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Laplacian flow for closed G_2 structures: real analyticity

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Let $\varphi(t), t \in [0, T_0]$ be a solution to the Laplacian flow for closed G_2 structures on a compact 7-manifold M. We show that for each fixed time $t \in (0, T_0]$, $(M, \varphi(t), g(t))$ is real analytic, where $g(t)$ is the metric induced by $\varphi(t)$. Consequently, any Laplacian soliton is real analytic and we obtain unique continuation results for the flow.

1. Introduction

Let M be a compact 7-manifold and let φ_0 be a closed G_2 structure on M. We consider solutions $\varphi(t), t \in [0, T_0]$, to the Laplacian flow for closed G_2

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structures:

$$
\begin{cases}
\frac{\partial}{\partial t}\varphi = \Delta_\varphi \varphi, \\
d\varphi = 0, \\
\varphi(0) = \varphi_0,
\end{cases}
\tag{1.1}
$$

where $\Delta_\varphi \varphi = dd^*\varphi + d^*d\varphi$ is the Hodge Laplacian of $\varphi(t)$ with respect to the metric $g(t)$ determined by $\varphi(t)$.

The flow (1.1) was introduced by Bryant (see [2, §5]) as a potential way to study the challenging problem of existence of torsion-free G_2 structures and thus Ricci-flat metrics with exceptional holonomy G_2, since stationary points of the flow are the G_2 structures φ satisfying $d\varphi = d^*\varphi = 0$, which is the torsion-free condition. (Although this statement about the stationary points is true for compact manifolds by integration by parts, we gave an alternative argument in [13] which shows that stationary points of the flow are always torsion-free, even in the non-compact setting.) Moreover, the flow moves within the cohomology class of φ_0 and has a variational interpretation due to Hitchin [4, 9]. The primary goal in the field is to find conditions on an initial closed G_2 structure φ_0 such that the flow (1.1) will exist for all time and converge to a torsion-free G_2 structure. Situations under which this occurs were proved by the authors in [14].

As M is compact, the Laplacian flow starting from any closed G_2 structure φ_0 is guaranteed to have a unique solution $\varphi(t)$ for a short time $t \in [0, \epsilon)$, where ϵ depends on φ_0 (see [2, 4]). In our previous papers [13, 14], we studied various foundational analytical and geometric properties of the flow (1.1), including Shi-type derivative estimates, uniqueness theorems, compactness results, soliton solutions, long-time existence results and stability of torsion-free G_2 structures along the flow.

On the face of it these analytic results are somewhat surprising because the velocity of the flow (1.1) is defined by the Hodge Laplacian, which we would usually think of as a positive operator, and thus the flow appears to look like a backwards heat equation. In spite of this, the Laplacian flow is actually weakly parabolic in a certain non-standard sense: it is parabolic in the direction of closed forms, modulo the action of diffeomorphisms. It is this fact that enables the analysis of the flow to proceed. The reader is referred to [13, 14] for more detailed information about the Laplacian flow.

In this paper, we continue to analyze the Laplacian flow (1.1) and investigate the regularity of the solution $\varphi(t)$ for each positive time t. Our main result is the following.

Theorem 1.1. *If $\varphi(t), t \in [0, T_0]$ is a smooth solution to the Laplacian flow* (1.1) *for closed* G_2 *structures on an open set $U \subset M$, then for each time $t \in (0, T_0]$, $(U, \varphi(t), g(t))$ is real analytic.*

Readers are referred to §3 for the definition and criterion for a G_2 structure to be real analytic. Real analyticity for positive times is well known for linear parabolic PDE (such as the heat equation) and some weakly parabolic nonlinear PDE (such as Ricci flow [1]). However, as we have indicated, the Laplacian flow is not weakly parabolic in a standard manner, and so one should not immediately expect such a regularity result.

Since any Laplacian soliton corresponds to a local self-similar solution to the Laplacian flow (1.1), we have the following corollary to Theorem 1.1.

Corollary 1.2. *Suppose (M, φ, X, λ) is a Laplacian soliton (not necessarily compact), i.e., $d\varphi = 0$ and*

$$\Delta_\varphi \varphi = \lambda \varphi + \mathcal{L}_X \varphi \tag{1.2}$$

for some smooth vector field X and constant λ. Then (M, φ) is real analytic.

The real analyticity of a torsion-free G_2 structure, i.e., the case $X = 0$, $\lambda = 0$ in (1.2), is already well-known (see [3] for example). Moreover, real analyticity plays a significant role in G_2 geometry, as can be seen in [3].

For convenience we say a G_2 structure φ on M is complete if its associated metric is complete. By modifying the argument in the proof of [12, Corollary 6.4, p.256], Theorem 1.1 immediately implies the following unique continuation results.

Corollary 1.3. *Suppose that M^7 is connected and simply connected, and $\varphi(t)$, $\tilde{\varphi}(t)$ are smooth complete solutions to the Laplacian flow* (1.1) *on $M \times [0, T_0]$. Then, for any $t \in (0, T_0]$, the following hold.*

(a) *If $\varphi(t) = \tilde{\varphi}(t)$ on some open set $U \subset M$, then there exists a diffeomorphism F of M such that $F^*\tilde{\varphi}(t) \equiv \varphi(t)$.*

(b) *Any local diffeomorphism $F : U \to V$ between connected open sets $U, V \subset M$ such that $F^*(\varphi(t)|_V) = \varphi(t)|_U$ can be uniquely extended to a global diffeomorphism F of M with $F^*\varphi(t) = \varphi(t)$.*

Corollary 1.4. *Suppose that M^7 is connected and simply-connected and (φ, X, λ) and $(\tilde{\varphi}, \tilde{X}, \tilde{\lambda})$ are complete Laplacian solitons on M. If $\varphi = \tilde{\varphi}$ on some connected open set $U \subset M$, then there exists a diffeomorphism F of M such that $F^*\tilde{\varphi} \equiv \varphi$.*

Since a G_2 structure φ determines a unique metric g_φ, any diffeomorphism $F : (M,\varphi)\to(M,\tilde{\varphi})$ such that $F^*\tilde{\varphi}=\varphi$ is an isometry between (M,g_φ) and $(M,g_{\tilde{\varphi}})$. The converse is clearly not always true, since the G_2 structure encodes strictly more information than the metric.

Our approach to prove Theorem 1.1 is similar to Bando's [1] proof of the real analyticity of Ricci flow, namely to use derivative estimates for the Riemann curvature tensor Rm, the torsion tensor T and φ along the flow. In our previous paper [13], we derived Shi-type derivative estimates along the Laplacian flow, which take the form

$$t^{\frac{k}{2}}\left(|\nabla^k Rm(x,t)|+|\nabla^{k+1}T(x,t)|\right)\le C_kK,\quad x\in M,\ t\in[0,1/K], \tag{1.3}$$

where C_k is a constant depending on the order k and K is the bound on

$$\Lambda(x,t)=(|Rm|^2(x,t)+|\nabla T|^2(x,t))^{1/2}. \tag{1.4}$$

However, in [13], we do not analyze how C_k depends on k, which is particularly relevant when k is large.

When one applies the heat operator to $|\nabla^k Rm(x,t)|+|\nabla^{k+1}T(x,t)|$, lower order terms are generated during the computation, and the number of these terms grows with the order k of differentiation, which then contributes to the growth of the constants C_k. By showing that the C_k are of sufficiently slow growth in the order k, we may deduce that the G_2 structure $\varphi(t)$ and associated metric $g(t)$ are real analytic at each fixed time $t>0$. The key step is to revisit the derivation of the derivative estimates (1.3) from [13] and obtain the following much more refined estimates:

$$\sum_{k=0}^{n}\frac{t^k}{(k+1)!^2}\left(|\nabla^k Rm|^2(x,t)+|\nabla^{k+2}\varphi|^2(x,t)\right)\le C(T_0,K_0) \tag{1.5}$$

on $M\times[0,\alpha/K_0]$ for all $n\in\mathbb{N}$ (we assume $\mathbb{N}$ to include 0), where $K_0=\sup_M|\Lambda(x,0)|$, $\alpha, C(T_0,K_0)$ are constants. As we will see in §3, the estimate (1.5) leads to the real analyticity of $(M,\varphi(t),g(t))$ for each time $t>0$.

2. Preliminaries

We collect some facts on closed G_2 structures, mainly based on [2, 10, 13].

Let $\{e_1,e_2,\dots,e_7\}$ be the standard basis of $\mathbb{R}^7$ and let $\{e^1,e^2,\dots,e^7\}$ be its dual basis. For simplicity we write $e^{ijk}=e^i\wedge e^j\wedge e^k$ and define a 3-form

ϕ by:

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}.$$

The subgroup of $GL(7,\mathbb{R})$ fixing ϕ is the exceptional Lie group G_2, which is a compact, connected, simple Lie subgroup of $SO(7)$ of dimension 14. It is well-known that G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which $\{e_1, e_2, \ldots, e_7\}$ is an oriented orthonormal basis.

Let M be a 7-manifold. For $x \in M$ we let

$$\Lambda^3_+(M)_x = \{\varphi_x \in \Lambda^3 T^*_x M \,|\, \exists u \in \mathrm{Hom}(T_xM, \mathbb{R}^7), u^*\phi = \varphi_x\}.$$

The bundle $\Lambda^3_+(M) = \bigsqcup_x \Lambda^3_+(M)_x$ is an open subbundle of $\Lambda^3 T^*M$. We call a section φ of $\Lambda^3_+(M)$ a G_2 structure on M and denote the space of G_2 structures on M by $\Omega^3_+(M)$. The notation is motivated by the fact that there is a 1-1 correspondence between G_2 structures in the sense of subbundles of the frame bundle and $\Omega^3_+(M)$. The bundle $\Lambda^3_+(M)$ has sections, which means that G_2 structures exist, if and only if M is oriented and spin.

A G_2 structure φ induces a unique metric g and orientation (given by a volume form vol_g of g) which satisfy

$$g(u, v)\,\mathrm{vol}_g = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi.$$

The metric and orientation determine the Hodge star operator $*_\varphi$, so we can define $\psi = *_\varphi \varphi$. Notice that the relationship between g and φ, and hence between ψ and φ, is nonlinear.

Although G_2 acts irreducibly on $\mathbb{R}^7$ (and hence on $\Lambda^1(\mathbb{R}^7)^*$ and $\Lambda^6(\mathbb{R}^7)^*$), it acts reducibly on $\Lambda^k(\mathbb{R}^7)^*$ for $2 \le k \le 5$. Hence a G_2 structure φ induces splittings of the bundles $\Lambda^k T^*M$ $(2 \le k \le 5)$, which we denote by $\Lambda^k_l(T^*M)$ so that l indicates the rank of the bundle, and we let the space of sections of $\Lambda^k_l(T^*M)$ be $\Omega^k_l(M)$. Explicitly, we have that

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M) \quad \text{and} \quad \Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M),$$

where (using the orientation in [2] rather than [10])

$$\begin{aligned} \Omega^2_7(M) &= \{\beta \in \Omega^2(M) \,|\, \beta \wedge \varphi = 2 *_\varphi \beta\} = \{X \lrcorner \varphi \,|\, X \in C^\infty(TM)\}, \\ \Omega^2_{14}(M) &= \{\beta \in \Omega^2(M) \,|\, \beta \wedge \varphi = - *_\varphi \beta\} = \{\beta \in \Omega^2(M) \,|\, \beta \wedge \psi = 0\}, \end{aligned}$$

and

$$\Omega^3_1(M)=\{f\varphi\,|\,f\in C^\infty(M)\},\quad \Omega^3_7(M)=\{X\lrcorner\psi\,|\,X\in C^\infty(TM)\},$$
$$\Omega^3_{27}(M)=\{\gamma\in\Omega^3(M)\,|\,\gamma\wedge\varphi=0=\gamma\wedge\psi\}.$$

Hodge duality gives corresponding decompositions of $\Omega^4(M)$ and $\Omega^5(M)$.

In our study it is convenient to write key quantities with respect to local coordinates $\{x^1,\dots,x^7\}$ on M. We write a k-form α locally as

$$\alpha=\frac{1}{k!}\alpha_{i_1i_2\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k},$$

where $\alpha_{i_1i_2\cdots i_k}$ is totally skew-symmetric in its indices. In particular, we write φ and ψ locally as

$$\varphi=\frac{1}{6}\varphi_{ijk}dx^i\wedge dx^j\wedge dx^k,\quad \psi=\frac{1}{24}\psi_{ijkl}dx^i\wedge dx^j\wedge dx^k\wedge dx^l.$$

As in [2] (up to a constant factor), we define an operator $i_\varphi:S^2T^*M\to\Lambda^3T^*M$ locally by

$$(i_\varphi(h))_{ijk}=h_i^l\varphi_{ljk}-h_j^l\varphi_{lik}-h_k^l\varphi_{lji}$$

where $h=h_{ij}dx^idx^j$. Then $\Lambda^3_{27}(T^*M)=i_\varphi(S^2_0T^*M)$, where $S^2_0T^*M$ denotes the bundle of trace-free symmetric 2-tensors on M, and $i_\varphi(g)=3\varphi$.

