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STABILIZED NONCONFORMING FINITE ELEMENT METHODS FOR DATA ASSIMILATION IN INCOMPRESSIBLE FLOWS

ERIK BURMAN AND PETER HANSBO

ABSTRACT. We consider a stabilized nonconforming finite element method for data assimilation in incompressible flow subject to the Stokes' equations. The method uses a primal dual structure that allows for the inclusion of nonstandard data. Error estimates are obtained that are optimal compared to the conditional stability of the ill-posed data assimilation problem.

1. INTRODUCTION

The design of computational methods for the numerical approximation of the Stokes' system of equations modelling creeping incompressible flow is by and large well understood in the case where the underlying problem is well-posed. Indeed, provided suitable boundary conditions are set, the system of equations are known to satisfy the hypotheses of the Lax-Milgram lemma and Brezzi's theorem ensuring well-posedness of velocities and pressure. These theoretical results then underpin much of the theory for the design of stable and accurate finite element methods for the Stokes system [16, 5].

In many cases of interest in applications, however, the necessary data for the theoretical results to hold are not known; this is the case for instance in data assimilation in atmospheric sciences or oceanography. Instead of knowing the solution on the boundary, data in the form of measured values of velocities may be known in some other set. It is then not obvious how best to apply the theory developed for the well-posed case. A classical approach is to rewrite the system as an optimisation problem and add some regularization, making the problem well-posed on the continuous level and then approximate the well-posed problem using known techniques. For examples of methods using this framework see [4] and [7].

In this paper we advocate a different approach in the spirit of [8, 9]. The idea is to formulate the optimization problem on the continuous level, but without any regularization. We then discretize the ill-posed continuous problem and instead regularize the discrete system. This leads to a method in the spirit of stabilized finite element methods where the properties of the different stabilizing operators are well studied. An important feature of this approach is that it eliminates the need for a perturbation analysis on the continuous level taking into account the Tikhonov regularization and perturbations in data, that the discretization error then has to match. In our case we are only interested in the discretization error and the perturbations in data. This allows us to derive error estimates that are

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optimal in the case of unperturbed data in a similar fashion as for the well-posed case.

We exemplify the theory in a model case for data assimilation where data is given in some subset of the computational domain instead of the boundary, and we obtain error estimates using a conditional stability result in the form of a three ball inequality due to Lin, Uhlmann, and Wang [22]. A particular feature of the method formulated for the integration of data in the bulk (and not on the boundary), is that the dual adjoint problem does not require any regularization on the discrete level. Indeed, the adjoint equation is inf–sup stable, similarly to the case of elliptic problems on non-divergence form discussed in [24].

The rest of the paper can be outlined as follows. First, in Section 2, we introduce the Stokes' problem that we are interested in and propose the continuous minimization problem. Then, in Section 3, we present the non-conforming finite element method and prove some preliminary results. In Section 4 we prove the fundamental stability and convergence results of the formulation. Finally we show the performance of the approach on some numerical examples.

2. Stokes equations

Let Ω be a polygonal (polyhedral) domain in $\mathbb{R}^d$, $d = 2$ or 3. We are interested in computing solutions to the Stokes' system

$$\begin{aligned} -\Delta u + \nabla p &= \mathfrak{f} \quad \text{in } \Omega \\ \nabla \cdot u &= \mathfrak{g} \quad \text{in } \Omega. \end{aligned} \tag{2.1}$$

Typically these equations are then equipped with suitable boundary conditions and are known to be well-posed using the Lax-Milgram Lemma for the velocities and Brezzi's theorem for the pressures. It is also known that the following continuous dependence estimate holds, here given under the assumption of homogeneous Dirichlet conditions on the boundary.

$$\|u\|_{H^1(\Omega)} + \|p\|_\Omega \lesssim \|\mathfrak{f}\|_{H^{-1}(\Omega)} + \|\mathfrak{g}\|_\Omega, \tag{2.2}$$

where we used the notation $\|x\|_\Omega := \|x\|_{L^2(\Omega)}$ and $a \lesssim b$ for $a \leq Cb$ with $C > 0$.

Observe that for any solution to the equations (2.1) and in any closed ball $B_R \subset \Omega$ there holds

$$(u, p)|_{B_R} \in [H^2(B_R)]^d \times H^1(B_R) \tag{2.3}$$

provided $\mathfrak{f} \in [L^2(\Omega)]^d$ and $\mathfrak{g} \in H^1(\Omega)$. See for instance [23, Proposition 3.2].

We will in the following make the stronger assumption that $(u, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$. Observe that this is not a strong assumption for the particular problem we will study below, since the domain Ω here is somewhat arbitrary and not necessarily determined by a physical geometry. Indeed the only situation in which this assumption can fail is when the boundary of Ω coincides with a physical boundary with a corner. Even though we assume $\mathfrak{f} \in [L^2(\Omega)]^d$ and $\mathfrak{g} = 0$ for the physical problem, we need the general form of equation (2.1), with the stability estimate (2.2) for the error analysis below.

Herein the main focus will be on methods that allow for the accurate approximation of the solution under the much weaker stability estimates that remain valid in the case of ill-posed problems where (2.2) fails.

A situation of particular interest is the case where the boundary data g_D is known only on a portion Γ_D of $\partial\Omega$ and nothing is known of the boundary conditions on the remaining part $\Gamma_D' := \partial\Omega \setminus \Gamma_D$. This lack of boundary information makes the problem ill-posed and we assume that some other data is known such as:

- The normal stress in some part of the boundary $\Gamma_N \subset \partial\Omega$ and $\Gamma_N \cap \Gamma_D \neq \emptyset$,
$$(-n \cdot \nabla u + pn) \cdot n = \psi. \tag{2.4}$$
We will refer to this problem as the *Cauchy problem* below.
- The measured value of (u, p) in some subdomain $\omega \subset \Omega$. We will refer to this problem as the *data assimilation problem* below.

In the first case it is known that if a solution exists, then $g_D = \psi = 0$ implies $u = 0$, $p = 0$ in Ω by unique continuation [15]. Only recently quantitiative estimates have been proved for the pure Cauchy problem [3]; see also [6] for results using additional measurements on the boundary. It is however not clear how to use these estimates in the present framework, since they require stronger norms on boundary data than the natural ones.

In the second case stability may be proven in the form of a three balls inequality and associated local stability estimates, see [22, 6]. For completeness of the analysis we focus on the second case for the error estimates below. In particular we consider the case where no data are known on the boundary, i.e. $\Gamma_D = \Gamma_N = \emptyset$. In the data assimilation case the following Theorem from [22] provides us with a conditional stability estimate. Assuming an optimal conditional stability estimate for the Cauchy problem in the spirit of [2], it is straightforward to extend the anaysis to this case following [10].

Theorem 2.1. *(Conditional stability for the Stokes' problem) There exists a positive number $\tilde{R} < 1$ such that if $0 < R_1 < R_2 < R_3 \leq R_0$ and $R_1/R_3 < R_2/R_3 < \tilde{R}$, then if $B_{R_0}(x_0) \subset \Omega$*

$$\int_{B_{R_2}(x_0)} |u|^2 \, dx \leq C \left(\int_{B_{R_1}(x_0)} |u|^2 \, dx \right)^{\tau} \left(\int_{B_{R_3}(x_0)} |u|^2 \, dx \right)^{1-\tau}$$

for $(u, p) \in [H^1(B_{R_0}(x_0))]^{d+1}$, satisfying (2.1) with $\mathfrak{f} = \mathfrak{g} = 0$ in $B_{R_0}(x_0)$, where the constant C depends on R_2/R_3 and $0 < \tau < 1$ depends on R_1/R_3, R_2/R_3 and d. For fixed R_2 and R_3, the exponent τ behaves like $1/(-\log(R_1))$ when R_1 is sufficiently small.

Proof. For the proof we refer to [22]. □

In the data assimilation problem corresponding to Theorem 2.1 measured data $u_M : \omega \mapsto \mathbb{R}^d$ are available in ω such that u_M satisfies (2.1) in ω and there exists u defined on Ω satisfying (2.1) such that $u|_\omega = u_M$. Our objective is to design a method for the reconstruction of u, given $\tilde{u}_M := u_M + \delta u$, where $\delta u \in [L^2(\omega)]^d$ is a perturbation of the exact data resulting from measurement error or interpolation of pointwise measurements inside ω. Observe that the considered configuration is also closely related to a pure boundary control problem, where we look for data on the boundary such that $u = u_M$ in the subset ω.

We will first cast the problem (2.1), with the notation $\mathfrak{f} = f$ and with $\mathfrak{g} = 0$, on weak form. For the derivation of the weak formulation we introduce the spaces $V := \{v \in [H^1(\Omega)]^d\}$ and $W := \{v \in [H^1_0(\Omega)]^d\}$ for velocities and $Q := L^2(\Omega)$ and

$Q_0 := L^2_0(\Omega)$, where the zero–subscript in the second case as usual indicates that the functions have zero integral over Ω.

