Errata:
Mathematical Theory of Shells on Elastic Foundations:

An Analysis of Boundary Forms, Constraints, and Applications to Friction and Skin Abrasion

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The proof of the derivation of the governing equation for the limiting-equilibrium case from Section 4.3.3 of Jayawardana [7] is incorrect. Thus, in this document, we provide the correct proof. Also, we present the corrections to minor typos and missing text of Jayawardana's thesis [7]. Sometimes we may rephrase entire passages of text from the thesis or present detailed explanations for the reader's benefit.

Pages 3 to **182**: most occurrences of the terms *Young's modulus*, *Poisson's ratio*, *Lamé's parameters*, *Gaussian curvature* and *Christoffel symbols* should follow the word *the*.

Page 4: in Section 1.1, Notations and Conventions should include the following:
By convention, we have $(u_\alpha u^\alpha)^{\frac{1}{2}} = \sqrt{u_1 u^1 + u_2 u^2}$.

Also, the first Lamé's parameters of the shell (or membrane) and the foundation should respectively be

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \text{ and}$$
$$\bar{\lambda} = \frac{\bar{\nu}\bar{E}}{(1+\bar{\nu})(1-2\bar{\nu})} .$$

Furthermore, we use δb and the critical parametric-latitude, β_δ, as proxies for the curvature of the contact region. Justification: In our numerical analysis, our contact region is often modelled as a semi-elliptical prism that is parametrised by the map $(x^1, a\sin(x^2), b\cos(x^2))_{\text{E}}$, where $|x^1| < \infty$, $|x^2| \leq \frac{1}{2}\pi - \epsilon$ and $\epsilon > 0$. Now, the mean curvature of this surface can be expressed as $H(x^2) = \frac{1}{2}ab\varphi(x^2)^{-3}$, where $\varphi(x^2) = (b^2\sin^2(x^2) + a^2\cos^2(x^2))^{\frac{1}{2}}$. Noticing that both δb and β_δ are positively correlated with $H(x^2)$, $\forall x^2 \in [\epsilon - \frac{1}{2}\pi, \frac{1}{2}\pi - \epsilon]$, justifies our definition.

Pages 11, 93, 97, 98, 99, 122 and **123**: the word *inequity* should be *inequality*.

Page 16: second equation of the last set of equations should be

$$\frac{Eh\kappa}{2(1+\nu)}(\Delta_{\mathsf{E}} u^3 - \boldsymbol{\nabla}_{\mathsf{E}} \cdot \boldsymbol{\phi}) + q = 0 .$$

Pages 16, 17 and **20**: the word *traverse* should be *transverse*.

Page 19: in the first sentence, the word *defection* should be *deflection*.

Page 21: in the last sentence, the word *isotopic* should be *isotropic*.

Page 25: in the last paragraph, the word *Further* should be *Another*.

Page 26: in the last paragraph, second to last sentence, the word *through* should be *though*.

Page 32: in the second paragraph, first sentence, the word *micromere* should be *micrometre*.

Page 34: in the first paragraph, the word *subtable* should be *suitable*. Also, in the first paragraph, fourth to last sentence should read as *Also, the very idea of a convex surface is nonsensical ....* Furthermore, the last sentence should read as *It is unclear ....*

Pages 35 and **36**: the word *perseveres* should be *preserves*.

Page 36: in the last paragraph, second to last sentence, ignore the duplicate word *by*.

Pages 36, 136 and **137**: the word *exit* should be *exist*.

Page 38: in the second to last paragraph, third sentence, *right-circler* should be *right-circular*.

Page 39: in the second sentence, note the following:
Regarding Cottenden's [4] assertion that a 3×2 matrix is invertible (see equation 5.15 of Cottenden [4]), upon further examination, it is clear that the author failed to understand the difference between the inverse of a bijective mapping and a preimage (which need not be bijective) as they both use the same mathematical notation, $\boldsymbol{\lambda}^{-1}$, in Pressley's publication [11] (note that Cottenden [4] accredits Pressley [11] for his differential geometry results). This misunderstanding of Pressley's work [11] leads to a substantial part of Cottenden's work [4] being incorrect, as Section 5.4 of Cottenden's thesis [4] is based on an assumption that a 3×2 matrix is invertible.