We have contraction identities for φ and ψ in index notation (see [2, 10]):

$$\varphi_{ijk}\varphi_{abl}g^{ia}g^{jb}=6g_{kl},\tag{2.1}$$
$$\varphi_{ijq}\psi_{abkl}g^{ia}g^{jb}=4\varphi_{qkl},\tag{2.2}$$
$$\varphi_{ipq}\varphi_{ajk}g^{ia}=g_{pj}g_{qk}-g_{pk}g_{qj}+\psi_{pqjk},\tag{2.3}$$
$$\begin{aligned}\varphi_{ipq}\psi_{ajkl}g^{ia}=&\;g_{pj}\varphi_{qkl}-g_{jq}\varphi_{pkl}+g_{pk}\varphi_{jql}\\&-g_{kq}\varphi_{jpl}+g_{pl}\varphi_{jkq}-g_{lq}\varphi_{jkp}.\end{aligned}\tag{2.4}$$

Given any G_2 structure $\varphi\in\Omega^3_+(M)$, there exist unique differential forms (called the intrinsic torsion forms) $\tau_0\in\Omega^0(M),\tau_1\in\Omega^1(M),\tau_2\in\Omega^2_{14}(M)$ and $\tau_3\in\Omega^3_{27}(M)$ such that $d\varphi$ and $d\psi$ can be expressed as follows (see [2]):

$$d\varphi=\tau_0\psi+3\tau_1\wedge\varphi+*_\varphi\tau_3\quad\text{and}\quad d\psi=4\tau_1\wedge\psi+\tau_2\wedge\varphi.$$

We shall only consider closed G_2 structures φ in this article. In this case $d\varphi=0$ forces $\tau_0=\tau_1=\tau_3=0$, and hence the only non-zero torsion form is

τ_2. We therefore from now on set $\tau = \tau_2 \in \Omega^2_{14}(M)$ and reiterate that

$$d\varphi = 0 \quad \text{and} \quad d\psi = \tau \wedge \varphi = -*_\varphi \tau.$$

We see immediately that

$$d^*\tau = *_\varphi d *_\varphi \tau = 0,$$

which is given in local coordinates by $g^{mi}\nabla_m \tau_{ij} = 0$.

The full torsion tensor is a 2-tensor T satisfying (see [10])

$$\nabla_i \varphi_{jkl} = T_i{}^m \psi_{mjkl}, \qquad T_i{}^j = \frac{1}{24}\nabla_i \varphi_{lmn}\psi^{jlmn}, \tag{2.5}$$

$$\nabla_m \psi_{ijkl} = -\Big(T_{mi}\varphi_{jkl} - T_{mj}\varphi_{ikl} - T_{mk}\varphi_{jil} - T_{ml}\varphi_{jki}\Big), \tag{2.6}$$

where $T_{ij} = T(\partial_i, \partial_j)$ and $T_i{}^j = T_{ik}g^{jk}$. In our setting we may compute that

$$T = -\frac{1}{2}\tau,$$

so T is divergence-free as $d^*\tau = 0$.

Given these formulae we can compute the Hodge Laplacian of φ, which is the velocity of the Laplacian flow, as in [2, 13].

Proposition 2.1. *For a closed* G_2 *structure* φ*, the Hodge Laplacian of* φ *satisfies*

$$\Delta_\varphi \varphi = d\tau = i_\varphi(h) \;\in\; \Omega^3_1(M) \oplus \Omega^3_{27}(M),$$

where h *is a symmetric* 2*-tensor on* M*, locally given by*

$$h_{ij} = -\nabla_m T_{ni}\varphi_j{}^{mn} - \frac{1}{3}|T|^2 g_{ij} - T_i{}^l T_{lj}. \tag{2.7}$$

Since φ determines a unique metric g on M, we then have the Riemann curvature tensor Rm of g on M, which in our convention is given by

$$Rm(X,Y,Z,W) = g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W)$$

for vector fields X, Y, Z, W on M. In local coordinates, we denote the components of Rm by $R_{ijkl} = Rm(\partial_i, \partial_j, \partial_k, \partial_l)$. The Ricci curvature Rc and scalar curvature R are given locally by $R_{ik} = g^{jl}R_{ijkl}$ and $R = g^{ij}R_{ij}$, and may be computed in terms of the torsion tensor as follows (see e.g. [13]).

Proposition 2.2. *The Ricci tensor and the scalar curvature of the associated metric g of a closed* G_2 *structure φ are given as*

$$R_{ik} = \nabla_j T_{li}\varphi_k{}^{jl} - T_i{}^j T_{jk} \quad \textit{and} \quad R = -|T|^2 = -T_{ik}T_{jl}g^{ij}g^{kl}. \tag{2.8}$$

Notice that Rm and ∇T are both second order in φ, and T is essentially $\nabla\varphi$, so we might expect Rm and ∇T to be related. The next proposition from [13] says that ∇T can be expressed using T and Rm.

Proposition 2.3. *For a closed* G_2 *structure φ, we have*

$$\begin{aligned} 2\nabla_i T_{jk} = \frac{1}{2}R_{ijmn}\varphi_k{}^{mn} + \frac{1}{2}R_{kjmn}\varphi_i{}^{mn} - \frac{1}{2}R_{ikmn}\varphi_j{}^{mn} \\ - T_{im}T_{jn}\varphi_k{}^{mn} - T_{km}T_{jn}\varphi_i{}^{mn} + T_{im}T_{kn}\varphi_j{}^{mn}. \end{aligned}$$

3. Criterion for a G_2 structure to be real analytic

Given a 7-manifold M, a real analytic structure on M is an atlas

$$\left\{(U_j, \{x_j^i\}_{i=1}^7)\right\}_{j\in J},$$

where J is some indexing set, such that the transition functions are real analytic. A Riemannian metric g on a real analytic manifold M is then real analytic if the components g_{ij} of g are real analytic functions with respect to a subatlas of real analytic coordinates.

Let M be an orientable and spinnable 7-manifold, let φ be a G_2 structure on M and let g be its associated Riemannian metric. Suppose further that there is a subatlas of normal coordinate systems on M such that the components of g are real analytic functions in each of these coordinate systems. By [6, Lemma 13.20], (M, g) is then a real analytic Riemannian manifold with respect to this subatlas and, in particular, an atlas for M can be found with real analytic transition functions. In fact, by [7, Lemma 1.2 & Theorem 2.1], for such (M, g) there exists an atlas of harmonic coordinates which are real analytic functions of the normal coordinates and so that the metric g is real analytic in these harmonic coordinates. Real analyticity of the transition functions for the atlas of harmonic coordinates then follows from the fact that the coordinates are harmonic and the metric is real analytic. If in addition the components φ_{ijk} of φ are real analytic with respect to the normal coordinates, which implies that φ is also real analytic in the harmonic coordinates by [7, Corollary 1.4], then we say that (M, φ) is real analytic.

Let $\mathrm{inj}_g(p)$ denote the injectivity radius of g at $p\in M$, and if $\{x^i\}_{i=1}^7$ are coordinates centred at p we denote the Christoffel symbols of the Levi-Civita connection of g by Γ_{ij}^l as usual and let $\partial^k=\sum_{k_1+\cdots+k_n=k}\frac{\partial^k}{(\partial x^1)^{k_1}\cdots(\partial x^n)^{k_n}}$. With this notation in hand, we have the following derivative estimates of φ and g in normal coordinates.

Lemma 3.1. *Let φ be a G_2 structure on M and let g be its associated metric. Let $p\in M$ and suppose there exist constants C_1 and $r>0$ such that*

$$|\nabla^k Rm|(x)+|\nabla^{k+2}\varphi|(x)\le C_1 k! r^{-k-2} \tag{3.1}$$

in a geodesic ball $B(p,r)$ for all $k\in\mathbb{N}$.

There exist constants $C_2,C_3,C_4,r_1=r_1(r),r_2=r_2(r)$ such that if we set $\rho=\min\{\frac{C_2}{\sqrt{C_1}}r,\mathrm{inj}_g(p)\}$ then, for all $x\in B(p,\rho)$ and $k\in\mathbb{N}$ we have in normal coordinates centred at p:

$$\frac{1}{2}\delta_{ij}\le g_{ij}(x)\le 2\delta_{ij},\qquad |\partial^k g_{ij}|(x)\le C_3 k! r_1^{-k} \tag{3.2}$$

$$|\partial^k\Gamma_{ij}^l|(x)\le C_3 k! r_1^{-k-1},\quad |\partial^k\varphi_{ijl}|(x)\le C_4 k! r_2^{-k}. \tag{3.3}$$

Proof. The assumption (3.1) implies $|\nabla^k Rm|(x)\le C_1 k! r^{-k-2}$ in $B(p,r)$. The proof of [8, Corollary 4.12] (see also [6, Lemma 13.31]) gives the existence of constants $C_2,C_3,r_1=r_1(r)>0$ such that for any $x\in B(p,\rho)$, where ρ is as stated, we have the derivative estimates for g_{ij} and Γ_{ij}^l in (3.2)–(3.3) for all $k\in\mathbb{N}$. Thus it remains to show that, under the assumption (3.1), there are constants C_4 and $r_2(r)>0$ such that for all $k\in\mathbb{N}$ we have

$$|\partial^k\varphi_{ijl}|(x)\le C_4 k! r_2^{-k}.$$

In the following, we will prove a slightly stronger estimate:

$$|\partial^l\nabla^{k-l}\varphi|(x)\le C_4 k! 2^{-(k-l)} r_2^{-k} \tag{3.4}$$

for all $0\le l\le k\le m$, where $k,l,m\in\mathbb{N}$. We prove (3.4) by induction on m. The case $m=0$ of (3.4) is trivial as $|\varphi|^2=7$. Suppose now that $m>1$ and (3.4) holds for all $0\le l\le k\le m-1$. We therefore only need to deal with the case where $k=m$ and we can perform an induction on l. Again, the case $k=m,l=0$ is trivial if we take $r_2\le r/2$, as the condition (3.1) gives that

$$|\nabla^m\varphi|\le C_1(m-2)! r^{-m}\le C_1 m! r^{-m}\le C_1 m! 2^{-m} r_2^{-m}.$$

So we now suppose that (3.4) holds for all $0 \le l < s$ for some $s \le k = m$ and consider the case $l = s$. Since $\nabla^{(k-s)}\varphi$ is a $(k-s+3)$-tensor, we have

$$\begin{aligned}
&|\partial^s \nabla^{(k-s)}\varphi(x)| \\
&= \left|\partial^{(s-1)}\left(\nabla^{(k-s+1)}\varphi(x) + (k-s+3)\Gamma(x) * \nabla^{(k-s)}\varphi(x)\right)\right| \\
&\le \left|\partial^{(s-1)}\nabla^{(k-s+1)}\varphi(x)\right| \\
&\quad + (k-s+3)\sum_{i=0}^{s-1}\binom{s-1}{i}\left|\partial^i\Gamma(x)\right|\left|\partial^{(s-1-i)}\nabla^{(k-s)}\varphi(x)\right| \\
&\le C_4 k! 2^{-(k-s+1)} r_2^{-k} \\
&\quad + (k-s+3)2^{-(k-s)}C_3C_4\sum_{i=0}^{s-1}\binom{s-1}{i} i!(k-1-i)! r_1^{-i-1} r_2^{-(k-i-1)} \\
&\le C_4 k! 2^{-(k-s)} r_2^{-k} \underbrace{\left(\frac{1}{2} + (k-s+3)C_3\sum_{i=0}^{s-1}\frac{(s-1)!}{(s-1-i)!}\frac{(k-1-i)!}{k!}\frac{r_2^{1+i}}{r_1^{1+i}}\right)}_{I}
\end{aligned}$$

To estimate the term I in the bracket above, by choosing $r_2 \le r_1$ we have

$$I \le \frac{1}{2} + C_3\frac{k-s+3}{k}\frac{\binom{s-1}{i}}{\binom{k-1}{i}}\frac{r_2}{r_1} \;\le\; \frac{1}{2} + 4C_3\frac{r_2}{r_1},$$

as $k \ge 1$. Thus we can choose

$$r_2 = r_2(r) = \min\left\{\frac{r_1}{8C_3}, r_1, \frac{r}{2}\right\} > 0$$

such that $I \le 1$ and then

$$|\partial^s \nabla^{(k-s)}\varphi(x)| \le C_4 k! 2^{-(k-s)} r_2^{-k}.$$

This completes the induction. □

It is a routine exercise to show that (3.2) and (3.3) imply that the coefficients of g and φ are real analytic with respect to normal coordinates. Hence, by Lemma 3.1, if we have the derivative estimates (3.1) for Rm and φ, then we can conclude that (M, φ, g) is real analytic.

