Growth of meromorphic solutions of delay differential equations

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Abstract

Necessary conditions are obtained for certain types of rational delay differential equations to admit a non-rational meromorphic solution of hyper-order less than one. The equations obtained include delay Painlevé equations and equations solved by elliptic functions.

1 Introduction

There have been many studies of the discrete (or difference) Painlevé equations. One way in which difference Painlevé equations arise is in the study of difference equations admitting meromorphic solutions of slow growth in the sense of Nevanlinna. The idea that the existence of sufficiently many finite-order meromorphic solutions could be considered as a version of the Painlevé property for difference equations was first advocated in [1]. This is a very restrictive property, as demonstrated by the relatively short list of possible equations obtained in [3] of the form $w(z+1)+w(z-1) = R(z, w(z))$, where R is rational in w with meromorphic coefficients in z, and w is assumed to have finite order but to grow faster than the coefficients. It was later shown in [4] that the same list is obtained by replacing the finite order assumption with the weaker assumption of hyper-order less than one.

Some reductions of integrable differential-difference equations are known to yield delay differential equations with formal continuum limits to (differential) Painlevé equations. For example, Quispel, Capel and Sahadevan [8] obtained the equation

$$w(z)\left[w(z+1) - w(z-1)\right] + aw'(z) = bw(z), \tag{1.1}$$

where a and b are constants, as a symmetry reduction of the Kac-van Moerbeke equation. They showed that equation (1.1) has a formal continuum limit to the first Painlevé equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 6y^2 + t. \tag{1.2}$$

Furthermore, they obtained an associated linear problem for equation (1.1) by extending the symmetry reduction to the Lax pair for the Kac-van Moerbeke equation.

Painlevé-type delay differential equations were also considered in Grammaticos, Ramani and Moreira [2] from the point of view of a kind of singularity confinement. More recently, Viallet [10] has introduced a notion of algebraic entropy for such equations.

We will assume that the reader is familiar with the standard notation and basic results of Nevanlinna theory (see, e.g., [5]). Let $w(z)$ be a meromorphic function. The hyper-order (or the iterated order) of $w(z)$ is defined by

$$\rho_2(w) = \limsup_{r\to\infty} \frac{\log\log T(r,w)}{\log r},$$

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where $T(r,w)$ is the Nevanlinna characteristic function of w. Most of the present paper is devoted to a proof of the following.

Theorem 1.1. *Let $w(z)$ be a non-rational meromorphic solution of*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = R(z,w(z)) = \frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.3}$$

where $a(z)$ is rational, $P(z,w)$ is a polynomial in w having rational coefficients in z, and $Q(z,w)$ is a polynomial in $w(z)$ with roots that are nonzero rational functions of z and not roots of $P(z,w)$. If the hyper-order of $w(z)$ is less than one, then

$$\deg_w(P) = \deg_w(Q) + 1 \leq 3 \quad \textit{or} \quad \deg_w(R) \leq 1. \tag{1.4}$$

We have used the notation $\deg_w(P) = \deg_w(P(z,w))$ for the degree of P as a polynomial in w and $\deg_w(R) = \max\{\deg_w(P), \deg_w(Q)\}$ for the degree of R as a rational function of w.

If $\deg_w(R(z,w)) = 0$ then equation (1.3) becomes

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = b(z), \tag{1.5}$$

where $a(z)$ and $b(z)$ are rational. Note that if $b(z) \equiv p\pi i a(z)$, where $p \in \mathbb{N}$, then $w(z) = C\exp(p\pi i z)$, $C \neq 0$, is a one-parameter family of zero-free entire transcendental finite-order solutions of (1.5) for any rational $a(z)$. In the following theorem we will single out the equation (1.1) from the class (1.5) by introducing an additional assumption that the meromorphic solution has sufficiently many simple zeros.

In value distribution theory the notation $S(r,w)$ usually means a quantity of magnitude $o(T(r,w))$ as $r \to \infty$ outside of an exceptional set of finite linear measure. In what follows we use a slightly modified definition with a larger exceptional set of finite logarithmic measure. We use the notation $N(r,w)$ to denote the integrated counting function of poles counting multiplicities and $\overline{N}(r,w)$ to denote the integrated counting function of poles ignoring multiplicities.

Theorem 1.2. *Let $w(z)$ be a non-rational meromorphic solution of equation* (1.5)*, where $a(z) \not\equiv 0$ and $b(z)$ are rational. If the hyper-order of $w(z)$ is less than one and for any $\epsilon > 0$*

$$\overline{N}\left(r, \frac{1}{w}\right) \geq \left(\frac{3}{4} + \epsilon\right) T(r,w) + S(r,w), \tag{1.6}$$

then the coefficients $a(z)$ and $b(z)$ are both constants.

Finally, we consider an equation outside the class (1.3).

Theorem 1.3. *Let $w(z)$ be a non-rational meromorphic solution of*

$$w(z+1) - w(z-1) = \frac{a(z)w'(z) + b(z)w(z)}{w(z)^2} + c(z), \tag{1.7}$$

where $a(z) \not\equiv 0$, $b(z)$ and $c(z)$ are rational. If the hyper-order of $w(z)$ is less than one and for any $\epsilon > 0$

$$\overline{N}\left(r, \frac{1}{w}\right) \geq \left(\frac{3}{4} + \epsilon\right) T(r,w) + S(r,w), \tag{1.8}$$

then (1.7) *has the form*

$$w(z+1) - w(z-1) = \frac{(\lambda + \mu z)w'(z) + (\nu\lambda + \mu(\nu z - 1))w(z)}{w(z)^2}, \tag{1.9}$$

where λ, μ and ν are constants.

