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Equivalence between contextuality and negativity of the Wigner function for qudits

Nicolas Delfosse[1,2,7], **Cihan Okay**[3], **Juan Bermejo-Vega**[4], **Dan E Browne**[5] and **Robert Raussendorf**[6]

[1] IQIM, California Institute of Technology, Pasadena, CA, United States of America
[2] Department of Physics and Astronomy, University of California, Riverside, CA, United States of America
[3] Department of Mathematics, University of Western Ontario, London, Ontario, Canada
[4] Dahlem Center for Complex Quantum Systems, Freie Universität Berlin, Berlin, Germany
[5] Department of Physics and Astronomy, University College London, Gower Street, London, United Kingdom
[6] Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada
[7] Author to whom any correspondence should be addressed.

E-mail: ndelfoss@caltech.edu

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Abstract

Understanding what distinguishes quantum mechanics from classical mechanics is crucial for quantum information processing applications. In this work, we consider two notions of non-classicality for quantum systems, negativity of the Wigner function and contextuality for Pauli measurements. We prove that these two notions are equivalent for multi-qudit systems with odd local dimension. For a single qudit, the equivalence breaks down. We show that there exist single qudit states that admit a non-contextual hidden variable model description and whose Wigner functions are negative.

1. Introduction

Understanding what distinguishes quantum mechanics from classical mechanics and probabilistic models is a central question of physics. Besides its foundamental aspects, this question is crucial for quantum information processing applications since the features that set quantum and classical mechanics appart are precisely the properties that we must exploit in order to obtain a quantum superiority for certain tasks [1–20]. In the present work, we compare two notions of non-classicality: contextuality [21–27] and negativity of the Wigner function [28–30].

The Wigner function shares several properties of probability distributions with the difference that it can take negative values. This phenomenon is generally considered as an indicator of non-classicality of quantum states [31–36] (see also the discussion in [37]).

The ressemblance between contextuality and negativity was exploited by Spekkens who generalized these two notions in order to prove that they coincide [38]. However, this result remains difficult to apply, since a large number of Wigner functions must be probed to identify contextuality. Howard *et al* [18] showed that, if one restricts to a particular class of measurements, namely Pauli measurements, one can select a particular Wigner function which allows by itself to characterize contextuality in the traditional Kochen–Specker sense [22]. They proved that contextuality for stabilizer measurements and negativity of Gross' Wigner function [30] coincide for quopits, i.e. odd prime dimensional qudits. Namely, they showed that a single quopit state ρ has a negative Wigner function if and only if its tensor product with any other single-quopit state violates a two-quopit non-contextuality inequality.

In the present work, we establish the equivalence between contextuality for stabilizer measurements and negativity of Gross' Wigner function [30] for any multi-qudit state with odd local dimension. Such a neat equivalence between these features introduced in different fields is quite unexpected. Indeed, while contextuality is grounded in the foundations of quantum physics, Wigner functions originate from quantum optics. In addition, our proof of this equivalence is much more straightforward. Indeed, we directly compute the value of the Wigner

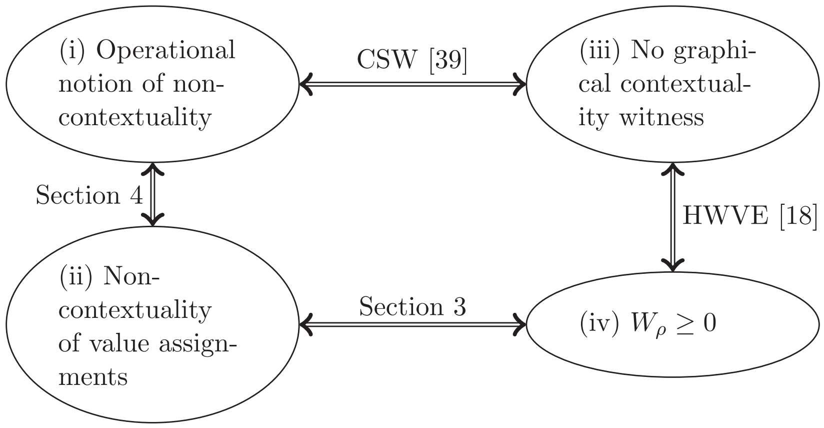

Figure 1. Relation between different notions of non-classicality. The equivalence HWVE [18] is only proven for product states when $n = 2$ and d is an odd prime number.

function in terms of the hidden variable model (HVM) and we observe that if the HVM is non-contextual then the Wigner function is non-negative. Howard *et al* required a two-qudit experiment to demonstrate contextuality of a single qudit state. We elucidate this by showing explicitly that in a single qudit experiment, the equivalence between contextuality and negativity does not hold. We show this by constructing quantum states that admit a non-contextual HVM (NCHVM) description although their Wigner functions take negative values.

Our proof of this equivalence relies on the choice of a simple definition of contextuality based on value assignments introduced in the work of Kochen and Specker [22], whereas the work of Howard *et al* is based on the graphical formalism of Cabello *et al* [39]. The relations between these different notions of non-classicality are depicted in figure 1. Following [22–24], we consider only contextuality of stabilizer measurements and HVMs are assumed to be deterministic. This last assumption is also present in the graphical formalism of Cabello *et al* [39] and in the work of Howard *et al* [18]. Our argument does not apply to the generalized notion of HVMs considered for instance by Spekkens [38, 40].

Our work clarifies the relationship between discrete Wigner function and stabilizer contextuality for odd-dimensional qudits. It cannot be extended to systems of qubits due to the presence of state-independent contextuality [23–25]. In addition, no qubit Wigner function that satisfies all the properties of Gross' qudit Wigner function seems to exist [41–43].

This article is organized as follows. The necessary stabilizer formalism is recalled in section 2. Section 3 introduces a notion contextuality based on value assignment and a proof of the equivalence between this notion and the negativity of Gross' discrete Wigner function, i.e. (ii) $\Leftrightarrow$ (iv). The purpose of section 4 is to prove the equivalence between the two notions of contextuality (i) and (ii), completing the square in figure 1.

2. Background on the stabilizer formalism

In what follows, we consider the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$, where d is an odd integer and n is a non-negative integer. We consider an orthonormal basis $(|a\rangle)_{a\in\mathbb{Z}_d}$ of $\mathbb{C}^d$. The n-fold tensor products of these vectors provide an orthonormal basis $(|\mathbf{a}\rangle)_{\mathbf{a}\in\mathbb{Z}_d^n}$ of the Hilbert space $\mathcal{H}$ indexed by $\mathbb{Z}_d^n$. This section recalls standard tools of the stabilizer formalism for qudits [44].

The space $V = \mathbb{Z}_d^n \times \mathbb{Z}_d^n$ that is called the *phase space* and will be used to index generalized Pauli operators acting on $\mathcal{H}$. Vectors in V are denoted $(\mathbf{u}_\mathrm{Z}, \mathbf{u}_\mathrm{X})$, where both $\mathbf{u}_\mathrm{Z}$ and $\mathbf{u}_\mathrm{X}$ live in $\mathbb{Z}_d^n$. The space $\mathbb{Z}_d^n$ is equipped with the standard inner product $(\mathbf{a}|\mathbf{b}) = \sum_{i=1}^n a_i b_i$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_d^n$, whereas the phase space V is equipped with the symplectic inner product defined by

$$[\mathbf{u}, \mathbf{v}] = (\mathbf{u}_\mathrm{Z}|\mathbf{v}_\mathrm{X}) - (\mathbf{u}_\mathrm{X}|\mathbf{v}_\mathrm{Z}) \mod d,$$

where $\mathbf{u} = (\mathbf{u}_\mathrm{Z}, \mathbf{u}_\mathrm{X}) \in V$ and $\mathbf{v} = (\mathbf{v}_\mathrm{Z}, \mathbf{v}_\mathrm{X}) \in V$.

