large r limit, such that the inner bubble is effectively "screened".

Using the notation

$$\mathbf{U}_2 = U_r \mathbf{e}_r + U_\psi \mathbf{e}_\psi \quad (44)$$

we see that (8, 9) reduce to the coupled ODEs,

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, U_r) + \frac{1}{r\sin\psi}\frac{\partial}{\partial\psi}(\sin\psi\, U_\psi) &= 0 \quad (45)\\ \left[\frac{\partial}{\partial r}(r\, U_\psi) - \frac{\partial U_r}{\partial\psi}\right] &= \frac{s_{b,c}(\psi)}{r^{n-1}} \quad (46) \end{aligned}$$

$$\begin{aligned} s_{b,c} &= -\frac{3a}{b}\frac{2^{3b(1/2-c)}(1+3\cos 2\psi)^{bc}\sin 2\psi}{(5+3\cos 2\psi)^{1+b/2}} \quad (47)\\ n &= b\left(1-\frac{2c}{c+1}\right) \quad (48) \end{aligned}$$

which suggests a separable ansatz

$$\begin{aligned} \mathbf{U}_2 &= \left(\frac{r_0}{r}\right)^{n-1}\left(\frac{C_\rho}{4\pi}|c+1|\right)^{bc/(c+1)}\mathbf{B}(\psi) \quad (49)\\ \mathbf{B}(\psi) &= F_{b,c}(\psi)\,\mathbf{e}_r + G_{b,c}(\psi)\,\mathbf{e}_\psi \quad (50) \end{aligned}$$

where $F_{b,c}$, $G_{b,c}$ satisfy

$$\begin{aligned} &F_{b,c}\,(n-2)(n-3)\\ &{F_{b,c}}'\cot\psi + {F_{b,c}}'' = -({s_{b,c}}' + s_{b,c}\cot\psi) \quad (51)\\ &(2-n)\,G_{b,c} = s_{b,c} + F'_{b,c} \quad (52) \end{aligned}$$

The relative depreciation (for $c > 0$) or amplification of signal (for $c < 0$) is evident in the second factor of Equation (49). Once we characterise the $\mathbf{U}$ we may find the total contribution of the scalar *dof*

$$-\nabla\phi \simeq \frac{4\pi\alpha}{C_\rho}\left(\frac{v^2}{\phi}\right)^c \mathbf{U}\left(1+\frac{a}{b}\left(\frac{\phi}{v^2}\right)^{bc}\frac{1}{U^b}+\dots\right) \quad (53)$$

Naturally we can use the form of the $\mathbf{U}$ to find the separable form of the anomalous force, but for brevity we leave this for Appendix B. We see from the form of Equation (B1) that a much richer structure is present for models with $c \neq 0$. Depending on the model parameters, (C_ρ, c) the additional exterior bubble force may become significantly smaller than the Newtonian background within a short distance.

By way of a an extreme example, for $c = 1$ (and so $n = 0$), the additional force will have no r dependence (recall this expansion is valid for $r/r_0 \gg 1$ and so at leading order,

$$\frac{|\nabla\phi|}{|\nabla\Phi_N|} \simeq \frac{r_0}{r}\left(\frac{4\pi}{C_\rho\, N_r}\right)^{1/2} + \dots \quad (54)$$

Likewise for $c \simeq 0$

$$\frac{|\nabla\phi|}{|\nabla\Phi_N|} \simeq \left(\frac{8\pi}{C_\rho N_r}\right)^{c} + \dots \quad (55)$$