Also, in the last sentence, note the following:
Regarding Cottenden *et al.*'s [3] derivation of an arc-length, although the formula d[arc length] $= \sqrt{R(\theta)^2 + \frac{\mathrm{d}R(\theta)}{d\theta}}\mathrm{d}\theta$ holds true when calculating an arc-length of a curve (which can be derived with simple differential geometry techniques), the term d[arc length]$^2 = (R\mathrm{d}\theta)^2 + \mathrm{d}R^2$ (see directly above equation 12 of Cottenden *et al.* [3]) does not imply the former equation nor does it have any mathematical context.

Page 46: in the last paragraph, first sentence, *coordinators* should be *coordinates*.

Page 49: the word *precession* should be *precision*. Also, in the second paragraph, first sentence, ignore the duplicate word *coordinates*.

Page 50: in the title of Section 1.1, ignore the duplicate word *and*.

Page 54: in the paragraph, the equation should read as $\bar{\boldsymbol{u}}(x^1, x^2, 0) \neq \boldsymbol{w}(x^1, x^2, 0)$. Also, errors of theorem 2 of Baldelli and Bourdin [2] are mainly caused by mismatching of asymptotic scalings (i.e. mismatching ε terms), most notably in the $\epsilon_{3j}(\cdot)$ tensors, where $\forall j \in \{1, 2, 3\}$.

Pages 65, 66, 87, 119 and **176**: the words *stain* and *stained* should be *strain* and *strained*, respectively.

Page 67: in Chapter 2, note the following:
Membranes supported by rigid foundations (generalised capstan equations) are both modified and special cases of the work of Konyukhov's [9] and Konyukhov's and Izi's [10], but derived independently. Also, in our analysis, an elastic string may have an arbitrary Poisson's ratio, within the limits $(-1, \frac{1}{2})$, of course.

Pages 68, 70, 71, 74, 75, 76, 77, 88, 153, 154 and **166**: the word *parameterised* should be *parametrised*.

Page 70: in Section 2.3, Theorem 1, the field $(0, g_r^2, g_r^3)$ is an external force field in the curvilinear space where g_r^j are Lipschitz continuous.

Page 71: in Section 2.3, Corollary 1, the field $(0, g_r^2, g_r^3)$ is an external stress field in the curvilinear space where g_r^j are Lipschitz continuous.

Page 80: in the last paragraph, incremental azimuthal length should be $\Delta x^2 = \frac{1}{N-1}\pi$.

Page 84: in the first paragraph, the last sentence should be:
This is an intuitive result and analogous results are found in Section 2.5.3 for the modified capstan equation, given the contact interval is $[\varepsilon - \frac{1}{2}\pi, \frac{1}{2}\pi - \varepsilon]$, where $\varepsilon > 0$.

Page 86: the word *equitation* should be *equation*.

Page 86: in Section 2.7, Conclusions, note the following:
Our numerical results indicate that increasing the curvature of the contact region, the Poisson's ratio or the thickness of the elastic body (membrane or otherwise) increases the frictional force, for a constant coefficients of friction, and incompressible materials such as rubber can have a high frictional forces. Our analysis imply that coefficient of friction is model dependent.

Page 88: in Section 3.2,
Assertion 1, note the following:
$K = (F_{[\mathrm{II}]1}^{\ 1} F_{[\mathrm{II}]2}^{\ 2} - F_{[\mathrm{II}]1}^{\ 2} F_{[\mathrm{II}]2}^{\ 1})$ is the Gaussian curvature and $H = -\frac{1}{2} F_{[\mathrm{II}]\alpha}^{\ \alpha}$ is the mean curvature.

Also, the term $J(\boldsymbol{u})_{\text{shell}}$ should read as $J_{\text{shell}}(\boldsymbol{u})$.

Page 89: in Section 3.2, Derivation, note the following:
Assume for the time being that we are dealing with a shell with a thickness $2h$ and where the mid-surface of the shell is described by $\boldsymbol{\sigma}(\omega)$, and thus, we may express the energy functional of this

shell as

$$J_{2h}(\boldsymbol{u}) = \int_\omega \left[B^{\alpha\beta\gamma\delta}\left(h\epsilon_{\alpha\beta}(\boldsymbol{u})\epsilon_{\gamma\delta}(\boldsymbol{u}) + \frac{1}{3}h^3\rho_{\alpha\beta}(\boldsymbol{u})\rho_{\gamma\delta}(\boldsymbol{u})\right) - 2hf^iu_i\right]d\omega - \int_{\partial\omega} 2h\tau_0^iu_i\ d(\partial\omega)\ ,$$

