HYPERBOLIC INVERSE PROBLEM WITH DATA ON DISJOINT SETS

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ABSTRACT. We consider a restricted Dirichlet-to-Neumann map $\Lambda^T_{\mathcal{S},\mathcal{R}}$ associated to the operator $\partial_t^2 - \Delta_g + A + Q$ where Δ_g is the Laplace-Beltrami operator of a Riemannian manifold (M, g), and A and Q are a vector field and a function on M. The restriction $\Lambda^T_{\mathcal{S},\mathcal{R}}$ corresponds to the case where the Dirichlet traces are supported on $(0,T)\times\mathcal{S}$ and the Neumann traces are restricted on $(0,T)\times\mathcal{R}$. Here $\mathcal{S}$ and $\mathcal{R}$ are open sets on the boundary of M. We show that $\Lambda^T_{\mathcal{S},\mathcal{R}}$ determines the geometry and the lower order terms A and Q up the natural gauge invariances in a neighborhood of the set $\mathcal{R}$ assuming that $\mathcal{R}$ is strictly convex and that the wave equation is exactly controllable from $\mathcal{S}$ in time $T/2$. We give also a global result under a convex foliation condition. The main novelty is the recovery of A and Q when the sets $\mathcal{R}$ and $\mathcal{S}$ are disjoint. We allow A and Q to be non-self-adjoint, and in particular, the corresponding physical system may have dissipation of energy.

1. Introduction

Let (M, g) be a smooth, connected and compact Riemannian manifold of dimension n with nonempty boundary ∂M, let A be a smooth complex valued vector field on M, and let Q be a smooth complex valued function on M. We consider the wave equation with Dirichlet data $f \in C_0^\infty((0,\infty)\times\partial M)$,

$$(1)\qquad \begin{cases} (\partial_t^2 - \Delta_g + A + Q)u(t,x) = 0, & \text{in } (0,\infty)\times M,\\ u|_{(0,\infty)\times\partial M} = f, & \text{in } (0,\infty)\times\partial M,\\ u|_{t=0} = \partial_t u|_{t=0} = 0, & \text{in } M,\end{cases}$$

and denote by $u^f = u(t,x)$ the solution of (1). For open and nonempty sets $\mathcal{S},\mathcal{R}\subset\partial M$ and $T\in(0,\infty]$ we define the response operator,

$$\Lambda^T_{\mathcal{S},\mathcal{R}} : f \mapsto (\partial_\nu u^f - \frac{1}{2}(A,\nu)_g u^f)|_{(0,T)\times\mathcal{R}}, \quad f\in C_0^\infty((0,T)\times\mathcal{S}).$$

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Here ν is the interior unit normal vector field on ∂M, and $(A,\nu)_g$ is the inner product of A and ν. We use real inner products throughout the paper. If $A(x) = \sum_{j=1}^n A^j(x)\partial_j$ in local coordinates, then $(A,\nu)_g$ is given by $g_{jk}A^j\nu^k$ locally.

When f is regarded as a boundary source, the operator $\Lambda^T_{\mathcal{S},\mathcal{R}}$ models boundary measurements for the wave equation with sources on the set $(0,T)\times\mathcal{S}$ and the waves being observed on $(0,T)\times\mathcal{R}$. We consider the inverse boundary value problem to determine the manifold (M,g), the vector field A and the potential Q from $\Lambda^T_{\mathcal{S},\mathcal{R}}$.

We have studied previously the determination of the geometry (M,g) [24], in the case that $A=0$ and $Q=0$, and the main focus of the present paper is on the recovery of the lower order terms A and Q. In order to recover A and Q, we construct boundary sources f such that, at time $t=T$, the corresponding solutions u^f are essentially localized at a point near $\mathcal{R}$. This differs from the construction in [24] which does not use localized waves.

The lower order terms A and Q can be determined only up to the action of a group gauge transformation, that we will describe next. Let κ be a smooth nowhere vanishing complex valued function on M satisfying $\kappa=1$ on $\mathcal{R}$. The response operator $\Lambda^T_{\mathcal{S},\mathcal{R}}$ does not change under the transformation $(A,Q)\mapsto(A_\kappa,Q_\kappa)$ where

$$A_\kappa = A + 2\kappa^{-1}\operatorname{grad}_g\kappa, \quad Q_\kappa = Q + \kappa(A-\Delta_g)\kappa^{-1}, \tag{2}$$

and grad_g is the gradient on (M,g). We refer to [19] for a similar computation in the self-adjoint case. When $\mathcal{U}\subset M$ is a neighborhood of $\mathcal{R}$, we write

$$\mathcal{G}_{\mathcal{U},\mathcal{R}}(A,Q) = \{(A_\kappa|_{\mathcal{U}}, Q_\kappa|_{\mathcal{U}});\ \kappa\in C^\infty(\overline{\mathcal{U}}),\ \kappa\neq 0,\ \kappa|_{\mathcal{R}}=1\}$$

for the orbit of the group of gauge transformations on $\mathcal{U}$.

We allow also the geometry (M,g) to be apriori unknown, and there is a second group of transformations leaving $\Lambda^T_{\mathcal{S},\mathcal{R}}$ invariant. Namely, if $\Phi: M\to M$ is a diffeomorphism fixing $\mathcal{S}\cup\mathcal{R}$, that is, Φ is the identity on $\mathcal{S}\cup\mathcal{R}$, then $\Lambda^T_{\mathcal{S},\mathcal{R}}$ does not change if g, A and Q are replaced with their pullbacks under Φ.

We recall that the wave equation (1) is said to be exactly controllable from $\mathcal{S}$ in time T if the map

$$f\mapsto(u^f(T),\partial_t u^f(T)) : L^2((0,T)\times\mathcal{S})\to L^2(M)\times H^{-1}(M), \tag{3}$$

is surjective. If there is such $T>0$, then we say that (1) is exactly controllable from $\mathcal{S}$. The exact controllability can be characterized in terms of the billiard flow of the manifold (M,g) [2, 6]. The geometric characterization says roughly that all unit speed geodesics, continued by reflection on $\partial M \setminus \mathcal{S}$, must exit M through $\mathcal{S}$ during time T. In particular, the geometric characterization implies that the exact controllability does not depend on the lower order terms A and Q.

In this paper we show the following theorem:

Theorem 1. *Let $\mathcal{S} \subset \partial M$ be open and suppose that the wave equation (1) is exactly controllable from $\mathcal{S}$ in time $T > 0$. Let $\mathcal{R} \subset \partial M$ be open and strictly convex. Then there is a neighborhood $\mathcal{U} \subset M$ of $\mathcal{R}$ such that $\Lambda^{2T}_{\mathcal{S},\mathcal{R}}$ determines the Riemannian manifold $(\mathcal{U}, g)$, up to an isometry, and the orbit $\mathcal{G}_{\mathcal{U},\mathcal{R}}(A, Q)$.*

We show also a global uniqueness result under the assumption that there is a convex foliation similar to that in [29]. We assume that Σ_s, $s \in (0, 1]$, satisfy the following:

- (F1) $\Sigma_s \subset M^{\text{int}}$ is a smooth manifold of codimension one.
- (F2) The union $\Omega_s = \bigcup_{r \in (0,s)} \Sigma_r \subset M^{\text{int}}$ is open and connected, and $\Omega_r \subset \Omega_s$ when $r < s$.
- (F3) $\partial\Omega_s = \Sigma_s \cup \mathcal{R}_s$ and $\mathcal{R}_s \subset \mathcal{R}$ where $\mathcal{R}_s = \overline{\Omega_s} \cap \partial M$.
- (F4) Σ_s is strictly convex as a subset of ∂M_s where
$$M_s = M \setminus (\Omega_s \cup \mathcal{R}_s).$$
- (F5) The Hausdorff distances satisfy $\text{dist}(\Omega_r, \Omega_s) \to 0$ as $r \to s$.
- (F6) There is a set $\mathcal{R}_0 \subset \mathcal{R}$ such that $\text{dist}(\Omega_s, \mathcal{R}_0) \to 0$ as $s \to 0$.

Furthermore, to simplify the notation, we assume

- (F7) $\mathcal{R} = \bigcup_{s \in (0,1]} \mathcal{R}_s$.

Theorem 2. *Let $\mathcal{S} \subset \partial M$ be open and suppose that the wave equation (1) is exactly controllable from $\mathcal{S}$. Let $\mathcal{R} \subset \partial M$ be open and strictly convex and let Σ_s, $s \in (0, 1]$, be a convex foliation satisfying (F1)-(F7). Then $\Lambda^{\infty}_{\mathcal{S},\mathcal{R}}$ determines the Riemannian manifold (Ω_1, g), up to an isometry, and the orbit $\mathcal{G}_{\Omega_1,\mathcal{R}}(A, Q)$.*

In Section 6 we show that, in the above theorem, exact controllability from $\mathcal{S}$ can be replaced with exact controllability from $\mathcal{R}$. Our result is new even in the following case:

Example 1. *Let (M, g) be the Euclidean unit disk $\{z \in \mathbb{C};\ |z| \le 1\}$. Let $\epsilon > 0$ and define $\mathcal{R} = \{e^{i\theta}; \theta \in (-\epsilon, \pi + \epsilon)\}$. Let $\mathcal{S} \subset \partial M$ be open and nonempty. Then $\Lambda^{\infty}_{\mathcal{S},\mathcal{R}}$ determines A and Q, up to the gauge transformations, in the convex hull of $\mathcal{R}$.*

Let us also point out that we could use a time continuation argument analogous to [24, Lemma 4] and prove Theorem 2 also for measurements on a long enough but finite time interval.

Our proof is based on the Boundary Control (BC) method. The BC method was introduced by Belishev [3], and it was first used in a geometric context in [4]. Stability properties of the method are discussed in [1]. First order perturbations have been considered in the self-adjoint case in [16], and in the non-self-adjoint case in [20, 22]. All the above results assume that $\mathcal{S} = \mathcal{R}$. The case of disjoint $\mathcal{S}$ and $\mathcal{R}$ was first considered in the above mentioned [24] where no first order perturbation was present.

In addition to [24], we are aware of only two results on inverse boundary value problems with disjoint data analogous to the case $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} = \emptyset$. Rakesh [27] considers a wave equation on a one-dimensional interval with sources supported on one end of the interval and the waves observed on the other end, and Imanuvilov, Uhlmann, and Yamamoto [15] proved that a zeroth order term in a Schrödinger equation on a two-dimensional domain homeomorphic to a disk, whose boundary is partitioned into eight parts $\Gamma_1, \Gamma_2, \dots, \Gamma_8$ in the clockwise order, is determined by boundary measurements with Dirichlet data supported on $\mathcal{S} = \Gamma_2 \cup \Gamma_6$ and the Neumann trace observed on $\mathcal{R} = \Gamma_4 \cup \Gamma_8$.