We may thne multiply the first equation of (2.1) by $w \in W$ and first integrate over Ω and then apply Green's formula to obtain

$$\int_\Omega \nabla u : \nabla w \ \mathrm{d}x - \int_\Omega p \nabla \cdot w \ \mathrm{d}x = \int_\Omega f w \ \mathrm{d}x, \quad \forall w \in W$$

similarly we may multiply the second equation by $q \in L^2(\Omega)$ and integrate over Ω to get

$$\int_\Omega q \nabla \cdot u \ \mathrm{d}x = 0, \quad \forall q \in L^2(\Omega).$$

Introducing the forms

$$a(u, w) := \int_\Omega \nabla u : \nabla w \ \mathrm{d}x,$$

$$b(p, w) = - \int_\Omega p \, \nabla \cdot w \ \mathrm{d}x$$

and

$$l(w) := \int_\Omega f w \ \mathrm{d}x$$

we may formally write the problem as: find $(u, p) \in V \times Q_0$ such that $u|_\omega = u_M$ and

$$a(u, w) + b(p, w) = l(w), \quad \forall w \in W \tag{2.5}$$

$$b(y, u) = 0, \quad \forall y \in Q. \tag{2.6}$$

Observe that this problem is ill-posed. In particular observe that we are not allowed to test with $w = u$ because of the homogeneous Dirichlet conditions set on the functions in W. Another consequence of this is that the pressure p, is determined only up to a constant. The problem however is conditionally wellposed, meaning that if there is a solution it is unique, as we shall show below. To obtain a formulation where this property can be exploited, we cast the problem on the form of a minimization problem. We wish to find $(u, p) \in V \times Q_0$ such that $\|u - \tilde{u}_M\|^2_\omega$ is minimized under the pde constraint of (2.5)-(2.6). First we write

$$A[(u, p), (w, y)] := a(u, w) + b(p, w) - b(y, u)$$

and then we introduce the Lagrangian

$$\mathcal{L}[(u, p), (z, x)] := \frac{1}{2} \gamma_M \|u - \tilde{u}_M\|^2_\omega + A[(u, p), (z, x)] - l(z), \tag{2.7}$$

where

$$(u, v)_\omega := \int_\omega u v \ \mathrm{d}x,$$

γ_M is a free parameter. The optimality system of the constrained minimization problem then takes the form

$$A[(u, p), (w, y)] = l(w), \quad \forall (w, y) \in W \times Q \tag{2.8}$$

$$A[(v, q), (z, x)] + \gamma_M (u, v)_\omega = \gamma_M (\tilde{u}_M, v)_\omega, \quad \forall (v, q) \in V \times Q_0. \tag{2.9}$$

This problem is ill-posed in general, but, as anticipated above, if a solution exists it is unique. We now prove this result.

Proposition 2.2. *Assume that* $\tilde{u}_M$ *in* (2.9) *is such that* (2.5)-(2.6) *admits a solution* $(u,p) \in V \times Q_0$ *such that* $u|_\omega = \tilde{u}_M$. *Then this solution is the unique solution of* (2.8)-(2.9), *together with* $z = 0$ *and* $x = 0$.

Proof. To show this assume that there are two solutions $(u_1, p_1) \in V \times Q_0$ and $(u_2, p_2) \in V \times Q_0$ that solve (2.8)–(2.9), then $\nu = u_1 - u_2 \in V$ solves the homogenous Stokes' equation and has $\nu|_\omega = 0$ and the uniqueness is a consequence of unique continuation [15, Proposition 1.1]. Setting $\mu = p_1 - p_2$, it follows by the surjectivity of the divergence operator that there exists $v_q \in W$ such that $\nabla \cdot v_q = \mu$. As a consequence, taking $w = v_q$ in equation (2.5) for (ν, μ) we have

$$\|\mu\|_\Omega^2 = b(\mu, v_q) = -a(\nu, v_q) = 0.$$

To see that the multiplier $(z, x) \in W \times Q$ is zero, we recall that $(u - \tilde{u}_M, v)_\omega = 0$ and take $v = z \in V$ and $q = x - |\Omega|^{-1} \int_\Omega x \in Q_0$ in (2.9). For this choice we have, since $z|_{\partial\Omega} = 0$,

$$0 = a(v,z) - b(x,v) + b(q,z) = \|\nabla z\|_\Omega^2 - (x, \nabla \cdot z)_\Omega + (x - |\Omega|^{-1} \int_\Omega x, \nabla \cdot z)_\Omega = \|\nabla z\|_\Omega^2.$$

It follows by Poincaré 's inequality that $z = 0$. Exactly as for the primal equation we may show that $x - |\Omega|^{-1} \int_\Omega x = 0$, since $W \subset V$. It follows that x is a global constant. To show that this constant is zero we once again use eqation (2.9),

$$0 = b(x, v) = x \int_{\partial\Omega} v \cdot n \, \mathrm{d}s.$$

It is then enough to chose a $v \in V$ such that

$$\int_\Gamma v \cdot n \, \mathrm{d}s = x$$

for some side Γ of $\partial\Omega$ and $v = 0$ on $\partial\Omega \setminus \Gamma$. □

Below we will assume that there exists a unique solution $(u, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ that satisfies (2.1) in Ω with $u = u_M$ in ω.

The classical way of obtaining a wellposed system from (2.8)–(2.9), is to add Tikhonov regularization terms to the functional (2.7) and then discretize the regularized system. This approach however leads to an $O(1)$ perturbation of the original system and the solution of the regularized system will only converge to the solution of the unperturbed system in the limit of vanishing regularization. In this work we will instead follow the approach proposed in [10], where regularization was introduced only after discretization, in the spirit of stabilized finite element methods.

3. The nonconforming stabilized method

Let $\{\mathcal{T}_h\}_h$ denote a family of shape regular and quasi uniform tesselations of Ω into nonoverlapping simplices, such that for any two different simplices κ, $\kappa' \in \mathcal{T}_h$, $\kappa \cap \kappa'$ consists of either the empty set, a common face, a common edge or a common vertex. The outward pointing normal of a simplex κ will be denoted n_κ. We denote the set of element faces in $\mathcal{T}_h$ by $\mathcal{F}$ and let $\mathcal{F}_i$ denote the set of interior faces F in $\mathcal{F}$. To each face F we associate a unit normal vector, n_F. For interior faces its orientation is arbitrary, but fixed. On the boundary $\partial\Omega$ we identify n_F with the outward pointing normal of Ω. We let h_κ denote the diameter of the smallest sphere circumscribing the element κ and similarly let h_F denote the diameter of the face F. Also define the global mesh parameter $h := \max_{\kappa \mathcal{T}_h} h_\kappa$. For each face $F \in \mathcal{F}_i$ we introduce the associated element patch, $\Delta_F := \kappa \cup \kappa'$ where κ, $\kappa' \in \mathcal{T}_h$ such $\kappa \cap \kappa' = F$. The set of faces F' such that $F' \in \partial\kappa \cup \partial\kappa'$ will be denoted $\mathcal{F}_{\Delta_F}$.

We define the jump over interior faces $F \in \mathcal{F}_i$ by $[v]|_F := \lim_{\epsilon\to 0^+}(v(x|_F - \epsilon n_F) - v(x|_F + \epsilon n_F))$ and for faces on the boundary, $F \in \partial\Omega$, we let $[v]|_F := v|_F$. Similarly we define the average of a function over an interior face F by $\{v\}|_F := \frac{1}{2}\lim_{\epsilon\to 0^+}(v(x|_F - \epsilon n_F) + v(x|_F + \epsilon n_F))$ and for F on the boundary we define $\{v\}|_F := v|_F$. The classical nonconforming space of piecewise affine finite element functions (see [12]) then reads

$$\mathcal{CR} := \{v_h \in L^2(\Omega) : \int_F [v_h] \, \mathrm{d}s = 0, \, \forall F \in \mathcal{F}_i \text{ and } v_h|_\kappa \in \mathbb{P}_1(\kappa), \, \forall \kappa \in \mathcal{T}_h\}$$

where $\mathbb{P}_1(\kappa)$ denotes the set of polynomials of degree less than or equal to one restricted to the element κ. With homogeneous (elementwise average) Dirichlet boundary conditions, the space takes the form

$$\mathcal{CR}^0 := \{v_h \in L^2(\Omega) : \int_F [v_h] \, \mathrm{d}s = 0, \, \forall F \in \mathcal{F} \text{ and } v_h|_\kappa \in \mathbb{P}_1(\kappa), \, \forall \kappa \in \mathcal{T}_h\}.$$

We may then define the spaces $V_h := [\mathcal{CR}]^d$ and $W_h := [\mathcal{CR}^0]^d$. For the pressure spaces we define

$$Q_h := \{q_h \in L^2(\Omega) : q_h|_\kappa \in \mathbb{R}, \forall \kappa \in \mathcal{T}_h\} \text{ and } Q_h^0 := Q_h \cap L^2_0(\Omega).$$

To make the notation more compact we introduce the composite spaces $\mathcal{V}_h := V_h \times Q_h^0$ and $\mathcal{W}_h := W_h \times Q_h$.

3.1. **Finite element formulation.** Introducing the discrete bilinear form $A_h : \mathcal{V}_h \times \mathcal{W}_h \mapsto \mathbb{R}$,

$$A_h[(u_h, p_h), (w_h, y_h)] := a_h(u_h, w_h) + b_h(p_h, w_h) - b_h(y_h, u_h) \tag{3.1}$$

where the forms are defined by

$$a_h(u_h, w_h) = \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u_h : \nabla w_h \, \mathrm{d}x$$

and, introducing the broken scalar product $(y_h, x_h)_h := \sum_{\kappa\in\mathcal{T}_h} \int_\kappa x_h y_h \, \mathrm{d}x$,

$$b_h(p_h, w_h) = -(p_h, \nabla \cdot w_h)_h$$

the discrete version of the Lagrangian (2.7) may be written, for $(u_h, p_h) \times (z_h, x_h) \in \mathcal{V}_h \times \mathcal{W}_h$

$$\mathcal{L}_h[(u_h, p_h), (z_h, x_h)] := \frac{1}{2}\gamma_M \|u - \tilde{u}_M\|^2_\omega + A_h[(u_h, p_h), (z_h, x_h)] - l(z_h). \tag{3.2}$$

The stationary points of the discrete Lagrangian (3.2) are characterized by the solution of the following system: find $(u_h, p_h) \times (z_h, x_h) \in \mathcal{V}_h \times \mathcal{W}_h$ such that,

$$A_h[(u_h, p_h), (w_h, y_h)] = l(w_h) \tag{3.3}$$

$$A_h[(v_h, q_h), (z_h, x_h)] + \gamma_M (u_h, v_h)_\omega = \gamma_M (\tilde{u}_M, v_h)_\omega. \tag{3.4}$$

for all $(v_h, q_h) \times (w_h, y_h) \in \mathcal{V}_h \times \mathcal{W}_h$.