Pauli matrices can be generalized to obtain matrices acting on $\mathbb{C}^d$ as follows. Let ω be the dth root of unity, $\omega = \mathrm{e}^{2\mathrm{i}\pi/d}$. The generalized Pauli matrices X and Z are defined by

$$X|a\rangle = |a+1\rangle, \quad Z|a\rangle = \omega^a|a\rangle$$

for all $a \in \mathbb{Z}_d$, where the addition of basis labels is considered modulo d. Tensor products of these matrices are denoted $Z^{\mathbf{a}} = Z^{a_1} \otimes \ldots Z^{a_n}$ and $X^{\mathbf{a}} = X^{a_1} \otimes \ldots X^{a_n}$ where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_d^n$. They satisfy $Z^{\mathbf{a}} X^{\mathbf{b}} = \omega^{(\mathbf{a}|\mathbf{b})} X^{\mathbf{b}} Z^{\mathbf{a}}$.

Generalized Pauli operators acting on this Hilbert space $\mathcal{H}$ are of the form $\omega^a Z^{\mathbf{u_Z}} X^{\mathbf{u_X}}$ where $\mathbf{u} = (\mathbf{u_Z}, \mathbf{u_X}) \in V$, and $a \in \mathbb{Z}_d$. Just as qubit Pauli operators, they form a group. We fix the phase of $Z^{\mathbf{u_Z}} X^{\mathbf{u_X}}$ to be $\omega^{-(\mathbf{u_Z}|\mathbf{u_X})/2}$. Recall that, since d is an odd integer, $1/2 = (d+1)/2$ exists in $\mathbb{Z}_d$ and the term $(\mathbf{u_Z}|\mathbf{u_X})/2$ is a well defined element of $\mathbb{Z}_d$. This defines *Heisenberg–Weyl operators*

$$T_{\mathbf{u}} = \omega^{-(\mathbf{u_Z}|\mathbf{u_X})/2} Z^{\mathbf{u_Z}} X^{\mathbf{u_X}} \tag{1}$$

for all $\mathbf{u} \in V$. This choice is well suited to describe measurement outcomes since $T_{\mathbf{u}} T_{\mathbf{v}} = \omega^{[\mathbf{u},\mathbf{v}]/2} T_{\mathbf{u}+\mathbf{v}}$, in particular two operators $T_{\mathbf{u}}$ and $T_{\mathbf{v}}$ can be measured simultaneously if and only if $[\mathbf{u}, \mathbf{v}] = 0$, which implies $T_{\mathbf{u}} T_{\mathbf{v}} = T_{\mathbf{u}+\mathbf{v}}$. Their commutation relation depends on the symplectic inner product as follows $T_{\mathbf{u}} T_{\mathbf{v}} = \omega^{[\mathbf{u},\mathbf{v}]} T_{\mathbf{v}} T_{\mathbf{u}}$. These operators satisfy $(T_{\mathbf{u}})^d = 1$ which proves that their eigenvalues belong to the group $U_d = \{\omega^s \mid s \in \mathbb{Z}_d\}$ of dth roots of unity.

Measuring a family of m mutually commuting operators $C = \{T_{\mathbf{a_1}}, \ldots, T_{\mathbf{a_m}}\}$ returns the outcome $\mathbf{s} = (s_1, \ldots, s_m) \in \mathbb{Z}_d^m$ with probability $Tr(\Pi_u^s \rho)$, where $\Pi_{\mathbf{u}}^{\mathbf{s}}$ is the projector onto the common eigenspace of the operators $T_{\mathbf{a_i}}$ with respective eigenvalue ω^{s_i}. When no confusion is possible, this projector is simply denoted $\Pi_C^{\mathbf{s}}$. Such a subset C of mutually commuting operators is called a *context*. The largest possible size of a context is d^n, which means that $m \leqslant d^n$. The set of all the contexts is denoted $\mathcal{C}$. For a single operator $T_{\mathbf{u}}$, we have $T_{\mathbf{u}} = \sum_{s \in \mathbb{Z}_d} \omega^s \Pi_{\mathbf{u}}^s$, by definition of these projectors. Moreover, $\Pi_{\mathbf{u}}^s$ can be obtained from the operator $T_{\mathbf{u}}$ as

$$\Pi_{\mathbf{u}}^s = \frac{1}{d} \sum_{k \in \mathbb{Z}_d} \omega^{-ks} T_{\mathbf{u}}^k. \tag{2}$$

Indeed, one can check that if $|\psi\rangle$ is an eigenvector of $T_{\mathbf{u}}$ with eigenvalue ω^a, we have $\frac{1}{d}\sum_{k \in \mathbb{Z}_d} \omega^{-ks} T_{\mathbf{u}}^k |\psi\rangle = \delta_{s,a} |\psi\rangle$. In order to generalize equation (2) to a family $C = \{T_{\mathbf{a_1}}, \ldots, T_{\mathbf{a_m}}\}$ of operators, introduce the $\mathbb{Z}_d$-linear subspace $M_{\mathbf{a}} = \langle \mathbf{a_1}, \ldots, \mathbf{a_m} \rangle$ of V generated by the vectors $\mathbf{a_i}$. Any measurement outcome $\mathbf{s} \in \mathbb{Z}_d^m$ induces a $\mathbb{Z}_d$-linear form $\ell_{\mathbf{s}} : M_{\mathbf{a}} \mapsto \mathbb{Z}_d$ defined by $\ell_{\mathbf{s}}(\sum_{i=1}^m x_i \mathbf{a_i}) = \sum_{i=1}^m x_i s_i$ where $x_i \in \mathbb{Z}_d$ for all $i = 1, \ldots, m$. This map $\ell_{\mathbf{s}}$ parametrizes the group generated by the operators $\omega^{s_i} T_{\mathbf{a_i}}$ as follows $\langle \omega^{s_i} T_{\mathbf{a_i}} \rangle = \{\omega^{\ell_{\mathbf{s}}(\mathbf{u})} T_{\mathbf{u}} \mid \mathbf{u} \in M_{\mathbf{a}}\}$. The projector $\Pi_{\mathbf{a}}^{\mathbf{s}}$ can be written

$$\Pi_{\mathbf{a}}^{\mathbf{s}} = \frac{1}{|M_{\mathbf{a}}|} \sum_{\mathbf{u} \in M_{\mathbf{a}}} \omega^{-\ell_{\mathbf{s}}(\mathbf{u})} T_{\mathbf{u}}. \tag{3}$$

This expression can be deduced from equation (2) by writing $\Pi_{\mathbf{a}}^{\mathbf{s}}$ as the product $\Pi_{\mathbf{a}}^{\mathbf{s}} = \prod_{i=1}^m \Pi_{\mathbf{a_i}}^{s_i}$. The image of the projector (3) is a stabilizer code [44].