Let us mention also the result on recovery of a conformal scaling factor in the metric tensor given the Dirichlet-to-Neumann map [31] that, analogously to our result, uses local convexity of the boundary. The proof [31] is based on a reduction to the boundary rigidity result [30] and this approach seems to require that $\mathcal{S} = \mathcal{R}$.

A vast majority of results on inverse boundary value problems assume that $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} \neq \emptyset$. For this type of non-disjoint, partial data results, we refer to [7, 8, 9, 11, 12, 13, 14, 18].

2. Tools for the reconstruction

In this section we present the two main components of the Boundary Control method: an integration by parts technique originating from Blagoveščenskiĭ's study of the $1+1$ dimensional wave equation [5], and a density result based on the hyperbolic unique continuation result by Tataru [32].

2.1. **Blagoveščenskiĭ's identity.** Let $\Gamma \subset \partial M$ and $B \subset M^{\text{int}}$ be open, and let $\kappa : B \to \mathbb{C}$. We define for $f \in C_0^\infty((0,\infty) \times \mathcal{S})$,

$$\Lambda_\Gamma f = (\partial_\nu u - \frac{1}{2}(A,\nu)_g u)|_{(0,\infty)\times\Gamma}, \quad \Lambda_{B,\kappa} f = \kappa u|_{(0,\infty)\times B},$$

where u is the solution of (1), and write $\Lambda_B = \Lambda_{B,\kappa}$ when considering a fixed κ. Note that $\Lambda_{\mathcal{R}}$ is just a shorthand notation for the response operator $\Lambda^\infty_{\mathcal{S},\mathcal{R}}$.

For $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$, we define

$$K_{\mathcal{V}}^T = J\Lambda_{\mathcal{V}} - R\Lambda_{\mathcal{V}} RJ, \tag{4}$$

where $R\psi(t) = \psi(T-t)$ and $J\psi(t) = \frac{1}{2}\int_t^{2T-t} \psi(s)ds$. We write also $U^T f = u(T)$.

Let us now suppose that $\kappa \in C^\infty(\overline{B})$ and consider the adjoint wave equation

$$(\partial_t^2 - \Delta_g - A + Q_{ad})v = \kappa H, \quad \text{in } (0,T)\times M, \tag{5}$$

where $H \in C_0^\infty(B)$. We impose the initial and boundary conditions

$$\begin{aligned} &v|_{(0,T)\times\partial M} = \phi, \quad \text{in } (0,T)\times \partial M,\\ &v|_{t=0} = 0, \quad \partial_t v|_{t=0} = 0, \quad \text{in } M, \end{aligned} \tag{6}$$

where $\phi \in C_0^\infty((0,\infty)\times\partial M)$. Here $Q_{ad} = Q - \mathrm{div}_g A$, where div_g is the divergence on (M,g). We define

$$W_\Gamma^T \phi = v(T), \quad W_\Gamma^T : C_0^\infty((0,\infty)\times\Gamma) \to C^\infty(M),$$

where v is the solution of (5), (6) with $H = 0$, and

$$W_B^T H = v(T) \quad W_B^T : C_0^\infty((0,\infty)\times B) \to C^\infty(M),$$

where v is the solution of (5), (6) with $\phi = 0$.

Lemma 1 (Blagoveščenskiĭ type identity)**.** *Let $T > 0$ and let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open. Let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Then for functions $f \in L^2((0,\infty)\times\mathcal{S})$ and $p \in L^2((0,\infty)\times\mathcal{V})$ we have*

$$\langle W_{\mathcal{V}}^T p, U^T f\rangle_{L^2(M)} = \langle p, K_{\mathcal{V}}^T f\rangle_{L^2((0,T)\times\mathcal{V})}. \tag{7}$$

Proof. Let u be the solution of (1) with $f \in C_0^\infty((0,\infty)\times\mathcal{S})$ and let us consider a smooth solution v of the adjoint equation (5). Then

$$\begin{aligned} &\langle u, \kappa H\rangle_{L^2((0,T)\times M)}\\ &\quad = \langle u, (\partial_t^2 - \Delta_g - A + Q_{ad})v\rangle_{L^2((0,T)\times M)}\\ &\qquad - \langle (\partial_t^2 - \Delta_g + A + Q)u, v\rangle_{L^2((0,T)\times M)}\\ &\quad = \langle u(T), \partial_t v(T)\rangle_{L^2(M)} - \langle \partial_t u(T), v(T)\rangle_{L^2(M)}\\ &\qquad + \langle u, \partial_\nu v + \frac{1}{2}(A,\nu)_g v\rangle_{L^2((0,T)\times\partial M)}\\ &\qquad - \langle \partial_\nu u - \frac{1}{2}(A,\nu)_g u, v\rangle_{L^2((0,T)\times\partial M)}. \end{aligned} \tag{8}$$

We define

$$L_\Gamma^T \phi = (\partial_\nu v + \frac{1}{2}(A,\nu)_g v)|_{(0,T)\times\mathcal{S}}, \quad \phi \in C_0^\infty((0,\infty)\times\Gamma),$$

where v is the solution of (5), (6) with $H = 0$, and

$$L_B^T H = (\partial_\nu v + \frac{1}{2}(A,\nu)_g v)|_{(0,T)\times\mathcal{S}}, \quad H \in C_0^\infty((0,\infty)\times B),$$

where v is the solution of (5), (6) with $\phi = 0$.

Note that if we replace v with Rv in the integration by parts (8), we see that

$$\langle \Lambda_\mathcal{V} f, p\rangle_{L^2((0,T)\times\mathcal{V})} = \langle f, RL_\mathcal{V}^T Rp\rangle_{L^2((0,T)\times\mathcal{S})}. \tag{9}$$

Hence $R\Lambda_\mathcal{V} R$ is the adjoint of $L_\mathcal{V}^T$.

Let $t \in (0,T)$ and $s \in (0,2T)$. Then

$$\begin{aligned}
&(\partial_t^2 - \partial_s^2)\langle u(s), v(t)\rangle_{L^2(M)} \\
&\quad = \langle u(s), (\Delta_g + A - Q_{ad})v(t)\rangle_{L^2(M)} \\
&\qquad - \langle (\Delta_g - A - Q)u(s), v(t)\rangle_{L^2(M)} \\
&\quad = \langle u(s), \kappa H(t)\rangle_{L^2(M)} - \langle u(s), \partial_\nu v(t) + \frac{1}{2}(A,\nu)_g v(t)\rangle_{L^2(\partial M)} \\
&\qquad + \langle \partial_\nu u(s) - \frac{1}{2}(A,\nu)_g u(s), v(t)\rangle_{L^2(\partial M)} \\
&\quad = \langle \Lambda_\mathcal{V} f(s), p(t)\rangle_{L^2(\mathcal{V})} - \langle f(s), L_\mathcal{V}^T p(t)\rangle_{L^2(\mathcal{S})},
\end{aligned}$$

where v is the solution of (5), (6) either with $H = 0$ and $\phi = p$ or with $H = p$ and $\phi = 0$. Thus the function $(t,s) \mapsto \langle u(s), v(t)\rangle_{L^2(M)}$ satisfies a $1+1$ dimensional wave equation with a known right-hand side. We solve this wave equation in the triangle with corners (T,T), $(0,0)$ and $(0,2T)$, and obtain

$$\begin{aligned}
\langle u(T), v(T)\rangle_{L^2(M)} &= \frac{1}{2}\int_0^T \int_t^{2T-t} \langle \Lambda_\mathcal{V} f(s), p(t)\rangle_{L^2(\mathcal{V})} ds dt \\
&\quad - \frac{1}{2}\int_0^T \int_t^{2T-t} \langle f(s), L_\mathcal{V}^T p(t)\rangle_{L^2(\mathcal{S})} ds dt,
\end{aligned} \tag{10}$$

where we have used the fact that $u(0) = \partial_t u(0) = 0$. The identity (7) for $f \in C_0^\infty((0,T)\times\mathcal{S})$ and $p \in C_0^\infty((0,T)\times\mathcal{V})$ follows from (10) by applying the identity (9) to Jf.

The map U^T is continuous $L^2((0,T)\times\mathcal{S}) \to L^2(M)$, and the analogous statement is true for the map $W_\mathcal{V}^T$ [23, 17]. Hence the operator $K_\mathcal{V}^T$ has a unique continuous extension as an operator from $L^2((0,T)\times\mathcal{S})$ to $L^2((0,T)\times\mathcal{V})$ and the identity (7) holds for the extension. □

2.2. **Approximate controllability.** Next we consider approximate controllability on a domain of influence. Let $\Gamma \subset \partial M$ and $B \subset M^{\text{int}}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $h : \overline{\mathcal{V}} \to \mathbb{R}$ be piecewise continuous, and define the domain of influence

$$M(\mathcal{V}, h) := \{x \in M;\ \inf_{y \in \mathcal{V}} (d(x, y) - h(y)) \le 0\},$$

where d is the distance function of M. Moreover, we write

$$\mathcal{B}(\mathcal{V}, h; T) := \{(t, y) \in (0, \infty) \times \mathcal{V};\ T - h(y) < t\}.$$

We extend the notations $M(\mathcal{V}, h)$ and $\mathcal{B}(\mathcal{V}, h; T)$ for constants $h \in \mathbb{R}$ by interpreting h as a constant function. Moreover, we define $M(x, h)$ by $M(\{x\}, h)$ for points $x \in \partial M$.

We have the following approximate controllability result that is analogous to [24, Lemma 5] and [21, Lemma 3.6].

Lemma 2. *Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $h : \overline{\mathcal{V}} \to \mathbb{R}$ to be piecewise continuous. In the case when $\mathcal{V} = B$ suppose, moreover, that $h > 0$ pointwise. Then*

$$W_{\mathcal{V}}^T \left(C_0^\infty(\mathcal{B}(\mathcal{V}, h; T))\right)$$

is dense in $L^2(M(\mathcal{V}, h)) = \{w \in L^2(M);\ supp\,(w) \subset M(\mathcal{V}, h)\}$. Moreover, for all $x \in M(\mathcal{V}, h)$ there is $p \in C_0^\infty(\mathcal{B}(\mathcal{V}, h; T))$ such that $W_{\mathcal{V}}^T p(x) \ne 0$.

3. Local reconstruction of the first order perturbation

In this section we prove Theorem 1. As we have established the key elements of the Boundary Control method, that is, Lemmas 1 and 2, in the present context, the reconstruction of the geometry near $\mathcal{R}$ is analogous to the local reconstruction step in [24]. We refer to [25] for the details, and focus here on the recovery of the lower order terms A and Q.