where $\boldsymbol{u}$ describes the displacement field with respect to ω. Note that the above energy functional can alternatively be expressed as

$$\frac{J_{2h}(\boldsymbol{u})}{2h} + \int_\omega f^iu_id\omega + \int_{\partial\omega}\tau_0^iu_i\ d(\partial\omega) = \int_\omega\left[\frac{1}{2}\mu\left(\frac{\lambda}{\lambda+2\mu}\epsilon^\alpha_\alpha(\boldsymbol{u})\epsilon^\gamma_\gamma(\boldsymbol{u}) + \epsilon^\gamma_\alpha(\boldsymbol{u})\epsilon^\alpha_\gamma(\boldsymbol{u})\right)\right.$$
$$\left. + \frac{1}{6}\mu\left(\frac{\lambda}{\lambda+2\mu}\rho^\alpha_\alpha(\boldsymbol{u})\rho^\gamma_\gamma(\boldsymbol{u}) + \rho^\gamma_\alpha(\boldsymbol{u})\rho^\alpha_\gamma(\boldsymbol{u})\right)h^2\right]d\omega\ .$$

Note that as $\boldsymbol{\epsilon}(\cdot)$ is the half of the change in the first fundamental form tensor, we have $\epsilon^\alpha_\alpha(\boldsymbol{u})\epsilon^\gamma_\gamma(\boldsymbol{u}) \ll F_{[\mathrm{I}]}{}^\alpha_\alpha F_{[\mathrm{I}]}{}^\gamma_\gamma = 4$ and $\epsilon^\gamma_\alpha(\boldsymbol{u})\epsilon^\alpha_\gamma(\boldsymbol{u}) \ll \frac{1}{4}F_{[\mathrm{I}]}{}^\gamma_\alpha F_{[\mathrm{I}]}{}^\alpha_\gamma = \frac{1}{2}$. Also, as $\boldsymbol{\rho}(\cdot)$ is the half of the change in the second fundamental form tensor, we have

$$h^2\rho^\alpha_\alpha(\boldsymbol{u})\rho^\gamma_\gamma(\boldsymbol{u}) \ll h^2F_{[\mathrm{II}]}{}^\alpha_\alpha F_{[\mathrm{II}]}{}^\gamma_\gamma = (2hH)^2 \ll 1 \text{ and}$$
$$h^2\rho^\gamma_\alpha(\boldsymbol{u})\rho^\alpha_\gamma(\boldsymbol{u}) \ll h^2F_{[\mathrm{II}]}{}^\gamma_\alpha F_{[\mathrm{II}]}{}^\alpha_\gamma = 2h^2(2H^2-K) \ll 1\ ,$$

given that this shell satisfies the conditions described by Assertion 1. Therefore, we may assume that $h^2\rho^\alpha_\alpha(\boldsymbol{u})\rho^\gamma_\gamma(\boldsymbol{u}) \ll \epsilon^\alpha_\alpha(\boldsymbol{u})\epsilon^\gamma_\gamma(\boldsymbol{u})$ and $h^2\rho^\gamma_\alpha(\boldsymbol{u})\rho^\alpha_\gamma(\boldsymbol{u}) \ll \epsilon^\gamma_\alpha(\boldsymbol{u})\epsilon^\alpha_\gamma(\boldsymbol{u})$, and thus, the above equation implies that one can expect $J_{2h}(\boldsymbol{u})$ to behave approximately linear in h, despite its cubic h dependence. This, in turn, implies that the energy stored in the shell's upper and lower halves may be approximated by dividing the energy functional of the shell by 2. To be more precise, if $J_{2h}(\boldsymbol{u}) = J_{\mathrm{upper}}(\boldsymbol{u}) + J_{\mathrm{lower}}(\boldsymbol{u})$, then assume that $\frac{1}{2}J_{2h}(\boldsymbol{u}) \approx J_{\mathrm{upper}}(\boldsymbol{u}) \approx J_{\mathrm{lower}}(\boldsymbol{u})$. Now, take the upper half and assert that this is the form of an overlying shell equation.

Pages 92 and **124**: the word *action* should be *acting*.