To obtain a stable formulation we need to add stabilizing terms to (3.2). This can be done in several different ways, resulting in different methods with different stability, accuracy and conservation properties. Our choice herein has been guided by the principle that stabilization is added only if it is necessary for accuracy and has minimal influence on the conservation properties of the scheme. We will also comment on some variants. The key feature here is that the regularization uses the structure of the discrete approximation, and has some consistency properties. For the primal velocities we suggest to use the standard jump stabilization that has been shown to stabilize the Crouzeix-Raviart element in a number of applications [17, 18, 11],

$$s_{j,t}(u_h, v_h) := \sum_{F \in \mathcal{F}_i} \int_F h_F^t [u_h][v_h] \, \mathrm{d}s. \tag{3.5}$$

For the pressure on the other hand we propose to use the following weak penalty term

$$s_{p,t}(p_h, q_h) := \int_\Omega h^t p_h q_h \, \mathrm{d}x. \tag{3.6}$$

This leads to the perturbed Lagrangian,

$$\mathcal{L}_h[U_h, Z_h] + \mathcal{S}_h[(U_h, Z_h), (U_h, Z_h)],$$

where $U_h := (u_h, p_h) \in V_h \times Q_h^0$ and $Z_h := (z_h, x_h) \in W_h \times Q_h$ and

$$\mathcal{S}_h[(U_h, Z_h), (X_h, Y_h)] := S_p[U_h, U_h] - S_a[Z_h, Z_h], \tag{3.7}$$

with S_a and S_p positive semi-definite, symmetric bilinear forms to be defined. The precise design of the regularization is problem dependent. For the Cauchy problem, the velocities must be stabilized both for the forward and the adjoint problems. This is not necessary in the data assimilation case, where the stabilizing terms take the form

$$S_p[U_h, U_h] := \gamma_u s_{j,-1}(u_h, u_h) + \gamma_p s_{p,2}(p_h, p_h), \; \gamma_u > 0, \; \gamma_p \geq 0 \tag{3.8}$$

and

$$S_a[Z_h, Z_h] := \gamma_x s_{p,0}(x_h, x_h), \; \gamma_x \geq 0. \tag{3.9}$$

The Euler-Lagrange equations may then be written on the compact form: find $(U_h, Z_h) \in \mathcal{V}_h \times \mathcal{W}_h$, such that,

$$\begin{aligned}\mathcal{A}_h[(U_h, Z_h), (X_h, Y_h)] + \mathcal{S}_h[(U_h, Z_h), (X_h, Y_h)] + \gamma_M (u_h, v_h)_\omega \\ = l(w_h) + \gamma_M (\tilde{u}_M, v_h)_\omega\end{aligned} \tag{3.10}$$

for all $(X_h, Y_h) \in \mathcal{V}_h \times \mathcal{W}_h$, $X_h := (v_h, q_h)$ and $Y_h := (w_h, y_h)$. The bilinear forms are then given by

$$\mathcal{A}_h[(U_h, Z_h), (X_h, Y_h)] := A_h[(u_h, p_h), (w_h, y_h)] + A_h[(v_h, q_h), (z_h, x_h)]. \tag{3.11}$$

For an even more compact notation, we shall in the following also make use of the following bilinear form

$$\begin{aligned}\mathcal{G}[(U_h, Z_h), (X_h, Y_h)] :=& \mathcal{A}_h[(U_h, Z_h), (X_h, Y_h)] \\ &+ \mathcal{S}_h[(U_h, Z_h), (X_h, Y_h)] + \gamma_M (u_h, v_h)_\omega.\end{aligned} \tag{3.12}$$

Observe that the minimal stabilization that allows for optimal error estimates is $\gamma_u > 0$, $\gamma_p = \gamma_x = 0$. In the analysis below we will focus on this case, noting that the case with added pressure stabilization follows in a similar way, but is slightly more elementary. From the theoretical point of view the choice $\gamma_p > 0$ has no detrimental effect, neither on conservation nor on the accuracy of the primal solution. The choice $\gamma_x > 0$ on the other hand perturbs both local and global conservation of mass, but still allows for optimal error estimates. The interest of the addition of the pressure stabilization stems from the possibility of eliminating the pressure and we briefly discuss the resulting formulation before proceeding with the analysis.

3.2. **Elimination of the pressure.** Consider the dual mass conservation equation in the formulation (3.10) with the stabilization given by (3.8) and (3.9) and $\gamma_p > 0$,

$$b_h(q_h, z_h) + \gamma_p s_{p,2}(p_h, q_h) = 0, \forall q_h \in Q_h^0.$$

Observing that for $w_h \in W_h$, $\gamma_p^{-1} h^{-2} \nabla \cdot w_h \in Q_h^0$ we may eliminate the physical pressure from the formulation, since

$$\begin{aligned} b_h(p_h, w_h) = -(p_h, \nabla \cdot w_h)_h = -\gamma_p s_{p,2}(p_h, \gamma_p^{-1} h^{-2} \nabla \cdot w_h) = b_h(\gamma_p^{-1} h^{-2} \nabla \cdot w_h, z_h) \\ = -(\gamma_p^{-1} h^{-2} \nabla \cdot w_h, \nabla \cdot z_h)_h. \end{aligned}$$

Similarly, for $\gamma_x > 0$ the dual pressure x_h may be eliminated. Starting from the mass conservation equation

$$-b_h(y_h, u_h) - \gamma_x s_{x,0}(x_h, y_h) = 0$$

we use that for all $v_h \in V_h$, $y_h = \gamma_x^{-1} \nabla \cdot v_h \in Q_h$ is a valid test function to deduce

$$\begin{aligned} - b_h(x_h, v_h) = (x_h, \nabla \cdot v_h)_h = \gamma_x s_{x,0}(x_h, \gamma_x^{-1} \nabla \cdot v_h) \\ = -b_h(\gamma_x^{-1} \nabla \cdot v_h, u_h) = \gamma_x^{-1} (\nabla \cdot v_h, \nabla \cdot u_h)_h. \end{aligned}$$

The resulting formulation is an equal order interpolation formulation for the Stokes' system using the nonconforming Crouzeix-Raviart element for both the forward and the dual system. Find $(u_h, z_h) \in V_h \times W_h$ such that

$$\begin{aligned} a_h(u_h, w_h) - (\gamma_p^{-1} h^{-2} \nabla \cdot w_h, \nabla \cdot z_h)_h = l(w_h) \\ a_h(z_h, v_h) + \gamma_x^{-1} (\nabla \cdot u_h, \nabla \cdot v_h)_h + \gamma_u s_{j,-1}(u_h, v_h) + \gamma_M (u_h, v_h)_\omega = \gamma_M (\tilde{u}_M, v_h)_\omega \end{aligned} \tag{3.13}$$

for all $(v_h, w_h) \in V_h \times W_h$. We identify this scheme as a discretization of the continuous regularization of the Stokes' Cauchy problem proposed in [7]. It follows that the analysis below also covers that method in the special case that the discretization uses the nonconforming space $\mathcal{CR}$. Note that the pressure approximation in this case is given by the piecewise constant function $\gamma_p^{-1} h^{-2} \nabla \cdot z_h$.

3.3. **Technical Lemmas.** We will end this section by proving some elementary Lemmas that will be useful in the analysis below. We will use $\| \cdot \|_X$ to denote the L^2–norm over X, subset of $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$. It is also conventient to introduce the broken norms

$$\|x\|_h^2 := (x, x)_h \text{ and } \|x\|_{1,h}^2 := \|x\|_h^2 + a_h(x, x).$$

We recall the interpolation operator $r_h : [H^1(\Omega)]^d \to [\mathcal{CR}]^d$ defined by the (component wise) relation

$$\overline{\{r_h v\}}|_F := |F|^{-1} \int_F \{r_h v\} \, \mathrm{d}s = |F|^{-1} \int_F v \, \mathrm{d}s$$

for every $F \in \mathcal{F}$ and with $|F|$ denoting the $(d-1)$-measure of F.

The following inverse and trace inequalities are well known (see for instance [13, Section 1.4.3],

$$\begin{aligned} \|v\|_{\partial\kappa} &\le C_t(h^{-\frac{1}{2}}\|v\|_\kappa + h^{\frac{1}{2}}\|\nabla v\|_\kappa), \quad \forall v \in H^1(\kappa), \\ h_\kappa\|\nabla v_h\|_\kappa + h_\kappa^{\frac{1}{2}}\|v_h\|_{\partial\kappa} &\le C_i\|v_h\|_\kappa, \quad \forall v_h \in \mathcal{CR}. \end{aligned} \tag{3.14}$$

Using the inequalities of (3.14) and standard approximation results from [12] it is straightforward to show the following approximation results of the interpolant r_h

$$\begin{aligned} \|u - r_h u\|_\Omega + h\|\nabla(u - r_h u)\|_h &\le Ch^t|u|_{H^t(\Omega)} \\ \|h^{-\frac{1}{2}}(u - r_h u)\|_{\mathcal{F}} + \|h^{\frac{1}{2}}\nabla(u - r_h u)\cdot n_F\|_{\mathcal{F}} &\le Ch^{t-1}|u|_{H^t(\Omega)} \end{aligned} \tag{3.15}$$

where $t \in \{1, 2\}$. Also recall the projection onto piecewise constant functions, $\pi_0 : L^2(\Omega) \mapsto Q_h$, defined for $v \in L^2(\Omega)$ by $\pi_0 v|_\kappa := |\kappa|^{-1}\int_\kappa v \, \mathrm{d}x$, where $|\kappa|$ denotes the d-measure of the simplex κ, and for which there holds,

$$\|\pi_0 v\|_\Omega \le \|v\|_\Omega, \forall v \in L^2(\Omega) \quad \text{and} \quad \|\pi_0 v - v\|_\Omega \le Ch|v|_{H^1(\Omega)}, \forall v \in H^1(\Omega) \tag{3.16}$$

It will also be useful to bound the L^2-norm of the interpolant r_h by its values on the element faces. To this end we prove a technical lemma.