3. Contextuality of value assignments and negativity

In this section, we prove that the notion of non-contextuality based on the existence of non-contextual value assignments [22–25] is equivalent to the non-negativity of the discrete Wigner function. A special case of this equivalence was proven by Howard *et al* [18] in odd prime dimension. We propose a simple proof of this result which allows us to generalize this equivalence to any system of multiple qudits and to any odd local dimension.

Recall that contextuality refers to the fact that measurement outcomes cannot be described in a deterministic way [22]. One cannot associate a fixed outcome $\lambda(A)$ with each observable A in such a way that this value is simply revealed after measurement. The algebraic relations between compatible observables must be satisfied by outcomes as well, making the existence of such pre-existing outcomes impossible. For instance, given two commuting operators A and B, the three observables A, B and $C = AB$ can be measured simultaneously and the values $\lambda(A)$, $\lambda(B)$ and $\lambda(C)$ associated with these operators must satisfy $\lambda(C) = \lambda(A)\lambda(B)$. No value assignment satisfying all these algebraic constraints exists in general.

The Wigner function of a state is a description of this state that was introduced in quantum optics in order to identify states with a classical behavior. Quantum states with a non-negative Wigner function are considered as quasi-classical states. The non-negativity of the Wigner function allows to describe the statistics of the outcomes of a large class of measurements in a classical way. The success of this representation motivated its generalization to finite dimensional Hilbert spaces, called discrete Wigner function. Different generalizations have been considered and finding a finite-dimensional Wigner representation that behaves as nicely as its original infinite-dimensional version [28] of use in quantum optics is a non-trivial question. The qubit case illustrates the difficulty of this task [41, 42]. In this work, we restrict ourselves to systems of qudits with odd local dimension and we consider Gross' discrete Wigner function which encloses most of the features of its quantum optics counterpart [30].

3.1. Value assignments are characters in odd local dimension

In this definition, the HVM associates a deterministic eigenvalue $\lambda_\nu(\mathbf{a}) \in U_d$ with each operator $T_\mathbf{a}$. Measurements only reveal these pre-existing values that are independent on other compatible measurements being performed. Formally, non-contextual value assignments are defined as follows.

Definition 1. Let ρ be a density matrix over $\mathcal{H}$. A *set of NC value assignments* for the state ρ is a triple (S, q_ρ, λ) where S is a finite set, q_ρ is a probability distribution over S and λ is a collection of maps $\lambda_\nu : V \to U_d$, for each $\nu \in S$, such that

1. for all $\mathbf{u}, \mathbf{v} \in V$, $[\mathbf{u}, \mathbf{v}] = 0$ implies $\lambda_\nu(\mathbf{u} + \mathbf{v}) = \lambda_\nu(\mathbf{u})\lambda_\nu(\mathbf{v})$,
2. for all $\mathbf{u} \in V$, $\mathrm{Tr}(T_\mathbf{u}\rho) = \sum_{\nu\in S}\lambda_\nu(\mathbf{u})q_\rho(\nu)$.

This definition corresponds to the notion of value assignment introduced by Kochen and Specker [22] restricted to measurements associated with Heisenberg–Weyl operators. The set S represents the hidden variable states or ontic states. Without loss of generality, one can assume that the value assignment associated with distinct states $\nu \in S$ are distinct. Then, S is necessarily a finite set since there is only a finite number of distinct maps λ_ν from V to U_d. A map λ_ν which satisfies $\lambda_\nu(\mathbf{u} + \mathbf{v}) = \lambda_\nu(\mathbf{u})\lambda_\nu(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ such that $[\mathbf{u}, \mathbf{v}] = 0$, is called a *value assignment*. Recall that the value $\lambda_\nu(\mathbf{u})$ is associated with the operator $T_\mathbf{u}$ defined in equation (1). With this phase convention, a value assignment is defined in such a way that the value of the product $T_{\mathbf{u}+\mathbf{v}} = T_\mathbf{u} T_\mathbf{v}$ of two compatible operators is the products of their values. As we will see later, multiplicativity of λ_ν whenever $[\mathbf{u}, \mathbf{v}] = 0$ is the non-contextuality assumption whereas the condition $\mathrm{Tr}(T_\mathbf{u}\rho) = \sum_{\nu\in S}\lambda_\nu(\mathbf{u})q_\rho(\nu)$ ensures that this triple is sufficient to recover the prediction of quantum mechanics for the measurement of $T_\mathbf{u}$.

A value assignment is a map λ_ν that satisfies the constraint $\lambda_\nu(\mathbf{u} + \mathbf{v}) = \lambda_\nu(\mathbf{u})\lambda_\nu(\mathbf{v})$, for all pairs of vectors such that $[\mathbf{u}, \mathbf{v}] = 0$. This enourages us to consider the *characters* of V. Recall that a character of V is defined to be group morhism from V to the multiplicative group $\mathbb{C}^*$. In other words, it is a map $\lambda : V \to \mathbb{C}^*$ that satisfies the constraint $\lambda(\mathbf{u} + \mathbf{v}) = \lambda(\mathbf{u})\lambda(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$. There exist $|V| = d^{2n}$ characters of V and they are of the form $\lambda_\mathbf{a}(\mathbf{u}) = \omega^{[\mathbf{a},\mathbf{u}]}$ for some $\mathbf{a} \in V$. These d^{2n} characters provides d^{2n} value assignments. Conversely, any value assignment λ_ν: $V \to \mathbb{C}^*$ coincindes with a character of V over any isotropic subspace. However, nothing in definition 1 guarantees that these assignments are actually characters of V. The next lemma proves this property. As a consequence, the only consistent value assignments of the HVM are given by the d^{2n} characters of V.

Lemma 1. *For any odd integer $d > 1$ and for any integer $n \geqslant 2$, value assignments λ_ν are characters of V.*

Proof. To make the proof easier to follow, we regard λ as a map from V to $\mathbb{Z}_d$ (through the group isomorphism between U_d and $\mathbb{Z}_d$) and we use the additive notation $\lambda(\mathbf{u}) + \lambda(\mathbf{v})$ instead of $\lambda(\mathbf{u})\lambda(\mathbf{v})$ in $\mathbb{C}$. We already know that $\lambda(\mathbf{u} + \mathbf{v}) = \lambda(\mathbf{u}) + \lambda(\mathbf{v})$ whenever $[\mathbf{u}, \mathbf{v}] = 0$. In particular for all $\mathbf{u} \in V$ and for all $k \in \mathbb{Z}_d$ we have $\lambda(k\mathbf{u}) = k\lambda(\mathbf{u})$.

Consider the canonical basis of $\mathbb{Z}_d^n \times \mathbb{Z}_d^n$ that we denote $(\mathbf{e_1}, \ldots, \mathbf{e_n}, \mathbf{f_1}, \ldots \mathbf{f_n})$. Clearly, $[\mathbf{e_i}, \mathbf{f_i}] = 1$ and the planes $P_i = \langle \mathbf{e_i}, \mathbf{f_i} \rangle$ are pairwise orthogonal with respect to the symplectic inner product for all $i = 1, \ldots, n$. The orthogonality between these planes allows us to write

$$\lambda(\mathbf{u}) = \sum_{i=1}^{n} \lambda(\alpha_i\mathbf{e_i} + \beta_i\mathbf{f_i})$$

for all $\mathbf{u} = \sum_{i=1}^{n}(\alpha_i\mathbf{e_i} + \beta_i\mathbf{f_i})$. It remains to prove that the restriction of λ to any plane P_i is additive.