2. $z \ll 1$

In this inner bubble regime

$$\begin{aligned} a_1 &\simeq \left(\frac{\phi}{v^2}\right)^c z^a \quad (56)\\ U &\simeq \left(\frac{\phi}{v^2}\right)^c z^{a+1} \quad (57)\\ 4m &\to \frac{2(a+1)}{a} \quad (58) \end{aligned}$$

which can be solved with the $U \ll 1$ solutions of the $c = 0$ case. Putting this together with our ansatz for $\mathbf{U}$ in (25)

$$\begin{aligned} -\nabla\phi &\simeq \frac{4\pi\alpha}{C_\rho}\frac{\mathbf{U}}{z^a}\left(\frac{v^2}{\phi}\right)^c \simeq \mathbf{U}\left(\frac{v^{2c}}{U^a\,\phi^c}\right)^{1/(a+1)} \quad (59)\\ -\nabla\phi^{a_c/(a+1)} &\simeq \frac{4\pi\alpha\,\mathbf{U}}{C_\rho}\left(\frac{a_c}{a+1}\right)\left(\frac{v^{2c}}{U^a}\right)^{1/(a+1)} \quad (60) \end{aligned}$$

and again we leave the full separable form of the anomalous force to Appendix B

$$\begin{aligned} H(\psi) &= \left(\frac{F_a}{({F_a}^2+{G_a}^2)^{1/2}}\right)^{a/a_c} \quad (61)\\ a_c &= a+c+1\\ a_\gamma &= a+\gamma+1 \end{aligned}$$

The condition for divergent tidal stresses becomes,

$$\frac{\gamma-c}{a_c} < 1 \Longrightarrow c > \frac{\gamma-a-1}{2} \quad (62)$$

3. An Example

To clarify matters, we will specialise to a model with $a = 1$, $b = 2$ [10], which for the inner bubble presents solutions of the form

$$\begin{aligned} \mathbf{U} &= C_M\left(\frac{r}{r_0}\right)^\gamma (F_1(\psi)\,\mathbf{e}_r + G_1(\psi)\,\mathbf{e}_\psi)\\ F_1 &\simeq 0.2442 + 0.7246\cos 2\psi + 0.0472\cos 4\psi\\ G_1 &\simeq -0.8334\sin 2\psi - 0.0368\sin 4\psi\\ C_M &\simeq 1.3163\\ \gamma &\simeq 1.5256 \quad (63) \end{aligned}$$

Thus in this model, divergent tidal stress are found for $c > -0.237$. On the other hand investigating anomalous time delays requires

$$\frac{a_\gamma}{a_c} < 0 \quad (64)$$

This give rise to two branches of solutions

$$\begin{aligned} \gamma < -(a+1) &\iff c > -(a+1) \quad (65)\\ \gamma > -(a+1) &\iff c < -(a+1) \quad (66) \end{aligned}$$

We see therefore that divergent solutions are feasible in both branches, contrast to just the $a<0$ solutions predicted for $c=0$.

III. TYPE II & III THEORIES

Ultimately we may consider other types of non-relativistic effective theory with these more general free functions, particular those characterised as types II and III. Type II theories follow relations of the form

$$\nabla^2\phi = \frac{C_\rho}{4\pi}\nabla\cdot(b_1\nabla\Phi_N) \tag{67}$$

where once again C_ρ is some coupling to matter and a_2 is a free function. Typically b_1 have been chosen as function of the Newtonian acceleration $|\nabla\Phi_N|$ however here we will attempt to relax such a condition to

$$b_1 = b_1(\Phi_N, |\nabla\Phi_N|) \tag{68}$$

By employing the Newtonian linear approximation and a choice of parameterised free function for illustration, we aim to find suggestive analytical solutions in this case

$$b_1 = \left(1+\frac{u^c}{w^b}\right)^{a/b} \tag{69}$$

$$w = \left(\frac{C_\rho}{4\pi}\right)^2\frac{|\nabla\Phi|}{\alpha} = \frac{r}{r_0}\mathbf{N} \tag{70}$$

$$u = \frac{|\Phi_N|}{v^2} = \frac{1}{2}\left(\frac{r}{r_0}\right)^2 N_r \tag{71}$$

where the $c=0$ case results in a typical Type II parametrised free function. The solutions are best expressed here in terms of the ratio r/r_0 which helps to distinguish inner and outer bubble solutions

$$\nabla^2\phi = a\,\nu^{1-b/a}\,A\left(\frac{r}{r_0}\right)\frac{u^c}{w^{b+1}}f(N,N_r,N_\psi) \tag{72}$$

$$f = NN_r + N\psi N' - 2\frac{c}{b}\frac{N}{N_r}\left(N_r^2+\frac{N_\psi N'}{2}\right) \tag{73}$$