Page 101: in the second to last paragraph, incremental azimuthal length should be $\Delta x^2 = \frac{1}{N-1}\pi$.

Page 103: in Section 3.6, the term $J(\boldsymbol{w})_{\mathrm{Baldelli}}$ should read as $J_{\mathrm{Baldelli}}(\boldsymbol{w})$.

Page 105: in the last sentence, the word *stranded* should be *standard* and the word *liner* should be *linear*.

Page 107: in the first sentence, incremental azimuthal length should be $\Delta x^2 = \frac{1}{N-1}\pi$.

Page 115: in Section 3.8, Conclusions, note the following:

Our analysis shows that the radial solution of our bonded shell model can approximate the displacement field of foundation with a significant degree of accuracy given that the Young's modulus of the shell is high, which is consistent with what is documented in the literature [1]. However, both our numerical and asymptotic analyses (i.e. the scaling $\phi_b \sim 1$) show that there exist optimal values of the Young's modulus, the Poisson's ratio and the thickness of the shell (with respect to the foundation), and the curvature of the contact region such that we observe a minimum azimuthal error. Our numerical modelling also implies that the radial error is a minimum for a shell if it has a relatively low Poisson's ratio and is relatively thin, and if the contact region (between the shell and the elastic foundation) has a low curvature, where the latter two conditions are consistent with our derivation of the overlying shell model (i.e. consistent with Assertion 1 and Hypothesis 2). Furthermore, it is often regarded in the field of stretchable and flexible electronics that the planar solution (where stretching effects are dominant) is mostly accurate when the stiffness of the plate/shell increases indefinitely. The significance of our work is that, as far as we are aware, this is the first analysis conducted on the planer solution, both by asymptotically and numerically showing that indefinitely increasing the stiffness of the membrane will not guarantee a more accurate solution as there exists an optimum Young's modulus (with respect to other variables) where the error between the unapproximated and approximated solution is a minimum.

Pages 118 and **149**: the word *referrer* should be *refer*.

Page 120: in Section 4.2, Derivation, note the following:
Assume that the shell is coupled to the elastic foundation with friction, where a portion of the foundation is satisfying the zero-Dirichlet boundary condition. Also, assume that one is applying forces to both the top and to a portion of the boundary of the shell to mimic compression and shear respectively at the contact region. Now, the higher the compression, then the higher the normal displacement is towards the bottom, i.e. $u^3|_{\omega^+} < 0$ (Assertion 1 can guarantee this condition for sensible boundary forces), and the higher the shear, then the higher the tangential displacement is in the direction of the applied tangential force, i.e. $(u_\alpha u^\alpha)^{\frac{1}{2}}|_{\omega^+} > 0$. Now, we consider Kikuchi and Oden's model for Coulomb's law of static friction for a three-dimensional elastic body [8] (i.e. not a shell), and once extended to curvilinear coordinates and taking the limit $\varepsilon \to 0$, we find

$$\left[T_3^\beta(\boldsymbol{v}) + \nu_F\,(g_{33})^{\frac{1}{2}}\,(v_\alpha v^\alpha)^{-\frac{1}{2}} v^\beta T_3^3(\boldsymbol{v})\right]|_{\omega^+} \leq 0\ ,$$

for $T_3^3(\boldsymbol{v})|_{\partial\omega^+} < 0$, where $\boldsymbol{v}$ is the displacement field and the volume $\{\omega \times [0,h]\}$ describes the reference configuration of this elastic body. Just as it is for Coulomb's friction case, where the bodies are in relative equilibrium, given that the magnitude of the normal stress is above a certain factor of the magnitude of the tangential stress, we assert that the bodies (the shell and the foundation) are in relative equilibrium given that the normal displacement is below a certain factor of the magnitude of the tangential displacement, i.e.

$$\left[(u_\alpha u^\alpha)^{\frac{1}{2}} + C\,(g_{33})^{\frac{1}{2}}\,u^3\right]|_{\omega^+} \leq 0\ ,$$

if $u^3|_{\omega^+} \leq 0$, for some dimensionless constant C.