We will use a second plane $P_j = \langle \mathbf{e_j}, \mathbf{f_j} \rangle$ with $j \neq i$ (which exists only when $n \geqslant 2$). Denote $\mathbf{u} = \alpha_i\mathbf{e_i}$ and $\mathbf{v} = \beta_i\mathbf{f_i}$ and define $\mathbf{u}' = \beta_i\mathbf{e_j}$ and $\mathbf{v}' = \alpha_i\mathbf{f_j}$ so that $[\mathbf{u}, \mathbf{v}] = [\mathbf{u}', \mathbf{v}']$. To conclude it suffices to prove that $\lambda(\mathbf{u} + \mathbf{v}) = \lambda(\mathbf{u}) + \lambda(\mathbf{v})$. Write

$$\mathbf{u} + \mathbf{v} = \frac{1}{2}((\mathbf{u} + \mathbf{v} + \mathbf{u}' + \mathbf{v}') + (\mathbf{u} + \mathbf{v} - \mathbf{u}' - \mathbf{v}'))\cdot$$

This decomposition is chosen in such a way that $(\mathbf{u} + \mathbf{v} + \mathbf{u}' + \mathbf{v}')$ and $(\mathbf{u} + \mathbf{v} - \mathbf{u}' - \mathbf{v}')$ are orthogonal:

$$[(\mathbf{u} + \mathbf{v} + \mathbf{u}' + \mathbf{v}'), (\mathbf{u} + \mathbf{v} - \mathbf{u}' - \mathbf{v}')] = [\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{u}] + [\mathbf{u}', -\mathbf{v}'] + [\mathbf{v}', -\mathbf{u}'] = 0\cdot$$

We will also use the orthogonality relations $[\mathbf{u} \pm \mathbf{v}', \mathbf{v} \pm \mathbf{u}'] = 0$ and between the planes P_i and P_j. We obtain

$$
\begin{aligned}
\lambda(\mathbf{u}+\mathbf{v}) &= \lambda\left(\frac{1}{2}(\mathbf{u}+\mathbf{v}+\mathbf{u}'+\mathbf{v}') + \frac{1}{2}(\mathbf{u}+\mathbf{v}-\mathbf{u}'-\mathbf{v}')\right)\\
&= \frac{1}{2}\lambda(\mathbf{u}+\mathbf{v}+\mathbf{u}'+\mathbf{v}') + \frac{1}{2}\lambda(\mathbf{u}+\mathbf{v}-\mathbf{u}'-\mathbf{v}')\\
&= \frac{1}{2}\lambda((\mathbf{u}+\mathbf{v}')+(\mathbf{v}+\mathbf{u}')) + \frac{1}{2}\lambda((\mathbf{u}-\mathbf{v}')+(\mathbf{v}-\mathbf{u}'))\\
&= \frac{1}{2}(\lambda(\mathbf{u}+\mathbf{v}') + \lambda(\mathbf{v}+\mathbf{u}') + \lambda(\mathbf{u}-\mathbf{v}') + \lambda(\mathbf{v}-\mathbf{u}'))\\
&= \frac{1}{2}(\lambda(\mathbf{u}) + \lambda(\mathbf{v}') + \lambda(\mathbf{v}) + \lambda(\mathbf{u}') + \lambda(\mathbf{u}) - \lambda(\mathbf{v}') + \lambda(\mathbf{v}) - \lambda(\mathbf{u}'))\\
&= \lambda(\mathbf{u}) + \lambda(\mathbf{v})\cdot
\end{aligned}
$$

This proves that λ is a character. □

3.2. Discrete Wigner functions

The purpose of this section is to recall that a set of NC value assignments can be derived from a discrete Wigner function of a state whenever this function is non-negative [18].

Quantum states are generally represented by their density matrix ρ. The *Wigner function* W_ρ of a state ρ is an alternative description of the state ρ. This representation is sometimes more convenient than the density matrix. Wigner functions have been introduced in quantum optics [28] and the negativity of the Wigner function of a state is regarded as an indicator of non-classicality of this quantum state. In the present work, we consider their finite dimensional generalization which is called *discrete Wigner function* [7, 29].

We focus on Gross' discrete Wigner function [30] $W_\rho : V \to \mathbb{R}$ defined by $W_\rho(\mathbf{u}) = d^{-n}\,\mathrm{Tr}(A_\mathbf{u}\rho)$ where $A_\mathbf{u} = d^{-n}\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u},\mathbf{v}]}T_\mathbf{v}$. The operators $A_\mathbf{u}$ are Hermitian. The family $(A_\mathbf{u})_{\mathbf{u}\in V}$ is an orthonormal basis of the space of $(d^n \times d^n)$-matrices equipped with the inner product $(A|B) = d^{-n}\,\mathrm{Tr}(A^\dagger B)$. The values $W_\rho(\mathbf{u})$ for $\mathbf{u} \in V$ are simply the coefficients of the decomposition of the matrix ρ in this basis:

$$\rho = \sum_{\mathbf{u}\in V} W_\rho(\mathbf{u})A_\mathbf{u}. \tag{4}$$

This proves that the Wigner function W_ρ fully describes the state ρ.

In order to describe measurement outcomes, the Wigner representation can be extended to POVM elements $(E_s)_s$. The *Wigner function of a POVM element* E_s is defined to be $W_{E_s}(\mathbf{u}) = \mathrm{Tr}(E_s A_\mathbf{u})$. This definition is chosen in such a way that the probability $\mathrm{Tr}(E_s\rho)$ of the outcome s is given in terms of the Wigner functions W_ρ and W_{E_s} by

$$\mathrm{Tr}(E_s\rho) = \sum_{\mathbf{u}\in V} W_{E_s}(\mathbf{u})W_\rho(\mathbf{u}). \tag{5}$$

This expression is obtained by replacing ρ by its decomposition (4).

Consider for instance the measurement of an operator $T_\mathbf{a}$. It corresponds to the POVM elements $(\Pi_\mathbf{a}^s)_{s\in\mathbb{Z}_d}$. Calculating the value of the Wigner function of the POVM element $\Pi_\mathbf{a}^s$, we find $W_{\Pi_\mathbf{a}^s}(\mathbf{u}) = \delta_{[\mathbf{a},\mathbf{u}],s}$, and injecting this in equation (5) yields $\mathrm{Tr}(\Pi_\mathbf{a}^s\rho) = \sum_{\mathbf{u}\in V}\delta_{[\mathbf{a},\mathbf{u}],s}W_\rho(\mathbf{u})$. Using the decomposition of $T_\mathbf{a}$ as a sum of projectors, this provides the expectation of $T_\mathbf{a}$.

Lemma 2. *For all Heisenberg–Weyl operators $T_\mathbf{a}$ given in equation (1), it holds that*

$$\mathrm{Tr}(T_\mathbf{a}\rho) = \sum_{\mathbf{u}\in V} W_\rho(\mathbf{u})\omega^{[\mathbf{a},\mathbf{u}]}.$$

Comparing lemma 2 with definition 1, we see that the triple $(V,\ W_\rho,\ (\lambda_\mathbf{u})_{\mathbf{u}\in V})$ is a set of NC value assignments for $\lambda_\mathbf{u}(\mathbf{a}) = \omega^{[\mathbf{a},\mathbf{u}]}$ whenever the Wigner function of the state ρ is non-negative. Note that all characters of V can be written as $\omega^{[\cdot,\mathbf{u}]}$ for some $\mathbf{u} \in V$.