Our proof is based on using the convexity of $\mathcal{R}$. Let us recall the definition of the boundary normal coordinates. Let $\Gamma \subset \partial M$ be open. Then the boundary normal coordinates adapted to Γ are given by the map

$$(s, y) \mapsto \gamma(s; y, \nu), \quad y \in \Gamma,\ s \in [0, \sigma_{\Gamma,M}(y)), \tag{11}$$

where the cut distance $\sigma_{\Gamma,M} : \Gamma \to (0, \infty)$ is defined by

$$\begin{aligned} \sigma_{\Gamma,M}(y) &= \max\{s \in (0, \tau_M(y)];\ d(\gamma(s; y, \nu), \Gamma) = s\}, \\ \tau_M(y) &= \sup\{s \in (0, \infty);\ \gamma(s; y, \nu) \in M^{\text{int}}\}. \end{aligned} \tag{12}$$

Here $\gamma(\cdot; x, \xi)$ is the geodesic with the initial data $(x, \xi) \in TM$, and we recall that ν is the interior unit normal on ∂M. We often write $\sigma_\Gamma = \sigma_{M,\Gamma}$. Note that $\sigma_\Gamma(y) > 0$, see e.g. [17, p. 50].

We define

$$M_\Gamma = \{\gamma(s; y, \nu);\ y \in \Gamma,\ s \in [0, \sigma_\Gamma(y))\}.$$

Then a point $x \in M_\Gamma$ is represented in the coordinates (11) by (s, y), where $s = d(x, \Gamma)$ and y is the unique closest point to x in Γ.

3.1. **A convexity argument.** Our aim is to construct a sequence of functions h_j, $j = 1, 2, \dots$ on $\mathcal{R}$ such that the difference of the domains of influences $M(\Gamma, s) \setminus M(\mathcal{R}, h_j)$ converges to a point as $j \to \infty$. Here $\Gamma \subset \mathcal{R}$ and $s > 0$ will be chosen suitably. We use the notation

$$B(p, r) = \{x \in M; d(x, p) < r\}, \quad p \in M,\ r > 0.$$

Lemma 3. *Let $\Gamma \subset \partial M$ be open and strictly convex, and let $\mathcal{K} \subset \Gamma$ be compact. Then there is $\delta(\mathcal{K}) > 0$ and a neighborhood $U(\mathcal{K}) \subset M_\Gamma$ of $\mathcal{K}$ such that, for all $p \in U(\mathcal{K})$ and $q \in B(p, \delta(\mathcal{K})) \setminus \{p\}$, there is $z \in \Gamma$ satisfying $d(z, q) < d(z, p)$.*

Proof. Let us consider a unit speed geodesic $\gamma(t) = (s(t), z(t))$ in coordinates (11) and denote the initial data of γ by

$$\gamma(0) = (s, y), \quad \dot\gamma(0) = (\rho, \eta),$$

where $s = s(0)$. We will first show that there is a neighborhood $U \subset M$ of $\mathcal{K}$ and $\rho_0 > 0$ such that, for all $(s, y) \in \overline{U}$ and $\rho \in [-1, \rho_0]$, the geodesic γ intersects Γ and is distance minimizing until the intersection.

To this end recall that, in coordinates (11), the metric tensor g is of the form

$$g(s, y) = \begin{pmatrix} 1 & 0 \\ 0 & h(s, y) \end{pmatrix},$$

and the Christoffel symbols Γ^l_{jk} satisfy

$$\Gamma^1_{j1} = \Gamma^j_{11} = 0, \quad g^{\alpha\gamma}\Gamma_{1\beta\gamma} = -\Gamma^\alpha_{\beta 1},$$

where g^{jk} is the inverse of g_{jk} and we are using the Einstein summation convention with the Greek indices running over $2, 3, \dots, n$. In particular,

$$s(t) = s + t\rho - \frac{t^2}{2}\Gamma^1_{\alpha\beta}(s, y)\eta^\alpha\eta^\beta + \mathcal{O}(t^3).$$

Let us also recall that the second fundamental form of Γ is given by $II(\partial_\alpha, \partial_\beta) = \Gamma^1_{\alpha\beta}(0, y)$, see e.g. [28, p. 113].

The strict convexity of Γ, the lower semi-continuity of the cut distance function σ_Γ and the compactness of $\mathcal{K}$ imply that there is a neighborhood $U_0 \subset M_\Gamma$ of $\mathcal{K}$ and $c > a > 0$ such that, for all $(s, y) \in \overline{U_0}$,

$$a|\eta|_h^2 \le \Gamma^1_{\alpha\beta}(s, y)\eta^\alpha\eta^\beta \le c|\eta|_h^2.$$

We will consider only the case $|\eta|_h^2 > 1/2$. Note that if $\rho_0 > 0$ is small and $|\eta|_h^2 \le 1/2$, then $\rho < \rho_0$ implies that $\rho < 0$ since (ρ, η) is an unit vector. For small $t > 0$, we have the bound

$$s + t\rho - ct^2 \le s(t) \le s + t\rho - \frac{at^2}{8}.$$

As $\mathcal{K} \subset \Gamma$ is closed and γ is unit speed, $z(t) \in \Gamma$ for small $t > 0$, if $s > 0$ or $s = 0$ but $\rho > 0$. It then follows that there is $\sigma > 0$ such that, for $s \le \sigma$, $y \in \mathcal{K}$, $\rho \le \sigma$ with $(s, y) \in U_0$, the geodesic $\gamma(t)$ intersects Γ at some $t = \tau(s, y; \rho, \eta)$. Moreover, $y(t) \in \Gamma$ for $0 \le t \le \tau(s, y; \rho, \eta)$ and $\gamma(t)$ is the distance minimizing up to $z_\gamma = \gamma(\tau)$. Thus, for $t > 0$,

$$d(z_\gamma, \gamma(t)) = \tau - t < d(z_\gamma, \gamma(0)). \tag{13}$$

Let us emphasize that the case $s = 0$ is also allowed in the above argument. We take $U = \{(s, y) \in U_0;\ s < \sigma\}$ and $\rho_0 = \sigma$.

Let $(s, y) \in U$, (ρ, η) be a unit vector and suppose that $\rho > \rho_0$. We may choose $\eta_0 = b\eta$, $0 < b < 1$, such that (ρ_0, η_0) is also a unit vector at (s, y). Then the geodesic $\gamma_0(s)$ with the initial data

$$\gamma_0(0) = (s, y), \quad \dot\gamma_0(0) = (\rho_0, \eta_0)$$

intersects Γ at z_{γ_0} and is distance minimizing until the intersection.

As $\rho_0 > 0$ we have that $\tau(s, y; \rho_0, \eta)$ is strictly positive for $(s, y) \in \overline{U}$ and $\eta_0 \in S := \{\eta \in \mathbb{R}^{n-1};\ |\eta|_h^2 + \rho_0^2 = 1\}$. Together with continuity of τ this implies

$$\tau_U := \min_{(s,y)\in\overline{U}, \eta_0 \in S} \tau(s, y; \rho_0, \eta_0)/2 > 0.$$

Moreover, the first variation formula, see e.g. [26, Prop. 10.2], implies

$$\partial_t d(\gamma_0(\tau_U), \gamma(t))|_{t=0} = -(\dot\gamma_0(0), \dot\gamma(0))_g = -\rho_0\rho - b|\eta_0|_h^2 \le -\rho_0^2.$$

It follows from the above inequality together with the relative compactness of U that there is $\delta > 0$ such that, if $t \in (0, \delta)$, $(s, y) \in U$, $\rho > \rho_0$, then

$$d(z_{\gamma_0}, \gamma(t)) \le d(z_{\gamma_0}, \gamma(0)) - t\rho_0^2/2. \tag{14}$$

The claim now follows from (13) and (14). □

Lemma 4. *Let $\Gamma \subset \partial M$ be open and let $p \in M_\Gamma$. Then, for all $q \in M(\Gamma, d(p, \Gamma)) \setminus \{p\}$, there is $z \in \overline{\Gamma}$ satisfying $d(z, q) < d(z, p)$.*

Proof. Let $p = (s, y)$, $s = d(p, \Gamma)$, in coordinates (11), and let z be a closest point to q in $\overline{\Gamma}$. If $z \ne y$ then

$$d(z, q) = d(q, \Gamma) \le d(p, \Gamma) < d(z, p),$$

since z is not the closest point to p in Γ. Suppose now that $z = y$ and write $r = d(y, q)$. Then $r \le d(p, \Gamma) = s$ and $q = (r, y)$ in coordinates (11). Moreover $q \ne p$, whence $r < s$. □

We define $B_{\partial M}(y, \epsilon) = \{x \in \partial M;\ d(x, y) < \epsilon\}$.

Lemma 5. *Let $\delta > 0$ and let $p \in M_\Gamma$ have the boundary normal coordinates (s, y). Then there is $\epsilon = \epsilon(p, \delta) > 0$ such that for all $q \in B(p, \epsilon)$,*

$$M(B_{\partial M}(y, \epsilon), s + \epsilon) \subset M(\Gamma, d(q, \Gamma)) \cup B(q, \delta). \tag{15}$$

Proof. To prove (15) we assume the contrary. Then there exist sequences $\epsilon_n \to 0$,

$$q_n = (r_n, z_n) \in B(p, \epsilon_n), \quad q_n' \in M(B_{\partial M}(y, \epsilon_n), s + \epsilon_n),$$

such that $d(q_n', \Gamma) > r_n$ and $d(q_n', q_n) \ge \delta$. Taking if necessary a subsequence, we may assume that $q_n' \to q'$. Then it follows from the above that

$$d(q', y) \le s, \quad d(q', \Gamma) \ge s, \quad d(q', p) \ge \delta.$$

This is a contradiction since the first two conditions imply $q' = p$. □

Lemma 6. *Let $\Gamma \subset \partial M$ be open and strictly convex and let $\mathcal{K} \subset \Gamma$ be compact. Let $\delta > 0$ and $U \subset M_\Gamma$ be as in Lemma 3. Let $p = (s, y) \in U$ and let $\epsilon = \epsilon(p, \delta) > 0$ be as in Lemma 5. We decrease $\epsilon > 0$, if necessary, so that $B_{\partial M}(y, \epsilon) \subset \Gamma$. Furthermore, we define*

$$C_p = ((s - \epsilon', s + \epsilon') \cap [0, \infty)) \times B_{\partial M}(y, \epsilon'),$$

where $\epsilon' \in (0, \epsilon)$ is chosen so that, in the coordinates (11), $C_p \subset B(p, \delta)$. Let $x \in C_p$ and define, for $j = 1, 2, \dots,$

$$\begin{aligned} X_j &= M(B_{\partial M}(y, \epsilon), s + \epsilon)^{int} \setminus M(\Gamma, h_j), \\ h_j(z) &= d(z, x) - 1/j, \quad z \in \Gamma. \end{aligned}$$

Then X_j is a neighborhood of x, and diam $(X_j) \to 0$ as $j \to \infty$.

The set X_j is visualized in Figure 1.