4. Laplacian flow and evolution equations

The goal of this paper is to prove the real analyticity of the solution to the Laplacian flow (1.1). From Proposition 2.1, (1.1) is equivalent to

$$\frac{\partial}{\partial t}\varphi(t) = i_{\varphi}(h(t)), \tag{4.1}$$

where $h(t)$ is the symmetric 2-tensor on M given locally in (2.7). By (2.8), we can also write h locally as

$$h_{ij} = -R_{ij} - \frac{1}{3}|T|^2 g_{ij} - 2T_i{}^k T_{kj}. \tag{4.2}$$

Notice that $T_i{}^k = T_{il}g^{kl}$ and $T_{il} = -T_{li}$.

Throughout the remainder of the article we will use the symbol Δ to denote the "analyst's Laplacian", which is a non-positive operator given in local coordinates as $\nabla^i\nabla_i$, in contrast to the Hodge Laplacian Δ_φ.

Under (4.1), the associated metric $g(t)$ of $\varphi(t)$ evolves by

$$\frac{\partial}{\partial t}g(t) = 2h(t). \tag{4.3}$$

Substituting (4.2) into this equation, we have that

$$\frac{\partial}{\partial t}g_{ij} = -2\left(R_{ij} + \frac{1}{3}|T|^2 g_{ij} + 2T_i{}^k T_{kj}\right). \tag{4.4}$$

Moreover, by (4.4), the inverse of the metric evolves by

$$\frac{\partial}{\partial t}g^{ij} = -2h^{ij} = 2g^{ik}g^{jl}\left(R_{kl} + \frac{1}{3}|T|^2 g_{kl} + 2T_k{}^m T_{ml}\right). \tag{4.5}$$

The next lemma describes the evolution equations of the torsion tensor T, $\nabla\varphi$ and the curvature tensor Rm along the Laplacian flow. Here, and for the rest of the article, if A, B are tensors and $k \in \mathbb{N}$, then $A * B$ denotes a contraction of tensors A, B using only the metric g (which is covariant constant) and we write a tensor $S \lessapprox kA * B$[1] if S is equal to the sum of at most k terms of the form $A * B$.

[1]Note that the inequality " $\lessapprox$ " can be differentiated, i.e., $\nabla S \lessapprox k\nabla A * B + kA * \nabla B$, unlike the usual inequality " $\leq$ " case.

Lemma 4.1. *Suppose that $\varphi(t), t\in[0,T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M. The evolution equations of the torsion tensor T, $\nabla\varphi$ and the curvature tensor Rm satisfy the following estimates:*

$$\left(\frac{\partial}{\partial t}-\Delta\right)T \lessapprox 8Rm*T+Rm*\nabla\varphi + 11\nabla T*T*\varphi+4T*T*T; \tag{4.6}$$

$$\left(\frac{\partial}{\partial t}-\Delta\right)\nabla\varphi \lessapprox 62\nabla T*T*\varphi+6\nabla T*\nabla\varphi*\varphi+29Rm*\nabla\varphi + Rm*T+Rm*\varphi*(T*\varphi+\nabla\varphi*\varphi) + 24T*T*\nabla\varphi \tag{4.7}$$

and

$$\left(\frac{\partial}{\partial t}-\Delta\right)Rm \lessapprox 33Rm*Rm+4Rm*T*T+35\nabla\left(\nabla T*T\right). \tag{4.8}$$

Proof. The estimates (4.6) and (4.8) follow directly from the evolution equations of T and Rm along the Laplacian flow, which have been derived in [13, §3]. To show (4.7), recall that $\nabla\varphi$ and T are related by

$$\nabla_i\varphi_{jkl}=T_{im}g^{mn}\psi_{njkl}.$$

Then we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\varphi_{jkl} &= \left(\frac{\partial}{\partial t}T_{im}\right)g^{mn}\psi_{njkl}+T_{im}\left(\frac{\partial}{\partial t}g^{mn}\right)\psi_{njkl} \\ &\quad + T_{im}g^{mn}\left(\frac{\partial}{\partial t}\psi_{njkl}\right) \\ &= I+II+III.\end{aligned} \tag{4.9}$$

For the first term I, recall that from [13, §3.2], we have

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij} &= \Delta T_{ij}+R_i{}^kT_{kj}+\frac{1}{2}R_{ijmk}T^{mk}+\frac{1}{2}R_{mpi}{}^k\nabla_k\varphi_j{}^{pm} \\ &\quad + \nabla_pT_{qi}\nabla_m\varphi_n{}^{pq}\varphi_j{}^{mn}+\frac{1}{3}\nabla_m|T|^2\varphi_{ij}{}^m \\ &\quad + \nabla_m(T_i{}^kT_{kn})\varphi_j{}^{mn}-T_i{}^k\nabla_pT_{qk}\varphi_j{}^{pq}-\frac{1}{3}|T|^2T_{ij}-T_i{}^kT_k{}^mT_{mj}.\end{aligned}$$

Then

$$I = \left(\frac{\partial}{\partial t}T_{im}\right) g^{mn}\psi_{njkl} \tag{4.10}$$
$$= \Delta T_{im}g^{mn}\psi_{njkl} + R_i^{\ p}T_{pm}g^{mn}\psi_{njkl} + \frac{1}{2}R_{impq}T^{pq}g^{mn}\psi_{njkl}$$
$$+ \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_m^{\ qp}g^{mn}\psi_{njkl} + \nabla_pT_{qi}\nabla_s\varphi_t^{\ pq}\varphi_m^{\ st}g^{mn}\psi_{njkl}$$
$$+ \frac{1}{3}\nabla_p|T|^2\varphi_{im}^{\ \ p}g^{mn}\psi_{njkl} + \nabla_p(T_i^{\ s}T_{sq})\varphi_m^{\ pq}g^{mn}\psi_{njkl}$$
$$- T_i^{\ s}\nabla_pT_{qs}\varphi_m^{\ pq}g^{mn}\psi_{njkl} - \frac{1}{3}|T|^2T_{im}g^{mn}\psi_{njkl}$$
$$- T_i^{\ p}T_p^{\ q}T_{qm}g^{mn}\psi_{njkl}.$$

Using (2.6),

$$\Delta\nabla_i\varphi_{jkl} = \Delta\left(T_{im}g^{mn}\psi_{njkl}\right)$$
$$= \left(\Delta T_{im}\right)g^{mn}\psi_{njkl} + T_{im}g^{mn}\Delta\psi_{njkl} + 2\left(\nabla_pT_{im}\right)g^{mn}g^{pq}\nabla_q\psi_{njkl}$$
$$= \left(\Delta T_{im}\right)g^{mn}\psi_{njkl} - T_i^{\ n}\nabla^p\left(T_{pn}\varphi_{jkl} + T_{pj}\varphi_{nkl} + T_{pk}\varphi_{jnl} + T_{pl}\varphi_{jkn}\right)$$
$$- 2\left(\nabla_pT_{im}\right)g^{mn}g^{pq}\left(T_{qn}\varphi_{jkl} + T_{qj}\varphi_{nkl} + T_{qk}\varphi_{jnl} + T_{ql}\varphi_{jkn}\right)$$
$$= \left(\Delta T_{im}\right)g^{mn}\psi_{njkl} - T_i^{\ n}(T_{pn}\nabla^p\varphi_{jkl} + T_{pj}\nabla^p\varphi_{nkl} + T_{pk}\nabla^p\varphi_{jnl}$$
$$+ T_{pl}\nabla^p\varphi_{jkn}) - 2\nabla_pT_i^{\ n}g^{pq}\left(T_{qn}\varphi_{jkl} + T_{qj}\varphi_{nkl} + T_{qk}\varphi_{jnl} + T_{ql}\varphi_{jkn}\right),$$

where in the last equality we used $\nabla^pT_{pk} = 0$. Using (2.5), the second term of (4.10) is equal to $R_i^{\ p}\nabla_p\varphi_{jkl}$ and the last two terms of (4.10) can be rewritten as

$$-\frac{1}{3}|T|^2\nabla_i\varphi_{jkl} - T_i^{\ p}T_p^{\ q}\nabla_q\varphi_{jkl}.$$

The third and fourth terms of (4.10) can be expressed using the contraction identity (2.3) as follows:

$$\frac{1}{2}R_{impq}T^{pq}g^{mn}\psi_{njkl} + \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_m^{\ qp}g^{mn}\psi_{njkl}$$
$$= \frac{1}{2}R_{impq}T^{pq}g^{mn}\left(\varphi_{snj}\varphi_{tkl}g^{st} - g_{nk}g_{jl} + g_{nl}g_{jk}\right)$$
$$+ \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_m^{\ qp}g^{mn}\left(\varphi_{snj}\varphi_{tkl}g^{st} - g_{nk}g_{jl} + g_{nl}g_{jk}\right)$$
$$= \frac{1}{2}R_{impq}T^{pq}g^{mn}\varphi_{snj}\varphi_{tkl}g^{st} + \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_m^{\ qp}g^{mn}\varphi_{snj}\varphi_{tkl}g^{st}$$
$$- \frac{1}{2}R_{ikpq}T^{pq}g_{jl} + \frac{1}{2}R_{ilpq}T^{pq}g_{jk} + \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_k^{\ qp}g_{jl} + \frac{1}{2}R_{pqi}{}^{s}\nabla_s\varphi_l^{\ qp}g_{jk}.$$

Thus, we obtain our expression for the first term I in (4.9):

$$
\begin{aligned}
(4.11)\quad I = {}& \Delta\nabla_i\varphi_{jkl} + 2\nabla_m T_i{}^n\left(T_{mn}\varphi_{jkl} + T_{mj}\varphi_{nkl} + T_{mk}\varphi_{jnl} + T_{ml}\varphi_{jkn}\right)\\
&+ T_i{}^n\left(T_{pn}\nabla^p\varphi_{jkl} + T_{pj}\nabla^p\varphi_{nkl} + T_{pk}\nabla^p\varphi_{jnl} + T_{pl}\nabla^p\varphi_{jkn}\right)\\
&+ R_i{}^p\nabla_p\varphi_{jkl} + \frac{1}{2}R_{impq}T^{pq}g^{mn}\varphi_{snj}\varphi_{tkl}g^{st}\\
&+ \frac{1}{2}R_{pqi}{}^s\nabla_s\varphi_m{}^{qp}g^{mn}\varphi_{snj}\varphi_{tkl}g^{st} - \frac{1}{2}R_{ikpq}T^{pq}g_{jl}\\
&+ \frac{1}{2}R_{ilpq}T^{pq}g_{jk} + \frac{1}{2}R_{pqi}{}^s\nabla_s\varphi_k{}^{qp}g_{jl}\\
&+ \frac{1}{2}R_{pqi}{}^s\nabla_s\varphi_l{}^{qp}g_{jk} + \nabla_p T_{qi}\nabla_s\varphi_t{}^{pq}\varphi_m^{st}g^{mn}\psi_{njkl}\\
&+ \frac{1}{3}\nabla_p|T|^2\varphi_{im}{}^p g^{mn}\psi_{njkl} + \nabla_p(T_i{}^sT_{sq})\varphi_m{}^{pq}g^{mn}\psi_{njkl}\\
&- T_i{}^s\nabla_p T_{qs}\varphi_m{}^{pq}g^{mn}\psi_{njkl} - \frac{1}{3}|T|^2\nabla_i\varphi_{jkl} - T_i{}^pT_p{}^q\nabla_q\varphi_{jkl}.
\end{aligned}
$$

Here we leave the terms involving $\varphi_{im}{}^p g^{mn}\psi_{njkl}$ and related expressions unchanged in (4.11), but observe that they can be expressed in terms of φ using the contraction identity (2.4).