Lemma 3.1. *For any function $v_h \in \mathcal{CR}$ there holds*

$$\|v_h\|_\Omega \le c_{\mathcal{T}}\left(\sum_{F\in\mathcal{F}} h_F\|\overline{\{v_h\}}\|_F^2\right)^{\frac{1}{2}}$$

Proof. It follows by norm equivalence of discrete spaces on the reference element and a scaling argument (under the assumption of shape regularity) that for all $\kappa \in \mathcal{T}_h$

$$\|v_h\|_\kappa^2 \le C\sum_{F\in\partial\kappa} h_F\|\overline{v}_h\|_F^2. \tag{3.17}$$

The claim follows by summing over the elements of $\mathcal{T}_h$ and recalling that $\|\overline{v}_h\|_F^2 = \|\overline{\{v_h\}}\|_F^2$. □

For the analysis below we also need a quasi-interpolation operator that maps piecewise linear, nonconforming functions into the space of piecewise linear conforming functions. Let $I_{\mathrm{cf}} : [Q_h]^d \cup V_h \mapsto V \cap V_h$ denote the quasi interpolation of u_h into $V_h \cap V$, [20, 1, 21] such that

$$\|I_{\mathrm{cf}}u_h - u_h\|_\Omega + h\|\nabla(I_{\mathrm{cf}}u_h - u_h)\|_h \lesssim \|h^{\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}$$

and for $v_h \in \mathcal{CR}$, such that the element-wise evaluated gradient $\nabla v_h \in [Q_h]^d$

$$\|I_{\mathrm{cf}}\nabla v_h - \nabla v_h\|_h \lesssim \|h^{\frac{1}{2}}[\nabla v_h]\|_{\mathcal{F}_i}. \tag{3.18}$$

In the following we will typically apply I_{cf} to ∇v_h with $v_h \in V_h$. The operator is then applied on each column of the gradient matrix. Below, the global conservation properties of this operator will be important and we therefore propose the following perturbed variant that satisfies a global conservation property. We define the modified interpolant by

$$\tilde{u}_h := I_{\mathrm{cf}}u_h + d_h \tag{3.19}$$

where the perturbation $d_h \in V_h \cap V$ is the solution to the following constrained problem, $\bar p \in \mathbb{R}$

$$(3.20)\qquad \begin{array}{rcl} (d_h, w_h)_h + (\nabla d_h, \nabla w_h)_h + (\bar p, \nabla \cdot w_h)_h &=& 0 \\ (\nabla \cdot d_h, \bar q)_h &=& (-\nabla \cdot I_{\rm cf} u_h, \bar q)_h, \end{array}$$

for all $(w_h, \bar q) \in (V_h \cap V) \times \mathbb{R}$. This implies that

$$\int_{\partial\Omega} \tilde u_h \cdot n \, \mathrm{d}s = \int_\Omega \nabla \cdot \tilde u_h \, \mathrm{d}x = |\Omega| \overline{\nabla \cdot (I_{\rm cf} u_h + d_h)} = 0$$

with $|\Omega|$ denoting the d-measure of Ω and

$$\overline{\nabla \cdot I_{\rm cf} u_h} := |\Omega|^{-1} \int_\Omega \nabla \cdot I_{\rm cf} u_h \, \mathrm{d}x.$$

Lemma 3.2. *The problem* (3.20) *is well-posed and the solution satisfies*

$$\|d_h\|_{H^1(\Omega)} \le \|\overline{\nabla \cdot I_{\rm cf} u_h}\|_\Omega \lesssim \|h^{-\frac12}[u_h]\|_{\mathcal{F}_i} + \|\overline{\nabla \cdot u_h}\|_h$$

Proof. Since the linear system corresponding to (3.20) is square, existence and uniqueness is a consequence of the stability estimate. Take $w_h = d_h + \alpha \bar p \mathbf{x}$, with $\alpha > 0$, $\bar q = \bar p$ in (3.20) and observe that for α small enough, there exists $c(\alpha) > 0$ such that

$$(3.21)\qquad \begin{aligned} \|d_h\|^2_{H^1(\Omega)} + c(\alpha)\|\bar p\|^2_\Omega \lesssim \|\overline{\nabla \cdot I_{\rm cf} u_h}\|^2_\Omega \lesssim \|\overline{\nabla \cdot (I_{\rm cf} u_h - u_h)}\|^2_h + \|\overline{\nabla \cdot u_h}\|^2_h \\ \lesssim \|h^{-\frac12}[u_h]\|^2_{\mathcal{F}_i} + \|\overline{\nabla \cdot u_h}\|^2_h. \end{aligned}$$

□

An immediate consequence of Lemma 3.2 is that $\tilde u_h$ satisfies similar approximation estimates as $I_{\rm cf} u_h$, but with improved global conservation. We collect these results, the proof of which is an immediate consequence of the discussion above, in a corollary.

Corollary 3.3. *The conforming approximation $\tilde u_h$ satisfies the discrete estimate,*

$$(3.22)\qquad \|\tilde u_h - u_h\|_{1,h} \lesssim \|h^{-\frac12}[u_h]\|_{\mathcal{F}_i} + \|\nabla \cdot u_h\|_h$$

and has the global conservation property

$$\int_{\partial\Omega} \tilde u_h \cdot n \, \mathrm{d}s = 0.$$

Using the regularity assumptions on the data in $l(w)$ it is straightforward to show that the formulation satisfies the following weak consistency

Lemma 3.4. *(Weak consistency) Let (u, p) be the solution of* (2.1)*, with $f \in L^2(\Omega)$, and let $(u_h, p_h) \in \mathcal{V}_h$ be the solution of* (3.10)*. Then, for all $w_h \in W_h$, there holds,*

$$(3.23)\qquad \begin{aligned} &|a_h(u_h - u, w_h) + b_h(p_h - p, w_h)| \\ &\qquad \le \inf_{(\nu_h, \eta_h) \in \mathcal{V}_h} \sum_{F \in \mathcal{F}} \int_F |n_F \cdot (\sigma(u,p) - \{\sigma(\nu_h, \eta_h)\})| |[w_h]| \, \mathrm{d}s. \end{aligned}$$

where $\sigma(u,p) := \nabla u - \mathcal{I} p$, with $\mathcal{I}$ the identity matrix.

Proof. Multiplying (2.1) with $w_h \in W_h$ and integrating by parts we have

$$\begin{aligned}\int_\Omega f w_h \,\mathrm{d}x &= -\int_\Omega \nabla\cdot(\nabla u - \mathcal{I}p)\cdot w_h \,\mathrm{d}x \\ &= -\sum_{\kappa\in\mathcal{T}_h}\sum_{F\in\partial\kappa}\int_F \sigma(u,p)\cdot n_\kappa \cdot w_h \,\mathrm{d}s + a_h(u,w_h) + b_h(p,w_h)\end{aligned} \tag{3.24}$$

or by rearranging terms

$$a_h(u,w_h) + b_h(p,w_h) = l(w_h) + \sum_{\kappa\in\mathcal{T}_h}\sum_{F\in\partial\kappa}\int_F \sigma(u,p)\cdot n_\kappa \cdot w_h \,\mathrm{d}s.$$

Using (3.10) we obtain

$$a_h(u_h - u, w_h) + b_h(p_h - p, w_h) = -\sum_{\kappa\in\mathcal{T}_h}\sum_{F\in\partial\kappa}\int_F \sigma(u,p)\cdot n_\kappa \cdot w_h \,\mathrm{d}s.$$

Since every internal face appears twice with different orientation of n_κ, and w_h has zero average on every boundary face we have for all $(\nu_h,\eta_h)\in\mathcal{V}_h$,

$$\begin{aligned}&\sum_{\kappa\in\mathcal{T}_h}\sum_{F\in\partial\kappa}\int_F \sigma(u,p)\cdot n_\kappa \cdot w_h \,\mathrm{d}s \\ &\qquad = \sum_{\kappa\in\mathcal{T}_h}\sum_{F\in\partial\kappa}\int_F n_\kappa \cdot (\sigma(u,p) - \{\sigma(\nu_h,\eta_h)\})\cdot w_h \,\mathrm{d}s.\end{aligned}$$

We now observe that by replacing w_h with the jump $[w_h]$ we may write the sum over the faces of the mesh, replacing n_κ by n_F. The conclusion follows by taking absolute values on both sides resulting in the desired inequality. □

We now use the previous result to give a characterisation of the consistency error for the full data assimilation problem.