Proof. It is clear that $X_{j+1} \subset X_j$ and that $x \in X_j$ for all j. Suppose that $q \in \overline{X}_j$ for all j. If $q \notin B(x, \delta)$ then (15) yields that $q \in M(\Gamma, d(x, \Gamma))$. Now Lemma 4 implies that $q \in M(\Gamma, h_j)^{\text{int}}$ for large j which is a contradiction with $q \in \overline{X}_j$. If, however, $q \in B(x, \delta) \setminus \{x\}$, then Lemma 3 implies that $q \in M(\Gamma, h_j)^{\text{int}}$ for large j which is again a contradiction. Thus $q = x$. As the sequence of sets X_j is decreasing and $\bigcap_{j \ge 1} \overline{X_j} = \{x\}$, we have that diam $(X_j) \to 0$ as $j \to \infty$. □

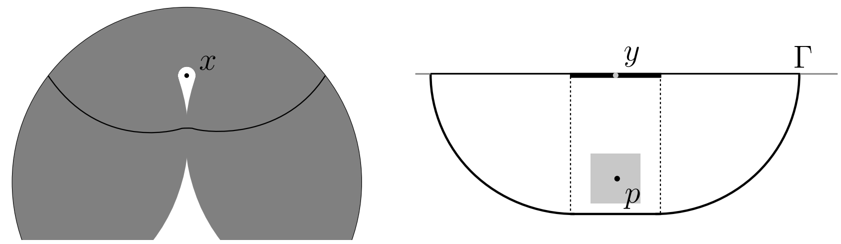

FIGURE 1. *Left.* A part of the domain of influence $M(\Gamma, h_j)$ in gray, where (M, g) is the Euclidean unit disk, $x = (0, 3/5)$, $j = 20$, and Γ is slightly less than the upper half circle. The black curve is the boundary of $M(B_{\partial M}(y, \epsilon), s + \epsilon)$ where $p = (s, y)$, $y = (0, 1)$, $s = 1/2$ and $\epsilon = 1/5$. The set X_j is the white region around x. *Right.* Schematic diagram of the sets $B_{\partial M}(y, \epsilon) \subset \Gamma$, in black around the gray point y, and C_p, in gray around the black point $p = (s, y)$. The black curve is the boundary of $M(B_{\partial M}(y, \epsilon), s+\epsilon)$.

3.2. **Localized solutions.** We denote by $|X|$ the Riemannian volume of a measurable set $X \subset M$.

Lemma 7. *Let $\mathcal{X} \subset M$ be open, $x \in \mathcal{X}$ and let $X_j \subset M$, $j = 1, 2, \dots$, be a sequence of neighborhoods of x satisfying $\lim_{j\to\infty}$ diam $(X_j) = 0$. Let $\psi_0 \in C_0^\infty(\mathcal{X})$ satisfy $\psi_0(x) \neq 0$. Let $T > 0$ and suppose that a sequence $(f_j)_{j=1}^\infty$ of functions in $L^2((0,T) \times \mathcal{S})$ satisfies*

(i) *there is $C > 0$ such that $\|f_j\|_{L^2((0,T)\times\mathcal{S})} \leq C|X_j|^{-1/2}$ for all j,*
(ii) *supp $(U^T f_j) \subset \overline{X}_j \cup (M \setminus \mathcal{X})$ for all j,*
(iii) *$(\langle U^T f_j, \psi_0\rangle_{L^2(M)})_{j=1}^\infty$ converges.*

Then there is $\kappa \in \mathbb{C}$ such that $\langle U^T f_j, \psi\rangle_{L^2(M)} \to \kappa\psi(x)$ for all functions $\psi \in C_0^\infty(\mathcal{X})$.

Furthermore, if the wave equation (1) is exactly controllable from $\mathcal{S}$ in time T, then there is a sequence $(f_j)_{j=1}^\infty$ that satisfies (i)-(iii) and for which $\kappa = 1$.

Let us emphasize that $\mathcal{X}$ may intersect ∂M in which case x may belong to ∂M.

Proof. To simplify the notation, we write $U = U^T$ and $u_j = Uf_j$. Let $\psi \in C_0^\infty(\mathcal{X})$. Then supp $(u_j\psi) \subset \overline{X}_j$ and

$$\langle u_j, \psi\rangle_{L^2(M)} = \psi(x)\langle u_j, 1\rangle_{L^2(M)} + R_j,$$

where, using some local coordinates $\widetilde{x}$ in X_j for all large enough j, the remainder term satisfies

$$\begin{aligned} |R_j| &\leq C\,\|\nabla\psi\|_{C(X_j)} \int_{X_j} |u_j(\widetilde{x})| d(\widetilde{x},x) d\widetilde{x} \\ &\leq C\,\|\nabla\psi\|_{C(X_j)} \|u_j\|_{L^2(M)} \left(\int_{X_j} d^2(\widetilde{x},x) d\widetilde{x} \right)^{1/2} \\ &\leq C\,\|\nabla\psi\|_{C(X_j)} \operatorname{diam}(X_j) \to 0. \end{aligned}$$

Notice that the constant $C > 0$ may increase between the inequalities and that, at the last inequality, we use $||u_j||_{L^2(M)} \leq C\,\|f_j\|_{L^2((0,T)\times\mathcal{S})}$, see [23], together with (i). We choose $\psi = \psi_0$ and see that $\lim_{j\to\infty}\langle u_j, 1\rangle_{L^2(M)}$ exists. We denote the limit by κ. Thus for any $\psi \in C_0^\infty(\mathcal{X})$ it holds that $\langle u_j, \psi\rangle_{L^2(M)} \to \kappa\psi(x)$ as $j \to \infty$.

Let us now assume that (1) is exactly controllable from $\mathcal{S}$ in time T. By exact controllability,

$$U : L^2((0,T)\times\mathcal{S}) \to L^2(M),$$

is surjective. Hence its pseudoinverse

$$U^\dagger : L^2(M) \to L^2((0,T)\times\mathcal{S})$$

is continuous and the composition $UU^\dagger$ gives the identity map, see e.g. [10, pp. 33-34]. Now $f_j = U^\dagger 1_{X_j}/|X_j|$ has the required properties. □

Lemma 8. *Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $T > 0$ and let $h : \overline{\mathcal{V}} \to [0,T]$ be piecewise continuous. In the case when $\mathcal{V} = B$ suppose, moreover, that $h > 0$ pointwise. Let $\mathcal{C} \subset M(\mathcal{V},h) \cap M^{int}$ be open and let $\kappa : \mathcal{C} \to \mathbb{C}$. We define*

$$w_\phi(x) = \kappa(x)(W_\mathcal{V}^T\phi)(x), \quad \phi \in C_0^\infty(\mathcal{B}(\mathcal{V},h;T)),\ x \in \mathcal{C},$$

and have the following:

1. *If $w_\phi \in C^\infty(\mathcal{C})$ for all $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V},h;T))$ then $\kappa \in C^\infty(\mathcal{C})$.*
2. *If for all $x \in \mathcal{C}$ there is $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V},h;T))$ such that $w_\phi(x) \neq 0$, then $\kappa(x) \neq 0$ for all $x \in \mathcal{C}$.*

Moreover, in the case $\mathcal{V} = \Gamma$ we can enforce smoothness up to the boundary, that is, we define $\widetilde{\mathcal{C}} = \mathcal{C} \cup (S \cap \overline{\mathcal{C}})$ where S is an open set in ∂M such that $h > 0$ in S, and have the following:

3. *If $w_\phi \in C^\infty(\widetilde{\mathcal{C}})$ for all $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V},h;T))$ then $\kappa \in C^\infty(\widetilde{\mathcal{C}})$.*

Proof. Let $x \in \mathcal{C}$. By Lemma 2, there is a neighborhood U of x and $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V},h;T))$ such that $W_\mathcal{V}^T\phi$ is non-vanishing in U. We have that $\kappa = w_\phi/W_\mathcal{V}^T\phi$ in U, and (1) and (2) follow.

Suppose now that $\mathcal{V} = \Gamma$ and that $x \in S \cap \overline{\mathcal{C}}$. Then there is a neighborhood of $U \subset M$ of x and $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V}, h; T))$ such that $W_\Gamma^T \phi$ is non-vanishing in U, since on S we can choose $W_\Gamma^T \phi = \phi(T)$ to be non-vanishing. The function $W_\Gamma^T \phi$ is smooth up to ∂M, whence κ is smooth in U. □

The below lemma follows immediately from Lemma 2.

Lemma 9. *Let $\Gamma_j \subset \partial M$ and $B_j \subset M^{int}$ be open, and let $\mathcal{V}_j = \Gamma_j$ or $\mathcal{V}_j = B_j$, $j = 1, 2$. Let $T > 0$ and let $h_j : \overline{\mathcal{V}}_j \to [0, T]$ be piecewise continuous. In the case when $\mathcal{V}_j = B_j$ suppose, moreover, that $h_j > 0$ pointwise. We define $\mathcal{V} = \mathcal{V}_2 \cap \mathcal{V}_1$ and $h = \min(h_1, h_2)$ on $\mathcal{V}$. Let $\mathcal{C}_j \subset M(\mathcal{V}_j, h_j)$ be open in M^{int}, and let $\kappa_j \in C^\infty(\mathcal{C}_j)$. Suppose that $\mathcal{C}_1 \cap \mathcal{C}_2 \subset M(\mathcal{V}, h)$ and that*

$$\kappa_1 W_\mathcal{V}^T \phi = \kappa_2 W_\mathcal{V}^T \phi, \quad \phi \in C_0^\infty(\mathcal{B}(\mathcal{V}, h; T)),$$

on $\mathcal{C}_1 \cap \mathcal{C}_2$. Then $\kappa_1 = \kappa_2$ on $\mathcal{C}_1 \cap \mathcal{C}_2$.

Let $\mathcal{C} \subset M^{\text{int}}$ be open and let $\kappa \in C^\infty(\mathcal{C})$. We define $\mathcal{A}_{\mathcal{C},\kappa}$ as the restriction of $\kappa^{-1} \mathcal{A} \kappa$ on $\mathcal{C}$, where $\mathcal{A} = \Delta_g - A - Q$.

Lemma 10. *Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $T > 0$ and $h : \overline{\mathcal{V}} \to [0, T]$ be piecewise continuous. Let $\mathcal{C} \subset M(\mathcal{V}, h) \cap M^{int}$ be open and let $\kappa \in C^\infty(\mathcal{C})$ be nowhere vanishing. Then the functions*

$$\kappa W_\mathcal{V}^T \phi, \quad \phi \in C_0^\infty(\mathcal{B}(\mathcal{V}, h; T)),$$

on $\mathcal{C}$, together with the Riemannian structure $(\mathcal{C}, g)$, determine the coefficients of the operator $\mathcal{A}_{\mathcal{C},\kappa}$.