Likewise Type III theories can be viewed as a recasting Type I theories, with the caveat that the free function depends on just the physical potential $\nabla\Phi$,

$$c_1 = c_1(|\nabla\Phi|) \tag{74}$$

and the effective coupling takes the value $C_\rho = 4\pi$,

$$\nabla\cdot(c_1\nabla\Phi) = \nabla^2\Phi_N \tag{75}$$

where we are positing that

$$c_1 = c_1(\Phi, |\nabla\Phi|) \tag{76}$$

This would produce similar results as the type I theory, however we are restricted to models which have

$$\lim_{|\nabla\Phi|\gg a_{pref}} c_1 \;\rightarrow\; C_1 + C_2(\Phi,|\nabla\Phi|)+\ldots \tag{77}$$

where C_1 is a constant close to unity. Such a form is required to ensure Solar System tests are not violated.

IV. A TOY MODEL

Given that we may consider free functions of the form outlined in Section ??, it would be interested to understand how to motivate them from the point of view of a relativistic gravitational theory. Consider first the action for k-essence theories minimally coupled to gravity

$$S = \int\left(\frac{M_{pl}^2}{2}R + P(X,\phi) - V(\phi)\right)\sqrt{-g}\,d^4x \tag{78}$$

where $X = -\frac{1}{2}\partial_\mu\phi\partial_\mu\phi g^{\mu\nu}$ is the canonical kinetic term and obviously $P=X$ in the case of a simple canonical scalar field. Different cases of the forms of P are enumerated in the literature and these have been proposed as candidate theories for both inflation and latterly dark energy[1] (DE). If we consider an inflationary theory, where the choice of $V(\phi)$ will enforce a regime of a slow-rolling ϕ leading to acceleration expansion of the universe and then as $V(\phi)$ settles into a minima, inflation ends.

We can contrast this with a scalar preferred acceleration scale theory [13],

$$\begin{aligned} S &= \int \frac{M_{pl}^2}{2}R\sqrt{-g}\,d^4x \\ &+ \int\frac{1}{\kappa}\left(\frac{a_1}{\ell}X - V(\phi) - \alpha^2F(a_1)\right)\sqrt{-g}\,d^4x \\ &+ \int\mathcal{L}_M\sqrt{-\tilde{g}}\,d^4x \end{aligned} \tag{79}$$

$$\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu} \tag{80}$$

examining the equations of motion, both for the metric

$$G_{\mu\nu} = 8\pi G\,T^M{}_{\mu\nu} + \frac{16\pi G}{\kappa}T^\phi{}_{\mu\nu} \tag{81}$$

where $T^M{}_{\mu\nu}$ is the (Einstein frame) matter stress energy and $T^\phi{}_{\mu\nu}$ is the scalar field stress energy. Additionally for the scalar field,

$$\nabla_\mu(a_1\nabla_\nu\phi)g^{\mu\nu} = \kappa\ell(V_{,\phi} + A_{,\phi}\,A\,\mathcal{L}_M) \tag{82}$$

$$F_{,a_1} = \frac{X}{\alpha^2\ell} \tag{83}$$

where $_{,\phi}\equiv\frac{\partial}{\partial\phi}$ etc. This set up ensures that for some free function a_1,

$$F_{,a_1} = G(a_1) = \frac{X}{\alpha^2\ell} \Rightarrow a_1 = G^{-1}\left(\frac{X}{\alpha^2\ell}\right) \tag{84}$$

where G^{-1} denotes the inverse function of G and here enforces a_1 to be a function of just $|\nabla\phi|$. However we are at liberty to drop this condition and just take the