When $\mu = \nu = 0$ and $\lambda \neq 0$ then equation (1.9) has a multi-parameter family of elliptic function solutions:

$$w(z) = \alpha \left[\wp(\Omega z; g_2, g_3) - \wp(\Omega; g_2, g_3)\right],$$

where $\wp$ is the Weierstrass elliptic function, Ω, g_2 and g_3 are arbitrary (provided that $\wp'(\Omega; g_2, g_3) \neq 0$ or ∞) and $\alpha^2 = -\lambda\Omega/\wp'(\Omega; g_2, g_3)$. Furthermore, when $\mu = 0$, equation (1.9) has a formal continuum limit to the first Painlevé equation. Specifically, we take the limit $\epsilon \to 0$ for fixed $t = \epsilon z$, where $w(z) = 1 - \epsilon^2 y(t)$, $\lambda = 2 + O(\epsilon)$ and $\lambda\nu = -\frac{1}{3}\epsilon^5 + O(\epsilon^6)$. Then equation (1.9) becomes $\mathrm{d}^3y/\mathrm{d}t^3 = 12y\,\mathrm{d}y/\mathrm{d}t + 1$, which integrates to $\mathrm{d}^2y/\mathrm{d}t^2 = 6y^2 + t - t_0$, for some constant t_0. Replacing t with $t + t_0$ gives the first Painlevé equation (1.2). Finally, when $\mu = 0$ and $\lambda\nu \neq 0$, equation (1.9) is a symmetry reduction of the known integrable differential-difference modified Korteweg-de Vries equation

$$v_t(x,t) = v(x,t)^2 \left(v(x+1,t) - v(x-1,t)\right),$$

in which $v(x,t) = (-2\lambda\nu t)^{-1/2} w(z)$, where $z = x - (2\nu)^{-1}\log t$.

2 Value distribution of slow growth solutions

We begin by proving an important lemma, which relates the value distribution of meromorphic solutions of a large class of delay differential equations to the growth of these solutions. A differential difference polynomial in $w(z)$ is defined by

$$P(z,w) = \sum_{l\in L} b_l(z) w(z)^{l_{0,0}} w(z+c_1)^{l_{1,0}} \cdots w(z+c_\nu)^{l_{\nu,0}} w'(z)^{l_{0,1}} \cdots w^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}},$$

where $c_1, \ldots, c_\nu$ are distinct complex constants, L is a finite index set consisting of elements of the form $l = (l_{0,0}, \ldots, l_{\nu,\mu})$ and the coefficients $b_l(z)$ are rational functions of z for all $l \in L$.

Lemma 2.1. *Let $w(z)$ be a non-rational meromorphic solution of*

$$P(z,w) = 0 \tag{2.1}$$

where $P(z,w)$ is differential difference polynomial in $w(z)$ with rational coefficients, and let $a_1, \ldots, a_k$ be rational functions satisfying $P(z, a_j) \not\equiv 0$ for all $j \in \{1, \ldots, k\}$. If there exists $s > 0$ and $\tau \in (0,1)$ such that

$$\sum_{j=1}^{k} n\left(r, \frac{1}{w - a_j}\right) \leq k\tau\, n(r+s, w) + O(1), \tag{2.2}$$

then the hyper-order $\rho_2(w)$ of w is at least 1.

Proof. We suppose against the conclusion that $\rho_2(w) < 1$ aiming to obtain a contradiction. We first show that the assumption $P(z, a_j) \not\equiv 0$ implies that

$$m\left(r, \frac{1}{w - a_j}\right) = S(r,w). \tag{2.3}$$

This fact is an extension of Mohon'ko's theorem and its difference analogue (see [4, Remark 5.3]) for differential delay equations with meromorphic solutions of hyper-order less than one.

By substituting $w = g + a_j$ into (2.1) it follows that

$$Q(z,g) + R(z) = 0, \tag{2.4}$$

where $R(z) \not\equiv 0$ is a rational function, and

$$Q(z,g) = \sum_{l \in L} b_l(z) G_l(z,g) \tag{2.5}$$

is a differential difference polynomial in g such for all l in the finite index set L, $G_l(z,g)$ is a non-constant product of derivatives and shifts of $g(z)$. The coefficients b_l in (2.5) are all rational. Now, letting $E_1 = \{\theta \in [0,2\pi) : |g(re^{i\theta})| \leq 1\}$ and $E_2 = [0,2\pi) \setminus E_1$, we have

$$m\left(r, \frac{1}{w - a_j}\right) = m\left(r, \frac{1}{g}\right) = \int_{\theta \in E_1} \log^+ \left|\frac{1}{g(re^{i\theta})}\right| \frac{d\theta}{2\pi}. \tag{2.6}$$

Moreover, for all $z = re^{i\theta}$ such that $\theta \in E_1$,

$$\begin{aligned}
&\left|\frac{Q(z,g)}{g}\right| \\
&= \frac{1}{|g|} \left|\sum_{l \in L} b_l(z) g(z)^{l_{0,0}} g(z+c_1)^{l_{1,0}} \cdots g(z+c_\nu)^{l_{\nu,0}} g'(z)^{l_{0,1}} \cdots g^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}}\right| \\
&\leq \sum_{l \in L} |b_l(z)| \left|\frac{g(z+c_1)}{g(z)}\right|^{l_{1,0}} \cdots \left|\frac{g(z+c_\nu)}{g(z)}\right|^{l_{\nu,0}} \cdot \left|\frac{g'(z)}{g(z)}\right|^{l_{0,1}} \cdots \left|\frac{g^{(\mu)}(z+c_\nu)}{g(z)}\right|^{l_{\nu,\mu}},
\end{aligned}$$

since $\deg_g(G_l) \geq 1$ for all $l \in L$ with $l = (l_{0,0}, \ldots, l_{\nu,\mu})$. Now, since