3.3. Equivalence between NC value assignments and non-negativity

The previous section shows that non-negativity of W_ρ implies the existence of a set of NC value assignments for the state ρ. We now prove the converse statement.

Proposition 1. *Let $n \geqslant 2$ and let ρ be a n-qudits state of odd local dimension $d > 1$. If ρ admits a set of NC value assignments then $W_\rho \geqslant 0$.*

Proof. Given a set of NC value assignments, let us compute the value of the Wigner function of ρ to prove that it is non-negative. We have

$$\begin{aligned} W_\rho(\mathbf{u}) &= d^{-n}\,\mathrm{Tr}(A_\mathbf{u}\rho) \\ &= d^{-2n}\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u},\mathbf{v}]}\mathrm{Tr}(T_\mathbf{v}\rho) \\ &= d^{-2n}\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u},\mathbf{v}]}\sum_{\nu\in S}\lambda_\nu(\mathbf{v})q_\rho(\nu) \\ &= d^{-2n}\sum_{\nu\in S}\left(\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u},\mathbf{v}]}\lambda_\nu(\mathbf{v})\right)q_\rho(\nu). \end{aligned}$$

The fact that such a sum is positive or even real is not clear. However, lemma 1 shows that $\omega^{[\mathbf{u},\cdot]}\lambda_\nu(\cdot)$, which is a product of two characters, is also a character. Hence, any sum $\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u},\mathbf{v}]}\lambda_\nu(\mathbf{v})$ is either 0 or d^{2n} [45]. Since $q_\rho(\nu)\geqslant 0$, this proves that $W_\rho(\mathbf{u})\geqslant 0$. □

Actually the HVM derived from the discrete Wigner function is essentially unique. The following corollary shows that the distribution q_ρ of any set of NC value assignments can be identified with the Wigner function distribution W_ρ over V.

Corollary 1. *Let $n\geqslant 2$ and let ρ be a n-qudits state of odd local dimension $d>1$. If (S, q_ρ, λ) is a NC set of value assignments for* ρ *then there exists a bijective map $\sigma: S\to V$ such that*

$$q_\rho(\nu) = W_\rho(\sigma(\nu))$$

for all $\nu\in S$.

Proof. Let us refine the argument of the proof of proposition 1. By lemma 1 and since value assignments corresponding to distinct states of S are assumed to be different, there exists an injective map $\phi: S\to V$ such that $\lambda_\nu = \omega^{[\phi(\nu),\cdot]}$. Without loss of generality, we can assume that ϕ is sujective by adding extra elements to S corresponding to the missing characters of V. For these new elements $\nu\in S$, we simply set $q_\rho(\nu)=0$ to preserve the prediction of the triple (S, q_ρ, λ). With this notation, the expression of $W_\rho(\mathbf{u})$ obtained in the previous proof becomes

$$W_\rho(\mathbf{u}) = d^{-2n}\sum_{\nu\in S}\left(\sum_{\mathbf{v}\in V}\omega^{[\mathbf{u}+\phi(\nu),\mathbf{v}]}\right)q_\rho(\nu) = \sum_{\nu\in S}\delta_{\mathbf{u}+\phi(\nu),0}\cdot q_\rho(\nu). \quad (6)$$

For all vectors $\mathbf{u}$, there exists a unique state $\nu\in S$ such that $\mathbf{u}+\phi(\nu)=0$. Denote by $\nu_\mathbf{u}$ this state. Then, equation (6) becomes $W_\rho(\mathbf{u}) = q_\rho(\nu_\mathbf{u})$. To conclude the proof, note that the map $\mathbf{u}\to\nu_\mathbf{u}$ is invertible, by bijectivity of ϕ. Its inverse is the map σ of the corollary. □

3.4. The single qudit case

This section shows that the previous equivalence breaks down for single qudits.

Lemma 1 breaks down for $n=1$. For instance, for a prime dimension d, there exists d^{d+1} value assignments while there is only d^2 characters. Let us prove this property. By straightforward calculation we can check that two non-unit Pauli operators $T_\mathbf{a}$ and $T_\mathbf{b}$ commute if and only if they are integer powers of one another. Therefore, each maximal context consists of a picked operator and its powers. Each non-unit observable thus appears in a single maximal context, and there are $d-1$ non-unit operators $T_\mathbf{u}$ in each maximal context. The total number of non-unit Pauli operator is d^2-1. Therefore, there are $d+1$ contexts. In each context, we can freely choose the value of one non-unit Pauli observable, and the other values in the context then follow. There are thus d choices per context, and the total number of value assignments is d^{d+1}.

Breakdown of proposition 1 for $n=1$ and $d=3$. We consider the case of a single qutrit. Our strategy is to show that there is a valid quantum state that admits a set of NC value assignments, whose Wigner function is negative. To this end, we loop through all 81 consistent value assignments λ, and construct the density matrix ρ_λ and Wigner function W_λ corresponding to each value assignment λ. The density matrix ρ_λ is given by

$$\rho_\lambda = \sum_{u\in V}\frac{\lambda(u)}{3}T_u^\dagger$$

ρ_λ reproduces the measurement statistics of the value assignment λ, i.e. in the measurement of T_a, the outcome $\lambda(a)$ is certain. Further, ρ_λ is Hermitian and has unit trace, but is in general not positive semidefinite.

To mitigate the latter shortcoming, we add an admixture of the completely mixed state to ρ_λ, with increasing probability p,

$$\rho_{\lambda,p} = (1-p)\rho_\lambda + \frac{p}{3}I.$$

We check whether the resulting state $\rho_{\lambda,p}$ becomes positive semidefinite before the corresponding Wigner function $W_{\lambda,p} = (1-p)W_\lambda + \frac{p}{9}\mathbf{1}$ becomes positive, where $\mathbf{1}$ denotes the constant Wigner function taking the value 1.

Denote the smallest eigenvalue of ρ_λ by ν_λ, and the height of the deepest valley of W_λ by w_λ. Because of the special form of the admixture, the smallest eigenvalue $\nu_{\lambda,p}$ of $\rho_{\lambda,p}$ and the height $w_{\lambda,p}$ of the deepest valley of $W_{\lambda,p}$ are

$$\nu_{\lambda,p} = (1-p)\nu_\lambda + \tfrac{p}{3},$$
$$w_{\lambda,p} = (1-p)w_\lambda + \tfrac{p}{9}.$$

Therefore, there is a region of p for which $\rho_{\lambda,p}$ is positive semidefinite while $W_{\lambda,p} < 0$ if and only if

$$3w_\lambda < min(\nu_\lambda, 0). \tag{7}$$

Checking all 81 value assignments, we find that there are two cases.

Case I: value assignments that are characters of V (occurs 9 times).

$$w_\lambda = 0, \quad \nu_\lambda = -1.$$

The condition equation (7) is not satisfied, as expected for linear value assignments.

Case II: value assignments that are not characters of V (occurs 72 times).

$$w_\lambda = -\frac{1}{3}, \quad \nu_\lambda = -0.618.$$

The condition equation (7) is satisfied, and we thus know that there are positive semidefinite density matrices with negative Wigner functions. (The value of ν_λ above is minus the golden ratio.)