To determine the constant C, consider Coulomb's law of static friction for the limiting equilibrium case (i.e. at the point of slipping) and rearranging to obtain equation

$$\left[\bar{\nabla}_3\left((v_\alpha v^\alpha)^{\frac{1}{2}} + 2\nu_F\left(1+\frac{\gamma}{1-2\gamma}\right)(g_{33})^{\frac{1}{2}}\,v^3\right) + \left(\frac{2\nu_F\gamma}{1-2\gamma}\right)(g_{33})^{\frac{1}{2}}\,\bar{\nabla}_\alpha v^\alpha + \frac{v^\delta \bar{\nabla}_\delta v_3}{(v_\alpha v^\alpha)^{\frac{1}{2}}}\right]|_{\omega^+} = 0\ .$$

Now, the above equation must hold for all elastic conditions, even under extreme conditions such as the *incompressible elasticity* condition, i.e. $(\frac{\gamma}{1-2\gamma})\bar{\nabla}_i v^i = p(x^1,x^2,x^3)$, where $p(\cdot)$ is a finite function

[6]. Thus, we may assume the following equation,

$$\left[\bar{\nabla}_3\left((v_\alpha v^\alpha)^{\frac{1}{2}}+2\nu_F\,(g_{33})^{\frac{1}{2}}\,v^3\right)+\frac{v^\delta\bar{\nabla}_\delta v_3}{(v_\alpha v^\alpha)^{\frac{1}{2}}}+2\nu_F\,(g_{33})^{\frac{1}{2}}\,p(x^1,x^2,x^3)\right]|_{\omega^+}=0\ .$$

Now, nondimensionalise the above equation by making the transformations $v^i = Lw^i$, $x^\alpha = Ly^\alpha$ and $x^3 = hy^3$, where $L = \sqrt{\mathrm{meas}(\omega;\mathbb{R}^2)}$, to obtain

$$\left[\left(\frac{L}{h}\right)(g^{33})^{\frac{1}{2}}\,\frac{\partial}{\partial y^3}\left((w_\alpha w^\alpha)^{\frac{1}{2}}+2\nu_F\,(g_{33})^{\frac{1}{2}}\,w^3\right)\right.$$
$$\left.+\frac{(g_{33})^{\frac{1}{2}}\,w^\delta}{(w_\alpha w^\alpha)^{\frac{1}{2}}}\left(\frac{\partial w^3}{\partial y^\delta}+\Gamma^3_{\delta i}w^i\right)+2\nu_F p(Ly^1,Ly^2,hy^3)\right]|_{\omega^+}=0\ .$$

As our goal is to study shells, we consider the limit $(h/L)\to 0$. As we also require Coulomb's law of static friction for the limiting equilibrium to stay finite in this limit, the above equation implies that

$$\left[(w_\alpha w^\alpha)^{\frac{1}{2}}+2\nu_F\,(g_{33})^{\frac{1}{2}}\,w^3\right]|_{\{(\frac{\omega}{L^2})\times[0,1]\}}=q(y^1,y^2)+\mathcal{O}\left(\left(\frac{h}{L}\right),y^3\right)\ ,$$

where $q(\cdot)$ is some finite function. As we are seeking a relation of the form of Coulomb's law, we may assume $q(\cdot)=0$, and thus, letting $C=2\nu_F$ is a sound approximation. Finally, assuming that $\boldsymbol{u}$ is continuous on $\bar{\Omega}$ and noting that we have $g_{33}=1$ in a shell, we arrive at Hypothesis 3.

Page 120: in Section 4.2, Hypothesis 3 can be better expressed as:
A shell supported by an elastic foundation with a rough contact area that is in agreement with Assertion 1 satisfies the following displacement-based friction condition

$$\left[2\nu_F u^3+(u_\alpha u^\alpha)^{\frac{1}{2}}\right]|_\omega\leq 0\ ,$$

where ν_F is the coefficient of friction between the shell and the foundation, and $\boldsymbol{u}$ is the displacement field of the shell with respect to the contact region ω. If $[2\nu_F u^3+(u_\alpha u^\alpha)^{\frac{1}{2}}]|_\omega<0$, then we say that the shell is *bonded* to the foundation, and, if $[2\nu_F u^3+(u_\alpha u^\alpha)^{\frac{1}{2}}]|_\omega=0$, then we say that the shell is at *limiting-equilibrium*.