$$\begin{aligned}
\log^+ \left|\frac{1}{g(z)}\right| &\leq \log^+ \left|\frac{R(z)}{g(z)}\right| + \log^+ \left|\frac{1}{R(z)}\right| \\
&= \log^+ \left|\frac{Q(z,g)}{g(z)}\right| + \log^+ \left|\frac{1}{R(z)}\right|
\end{aligned}$$

by equation (2.4), it follows from (2.6) by defining $c_0 = 0$ that

$$\begin{aligned}
m\left(r, \frac{1}{w - a_j}\right) &\leq \int_{\theta \in E_1} \log^+ \left|\frac{Q(z,g)}{g(z)}\right| \frac{d\theta}{2\pi} + O(\log r) \\
&\leq \sum_{n=0}^{\nu} \sum_{m=0}^{\mu} l_{n,m} m\left(r, \frac{g^{(m)}(z+c_n)}{g(z)}\right) + O(\log r) \\
&\leq \sum_{n=0}^{\nu} \sum_{m=0}^{\mu} l_{n,m} \left(m\left(r, \frac{g^{(m)}(z+c_n)}{g(z+c_n)}\right) + m\left(r, \frac{g(z+c_n)}{g(z)}\right)\right) + O(\log r).
\end{aligned} \tag{2.7}$$

The claim that (2.3) holds follows by applying the lemma on the logarithmic derivative, its difference analogue [4, Theorem 5.1] and [4, Lemma 8.3], to the right hand side of (2.7).

To finish the proof, we observe that from the assumption (2.2) it follows that

$$\sum_{j=1}^{k} N\left(r, \frac{1}{w - a_j}\right) \leq (\tau + \varepsilon) k\, N(r+s, w) + O(\log r) \tag{2.8}$$

where $\varepsilon > 0$ is chosen so that $\tau + \varepsilon < 1$. The first main theorem of Nevanlinna theory now yields

$$kT(r,w) = \sum_{j=1}^{k} \left(m\left(r, \frac{1}{w - a_j}\right) + N\left(r, \frac{1}{w - a_j}\right)\right) + O(\log r). \tag{2.9}$$

By combining (2.3), (2.8) and (2.9) it follows that

$$kT(r,w) \le (\tau+\varepsilon)kN(r+s,w) + S(r,w) \le (\tau+\varepsilon)k\,T(r+s,w) + S(r,w). \tag{2.10}$$

An application of [4, Lemma 8.3] yields $T(r+s,w) = T(r,w) + S(r,w)$, and so (2.10) becomes

$$T(r,w) \le (\tau+\varepsilon)T(r,w) + S(r,w),$$

which gives us the desired contradiction $T(r,w) = S(r,w)$ since $\tau+\varepsilon < 1$. We conclude that $\rho_2(w) \ge 1$. □

3 The proof of Theorem 1.1

Before proving Theorem 1.1 we first prove three lemmas related to equations of the form (1.3). The first bounds the degree of R.

Lemma 3.1. *Let $w(z)$ be a non-rational meromorphic solution of equation* (1.3) *where a and R are rational functions of one and two variables respectively. If the hyper-order of w is less than one then $\deg_w(R) \le 4$. Furthermore, if the hyper-order of w is less than one and $\deg_w(R) = 4$ then $\overline{N}(r,1/w) = T(r,w) + S(r,w)$.*

Proof. Taking the Nevanlinna characteristic function of both sides of (1.3) and applying an identity due to Valiron [9] and Mohon'ko [7] (see also [6, Theorem 2.2.5]), we have

$$\begin{aligned} T\left(r, w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)}\right) &= T(r, R(z,w(z))) \\ &= \deg_w(R)T(r,w(z)) + O(\log r). \end{aligned}$$

Thus by using the lemma on the logarithmic derivative and its difference analogue [4, Theorem 5.1], it follows that

$$\begin{aligned} &\deg_w(R)T(r,w(z)) \\ &\le T\left(r, w(z+1) - w(z-1)\right) + T\left(r, \frac{w'(z)}{w(z)}\right) + O(\log r) \\ &\le N\left(r, w(z+1) - w(z-1)\right) + m(r,w(z)) + \overline{N}(r,w(z)) \\ &\quad + \overline{N}\left(r, \frac{1}{w(z)}\right) + S(r,w). \end{aligned} \tag{3.1}$$

On using [4, Lemma 8.3] to obtain

$$\begin{aligned} N\left(r, w(z+1) - w(z-1)\right) &\le N\left(r, w(z+1)\right) + N\left(r, w(z-1)\right) \\ &\le 2N(r+1, w(z)) = 2N(r,w(z)) + S(r,w), \end{aligned}$$

inequality (3.1) becomes

$$\begin{aligned} \deg_w(R)T(r,w(z)) \le T(r,w(z)) &+ N(r,w(z)) + \overline{N}(r,w(z)) \\ &+ \overline{N}\left(r, \frac{1}{w(z)}\right) + S(r,w). \end{aligned} \tag{3.2}$$

Therefore

$$(\deg_w(R) - 3)T(r,w(z)) \le \overline{N}\left(r, \frac{1}{w(z)}\right) + S(r,w), \tag{3.3}$$

which implies the conclusions of the lemma. □

Next we consider the case in which $R(z,w)$ is a polynomial in w.

Lemma 3.2. *Let w be a non-rational meromorphic solution of the equation*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = P(z, w(z)), \tag{3.4}$$

where $a(z)$ is rational in z and $P(z,w)$ is a polynomial in w and rational in z. If the hyper-order of w is less than one then $\deg_w(P) \le 1$.