Proof. Taking into account the translation invariance in time of (5), we can differentiate $\kappa W_\mathcal{V}^T \phi|_\mathcal{C}$ twice in time to get $\kappa \mathcal{A}^* W_\mathcal{V}^T \phi|_\mathcal{C}$. Let ψ be a function in $C_0^\infty(\mathcal{C})$. As we know the Riemannian structure on $\mathcal{C}$, we can compute

$$\langle \kappa \mathcal{A}^* W_\mathcal{V}^T \phi, \psi \rangle_{L^2(\mathcal{C})} = \langle \kappa W_\mathcal{V}^T \phi, \kappa^{-1} \mathcal{A} \kappa \psi \rangle_{L^2(\mathcal{C})}.$$

As the functions $W_\mathcal{V}^T \phi|_\mathcal{C}$, $\phi \in C_0^\infty(\mathcal{B}(\mathcal{V}, h; T))$, are dense on $L^2(\mathcal{C})$, and we know $\kappa W_\mathcal{V}^T \phi|_\mathcal{C}$ and $\kappa \mathcal{A}^* W_\mathcal{V}^T \phi|_\mathcal{C}$, we can recover $\kappa^{-1} \mathcal{A} \kappa \psi$. This allows us to determine the coefficients of the operator $\kappa^{-1} \mathcal{A} \kappa$ on $\mathcal{C}$. □

3.3. **Local reconstruction near the set $\mathcal{R}$.** We are ready to prove the local result formulated in the introduction.

Proof of Theorem 1. As the wave equation is exactly controllable from $\mathcal{S}$ in time T, [2, Theorem 3.2] implies that $\sigma_\mathcal{R} \leq T$ pointwise on $\mathcal{R}$. We recall that $\sigma_\mathcal{R}$ is defined by (12).

Let $\mathcal{K} \subset \mathcal{R}$ be compact, and consider the sets defined in Lemma 6. We write $C_p(\mathcal{K}) = C_p$ to emphasize the dependence on $\mathcal{K}$, and use an analogous notation also for other quantities in Lemma 6. Furthermore, we write $\Gamma_p(\mathcal{K}) = B_{\partial M}(y, \epsilon)$.

As discussed in the beginning of Section 3, by using the method of [24, 25], we recover first $(M_\mathcal{R}, g)$. Then we can construct the functions h_j and the sets X_j for any $x \in C_p(\mathcal{K})$ and $p \in U(\mathcal{K})$. We choose $\mathcal{X} = M(\Gamma_p(\mathcal{K}), s + \epsilon)^{\text{int}}$ in Lemma 7. As $(M_\mathcal{R}, g)$ is known, we can determine if the condition (i) of Lemma 7 holds. Lemmas 1 and 2 imply that the condition (ii) holds if and only if

$$\langle \phi, K_\mathcal{R}^T f_j \rangle_{L^2((0,T)\times\mathcal{R})} = 0, \quad \phi \in C_0^\infty(\mathcal{B}(\mathcal{R}, h_j; T)),$$

where $K_\mathcal{R}^T$ is defined by (4). Note that $(M_\mathcal{R}, g)$ determines the Riemannian volume measure of $(\mathcal{R}, g)$, whence we can compute the above inner products. The condition (iii) holds if $\lim_{j\to\infty} \langle \phi, K_\mathcal{R}^T f_j \rangle_{L^2((0,T)\times\mathcal{R})}$ exists for all $\phi \in C_0^\infty((0,\infty) \times \mathcal{R})$. Indeed, by Lemma 2 there is $\phi \in C_0^\infty((0,\infty) \times \mathcal{R})$ such that $W_\mathcal{R}^T \phi$ does not vanish at a given point in $M_\mathcal{R}$, and we can apply Lemma 7 with $\psi_0 = W_\mathcal{R}^T \phi$.

Let us use the shorthand notation $\mathcal{B}_p(\mathcal{K}) = \mathcal{B}(\Gamma_p(\mathcal{K}), s+\epsilon, T)$. As we can verify if all the conditions (i)-(iii) in Lemma 7 hold, we can choose for all $p \in U(\mathcal{K})$ and $x \in C_p(\mathcal{K})$ a sequence

$$F(x; p, \mathcal{K}) = (f_j)_{j=1}^\infty \subset L^2((0,T) \times \mathcal{S})$$

such that $\langle \phi, K_\mathcal{R}^T f_j \rangle_{L^2(M)}$ converges to

$$\kappa(x; F, p, \mathcal{K})(W_\mathcal{R}^T \phi)(x), \quad \phi \in C_0^\infty(\mathcal{B}_p(\mathcal{K})),$$

where the factors $\kappa(x; F, p, \mathcal{K})$ remain unknown and depend on x, p, $\mathcal{K}$ and the choice of the sequence $F = F(x; p, \mathcal{K})$. We will next impose further conditions on the choice of F that enforce the functions $\kappa(x; F, p, \mathcal{K})$ to be restrictions of a smooth function defined in a neighborhood of $\mathcal{R}$.

Using Lemma 8 we choose $F = F(x)$ such that

$$\kappa_{p,\mathcal{K}}(x) = \kappa(x; F(x), p, \mathcal{K})$$

is smooth and nowhere vanishing on $C_p(\mathcal{K})$. Note that the second claim in Lemma 7 implies that there exists such a choice of $F(x)$. We choose a collection J of compact sets in $\mathcal{R}$ such that $\bigcup_{\mathcal{K}\in J} \mathcal{K} = \mathcal{R}$. Let us now use Lemma 9 to enforce the functions

$$\kappa_{p,\mathcal{K}} \in C^\infty(C_p(\mathcal{K})), \quad p \in U(\mathcal{K}),\ \mathcal{K} \in J, \tag{16}$$

to be restrictions of a smooth function defined on the set $U(\mathcal{K})$. Let $\mathcal{K}_j \in J$ and $p_j = (s_j, y_j) \in U(\mathcal{K}_j)$, $j = 1, 2$, and define

$$C = C_{p_1}(\mathcal{K}_1) \cap C_{p_2}(\mathcal{K}_2).$$

Note that since $C_{p_j}(\mathcal{K}_j)$, $j = 1, 2$, are cylinders so is C, in fact,

$$C = ((a, b) \cap [0, \infty)) \times \Gamma,$$

where $a = \max_{j=1,2} s_j - \epsilon'_{p_j}(\mathcal{K}_j)$, $b = \min_{j=1,2} s_j + \epsilon'_{p_j}(\mathcal{K}_j)$ and

$$\Gamma = \bigcap_{j=1,2} B_{\partial M}(y_j, \epsilon'_{p_j}(\mathcal{K}_j)).$$

Note that $C \subset M(\Gamma, b)$. We require that

$$\kappa_{p_1,\mathcal{K}_1} W^T_{\mathcal{R}} \phi = \kappa_{p_2,\mathcal{K}_2} W^T_{\mathcal{R}} \phi, \quad \phi \in C^\infty_0(\bigcap_{j=1,2} \mathcal{B}_{p_j}(\mathcal{K}_j)),$$

on C. Then $\kappa_{p_1,\mathcal{K}_1} = \kappa_{p_2,\mathcal{K}_2}$ on C by Lemma 9. It follows that the functions (16) fit together on $\mathcal{U} = \bigcup_{\mathcal{K}\in J} U(\mathcal{K})$ and form a function $\kappa \in C^\infty(\mathcal{U})$. We require, furthermore, that

$$\kappa(x) W^T_{\mathcal{R}} \phi(x) = \phi(T, x), \quad x \in \overline{C_p(\mathcal{K})} \cap \mathcal{K},\ \phi \in C^\infty_0(\mathcal{B}_p(\mathcal{K})),$$

whenever $\mathcal{K} \in J$ and $p \in U(\mathcal{K})$ satisfy $\overline{C_p(\mathcal{K})} \cap \mathcal{K} \neq \emptyset$. By varying p and $\mathcal{K}$, we get $\kappa = 1$ on $\mathcal{R}$. We may choose $\mathcal{U}$ to be a slightly smaller neighborhood of $\mathcal{R}$ to guarantee that $\kappa \in C^\infty(\overline{\mathcal{U}})$.

We apply Lemma 10 to recover $\kappa^{-1}\mathcal{A}\kappa$ on each $C_p(\mathcal{K}_j)$, $p \in U(\mathcal{K}_j)$. This gives us $\kappa^{-1}\mathcal{A}\kappa = \Delta_g - A_\kappa - Q_\kappa$ on $\mathcal{U}$, and $(A_\kappa|_{\mathcal{U}}, Q_\kappa|_{\mathcal{U}})$ belongs to the orbit $\mathcal{G}_{\mathcal{U},\mathcal{R}}(A, Q)$. Hence we can determine the orbit $\mathcal{G}_{\mathcal{U},\mathcal{R}}(A, Q)$. □

4. Reconstruction of the first order perturbation along a convex foliation

In this section we prove the following global result:

Theorem 3. *Let $\mathcal{S} \subset \partial M$ be open and suppose that the wave equation (1) is exactly controllable from $\mathcal{S}$. Let $\mathcal{R} \subset \partial M$ be open and strictly convex and let Σ_s, $s \in (0, 1]$, be a convex foliation satisfying (F1)-(F7). Then $\Lambda^\infty_{\mathcal{S},\mathcal{R}}$, together with (Ω_1, g), determines the orbit $\mathcal{G}_{\Omega_1,\mathcal{R}}(A, Q)$.*

The proof is based on iterating the local reconstruction method of the previous section along the convex foliation.

4.1. **Local reconstruction near the set Σ_s.** Let Σ_s, $s \in (0,1]$, be a convex foliation satisfying (F1)-(F7). Let $\Gamma \subset \Sigma_s$ be open and let $h : \overline{\Gamma} \to \mathbb{R}$ be piecewise continuous. We recall that M_s is defined in (F4), and consider the domain of influence on M_s,

$$M_s(\Gamma, h) := \{x \in M_s;\ \inf_{y\in\Gamma}(d_{M_s}(x,y) - h(y)) \le 0\}.$$

Here $d_{M_s}(x,y)$ is the distance function on (M_s, g). We will also use the notation $d_{\overline{\Omega}_s}(x,y)$ for the distance function on $(\overline{\Omega}_s, g)$.

Lemma 11. *Let Σ_s, $s \in (0,1]$, be a convex foliation satisfying (F1)-(F7), and let $s \in (0,1]$. Let $h : \overline{\Sigma}_s \to \mathbb{R}$ be piecewise continuous. Then*

$$M_s(\Sigma_s, h) \cup \overline{\Omega}_s = M(\Omega_s, \widetilde{h}), \tag{17}$$

where $\widetilde{h}(y) = \max(\sup_{z\in\Sigma_s}(h(z) - d_{\overline{\Omega}_s}(z,y)), d_{\overline{\Omega}_s}(y, \partial\Omega_s))$.