It remains to show that the density matrices $\rho_{\lambda,p}$ discussed in Case II have a representation in terms of NC value assignments. To this end, note that each ρ_λ has an NC value assignments representation, namely the probability distribution peaked at the assignment λ. The completely mixed state $I/3$ also has a NC value assignments description, namely it is the equal mixture of all value assignments that are characters of V. Hence, the probabilistic mixture of these two density matrices also is represented by a set of NC value assignments. □

To summarize, we have shown that there are well-defined quantum states whose Wigner function is negative but which have an NCHVM description. These states are counterexamples to an extension of proposition 1 to $n = 1$. Our argument is presented for a qutrit but we believe that such single qudit states exist for all odd dimensions. We leave the general case as an open question. More generally, it would be interesting to provide a simple characterization of the single qudit states that admit a set of NC value assignments.

In [18], the equivalence between Wigner function negativity and contextuality is established for single qudit states, and the question arises whether their result and the above argument are in contradiction. This is not the case. The construction of [18] uses an auxiliary qudit that can be in any state σ, for instance the completely mixed state. The desired equivalence between Wigner function negativity and contextuality is established for one qudit states ρ by considering the tensor product state $\rho \otimes \sigma$ on the contextuality side. Since σ is merely a by-stander, Howard *et al* count the setting as $n = 1$. By our counting, since the involved observables act on two qudits, the setting is $n = 2$. There is thus no contradiction between the result of [18] and the above argument that proposition 1 does not extend to $n = 1$.

4. Operational definition of non-contextuality

Our proof of the equivalence between non-contextuality and non-negativity of the Wigner function relies on the choice of a simple definition of contextuality (definition 1). The purpose of this section is to prove that, although simple, this definition is sufficient to capture the same notion of contextuality as the one considered in previous work [38, 40]. Namely, we prove that the notion of NC value assignments is equivalent to the operational definition of contextuality given below, inspired by the work of Spekkens with the restriction to deterministic models. All the results of this section are true for both multi-qudit systems and single qudit systems. They also remain valid for even local dimension d.

4.1. Deterministic hidden variable model

The role of a HVM is to describe the outcome distribution of Pauli measurements for a state ρ. In what follows, an ordered family of commuting operators $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_m}\}$ is called a *context*. This guarantees that they can be measured simultaneously. We denote by $\mathcal{C}$ the set of all the contexts, that is of all ordered tuples of commuting Heisenberg–Weyl operators $T_{\boldsymbol{u}}$.

The HVM considered in definition 1 is designed to predict the expectation of single Pauli operators. We now consider a framework which is *a priori* more general. The HVM is required to predict the outcome distribution of the measurement of any Pauli context $C \in \mathcal{C}$.

Definition 2. A HVM for a state ρ is defined to be a triple (S, q_ρ, p) such that S is a set, q_ρ is a distribution over S and $p = (p_C)_{C\in\mathcal{C}}$ is a family of conditional probability distributions $p_C(\mathbf{s}|\nu)$ over the possible outcome $\mathbf{s}$ for any context $C \in \mathcal{C}$ and for any state $\nu \in S$. We require that the prediction of the HVM matches the quantum mechanical value, i.e.

$$\mathrm{Tr}(\Pi_C^{\mathbf{s}}\rho) = \sum_{\nu\in S} p_C(\mathbf{s}|\nu)q_\rho(\nu), \tag{8}$$

for every context $C \in \mathcal{C}$ and for every outcome $\mathbf{s} \in \mathbb{Z}_d^m$.

The set S is the set of states of the HVM. Note that this set is not *a priori* the same as the set used in definition 1. We understand the probability $q_\rho(\nu)$ as the probability that the system is in the state $\nu \in S$. As suggested by the notation, the probability $p_C(s|\nu)$ can be interpreted as the probability of an outcome $\mathbf{s}$ when measuring the operators of the context C and when the system is in the state ν of the HVM. It is then natural to define the prediction of the HVM as in equation (8).

In the present work, we consider HVM that are deterministic in the following sense. We can associate a fixed measurement outcome $\mathbf{s}$ with each state ν of the model. In other words, conditional probabilities $p_C(\mathbf{s}|\nu)$ are delta functions.

Definition 3. A HVM (S, q_ρ, p) is said to be *deterministic* if conditional distributions p_C are all delta functions, i.e. for all $C \in \mathcal{C}$ and for all outcome $\mathbf{s}$, we have $p_C(\mathbf{s}|\nu) = \delta_{\alpha_\nu(C),\mathbf{s}}$ for some map α_ν that associates a value $\alpha_\nu(C) \in \mathbb{Z}_d^{|C|}$ with each context.

We often denote such a HVM by the triple (S, q, α). When C contains m operators, both $\alpha_\nu(C)$ and $\mathbf{s}$ are m-tuples. If $\alpha_\nu(C) = (x_1, \ldots, x_m)$, then $\delta_{\alpha_\nu(C),\mathbf{s}}$ is the product of the delta functions δ_{x_i,s_i}.

4.2. Operational definition of non-contextuality

Within our formalism restricted to measurement of Pauli operators $T_{\mathbf{a}}$, there exist different ways to realize a measurement. The operational notion of contextuality refers to the fact that the conditional distribution of outcomes in the HVM may depend on the way the measurement is implemented [40]. This section presents a formal definition of this notion.

To illustrate what should be the right definition of an implementation, we start with some examples. We can measure the operators of a context $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_\ell}\} \in \mathcal{C}$ and return the corresponding outcome $\mathbf{s} = (s_1, \ldots, s_\ell)$. This realizes the measurement defined by the family of orthogonal projectors $(\Pi_{\mathbf{a}}^{\mathbf{s}})_{\mathbf{s}}$ for $\mathbf{s} \in \mathbb{Z}_d^m$. Different contexts C may produce the same family of projectors, that is the same measurement. For instance, the 2-qudit measurement defined by the projectors $P^{x_1,x_2} = |x_1\rangle\langle x_1| \otimes |x_2\rangle\langle x_2|$, indexed by $(x_1, x_2) \in \mathbb{Z}_d^2$, can be implemented via the contexts $C = \{X\otimes I, I\otimes X\}$ or alternatively via $C' = \{X\otimes I, X\otimes X\}$.

We can also measure of a family of operators but read only a subset of the outcomes or even of function of the outcomes. Such a classical postprocessing extends the set of projectors that can be reached. For instance, for $\mathbf{u} \in V$ consider a pair of projectors $\{\Pi^0 = \Pi_{\mathbf{u}}^0, \Pi^1 = I - \Pi_{\mathbf{u}}^0\}$. This measurement is realized by measuring $T_{\mathbf{u}}$ and returning 0 if the outcome is 0 and returning 1 for all other outcomes $s \in \mathbb{Z}_d\setminus\{0\}$. To provide an example with a less trivial postprocessing consider the measurement $(\Pi_{\mathbf{u}+\mathbf{v}}^s)$ with outcome $s \in \mathbb{Z}_d$, for $\mathbf{u}, \mathbf{v} \in V$ such that $[\mathbf{u}, \mathbf{v}] = 0$. We can realize this measurement by measuring the pair $C = \{T_{\mathbf{u}}, T_{\mathbf{v}}\}$ and by returning only the sum $s = s_{\mathbf{u}} + s_{\mathbf{v}} \in \mathbb{Z}_d$ of the two outcomes.

These examples motivate our definition of an implementation of a measurement. Consider a measurement defined by a family of stabilizer projectors $(\Pi^s)_{s\in O}$, that sum up to identity, indexed by the elements of a finite set O.