Proof. Assume that $\deg_w(P) \ge 2$, and suppose first that $w(z)$ has either infinitely many zeros or poles (or both). Let $w(z)$ have either a zero or a pole at $z = \hat{z}$. Then either there is a cancelation with a zero or pole of some of the coefficients in (3.4), or $w(z)$ has a pole of order at least 1 at $z = \hat{z}+1$, or at $z = \hat{z}-1$. Since the coefficients of (3.4) are rational, we can always choose a zero or a pole of $w(z)$ in such a way that there is no cancelation with the coefficients. Suppose, without loss of generality, that there is a pole of $w(z)$ at $z = \hat{z}+1$. By shifting (3.4) up we obtain

$$w(z+2) - w(z) + a(z+1)\frac{w'(z+1)}{w(z+1)} = P(z+1, w(z+1)),$$

from which it follows that $w(z)$ has a pole at $z = \hat{z}+2$ of order at least $\deg_w(P)$, and a pole of order at least $(\deg_w(P))^2$ at $z = \hat{z}+3$, and so on. The only way that this string of poles with exponential growth in the multiplicity can terminate, or that there can be a drop in the orders of poles, is if there is a cancelation with a suitable zero or pole of a coefficient of (3.4). But since the coefficients are rational and thus have finitely many zeros and poles, and $w(z)$ has infinitely many zeros or poles, we can choose the starting point $\hat{z}$ of the iteration from outside a sufficiently large disc in such a way that no cancelation occurs. Thus,

$$n(d + |\hat{z}|, w) \ge (\deg_w(P))^d$$

for all $d \in \mathbb{N}$, and so

$$\begin{aligned}\lambda_2(1/w) &= \limsup_{r\to\infty} \frac{\log\log n(r,w)}{\log r} \\ &\ge \limsup_{d\to\infty} \frac{\log\log n(d+|\hat{z}|, w)}{\log(d+|\hat{z}|)} \\ &\ge \limsup_{d\to\infty} \frac{\log\log(\deg_w(P))^d}{\log(d+|\hat{z}|)} = 1.\end{aligned}$$

Therefore, $\rho_2(w) \ge \lambda_2(1/w) \ge 1$.

Suppose now that $w(z)$ has finitely many poles and zeros, and that $\rho_2(w) < 1$. Then

$$w(z) = f(z)\exp(g(z)), \tag{3.5}$$

where $f(z)$ is a rational function and $g(z)$ is entire. By substituting (3.5) into (3.4), it follows that

$$f(z+1)e^{g(z+1)} - f(z-1)e^{g(z-1)} + a(z)\left(\frac{f'(z)}{f(z)} + g'(z)\right) = P(z, f(z)\exp(g(z))). \tag{3.6}$$

Now, since $\rho_2(\exp(g(z))) < 1$, it follows from the difference analogue of the lemma on the logarithmic derivatives, [4, Theorem 5.1], that

$$T\left(r, e^{g(z+1)-g(z)}\right) = m\left(r, e^{g(z+1)-g(z)}\right) = S(r, e^g),$$

and similarly

$$T\left(r, e^{g(z-1)-g(z)}\right) = m\left(r, e^{g(z-1)-g(z)}\right) = S(r, e^g).$$

Hence, by writing (3.6) in the form

$$\begin{aligned} &e^{g(z)}\left(f(z+1)e^{g(z+1)-g(z)} - f(z-1)e^{g(z-1)-g(z)}\right) + a(z)\left(\frac{f'(z)}{f(z)} + g'(z)\right) \\ &= P(z, f(z)\exp(g(z))), \end{aligned}$$

and taking Nevanlinna characteristic from both sides, we arrive at the equation

$$\deg_w(P)T(r, e^g) = T(r, e^g) + S(r, e^g) + O(\log r).$$

Since $\deg_w(P) \geq 2$ by assumption, this implies that g is a constant. But this means that w is rational, which is a contradiction. Thus $\rho_2(w) \geq 1$. □

In our final lemma we consider the case in which $Q(z, w)$ has a repeated root as a polynomial in w.

Lemma 3.3. *Let w be a non-rational meromorphic solution of the equation*

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{(w(z) - b_1(z))^\kappa \check{Q}(z, w(z))}, \tag{3.7}$$

where a and b_1 are rational functions of z, $P(z, w)$ and $\check{Q}(z, w)$ are polynomials in w and rational in z, and κ is an integer greater than one. Furthermore we assume that $w - b_1(z)$, $P(z, w)$ and $\check{Q}(z, w)$ are pairwise co-prime. Then w has hyper-order at least one.

Proof. Notice that $b_1(z)$ is not a solution of (3.7), even if $b_1 \equiv 0$, thus the first condition of Lemma 2.1 is satisfied for b_1. Suppose that $\hat{z}$ is a zero of $w(z) - b_1(z)$ of order p and that neither $a(z)$, $b_1(z)$ nor any of the coefficient functions in $P(z, w)$ nor $\check{Q}(z, w)$ has a zero or a pole at $\hat{z}$. We will also require that these coefficient functions do not have zeros or poles at points of the form $\hat{z} + j$ for a finite number of integers j (in this particular case, we take $-4 \leq j \leq 4$). We will call such a point $\hat{z}$ a *generic zero of order p*. We will assume, often without further comment, that in similar situations we are only considering generic zeros. Since the coefficients are rational, when estimating the corresponding unintegrated counting functions, the contribution from the non-generic zeros can be included in a bounded error term, leading to an error term of the type $O(\log r)$ in the integrated estimates involving $T(r, w)$.