Lemma 3.5. *Let $U := (u,p) \in V \times Q^0$ denote the solution to* (2.8)-(2.9) *with $\delta u = 0$. Then there holds*

$$\begin{aligned}&|\mathcal{A}[(U-U_h, Z_h),(X_h,Y_h)] - \mathcal{S}_h[(U_h,Z_h),(X_h,Y_h)] + \gamma_M(\tilde{u}_{\mathcal{M}} - u_h, v_h)_\omega| \\ &\qquad \le \inf_{(\nu_h,\eta_h)\in\mathcal{V}_h}\sum_{F\in\mathcal{F}}\int_F |n_F\cdot(\sigma(u,p) - \{\sigma(\nu_h,\eta_h)\})||[w_h]| \; ds\end{aligned}$$

for all $(X_h,Y_h) := ([w,y_h],[v_h,q_h]) \in \mathcal{V}_h\times\mathcal{W}_h$.

Proof. Subtract (3.10) from (2.8)-(2.9) and apply Lemma 3.4 to the equation for the primal variable U. □

For the analysis it is useful to establish the following elementary properties of the interpolant r_h and the projection π_0.

Lemma 3.6. *For any $v \in [H^1(\Omega)]^d$, $y \in L^2(\Omega)$ and for all $w_h \in W_h$, $q_h \in Q_h$ there holds*

$$a_h(v - r_h v, w_h) = 0,\ b_h(q_h, v - r_h v) = 0 \text{ and } b_h(y - \pi_0 y, w_h) = 0$$

Proof. By integration by parts we have, using the orthogonality property on the faces of r_h,

$$a_h(v - r_h v, w_h) = \sum_{\kappa \in \mathcal{T}_h} \sum_{F \in \partial\kappa} \int_F (v - r_h v) \cdot (n_\kappa \cdot \nabla w_h) \, \mathrm{d}s = 0,$$

$$b_h(q_h, v - r_h v) = \sum_{\kappa \in \mathcal{T}_h} \sum_{F \in \partial\kappa} \int_F (v - r_h v) \cdot n_\kappa q_h \, \mathrm{d}s = 0,$$

and by definition

$$b_h(y - \pi_0 y, w_h) = (y - \pi_0 y, \nabla \cdot w_h)_h = 0.$$

□

For the discrete stability of the system we will need to prove that the jumps of the gradient of the velocities and the jumps of the pressures can be bounded. The following technical result is therefore a key ingredient in the coming stability analysis.

Lemma 3.7. *Let* $(u_h, p_h) \in \mathcal{V}_h$ *then there holds*

$$\|h^{\frac{1}{2}} n_F \cdot [\nabla u_h]\|_{\mathcal{F}_i} + \|h^{\frac{1}{2}} [p_h]\|_{\mathcal{F}_i} \lesssim \|h^{\frac{1}{2}} n_F \cdot [\nabla u_h - \mathcal{I} p_h]\|_{\mathcal{F}_i} + \|\nabla \cdot u_h\|_\Omega + \|h^{-\frac{1}{2}} [u_h]\|_{\mathcal{F}_i}.$$

Proof. Let u_i, $i = 1, \dots, d$ denote the components of u_h and define the tangential projection of the gradient matrix on the face F by $T\nabla u_h := (I - n_F \otimes n_F)\nabla u_h$ where $\otimes$ denotes outer product. Considering one face $F \in \mathcal{F}_i$ we have

$$\|h^{\frac{1}{2}} n_F \cdot [\nabla u_h - \mathcal{I} p_h]\|_F^2 = \|h^{\frac{1}{2}} n_F \cdot [\nabla u_h]\|_F^2 + \|h^{\frac{1}{2}} [p_h]\|_F^2 - 2 \int_F h_F n_F \cdot [\nabla u_h] \cdot (n_F \cdot [\mathcal{I} p_h]) \mathrm{d}s.$$

The integrand of the last term of the right hand side may be written

$$n_F \cdot [\nabla u_h] \cdot (n_F \cdot [\mathcal{I} p_h]) = [p_h] \sum_{i=1}^d \sum_{j=1}^d n_{F,i} n_{F,j} [\partial_{x_j} u_i].$$

By applying the following identity

$$\sum_{i=1}^d \sum_{j=1}^d n_{F,i} n_{F,j} \partial_{x_j} u_i = \nabla \cdot u_h - \operatorname{tr}(T\nabla u_h),$$

where $\operatorname{tr}(T\nabla u_h)$ denotes the trace of $T\nabla u_h$, we may write

$$[p_h] \left(\sum_{i=1}^d \sum_{j=1}^d n_{F,i} n_{F,j} [\partial_{x_j} u_i] \right) = [p_h] \left([\nabla \cdot u_h] - [\operatorname{tr}(T\nabla u_h)] \right).$$

Observe that since the tangential component of the gradient of the conforming approximation $I_{\mathrm{cf}} u_h$ does not jump we have

$$[\operatorname{tr}(T\nabla u_h)] = [\operatorname{tr}(T(\nabla u_h - \nabla I_{\mathrm{cf}} u_h)].$$

Collecting these identities we obtain

$$\begin{aligned} \int_F h_F n_F \cdot [\nabla u_h] \cdot n_F \cdot [\mathcal{I} p_h] \mathrm{d}s &= \int_F h_F [p_h] \Big([\nabla \cdot u_h] - [\operatorname{tr}(T\nabla(u_h - I_{\mathrm{cf}} u_h))] \Big) \mathrm{d}s \\ &\leq \|h^{\frac{1}{2}} [p_h]\|_F C_i (\|\nabla(u_h - I_{\mathrm{cf}} u_h)\|_{\Delta_F} + \|\nabla \cdot u_h\|_{\Delta_F}). \end{aligned}$$

Consequently

$$2\int_F h_F n_F \cdot [\nabla u_h] \cdot n_F \cdot [\mathcal{I} p_h] \mathrm{d}s \leq \frac{1}{2} \|h^{\frac{1}{2}}[p_h]\|_F^2 + C\|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_{\Delta_F}}^2 + C\|\nabla \cdot u_h\|_{\Delta_F}^2.$$

Summing over $F \in \mathcal{F}_i$ we see that

$$\|h^{\frac{1}{2}} n_F \cdot [\nabla u_h]\|_{\mathcal{F}_i}^2 + \frac{1}{2}\|h^{\frac{1}{2}}[p_h]\|_{\mathcal{F}_i}^2 \lesssim \|h^{\frac{1}{2}} n_F \cdot [\nabla u_h - \mathcal{I} p_h]\|_{\mathcal{F}_i}^2 + \|\nabla \cdot u_h\|_\Omega^2 + C\|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}^2$$

which proves the claim. □

Lemma 3.8. *(Discrete Poincaré inequality) For all $(u_h, p_h) \in V_h \times Q_h^0$ there holds*

$$\|hu_h\|_{1,h} \lesssim \|h^{\frac{1}{2}} n_F \cdot [\nabla u_h]\|_{\mathcal{F}_i} + \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} + \|u_h\|_\omega$$

and

$$\|hp_h\|_\Omega \lesssim \|h^{\frac{1}{2}}[p_h]\|_{\mathcal{F}_i}.$$

Proof. For the first inequality use the Poincaré inequality for nonconforming finite elements and a triangle inequality

$$\|hu_h\|_{1,h} \lesssim \|h(\nabla u_h - I_{\mathrm{cf}} \nabla u_h)\|_h + \|hI_{\mathrm{cf}} \nabla u_h\|_h.$$

Then observe that for $I_{\mathrm{cf}} \nabla u_h$ constant, $\|u_h\|_\omega = 0$ implies that $I_{\mathrm{cf}} \nabla u_h = 0$ and therefore [14, Lemma B.63]

$$\|hI_{\mathrm{cf}} \nabla u_h\|_h \leq \|h\nabla(I_{\mathrm{cf}} \nabla u_h - \nabla u_h)\|_h + \|u_h\|_\omega.$$

Using the second inequality of (3.14) and then (3.18) componentwise twice we then have

$$\|hu_h\|_{1,h} \lesssim \|h^{\frac{1}{2}}[\nabla u_h]\|_{\mathcal{F}_i} + \|u_h\|_\omega.$$

Finally each component of ∇u_h is decomposed on the normal and tangential component on each face F and we observe that using an elementwise trace inequality,

$$\begin{aligned}\|h^{\frac{1}{2}}(\mathcal{I} - n_F \otimes n_F)[\nabla u_h]\|_{\mathcal{F}_i} &= \|h^{\frac{1}{2}}(\mathcal{I} - n_F \otimes n_F)[\nabla(u_h - I_{\mathrm{cf}} u_h)]\|_{\mathcal{F}_i} \\ &\lesssim \|\nabla(u_h - I_{\mathrm{cf}} u_h)\|_h \lesssim \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}.\end{aligned}$$

Similarly for the proof of the second inequality observe that since (redefining I_{cf} to act on a scalar variable, and once again by [14, Lemma B.63]) $\|hI_{\mathrm{cf}} p_h\|_\Omega \lesssim \|h\nabla I_{\mathrm{cf}} p_h\|_\Omega + \int_\Omega hI_{\mathrm{cf}} p_h \, \mathrm{d}x$ there holds

$$\|hp_h\|_h \lesssim \|h(p_h - I_{\mathrm{cf}} p_h)\|_h + \|h\nabla(I_{\mathrm{cf}} p_h - p_h)\|_h + \int_\Omega h(I_{\mathrm{cf}} p_h - p_h) \, \mathrm{d}x.$$

It then follows using an inverse inequality that

$$\|hp_h\|_h \lesssim \|p_h - I_{\mathrm{cf}} p_h\|_\Omega \lesssim \|h^{\frac{1}{2}}[p_h]\|_{\mathcal{F}_i}$$

and the proof is complete. □

4. Stability estimates

We will now focus on the formulation (3.10) with $\gamma_p = \gamma_x = 0$. An immediate consequence of this choice is that any solution to the system must satisfy

$$\nabla \cdot u_h|_\kappa = \nabla \cdot z_h|_\kappa = 0, \ \forall \kappa \in \mathcal{T}_h. \tag{4.1}$$

To see this we note that for the forward problem the mass conservation equations reads

$$b_h(u_h, y_h) = 0, \quad \forall y_h \in Q_h.$$

Since $\nabla \cdot u_h \in Q_h$ this is an admissible choice for the textfunction y_h and we conclude that $\|\nabla \cdot u_h\|_h = 0$. For the dual velocities on the other hand we have that

$$b_h(x_h, q_h) = 0, \quad \forall q_h \in Q_h^0.$$

Since $x_h \in W_h$ there holds (using the zero mean value property of the non-conforming space)

$$\sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla \cdot x_h \ \mathrm{d}x = \int_{\partial\Omega} x_h \cdot n \ \mathrm{d}s = 0.$$

Then $\nabla \cdot x_h \in Q_h^0$ and by taking $q_h = \nabla \cdot x_h$ we see that $\|\nabla \cdot x_h\|_h = 0$.