[1] Although there exists a no-go theorem for many classes of these theories as a DE candidate, see [12]

necessary equations of motion from an action of the form (78). Thus we set up a hybrid action,

$$\begin{aligned} S &= \int \left(\frac{M_{pl}}{2} R + \frac{1}{\kappa} \left[P(X,\phi) - V(\phi) \right] \right) \sqrt{-g}\, d^4x \\ &+ \int \mathcal{L}_m \sqrt{-\tilde{g}}\, d^4x \end{aligned} \tag{85}$$

We draw the reader to the similarities between this action and that of Chameleon theories [14], where $P(X,\phi) = X$ and here we present this theory as a simple extension. Making the association in the equations of motion

$$\frac{P(X,\phi)}{X} \longrightarrow \frac{a_1(X,\phi)}{\ell} \tag{86}$$

produces a free function that is now a function of both $X = -\frac{1}{2}|\nabla\phi|^2$ and ϕ. Likewise we can attempt to expand these perturbatively in X.

Cosmological solutions in the inflationary regime obviously neglect $\mathcal{L}_m$, whilst the spectrum of solutions in the weak-field limit will be regulated by the choice of $P(X,\phi)$ and the magnitude of the effective model parameters. Taking the equation of motion (82) in the weak field limit, assuming a pressureless matter stress energy and the quasi-static limit,

$$T^\mu{}_\mu = \rho \tag{87}$$

$$g_{\mu\nu} \to \eta_{\mu\nu} \tag{88}$$

$$|\phi| \ll 1 \tag{89}$$

Additionally we pick a representative conformal factor for illustration,

$$A = e^{m\phi} \tag{90}$$

and assume that since inflation has ended $V_{,\phi} \to 0$, thus the effective weak field equations of motion become

$$\nabla \cdot (a_1 \nabla\phi) = \kappa\ell m\, \rho = \frac{\kappa\ell m}{4\pi G} \nabla^2 \Phi_N \tag{91}$$

Setting up a dictionary to bridge these notions, we find

$$C_\rho = \kappa\ell m \tag{92}$$

$$X = -\frac{z^2}{2}\left(\frac{4\pi\,\alpha}{C_\rho}\right)^2 = -z^2\, X_0 \tag{93}$$

$$U = a_1 \sqrt{-\frac{X}{X_0}} = a_1 \sqrt{\tilde{X}} \tag{94}$$

where $\tilde{X}$ is a dimensionless counterpart for X. Likewise

$$4m = 1 \Bigg/ \frac{\partial \ln a_1}{\partial \ln U^2} = \frac{2}{1 - a_1^2 \tilde{X}_{,U^2}} \tag{95}$$

A. An Example From k-essence

We can illustrate our techniques with a example inspired by k-essence [15],

$$P = \frac{C_1\sqrt{1+X} - C_2}{\phi^2} + \ldots \tag{96}$$

where C_1, C_2 are constants and P contains higher order terms that will be subdominant here. Firstly identifying

$$a_1 \simeq \frac{C_2 - C_1\sqrt{1-\tilde{X}}}{\tilde{X}\,\phi^2} \tag{97}$$

and then examining the near SP limit, $|\tilde{X}| \ll 1$

$$a_1 \simeq \frac{C_2 - C_1}{\tilde{X}\phi^2}\left(1 + \frac{C_1}{C_2 - C_1}\frac{\tilde{X}}{2} + \ldots\right) \tag{98}$$

$$U^2 \simeq \frac{(C_2 - C_1)^2}{\tilde{X}\phi^4} + \ldots \tag{99}$$

$$4m \to 1 \tag{100}$$

These results suggest that in this model, we transition from one "inner bubble" like regime (as seen in (25)) to another. This means that in these models, provided the outer bubble effects are consistently screened, these models would survive Solar System tests, as well as appearing in exotic inflationary or DE theories.

B. More General $P(X,\phi)$

Returning to the different forms of generalised pressure, we find that in each case, a simple scheme for predicting effects can be found

- **Purely Kinetic Function** $P = P(X)$, this case is simply a restating of a preferred scale model such that

$$a_1(\tilde{X}) = \frac{P(X)}{X} = \tilde{P}(\tilde{X}) \tag{101}$$

  which easily allows analysis using the techniques of Section II A 1.