Now either $w(z+1)$ or $w(z-1)$ has a pole of order $q \geq \kappa p$ at $z = \hat{z}$, and we suppose without loss of generality that $\hat{z} + 1$ is such a pole. Suppose next that

$$\deg_w(P) \leq \kappa + \deg_w(\check{Q}). \tag{3.8}$$

Then $w(z)$ has a pole of order one at $\hat{z}+2$ and a pole of order q at $\hat{z}+3$. By continuing the iteration, it follows that $w(z)$ has either a simple pole or a finite value at $\hat{z} + 4$. Therefore it may be that $w(\hat{z}+4) = b_1(\hat{z}+4)$, and so it is at least in principle possible that $w(\hat{z}+5)$ is a finite value. This can only happen if the order of the zero of $w(z) - b_1(z)$ at $z = \hat{z}+4$ is $p' = q/\kappa \geq p$. But even so, by considering the multiplicities of zeros and poles of $w - b_1$ in the set $\{\hat{z}, \ldots, \hat{z}+4\}$, we find that there are $2q+1 > 2q \geq \kappa p + \kappa p'$ poles of w for $p + p'$ zeros of $w - b_1$. This is the "worst case scenario" in the sense that if $w(\hat{z}+4) \neq b_1(\hat{z}+4)$, or a zero of $\check{Q}(z, w(z))$, then $\hat{z} + 5$ is a pole of w of order $q \geq \kappa p$, and we have even more poles for every zero of $w - b_1$. By adding up the contribution from all points $\hat{z}$ to the corresponding counting functions, it follows that

$$n\left(r, \frac{1}{w-b_1}\right) \leq \frac{1}{\kappa} n(r+4, w) + O(1).$$

Thus both conditions of Lemma 2.1 are satisfied, and so the hyper-order of w is at least one.

Assume now that

$$\deg_w(P) \geq \kappa + \deg_w(\check{Q}) + 1. \tag{3.9}$$

Suppose again that $\hat{z}$ is a generic zero of $w(z) - b_1(z)$ of order p. Then, as in the case (3.8), $w(z)$ has a pole of order $q \geq \kappa p$ at either $\hat{z}+1$ or $\hat{z}-1$, say $\hat{z}+1$. This implies that $w(z)$ has a pole of order $q' \geq q$ at $\hat{z}+2$, and so, the only way that $w(\hat{z}+4)$ can be finite is that $w(z) - b_1(z)$ has a zero at $\hat{z}+3$ with multiplicity $p' = q'/\kappa$, or $w(\hat{z}+3)$ is a zero of $\check{Q}(z, w(z))$. Even if this would be the case, we have found at least $\kappa p + \kappa p'$ poles, taking into account multiplicities, that correspond uniquely to at most $p + p'$ zeros of $w - b_1$. Therefore, we have

$$n\left(r, \frac{1}{w-b_1}\right) \leq \frac{1}{\kappa} n(r+3, w) + O(1)$$

by going through all zeros of $w - b_1$ in this way. Lemma 2.1 thus implies that the hyper-order of w is at least one. □

Proof of Theorem 1.1.
From lemmas 3.1, 3.2 and 3.3, it follows that $\deg_w(P) \leq 4$ and $1 \leq \deg_w(Q) \leq 4$ and that $Q(z, w)$ has only simple roots as a polynomial in w. We will begin with the case in which $\deg_w(Q) = 1$. We can therefore without loss of generality write the denominator of the right hand side of (1.3) in the form $Q(z, w) = w - b_1$.

Assume first that $\deg_w(P) \geq 3$. Let $\hat{z}$ be a generic zero of $w(z) - b_1(z)$ of order p. Then $w(z)$ has a pole of order at least p at $\hat{z}+1$ or $\hat{z}-1$. We assume without loss of generality that $\hat{z}+1$ is such a pole. Then $\hat{z}+2$ is a pole of order at least $2p$ and $\hat{z}+3$ is a pole of order at least $4p$, and so on. In this case we therefore have

$$n\left(r, \frac{1}{w-b_1}\right) \leq \frac{1}{3} n(r+2, w) + O(1).$$

Lemma 2.1 thus implies that the hyper-order of w is at least one.

Assume now that $Q(z, w) = w - b_1$ and $\deg_w(P) \leq 2$. If $\deg_w(P) = 2$, then $\deg_w(P) = \deg_w(Q) + 1$ and if $\deg_w(P) \leq 1$, then $\deg_w(R) = 1$, thus the assertion (1.4) holds.

Suppose now that the denominator of $R(z, w(z))$ has at least two simple non-zero rational roots for w as a function of z, say $b_1(z) \not\equiv 0$ and $b_2(z) \not\equiv 0$. Then we may write equation (1.3) in the form

$$w(z+1) - w(z-1) + a(z)\frac{w'(z)}{w(z)} = \frac{P(z, w(z))}{(w(z) - b_1(z))(w(z) - b_2(z))\widetilde{Q}(z, w(z))}, \tag{3.10}$$

where $P(z, w(z)) \not\equiv 0$ and $\widetilde{Q}(z, w(z)) \not\equiv 0$ are polynomials in $w(z)$ of at most degree 4 and 2 respectively, with no common factors. Then neither $b_1(z)$, nor $b_2(z)$ is a solution of (3.10), and so they satisfy the first condition of Lemma 2.1. Let $z = \hat{z}$ be a generic zero of order p of $w - b_1$.