Proof. Let us show first that

$$d(x,z) = d_{M_s}(x,z), \quad x, z \in M_s.$$

It is enough to show that a shortest path γ between x and z stays in M_s. To get a contradiction suppose that $S < s$, where

$$S = \inf\{r \in [0,s];\ \gamma \cap \Sigma_r \ne \emptyset\},$$

and we have used the notation $\Sigma_0 = \mathcal{R}_0$. Let $p \in \gamma \cap \Sigma_S$. Let us consider first the case $S > 0$. Then γ is a geodesic near p. As $\gamma \cap \Omega_S = \emptyset$, the intersection is tangential. But then the strict convexity of Σ_S implies that γ is in Ω_S near p, which is a contradiction. On the other hand, if $S = 0$ then the intersection must be tangential again, since a shortest path is C^1. But this is impossible by the strict convexity of $\Sigma_0 \subset \mathcal{R}$.

Let us now show (17). Note that $\widetilde{h}(y) \ge h(y)$ for $y \in \Sigma_s$ and that $\widetilde{h} > 0$ on Ω_s. Hence $M_s(\Sigma_s, h) \cup \overline{\Omega}_s \subset M(\Omega_s, \widetilde{h})$. On the other hand, if $x \in M(\Omega_s, \widetilde{h}) \setminus \overline{\Omega}_s$ then there is $y \in \overline{\Omega}_s$ such that $d(x,y) - \widetilde{h}(y) \le 0$ and $z \in \overline{\Sigma}_s$ such that $\widetilde{h}(y) = h(z) - d_{\overline{\Omega}_s}(z,y)$. Thus

$$d_{M_s}(x,z) - h(z) = d(x,z) - d_{\overline{\Omega}_s}(z,y) - \widetilde{h}(y) \le d(x,y) - \widetilde{h}(y) \le 0,$$

and $x \in M_s(\Sigma_s, h)$. $\square$

Let us prove next the following analogue of Theorem 1 with data on Ω_s. Note that contrary to Theorem 1, we do not require κ to have a specific value on Σ_s. We recall that for and open set $U \subset M^{\mathrm{int}}$, $\Lambda_{U,\kappa} f = \kappa u|_{(0,\infty)\times U}$, where u is the solution of (1).

Lemma 12. *Let $\mathcal{S} \subset \partial M$ be open and suppose that the wave equation (1) is exactly controllable from $\mathcal{S}$ in time $T > 0$. Let Σ_s, $s \in (0,1]$, be a convex foliation satisfying (F1)-(F7), let $s \in (0,1]$, and let $\kappa_0 \in C^\infty(\overline{\Omega_s})$ be nowhere vanishing. Then there is a neighborhood $\mathcal{U}_s \subset M_s$ of Σ_s such that $\Lambda_{\Omega_s,\kappa_0}$ determines the family of operators*

$$\{\mathcal{A}_{\mathcal{U}_s,\kappa};\ \kappa \in C^\infty(\mathcal{U}_s),\ \kappa(x) \neq 0, x \in \mathcal{U}_s\}. \tag{18}$$

Proof. Let $\mathcal{K} \subset \Sigma_s$ be compact, and let us consider the sets defined in Lemma 6 where M is replaced with M_s. We define $\mathcal{X} = M_s(\Omega_s, \widetilde{h})^{\text{int}}$ and $\mathcal{B}_p(\mathcal{K}) = \mathcal{B}(\Omega_s, \widetilde{h}, T)$, where $\widetilde{h}$ is as in Lemma 11 with the choice $h = (s+\epsilon)1_{\Gamma_p(\mathcal{K})}$, and s and ϵ are as in Lemma 6.

Analogously to the proof of Theorem 1, we use Lemma 2 together with Lemma 11 to determine if the conditions (i)-(iii) of Lemma 7 hold. We can choose for all $p \in U(\mathcal{K})$ and $x \in C_p(\mathcal{K})$ a sequence

$$F(x;p,\mathcal{K}) = (f_j)_{j=1}^\infty \subset L^2((0,T) \times \mathcal{S})$$

such that $\langle \phi, K_{\Omega_s}^T f_j \rangle_{L^2(M)}$ converges to

$$\kappa(x;F,p,\mathcal{K})(W_{\Omega_s}^T \phi)(x), \quad \phi \in C_0^\infty(\mathcal{B}_p(\mathcal{K})),$$

where the factors $\kappa(x;F,p,\mathcal{K})$ remain again unknown and depend on x, p, $\mathcal{K}$ and the choice of the sequence $F = F(x;p,\mathcal{K})$.

We choose a collection J of compact sets in Σ_s such that they cover Σ_s, and use again Lemmas 8 and 9 to enforce

$$\kappa(x;F,p,\mathcal{K}), \quad x \in C_p(\mathcal{K}),\ p \in U(\mathcal{K}),\ \mathcal{K} \in J,$$

to form a smooth nowhere vanishing function κ on $\mathcal{U}_s = \bigcup_{\mathcal{K} \in J} U(\mathcal{K})$. Finally, we apply Lemma 10 to recover the orbit (18). □

4.2. **Gluing of the gauges.** Let $\mathcal{S}, \mathcal{R} \in \partial M$ satisfy the assumptions of Theorem 3, and let Σ_s, $s \in (0,1]$, be a convex foliation satisfying (F1)-(F7). We define

$$J = \{s \in (0,1];\ \ \Lambda_{\mathcal{R}} \text{ determines the orbit } \mathcal{G}_{U,\mathcal{R}}(A,Q) \text{ for open } U \subset M \text{ containing } \overline{\Omega}_s\}. \tag{19}$$

The set J is nonempty by (F6), since for small enough s we have that $\Omega_s \subset \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of $\mathcal{R}$ as in Theorem 1. Moreover, the continuity condition (F5) implies that J is open. Theorem 3 follows after we have shown that J is closed.

Lemma 13. *Let $U \subset M^{int}$ be open and suppose that $\overline{U} \cap \partial M \subset \mathcal{R}$. Suppose that $\kappa : \overline{U} \to \mathbb{C}$ is smooth near $\mathcal{R}$ and that $\Lambda_{U,\kappa} f \in C^\infty(\overline{U})$ for all $f \in C_0^\infty((0,\infty) \times \mathcal{S})$. Then $\kappa \in C^\infty(\overline{U})$.*

Proof. Let $x \in \overline{U} \cap M^{\text{int}}$. By Lemma 2 there is $f \in C_0^\infty((0,\infty)\times\mathcal{S})$ and a neighborhood $B \subset M^{\text{int}}$ of x such that $u(T) \neq 0$ in B. This implies that κ is smooth in $B \cap \overline{U}$. □

Let $U \subset M^{\text{int}}$ be open. We define

$$\mathbb{K}(U) := \{\kappa \in C^\infty(\overline{U});\ \kappa|_{\overline{U}\cap\mathcal{R}} = 1,\ \kappa(x) \neq 0,\ x \in \overline{U}\}.$$

Lemma 14. *Let $U \subset M^{int}$ be open and connected and suppose that $\overline{U} \cap \partial M \subset \mathcal{R}$ and that the interior of $\overline{U} \cap \mathcal{R}$ in ∂M is nonempty. We consider the family of operators $\mathcal{F} = \{\mathcal{A}_{U,\kappa};\ \kappa \in \widetilde{\mathbb{K}}(U)\}$, where $\widetilde{\mathbb{K}}(U)$ is the set of piecewise smooth functions $\kappa : \overline{U} \to \mathbb{C}$ such that κ is nowhere vanishing, $\kappa = 1$ in $\overline{U} \cap \mathcal{R}$, κ is smooth near $\mathcal{R}$, and that $\mathcal{A}_{U,\kappa}$ has smooth coefficients. Then $\mathcal{F}$ and $\Lambda_\mathcal{R}$ determine the family $\{(\mathcal{A}_{U,\kappa}, \Lambda_{U,\kappa});\ \kappa \in \mathbb{K}(U)\}$.*

Proof. Let $\kappa^{-1}\mathcal{A}\kappa \in \mathcal{F}$. Let $p \in U$ be such that $p \in M(\Gamma, r) \subset U$ for some open set $\Gamma \subset \overline{U} \cap \mathcal{R}$ and $r > 0$. We solve the wave equation corresponding to the adjoint of $\kappa^{-1}\mathcal{A}\kappa$,

$$\begin{aligned} \partial_t^2 w - \kappa\mathcal{A}^*\kappa^{-1} w = 0, &\quad \text{in } (0,T)\times M,\\ w|_{(0,\infty)\times\partial M} = \phi, &\quad \text{in } (0,T)\times\partial M,\\ w|_{t=0} = \partial_t w|_{t=0} = 0, &\quad \text{in } M. \end{aligned}$$

where $\phi \in C_0^\infty((T-r,T)\times\Gamma)$. Then $\kappa^{-1} w = v$, where v solves (5), (6) with $H = 0$. By Lemma 1, $\Lambda_\mathcal{R}$ determines the inner products

$$\langle W_\mathcal{R}^T \phi, U^T f\rangle_{L^2(M)} = \langle w(T), \kappa^{-1} U^T f\rangle_{L^2(M)}.$$

The functions $w(T)$, $h \in C_0^\infty((T-r,T)\times\Gamma)$, are dense in $L^2(M(\Gamma,r))$ by Lemma 2, and thus we can recover $\kappa^{-1}U^T f$ in $M(\Gamma, r)$. By using the translation invariance in time, we recover the operator $\Lambda_{B,\kappa}$, where $B = M(\Gamma,r)^{\text{int}}$.

Let us now suppose that $p \in U$ and $\epsilon > 0$ satisfy $B(p, 2\epsilon) \subset U$. We will show that the maps $\Lambda_{B(p,\epsilon),\kappa}$ and $\kappa\mathcal{A}^*\kappa^{-1}$ on U determine the map $\Lambda_{B(p,2\epsilon),\kappa}$.

We can compute the inner products

$$\langle W_{B(p,\epsilon)}^T H, U^T f\rangle_{L^2(M)}, \quad H \in C_0^\infty((0,\infty)\times B(p,\epsilon)),\ f \in C_0^\infty((0,\infty)\times\mathcal{S}),$$

by using Lemma 1, and we can solve the wave equation

$$\begin{aligned} \partial_t^2 w - \kappa\mathcal{A}^*\kappa^{-1} w = H, &\quad \text{in } (0,T)\times M,\\ w|_{(0,\infty)\times\partial M} = 0, &\quad \text{in } (0,T)\times\partial M,\\ w|_{t=0} = \partial_t w|_{t=0} = 0, &\quad \text{in } M. \end{aligned}$$

whenever $H \in C_0^\infty((T-\epsilon,T)\times B(p,\epsilon))$. Moreover, $W^T_{B(p,\epsilon)}H = \kappa^{-1}w(T)$, whence $\kappa^{-1}U^T f|_{B(p,2\epsilon)}$ can be determined from the inner products

$$\langle W^T_{B(p,\epsilon)}H, U^T f\rangle_{L^2(M)} = \langle \kappa^{-1}w(T), U^T f\rangle_{L^2(M)}.$$

By using the translation invariance in time, we recover the operator $\Lambda_{B(p,2\epsilon),\kappa}$.