The second term II in (4.9) can be estimated using (4.5). For the third term III in (4.9), recall from the contraction identity (2.3), we have

$$\psi_{ijkl} = \varphi_{mij}\varphi_{nkl}g^{mn} - g_{ik}g_{jl} + g_{il}g_{jk}.$$

By (4.1), (4.4) and (4.5), we can then derive that (see e.g. [10])

$$(4.12)\qquad \frac{\partial}{\partial t}\psi_{ijkl} = h_i^m\psi_{mjkl} + h_j^m\psi_{imkl} + h_k^m\psi_{ijml} + h_l^m\psi_{ijkm},$$

where h is given in (2.7) (and equivalently in (4.2)). Then using (4.5) and (4.12), we have that $II + III$ is equal to

$$
\begin{aligned}
(4.13)\quad & - 2T_{im}h^{mn}\psi_{njkl} + T_{im}g^{mn}\left(h_n^s\psi_{sjkl} + h_j^s\psi_{nskl} + h_k^s\psi_{njsl} + h_l^s\psi_{njks}\right)\\
&= T_{im}g^{mn}\left(-h_n^s\psi_{sjkl} + h_j^s\psi_{nskl} + h_k^s\psi_{njsl} + h_l^s\psi_{njks}\right).
\end{aligned}
$$

By (2.4) and (2.7), the first term on the right-hand side of (4.13) is

$$\begin{aligned}-T_{im}g^{mn}h_n^s\psi_{sjkl} &= T_{im}g^{mn}\left(\nabla_p T_{qn}\varphi^{spq} + \frac{1}{3}|T|^2\delta_{ns} + T_n{}^p T_p{}^s\right)\psi_{sjkl}\\ &= T_i{}^n\nabla_j T_{qn}\varphi_{kl}{}^q - T_i{}^n\nabla_p T_{jn}\varphi_{kl}{}^p - T_i{}^n\nabla_k T_{qn}\varphi_{jl}{}^q\\ &\quad + T_i{}^n\nabla_p T_{kn}\varphi_{jl}{}^q + T_i{}^n\nabla_l T_{qn}\varphi_{jk}{}^q - T_i{}^n\nabla_p T_{ln}\varphi_{jk}{}^p\\ &\quad + \frac{1}{3}|T|^2\nabla_i\varphi_{jkl} + T_i{}^n T_n{}^p\nabla_p\varphi_{jkl}.\end{aligned}$$

By (2.5) and (4.2), the remaining three terms of (4.13) are equal to

$$\begin{aligned}&T_{im}g^{mn}\big(h_j^s\psi_{nskl} + h_k^s\psi_{njsl} + h_l^s\psi_{njks}\big)\\ &= h_j^s\nabla_i\varphi_{skl} + h_k^s\nabla_i\varphi_{jsl} + h_l^s\nabla_i\varphi_{jks}\\ &= -\left(R_{jp} + \frac{1}{3}|T|^2 g_{jp} + 2T_j{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{qkl}\\ &\quad - \left(R_{kp} + \frac{1}{3}|T|^2 g_{kp} + 2T_k{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{jql}\\ &\quad - \left(R_{lp} + \frac{1}{3}|T|^2 g_{lp} + 2T_l{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{jkq}.\end{aligned}$$

Therefore,

$$\begin{aligned}II + III =\, &T_i{}^n\nabla_j T_{qn}\varphi_{kl}{}^q - T_i{}^n\nabla_p T_{jn}\varphi_{kl}{}^p - T_i{}^n\nabla_k T_{qn}\varphi_{jl}{}^q\\ &+ T_i{}^n\nabla_p T_{kn}\varphi_{jl}{}^q + T_i{}^n\nabla_l T_{qn}\varphi_{jk}{}^q - T_i{}^n\nabla_p T_{ln}\varphi_{jk}{}^p\\ &+ \frac{1}{3}|T|^2\nabla_i\varphi_{jkl} + T_i{}^n T_n{}^p\nabla_p\varphi_{jkl}\\ &- \left(R_{jp} + \frac{1}{3}|T|^2 g_{jp} + 2T_j{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{qkl}\\ &- \left(R_{kp} + \frac{1}{3}|T|^2 g_{kp} + 2T_k{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{jql}\\ &- \left(R_{lp} + \frac{1}{3}|T|^2 g_{lp} + 2T_l{}^s T_{kp}\right)g^{pq}\nabla_i\varphi_{jkq}.\end{aligned} \tag{4.14}$$

The estimate (4.7) then follows from (4.9), (4.11) and (4.14). □

Remark 4.2. Although $\nabla\varphi$ can be expressed using T via (2.5), it is not straightforward to write $\nabla^k\varphi$ in terms of $\nabla^j T, j = 0, 1, \dots, k-1$. The evolution equation (4.7) for $\nabla\varphi$ is thus useful in §5 to estimate $\nabla^k\varphi$.

5. Global real analyticity

In this section, we first prove the key derivative estimates for Rm, T and φ, and then deduce Theorem 1.1 in the special case when $U = M$ is compact.

5.1. Commutator formula

First, we have the following commutator formula for ∇^k and Δ, which can be proved using the Ricci identity for commuting the covariant derivatives of a tensor, i.e., for a k-tensor A on M:

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)A_{i_1 i_2\cdots i_k} = \sum_{l=1}^{k} R_{iji_l}{}^{m} A_{i_1\cdots i_{l-1} m i_{l+1}\cdots i_k}.$$

Lemma 5.1 ([1]; [6, Lemma 13.24]). *For any p-tensor A with $p \geq 1$ and any integer $k \in \mathbb{N}$, we have*

$$\nabla^k\Delta A - \Delta\nabla^k A \lessapprox 14(p+1)\sum_{i=0}^{k}\binom{k+2}{i+2}\nabla^i Rm * \nabla^{k-i}A.$$

We also have the following commutator formula for ∇^k and $\frac{\partial}{\partial t}$ acting on a tensor along the Laplacian flow.

Lemma 5.2. *If $\varphi(t), t \in [0, T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M, then for any p-tensor A with $p \geq 1$ and any integer $k \in \mathbb{N}$, we have*

$$(5.1)\qquad \nabla^k\frac{\partial}{\partial t}A - \frac{\partial}{\partial t}\nabla^k A \lessapprox 21(p+1)\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i Rm * \nabla^{k-i}A + 13(p+1)\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i(T*T) * \nabla^{k-i}A.$$

Proof. First, by a trivial adjustment to the proof of [6, Lemma 13.26], for any smooth one-parameter family of metrics $g(t)$ on M evolving by (4.3) for any smooth family of symmetric 2-tensors $h(t)$, we have

$$(5.2)\qquad \nabla^k\frac{\partial}{\partial t}A - \frac{\partial}{\partial t}\nabla^k A \lessapprox 3(p+1)\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i h * \nabla^{k-i}A.$$

Under the Laplacian flow, $g(t)$ evolves by (4.4), so

$$\nabla^i h(t) \lessapprox 7\nabla^i Rm + \frac{13}{3}\nabla^i (T * T). \tag{5.3}$$

The commutator formula (5.1) follows by substituting (5.3) into (5.2). $\square$

Combining Lemmas 5.1 and 5.2, we have the following commutator formula of ∇^k and the heat operator $\frac{\partial}{\partial t} - \Delta$ acting on a tensor.

Proposition 5.3. *If $\varphi(t), t \in [0, T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M, then for any p-tensor A with $p \geq 1$ and any integer $k \in \mathbb{N}$, we have*

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k A - \nabla^k \left(\frac{\partial}{\partial t} - \Delta\right) A \\ &\lessapprox 21(p+1)\sum_{i=0}^{k} \frac{i+k+4}{i+2}\binom{k+1}{i+1}\nabla^i Rm * \nabla^{k-i} A \\ &\quad + 13(p+1)\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i (T*T) * \nabla^{k-i} A. \end{aligned} \tag{5.4}$$

5.2. Main derivative estimate

Our main estimate is the following, recalling the quantity $\Lambda(x,t)$ given in (1.4).

Theorem 5.4. *Suppose that $\varphi(t), t \in [0, T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M. There exists a universal positive constant α and a positive constant $C_* = C_*(T_0, K_0)$, where $K_0 = \sup_M |\Lambda(x, 0)|$, such that*

$$\sum_{k=0}^{N} \frac{t^k}{(k+1)!^2}\left(|\nabla^k Rm|^2(x,t) + |\nabla^{k+1} T|^2(x,t) + |\nabla^{k+2}\varphi|^2(x,t)\right) \leq C_* \tag{5.5}$$

on $M \times [0, \min\{T_0, \alpha/K_0\}]$ for all $N \in \mathbb{N}$.

For convenience, we define

$$(5.6)\quad a_k=\frac{t^{\frac{k}{2}}|\nabla^k Rm|}{(k+1)!},\ b_k=\frac{t^{\frac{k}{2}}|\nabla^{k+1}T|}{(k+1)!},\ c_k=\frac{t^{\frac{k}{2}}|\nabla^{k+2}\varphi|}{(k+1)!},\ \text{for } k\geq 0,$$

$$(5.7)\quad \tilde{a}_k=\frac{t^{\frac{k-1}{2}}|\nabla^k Rm|}{k!},\ \tilde{b}_k=\frac{t^{\frac{k-1}{2}}|\nabla^{k+1}T|}{k!},\ \tilde{c}_k=\frac{t^{\frac{k-1}{2}}|\nabla^{k+2}\varphi|}{k!},\ \text{for } k\geq 1.$$

By setting $k!=1$ for all $k\leq 0$, the above definition can cover

$$\begin{aligned} \tilde{a}_0&=t^{-\frac{1}{2}}|Rm|, & \tilde{b}_0&=t^{-\frac{1}{2}}|\nabla T|, & \tilde{c}_0&=t^{-\frac{1}{2}}|\nabla^2\varphi|,\\ b_{-1}&=t^{-\frac{1}{2}}|T|, & c_{-1}&=t^{-\frac{1}{2}}|\nabla\varphi|, & c_{-2}&=t^{-1}|\varphi|,\\ \tilde{b}_{-1}&=t^{-1}|T|, & \tilde{c}_{-1}&=t^{-1}|\nabla\varphi|, & \tilde{c}_{-2}&=t^{-\frac{3}{2}}|\varphi|. \end{aligned}$$

Note that $|\varphi|^2=7$ and $|\nabla\varphi|^2=|T|^2\leq|Rm|=a_0$. Next, we define

$$\begin{aligned} A_N&=\sum_{k=0}^N a_k^2, & B_N&=\sum_{k=0}^N b_k^2, & C_N&=\sum_{k=0}^N c_k^2,\\ \tilde{A}_N&=\sum_{k=1}^N \tilde{a}_k^2, & \tilde{B}_N&=\sum_{k=1}^N \tilde{b}_k^2, & \tilde{C}_N&=\sum_{k=1}^N \tilde{c}_k^2, \end{aligned}$$

and

$$\Phi_N=A_N+B_N+C_N,\qquad \Psi_N=\tilde{A}_N+\tilde{B}_N+\tilde{C}_N.$$

Then (5.5) is equivalent to showing that $\Phi_N\leq C_*$ for any $N\in\mathbb{N}$.

The approach to prove (5.5) is to establish an evolution inequality for Φ_N and then apply the maximum principle. Although the method is clear, the derivation of the evolution inequality is somewhat computationally involved, so we break it up into a sequence of lemmas which deals with each of the terms A_N, B_N and C_N in turn. Throughout the proofs we will use the same symbol C to denote a (finite) universal constant.

Lemma 5.5. *Suppose that $\varphi(t), t\in[0,T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M. There exists a universal constant C such that*

$$(5.8)\qquad \left(\frac{\partial}{\partial t}-\Delta\right)A_N\leq -\frac{7}{4}\tilde{A}_{N+1}+\frac{1}{4}\Psi_{N+1}+C(t\Phi_N^{\frac{1}{2}}+t^2\Phi_N)\Psi_N + C\Phi_N^{\frac{3}{2}}(1+t\Phi_N^{\frac{1}{2}}).$$

Proof. Applying (5.4) to $A = Rm$ (where $p = 4$), we have

$$
\begin{aligned}
(5.9)\quad \left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k Rm \lessapprox\ & \nabla^k\left(\frac{\partial}{\partial t} - \Delta\right)Rm \\
& + 105\sum_{i=0}^{k}\frac{i+k+4}{i+2}\binom{k+1}{i+1}\nabla^i Rm * \nabla^{k-i} Rm \\
& + 65\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i(T*T)*\nabla^{k-i}Rm.
\end{aligned}
$$

Applying ∇^k to (4.8) and substituting into (5.9), we obtain

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k Rm \lessapprox\ & 243\sum_{i=0}^{k}\binom{k+2}{i+2}\nabla^i Rm * \nabla^{k-i} Rm \\
& + 35\nabla^{k+1}(\nabla T * T) + 4T * T * \nabla^k Rm \\
& + 69\sum_{i=1}^{k}\binom{k+1}{i+1}\nabla^i(T*T)*\nabla^{k-i}Rm.
\end{aligned}
$$

Since

$$|\nabla^k Rm|^2 = (g^{-1})^{*(k+4)} * \nabla^k Rm * \nabla^k Rm,$$

we can use the evolution equation (4.5) of g^{-1} to compute

$$
\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla^k Rm|^2 \\
&= 2\left\langle \left(\frac{\partial}{\partial t} - \Delta\right)\nabla^k Rm, \nabla^k Rm\right\rangle - 2|\nabla^{k+1}Rm|^2 \\
&\quad + 2(k+4)\left(Rc + \frac{1}{3}|T|^2 g + 2T*T\right) * \nabla^k Rm * \nabla^k Rm \\
&\leq -2|\nabla^{k+1}Rm|^2 + C\sum_{i=0}^{k}\binom{k+2}{i+2}|\nabla^i Rm||\nabla^{k-i}Rm||\nabla^k Rm| \\
&\quad + C|\nabla^{k+1}(\nabla T * T)||\nabla^k Rm| \\
&\quad + C\sum_{i=0}^{k}\binom{k+1}{i+1}|\nabla^i(T*T)||\nabla^{k-i}Rm||\nabla^k Rm|.
\end{aligned}
$$