- **General Mixed Function** $P = P(X,\phi)$, this case suggests

$$a_1(\tilde{X},\phi) = \frac{P(X,\phi)}{X} = \tilde{P}(\tilde{X},\phi) \tag{102}$$

  Depending on the form of $\tilde{P}$ and regime in question, we are left with differing forms of fifth force which may be perturbative (with $\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_2$) or non-perturbative (with just $\mathbf{U}$).

$$a_1 \simeq \tilde{P}_0(\phi) + \tilde{P}_n(\phi)\,\tilde{X}^n + \ldots \tag{103}$$

$$U^2 \simeq \tilde{X}\left(\tilde{P}_0 + \tilde{P}_n\,\tilde{X}^n + \ldots\right)^2 \tag{104}$$

$$4m \simeq \frac{2n+1}{n} + \frac{\tilde{P}_0}{n\,\tilde{P}_n\,\tilde{X}^n} + \ldots \tag{105}$$

1. $\tilde{P}_0 \neq 0$ **case**,

$$\bar{P}_0(\phi_0) = \int \tilde{P}_0(\phi_0)\, d\phi_0 = \frac{C_\rho}{4\pi}\Phi_N \quad (106)$$

$$\phi_0 = \bar{P}_0^{\;-1}(\Phi_N) \quad (107)$$

$$-\nabla\phi \simeq \frac{4\pi\alpha}{C_\rho}\frac{\tilde{P}_0(\phi_0)^{2n}}{\tilde{P}_0(\phi_0)^{2n+1} + \tilde{P}_n(\phi_0)\,U^{2n}}\mathbf{U} \quad (108)$$

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{U}_2 \quad (109)$$

$$\nabla \wedge \mathbf{U}_2 = -\frac{n\tilde{P}_n(\phi_0)}{\tilde{P}_0(\phi_0)}\frac{\mathbf{U}_0 \wedge \nabla(U_0)^2}{(U_0)^{2-2n}} \quad (110)$$

2. $\tilde{P}_0 = 0$ **case**,

$$\bar{P}_n = \int \tilde{P}_n\,(\phi)^{\frac{1}{2n+1}}\, d\phi \quad (111)$$

$$\mathbf{U} = C_M\left(\frac{r}{r_0}\right)^{\gamma_n}(F_n\mathbf{e}_r + G_n\mathbf{e}_\psi) \quad (112)$$

$$-\nabla\bar{P}_n \simeq \frac{4\pi\alpha}{C_\rho}\frac{\mathbf{U}}{U^{\,2n/(2n+1)}} \quad (113)$$

$$\bar{P}_n = \frac{4\pi\alpha}{C_\rho\, n_\gamma}\frac{F_n(\psi)\, r^{n_\gamma}}{({F_n}^2 + {G_n}^2)^{1/2}}{C_M}^{1/(2n+1)} \quad (114)$$

$$n_\gamma = \frac{2n + \gamma_n + 1}{2n+1}$$

$$\phi = \bar{P}^{\;-1}(r,\psi) \quad (115)$$

where the value of $4m\,(n)$ will determine the parameter γ_n, C_M (which is usually an $\mathcal{O}(1)$ contribution) and the profile functions F_n, G_n.

- **Separable Function** $P = f(\phi)\,g(X)$, this case suggests

$$a_1(\tilde{X},\phi) = f(\phi)\,\frac{g(X)}{X} = f(\phi)\,\tilde{g}(\tilde{X}) \quad (116)$$

and so a background level calculation must be performed for each regime. In each case the leading order expansion in the requisite regimes may take two possible forms,

$$a_1 \simeq f(\phi)\,\left(\tilde{g}_0 + \tilde{g}_n\tilde{X}^n + \ldots\right) \quad (117)$$

$$U^2 \simeq f^2(\phi)\,\tilde{X}\,\left(\tilde{g}_0 + \tilde{g}_n\,\tilde{X}^n + \ldots\right)^2 \quad (118)$$

$$4m \simeq \frac{2n+1}{n} + \frac{\tilde{g}_0}{n\,\tilde{g}_n\,\tilde{X}^n} + \ldots \quad (119)$$