A point $p \in U$ can be connected to $\mathcal{R}$ with a path $\gamma : [0,1] \to \overline{U}$ such that γ can be covered by a domain of influence $M(\Gamma, r) \subset U$, where $\Gamma \subset \mathcal{R}$ is open and $r > 0$, and sets $B(\gamma(t),\epsilon)$ such that $B(\gamma(t), 2\epsilon) \subset U$. Here $t \in [t_0, 1]$, $\epsilon > 0$ and $B(\gamma(t_0),\epsilon) \subset M(\Gamma, r)$. Now we can iteratively move the data $\Lambda_{B(\gamma(t),\epsilon),\kappa}$ along γ. This gives us the operator $\Lambda_{B(\gamma(t),\epsilon),\kappa}$, and as $p \in U$ can be chosen arbitrarily, the operator $\Lambda_{U,\kappa}$ is determined. Finally, by using Lemma 13, we can enforce κ to be smooth in $\overline{U}$. □

Corollary 1. *Let $s \in J$ where J is defined by (19). Then $\Lambda_{\mathcal{R}}$ determines the family $\{(\mathcal{A}_{\Omega_s,\kappa}, \Lambda_{\Omega_s,\kappa});\ \kappa \in \mathbb{K}(\Omega_s)\}$.*

Lemma 15. *Let $s_1, s_2 \in J$ where J is defined by (19), and suppose that $s_1 < s_2$. Then $\Lambda_{\mathcal{R}}$ determines the family*

$$\{((\mathcal{A}_{\Omega_{s_j},\kappa_j}, \Lambda_{\Omega_{s_j},\kappa_j}))_{j=1}^2;\ \kappa_2|_{\Omega_{s_1}} = \kappa_1,\ \kappa_j \in \mathbb{K}(\Omega_{s_j}), j = 1,2\}.$$

Proof. By Corollary 1 we can determine the two families of operators $\mathcal{F}_j = \{(\mathcal{A}_{\Omega_{s_j},\kappa_j}, \Lambda_{\Omega_{s_j},\kappa_j});\ \kappa_j \in \mathbb{K}(\Omega_{s_j})\}$, $j = 1, 2$. Let $(\mathcal{A}_{\Omega_{s_j},\kappa_j}, \Lambda_{\Omega_{s_j},\kappa_j})$ be in $\mathcal{F}_j$, $j = 1, 2$. We require that on Ω_{s_1}

$$\Lambda_{\Omega_{s_1},\kappa_1} f = \Lambda_{\Omega_{s_2},\kappa_2} f, \quad f \in C_0^\infty((0,\infty)\times\mathcal{S}).$$

Let $x \in \Omega_{s_1}$. By Lemma 2 there is $f \in C_0^\infty((0,\infty)\times\mathcal{S})$ and a neighborhood $B \subset M^{\text{int}}$ of x such that $u(T) \neq 0$ in B, where u is the solution of (1). Thus $\kappa_1 = \kappa_2$ in B. □

Lemma 16. *Let $s_j \in J$, $j = 1, 2, \dots$, form a strictly increasing sequence, and suppose that $\lim_{j\to\infty} s_j = s$. Here J is defined by (19). Then $\Lambda_{\mathcal{R}}$ determines the family $\{(\mathcal{A}_{\Omega_s,\kappa}, \Lambda_{\Omega_s,\kappa});\ \kappa \in \mathbb{K}(\Omega_s)\}$.*

Proof. An induction using Corollary 1 and Lemma 15 shows that $\Lambda_{\mathcal{R}}$ determines the family

$$\{((\mathcal{A}_{\Omega_{s_j},\kappa_j}, \Lambda_{\Omega_{s_j},\kappa_j}))_{j=1}^\infty;\ \kappa_{j+1}|_{\Omega_{s_j}} = \kappa_j,\ \kappa_j \in \mathbb{K}(\Omega_{s_j}),\ j = 1,2,\dots\}.$$

The functions κ_j, $j = 1, 2, \dots$, fit together and give a function Ω_s. Thus $\Lambda_{\mathcal{R}}$ determines the families of $(\mathcal{A}_{\Omega_s,\kappa}, \Lambda_{\Omega_s,\kappa})$, where κ is smooth in Ω_s, up to $\overline{\Omega_s} \cap \mathcal{R}$, κ is nowhere vanishing and satisfies $\kappa = 1$ in $\overline{\Omega_s} \cap \mathcal{R}$. By using Lemma 13 we can enforce κ to be smooth up to $\overline{\Omega_s} \cap M^{\text{int}}$. □

We are now ready to prove the global result.

Proof of Theorem 3. It remains to show that J is closed. Let $s_j \in J$, $j = 1, 2, \dots$, form a strictly increasing sequence, and suppose that $\lim_{j\to\infty} s_j = s$. We will show that $s \in J$. By Lemma 16 we can determine $\mathcal{A}_{\Omega_s,\kappa_0}$ and $\Lambda_{\Omega_s,\kappa_0}$ for some $\kappa_0 \in \mathbb{K}(\Omega_s)$. By Lemma 12, $\Lambda_{\Omega_s,\kappa_0}$ determines the family (18). Let $\mathcal{A}_{\mathcal{U}_s,\kappa}$ be in the family (18), and require furthermore that $\mathcal{A}_{\Omega_s,\kappa_0}$ and $\mathcal{A}_{\mathcal{U}_s,\kappa}$ fit together in the sense that they are restrictions of an operator with smooth coefficients on $U = \Omega_s \cup \mathcal{U}_s$. Then (2) implies that the function

$$\widetilde{\kappa}(x) = \begin{cases} \kappa_0(x), & x \in \Omega_s, \\ \kappa(x), & x \in \mathcal{U}_s, \end{cases}$$

is smooth except possibly on Σ_s. Now Lemma 14 allows us to restrict the choice of κ so that $\widetilde{\kappa}$ is in $\mathbb{K}(U)$. □

5. Reconstruction of the geometry

In this section we briefly explain how the method of [24] can be adapted to the reconstruction of (Ω_1, g) in the context of Theorem 2.

The lemma below follows immediately from the identity (7) and the definition of the exact controllability.

Lemma 17. *Suppose that the wave equation (1) is exactly controllable from $\mathcal{S}$ in time T. Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $(f_j)_{j=1}^\infty \subset C_0^\infty((0,\infty) \times \mathcal{V})$. Then $(W_\mathcal{V}^T f_j)_{j=1}^\infty$ converges weakly to zero in $L^2(M)$ if and only if*

$$\lim_{j\to 0} \left\langle f_j, K_\mathcal{V}^T \psi \right\rangle_{L^2((0,T)\times\mathcal{V})} = 0, \quad \psi \in L^2((0,T) \times \mathcal{S}).$$

The lemma below allows us to extract geometric information from the knowledge of weakly convergent sequences. The lemma follows from Lemma 2 and we refer to [24, Lemma 6] for a proof.

Lemma 18. *Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $h_\ell : \overline{\mathcal{V}} \to [0, T]$, $\ell = 1, 2$, be piecewise continuous functions. In the case that $\mathcal{V}$ is B suppose, moreover, that $h_1 > 0$ and $h_2 > 0$ pointwise. Then the following properties are equivalent:*

(i) *$M(\mathcal{V}, h_1) \subset M(\mathcal{V}, h_2)$.*
(ii) *For all $f_0 \in C_0^\infty(\mathcal{B}(\mathcal{V}, h_1; T))$ there is $(f_j)_{j=1}^\infty \subset C_0^\infty(\mathcal{B}(\mathcal{V}, h_2; T))$ such that $W_\mathcal{V}^T(f_0 - f_j)$ converges weakly to zero in $L^2(M)$.*

By combining Lemmas 1, 11,17 and 18 we see that Λ_{Ω_s} and (Ω_s, g) determine if

$$M_s(\Sigma_s, h_1) \subset M_s(\Sigma_s, h_2)$$

holds for piecewise continuous h_1 and h_2 on $\overline{\Sigma}_s$. Now [24] implies that this relation determines (M_{Σ_s}, g), where M_{Σ_s} is the image of

$$\{(r,y) \in (0,\infty) \times \Sigma_s;\ r < \sigma_{M_s,\Sigma_s}(y)\}$$

under the map $(r,y) \mapsto \gamma(r; y, \nu_s)$. Here ν_s is the interior unit normal of M_s.

The above step recovering (M_{Σ_s}, g) given Λ_{Ω_s} can be iterated similarly to the iteration in Section 4. We refer to [21, Section 4] for a detailed exposition of gluing arguments that can be used to construct an isometric copy of (Ω_1, g).

6. Complementary results

In this section we show that instead of assuming exact controllability from $\mathcal{S}$ and strict convexity of $\mathcal{R}$, we may assume that exact controllability holds from $\mathcal{S}$ or $\mathcal{R}$ and that one of them is strictly convex. Then we can determine the geometry and the lower order terms near the strictly convex set $\mathcal{R}$ or $\mathcal{S}$.

Observe first that the adjoint of $\Lambda^T_{\mathcal{S},\mathcal{R}}$ is $R\Lambda^T_{\mathcal{R},\mathcal{S}}R$ where R is the time-reversal $R\phi(t) = \phi(T-t)$. Thus Theorem 1 implies that we can determine the geometry and the lower order terms near $\mathcal{S}$ if it is strictly convex and exact controllability holds from $\mathcal{R}$.

Let us show that the conclusion of Theorem 1 holds when $\mathcal{R}$ is strictly convex and the wave equation (1) is exactly controllable from $\mathcal{R}$. The fourth case, that is, $\mathcal{S}$ is strictly convex and exact controllability holds from there, follows then again by transposition.

We used the exact controllability twice in the proof of Theorem 1, namely in Lemma 17 and when we invoked Lemma 7 in the proof of Theorem 1. We will give next the analogies of Lemmas 17 and 7 in the case when the exact controllability holds from $\mathcal{R}$ instead of from $\mathcal{S}$, and outline how this change affects the proof of Theorem 1.

