A combinatorial criterion for k-separability of multipartite Dicke states

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We derive a combinatorial criterion for detecting k-separability of N-partite Dicke states. The criterion is efficiently computable and implementable without full state tomography. We give examples in which the criterion succeeds, where known criteria fail.

I. INTRODUCTION

While the structure of bipartite entangled states is fairly well-understood, the study of multipartite entanglement still presents a number of partial successes and difficult open problems (see the recent reviews [1, 2]). General criteria for multipartite states of any dimension were recently proposed in [3–11, 13].

Here, we derive a criterion for detecting k-nonseparability in Dicke states based on various ideas developed in [3–6, 8, 9, 12]. The criterion can be seen as a generalization of a method for detecting genuine multipartite entanglement in Dicke states detailed in [7]. The criterion have the advantages of being computationally efficient and implementable without the need of state tomography. We give examples in which the criterion is stronger than the ones proposed in [13].

Let us recall some standard terminology and the definition of a Dicke state. An N-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ (dim $\mathcal{H}_i = d_i \geq 2, 1 \leq i \leq N$) is said to be *$k$-separable* if there is a k-partition [8, 9]

$$j_1^1 \cdots j_{m_1}^1 | j_1^2 \cdots j_{m_2}^2 | \cdots | j_1^k \cdots j_{m_k}^k \text{ such that } |\psi\rangle = |\psi_1\rangle_{j_1^1 \cdots j_{m_1}^1} |\psi_2\rangle_{j_1^2 \cdots j_{m_2}^2} \cdots |\psi_k\rangle_{j_1^k \cdots j_{m_k}^k},$$

where $|\psi_i\rangle_{j_1^i \cdots j_{m_i}^i}$ is the state of the subsystems $j_1^i, j_2^i, ..., j_{m_i}^i$, and $\bigcup_{i=1}^k \{j_1^i, j_2^i, \cdots, j_{m_i}^i\} = \{1, 2, \cdots, N\}$. More generally, an N-partite mixed state ρ is said to be *k-separable* if it can be written as a convex combination of k-separable pure states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle$ is possibly k-separable under different partitions. An n-partite state is said to be *fully separable* when it is N-separable and *N-partite entangled* if it is not 2-separable. A k-separable mixed state might not be separable with regard to any specific k-partition, which makes k-separability difficult to deal with. We shall consider pure states as a special case.

The *N-qubits Dicke state with m excitations* (see [14]) is defined as

$$|D_m^N\rangle = \frac{1}{\sqrt{C_N^m}} \sum_{1 \leq i_j \neq i_l \leq N} |\phi_{i_1,\ldots,i_m}\rangle, \text{ where } |\phi_{i_1,\ldots,i_m}\rangle = \bigotimes_{i \notin \{i_1,\ldots,i_m\}} |0\rangle_i \bigotimes_{i \in \{i_1,\ldots,i_m\}} |1\rangle_i,$$

where $C_N^m := \binom{N}{m}$ is the binomial coefficient. For instance,

$$|D_2^4\rangle = 6^{-1/2} \left(|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0011\rangle + |0101\rangle\right),$$

when $N = 4$ and $m = 2$.

Section II contains the statements and proofs of the results. Examples are in Section III.

II. RESULTS

We construct a set of inequalities which are optimally suited for testing whether a given Dicke state is N-partite entangled:

Theorem 1 *Suppose that ρ is an N-partite density matrix acting on a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$. Let*

$$\mathcal{F}(\rho, \phi) := A(\rho, \phi) - B(\rho, \phi),$$

where

$$A(\rho,\phi) := \sum_{1\le i\neq j\neq j'\le N}\left(|\langle\phi_{i,j}|\rho|\phi_{i,j'}\rangle| - \sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j'}|\Pi_j\rho^{\otimes 2}\Pi_j|\phi_{i,j}\rangle\otimes|\phi_{i,j'}\rangle}\right),$$

and

$$B(\rho,\phi) := N_k\sum_{1\le i\neq j\le N}\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle.$$

Here, $|\phi_{i,j}\rangle := |0...010...010...0\rangle \in \mathcal{H}$*, with the* 1*s in the subspaces* $\mathcal{H}_i$ *and* $\mathcal{H}_j$*,* $N_k := \max\{2(N-k-1), N-k\}$ *and* Π_j*, is the operator swapping the two copies of* $\mathcal{H}_j$ *in* $\mathcal{H}\otimes\mathcal{H}$*, for* $1\le i\neq j\le N$*. If the density matrix* ρ *is* k*-separable then*

$$\mathcal{F}(\rho,\phi)\le 0.$$

Proof. We start with a 4-qubit state to get an intuition. Note that for a four-qubit pure state $\rho=|\psi\rangle\langle\psi|$, we have

$$\begin{aligned}\mathcal{F}(\rho,\phi) = 2(&|\rho_{4,6}| - \sqrt{\rho_{2,2}\rho_{8,8}} + |\rho_{4,7}| - \sqrt{\rho_{3,3}\rho_{8,8}} + |\rho_{4,10}| - \sqrt{\rho_{2,2}\rho_{12,12}} + |\rho_{4,11}| - \sqrt{\rho_{3,3}\rho_{12,12}}\\ &+ |\rho_{6,7}| - \sqrt{\rho_{5,5}\rho_{8,8}} + |\rho_{6,10}| - \sqrt{\rho_{2,2}\rho_{14,14}} + |\rho_{6,13}| - \sqrt{\rho_{5,5}\rho_{14,14}} + |\rho_{7,11}| - \sqrt{\rho_{3,3}\rho_{15,15}}\\ &+ |\rho_{7,13}| - \sqrt{\rho_{5,5}\rho_{15,15}} + |\rho_{10,11}| - \sqrt{\rho_{9,9}\rho_{12,12}} + |\rho_{10,13}| - \sqrt{\rho_{9,9}\rho_{14,14}}\\ &+ |\rho_{11,13}| - \sqrt{\rho_{9,9}\rho_{15,15}}) - N_k(\rho_{4,4}+\rho_{6,6}+\rho_{7,7}+\rho_{10,10}+\rho_{11,11}+\rho_{13,13}).\end{aligned} \tag{1}$$