Then from the definition (5.6) of a_k, we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)a_k^2 \leq -2\tilde{a}_{k+1}^2+\frac{k\tilde{a}_k^2}{(k+1)^2}+I_1(k)+I_2(k)+I_3(k), \tag{5.10}$$

where

$$\begin{aligned}
I_1(k) &= \frac{Ct^k}{(k+1)!^2}\sum_{i=0}^{k}\binom{k+2}{i+2}|\nabla^i Rm||\nabla^{k-i}Rm||\nabla^k Rm|,\\
I_2(k) &= \frac{Ct^k}{(k+1)!^2}|\nabla^{k+1}(\nabla T * T)||\nabla^k Rm|,\\
I_3(k) &= \frac{Ct^k}{(k+1)!^2}\sum_{i=0}^{k}\binom{k+1}{i+1}|\nabla^i (T*T)||\nabla^{k-i}Rm||\nabla^k Rm|.
\end{aligned}$$

To obtain (5.8) we sum (5.10) from $k=0$ to N. First, for $k=0$, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)a_0^2 &\leq -2\tilde{a}_1^2+C\left(a_0^3+b_0^2a_0+\tilde{b}_1a_0^{\frac{3}{2}}\right)\\
&\leq -2\tilde{a}_1^2+C\Phi_N^{\frac{3}{2}}+C_n\tilde{B}_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}.
\end{aligned} \tag{5.11}$$

For $k=1$ to N, we estimate the sum over k of the three terms $I_1(k)$, $I_2(k)$, $I_3(k)$ separately. For $I_1(k)$ we have

$$\begin{aligned}
\sum_{k=1}^{N}I_1(k) &= \sum_{k=1}^{N}\left(Ca_0a_k^2+Ct\sum_{i=0}^{k-1}\frac{(k+2)a_i\tilde{a}_{k-i}\tilde{a}_k}{(k+1)(i+2)}\right)\\
&\leq CA_N^{\frac{3}{2}}+Ct\sum_{k=1}^{N}\left(\sum_{i=0}^{k-1}\frac{1}{(i+2)^2}\sum_{i=0}^{k-1}a_i^2\tilde{a}_{k-i}^2\right)^{\frac{1}{2}}\tilde{a}_k\\
&\leq CA_N^{\frac{3}{2}}+Ct\left(\sum_{k=1}^{N}\sum_{i=0}^{k-1}a_i^2\tilde{a}_{k-i}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{N}\tilde{a}_k^2\right)^{\frac{1}{2}}\\
&\leq CA_N^{\frac{3}{2}}+CtA_N^{\frac{1}{2}}\tilde{A}_N\\
&\leq C\Phi_N^{\frac{3}{2}}+Ct\Phi_N^{\frac{1}{2}}\Psi_N,
\end{aligned} \tag{5.12}$$

where we used $a_0 \le A_N^{\frac{1}{2}}$, the Cauchy–Schwarz inequality and the elementary fact that $\sum_{i=0}^{\infty} \frac{1}{(i+2)^2} < 1$. For the sum of $I_2(k)$,

$$
\begin{aligned}
\sum_{k=1}^{N} I_2(k) &= \sum_{k=1}^{N} \left(C\tilde{b}_{k+1} a_0^{\frac{1}{2}} a_k + C b_0 b_k a_k + Ct \sum_{i=1}^{k} \frac{\tilde{b}_i b_{k-i} \tilde{a}_k}{k+1} \right) \\
&\le C\tilde{B}_{N+1}^{\frac{1}{2}} A_N^{\frac{3}{4}} + C B_N A_N^{\frac{1}{2}} + Ct \sum_{k=1}^{N} \frac{k^{\frac{1}{2}}}{k+1} \left(\sum_{i=1}^{k} \tilde{b}_i^2 b_{k-i}^2 \right)^{\frac{1}{2}} \tilde{a}_k \\
&\le C\tilde{B}_{N+1}^{\frac{1}{2}} A_N^{\frac{3}{4}} + C B_N A_N^{\frac{1}{2}} + Ct \tilde{B}_N^{\frac{1}{2}} B_N^{\frac{1}{2}} \tilde{A}_N^{\frac{1}{2}} \\
&\le C\tilde{B}_{N+1}^{\frac{1}{2}} \Phi_N^{\frac{3}{4}} + C\Phi_N^{\frac{3}{2}} + Ct\Phi_N^{\frac{1}{2}} \Psi_N,
\end{aligned}
\tag{5.13}
$$

where we used the elementary inequality

$$
\left(\sum_{i=1}^{k} \alpha_i \right)^2 \le k \sum_{i=1}^{k} \alpha_i^2, \quad \text{for } \alpha_i \ge 0.
$$

We can similarly estimate the sum of $I_3(k)$:

$$
\begin{aligned}
\sum_{k=1}^{N} I_3(k) &= C \sum_{k=1}^{N} \frac{t^k}{(k+1)!^2} |\nabla^k (T * T)||Rm||\nabla^k Rm| \\
&\quad + Ct \sum_{k=1}^{N} \sum_{i=0}^{k-1} \frac{\tilde{a}_{k-i}}{k+1} \left(\frac{t^{\frac{i}{2}} |\nabla^i (T * T)|}{(i+1)!} \right) \tilde{a}_k \\
&\le Cta_0 \sum_{k=1}^{N} \sum_{i=0}^{k} \frac{a_k}{k+1} b_{i-1} b_{k-i-1} \\
&\quad + Ct^2 \sum_{k=1}^{N} \sum_{i=0}^{k-1} \frac{\tilde{a}_{k-i}}{k+1} \left(\sum_{j=0}^{i} \frac{b_{j-1} b_{i-j-1}}{i+1} \right) \tilde{a}_k \\
&\le Cta_0 (B_N + b_{-1}^2) A_N^{\frac{1}{2}} + Ct^2 (B_N + b_{-1}^2) \tilde{A}_N \\
&\le Ct\Phi_N^2 + C\Phi_N^{\frac{3}{2}} + Ct^2 \Phi_N \Psi_N + Ct\Phi_N^{\frac{1}{2}} \Psi_N,
\end{aligned}
\tag{5.14}
$$

where in the third inequality we used that $a_0 \le A_N^{\frac{1}{2}}$ and $b_{-1}^2 \le t^{-1}A_N^{\frac{1}{2}}$. Combining (5.10)–(5.14), we conclude that

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)A_N &\le -2\tilde{A}_{N+1}+\sum_{k=1}^{N}\frac{k\tilde{a}_k^2}{(k+1)^2}+C\tilde{B}_{N+1}^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}+C\tilde{B}_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}\\ &\quad + C\Phi_N^{\frac{3}{2}}(1+t\Phi_N^{\frac{1}{2}})+C(t\Phi_N^{\frac{1}{2}}+t^2\Phi_N)\Psi_N\\ &\le -\frac{7}{4}\tilde{A}_{N+1}+\frac{1}{4}\Psi_{N+1}+C(t\Phi_N^{\frac{1}{2}}+t^2\Phi_N)\Psi_N\\ &\quad + C\Phi_N^{\frac{3}{2}}(1+t\Phi_N^{\frac{1}{2}}),\end{aligned}$$

where we used $\frac{k}{(k+1)^2}\le 1/4$, $\tilde{B}_N\le\tilde{B}_{N+1}\le\Psi_{N+1}$ and Cauchy–Schwarz. □

Lemma 5.6. *Suppose that $\varphi(t), t\in[0,T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M. There exists a universal constant C such that*

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)B_N &\le -\frac{7}{4}\tilde{B}_{N+1}+\frac{1}{4}\Psi_{N+1}\\ &\quad + C\left(t\Phi_N^{\frac{1}{2}}+t^2\Phi_N+t^3\Phi_N^{\frac{3}{2}}\right)\Psi_N\\ &\quad + C\Phi_N^{\frac{3}{2}}\left(1+t\Phi_N^{\frac{1}{2}}\right).\end{aligned}\tag{5.15}$$

Proof. By (4.6) and (5.4) (with $A=T$ so $p=2$), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^{k+1}T &\lessapprox 150\sum_{i=0}^{k+1}\binom{k+3}{i+2}\nabla^i Rm*\nabla^{k+1-i}T\\ &\quad+\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i Rm*\nabla^{k+2-i}\varphi\\ &\quad+11\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i(\nabla T*T)*\nabla^{k+1-i}\varphi\\ &\quad+43\sum_{i=1}^{k+1}\binom{k+2}{i+1}\nabla^i(T*T)*\nabla^{k+1-i}T\\ &\quad+4T*T*\nabla^{k+1}T.\end{aligned}\tag{5.16}$$

From the definition (5.6) of b_k and using $|T|^2 \leq |Rm|$ we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)b_k^2 &= \frac{t^k}{(k+1)!^2}\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla^{k+1}T|^2+\frac{k}{(k+1)^2}\tilde{b}_k^2\\ &= \frac{2t^k}{(k+1)!^2}\left\langle\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^{k+1}T,\nabla^{k+1}T\right\rangle-2\tilde{b}_{k+1}^2+\frac{k}{(k+1)^2}\tilde{b}_k^2\\ &\quad+\frac{2t^k(k+3)}{(k+1)!^2}\left(Rc+\frac{1}{3}|T|^2g+2T*T\right)*\nabla^{k+1}T*\nabla^{k+1}T\\ &\leq \frac{2t^k}{(k+1)!^2}\left\langle\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^{k+1}T,\nabla^{k+1}T\right\rangle-2\tilde{b}_{k+1}^2\\ &\quad+\frac{k}{(k+1)^2}\tilde{b}_k^2+C(k+3)|Rm|b_k^2. \end{aligned}\tag{5.17}$$

Substituting (5.16) into (5.17) and rearranging terms gives:

$$\left(\frac{\partial}{\partial t}-\Delta\right)b_k^2 \leq -2\tilde{b}_{k+1}^2+\frac{k\tilde{b}_k^2}{(k+1)^2}+II_1(k)+\cdots+II_4(k),\tag{5.18}$$

where

$$\begin{aligned} II_1(k) &= \frac{Ct^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+3}{i+2}|\nabla^iRm||\nabla^{k+1-i}T||\nabla^{k+1}T|,\\ II_2(k) &= \frac{t^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+1}{i}|\nabla^iRm||\nabla^{k+2-i}\varphi||\nabla^{k+1}T|,\\ II_3(k) &= \frac{Ct^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+1}{i}|\nabla^i(\nabla T*T)||\nabla^{k+1-i}\varphi||\nabla^{k+1}T|,\\ II_4(k) &= \frac{Ct^k}{(k+1)!^2}\sum_{i=1}^{k+1}\binom{k+2}{i+1}|\nabla^i(T*T)||\nabla^{k+1-i}T||\nabla^{k+1}T|. \end{aligned}$$

Note that we have absorbed $C(k+3)|Rm|b_k^2$ in (5.17) into $II_1(k)$ of (5.18). To derive the evolution inequality of B_N, we start with (5.18) for $k=0$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)b_0^2 &\leq -2\tilde{b}_1^2+C\left(a_0b_0^2+\tilde{a}_1a_0^{\frac{1}{2}}b_0+a_0b_0c_0+\tilde{b}_1a_0^{\frac{1}{2}}b_0+b_0^3\right)\\ &\leq -2\tilde{b}_1^2+C\Phi_N^{\frac{3}{2}}+C\Psi_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}.\end{aligned}\tag{5.19}$$

By summing $II_1(k), II_2(k), II_3(k)$ over $k = 1, \dots, N$, we have the following estimates:

$$
\begin{aligned}
(5.20)\quad \sum_{k=1}^{N} II_1(k) &= C\sum_{k=1}^{N}\left(\tilde{a}_{k+1}a_0^{\frac12}b_k + t^{\frac12}b_0a_k\tilde{b}_k\right)\\
&\quad + Ct\sum_{k=1}^{N}\sum_{i=0}^{k-1}\left(\frac{1}{i+2}+\frac{1}{k+1-i}\right)a_i\tilde{b}_{k-i}\tilde{b}_k\\
&\le C\tilde{A}_{N+1}^{\frac12}A_N^{\frac14}B_N^{\frac12} + Ct^{\frac12}B_N^{\frac12}A_N^{\frac12}\tilde{B}_N^{\frac12} + CtA_N^{\frac12}\tilde{B}_N\\
&\le C\Psi_{N+1}^{\frac12}\Phi_N^{\frac34} + Ct^{\frac12}\Phi_N\Psi_N^{\frac12} + Ct\Phi_N^{\frac12}\Psi_N;
\end{aligned}
$$

$$
\begin{aligned}
(5.21)\quad \sum_{k=1}^{N} II_2(k) &= C\sum_{k=1}^{N}\left(a_0b_kc_k + a_0^{\frac12}\tilde{a}_{k+1}b_k\right) + Ct\sum_{k=1}^{N}\sum_{i=1}^{k}\frac{\tilde{a}_ic_{k-i}\tilde{b}_k}{k+1}\\
&\le CA_N^{\frac12}B_N^{\frac12}C_N^{\frac12} + CA_N^{\frac14}\tilde{A}_{N+1}^{\frac12}B_N^{\frac12} + Ct\tilde{A}_N^{\frac12}C_N^{\frac12}\tilde{B}_N^{\frac12}\\
&\le C\Phi_N^{\frac32} + C\Psi_{N+1}^{\frac12}\Phi_N^{\frac34} + Ct\Phi_N^{\frac12}\Psi_N;
\end{aligned}
$$