Lemma 19. *Suppose (1) is exactly controllable from $\mathcal{R}$ in time T and that $T > \max_{x\in M} d(x,\mathcal{S})$. Let $\Gamma \subset \partial M$ and $B \subset M^{int}$ be open, and let $\mathcal{V} = \Gamma$ or $\mathcal{V} = B$. Let $(f_j)_{j=1}^\infty \subset C_0^\infty((0,\infty)\times\mathcal{V})$. Then $(W^T_{\mathcal{V}} f_j)_{j=1}^\infty$ converges weakly to zero in $L^2(M)$ if and only if both (a) and (b) hold where*

- (a) $\sup_{j,k\in\mathbb{N}} |\langle f_j, K^T_{\mathcal{V}}\psi_k\rangle_{L^2((0,T)\times\mathcal{V})}| < \infty$ *for all sequences* $(\psi_k)_{k=1}^\infty$ *in* $C_0^\infty((0,T)\times\mathcal{S})$ *such that* $(K^T_{\mathcal{V}}\psi_k)_{k=1}^\infty$ *is bounded in the space* $L^2((0,T)\times\mathcal{V})$.
- (b) $\lim_{j\in\mathbb{N}} \langle f_j, K^T_{\mathcal{V}}\psi\rangle_{L^2((0,T)\times\mathcal{V})} = 0$ *for all* $\psi \in C_0^\infty((0,T)\times\mathcal{S})$.

Proof. We begin by showing that $(K_{\mathcal{R}}^T \psi_k)_{k=1}^\infty$ is bounded in the space $L^2((0,T)\times\mathcal{R})$ if and only if $(U^T\psi_k)_{k=1}^\infty$ is bounded in the space $L^2(M)$. Suppose that $(K_{\mathcal{R}}^T \psi_k)_{k=1}^\infty$ is bounded and let $w \in L^2(M)$. As (1) is exactly controllable from $\mathcal{R}$ in time T, there is $f \in L^2((0,T)\times\mathcal{R})$ such that $W_{\mathcal{R}}^T f = w$. Thus

$$\left\langle w, U^T\psi_k\right\rangle_{L^2(M)} = \left\langle f, K_{\mathcal{R}}^T\psi_k\right\rangle_{L^2((0,T)\times\mathcal{R})}, \quad k = 1,2,\dots$$

is bounded. This shows that $(U^T\psi_k)_{k=1}^\infty$ is weakly bounded, and therefore bounded in norm, in $L^2(M)$. On the other hand if $(U^T\psi_k)_{k=1}^\infty$ is bounded, then

$$\left\langle f, K_{\mathcal{R}}^T\psi_k\right\rangle_{L^2((0,T)\times\mathcal{R})} = \left\langle W_{\mathcal{R}}^T f, U^T\psi_k\right\rangle_{L^2(M)}, \quad k = 1,2,\dots$$

is bounded for any $f \in L^2((0,T)\times\mathcal{R})$, and we see that also $(K_{\mathcal{R}}^T \psi_k)_{k=1}^\infty$ is bounded.

Suppose that $(W_{\mathcal{V}}^T f_j)_{j=1}^\infty$ converges weakly to zero in $L^2(M)$. Then (b) follows from Lemma 1, and (a) follows from Lemma 1 together with the fact that both the sequences $(W_{\mathcal{V}}^T f_j)_{j=1}^\infty$ and $(U^T\psi_k)_{k=1}^\infty$ are bounded in $L^2(M)$.

Suppose now that (a) and (b) hold, and let $w \in L^2(M)$. The assumption $T > \max_{x\in M} d(x,\mathcal{S})$ together with the analogue of Lemma 2 for U^T imply that there is a sequence $(\psi_k)_{k=1}^\infty$ in $C_0^\infty((0,T)\times\mathcal{S})$ such that $\lim_{k\to\infty} U^T\psi_k = w$ in $L^2(M)$. Note that $K_{\mathcal{V}}^T\psi_k = (W_{\mathcal{V}}^T)^* U^T\psi_k$, $k = 1,2,\dots$, is then bounded. By (a) it holds that

$$\sup_{j\in\mathbb{N}} |\left\langle W_{\mathcal{V}}^T f_j, w\right\rangle_{L^2(M))}| = \sup_{j\in\mathbb{N}} \lim_{k\to\infty} |\left\langle f_j, K_{\mathcal{V}}^T\psi_k\right\rangle_{L^2((0,T)\times\mathcal{V})}| < \infty.$$

Hence $(W_{\mathcal{V}}^T f_j)_{j=1}^\infty$ is bounded. Moreover,

$$\begin{aligned} |\left\langle W_{\mathcal{V}}^T f_j, w\right\rangle_{L^2(M))}| \quad &\le \sup_{j\in\mathbb{N}} \left\|W_{\mathcal{V}}^T f_j\right\|_{L^2(M)} \left\|w - U^T\psi_k\right\|_{L^2(M)} \\ &\quad + |\left\langle W_{\mathcal{V}}^T f_j, U^T\psi_k\right\rangle_{L^2(M))}|, \end{aligned}$$

where the first term on the right-hand side is small when we choose large k, and the second term converges to zero for fixed k. □

Lemma 20. *Let $\mathcal{X} \subset M$ be open, $x \in \mathcal{X}$ and let $X_j \subset M$, $j = 1,2,\dots$, be a sequence of neighborhoods of x satisfying $\lim_{j\to\infty} diam\,(X_j) = 0$. Let $\psi_0 \in C_0^\infty(\mathcal{X})$ satisfy $\psi_0(x) \neq 0$. Let $T > 0$ and suppose that a sequence $(f_{jk})_{j,k=1}^\infty$ of functions in $L^2((0,T)\times\mathcal{S})$ satisfies*

(0) *for all j, there is $u_j \in L^2(M)$ such that the sequence $(U^T f_{jk})_{k=1}^\infty$ converges weakly to u_j in $L^2(M)$,*
(i) *there is $C > 0$ such that $\left\|U^T f_{jk}\right\|_{L^2(M)} \le C|X_j|^{-1/2}$ for all j and k,*

(ii) *supp* $(u_j) \subset \overline{X}_j \cup (M \setminus \mathcal{X})$ *for all* j,

(iii) $(\langle u_j, \psi_0\rangle_{L^2(M)})_{j=1}^\infty$ *converges.*

Then there is $\kappa \in \mathbb{C}$ *such that* $\lim_{j\to\infty} \lim_{k\to\infty} \langle U^T f_{jk}, v\rangle_{L^2(M)} = \kappa v(x)$ *for all* $v \in C_0^\infty(\mathcal{X})$.

Furthermore, if $T > \max_{x\in M} d(x, \mathcal{S})$, *then there is a sequence* $(f_{jk})_{j,k=1}^\infty$ *that satisfies (i)-(iii) and for which* $\kappa = 1$.

Before giving a proof of the lemma, let us show that the conditions (0)-(iii) of Lemma 20 can be verified, when $\mathcal{X}$ and X_j are chosen as in the proof of Theorem 1, given $\Lambda_{\mathcal{S},\mathcal{R}}^{2T}$ and assuming that (1) is exactly controllable from $\mathcal{R}$ in time T and that $(M_\mathcal{R}, g)$ is known. This should be compared with the second paragraph of the proof of Theorem 1.

Let $\mathcal{K} \subset \mathcal{R}$ be compact. We construct h_j and the sets X_j for each $x \in C_p(\mathcal{K})$ and $p \in U(\mathcal{K})$ as in the proof of Theorem 1, see also Lemma 6, and define $\mathcal{X} = M(\Gamma_p(\mathcal{K}), s + \epsilon)^{\text{int}}$.

As $(M_\mathcal{R}, g)$ is known, we know also the Riemannian volume and surface measures on $M_\mathcal{R}$ and $\mathcal{R}$, respectively. In particular, we can compute the volumes $|X_j|$.

The exact controllability from $\mathcal{R}$ in time T implies that the sequence $(U^T f_{jk})_{k=1}^\infty$ converges weakly in $L^2(M)$ if and only if

$$\langle \phi, K_\mathcal{R}^T f_{jk}\rangle_{L^2((0,T)\times\mathcal{R})}, \quad k = 1, 2, \dots$$

converges for all $\phi \in L^2((0, T) \times \mathcal{R})$. Thus the condition (0) can be verified.

The condition (i) holds if and only if the sequence $(|X_j|^{1/2} U^T f_{jk})_{j,k=1}^\infty$ is weakly bounded in $L^2(M)$. As in the proof Lemma 19, this holds if and only if $(|X_j|^{1/2} K_\mathcal{R}^T f_{jk})_{j,k=1}^\infty$ is bounded in $L^2((0, T) \times \mathcal{R})$. Thus the condition (i) can be verified.

Lemma 2 implies that the condition (ii) holds if and only if

$$\lim_{k\to\infty} \langle \phi, K_\mathcal{R}^T f_{jk}\rangle_{L^2((0,T)\times\mathcal{R})} = 0, \quad \phi \in C_0^\infty(\mathcal{B}(\mathcal{R}, h_j; T)).$$

Finally, the condition (iii) holds if $\lim_{j\to\infty} \lim_{k\to\infty} \langle \phi, K_\mathcal{R}^T f_{jk}\rangle_{L^2((0,T)\times\mathcal{R})}$ exists for all $\phi \in C_0^\infty((0, \infty) \times \mathcal{R})$.

We can also prove Theorem 2 in the case where the exact controllability holds from $\mathcal{R}$ instead of from $\mathcal{S}$, since we can determine if the conditions (0)-(iii) hold when Lemma 20 replaces Lemma 7 in the proof of Lemma 12. In this case we use $\Lambda_\mathcal{R}$ to determine if the conditions (0) and (i) hold and $\Lambda_{\Omega_s,\kappa_0}$ to determine if the conditions (ii) and (iii) hold.

Proof of Lemma 20. Let $\psi \in C_0^\infty(\mathcal{X})$. Then supp $(u_j\psi) \subset \overline{X}_j$ and

$$\langle u_j, \psi\rangle_{L^2(M)} = \psi(x)\langle u_j, 1\rangle_{L^2(M)} + R_j,$$

where the remainder term R_j converges to zero as $j \to \infty$. This can be seen as in the proof of Lemma 7 since (i) implies that $\|u_j\| \le C|X_j|^{-1/2}$ for all j. We choose $\psi = \psi_0$ and see that $\lim_{j\to\infty}\langle u_j, 1\rangle_{L^2(M)}$ exists. We denote the limit by κ. Thus for any $\psi \in C_0^\infty(\mathcal{X})$ it holds that

$$\lim_{j\to\infty}\lim_{k\to\infty}\langle K_{\mathcal{R}}^T f_{jk}, \psi\rangle_{L^2((0,T)\times\mathcal{R})} = \lim_{j\to\infty}\langle u_j, \psi\rangle_{L^2(M)} = \kappa\psi(x).$$

Let us now assume that $T > \max_{x\in M} d(x, \mathcal{S})$. Then the analogue of Lemma 2 for U^T implies that for each j there is a sequence $(f_{jk})_{k=1}^\infty$ in $L^2((0,T)\times\mathcal{S})$ such that $(U^T f_{jk})_{k=1}^\infty$ converges to $1_{X_j}/|X_j|$ in $L^2(M)$. Then the conditions (0), (ii) and (iii) hold. By considering a suitable tail of each of the sequences $(f_{jk})_{k=1}^\infty$ we see that the sequence $(f_{jk})_{j,k=1}^\infty$ can be chosen so that (i) holds with $C = 2$. □

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