If a 4-qubit pure state $\rho=|\psi\rangle\langle\psi|$ is biseparable then $\mathcal{F}(\rho,\phi)\le 0$ by the criterion in [15].

Suppose that a 4-qubit pure state $\rho=|\psi\rangle\langle\psi|$ with $|\psi\rangle = \sum_{i_1i_2i_3i_4}\psi_{i_1i_2i_3i_4}|i_1i_2i_3i_4\rangle$ $(i_1,i_2,i_3,i_4=0,1)$ is k-separable, where $k=3,4$. Then,

$$\begin{aligned}\mathcal{F}(|\psi\rangle,\phi) = 2(&|\psi_{0011}\psi_{0101}| - |\psi_{0001}\psi_{0111}| + |\psi_{0011}\psi_{0110}| - |\psi_{0010}\psi_{0111}| + |\psi_{0011}\psi_{1001}| - |\psi_{0001}\psi_{1011}|\\ &+ |\psi_{0011}\psi_{1010}| - |\psi_{0010}\psi_{1011}| + |\psi_{0101}\psi_{0110}| - |\psi_{0100}\psi_{0111}| + |\psi_{0101}\psi_{1001}| - |\psi_{0001}\psi_{1101}|\\ &+ |\psi_{0101}\psi_{1100}| - |\psi_{0100}\psi_{1101}| + |\psi_{0110}\psi_{1010}| - |\psi_{0010}\psi_{1110}| + |\psi_{0110}\psi_{1100}| - |\psi_{0100}\psi_{1110}|\\ &+ \psi_{1001}\psi_{1010}| - |\psi_{1011}\psi_{1000}| + |\psi_{1001}\psi_{1100}| - |\psi_{1000}\psi_{1101}| + |\psi_{1010}\psi_{1100}| - |\psi_{1000}\psi_{1110}|)\\ &- N_k(|\psi_{0011}|^2 + |\psi_{0101}|^2 + |\psi_{0110}|^2 + |\psi_{1001}|^2 + |\psi_{1010}|^2 + |\psi_{1100}|^2).\end{aligned}$$

Note that there are six 3-partitions 1|2|34, 1|3|24, 1|4|23, 2|3|14, 2|4|13, and 3|4|12. WLOG we prove that $\mathcal{F}(|\psi\rangle,\phi)\le 0$ holds for a 4-qubit pure state $\rho=|\psi\rangle\langle\psi|$ which is 3-separable under the partition 1|2|34. Suppose that

$$\begin{aligned}|\psi\rangle &= (a_1|0\rangle + a_2|1\rangle)_1\otimes(b_1|0\rangle + b_2|1\rangle)_2\otimes(c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle)_{34}\\ &= a_1b_1c_1|0000\rangle + a_1b_1c_2|0001\rangle + a_1b_1c_3|0010\rangle + a_1b_1c_4|0011\rangle + a_1b_2c_1|0100\rangle + a_1b_2c_2|0101\rangle\\ &+ a_1b_2c_3|0110\rangle + a_1b_2c_4|0111\rangle + a_2b_1c_1|1000\rangle + a_2b_1c_2|1001\rangle + a_2b_1c_3|1010\rangle + a_2b_1c_4|1011\rangle\\ &+ a_2b_2c_1|1100\rangle + a_2b_2c_2|1101\rangle + a_2b_2c_3|1110\rangle + a_2b_2c_4|1111\rangle,\end{aligned}$$

then

$$\begin{aligned}A_1 &= 2(|\psi_{0011}\psi_{0101}| - |\psi_{0001}\psi_{0111}| + |\psi_{0011}\psi_{0110}| - |\psi_{0010}\psi_{0111}| + |\psi_{0011}\psi_{1001}| - |\psi_{0001}\psi_{1011}|\\ &+ |\psi_{0011}\psi_{1010}| - |\psi_{0010}\psi_{1011}| + |\psi_{0101}\psi_{1001}| - |\psi_{0001}\psi_{1101}| + |\psi_{0101}\psi_{1100}| - |\psi_{0100}\psi_{1101}| + |\psi_{0110}\psi_{1010}|\\ &- |\psi_{0010}\psi_{1110}| + |\psi_{0110}\psi_{1100}| - |\psi_{0100}\psi_{1110}| + |\psi_{1001}\psi_{1100}| - |\psi_{1000}\psi_{1101}| + |\psi_{1010}\psi_{1100}| - |\psi_{1000}\psi_{1110}|)\\ &= 0;\end{aligned}$$

and

$$\begin{aligned}A_2 &= 2(|\psi_{0101}\psi_{0110}| - |\psi_{0100}\psi_{0111}| + \psi_{1001}\psi_{1010}| - |\psi_{1011}\psi_{1000}|)\\ &- (|\psi_{0011}|^2 + |\psi_{0101}|^2 + |\psi_{0110}|^2 + |\psi_{1001}|^2 + |\psi_{1010}|^2 + |\psi_{1100}|^2)\\ &\le 0.\end{aligned}$$

It follows that $\mathcal{F}(|\psi\rangle,\phi) = A_1 + A_2 \le 0.$ if $|\psi\rangle$ is 3-separable.