Page 123: in Section 4.3.3, Governing Equations of the Overlying Shell, derivation can be better expressed as:
The set $\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)$ is not a linear set as it violates the homogeneity property. However, it can be shown that for any field $\boldsymbol{u}\in\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)$ there exists a field $\boldsymbol{w}\in\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)\backslash\{\boldsymbol{u}\}$ and a constant $\varepsilon>0$ such that $\boldsymbol{u}+s\boldsymbol{w}\in\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)$, $\forall\ s\in(-\varepsilon,1]$, i.e.

$$\int_U[2\nu_F(u^3+sw^3)+(u_\alpha u^\alpha+2su_\alpha w^\alpha+s^2w_\alpha w^\alpha)^{\frac{1}{2}}]\ dx^1dx^2\leq 0\ ,\ \forall\ U\in\mathcal{M}(\omega)\text{ with }\mathrm{meas}(U;\omega)>0\ .$$

To find the governing equations for the $[2\nu_F u^3+(u_\alpha u^\alpha)^{\frac{1}{2}}]|_\omega<0$ case consider a unique minimiser $\boldsymbol{u}\in\boldsymbol{V}_{\mathscr{O}}(\omega,\Omega)$, where $\boldsymbol{V}_{\mathscr{O}}(\omega,\Omega)=\{\boldsymbol{v}\in\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)\ |\ [2\nu_F v^3+(v_\alpha v^\alpha)^{\frac{1}{2}}]|_{\omega_{\mathscr{O}}}<0\text{ a.e.}\}$ and where $\omega_{\mathscr{O}}=\{V\in\mathcal{M}(\omega)\ |\ [2\nu_F v^3+(v_\alpha v^\alpha)^{\frac{1}{2}}]|_V<0\text{ a.e., }\mathrm{meas}(V;\omega)>0\}$. Now, given a $\boldsymbol{w}\in\boldsymbol{V}_{\mathscr{O}}(\omega,\Omega)$, there exists an $\varepsilon>0$ such that we get $\boldsymbol{u}+s\boldsymbol{w}\in\boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)$, $\forall\ s\in(-\varepsilon,1]$ where

$$\varepsilon<\frac{\left(2\nu_F||u^3||_{L^1(U)}-||(u_\gamma u^\gamma)^{\frac{1}{2}}||_{L^1(U)}\right)}{\left(2\nu_F||w^3||_{L^1(U)}+||(w_\alpha w^\alpha)^{\frac{1}{2}}||_{L^1(U)}\right)}$$

for some $U \in \mathcal{M}(\omega_{\mathscr{O}})$. Now, simply let $\boldsymbol{v} = \boldsymbol{u} + s\boldsymbol{w}$ in Corollary 4 to obtain $0 \leq J'(\boldsymbol{u})(s\boldsymbol{w})$, $\forall\ s \in (-\varepsilon, 1]$ for this $\boldsymbol{w} \in \boldsymbol{V}_{\mathscr{O}}(\omega,\Omega)$. Finally, noticing that $0 \leq J'(\boldsymbol{u})(\mathrm{sign}(s)|s|\boldsymbol{w})$, $\forall\ \boldsymbol{w} \in \boldsymbol{V}_{\mathscr{O}}(\omega,\Omega)$, we get the governing equations for the *bonded* case.

For the limiting-equilibrium case, an incorrect proof is given in the thesis. The correct proof is as follows: To find the governing equations for the $[2\nu_F u^3 + (u_\alpha u^\alpha)^{\frac{1}{2}}]|_\omega = 0$ case consider a unique minimiser $\boldsymbol{u} \in \boldsymbol{V}_{\mathscr{C}}(\omega,\Omega)$, where $\boldsymbol{V}_{\mathscr{C}}(\omega,\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_{\mathscr{F}}(\omega,\Omega) \mid [2\nu_F v^3 + (v_\alpha v^\alpha)^{\frac{1}{2}}]|_{\omega_{\mathscr{C}}} = 0 \text{ a.e.}\}$ and where $\omega_{\mathscr{C}} = \{V \in \mathcal{M}(\omega) \mid [2\nu_F v^3 + (v_\alpha v^\alpha)^{\frac{1}{2}}]|_V = 0 \text{ a.e., } \mathrm{meas}(V;\omega) > 0\}$. Now, noticing that $u^j|_{\omega_{\mathscr{C}}}$ are not independent and related by the condition $u^3|_{\omega_{\mathscr{C}}} = -\frac{1}{2}\nu_F^{-1}(u_\alpha u^\alpha)^{\frac{1}{2}}|_{\omega_{\mathscr{C}}}$, we get $\delta u^3|_{\omega_{\mathscr{C}}} = -\frac{1}{2}\nu_F^{-1}(u_\alpha u^\alpha)^{-\frac{1}{2}}(u_\gamma \delta u^\gamma)|_{\omega_{\mathscr{C}}}$. Let