The issue of stability of the discrete formulation is crucial since we have no coercivity or inf–sup stability of the continuous formulation (2.8)–(2.9) to rely on. Indeed here the regularization plays an important part, since it defines a semi-norm on the discrete space. We introduce a mesh-dependent norm for the primal variable $X_h := (v_h, q_h) \in \mathcal{V}_h$

$$|||X_h|||_{V,Q} := \|h^{\frac{1}{2}} n_F \cdot [\nabla v_h]\|_{\mathcal{F}_i} + \|h^{\frac{1}{2}}[q_h]\|_{\mathcal{F}_i} + \gamma_M^{\frac{1}{2}} \|v_h\|_\omega + \|h^{-\frac{1}{2}}[v_h]\|_{\mathcal{F}_i}, \tag{4.2}$$

We will also use the following triple norm with control of both the dual pressure variable x_h and the dual velocities z_h.

$$|||(U_h, Z_h)||| := |||U_h|||_{V,Q} + \|x_h\|_\Omega + \|\nabla z_h\|_h.$$

Since Dirichlet boundary conditions are set weakly on W_h, $|||(U_h, Z_h)|||$ can be shown to be a norm on $V_h \times Q_h^0$ using Lemmata 3.7–3.8. We now prove a fundamental stability result for the discretization (3.10).

Theorem 4.1. *Let $\gamma_u, \gamma_M > 0$, $\gamma_p = \gamma_x = 0$ in (3.10)–(3.7). There exists a positive constant c_s, that is independent of h, but not of γ_u, γ_M or the local mesh geometry, such that for all $(U_h, Z_h) \in \mathcal{V}_h \times \mathcal{W}_h$ there holds*

$$c_s |||(U_h, Z_h)||| \le \sup_{(X_h, Y_h) \in \mathcal{V}_h \times \mathcal{W}_h} \frac{\mathcal{G}[(U_h, Z_h), (X_h, Y_h)]}{|||(X_h, Y_h)|||}$$

Proof. First we observe that by testing with $X_h = U_h$ and $Y_h = -Z_h$ we have

$$\gamma_u \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}^2 + \gamma_M \|u_h\|_\omega^2 = \mathcal{G}[(U_h, Z_h), (U_h, -Z_h)].$$

Then observe that by integrating by parts in the bilinear form $a_h(\cdot, \cdot)$ and using the zero mean value property of the approximation space we have

$$a_h(u_h, w_h) + b_h(p_h, w_h) = \sum_{F \in \mathcal{F}_i} \int_F [n_F \cdot \nabla u_h - p_h n_F] \cdot \{w_h\} \ \mathrm{d}s.$$

Define the function $\xi_h \in W_h$ such that for every face $F \in \mathcal{F}_i$

$$\overline{\{\xi_h\}}|_F := h_F [n_F \cdot \nabla u_h - p_h n_F]|_F.$$

This is possible in the nonconforming finite element space since the degrees of freedom may be identified with the average value of the finite element function on an element face. Using Lemma 3.1 we have

$$\|\xi_h\|_\Omega^2 \leq c_{\mathcal{T}} \sum_{F \in \mathcal{F}_i} h_F^2 \|h_F^{\frac{1}{2}}[n_F \cdot \nabla u_h - p_h n_F]\|_F^2. \tag{4.3}$$

Testing with $Y_h = (\xi_h, 0)$ and $X_h = (0,0)$ we get

$$\|h^{\frac{1}{2}}[n_F \cdot \nabla u_h - p_h n_F]\|_{\mathcal{F}_i}^2 = \mathcal{G}[(U_h, Z_h), (0, Y_h)].$$

By testing with $X_h = (z_h + \alpha r_h v_x, x_h)$, where $\alpha > 0$ and $v_x \in [H^1(\Omega)]^d$ is a function such that $\nabla \cdot v_x = x_h$ and $\|v_x\|_{H^1(\Omega)} \leq c_x \|x_h\|_\Omega$, we have

$$\begin{aligned} \|\nabla z_h\|_h^2 + \alpha \|x_h\|^2 + a_h(z_h, \alpha r_h v_x) & \\ + \gamma_u s_{j,-1}(u_h, z_h + \alpha r_h v_x) + \gamma_M (u_h, z_h + \alpha r_h v_x)_\omega & \\ &= \mathcal{G}[(U_h, Z_h), (X_h, 0)]. \end{aligned}$$

Observe now that by the Cauchy-Schwarz inequality, the arithmetic-geometric inequality and the stability of $r_h v_x$ there holds

$$a_h(z_h, \alpha r_h v_x) \leq \frac{1}{4} \|\nabla z_h\|_h^2 + c_x^2 \alpha^2 \|x_h\|_\Omega^2.$$

Then by the trace inequality and Poincaré's inequality

$$\begin{aligned} \gamma_u s_{j,-1}(u_h, z_h + \alpha r_h v_x) + \gamma_M (u_h, z_h + \alpha r_h v_x)_\omega & \\ \lesssim (\gamma_u \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} + \gamma_M \|u_h\|_\omega)(\|\nabla z_h\|_h + \|v_x\|_{H^1(\Omega)}) & \\ \leq C_\gamma (\gamma_u \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}^2 + \gamma_M \|u_h\|_\omega^2) + \frac{1}{4}(\|\nabla z_h\|_h^2 + \alpha^2 \|x_h\|_\Omega^2). & \end{aligned}$$

The consequence of this is that for $\alpha, \beta > 0$ sufficiently small there exists c such that

$$\begin{aligned} c\Big(\|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}^2 + \|\nabla z_h\|_h^2 + \|u_h\|_\omega^2 & \\ + \|x_h\|_\Omega^2 + \|h^{\frac{1}{2}}[n_F \cdot \nabla u_h - p_h n_F]\|_{\mathcal{F}_i}^2 \Big) & \\ \leq \mathcal{G}[(U_h, Z_h), (X_{UZ}, Y_{UZ})], & \end{aligned} \tag{4.4}$$

where $X_{UZ} = U_h + (\beta(z_h + \alpha r_h v_x), x_h)$, $Y_{UZ} = -Z_h + (\xi_h, 0)$. Applying Lemma 3.7, recalling that $\|\nabla \cdot u_h\|_h = 0$ we deduce that there exists $C > 0$ such that

$$C|||(U_h, Z_h)|||^2 \leq \mathcal{G}[(U_h, Z_h), (X_{UZ}, Y_{UZ})]. \tag{4.5}$$

It remains to prove that $|||(X_{UZ}, Y_{UZ})||| \lesssim |||(U_h, Z_h)|||$. This follows by observing that

$$\begin{aligned} &|||(X_{UZ}, Y_{UZ})||| \leq |||(U_h, Z_h)||| \\ &+ \beta(\|h^{\frac{1}{2}}[\nabla(z_h + \alpha r_h v_x)]\|_{\mathcal{F}_i} + \|(z_h + \alpha r_h v_x)\|_\omega + \|h^{-\frac{1}{2}}[z_h + \alpha r_h v_x]\|_{\mathcal{F}_i}) + \|\nabla \xi_h\|_h \end{aligned}$$

and

$$\begin{aligned} \|h^{\frac{1}{2}}[\nabla(z_h + \alpha r_h v_x)]\|_{\mathcal{F}_i} + \|(z_h + \alpha r_h v_x)\|_\omega + \|h^{-\frac{1}{2}}[z_h + \alpha r_h v_x]\|_{\mathcal{F}_i} &\lesssim \|z_h\|_{1,h} + \|r_h v_x\|_{1,h} \\ &\lesssim \|\nabla z_h\|_h + \|x_h\|_\Omega. \end{aligned}$$

Finally we use an inverse inequality and the bound (4.3) to obtain the bound

$$\|\nabla \xi_h\|_h \lesssim \|h_F^{\frac{1}{2}}[n_F \cdot \nabla u_h - p_h n_F]\|_{\mathcal{F}_i} \lesssim |||(U_h, 0)|||$$

which completes the proof. $\square$

Corollary 4.2. *The formulation* (3.10) *admits a unique solution* $(u_h, p_h) \in \mathcal{V}_h$ *and* $(z_h, x_h) \in \mathcal{W}_h$.

Proof. The system matrix corresponding to (3.10) is a square matrix and we only need to show that there are no zero eigenvalues. Assume that $l(w_h) = 0$. It then follows by Theorem 4.1 that for any solution (u_h, p_h) there holds

$$c_s |||(U_h, Z_h)||| \leq \sup_{(X_h, Y_h) \in \mathcal{V}_h \times \mathcal{W}_h} \frac{\mathcal{G}[(U_h, Z_h), (X_h, Y_h)]}{|||(X_h, Y_h)|||} = 0.$$

Recalling Lemma 3.8 this implies that $u_h = p_h = z_h = x_h = 0$ showing that the solution is unique. $\square$

Remark 4.3. Observe that the proof of Theorem 4.1 works also for $\gamma_p > 0$ and $\gamma_x > 0$, the only modification in this case is that the contribution $\|\nabla \cdot u_h\|_h$ must be added to the norm $|||(u_h, p_h)|||_V$ and stability must be proven by testing with $y_h = \nabla \cdot u_h$.