Now, by (3.10), it follows that either $w(z+1)$ or $w(z-1)$ has a pole at $z=\hat{z}$ of order at least p. Without loss of generality we may assume that $w(z+1)$ has such a pole at $z=\hat{z}$. Then, by shifting the equation (3.10), we have

$$\begin{aligned} w(z+2) - w(z) + a(z+1)\frac{w'(z+1)}{w(z+1)} \\ = \frac{P(z+1, w(z+1))}{(w(z+1)-b_1(z+1))(w(z+1)-b_2(z+1))\widetilde{Q}(z+1, w(z+1))}, \end{aligned} \tag{3.11}$$

which implies that $w(z+2)$ has a pole of order one at $z=\hat{z}$ provided that

$$\deg_w(P) \leq \deg_w(\widetilde{Q}) + 2. \tag{3.12}$$

We suppose first that (3.12) is valid. By iterating (3.10) one more step, we have

$$\begin{aligned} w(z+3) - w(z+1) + a(z+2)\frac{w'(z+2)}{w(z+2)} \\ = \frac{P(z+2, w(z+2))}{(w(z+2)-b_1(z+2))(w(z+2)-b_2(z+2))\widetilde{Q}(z+2, w(z+2))}. \end{aligned} \tag{3.13}$$

Now, if $p>1$ then w must be a pole of order at least p at $\hat{z}+3$. Hence, in this case, we can pair up the zero of $w-b_1$ at $z=\hat{z}$ together with the pole of w at $\hat{z}+1$ without the possibility of a similar sequence of iterates starting from another point, say $z=\hat{z}+3$, and resulting in pairing the pole at $\hat{z}+1$ with another zero of $w-b_1$, or of $w-b_2$. Therefore, we have found a pole of order at least p which can be uniquely associated with the zero of $w-b_1$ at $\hat{z}$. If, on the other hand, $p=1$ it may in principle be possible that there is another zero of $w-b_1$ or of $w-b_2$ at $z=\hat{z}+3$ which needs to be paired with the pole of w at $z=\hat{z}+2$. But since now all of the poles in the iteration are simple, we may still pair up the zero of $w-b_1$ at $z=\hat{z}$ and the pole of w at $z=\hat{z}+1$. If there is another zero of, say, $w-b_1$ at $z=\hat{z}+3$ such that $w(\hat{z}+4)$ is finite, we can pair it up with the pole of w at $z=\hat{z}+2$. Thus for any $p\geq 1$ there is a pole of multiplicity at least p which can be paired up with the zero of $w-b_1$ at $z=\hat{z}$.

We can repeat the argument above for zeros of $w-b_2$ in a completely analogous fashion without any possible overlap in the pairing of poles with the zeros of $w-b_1$ and $w-b_2$. By considering all generic zeros of $w(z)-b_1(z)$, and similarly for $w(z)-b_2(z)$, it follows that

$$n\left(r, \frac{1}{w-b_1}\right) + n\left(r, \frac{1}{w-b_2}\right) \leq n(r+1, w) + O(1). \tag{3.14}$$

Therefore the remaining condition (2.2) of Lemma 2.1 is satisfied, and so w must be of hyper-order at least one.

We consider now the case where the opposite inequality to (3.12) holds, i.e.,

$$\deg_w(P) > \deg_w(\widetilde{Q}) + 2.$$

If $\deg_w(P)=3$, it immediately follows that $\deg_w(Q)=2$, and so the first part of assertion (1.4) holds in this case. Now assume that

$$4 = \deg_w(P) > \deg_w(\widetilde{Q}) + 2 = 2 \tag{3.15}$$

and suppose that $\hat{z}$ is a generic zero of $w(z)-b_1(z)$ of order p. Then again, by (3.10), either $w(z+1)$ or $w(z-1)$ must have a pole at $z=\hat{z}$ of order at least p, and we suppose as above that $w(z+1)$ has the pole at $\hat{z}$. Then, it follows that $w(z+2)$ has a pole of order

$2p$, and $w(z+3)$ a pole of order $4p$ at $z=\hat{z}$. Hence we can pair the zero of $w-b_1$ at $z=\hat{z}$ and the pole of w at $z=\hat{z}+1$ the same way as in the case (3.12). Identical reasoning holds also for the zeros of $w-b_2$, and so (3.14) holds. Lemma 2.1 therefore yields that w is of hyper-order at least one.

Suppose then that

$$4=\deg_w(P)>\deg_w(\widetilde{Q})+2=3, \tag{3.16}$$

and that $\hat{z}$ is a generic zero of order p of $w(z)-b_1(z)$. Since now $\deg_w(\widetilde{Q})=1$, we have

$$\widetilde{Q}(z,w(z))=w(z)-b_3(z),$$

where $b_3(z)\not\equiv b_j(z)$ for $j\in\{1,2\}$ is a rational function of z. Also, it follows by an assumption of the theorem that $b_3\not\equiv 0$. As before, we see from (3.10) that either $w(z+1)$ or $w(z-1)$ has a pole of order at least p at $z=\hat{z}$, and we may again suppose that $w(z+1)$ has that pole. If $p>1$ then (3.11) implies that $w(z+2)$ has a pole of order at least p at $z=\hat{z}$. Even if $w-b_j$ has a zero at $z=\hat{z}+3$ for some $j\in\{1,2,3\}$, we have found at least one pole for each zero of $w-b_j$ in this iteration sequence, taking multiplicities into account. Hence we can pair the zero of $w-b_1$ at $z=\hat{z}$ and the pole of w at $z=\hat{z}+1$ the same way as in cases (3.12) and (3.15). However, if $p=1$ it may in principle be possible that the pole of the right hand side of (3.11) at $z=\hat{z}$ cancels with the pole of the term

$$a(z+1)\frac{w'(z+1)}{w(z+1)}$$

at $z=\hat{z}$ in such a way that $w(\hat{z}+2)$ remains finite. If $w(\hat{z}+2)\neq b_j(\hat{z}+2)$ for $j\in\{1,2,3\}$, then it follows from (3.13) that $w(z+3)$ has a pole at $z=\hat{z}$, and we can pair up the zero of $w-b_1$ at $z=\hat{z}$ and the pole of w at $z=\hat{z}+1$. If $w(\hat{z}+2)=b_j(\hat{z}+2)$ for some $j\in\{1,2,3\}$, it may happen that also $w(\hat{z}+3)$ stays finite. If all points $\hat{z}$ such that $w(\hat{z})=b_j(\hat{z})$ are a part of an iteration sequence of this form, i.e., that

$$w(\hat{z})=b_{j_1}(\hat{z}),\qquad w(\hat{z}+1)=\infty,\qquad w(\hat{z})=b_{j_2}(\hat{z}),\qquad j_1,j_2\in\{1,2,3\},$$

then by considering the multiplicities of all zeros of $w-b_j$, $j\in\{1,2,3\}$, we have the inequality

$$n\left(r,\frac{1}{w-b_1}\right)+n\left(r,\frac{1}{w-b_2}\right)+n\left(r,\frac{1}{w-b_3}\right)\leq 2n(r+1,w)+O(1).$$