If $|\psi\rangle$ is fully separable, then

$$\begin{aligned}|\psi\rangle &= (a_1|0\rangle + a_2|1\rangle)_1 \otimes (b_1|0\rangle + b_2|1\rangle)_2 \otimes (c_1|0\rangle + c_2|1\rangle)_3 \otimes (d_1|0\rangle + d_2|1\rangle)_4 \\ &= a_1b_1c_1d_1|0000\rangle + a_1b_1c_1d_2|0001\rangle + a_1b_1c_2d_1|0010\rangle + a_1b_1c_2d_2|0011\rangle + a_1b_2c_1d_1|0100\rangle + a_1b_2c_1d_2|0101\rangle \\ &+ a_1b_2c_2d_1|0110\rangle + a_2b_2c_2d_2|0111\rangle + a_2b_1c_1d_1|1000\rangle + a_2b_2c_1d_2|1001\rangle + a_2b_1c_2d_1|1010\rangle + a_2b_1c_2d_2|1011\rangle \\ &+ a_2b_2c_1d_1|1100\rangle + a_2b_2c_1d_2|1101\rangle + a_2b_2c_2d_1|1110\rangle + a_2b_2c_2d_2|1111\rangle,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(|\psi\rangle, \phi) &= 2(|\psi_{0011}\psi_{0101}| - |\psi_{0001}\psi_{0111}| + |\psi_{0011}\psi_{0110}| - |\psi_{0010}\psi_{0111}| + |\psi_{0011}\psi_{1001}| - |\psi_{0001}\psi_{1011}| \\ &+ |\psi_{0011}\psi_{1010}| - |\psi_{0010}\psi_{1011}| + |\psi_{0101}\psi_{0110}| - |\psi_{0100}\psi_{0111}| + |\psi_{0101}\psi_{1001}| - |\psi_{0001}\psi_{1101}| \\ &+ |\psi_{0101}\psi_{1100}| - |\psi_{0100}\psi_{1101}| + |\psi_{0110}\psi_{1010}| - |\psi_{0010}\psi_{1110}| + |\psi_{0110}\psi_{1100}| - |\psi_{0100}\psi_{1110}| \\ &+ |\psi_{1001}\psi_{1010}| - |\psi_{1000}\psi_{1011}| + |\psi_{1001}\psi_{1100}| - |\psi_{1000}\psi_{1101}| + |\psi_{1010}\psi_{1100}| - |\psi_{1000}\psi_{1110}|) \\ &= 0.\end{aligned}$$

The equalities above confirms the statement in Eq.(1), when restricted to 4-qubit pure states.

For the general case, we use the notation and proof method given in [5, 9].

Suppose that $\rho = |\psi\rangle\langle\psi|$ is a k-separable pure state under the partition of $\{1, 2, \cdots, N\}$ into k pairwise disjoint subsets: $\{1, 2, \cdots, N\} = \bigcup_{l=1}^{k} A_l$, with $A_l = \{j_1^l, j_2^l, \cdots, j_{m_l}^l\}$ and

$$\begin{aligned}|\psi\rangle &= |\psi_1\rangle_{j_1^1\cdots j_{m_1}^1} \cdots |\psi_k\rangle_{j_1^k\cdots j_{m_k}^k} \\ &= \left(\sum_{i_1^1,\cdots,i_{m_1}^1} a_{i_1^1\cdots i_{m_1}^1} |i_1^1 \cdots i_{m_1}^1\rangle\right)_{j_1^1\cdots j_{m_1}^1} \cdots \left(\sum_{i_1^k,\cdots,i_{m_k}^k} a_{i_1^k\cdots i_{m_k}^k} |i_1^k \cdots i_{m_k}^k\rangle\right)_{j_1^k\cdots j_{m_k}^k} \\ &\sum_{i_1^1,\cdots,i_{m_1}^1,\cdots,i_1^k,\cdots,i_{m_k}^k} a_{i_1^1\cdots i_{m_1}^1} \cdots a_{i_1^k\cdots i_{m_k}^k} |i_1^1 \cdots i_{m_1}^1 \cdots i_1^k \cdots i_{m_k}^k\rangle_{j_1^1\cdots j_{m_1}^1\cdots j_1^k\cdots j_{m_k}^k}.\end{aligned}$$

Hence,

$$\rho_{\sum\limits_{s,t} i_t^s d_{j_t^s+1} d_{j_t^s+2}\cdots d_N d_{N+1}+1, \sum\limits_{s,t} \tilde{i}_t^s d_{j_t^s+1} d_{j_t^s+2}\cdots d_N d_{N+1}+1} = a_{i_1^1\cdots i_{m_1}^1} \cdots a_{i_1^k\cdots i_{m_k}^k} a^*_{\widetilde{i_1^1\cdots i_{m_1}^1}} \cdots a^*_{\widetilde{i_1^k\cdots i_{m_k}^k}}.$$

The sum is over all possible values of $\{i_t^s | s \in \{1, 2, \cdots, k\}, t \in \{1, 2, \cdots, m_s\}\}$, $d_i = 2$, when $i \neq N+1$ and $d_{N+1} = 1$.