Definition 4. An *implementation* of a measurement $(\Pi^{\mathbf{s}})_{\mathbf{s}\in O}$, is defined to be a pair (C, σ_C) where $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_\ell}\} \in \mathcal{C}$ and $\sigma_C : \mathbb{Z}_d^\ell \to O$ is a surjective postprocessing map such that

$$\Pi^{\mathbf{s}} = \sum_{\substack{\mathbf{t}\in\mathbb{Z}_d^\ell \\ \sigma_C(\mathbf{t})=\mathbf{s}}} \Pi_{\mathbf{a}}^{\mathbf{t}}$$

for all $\mathbf{s} \in O$.

Neither the choice of the context C nor the corresponding map σ_C is unique. The postprocessing map is assumed to be surjective only to ensure that all the projectors of the family $(\Pi^{\mathbf{s}})_{\mathbf{s}\in O}$ are reached.

Let (S, q_ρ, ν) be a HVM describing measurement outcomes for a state ρ. Consider an implementation (C, σ_C) of a measurement $(\Pi^{\mathbf{s}})_{\mathbf{s}\in O}$. By definition of the projectors $\Pi^{\mathbf{s}}$, the HVM predicts that the outcome $\mathbf{s}$ occurs with probability

$$p_C(\sigma_C^{-1}(\mathbf{s})|\nu) = \sum_{\substack{\mathbf{t}\in\mathbb{Z}_d^{|C|}\\ \sigma_C(\mathbf{t})=\mathbf{s}}} p_C(\mathbf{t}|\nu)$$

when the system is in position ν. Quantum mechanics predicts that the distribution of the outcome of a measurement only depends on the projectors $\Pi^{\mathbf{s}}$ and not on the implementation. That means that for any two implementations (C, σ_C) and $(C', \sigma_{C'})$ of a measurement $\Pi^{\mathbf{s}}$, we have

$$\begin{aligned}\mathrm{Tr}(\Pi^{\mathbf{s}}\rho) &= \sum_{\nu\in S} p_C(\sigma_C^{-1}(\mathbf{s})|\nu)q_\rho(\nu)\\ &= \sum_{\nu\in S} p_{C'}(\sigma_{C'}^{-1}(\mathbf{s})|\nu)q_\rho(\nu).\end{aligned}$$

However, nothing in definition 2 requires that the probabilities $p_C(\sigma_C^{-1}(\mathbf{s})|\nu)$ and $p_{C'}(\sigma_{C'}^{-1}(\mathbf{s})|\nu)$ coincide for all $\nu \in S$. This assumption is a notion of non-contextuality. It leads to the following definition of a NCHVM. This definition is based on the work of Spekkens [40] with the restriction to deterministic models and to systems of qudits. Spekkens approach differs also in the fact that it does not refer to the internal structure of quantum mechanics at all. The relationship between Spekkens' form of contextuality and Kochen–Specker contextuality is examined here [46].

Definition 5. A HVM (S, q_ρ, p) is said to be *non-contextual* if for all implementations (C, σ_C) of a measurement $(\Pi^{\mathbf{s}})_{\mathbf{s}\in O}$, for all $\nu \in S$, the conditional probability $p_C(\sigma_C^{-1}(\mathbf{s})|\nu)$ depends only on the projector $\Pi^{\mathbf{s}}$ and not on the implementation (C, σ_C).

For instance, we saw that the measurement $(\Pi^s_{\mathbf{u}+\mathbf{v}})_{s\in\mathbb{Z}_d}$ can be implemented by $C = \{T_{\mathbf{u}+\mathbf{v}}\}$ with a trivial map σ_C but also using $C' = \{T_{\mathbf{u}}, T_{\mathbf{v}}\}$ with the postprocessing map $\sigma_{C'}(s_{\mathbf{u}}, s_{\mathbf{v}}) = s_{\mathbf{u}} + s_{\mathbf{v}}$. For a non-contextual model, the corresponding conditional probabilities $p_C(s|\nu)$ and $p_{C'}(\sigma_{C'}^{-1}(s)|\nu)$ coincide for all states ν of the HVM. In this example, we have $\sigma_{C'}^{-1}(s) = \{(t, s-t) \mid t \in \mathbb{Z}_d\}$.

4.3. Equivalence of the two definitions of non-contextuality

In this section we prove that the existence of a NCHMV as given in definition 5 and the existence of a set of NC value assignments as in definition 1 are two equivalent notions of non-contextuality. This is the equivalence (i) $\Leftrightarrow$ (ii) in figure 1.

The following proposition shows that the implication (ii) $\Rightarrow$ (i) holds.

Proposition 2. *Let ρ be a n-qudit system with $d > 1$ and $n \geqslant 1$. If (S, q_ρ, λ) be a set of NC value assignments, then (S, q_ρ, p) defines a deterministic NCHVM where p is defined by*

$$p_C(\mathbf{s}|\nu) = \frac{1}{|M_{\mathbf{a}}|}\sum_{\mathbf{u}\in M_{\mathbf{a}}} \omega^{-\ell_s(\mathbf{u})}\cdot\lambda_\nu(\mathbf{u}) \tag{9}$$

for all $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_m}\} \in \mathcal{C}$.

The notations $M_{\mathbf{a}}$ and $\ell_{\mathbf{s}}$ used in equation (9) were introduced in equation (3). Recall that, by deterministic, we mean $p_C(\mathbf{s}|\nu) \in \{0, 1\}$.

Proof. First, let us prove that the HVM (S, q_ρ, p) defined in this proposition produces the same predictions as quantum mechanics. By equation (3), the probability of an outcome $\mathbf{s} = (s_1, \ldots, s_m)$ when measuring $\{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_m}\}$ is given by

$$\mathrm{Tr}(\Pi^{\mathbf{s}}_{\mathbf{a}}\rho) = \frac{1}{|M_{\mathbf{a}}|}\sum_{\mathbf{u}\in M_{\mathbf{a}}} \omega^{-\ell_s(\mathbf{u})}\mathrm{Tr}(T_{\mathbf{u}}\rho).$$

Replacing $\mathrm{Tr}(T_{\mathbf{u}}\rho)$ by its value in terms of the value assignments, we obtain

$$\mathrm{Tr}(\Pi^{\mathbf{s}}_{\mathbf{a}}\rho) = \sum_{\nu\in S}\left(\frac{1}{|M_{\mathbf{a}}|}\sum_{\mathbf{u}\in M_{\mathbf{a}}} \omega^{-\ell_s(\mathbf{u})}\cdot\lambda_\nu(\mathbf{u})\right)q_\rho(\nu).$$

This proves that the HVM (S, q_ρ, p) defined by equation (9) reproduces the quantum mechanical predictions.

It is deterministic since the sum $\sum_{u \in M_a} \omega^{-\ell_s(\mathbf{u})} \cdot \lambda_\nu(\mathbf{u})$, which is the sum of the values of a character, is either 0 or $|M_\mathbf{a}|$, implying that $p_C(\mathbf{s}|\nu) \in \{0, 1\}$. This HVM is also non-contextual. Indeed, conditional probabilities are defined in such a way that they do not depend on the particular choice of generators $\mathbf{a_i}$ for the subspace $M_\mathbf{a}$, that is they do not depend on the context. □

We now prove the converse statement. Together with the non-contextuality assumption, determinism of the HVM yields extra compatibility contraints on the functions α_ν. The following proposition proves that α_ν is completely determined by its value $\alpha_\nu(\{T_\mathbf{u}\})$ over single operator contexts $\{T_\mathbf{u}\}$. We shorten the notation $\alpha_\nu(\{T_\mathbf{u}\})$ by $\alpha_\nu(\mathbf{u})$. Moreover, we show that α_ν is additive when $[\mathbf{u}, \mathbf{v}] = 0$. Then, we will prove that we can construct a set of NC value assignments from the maps α_ν.