$$
\begin{aligned}
(5.22)\quad \sum_{k=1}^{N} II_3(k) &\le Ct\sum_{k=1}^{N}\frac{\tilde{b}_k}{k+1}\sum_{i=0}^{k+1}\tilde{b}_ib_{k-i}\\
&\quad + Ct^2\sum_{k=1}^{N}\frac{\tilde{b}_k}{k+1}\sum_{i=0}^{k}\left(\sum_{j=0}^{i}\frac{\tilde{b}_jb_{i-j-1}}{k+1-i}\right)c_{k-i-1}\\
&\le Ct(\tilde{B}_{N+1}+\tilde{b}_0^2)^{\frac12}(B_N+b_{-1}^2)^{\frac12}\tilde{B}_N^{\frac12}\\
&\quad + Ct^2\sum_{k=1}^{N}\left(\sum_{i=0}^{k}\sum_{j=0}^{i}\tilde{b}_j^2b_{i-j-1}^2c_{k-i-1}^2\right)^{\frac12}\tilde{b}_k\\
&\le Ct(\tilde{B}_{N+1}+\tilde{b}_0^2)^{\frac12}(B_N+b_{-1}^2)^{\frac12}\tilde{B}_N^{\frac12}\\
&\quad + C_nt^2(\tilde{B}_N+\tilde{b}_0^2)^{\frac12}(B_N+b_{-1}^2)^{\frac12}(C_N+c_{-1}^2)^{\frac12}\tilde{B}_N^{\frac12}\\
&\le C_n\Psi_{N+1}^{\frac12}\Psi_N^{\frac12}t^{\frac12}\Phi_N^{\frac14}(1+t\Phi_N^{\frac12})^{\frac12} + C_n\Phi_N^{\frac34}(1+t\Phi_N^{\frac12})^{\frac12}\Psi_N^{\frac12}\\
&\quad + C_n(t^2\Phi_N + t\Phi_N^{\frac12})\Psi_N + C_nt^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})\Psi_N^{\frac12},
\end{aligned}
$$

where we used $b_{-1}^2 \le t^{-1}a_0$ and $c_{-1}^2 \le t^{-1}a_0$. Finally,

$$
(5.23)\quad \sum_{k=1}^{N} II_4(k) = C\sum_{k=1}^{N}\sum_{i=1}^{k+1}\frac{(k+2)b_{k-i}\tilde{b}_k}{(k+1)(i+1)}\left(ta_0^{\frac12}b_{i-1} + t^2\sum_{j=1}^{i}\frac{1}{j}\tilde{b}_{j-1}b_{i-j-1}\right)
$$

$$
\begin{aligned}
&\le Cta_0^{\frac12}B_N^{\frac12}(B_N+b_{-1}^2)^{\frac12}\tilde{B}_N^{\frac12}+Ct^2\sum_{k=1}^{N}\left(\sum_{i=1}^{k+1}\sum_{j=1}^{i}\tilde{b}_{j-1}^2b_{i-j-1}^2b_{k-i}^2\right)^{\frac12}\tilde{b}_k\\
&\le Cta_0^{\frac12}B_N^{\frac12}(B_N+b_{-1}^2)^{\frac12}\tilde{B}_N^{\frac12}+Ct^2\left(\tilde{B}_N+\tilde{b}_0^2\right)^{\frac12}(B_N+b_{-1}^2)\tilde{B}_N^{\frac12}\\
&\le Ct^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})^{\frac12}\Psi_N^{\frac12}+Ct^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})\Psi_N^{\frac12}+C(t^2\Phi_N+t\Phi_N^{\frac12})\Psi_N.
\end{aligned}
$$

Combining the above inequalities (5.19)–(5.23), we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)B_N \le & -\frac74\tilde{B}_{N+1}+C\Phi_N^{\frac32}+C\Psi_N^{\frac12}\Phi_N^{\frac34}+C\Psi_{N+1}^{\frac12}\Phi_N^{\frac34}\\
&+C\Psi_{N+1}^{\frac12}\Psi_N^{\frac12}t^{\frac12}\Phi_N^{\frac14}(1+t\Phi_N^{\frac12})^{\frac12}\\
&+C\Phi_N^{\frac34}(1+t\Phi_N^{\frac12})^{\frac12}\Psi_N^{\frac12}+C(t^2\Phi_N+t\Phi_N^{\frac12})\Psi_N\\
&+Ct^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})\Psi_N^{\frac12}+Ct^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})^{\frac12}\Psi_N^{\frac12}.
\end{aligned}
$$

Noting that $\Psi_N\le\Psi_{N+1}$ and applying Cauchy–Schwarz to the above inequality gives (5.15). □

Lemma 5.7. *Suppose that $\varphi(t), t\in[0,T_0]$ is a solution to the Laplacian flow* (1.1) *on a compact manifold M. There exists a universal constant C such that*

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)C_N \le & -\frac74\tilde{C}_{N+1}+\frac14\Psi_{N+1}+C\Phi_N^{\frac32}\left(1+t\Phi_N^{\frac12}+t^2\Phi_N\right)\\
&+C\Psi_N\left(t\Phi_N^{\frac12}+t^2\Phi_N+t^3\Phi_N^{\frac32}+t^4\Phi_N^2\right).
\end{aligned}
\tag{5.24}
$$

Proof. By (4.7) and (5.4) (with $A=\nabla\varphi$ so $p=4$), we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^{k+2}\varphi \lessapprox\ & 62\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i(\nabla T*T)*\nabla^{k+1-i}\varphi\\
&+239\sum_{i=0}^{k+1}\binom{k+3}{i+2}\nabla^iRm*\nabla^{k+2-i}\varphi\\
&+\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^iRm*\nabla^{k+1-i}T
\end{aligned}
\tag{5.25}
$$

$$+\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i(Rm*\varphi)*\Big(\nabla^{k+1-i}(T*\varphi)+\nabla^{k+1-i}(\nabla\varphi*\varphi)\Big)$$
$$+89\sum_{i=1}^{k+1}\binom{k+2}{i+1}\nabla^i(T*T)*\nabla^{k+2-i}\varphi+24T*T*\nabla^{k+2}\varphi$$
$$+6\sum_{i=0}^{k+1}\binom{k+1}{i}\nabla^i(\nabla T*\nabla\varphi)*\nabla^{k+1-i}\varphi.$$

By the the definition (5.6) of c_k and noting that $\nabla^{k+2}\varphi$ is an $(k+5)$-tensor, we have the following:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)c_k^2&=\frac{t^k}{(k+1)!^2}\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla^{k+2}\varphi|^2+\frac{k}{(k+1)^2}\tilde{c}_k^2\\&\leq\frac{2t^k}{(k+1)!^2}\left\langle\left(\frac{\partial}{\partial t}-\Delta\right)\nabla^{k+2}\varphi,\nabla^{k+2}\varphi\right\rangle-2\tilde{c}_{k+1}^2\\&\quad+\frac{k}{(k+1)^2}\tilde{c}_k^2+C(k+5)|Rm|c_k^2.\end{aligned}\tag{5.26}$$

Substituting (5.25) into (5.26), we compute

$$\left(\frac{\partial}{\partial t}-\Delta\right)c_k^2\leq-2\tilde{c}_{k+1}^2+\frac{k}{(k+1)^2}\tilde{c}_k^2+III_1(k)+\cdots+III_6(k),\tag{5.27}$$

where

$$III_1(k)=C\frac{t^k}{(k+1)!^2}\sum_{i=1}^{k+1}\binom{k+1}{i}|\nabla^i(\nabla T*T)||\nabla^{k+1-i}\varphi||\nabla^{k+2}\varphi|,$$
$$III_2(k)=C\frac{t^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+3}{i+2}|\nabla^iRm||\nabla^{k+2-i}\varphi||\nabla^{k+2}\varphi|,$$
$$III_3(k)=\frac{2t^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+1}{i}|\nabla^iRm||\nabla^{k+1-i}T||\nabla^{k+2}\varphi|,$$
$$\begin{aligned}III_4(k)=\frac{2t^k}{(k+1)!^2}&\sum_{i=0}^{k+1}\binom{k+1}{i}|\nabla^i(Rm*\varphi)|\Big(|\nabla^{k+1-i}(T*\varphi)|\\&+|\nabla^{k+1-i}(\nabla\varphi*\varphi)|\Big)|\nabla^{k+2}\varphi|,\end{aligned}$$
$$III_5(k)=C\frac{t^k}{(k+1)!^2}\sum_{i=1}^{k+1}\binom{k+2}{i+1}|\nabla^i(T*T)||\nabla^{k+2-i}\varphi||\nabla^{k+2}\varphi|,$$

$$III_6(k) = \frac{12t^k}{(k+1)!^2}\sum_{i=0}^{k+1}\binom{k+1}{i}|\nabla^i(\nabla T*\nabla\varphi)||\nabla^{k+1-i}\varphi||\nabla^{k+2}\varphi|.$$

We now follow similar calculations to the proofs of Lemmas 5.5 and 5.6. For $k=0$ in (5.27):

$$\begin{aligned}(5.28)\qquad \left(\frac{\partial}{\partial t}-\Delta\right)c_0^2 &\le -2\tilde{c}_1^2 + C\Big(\tilde{b}_1 a_0^{\frac12}c_0 + b_0^2c_0 + b_0c_0^2 + a_0c_0^2 \\ &\quad + \tilde{a}_1a_0^{\frac12}c_0 + a_0b_0c_0 + a_0^2c_0\Big) \\ &\le -2\tilde{c}_1^2 + C\Phi_N^{\frac32} + C\Psi_N^{\frac12}\Phi_N^{\frac34}.\end{aligned}$$

We next estimate the sums of each of the six terms $III_1(k),\dots,III_6(k)$ in (5.27). Starting with $III_1(k)$ and using $\tilde{b}_0 = t^{-\frac12}b_0$ and $b_{-1}^2 = c_{-1}^2 \le t^{-1}a_0$:

$$\begin{aligned}(5.29)\qquad &\sum_{k=1}^N III_1(k) \\ &\le C\sum_{k=1}^N \frac{t^k}{(k+1)!^2}|\nabla^{k+1}(\nabla T*T)||\varphi||\nabla^{k+2}\varphi| \\ &\quad + C\sum_{k=1}^N\frac{t^k}{(k+1)!^2}\sum_{i=1}^k\binom{k+1}{i}|\nabla^i(\nabla T*T)||\nabla^{k+1-i}\varphi||\nabla^{k+2}\varphi| \\ &\le Ct\sum_{k=1}^N\frac{1}{k+1}\sum_{i=0}^{k+1}\tilde{b}_ib_{k-i}\tilde{c}_k \\ &\quad + Ct^2\sum_{k=1}^N\frac{1}{k+1}\sum_{i=1}^k\frac{1}{k+1-i}\sum_{j=0}^i\tilde{b}_jb_{i-j-1}c_{k-i-1}\tilde{c}_k \\ &\le Ct(\tilde{B}_{N+1}+\tilde{b}_0^2)^{\frac12}(B_N+b_{-1}^2)^{\frac12}\tilde{C}_N^{\frac12} \\ &\quad + Ct^2(\tilde{B}_N+\tilde{b}_0^2)^{\frac12}(B_N+b_{-1}^2)^{\frac12}(C_N+c_{-1}^2)^{\frac12}\tilde{C}_N^{\frac12} \\ &\le C\Psi_{N+1}^{\frac12}\Psi_N^{\frac12}(t^2\Phi_N+t\Phi_N^{\frac12})^{\frac12} + C\Phi_N^{\frac34}(1+t\Phi_N^{\frac12})^{\frac12}\Psi_N^{\frac12} \\ &\quad + C(t^2\Phi_N+t\Phi_N^{\frac12})^2\Psi_N + Ct^{\frac12}\Phi_N(1+t\Phi_N^{\frac12})\Psi_N^{\frac12}.\end{aligned}$$

Using $|\nabla\varphi| \le a_0^{\frac12} \le A_N^{\frac14}$ for $III_2(k)$:

$$\begin{aligned}
(5.30)\qquad \sum_{k=1}^{N} III_2(k) &\le Ct\sum_{k=1}^{N}\sum_{i=0}^{k-1}\frac{k+2}{k+1}\left(\frac{1}{i+2}+\frac{1}{k+1-i}\right)a_i\tilde{c}_{k-i}\tilde{c}_k \\
&\quad + C\sum_{k=1}^{N}\left(\tilde{a}_{k+1}|\nabla\varphi|c_k+\frac{k+3}{k+1}t^{\frac{1}{2}}\tilde{a}_k c_0 c_k\right) \\
&\le CtA_N^{\frac{1}{2}}\tilde{C}_N + C\tilde{A}_{N+1}^{\frac{1}{2}}A_N^{\frac{1}{4}}C_N^{\frac{1}{2}} + Ct^{\frac{1}{2}}\tilde{A}_N^{\frac{1}{2}}C_N \\
&\le Ct\Phi_N^{\frac{1}{2}}\Psi_N + C\Psi_{N+1}^{\frac{1}{2}}\Phi_N^{\frac{3}{4}} + Ct^{\frac{1}{2}}\Phi_N\Psi_N^{\frac{1}{2}}.
\end{aligned}$$