We shall distinguish between the cases in which both indices j and j' correspond to different parts A_l and A_l', or the same parts A_l, $1 \leq l \neq l' \leq k$, with respect to $|\psi\rangle$. By direct calculation, one has the following:

$$|\langle\phi_{i,j}|\rho|\phi_{i,j'}\rangle| = \sqrt{\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle\langle\phi_{i,j'}|\rho|\phi_{i,j'}\rangle} \leq \frac{\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle + \langle\phi_{i,j'}|\rho|\phi_{i,j'}\rangle}{2}, \tag{2}$$

when j and j' are in the same part;

$$|\langle\phi_{i,j}|\rho|\phi_{i,j'}\rangle| = \sqrt{\langle\phi_i|\rho|\phi_i\rangle\langle\phi_{i,j,j'}|\rho|\phi_{i,j,j'}\rangle} = \sqrt{\langle\phi_{i,j}| \otimes \langle\phi_{i,j}|\Pi_{j'}^{+}\rho^{\otimes 2}\Pi_j|\phi_{i,j}\rangle \otimes |\phi_{i,j'}\rangle}, \tag{3}$$

when j and j' are in the different parts ($j \in A_l, j \in A_{l'}$ with $l \neq l'$). Here, $|\phi_i\rangle = |00\cdots010\cdots0\rangle$, with $|1\rangle$ in the i-th subspace H_i, and $|\phi_{i,j,j'}\rangle = |0\cdots010\cdots010\cdots010\cdots0\rangle$, such that all subspaces are in the state $|0\rangle$, except for the subspaces H_i, H_j and H_j', which are in the state $|1\rangle$.

For a given $|\phi_{i,j}\rangle$, the number of $|\phi_{i,j'}\rangle$'s, with j and j' in same part, is at most $\max\{2(N-k-1), N-k\}$. Notice that the maximal number of subsystems contained in a part of a k-partition is $N-k+1$. Suppose that $A_1|A_2|\cdots|A_k$ is a k-partition of $\{1, 2, \cdots, n\}$, where $A_l = \{j_1^l\}$, for $l = 1, 2, \cdots, k-1$, and $A_k = \{j_1^k, j_2^k, \cdots, j_{N-k+1}^k\}$. When i, j and j' are in the same part A_k, the number of $|\phi_{i,j'}\rangle$'s is $2(N-k-1)$. When i belongs to $A_1 \cup A_2 \cup \cdots \cup A_{k-1}$, while j and j' belong to A_k, the number of $|\phi_{i,j'}\rangle$'s is $N-k$. Therefore, the number of $|\phi_{i,j'}\rangle$'s satisfying j and j' in the same part is at most $\max\{2(N-k-1), N-k\}$. This number is denoted as N_k.

By using the inequalities in (2) and (3), we have

$$\sum_{1\leq i,j,j'\leq N}|\langle\phi_{i,j}|\rho|\phi_{i,j'}\rangle| = \sum_i \sum_{\substack{j\in A_l,j'\in A_{l'},l\neq l'\\ l,l'\in\{1,2,\cdots,k\}}}|\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle| + \sum_{\substack{j,j'\in A_l,j\neq j'\\ l\in\{1,2,\cdots,k\}}}|\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle|$$
$$\leq \sum_i \sum_{\substack{j\in A_l,j\in A_{l'},l\neq l'\\ l,l'\in\{1,2,\cdots,k\}}}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j'}|\Pi_j^{+}\rho^{\otimes 2}\Pi_j|\phi_{i,j}\rangle\otimes|\phi_{i,j'}\rangle}$$
$$+\sum_i \sum_{\substack{j,j'\in A_l,j\neq j'\\ l\in\{1,2,\cdots,k\}}}\left(\frac{\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle+\langle\phi_{i,j'}|\rho|\phi_{i,j'}\rangle}{2}\right)$$
$$\leq \sum_i\sum_{j\neq j'}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j'}|\Pi_j^{+}\rho^{\otimes 2}\Pi_j|\phi_{i,j}\rangle\otimes|\phi_{i,j'}\rangle}+N_k\sum_{i,j}\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle.$$

Thus, the inequality in the statement of the theorem is satisfied by all k-separable N-partite pure states.

It remains to show that the inequality holds if ρ is a k-separable N-partite mixed state. Indeed, the generalization of the inequality to mixed states is a direct consequence of the convexity of the first summation in $A(\rho,\phi)$, the concavity of $B(\rho,\phi)$, and the second summation in $A(\rho,\phi)$, which we can see as follows.

Suppose that

$$\rho=\sum_m p_m\rho_m=\sum_m p_m|\psi_m\rangle\langle\psi_m|$$

is a k-separable N-partite mixed state, where $\rho_m=|\psi_m\rangle\langle\psi_m|$ is k-separable. Then, by the Cauchy-Schwarz inequality, $(\sum_{k=1}^m x_k y_k)^2\leq(\sum_{k=1}^m x_k^2)(\sum_{k=1}^m y_k^2)$, we get