$$\boldsymbol{V}_{\mathscr{C}}(\boldsymbol{u};\omega,\Omega) = \{\boldsymbol{v} \in \boldsymbol{V}_{\mathscr{C}}(\omega,\Omega) \mid (v^1, v^2)|_{\omega_{\mathscr{C}}} = (cu^1, cu^2)|_{\omega_{\mathscr{C}}} \text{ a.e., } \forall c > 0,\ \boldsymbol{u} \in \boldsymbol{V}_{\mathscr{C}}(\omega,\Omega)\}\ ,$$

and now, given a $\boldsymbol{w} \in \boldsymbol{V}_{\mathscr{C}}(\boldsymbol{u};\omega,\Omega)$ there exists an $\varepsilon > 0$ such that we get $\boldsymbol{u} + s\boldsymbol{w} \in \boldsymbol{V}_{\mathscr{F}}(\omega,\Omega)$, $\forall\ s \in (-\varepsilon, 1]$, where $\varepsilon < ||(w_\alpha w^\alpha)^{\frac{1}{2}}||^{-1}_{L^1(U)}||(u_\gamma u^\gamma)^{\frac{1}{2}}||_{L^1(U)}$ for some $U \in \mathcal{M}(\omega_{\mathscr{C}})$. Now, simply let $\boldsymbol{v} = \boldsymbol{u} + s\boldsymbol{w}$ in Corollary 4 to obtain $0 \leq J'(\boldsymbol{u})(s\boldsymbol{w}|_\Omega + s(w^1, w^2)|_{\omega_{\mathscr{C}}})$, $\forall\ s \in (-\varepsilon, 1]$ for this $\boldsymbol{w} \in \boldsymbol{V}_{\mathscr{C}}(\omega,\Omega)$. Finally, noticing that $J'(\boldsymbol{u})(\boldsymbol{w}|_\Omega) = 0$ (this leads to the governing equations in the foundation) and $0 \leq J'(\boldsymbol{u})(\mathrm{sign}(s)|s|(w^1, w^2)|_{\omega_{\mathscr{C}}})$, $\forall\ \boldsymbol{w} \in \boldsymbol{V}_{\mathscr{C}}(\boldsymbol{u};\omega,\Omega) \subset \boldsymbol{V}_{\mathscr{C}}(\omega,\Omega)$, we get the governing equations for the *limiting-equilibrium* case (adapted from Section 8.4.2 of Evans [5]).

Page 129: in the first paragraph, relative displacement between the two elastic bodies should be $\Phi(\boldsymbol{v} - \boldsymbol{u}) = (\ell_\alpha(\boldsymbol{v},\boldsymbol{u})\ell^\alpha(\boldsymbol{v},\boldsymbol{u}))^{\frac{1}{2}}$, where $\boldsymbol{\ell}(\boldsymbol{v},\boldsymbol{u}) = (v^1, v^2)|_{\omega^+} - (u^1, u^2)|_{\omega^-}$.

Page 130: the first paragraph should be:
Note that in this framework, i.e. in the set $\{\omega \times [0, h)\}$, one has $g_{33} = 1$.

Page 132: in the first sentence, ignore the last (duplicate) word *model*.

Page 137: in Section 4.7, Conclusions, note the following:
In the literature, friction laws are represented in either force or stress based formulations. However, as far as we are aware, this is the first documented displacement-based friction condition that is compatible with shells and shell-membranes on elastic foundations, and can also can be expressed in a closed variational form. Note, however, its physical validity (as is also the case with other friction models) still remain as an open question.

Page 137: in Chapter 5, Abstract, the first sentence, the words *a question* should be *open questions*.

Page 151: in the third paragraph, last sentence, the words *a dancers* should be *dancers*.

Page 156: in the third paragraph, incremental azimuthal length is $\Delta x^2 = \frac{1}{N-1}\pi$.