Using $b_{-1}^2 = c_{-1}^2 \le t^{-1}a_0$ again for $III_3(k)$ and $III_4(k)$:

$$\begin{aligned}
(5.31)\qquad \sum_{k=1}^{N} III_3(k) &\le 2t\sum_{k=1}^{N}\sum_{i=0}^{k-1}\frac{1}{k+1}\tilde{a}_i b_{k-i}\tilde{c}_k \\
&\quad + 2\sum_{k=1}^{N}(\tilde{a}_{k+1}|T|c_k + t^{\frac{1}{2}}\tilde{a}_k b_0 c_k) \\
&\le 2t\tilde{A}_N^{\frac{1}{2}}B_N^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}} + 2\tilde{A}_{N+1}^{\frac{1}{2}}A_N^{\frac{1}{4}}C_N^{\frac{1}{2}} + 2t^{\frac{1}{2}}\tilde{A}_N^{\frac{1}{2}}B_N^{\frac{1}{2}}C_N^{\frac{1}{2}} \\
&\le 2t\Phi_N^{\frac{1}{2}}\Psi_N + 2\Psi_{N+1}^{\frac{1}{2}}\Phi_N^{\frac{3}{4}} + 2t^{\frac{1}{2}}\Phi_N\Psi_N^{\frac{1}{2}};
\end{aligned}$$

$$\begin{aligned}
(5.32)\qquad \sum_{k=1}^{N} III_4(k) &\le 2\sum_{k=1}^{N}\frac{t\tilde{c}_k}{k+1}\sum_{i=0}^{k+1}\left(\tilde{a}_i + t\sum_{j=0}^{i-1}\frac{\tilde{a}_j c_{i-j-2}}{i-j}\right) \\
&\quad \times\left(b_{k-i} + t\sum_{l=0}^{k-i}\frac{(b_{l-1}+c_{l-1})c_{k-i-l-1}}{k+1-i-l}\right) \\
&\le 2t(\tilde{A}_{N+1}+\tilde{a}_0^2)^{\frac{1}{2}}\left(B_N + b_{-1}^2\right)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}} \\
&\quad + 2t^2(\tilde{A}_N+\tilde{a}_0^2)^{\frac{1}{2}}(C_N+c_{-1}^2)^{\frac{1}{2}}(B_N+b_{-1}^2)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}} \\
&\quad + 2t^2(\tilde{A}_{N+1}+\tilde{a}_0^2)^{\frac{1}{2}}(C_N+c_{-1}^2)^{\frac{1}{2}} \\
&\qquad \times (B_N+b_{-1}^2+C_N+c_{-1}^2)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}} \\
&\quad + 2t^3(\tilde{A}_N+\tilde{a}_0^2)^{\frac{1}{2}}(C_N+c_{-1}^2) \\
&\qquad \times (B_N+b_{-1}^2+C_N+c_{-1}^2)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}} \\
&\le 2\Psi_{N+1}^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}} + 2\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}}+1)^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}
\end{aligned}$$

$$+ 2\Psi_N(t^2\Phi_N + t\Phi_N^{\frac{1}{2}}) + 2\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}} + 1)^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}$$
$$+ 2\sqrt{2}\Psi_{N+1}^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}}) + 2\sqrt{2}t^{\frac{1}{2}}\Phi_N(t\Phi_N^{\frac{1}{2}} + 1)\Psi_N^{\frac{1}{2}}$$
$$+ 2\Psi_N(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{3}{2}} + 2\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}} + 1)^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})\Psi_N^{\frac{1}{2}}.$$

Finally, for $III_5(k)$ and $III_6(k)$ we have:

$$(5.33)\quad \sum_{k=1}^{N} III_5(k) \le C\sum_{k=1}^{N}\sum_{i=1}^{k+1}\frac{1}{i+1}\left(tb_{i-1}a_0^{\frac{1}{2}} + t^2\sum_{j=1}^{i}\frac{1}{j}\tilde{b}_{j-1}b_{i-j-1}\right)c_{k-i}\tilde{c}_k$$
$$\le C\left(ta_0^{\frac{1}{2}} + t^2\left(\tilde{B}_N + \tilde{b}_0^2\right)^{\frac{1}{2}}\right)B_N^{\frac{1}{2}}(C_N + c_{-1}^2)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}}$$
$$\le C\Psi_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}} + 1) + C\Psi_N^{\frac{1}{2}}t^{\frac{1}{2}}\Phi_N(t\Phi_N^{\frac{1}{2}} + 1)$$
$$+ C(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})\Psi_N;$$

$$(5.34)\quad \sum_{k=1}^{N} III_6(k) \le 12\sqrt{7}t\sum_{k=1}^{N}\frac{1}{k+1}\sum_{l=0}^{k+1}\tilde{b}_l c_{k-l}\tilde{c}_k$$
$$+ 12t^2\sum_{k=1}^{N}\frac{1}{k+1}\sum_{i=0}^{k}\frac{c_{k-i-1}\tilde{c}_k}{k+1-i}\left(\sum_{j=0}^{i}\tilde{b}_j c_{i-j-1}\right)$$
$$\le Ct(\tilde{B}_{N+1} + \tilde{b}_0^2)^{\frac{1}{2}}(C_N + c_{-1}^2)^{\frac{1}{2}}\tilde{C}_N^{\frac{1}{2}}$$
$$+ Ct^2(\tilde{B}_N + \tilde{b}_0^2)^{\frac{1}{2}}(C_N + c_{-1}^2)\tilde{C}_N^{\frac{1}{2}}$$
$$\le C\Psi_{N+1}^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}} + C(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})\Psi_N$$
$$+ C\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}} + 1)^{\frac{1}{2}}\Psi_N^{\frac{1}{2}} + Ct^{\frac{1}{2}}\Phi_N(t\Phi_N^{\frac{1}{2}} + 1)\Psi_N^{\frac{1}{2}}.$$

Combining (5.27)–(5.34), we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)C_N \le -\frac{7}{4}\tilde{C}_{N+1} + C\Phi_N^{\frac{3}{2}} + C\Psi_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}} + C\Psi_{N+1}^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}$$
$$+ C\Psi_{N+1}^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}} + C\Phi_N^{\frac{3}{4}}(1 + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}$$
$$+ C(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^2\Psi_N + Ct^{\frac{1}{2}}\Phi_N(1 + t\Phi_N^{\frac{1}{2}})\Psi_N^{\frac{1}{2}}$$
$$+ C\Psi_N(t^2\Phi_N + t\Phi_N^{\frac{1}{2}}) + C\Psi_N(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{3}{2}}$$

$$
\begin{aligned}
&+ 2\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}}+1)^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})^{\frac{1}{2}}\Psi_N^{\frac{1}{2}} \\
&+ 2\sqrt{2}\Psi_{N+1}^{\frac{1}{2}}\Psi_N^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}}) + C\Psi_N^{\frac{1}{2}}\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}}+1) \\
&+ 2\Phi_N^{\frac{3}{4}}(t\Phi_N^{\frac{1}{2}}+1)^{\frac{1}{2}}(t^2\Phi_N + t\Phi_N^{\frac{1}{2}})\Psi_N^{\frac{1}{2}}.
\end{aligned}
$$

The result (5.24) follows by applying Cauchy–Schwarz. □

We can now combine our results to prove Theorem 5.4.

Proof of Theorem 5.4. The estimates in Lemmas 5.5–5.7 give the existence of a universal constant $C > 0$ such that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\Phi_N &\le -\Psi_N + C\left(t\Phi_N^{\frac{1}{2}} + t^2\Phi_N + t^3\Phi_N^{\frac{3}{2}} + t^4\Phi_N^2\right)\Psi_N \\
&\quad + C\Phi_N^{\frac{3}{2}}\left(1 + t\Phi_N^{\frac{1}{2}} + t^2\Phi_N\right) \\
&\le -\Psi_N + C\left(\left(1+t\Phi_N^{\frac{1}{2}}\right)^4 - 1\right)\Psi_N + C\Phi_N^{\frac{3}{2}}\left(1+t\Phi_N^{\frac{1}{2}}\right)^2.
\end{aligned}
$$

Let τ_N be the time

$$
\tau_N := \sup\{a \in [0, T_0] \mid t\Phi_N^{\frac{1}{2}}(x,t) \le \left(C^{-1}+1\right)^{\frac{1}{4}} - 1,\ \forall (x,t) \in M \times [0,a]\}.
$$

Then on $M \times [0, \tau_N]$, we have

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\Phi_N \le C\Phi_N^{\frac{3}{2}}\left(C^{-1}+1\right)^{\frac{1}{2}} \le (1+C)\Phi_N^{\frac{3}{2}}. \tag{5.35}
$$

Since the initial value of Φ_N is bounded by

$$
\begin{aligned}
\Phi_N(x,0) &\le |Rm|^2(x,0) + |\nabla T|^2(x,0) + |\nabla^2\varphi|^2(x,0) \\
&\le |Rm|^2(x,0) + |\nabla T|^2(x,0) + 2|\nabla T|^2|\psi|^2(x,0) + 32|T|^4|\varphi|^2(x,0) \\
&\le 225 \sup_M \Lambda(\cdot,0)^2 = 225K_0^2,
\end{aligned}
$$

and M is compact, applying the maximum principle to (5.35) gives

$$
\Phi_N(x,t) \le \frac{225K_0^2}{(1-\frac{1}{2}(1+C)15K_0t)^2} \tag{5.36}
$$

on $M \times [0, \tau_N]$. Let $\alpha > 0$ be the universal constant

$$
\alpha := \frac{1}{15}\left(\frac{1}{2}(1+C) + \left((C^{-1}+1)^{\frac{1}{4}} - 1\right)^{-1}\right)^{-1}.
$$

Denote

$$T_* = T_*(T_0, K_0) := \min\{T_0, \frac{\alpha}{K_0}\} > 0. \tag{5.37}$$

By definition, $\tau_N \geq T_*$ for all $N \in \mathbb{N}$. In particular, (5.36) holds on $M \times [0, T_*]$ for all $N \in \mathbb{N}$. When $t \leq T_*$, the right hand side of (5.36) is bounded above by a positive constant C_* depending only on T_0 and K_0. □

5.3. Completing the proof

Suppose $\varphi(t), t \in [0, T_0]$ solves the Laplacian flow (1.1) on a compact manifold M. Theorem 5.4 implies that

$$t^{\frac{k}{2}} \left(|\nabla^k Rm|(x,t) + |\nabla^{k+2}\varphi|(x,t) \right) \leq C_*(k+1)!$$

on $M \times [0, T_*]$, where T_* is given in (5.37). Since $k+1 \leq 2^k$, for any fixed $t \in (0, T_*]$, we have

$$|\nabla^k Rm|(x,t) + |\nabla^{k+2}\varphi|(x,t) \leq Ck!r^{-k-2},$$

where $r = \sqrt{t}/2$ and $C = C_* T_*/4$ are uniform constants. Thus by Lemma 3.1 and the discussion following it, we conclude that $(M, \varphi(t), g(t))$ is real analytic for each $t \in (0, T_*]$. Theorem 1.1 in the case when $U = M$ follows by iterating the above argument to cover the entire time interval $t \in (0, T_0]$.

6. Local real analyticity

In this section, we localize the discussion in §5 using a cut-off function to prove Theorem 1.1. First, we show the existence of the required function.

Lemma 6.1. *Suppose $\varphi(t), t \in [0, T_0]$ is a smooth solution to the Laplacian flow* (1.1) *on an open subset $U \subset M$. Let $p \in U$ and $r > 0$ so that $\overline{B}_{g(0)}(p, 2r) \subset U$ is compact. Let $\alpha > 0$ be a constant and suppose that*

$$\Lambda(x,t) = \left(|Rm|^2(x,t) + |\nabla T|^2(x,t)\right)^{\frac{1}{2}} \leq K \tag{6.1}$$

for all $(x,t) \in B_{g(0)}(p, 2r) \times [0, T_*]$, where $0 < T_* \leq \alpha/K$.

There exist constants $C_0 = C_0(\alpha, r), C_1 = C_1(\alpha, K, r)$, and a cut-off function $\eta : U \to [0,1]$ with support in $B_{g(0)}(p, r)$, and with $\eta = 1$ in $B_{g(0)}(p, r/2)$

such that

$$|\nabla\eta(x)|^2_{g(t)} \le C_0\eta(x), \tag{6.2}$$

$$-\Delta_{g(t)}\eta(x) \le C_1 \tag{6.3}$$

on $U \times [0, T_*]$.