5. Error estimates using conditional stability

In this section we will use the stability proven in the previous section to derive error estimates.

Proposition 5.1. *Let* $(u, p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ *be the solution of* (2.5)-(2.6), *with* $u|_\omega = u_M$ *and* $(u_h, p_h) \times Z_h$ *the solution to* (3.10)–(3.7), *with* $\gamma_u, \gamma_M > 0$ *and* $\gamma_p = \gamma_x = 0$. *Then there holds*

$$|||((r_h u - u_h, \pi_0 p - p_h), Z_h)||| \lesssim h(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}) + \gamma_M^{\frac{1}{2}} \|\delta u\|_\omega$$

Proof. First denote the discrete error $\Theta_h = (r_h u - u_h, \pi_0 p - p_h)$. Then by Theorem 4.1

$$c_s |||(\Theta_h, Z_h)||| \leq \sup_{(X_h, Y_h) \in \mathcal{V}_h \times \mathcal{W}_h} \frac{\mathcal{G}[(\Theta_h, Z_h), (X_h, Y_h)]}{|||(X_h, Y_h)|||}.$$

Then applying Lemma 3.5 and 3.6 we have

$$\begin{aligned}\mathcal{G}[(\Theta_h, Z_h), (X_h, Y_h)] \leq \inf_{(\nu_h, \eta_h) \in V_h \times W_h} \sum_{F \in \mathcal{F}} \int_F |(\sigma(u,p) - \{\sigma(\nu_h, \eta_h)\}) \cdot n_F| |[w_h]| \, \mathrm{d}s \\ - b_h(y_h, r_h u - u) + s_{j,-1}(r_h u, v_h) + \gamma_M (r_h u - u - \delta u, v_h)_\omega.\end{aligned}$$

First note that

$$\begin{aligned}&\inf_{(\nu_h, \eta_h) \in \mathcal{V}_h} \sum_{F \in \mathcal{F}} \int_F |(\sigma(u,p) - \{\sigma(\nu_h, \eta_h)\}) \cdot n_F| |[w_h]| \, \mathrm{d}s \\ &\quad \leq h^{\frac{1}{2}} (\inf_{(\nu_h, \eta_h) \in \mathcal{V}_h} \sum_{F \in \mathcal{F}} \|\sigma(u,p) - \{\sigma(\nu_h, \eta_h)\}\|_F^2)^{\frac{1}{2}} \|\nabla w_h\|_h \\ &\quad\quad \lesssim h(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}) |||(0, Y_h)|||,\end{aligned}$$

$$b_h(y_h, r_h u - u) = 0,$$

$$s_{j,-1}(r_h u, v_h) \leq C h \|u\|_{H^2(\Omega)} \|h^{-\frac{1}{2}} [v_h]\|_{\mathcal{F}_i} \leq C h \|u\|_{H^2(\Omega)} |||(X_h, 0)|||.$$

Finally, using a Cauchy-Schwarz inequality inequality in the data term and that $\gamma_M^{\frac{1}{2}}\|v_h\|_\omega \le |||(X_h,0)|||$, we have

$$\gamma_M(r_h u - u, v_h)_\omega \lesssim \gamma_M\|r_h u - u\|_\omega\|v_h\|_\omega \lesssim \gamma_M^{\frac{1}{2}} h^2\|u\|_{H^2(\Omega)}|||(X_h,0)|||.$$

For the perturbation we have

$$\gamma_M(\delta u, v_h)_\omega \le \gamma_M\|\delta u\|_\omega\|v_h\|_\omega \le \gamma_M^{\frac{1}{2}}\|\delta u\|_\omega|||(X_h,0)|||.$$

Collecting the above estimates ends the proof. □

Theorem 5.2. *Assume that $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$. Let $\tilde{u}_h$ be defined by* (3.19) *then*

$$\|\tilde{u}_h\|_{H^1(\Omega)} + \|p_h\|_\Omega \lesssim \|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} + \gamma_M^{\frac{1}{2}} h^{-1}\|\delta u\|_\omega$$

and, if $\delta u = 0$

$$\tilde{u}_h \rightharpoonup u \text{ in } [H^1(\Omega)]^d \text{ and } p_h \rightharpoonup p \text{ in } L^2\Omega).$$

Proof. For the pressure we immediately observe that

$$\begin{aligned}\|p_h\|_\Omega \lesssim \|p_h - \pi_0 p\|_\Omega + \|p\|_\Omega \lesssim h^{-1}|||(0, p_h - \pi_0 p)|||_{V,Q} + \|p\|_\Omega\\ \lesssim \|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} + \gamma_M^{\frac{1}{2}} h^{-1}\|\delta u\|_\omega\end{aligned}$$

Then observe that by Corollary 3.3 and the H^1-stability of the interpolation operator r_h there holds

$$\begin{aligned}\|\tilde{u}_h\|_{H^1(\Omega)} &\le \|\tilde{u}_h - u_h\|_{1,h} + \|u_h\|_{1,h}\\ &\le \|\tilde{u}_h - u_h\|_{1,h} + \|u_h - r_h u\|_{1,h} + \|r_h u\|_{1,h}\\ &\lesssim \|h^{-\frac{1}{2}}[u_h - r_h u]\|_{\mathcal{F}_i} + \|u_h - r_h u\|_\omega + \|\nabla(u_h - r_h u)\|_h + \|u\|_{H^1(\Omega)}.\end{aligned}$$

Therefore

$$\|\tilde{u}_h\|_{H^1(\Omega)} \lesssim h^{-1}|||(u_h - r_h u, 0)|||_V + \|u\|_{H^1(\Omega)}$$

and the first claim follows by applying Proposition 5.1.

It follows that for $\delta u = 0$ we may extract a subsequence of pairs $(\tilde{u}_h, p_h)$ that converges weakly in $[H^1(\Omega)]^d \times L^2(\Omega)$. By construction the divergence of the H^1-conforming part satisfies

$$\|\nabla\cdot\tilde{u}_h\|_\Omega \lesssim \|h^{-\frac{1}{2}}[u_h - r_h u]\|_{\mathcal{F}_i} + \underbrace{\|\nabla\cdot u_h\|_h}_{=0} + h\|u\|_{H^2(\Omega)} \lesssim h\|u\|_{H^2(\Omega)}$$

and hence $\|\nabla\cdot\tilde{u}_h\|_h \to 0$ for $h \to 0$. It remains to show that the weak limit is a weak solution of Stokes equation. To this end consider, with $w \in C_0^1(\Omega)$,

$$\begin{aligned}&|a(\tilde{u}_h, w) + b(p_h, w) - l(w)|\\ &\quad= |a_h(\tilde{u}_h - u_h, w) + a_h(u_h, w - r_h w) + b(p_h, w - r_h w) - l(w - r_h w)|\\ &\quad= |a_h(\tilde{u}_h - u_h, w) - l(w - r_h w)| \lesssim (\|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} + h\|f\|_\Omega)\|w\|_{H^1(\Omega)}\\ &\quad\lesssim h(\|u\|_{H^2(\Omega)} + \|f\|_\Omega)\|w\|_{H^1(\Omega)}\end{aligned}$$

We conclude by taking the limit $h \to 0$.

□

Theorem 5.3. *Assume that* $(u,p) \in [H^2(\Omega)]^d \times H^1(\Omega)$ *is the unique solution of* (2.5)-(2.6) *with* $u = u_M$ *in* ω *and the parameters* R_1, R_2 *and* R_3 *satisfy the assumptions of Theorem 2.1. If* u_h *is the solution of* (3.10)-(3.7)*, with* $\gamma_u, \gamma_M > 0$, $\gamma_p = \gamma_x = 0$ *and* $\|\delta u\|_\Omega \le h_0$, $h_0 > 0$*, then, for* $h > h_0$*, there holds*

$$\|u - u_h\|_{B_{R_2}(x_0)} \lesssim h^\tau$$

where τ *is the power from Theorem 2.1 and the hidden constant depends on* R_2/R_3*, the local mesh geometry and* $\|u\|_{H^2(\Omega)}$ *and* $\|p\|_{H^1(\Omega)}$.