Page 166: in Section 6.4, Hypothesis 4 can be better expressed as:

Consider a rectangular membrane over a rough elastic prism whose strained contact region is parametrised by the map $(x, f(\theta), g(\theta))_{\mathsf{E}}$, where $f(\cdot)$ and $g(\cdot)$ are $C^1([\theta_0, \theta_{\mathsf{max}}])$ 2π-periodic functions, $|x| \leq \infty$, and the contact interval $[\theta_0, \theta_{\mathsf{max}}]$ is chosen such that $g'f'' - f'g'' > 0, \forall\, x^2 \in [\theta_0, \theta_{\mathsf{max}}]$ (i.e. has a positive mean curvature). If T_{max} is the maximum applied-tension at θ_{max} and T_0 is the minimum applied-tension at θ_0, then there exists a regression curve $Y(\cdot)$ of the form $\xi T_{\mathsf{max}} + \varepsilon = Y(\xi T_0 + \varepsilon)$, such that $Y'(x)|_{x\to\infty}$ is positive, finite and invariant with respect to the quantity (mean radius of curvature of the contact region)

$$r_0 = \left(\arctan\left(\frac{g'(\theta_0)}{f'(\theta_0)}\right) - \arctan\left(\frac{g'(\theta_{\mathsf{max}})}{f'(\theta_{\mathsf{max}})}\right)\right)^{-1} \int_{\theta_0}^{\theta_{\mathsf{max}}} \left((f')^2 + (g')^2\right)^{\frac{1}{2}}\ d\theta\ ,$$

where the normalising constant ξ is chosen such that $\xi \leq \frac{1}{\mathsf{max}(T_{\mathsf{max}})}$ and the translating constant ε is chosen such that $Y(\varepsilon)$ is not singular. Furthermore, given such a regression curve $Y(\cdot)$, the coefficient of friction μ_F has the following relation,

$$\mu_F = \left(\arctan\left(\frac{g'(\theta_0)}{f'(\theta_0)}\right) - \arctan\left(\frac{g'(\theta_{\mathsf{max}})}{f'(\theta_{\mathsf{max}})}\right)\right)^{-1} \log\left(Y'(x)\right)|_{x\to\infty}\ ,$$

and, in particular, μ_F is invariant with respect to the quantity r_0.

Page 166: in the last paragraph, both regression curve and the coefficient of friction should be $Y(\xi T_0 + 1) = a + b(\xi T_0 + 1) + c\log(\xi T_0 + 1)$ and

$$\mu_F = \frac{\log(b)}{\left(\arctan\left(\frac{g'(\theta_0)}{f'(\theta_0)}\right) - \arctan\left(\frac{g'(\theta_{\mathsf{max}})}{f'(\theta_{\mathsf{max}})}\right)\right)}\ ,$$

where $(0, f, g)_{\mathsf{E}}$ is the cross section of the final deformed geometry and $[\theta_0, \theta_{\mathsf{max}}]$ is the final contact interval.

Page 168: in the first paragraph, the last word, *modules* should be *modulus*.

Page 171: in Section 6.6, the first sentence, the word *cylindrical* should be *cylinder*.

Page 172: in the first set of equations, following is the correct formulation:

$$\Lambda = 4\mu\frac{\lambda + \mu}{\lambda + 2\mu}\ .$$

Page 172: in the second to last paragraph, incremental azimuthal length should be $\Delta x^2 = \frac{1}{N-1}\pi$.

Page 177: right after the definition of δl, the word *how* should be *the*.

Page 179: in Section 6.8, Conclusions, note the following:

Both numerical-modelling and experiments conducted on human subjects imply that it is unwise to use belt-friction models (e.g. capstan equation) to calculate the friction between in-vivo skin and fabrics. This is because such models assume a rigid foundation while human soft-tissue is complaint, and thus, a portion of the applied force is expending on deforming the soft-tissue, which in turn leads to the illusion of a higher coefficient of friction when belt-friction models are used to calculate the coefficient of friction. We also found that both numerical-modelling and real-life experimental data imply that given a constant coefficient of friction, a higher volume of soft-tissue (high radius) and more compliant soft-tissue (lower Young's modulus) would result in higher deformation of the skin, and a higher volume of soft-tissue would result in more shear-stress generated on the skin, which in turn could lead to grater probability of skin damage.

We stress to the reader that our statistical modelling techniques used in this section are unsatisfactory, due to a clear lack of knowledge in the field of statistics during the time of this analysis. However, the conclusions implied by various correlations (as described above) still stand.

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