We see therefore that this case is just a reduction of the general mixed function case with

$$\tilde{P}_0 \equiv f(\phi)\,\tilde{g}_0$$

$$\tilde{P}_n \equiv f(\phi)\,\tilde{g}_n \quad (120)$$

V. CONCLUSIONS

To conclude, we have considered adaptations to the free functions in different variations of preferred acceleration modified gravity theories. Throughout we have set about to show that current techniques for characterising experimental observations can be extended to provide a series of concrete predictions for these theories. Centering on the low acceleration regions around gravitational saddle points, we have demonstrated that both divergent tidal stresses and anomalous time delays provide different ways to constrain these models. Bringing about a correspondence between these theories and those arising in k-essence theories, we posit that a hybrid action (with similarities with both scalar-tensor ideas e.g. chameleon theories) could explain both preferred scale behaviour and inflationary dynamics. Further we argue that these theories lie naturally within the framework for screened theories, with the anomalous stresses for such models potentially falling off very quickly outside of the SP bubble region, further illustrating why such behaviour has not been noticed previously.

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Appendix A: Non-Relativistic Limits

Preferred acceleration theories can be classified according to the differences in their equations of motion. We assume that the Newtonian potential satisfies the usual Poisson relation

$$\nabla^2\Phi_N = 4\pi G\rho \quad (A1)$$

and then additionally in each theory the gravitational potential follows the relations given in Table I. Broadly speaking the different non-relativistic limits can produce additional effects with the observed gravitational constant. In theories with two potentials, in the large acceleration limit

(A2)

Name	Potential	Equation of Motion	Free Function
Type 1	$\Phi = \Phi_N + \phi$	$\nabla \cdot (a_1 \nabla\phi) = \frac{C_\rho}{4\pi}\nabla^2\phi_N$	$a_1(\lvert\nabla\phi\rvert)$
Type 2A	$\Phi = \Phi_N + \phi$	$\nabla^2\phi = \frac{C_\rho}{4\pi}\nabla \cdot (b_1 \nabla\Phi_N)$	$b_1(\lvert\nabla\Phi_N\rvert)$
Type 2B	Φ	$\nabla^2\Phi = \nabla \cdot (b_2 \nabla\Phi_N)$	$b_2(\lvert\nabla\Phi_N\rvert)$
Type 3	Φ	$\nabla \cdot (c_1 \nabla\Phi) = \nabla^2\Phi_N$	$c_1(\lvert\nabla\Phi\rvert)$

TABLE I: Summary of non-relativistic limits

Appendix B: Full expressions for anomalous forces

$$-\nabla\phi \;\simeq\; \frac{4\pi\alpha}{C_\rho}\left(\frac{4\pi}{C_\rho|c+1|}\frac{2}{N_r}\right)^{\frac{c}{c+1}}\left(\frac{r_0}{r}\right)^{\frac{c-1}{c+1}}\left[\mathbf{N}+\left(\frac{C_\rho\,|c+1|}{4\pi}\right)^{\frac{bc}{c+1}}\left(\frac{r_0}{r}\right)^{n}\left(\mathbf{B}-\frac{a\,\mathbf{N}}{b\,|\mathbf{N}|^{b}}\left(\frac{N_r}{2}\right)^{\frac{bc}{c+1}}\right)\right]+\ldots \tag{B1}$$

$$-\nabla\phi \;=\; \alpha\,(C_M)^{a_c}\left(\frac{4\pi}{C_\rho}\right)^{1+\frac{c}{a_c}}\left(\frac{a_c}{a_\gamma}\right)^{1-\frac{c}{a_c}}\left(\frac{r}{r_0}\right)^{(\gamma-c)/a_c}\left[\frac{a_\gamma}{a_c}H(\psi)\mathbf{e}_r+H'(\psi)\mathbf{e}_\psi\right] \tag{B2}$$

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