$$\sum_i\sum_{j\neq j'}|\langle\phi_{i,j}|\rho|\phi_{i,j'}\rangle|\leq\sum_i\sum_m p_m\sum_{j\neq j'}|\langle\phi_{i,j}|\rho_m|\phi_{i,j}\rangle|$$
$$\leq\sum_m p_m\left(\sum_i\sum_{j\neq j'}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j}|\Pi_j^{+}\rho_m^{\otimes 2}\Pi_j|\phi_{i,j'}\rangle\otimes|\phi_{i,j'}\rangle}\right.$$
$$\left.+N_k\sum_{i,j}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j}|\Pi_j^{+}\rho_m^{\otimes 2}\Pi_j|\phi_{i,j}\rangle\otimes|\phi_{i,j}\rangle}\right)$$
$$=\sum_i\sum_{j\neq j'}\sum_m\sqrt{\langle\phi_i|p_m\rho_m|\phi_i\rangle}\sqrt{\langle\phi_{i,j,j'}|p_m\rho_m|\phi_{i,j,j'}\rangle}+N_k\sum_{i,j}\sum_m p_m\langle\phi_{i,j}|\rho_m|\phi_{i,j}\rangle$$
$$\leq\sum_i\sum_{j\neq j'}\sqrt{\sum_m\langle\phi_i|p_m\rho_m|\phi_i\rangle\sum_m\langle\phi_{i,j,j'}|p_m\rho_m|\phi_{i,j,j'}\rangle}+N_k\sum_{i,j}\langle\phi_{i,j}|\rho|\phi_{i,j}\rangle$$
$$=\sum_i\sum_{j\neq j'}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j}|\Pi_j^{+}\rho^{\otimes 2}\Pi_j|\phi_{i,j'}\rangle\otimes|\phi_{i,j'}\rangle}+N_k\sum_{i,j}\sqrt{\langle\phi_{i,j}|\otimes\langle\phi_{i,j}|\Pi_j^{+}\rho^{\otimes 2}\Pi_j|\phi_{i,j}\rangle\otimes|\phi_{i,j}\rangle},$$

as desired. This completes the proof. ■

We can choose $|\phi\rangle$ *ad hoc* to get different inequalities for detecting k-separability of different classes. For Theorem 2, we have chosen $|\phi\rangle$ to be an N-qubit product states with m excitations (*i.e.* m entries of $|\phi\rangle$ are $|1\rangle$, while the remaining $N-m$ entries are $|0\rangle$). The criterion performs well to detect k-separability for N qubit Dicke states with m excitations mixed with white noises.

Theorem 2 *Suppose that ρ is an N-partite density matrix acting on Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\cdots\otimes\mathcal{H}_N$, and $|\phi_{i_1 i_2,\cdots,i_m}\rangle=|0\cdots010\cdots010\cdots010\cdots0\rangle$ is a state of $\mathcal{H}$, where the local state in $\mathcal{H}_l$ is $|0\rangle$, for $l\neq i_1,i_2,\cdots,i_m$, and $|1\rangle$, for $l=i_1,i_2,\cdots,i_m$. Let*

$$\mathcal{F}(\rho,\phi):=A(\rho,\phi)-B(\rho,\phi),$$

with

$$A(\rho,\phi) := \sum_{i_1,\cdots,i_j,\cdots,i_m,i'_j} \Big(|\langle \phi_{i_1\cdots,i_j,\cdots,i_m}|\rho|\phi_{i_1,\cdots,i'_j\cdots,i_m}\rangle| \\ - \sqrt{\langle \phi_{i_1,\cdots,i_j,\cdots,i_m}| \otimes \langle \phi_{i_1,\cdots,i'_j,\cdots,i_m}|\Pi_{i_j}\rho^{\otimes 2}\Pi_{i_j}|\phi_{i_1,\cdots,i_j,\cdots,i_m}\rangle \otimes |\phi_{i_1,\cdots,i'_j,\cdots,i_m}\rangle} \Big),$$

and

$$B(\rho,\phi) := N_k \sum_{i_1,i_2,\cdots,i_m} \langle \phi_{i_1,\cdots,i_j,\cdots,i_m}|\rho|\phi_{i_1,\cdots,i_j,\cdots,i_m}\rangle.$$

Here, Π_{i_j} *is the operator swapping the two copies of* $\mathcal{H}_{i_j}$ *in the twofold copy Hilbert space* $\mathcal{H}^{\otimes 2} := \mathcal{H}\otimes\mathcal{H}$*, and*

$$N_k := \max\{m(N-k+1-m), (m-1)(N-k-m+2), \cdots, (N-k)\}.$$

If the density matrix ρ *is* k*-separable then*

$$\mathcal{F}(\rho,\phi) \le 0.$$

In the following statement, we consider a criterion which is suitable for any general quantum states. The states to be chosen are $|\chi\rangle$, $|\chi_\alpha\rangle$ and $|\chi_\beta\rangle$.

Theorem 3 *Let* $V = \{|\chi_1\rangle, ..., |\chi_m\rangle\}$ *be a set of product states in* $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$*. If* ρ *is* k*-separable then*

$$\begin{aligned}\mathcal{T}(\rho,\chi) = &\sum_{|\chi_\alpha\rangle\in V}\sum_{|\chi_\beta\rangle\in K_\alpha} \Big(|\langle\chi_\alpha|\rho|\chi_\beta\rangle| - \sqrt{\langle\chi_\alpha|\otimes\langle\chi_\beta|\Pi_{\alpha\beta}\rho^{\otimes 2}\Pi_{\alpha\beta}|\chi_\alpha\rangle\otimes|\chi_\beta\rangle}\Big) - N_k\sum_\alpha \langle\chi_\alpha|\rho|\chi_\alpha\rangle \\ \le &\, 0,\end{aligned} \tag{4}$$

where

$$K_\alpha := \{|\chi_\beta\rangle : ||\chi_\alpha\rangle \cap |\chi_\beta\rangle| = N-2 \text{ with } |\chi_\alpha\rangle, |\chi_\beta\rangle \in V\},$$