Proposition 3. *Let ρ be a n-qudit system with $d > 1$ and $n \geqslant 1$. If (S, p, α) is a deterministic NCHVM then,*

- *for all contexts $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_\mathbf{m}}\} \in \mathcal{C}$, we have*
$$\alpha_\nu(\{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_\mathbf{m}}\}) = (\alpha_\nu(\mathbf{a}_1), \ldots, \alpha_\nu(\mathbf{a}_\mathbf{m})),$$
- *if $T_\mathbf{u}$ and $T_\mathbf{v}$ commute, i.e. if $[\mathbf{u}, \mathbf{v}] = 0$, we have*
$$\alpha_\nu(\mathbf{u} + \mathbf{v}) = \alpha_\nu(\mathbf{u}) + \alpha_\nu(\mathbf{v})\cdot$$

Proof. To prove the first item, we consider two implementations of the measurement $(\Pi^s_{\mathbf{a_i}})_{s \in \mathbb{Z}_d}$ for some $\mathbf{a_i} \in V$. First, one can simply measure $\{T_{\mathbf{a_i}}\}$ and reveal the outcome s_i. A second implementation is obtained via the context $C = \{T_{\mathbf{a}_1}, \ldots, T_{\mathbf{a}_\mathbf{m}}\} \in \mathcal{C}$ and the map σ_C that sends a measurement outcome $\mathbf{t} = (t_1, \ldots, t_m)$ onto its ith component t_i. In other words, we measure these m operators but we only keep the outcome of $T_{\mathbf{a_i}}$. By non-contextuality, these two procedures yield the same conditional probabilities at the level of the HVM, that is $p_C(\sigma_C^{-1}(s)|\nu) = p_{T_{\mathbf{a_i}}}(s|\nu)$ for all $\nu \in S$, for all $s \in \mathbb{Z}_d$. Replacing the first term by its definition, we obtain
$$\sum_{\substack{(t_1, \ldots, t_m) \in \mathbb{Z}_d^m \\ t_i = s}} p_C(\mathbf{t}|\nu) = p_{T_{\mathbf{a_i}}}(s|\nu). \tag{10}$$
Fix $\nu \in S$ and denote $\alpha_\nu(C) = (x_1, \ldots, x_m)$. Sharpness of the measurements implies that $p_C(\mathbf{t}|\nu) = \prod_{j=1}^m \delta_{x_j, t_j}$ and $p_{T_{\mathbf{a_i}}}(s|\nu) = \delta_{\alpha_\nu(\mathbf{a_i}),\mathbf{s}}$. Injecting these expressions in equation (10) produces the equality
$$\sum_{\substack{(t_1, \ldots, t_m) \in \mathbb{Z}_d^m \\ t_i = s}} \prod_{j=1}^m \delta_{x_j, t_j} = \delta_{\alpha_\nu(\mathbf{a_i}), s} \tag{11}$$
that is satisfied for all $s \in \mathbb{Z}_d$. The only possibility to have a non-trivial product at the left-hand side is to pick $t_j = x_j$ for all $j \neq i$, leading to
$$\delta_{x_i, s} = \delta_{\alpha_\nu(\mathbf{a_i}), s}.$$
This equality is satisfied for all $s \in \mathbb{Z}_d$, proving that $\alpha_\nu(\mathbf{a_i}) = x_i$. This concludes the proof of the first property.

The second item is proven using two implementations of the measurement of $(\Pi^s_{\mathbf{u}+\mathbf{v}})_{s \in \mathbb{Z}_d}$. First, we consider the direct implementation by measuring $T_{\mathbf{u}+\mathbf{v}}$. Then, we use the context $C = \{T_\mathbf{u}, T_\mathbf{v}\}$ with the postprocessing map $\sigma_C(s_\mathbf{u}, s_\mathbf{v}) = s_\mathbf{u} + s_\mathbf{v}$. Non-contextuality leads to
$$\sum_{k \in \mathbb{Z}_d} p_C((k, s - k)|\nu) = p_{T_{\mathbf{u}+\mathbf{v}}}(s|\nu).$$
Using the first result, the delta function describing the conditional distribution for C is associated with $\alpha_\nu(\{T_\mathbf{u}, T_\mathbf{v}\}) = (\alpha_\nu(\mathbf{u}), \alpha_\nu(\mathbf{v}))$. This implies
$$\sum_{k \in \mathbb{Z}_d} \delta_{\alpha_\nu(\mathbf{u}), k} \cdot \delta_{\alpha_\nu(\mathbf{v}), s-k} = \delta_{\alpha_\nu(\mathbf{u}+\mathbf{v}), s}.$$
The left-hand side is equal to $\delta_{\alpha_\nu(\mathbf{v}), s - \alpha_\nu(\mathbf{u})}$ proving that $\alpha_\nu(\mathbf{u}) + \alpha_\nu(\mathbf{v}) = \alpha_\nu(\mathbf{u} + \mathbf{v})$. □

As a corollary, we prove that the maps $\mathbf{u} \mapsto \alpha_\nu(\mathbf{u})$ define a family of NC value assignments. This complete the proof of the equivalence between (ii) and (iii).

Corollary 2. *Let ρ be a n-qudit system with $d > 1$ and $n \geqslant 1$. If (S, q_ρ, α) is a deterministic NCHVM then the triple (S, q_ρ, λ) where the map $\lambda_\nu : V \to U_d$ is defined by $\lambda_\nu(\mathbf{u}) = \omega^{\alpha_\nu(\mathbf{u})}$ for all ν, is a set of NC value assignments.*

Proof. Additivity of the maps λ_ν was proven in proposition 3. It remains to prove that this value assignment provides a good prediction for the expectation of operators $T_{\mathbf{u}}$. Writing this operator as a linear combination of projectors $T_{\mathbf{u}} = \sum_{s\in\mathbb{Z}_d}\omega^s\Pi_{\mathbf{u}}^s$, we find

$$\mathrm{Tr}(T_{\mathbf{u}}\rho) = \sum_{s\in\mathbb{Z}_d}\omega^s\,\mathrm{Tr}(\Pi_{\mathbf{u}}^s\rho).$$

Using the prediction of the HVM and sharpness of the measurements, we obtain

$$\mathrm{Tr}(T_{\mathbf{u}}\rho) = \sum_{\nu\in S}\left(\sum_{s\in\mathbb{Z}_d}\omega^s\delta_{\alpha_\nu(\mathbf{u}),s}\right)q_\rho(\nu) = \sum_{\nu\in S}\omega^{\alpha_\nu(\mathbf{u})}q_\rho(\nu)$$

proving the corollary. □

5. Concluding remarks

Through this work we show that negativity of the Wigner function and contextuality are exactly equivalent for systems of multiple qudits with odd local dimension. We also show that there exist single qudits quantum states that are described by a NCHVM while their Wigner functions take negative values. The description of all single qudit states admitting a non-contextual description is left as an open question.

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ORCID iDs

Nicolas Delfosse https://orcid.org/0000-0002-3949-981X

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