As this is the "worst case scenario", this estimate remains true in general. Also, we have already noted that neither b_1, nor b_2 satisfy the equation (3.10). The same is true also for b_3, and so all conditions of Lemma 2.1 are satisfied. Hence the hyper-order of w is at least one also in the case (3.16).

4 The proof of Theorem 1.2

Let $z=\hat{z}$ be a generic zero of $w(z)$, then by (1.5) there is a pole of $w(z)$ at $z=\hat{z}+1$ or at $z=\hat{z}-1$ (or at both points). We need to consider two cases. Suppose first that there is a pole of $w(z)$ at both points $z=\hat{z}-1$ and $z=\hat{z}+1$. Then, from (1.5) it follows that there are poles of $w(z)$ at $z=\hat{z}-2$ and $z=\hat{z}+2$. Now, at least in principle we may have $w(\hat{z}-3)=0=w(\hat{z}+3)$. Hence, in this case we can find at least four poles of $w(z)$ (ignoring multiplicity) which correspond to three zeros (also ignoring multiplicity) of $w(z)$ and to no other zeros.

Assume now that there is a pole of $w(z)$ at only one of the points $z = \hat{z}+1$ and $z = \hat{z}-1$. Without loss of generality we can then suppose that $w(z)$ has a pole at $z = \hat{z}+1$ (the case where the pole is at $z = \hat{z}-1$ is completely analogous). We will begin by showing that we need only consider simple generic zeros of $w(z)$. Let $N_1(r, 1/w)$ denote the integrated counting function for the simple zeros of w and let $N_{[p}(r, 1/w)$ be the counting function for the zeros of w, which are of order p or higher. Then $N(r, 1/w) = N_1(r, 1/w) + N_{[2}(r, 1/w)$ and

$$\begin{aligned}\overline{N}\left(r, \frac{1}{w}\right) &= N_1\left(r, \frac{1}{w}\right) + \overline{N}_{[2}\left(r, \frac{1}{w}\right)\\ &\leq N_1\left(r, \frac{1}{w}\right) + \frac{1}{2}N_{[2}\left(r, \frac{1}{w}\right)\\ &\leq \frac{1}{2}N_1\left(r, \frac{1}{w}\right) + \frac{1}{2}N\left(r, \frac{1}{w}\right).\end{aligned}$$

Hence, using the assumption (1.6),

$$\begin{aligned}N_1\left(r, \frac{1}{w}\right) &\geq 2\overline{N}\left(r, \frac{1}{w}\right) - N\left(r, \frac{1}{w}\right)\\ &\geq \left(\frac{3}{2} + \epsilon\right) T(r, w) - N\left(r, \frac{1}{w}\right)\\ &\geq \left(\frac{1}{2} + \epsilon\right) T(r, w) + S(r, w).\end{aligned}$$

Thus there are at least "$(1/2 + \epsilon)T(r, w)$" worth of simple poles of w. So if we consider the case in which the zero of w at $\hat{z}$ is simple, we have

$$\begin{aligned}&w(z-1) = K + O(z-\hat{z}), \qquad K \in \mathbb{C}, \qquad \alpha \in \mathbb{C}\setminus\{0\}\\ &w(z) = \alpha(z-\hat{z}) + O((z-\hat{z})^2),\\ &w(z+1) = -\frac{a(z)}{z-\hat{z}} + O(1),\\ &w(z+2) = \frac{a(z+1)}{z-\hat{z}} + O(1),\\ &w(z+3) = \frac{a(z+2)-a(z)}{z-\hat{z}} + O(1)\end{aligned} \tag{4.1}$$

in a neighborhood of $\hat{z}$.

If $a(\hat{z}+2) - a(\hat{z}) \neq 0$ then $w(z+4) = (a(z+3) + a(z+1))/(z-\hat{z}) + O(z-\hat{z})$. Therefore either we have infinitely many points such that $a(z+2) = a(z)$ and therefore the rational function a is a constant, or we can find at least four poles for every two simple zeros of w. In the second case it follows that

$$\begin{aligned}T(r, w) &\leq \frac{1}{\frac{1}{2}+\epsilon}N_1\left(r, \frac{1}{w}\right) + O(\log r)\\ &\leq \frac{2}{1+2\epsilon}\frac{1}{2}N(r+2, w) + O(\log r)\\ &\leq \frac{1}{1+2\epsilon}T(r, w) + S(r, w).\end{aligned}$$

But this implies that $T(r, w) = S(r, w)$, which is a contradiction. Thus $a(z)$ must be a constant.

5 The proof of Theorem 1.3

Let $\hat{z}$ be a generic zero of $w(z)$ of order p. We need to consider two cases. Suppose first that there is a pole of $w(z)$ at both points $z = \hat{z} - 1$ and $z = \hat{z} + 1$. Then, even if there are zeros of $w(z)$ at both $z = \hat{z} - 2$ and $z = \hat{z} + 2$, we can group together three zeros of w (ignoring multiplicity) with at least four poles of w (counting multiplicity).