and $||\chi_\alpha\rangle\cap|\chi_\beta\rangle|$ *is the number of coordinates that are equal in both vectors (*i.e.*,* $|\chi_\alpha\rangle$ *and* $|\chi_\beta\rangle$ *have only two different local states, say the* $i_{\alpha\beta}$*-th and* $i'_{\alpha\beta}$*-th local states), while* $\Pi_{\alpha\beta}$ *is the operator swapping the two copies of* $\mathcal{H}_{i_{\alpha\beta}}$ *in* $\mathcal{H}^{\otimes 2}$*. Additionally,*

$$N_k := \max_{\alpha,i_1,i_2,\cdots,i_{N-k+1}} s_{\alpha,i_1,i_2,\cdots,i_{N-k+1}},$$

where $s_{\alpha,i_1,i_2,\cdots,i_{N-k+1}}$ *is the number of states* $|\chi_\beta\rangle$ *in* K_α *such that two of the states for the* $N-k+1$ *particles* $i_1, i_2, \cdots, i_{N-k+1}$ *in* $|\chi_\beta\rangle$ *are different from that of* $|\chi_\alpha\rangle$*, when* $K_\alpha \ne \emptyset$*.*

Proof. By using the same proof method as in [5, 9], we prove that (4) holds for any k-separable pure states $\rho = |\psi\rangle\langle\psi|$. Let $\mathcal{T}(\rho,\chi) = A_1 + A_2$, where A_1 is the sum of terms $|\langle\chi_\alpha|\rho|\chi_\beta\rangle| - \sqrt{\langle\chi_\alpha|\otimes\langle\chi_\beta|\Pi_{\alpha\beta}\rho^{\otimes 2}\Pi_{\alpha\beta}|\chi_\alpha\rangle\otimes|\chi_\beta\rangle}$ in the first summation in (4). In this expression, the two different bits of $|\chi_\alpha\rangle$ and $|\chi_\beta\rangle$ are in two different parts of a k-partition, while A_2 is the sum containing the summands in (4), such that the different bits of $|\chi_\alpha\rangle$ and $|\chi_\beta\rangle$ are in the same part of a k-partition.

We first prove that $\mathcal{T}(\rho,\chi) \le 0$ for any 4-partite pure state. Let $V = \{|\chi_1\rangle, ..., |\chi_4\rangle\}$ be a set of product states in $\mathcal{H}$, where $|\chi_1\rangle = |0011\rangle$, $|\chi_2\rangle = |0101\rangle$, $|\chi_3\rangle = |0110\rangle$, and $|\chi_4\rangle = |1010\rangle$. Then $K_1 = \{|\chi_2\rangle, |\chi_3\rangle, |\chi_4\rangle\}$, $K_2 = \{|\chi_1\rangle, |\chi_3\rangle\}$, $K_3 = \{|\chi_1\rangle, |\chi_2\rangle, |\chi_4\rangle\}$, and $K_4 = \{|\chi_1\rangle, |\chi_3\rangle\}$. Thus,

$$\begin{aligned}\mathcal{T}(\rho,\chi) = &\, 2\Bigg(\sum_{i=2}^4 \Big(|\langle\chi_1|\rho|\chi_i\rangle| - \sqrt{\langle\chi_1|\otimes\langle\chi_i|\Pi_{12}\rho^{\otimes 2}\Pi_{12}|\chi_1\rangle\otimes|\chi_i\rangle}\Big) \\ &+ |\langle\chi_2|\rho|\chi_3\rangle| - \sqrt{\langle\chi_2|\otimes\langle\chi_3|\Pi_{23}\rho^{\otimes 2}\Pi_{23}|\chi_2\rangle\otimes|\chi_3\rangle} \\ &+ |\langle\chi_3|\rho|\chi_4\rangle| - \sqrt{\langle\chi_3|\otimes\langle\chi_4|\Pi_{34}\rho^{\otimes 2}\Pi_{34}|\chi_3\rangle\otimes|\chi_4\rangle}\Bigg) - N_k\sum_{i=1}^4\langle\chi_i|\rho|\chi_i\rangle \\ = &\, 2(|\phi_{0011}\phi_{0101}| - |\phi_{0001}\phi_{0111}| + |\phi_{0011}\phi_{0110}| - |\phi_{0010}\phi_{0111}| + |\phi_{0011}\phi_{1010}| - |\phi_{0010}\phi_{1011}| \\ &+ |\phi_{0110}\phi_{0101}| - |\phi_{0100}\phi_{0111}| + |\phi_{0110}\phi_{1010}| - |\phi_{0010}\phi_{1110}|) \\ &- N_k(|\phi_{0011}|^2 + |\phi_{0101}|^2 + |\phi_{0110}|^2 + |\phi_{1010}|^2) \\ = &\, A_1 + A_2;\end{aligned}$$

When $k=3$, there are six 3-partitions, *i.e.*, 1|2|34, 1|3|24, 1|4|23, 2|3|14, 2|4|13, and 3|4|12. For χ_1, we have $s_{1,34}=0$, $s_{1,24}=1$, $s_{1,23}=1$, $s_{1,14}=1$, $s_{1,13}=0$, and $s_{1,12}=0$; for χ_2, we have $s_{2,34}=1$, $s_{2,24}=0$, $s_{2,23}=1$, $s_{2,14}=0$, $s_{2,13}=0$, and $s_{2,12}=0$; for χ_3, we have $s_{3,34}=1$, $s_{3,24}=1$, $s_{3,23}=0$, $s_{3,14}=0$, $s_{3,13}=0$, and $s_{3,12}=1$; for χ_4, we have $s_{4,34}=0$, $s_{4,24}=0$, $s_{4,23}=0$, $s_{4,14}=1$, $s_{4,13}=0$, and $s_{4,12}=1$. So, we get $N_3=1$.