Proof. First let $u - u_h = \underbrace{u - \tilde{u}_h}_{e_u \in [H^1(\Omega)]^d} + \underbrace{\tilde{u}_h - u_h}_{e_h \in V_h}$, where $\tilde{u}_h$ is defined by (3.19). We recall that

$$\|e_h\|_\Omega \lesssim \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} \le Ch\|u\|_{H^2(\Omega)}$$

so we only need to bound $\|e_u\|_{B_{R_2}(x_0)}$. Also introduce $e_p = p - p_h \in L^2(\Omega)$. It follows that (e_u, e_p) is a solution to the Stokes' equation on weak form with a particular right hand side. Indeed we have for all $(w,q) \in [H_0^1(\Omega)]^d \times Q$

$$a(e_u, w) + b(e_p, w) = l(w) - a(\tilde{u}_h, w) + b(p_h, w) =: \langle \mathfrak{f}, w\rangle_{V',V} \tag{5.1}$$

and

$$-b(q, e_u) = b(q, \tilde{u}_h) =: (\mathfrak{g}, q)_\Omega \tag{5.2}$$

where $\mathfrak{f} \in V'$ and $\mathfrak{g} \in L_0^2(\Omega)$. Now consider the problem (2.1) with homogeneous Dirichlet boundary conditions on $\partial\Omega$ and the right hand side $\mathfrak{f}$ and $\mathfrak{g}$ as defined above. This problem is well-posed and we call its solution $\{\mathcal{E}_u, \mathcal{E}_p\} \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$. By the well-posedness of the problem we know that

$$\|\mathcal{E}_u\|_{H^1(\Omega)} + \|\mathcal{E}_p\|_\Omega \le \|\mathfrak{f}\|_{H^{-1}(\Omega)} + \|\mathfrak{g}\|_\Omega$$

We know from equation (3.22), the fact that $\|\nabla \cdot u_h\|_h = 0$ and Proposition 5.1 that $\|\mathfrak{g}\|_\Omega \lesssim \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} \lesssim h + \|\delta u\|_\omega$ and for $\|\mathfrak{f}\|_{V'}$ we use the relations of Lemma 3.6, Corollary 3.3 and the approximation of Lemma 3.15 to derive the bound

$$\begin{aligned}\sup_{\substack{w \in [H_0^1]^d \\ \|w\|_1 = 1}} \langle \mathfrak{f}, w\rangle_{V'V} &= l(w) - a(\tilde{u}_h, w) - b(p_h, w) \\ &= l(w - r_h w) - a_h(\tilde{u}_h - u_h, w) \\ &\lesssim h\|f\|_\Omega + s_{j,-1}(u_h, u_h)^{\frac{1}{2}} \lesssim h + \|\delta u\|_\omega.\end{aligned} \tag{5.3}$$

It follows that

$$\|\mathcal{E}_u\|_{H^1(\Omega)} + \|\mathcal{E}_p\|_\Omega \lesssim h + \|\delta u\|_\omega. \tag{5.4}$$

Considering now the functions $U := u - \tilde{u}_h - \mathcal{E}_u$ and $P := p - p_h - \mathcal{E}_p$ we see that $\{U, P\}$ is a solution to (2.1) with $\mathfrak{f} = 0$ and $\mathfrak{g} = 0$. By equation (2.3) we have $\{U, P\} \in [H^2(\varpi)]^d \times H^1(\varpi)$ on every compact $\varpi \subset \Omega$. We may then apply Theorem 2.1 to U and obtain

$$\int_{B_{R_2}(x_0)} |U|^2 \,\mathrm{d}x \le C \left(\int_{B_{R_1}(x_0)} |U|^2 \,\mathrm{d}x\right)^\tau \left(\int_{B_{R_3}(x_0)} |U|^2 \,\mathrm{d}x\right)^{1-\tau}. \tag{5.5}$$

These results may now be combined in the following way to prove the theorem. First by the triangle inequality, writing $u - u_h = U + \mathcal{E}_u + \tilde{u}_h - u_h$,

$$\|u - u_h\|_{B_{R_2}(x_0)} \le \|\mathcal{E}_u\|_{B_{R_2}(x_0)} + \|\tilde{u}_h - u_h\|_{B_{R_2}(x_0)} + \|U\|_{B_{R_2}(x_0)} = I + II + III.$$

By (2.2) and (5.3) there holds for the first term using (5.4)

$$I \lesssim \|\mathcal{E}_u\|_{H^1(\Omega)} \lesssim h + \|\delta u\|_\omega$$

and using the discrete interpolation of Corollary 3.3 and Proposition 5.1

$$II = \|\tilde{u}_h - u_h\|_{B_{R_2}(x_0)} \lesssim \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i} \lesssim h + \|\delta u\|_\omega.$$

For the last term, using the inequality (5.5), we have

$$III \lesssim \left(\int_{B_{R_1}(x_0)} |U|^2 \, \mathrm{d}x\right)^{\tau/2} \left(\int_{B_{R_3}(x_0)} |U|^2 \, \mathrm{d}x\right)^{(1-\tau)/2}.$$

By the definition of U and since by assumption $B_{R_1}(x_0) \subset \omega$

$$\left(\int_{B_{R_1}(x_0)} |U|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \lesssim \|r_h u - u_h\|_\omega + \|r_h u - u\|_\omega + \|\tilde{u}_h - u_h\|_\omega + \|\mathcal{E}_u\|_{B_{R_1}(x_0)} \lesssim h + \|\delta u\|_\omega. \tag{5.6}$$

Here we applied discrete interpolation (3.22), Proposition 5.1, (3.15), and (5.4) applied to $\mathcal{E}_u$. Finally by the triangle inequality, the a priori assumption $u \in H^2(\Omega)$, (5.4) and the first claim of Theorem 5.2 we have

$$\left(\int_{B_{R_3}(x_0)} |U|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \leq \|u\|_{H^1(\Omega)} + \|\tilde{u}_h\|_{H^1(\Omega)} + \|\mathcal{E}_u\|_{H^1(\Omega)} \lesssim 1 + h^{-1}\|\delta u\|_\omega.$$

The claim follows by collecting the bounds on the terms $I - III$ and applying the assumption on the perturbations in data versus the mesh-size, $\|\delta u\|_\omega < h_0 < h$. □

Remark 5.4. It is straightforward to prove the Proposition 5.1 and the Theorems 5.2 and 5.3 also for $\gamma_p \geq 0$ and $\gamma_x \geq 0$ and thereby extending the analysis to include the method (3.13). We leave the details for the reader.

Remark 5.5. One may also introduce perturbations in the right hand side f. Provided these perturbations are in $[L^2(\Omega)]^d$ the same results holds. Details on the necessary modifications can be found in [10].

6. Numerical example

For all computations in this section we have used FreeFEM++, [19]. Our numerical example is set in the unit square $\Omega = (0,1)^2$ and data given in the disc $S_{1/2} := \{(x,y) \in \mathbb{R}^2 : \sqrt{(x-0.5)^2 + (y-0.5)^2} < 0.125\}$. The flow is nonsymmetric with the exact solution given by

$$u(x,y) = (20xy^3, 5x^4 - 5y^4) \qquad \text{and} \qquad p(x,y) = 60x^2y - 20y^3 - 5.$$

It is straightforward to verify that (u,p) is a solution to the homogeneous Stokes' problem. We consider the formulation (3.10)-(3.7), with $l(w_h) = 0$ For the parameters we chose, $\gamma_M = 800$ and $\gamma_u = 10^{-5}$, $\gamma_p = \gamma_z = \gamma_x = 0$. First we perform the computation with unperturbed data. The results are presented in the left graphic of Figure 1. We report the velocity error both in the global L^2-norm (open square markers), the local L^2-norm in the subdomain where $\sqrt{(x-0.5)^2 + (y-0.5)^2} <$

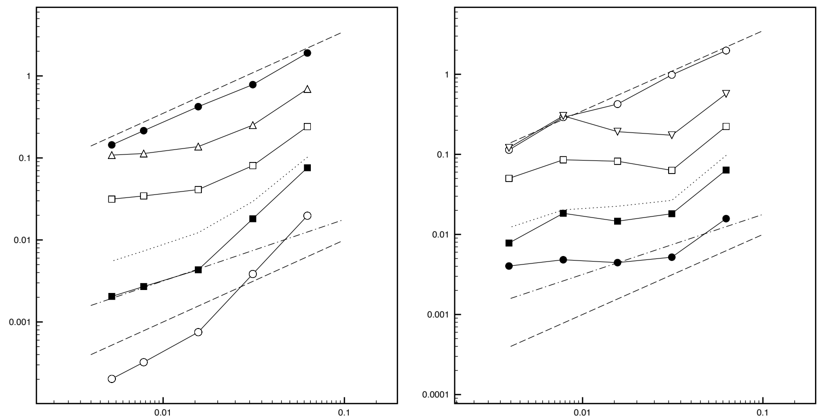

FIGURE 1. Relative L^2-error against mesh-size, left unperturbed data, right with 1% relative noise. Reference lines are the same in both plots and of orders, dashed lines $\approx O(h)$ with different constants, dash dot $\approx O(h^{\frac{3}{4}})$ and dotted $C_1\|e\|_\Omega^{0.3}(r_1+r_2)^{0.7}+10h^2$

0.375 (filled square markers) and in the residual quantities of (6.1) (circle markers, r_1 filled, r_2 open),

$$r_1 := \left(\int_{S_{1/2}} (u_h - u)^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \text{ and } r_2 := \|h^{-\frac{1}{2}}[u_h]\|_{\mathcal{F}_i}. \tag{6.1}$$

The global pressure is plotted with triangle markers. The error plots for this case are given in Figure 1. We observe the $O(h)$ convergence of the residual quantities (6.1). The global velocity and pressure L^2-errors appears to have approximately $O(|log(h)|^{-1})$ convergence. The local error matches the result of Theorem 5.3. Indeed the dotted line shows the behavior of the quantity $C_1\|e\|_\Omega^{0.3}(r_1+r_2)^{0.7}+10h^2$ illustrating the different components of the local error used in the proof of the theorem. We see that this quantity (with a properly chosen constant) gives a good fit with the local error and that for this computational configuration we have $\tau \sim 0.7$ for the error quantity indicated with filled square markers.

The same computation was repeated with a 1% relative random perturbation of data in the L^∞-norm. The results for this case is reported in the right plot of Figure 1. As predicted by theory the results are stable under perturbation of data as long as the discretization error is larger than the random perturbation (up to a constant). When the perturbations dominate the errors in all quantities appear to stagnate.

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Department of Mathematics, University College London, London, UK-WC1E 6BT, United Kingdom

E-mail address: e.burman@ucl.ac.uk

Department of Mechanical Engineering, Jönköping University, SE-55111 Jönköping, Sweden

E-mail address: Peter.Hansbo@ju.se