Assume now that there is a pole of $w(z)$ at only one of the points $z = \hat{z}+1$ and $z = \hat{z}-1$. Without loss of generality we can then suppose that $w(z)$ has a pole at $z = \hat{z}+1$ (the case where the pole is at $z = \hat{z} - 1$ is completely analogous). Consider first the case where the zero is simple, and suppose that $c(z) \not\equiv 0$. Then, in a neighborhood of $\hat{z}$,

$$\begin{aligned}
w(z-1) &= K + O(z-\hat{z}), \qquad K \in \mathbb{C},\\
w(z) &= \alpha(z-\hat{z}) + O((z-\hat{z})^2), \qquad \alpha \in \mathbb{C}\setminus\{0\}\\
w(z+1) &= \frac{a(z)}{\alpha(z-\hat{z})^2} + \frac{b(z)}{\alpha(z-\hat{z})} + c(z) + K + O(z-\hat{z}),\\
w(z+2) &= c(z+1) + O(z-\hat{z}),\\
w(z+3) &= \frac{a(z)}{\alpha(z-\hat{z})^2} + \frac{b(z)}{\alpha(z-\hat{z})} + O(1),
\end{aligned} \tag{5.1}$$

where there can be at most finitely many $\hat{z}$ such that $c(\hat{z}+1) = 0$. Hence there are two poles of $w(z)$ (counting multiplicity) corresponding to one zero (ignoring multiplicity) in this case.

Assume now that $c(z) \equiv 0$, $w(z)$ has a pole at $z = \hat{z} + 1$, and that $w(\hat{z} - 1)$ is finite. Then, in a neighborhood of $\hat{z}$,

$$\begin{aligned}
w(z-1) &= K + O(z-\hat{z}), \qquad K \in \mathbb{C},\\
w(z) &= \alpha(z-\hat{z}) + O((z-\hat{z})^2), \qquad \alpha \in \mathbb{C}\setminus\{0\}\\
w(z+1) &= \frac{a(z)}{\alpha(z-\hat{z})^2} + \frac{b(z)}{\alpha(z-\hat{z})} + O(1),\\
w(z+2) &= \alpha\left(1 - \frac{2a(z+1)}{a(z)}\right)(z-\hat{z}) + O((z-\hat{z})^2),\\
w(z+3) &= \frac{a(z)(a(z+2)-2a(z+1)+a(z))}{(a(z)-2a(z+1))\alpha(z-\hat{z})^2} + \frac{\gamma(z)}{\alpha(z-\hat{z})} + O(1),
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
\gamma(z) =& \frac{a(z)b(z+2) - (2a(z+1)-a(z))b(z)}{a(z)-2a(z+1)}\\
&- \frac{2a(z+2)[a(z)a'(z+1) - a(z+1)a'(z)]}{(a(z)-2a(z+1))^2}.
\end{aligned} \tag{5.3}$$

If $w(\hat{z}+3)$ is a pole of order two, then are at least four poles (counting multiplicities) in this sequence that can be uniquely grouped with the two zeros of $w(z)$ (ignoring multiplicities). The only way that $w(z)$ can have a simple pole at $z = \hat{z} + 3$ is that

$$a(\hat{z}+2) - 2a(\hat{z}+1) + a(\hat{z}) = 0 \tag{5.4}$$

and $\gamma(z) \not\equiv 0$. But in this case from equation (1.7) it follows that

$$w(z+4) = -\frac{\alpha a(z+3)}{\gamma(z)} + O(z-\hat{z})$$

for all z in a neighborhood of $\hat{z}$, and so $w(\hat{z}+4)$ is finite and non-zero with at most finitely many exceptions. Thus we can group together three poles of $w(z)$ (counting multiplicities) and two zeros of $w(z)$ (ignoring multiplicities). The only way that $w(\hat{z}+3)$ can be finite is that (5.4) holds together with $\gamma \equiv 0$.

If the order of the zero of $w(z)$ at $z = \hat{z}$ is $p \geq 2$, then there are always at least three poles of $w(z)$ (counting multiplicity) for each two zeros of $w(z)$ (ignoring multiplicity) in sequence (5.1) and (5.2).

If there are only finitely many zeros $\hat{z}$ of $w(z)$ such that (5.4) and $\gamma(z) \equiv 0$ both hold, then

$$\overline{n}\left(r, \frac{1}{w}\right) \leq \frac{3}{4} n(r+1, w) + O(1).$$

Hence, for any $\varepsilon > 0$,

$$\overline{N}\left(r, \frac{1}{w}\right) \leq \left(\frac{3}{4} + \frac{\varepsilon}{2}\right) N(r+1, w) + O(\log r),$$

and so by using [4, Lemma 8.3] to deduce that $N(r+1, w) = N(r, w) + S(r, w)$, we have

$$\overline{N}\left(r, \frac{1}{w}\right) \leq \left(\frac{3}{4} + \frac{\varepsilon}{2}\right) T(r, w) + S(r, w).$$

This is in contradiction with (1.8), and so there must be infinitely many points $\hat{z}$ such that (5.4) and $\gamma(z) \equiv 0$ are both satisfied. The only rational functions $a(z)$ satisfying (5.4) at infinitely many points have the form $a(z) = \lambda + \mu z$, for some constants λ and μ. Equation $\gamma(z) \equiv 0$ becomes

$$\frac{b(z+2)}{a(z+2)} - \frac{b(z)}{a(z)} = 2\frac{a(z+1)a'(z) - a(z)a'(z+1)}{a(z)a(z+2)} = \mu\left(\frac{1}{a(z)} - \frac{1}{a(z+2)}\right).$$

Hence $b(z) = ka(z) - \mu$, where k is a constant (since a and b are assumed to be rational).

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