For the case 1|2|34:

$$\begin{aligned}A_1 &= 2\left[\sum_{i=2}^{4}\left(|\langle\chi_1|\rho|\chi_i\rangle| - \sqrt{\langle\chi_1|\otimes\langle\chi_i|\Pi_{12}\rho^{\otimes 2}\Pi_{12}|\chi_1\rangle\otimes|\chi_i\rangle}\right) + |\langle\chi_3|\rho|\chi_4\rangle| - \sqrt{\langle\chi_3|\otimes\langle\chi_4|\Pi_{34}\rho^{\otimes 2}\Pi_{34}|\chi_3\rangle\otimes|\chi_4\rangle}\right]\\ &= 2(|\phi_{0011}\phi_{0101}| - |\phi_{0001}\phi_{0111}| + |\phi_{0011}\phi_{0110}| - |\phi_{0010}\phi_{0111}| + |\phi_{0011}\phi_{1010}| - |\phi_{0010}\phi_{1011}| + |\phi_{0110}\phi_{1010}| - |\phi_{0010}\phi_{1110}|)\\ &= 0;\end{aligned}$$

$$\begin{aligned}A_2 &= 2\left(|\langle\chi_2|\rho|\chi_3\rangle| - \sqrt{\langle\chi_2|\otimes\langle\chi_3|\Pi_{23}\rho^{\otimes 2}\Pi_{23}|\chi_2\rangle\otimes|\chi_3\rangle}\right) - N_3(|\phi_{0011}|^2 + |\phi_{0101}|^2 + |\phi_{0110}|^2 + |\phi_{1010}|^2)\\ &= 2(|\phi_{0110}\phi_{0101}| - |\phi_{0100}\phi_{0111}|) - (|\phi_{0011}|^2 + |\phi_{0101}|^2 + |\phi_{0110}|^2 + |\phi_{1010}|^2)\\ &\le 0.\end{aligned}$$

This implies that $\mathcal{T}(\rho,\chi) = A_1 + A_2 \le 0$. For the other 3-partitions, we can get the same result $\mathcal{T}(\rho,\chi) = A_1 + A_2 \le 0$, as 1|2|34.

When $k=4$, there is a single 4-partition, 1|2|3|4. Then, it is not possible for any two different bits to be in the same partition. It follows that $N_4=0$ and $\mathcal{T}(\rho,\chi) = A_1 = 0$.

For a k-separable 4-partite mixed state $\rho = \sum_m p_m\rho_m$, where $\rho_m = |\psi_m\rangle\langle\psi_m|$ is k-separable, we have

$$\begin{aligned}\mathcal{T}(\rho,\chi) &= 2\left[\sum_{i=2}^{4}\left(|\langle\chi_1|\sum_m p_m\rho_m|\chi_i\rangle| - \sqrt{\langle\chi_1|\otimes\langle\chi_i|\Pi_{12}(\sum_m p_m\rho_m)^{\otimes 2}\Pi_{12}|\chi_1\rangle\otimes|\chi_i\rangle}\right)\right.\\ &\quad + |\langle\chi_2|\sum_m p_m\rho_m|\chi_3\rangle| - \sqrt{\langle\chi_2|\otimes\langle\chi_3|\Pi_{23}(\sum_m p_m\rho_m)^{\otimes 2}\Pi_{23}|\chi_2\rangle\otimes|\chi_3\rangle}\\ &\quad \left. + |\langle\chi_3|\sum_m p_m\rho_m|\chi_4\rangle| - \sqrt{\langle\chi_3|\otimes\langle\chi_4|\Pi_{34}(\sum_m p_m\rho_m)^{\otimes 2}\Pi_{34}|\chi_3\rangle\otimes|\chi_4\rangle}\right]\\ &\quad - N_k\sum_{i=1}^{4}\langle\chi_i|\sum_m p_m\rho_m|\chi_i\rangle\\ &\le \sum_m 2p_m\left[\sum_{i=2}^{4}\left(|\langle\chi_1|\rho_m|\chi_i\rangle| - \sqrt{\langle\chi_1|\otimes\langle\chi_i|\Pi_{12}(\rho_m)^{\otimes 2}\Pi_{12}|\chi_1\rangle\otimes|\chi_i\rangle}\right)\right.\\ &\quad \left. + |\langle\chi_2|\rho_m|\chi_3\rangle| - \sqrt{\langle\chi_2|\otimes\langle\chi_3|\Pi_{23}(\rho_m)^{\otimes 2}\Pi_{23}|\chi_2\rangle\otimes|\chi_3\rangle}\right]\\ &= \sum_m p_m\mathcal{T}(\rho_m,\chi)\\ &\le 0\end{aligned}$$

which implies that the inequality (4) holds for k-separable 4-partite states.