Proof. Recall that along the Laplacian flow (1.1), the associated metric $g(t)$ evolves by (4.3), i.e. with velocity $2h(t)$, where $h(t)$ is given in (4.2). Let $\Theta(x,t) = tK^2|\nabla h(x,t)|^2$. Since

$$|Rm| \le \Lambda(x,t) \le K \quad \text{in } B_{g(0)}(p,r) \times [0, T_*], \tag{6.4}$$

by a straightforward adjustment to the proof of [6, Lemma 14.3], there exists a cut-off function $\eta : U \to [0,1]$ with support in $B_{g(0)}(p,r)$ and with $\eta = 1$ in $B_{g(0)}(p, r/2)$ such that

$$|\nabla\eta(x)|^2_{g(t)} \le \frac{C(\alpha)}{r^2}\eta(x), \tag{6.5}$$

$$-\Delta_{g(t)}\eta(x) \le \frac{C(\alpha, \sqrt{K}r)}{r^2} + \frac{C(\alpha)}{K^{\frac{3}{2}}r}\sup_{s\in[0,t]}(\eta\Theta)^{\frac{1}{2}}(x,s) \tag{6.6}$$

for some constants $C(\alpha)$ and $C(\alpha, \sqrt{K}r)$, for all $(x,t) \in B_{g(0)}(p,r) \times [0, T_*]$. We obtain (6.2) from (6.5) by defining $C_0 = C(\alpha, n)/r^2$.

The cut-off function η here is constructed by a composition of a scalar function with the Riemannian distance function $d_{g(0)}(x,p)$ with respect to the initial metric $g(0)$. The key is that the bound (6.4) and the fact $|T|^2 = -R$ imply that h in (4.2) is uniformly bounded in $B_{g(0)}(p,r) \times [0, T_*]$, which in turn implies the uniform equivalence of the metrics $g(t)$ for $t \in [0, T_*]$.

We next show (6.3). Under the assumption (6.1), the local Shi-type derivative estimates from [13, Theorem 4.3] for Rm and T give that

$$t^{\frac{1}{2}}\left(|\nabla Rm|(x,t) + |\nabla^2 T|(x,t)\right) \le C(\alpha, \sqrt{K}r)K, \tag{6.7}$$

for a constant $C(\alpha, \sqrt{K}r)$, for all $(x,t) \in B_{g(0)}(p,r) \times [0, T_*]$. (Note that in [13, Theorem 4.3], we only state the estimate when $\alpha = 1$, but a trivial adjustment of the proof gives (6.7) as stated.) We deduce that

$$\begin{aligned}\Theta(x,t) &= tK^2|\nabla h(x,t)|^2\\ &\leq CtK^2\left(|\nabla Rm|^2(x,t)+|T|^2|\nabla T|^2(x,t)\right)\\ &\leq CtK^2\left(|\nabla Rm|^2(x,t)+|Rm|^3(x,t)\right)\\ &\leq C\left(C(\alpha,\sqrt{K}r)^2+\alpha\right)K^4\end{aligned}\tag{6.8}$$

for all $(x,t)\in B_{g(0)}(p,r)\times[0,T_*]$, where in the third inequality we used Propositions 2.2–2.3 and $tK\leq\alpha$. Combining (6.8) and (6.6) gives (6.7). □

Remark 6.2. Although η in Lemma 6.1 is constructed by a composition with a Riemannian distance function, by the standard Calabi's trick (see, for example, [5, pp.453–456]), we can for the purpose of applying the maximum principle treat η as a smooth function.

Theorem 6.3. *Suppose that $\varphi(t), t\in[0,T_0]$ solves the Laplacian flow* (1.1) *on an open set $U\subset M$. Let $p\in U$ and $r>0$ be such that $\overline{B_{g(0)}(p,2r)}\subset U$ is compact. Let $K=\sup_{B_{g(0)}(p,2r)\times[0,T_0]}\Lambda(x,t)$, where $\Lambda(x,t)$ is given in* (1.4), *and let $\alpha>0$ be such that $\alpha\leq KT_0$.*

There exist positive constants L, C, T_* depending only on α, K, r such that

$$t^{\frac{k}{2}}\left(|\nabla^k Rm|(x,t)+|\nabla^{k+1}T|(x,t)+|\nabla^{k+2}\varphi|(x,t)\right)\leq CL^{\frac{k}{2}}(k+1)!\tag{6.9}$$

for all $k\in\mathbb{N}$ and $(x,t)\in B_{g(0)}(p,r/2)\times[0,T_*]$.

Remark 6.4. If $\varphi(t), t\in[0,T_0]$ is a smooth solution to the Laplacian flow (1.1) on an open set $U\subset M$, then a similar argument as in the proof of [13, Theorem 1.3] shows that $\Lambda(x,t)$ is bounded on $U\times[0,T_0]$. Thus, $K=\sup_{B_{g(0)}(p,2r)\times[0,T_0]}\Lambda(x,t)$ in Theorem 6.3 is finite.

Proof. We consider localized modifications of Φ_N,Ψ_N in §5, in a similar spirit to [11]. For $L>0$, to be determined later, we define

$$\begin{aligned}\alpha_k &= \frac{\eta^{\frac{k+1}{2}}}{L^{\frac{k}{2}}}a_k,\ \beta_k=\frac{\eta^{\frac{k+1}{2}}}{L^{\frac{k}{2}}}b_k,\ \gamma_k=\frac{\eta^{\frac{k+1}{2}}}{L^{\frac{k}{2}}}c_k,\quad \text{for } k\geq 0,\\ \tilde{\alpha}_k &= \frac{\eta^{\frac{k}{2}}}{L^{\frac{k-1}{2}}}\tilde{a}_k,\ \tilde{\beta}_k=\frac{\eta^{\frac{k}{2}}}{L^{\frac{k-1}{2}}}\tilde{b}_k,\ \tilde{\gamma}_k=\frac{\eta^{\frac{k}{2}}}{L^{\frac{k-1}{2}}}\tilde{c}_k,\quad \text{for } k\geq 1,\end{aligned}$$

where $a_k, b_k, c_k, \tilde{a}_k, \tilde{b}_k, \tilde{c}_k$ are defined in (5.6)–(5.7) and η is the cut-off function constructed in Lemma 6.1. We further define

$$\tilde{\Phi}_N = \sum_{k=0}^{N} \left(\alpha_k^2 + \beta_k^2 + \gamma_k^2\right) \quad \text{and} \quad \tilde{\Psi}_N = \sum_{k=1}^{N} \left(\tilde{\alpha}_k^2 + \tilde{\beta}_k^2 + \tilde{\gamma}_k^2\right). \tag{6.10}$$

We aim to estimate $\tilde{\Phi}_N$. We first compute an evolution inequality for $\tilde{\Phi}_N$, by looking at each of α_k^2, β_k^2 and γ_k^2 in turn. First,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\alpha_k^2 = \frac{\eta^{k+1}}{L^k}\left(\frac{\partial}{\partial t} - \Delta\right)a_k^2 + \frac{a_k^2}{L^k}\left(\frac{\partial}{\partial t} - \Delta\right)\eta^{k+1} \\ - \frac{2}{L^k}\langle \nabla\eta^{k+1}, \nabla a_k^2\rangle.\end{aligned} \tag{6.11}$$

By (6.3) and $t \le T_0$, the second term on the right hand side of (6.11) satisfies

$$\begin{aligned}\frac{a_k^2}{L^k}\left(\frac{\partial}{\partial t} - \Delta\right)\eta^{k+1} &= \frac{a_k^2}{L^k}\left(-k(k+1)\eta^{k-1}|\nabla\eta|^2 - (k+1)\eta^k\Delta\eta\right) \\ &\le (k+1)C_1\frac{a_k^2}{L^k}\eta^k = \frac{C_1 t}{L(k+1)}\tilde{\alpha}_k^2 \le \frac{C_1 T_0}{L(k+1)}\tilde{\alpha}_k^2\end{aligned} \tag{6.12}$$

on $U \times [0, \alpha/K]$. To estimate the third term of (6.11), we use (6.2), $t \le T_0$ and the Cauchy–Schwarz inequality:

$$\begin{aligned}-\frac{2}{L^k}\langle\nabla\eta^{k+1}, \nabla a_k^2\rangle &\le 4\frac{\eta^k|\nabla\eta|}{L^k}\frac{t^k|\nabla^k Rm||\nabla^{k+1}Rm|}{k!(k+1)!} \\ &\le \frac{1}{4}\tilde{\alpha}_{k+1}^2 + \frac{16C_0T_0}{L}\tilde{\alpha}_k^2\end{aligned} \tag{6.13}$$

on $U \times [0, \alpha/K]$. Substituting (5.10), (6.12) and (6.13) into (6.11), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\alpha_k^2 \le -\frac{7}{4}\tilde{\alpha}_{k+1}^2 + \frac{C}{L}\tilde{\alpha}_k^2 + \frac{\eta^{k+1}}{L^k}(I_1(k) + I_2(k) + I_3(k)), \tag{6.14}$$

where $C = C(\alpha, r, T_0)$. Using (5.12)–(5.14), we can estimate

$$\begin{aligned}&\sum_{k=1}^{N}\frac{\eta^{k+1}}{L^k}(I_1(k) + I_2(k) + I_3(k)) \\ &\le CK\tilde{\Phi}_N + CK^2\tilde{\Phi}_N^{\frac{1}{2}} + C\frac{t}{L}K\tilde{\Psi}_N \\ &\quad + C\frac{t}{L}\tilde{\Phi}_N^{\frac{1}{2}}\tilde{\Psi}_N + CK^{\frac{1}{2}}\tilde{\Psi}_{N+1}^{\frac{1}{2}}\tilde{\Phi}_N^{\frac{1}{2}} + C\frac{t^2}{L^2}\tilde{\Phi}_N\tilde{\Psi}_N.\end{aligned} \tag{6.15}$$

Combining (5.11) and (6.11)–(6.15), we obtain

$$\left(\frac{\partial}{\partial t}-\Delta\right)\sum_{k=0}^{N}\alpha_k^2 \le -\frac{3}{2}\tilde{\Psi}_{N+1}+C(\tilde{\Phi}_N+1) + C\left(\frac{1}{L}+\frac{t}{L}\tilde{\Phi}_N^{\frac{1}{2}}+\frac{t^2}{L^2}\tilde{\Phi}_N\right)\tilde{\Psi}_N \tag{6.16}$$

on $U\times[0,\alpha/K]$, again using Cauchy–Schwarz and $tK\le\alpha$, where the constant C depends only on α, r, K, T_0. We can deal with $\sum_{k=0}^N \beta_k^2$ and $\sum_{k=0}^N\gamma_k^2$ similarly and obtain the following estimate:

$$\left(\frac{\partial}{\partial t}-\Delta\right)\sum_{k=0}^{N}(\beta_k^2+\gamma_k^2) \le -\frac{3}{2}\tilde{\Psi}_{N+1}+C(\tilde{\Phi}_N+1) + C\left(\frac{1}{L}+\frac{t}{L}\tilde{\Phi}_N^{\frac{1}{2}}+\frac{t^2}{L^2}\tilde{\Phi}_N+\frac{t^4}{L^4}\tilde{\Phi}_N^4\right)\tilde{\Psi}_N. \tag{6.17}$$

Since $-\tilde{\Psi}_{N+1}\le-\tilde{\Psi}_N$, we may put together the estimates (6.16)–(6.17) and choose L large enough so that

$$\left(\frac{\partial}{\partial t}-\Delta\right)\tilde{\Phi}_N \le -\tilde{\Psi}_N+C_2(\tilde{\Phi}_N+1)+C_3\left(\left(1+t\tilde{\Phi}_N^{\frac{1}{2}}\right)^4-1\right)\tilde{\Psi}_N,$$

where C_2, C_3 depend only on α, r, K, T_0. Let

$$\tau_N=\{a\in[0,\alpha/K]\mid t\tilde{\Phi}_N^{\frac{1}{2}}\le(C_3^{-1}+1)^{\frac{1}{4}}-1, \forall(x,t)\in U\times[0,a]\}.$$

Then on $B_{g(0)}(p,r)\times[0,\tau_N]$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)\tilde{\Phi}_N\le C_2(\tilde{\Phi}_N+1).$$

As $\tilde{\Phi}_N=0$ on $\partial B_{g(0)}(p,r)\times[0,\tau_N]$ and $\tilde{\Phi}_N(\cdot,0)\le 225K^2$, the maximum principle gives that

$$\tilde{\Phi}_N\le(225K^2+1)e^{C_2T_0}:=C_4$$

for all $(x,t)\in B_{g(0)}(p,r)\times[0,\tau_N]$. Let

$$T_*=\min\{\alpha/K, C_4^{-\frac{1}{2}}((C_3^{-1}+1)^{\frac{1}{4}}-1)\}>0,$$

which depends only on α, r, K, T_0. Then $\tau_N \geq T_*$ for all $N \in \mathbb{N}$ and thus $\tilde{\Phi}_N \leq C_4$ for all $N \in \mathbb{N}$ and $(x,t) \in B_{g(0)}(p,r) \times [0,T_*]$. Since $\eta \equiv 1$ on $B_{g(0)}(p,r/2)$, the estimate (6.9) follows. □

Finally, Theorem 1.1 follows from Theorem 6.3 and the discussion in §3.

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