Notice that for any k-separable pure states $\rho = |\psi\rangle\langle\psi|$, if the two different bits of $|\chi_\alpha\rangle$ and $|\chi_\beta\rangle$ are in two different parts, then $|\langle\chi_\alpha|\rho|\chi_\beta\rangle| - \sqrt{\langle\chi_\alpha|\otimes\langle\chi_\beta|\Pi_{\alpha\beta}\rho^{\otimes 2}\Pi_{\alpha\beta}|\chi_\alpha\rangle\otimes|\chi_\beta\rangle} = 0$, otherwise $|\langle\chi_\alpha|\rho|\chi_\beta\rangle| - \frac{\langle\chi_\alpha|\rho|\chi_\alpha\rangle + \langle\chi_\beta|\rho|\chi_\beta\rangle}{2} \le 0$. This implies that inequality (4) holds for k-separable pure N-partite states ρ.

Suppose that $\rho = \sum\limits_m p_m\rho_m$ is a k-separable mixed N-partite state, where $\rho_m = |\psi_m\rangle\langle\psi_m|$ is k-separable. It follow that

$$\begin{aligned}\mathcal{T}(\rho,\chi) &= \sum_{|\chi_\alpha\rangle\in V}\sum_{|\chi_\beta\rangle\in K_\alpha}\left(|\langle\chi_\alpha|\sum_m p_m\rho_m|\chi_\beta\rangle| - \sqrt{\langle\chi_\alpha|\otimes\langle\chi_\beta|\Pi_{\alpha\beta}(\sum_m p_m\rho_m)^{\otimes 2}\Pi_{\alpha\beta}|\chi_1\rangle\otimes|\chi_i\rangle}\right)\\ &- N_k\sum_\alpha\langle\chi_\alpha|\sum_m p_m\rho_m|\chi_\alpha\rangle\\ &\le \sum_m p_m\left[\sum_{|\chi_\alpha\rangle\in V}\sum_{|\chi_\beta\rangle\in K_\alpha}\left(|\langle\chi_\alpha|\rho_m|\chi_\beta\rangle| - \sqrt{\langle\chi_\alpha|\otimes\langle\chi_\beta|\Pi_{\alpha\beta}(\rho_m)^{\otimes 2}\Pi_{\alpha\beta}|\chi_1\rangle\otimes|\chi_i\rangle}\right)\right.\\ &\left.- N_k\sum_\alpha\langle\chi_\alpha|\rho_m|\chi_\alpha\rangle\right]\\ &= \sum_m p_m\mathcal{T}(\rho_m,\chi),\\ &\le 0\end{aligned}$$

which completes the proof. ∎

III. EXAMPLES

Consider the family of N-qubit mixed states

$$\rho^{(D_2^N)} = a|D_2^N\rangle\langle D_2^N| + \tfrac{(1-a)I_N}{2^N},\ \text{where}\ |D_2^N\rangle = \tfrac{1}{\sqrt{C_N^2}}\sum_{1\le i\ne j\le N}|\phi_{i,j}\rangle.$$

By Theorem 1, if

$$a > \frac{2C_N^2(N-2)+N_kC_N^2}{2C_N^2(N-2)+N_kC_N^2-2^NN_k+2^{N+1}(N-2)}$$

then $\rho^{(D_2^N)}$ is k-nonseparable. Thus, if

$$a > \frac{C_N^2(2N-5)}{C_N^2(2N-5)+2^N}$$

then $\rho^{(D_2^N)}$ are genuine entangled, which is exactly the same as in [15]; if $a > \frac{9}{17}$ then $\rho^{(D_2^4)}$ is genuine entangled; if $a > \frac{5}{13}$ then $\rho^{(D_2^4)}$ is 3-nonseparable; if $a > \frac{3}{11}$ then $\rho^{(D_2^4)}$ is not fully-separable; if $a > \frac{5}{21} = 0.23$ then $\rho^{(D_2^5)}$ is not fully-separable. However, the inequality in [13] detects that, if $a > \frac{21}{29}$ then $\rho^{(D_2^4)}$ is genuine entangled; if $a > \frac{9}{13}$ then $\rho^{(D_2^4)}$ is 3-nonseparable; if $a > \frac{3}{11}$ then $\rho^{(D_2^4)}$ is not fully-separable; if $a > 0.27$ then $\rho^{(D_2^5)}$ is not a fully-separable 5-partite state.

Consider the N-qubit state

$$\rho^{(D_m^N)} = \tfrac{(1-a)I_N}{2^N} + a|D_m^N\rangle\langle D_m^N|,\ \text{where}\ |D_m^N\rangle = \tfrac{1}{\sqrt{C_N^m}}\sum_{1\le i_1\le i_2\le\cdots\le i_m\le N}|\phi_{i_1,i_2,\cdots,i_m}\rangle,$$

By Theorem 2, if

$$a > \frac{mC_N^m(N-m)+N_kC_N^m}{mC_N^m(N-m)+N_kC_N^m-2^NN_k+2^Nm(N-m)},$$

then $\rho^{(D_m^N)}$ is k-nonseparable. For $N=5$ and $m=3$, we get that if $a > \frac{5}{13}$ then $\rho^{(D_m^N)}$ is 3-nonseparable, while the method in [13] fails.

Consider the one-parameter four-qubit state

$$\rho = \tfrac{1-a}{16}I_{16} + a|\phi\rangle\langle\phi|,\ \text{where}\ |\phi\rangle = \tfrac{1}{2}(|0011\rangle + |0101\rangle + |0110\rangle + |1010\rangle).$$

By Theorem 3, if $a > \frac{7}{19}$ and $a > \frac{1}{5}$ then ρ is 3-nonseparable and not fully-separable, respectively, while in [13], if $a > \frac{9}{13}$ and $a > \frac{3}{11}$, then ρ is 3-nonseparable and not fully-separable.

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