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THE PENALTY-FREE NITSCHE METHOD AND NONCONFORMING FINITE ELEMENTS FOR THE SIGNORINI PROBLEM*

ERIK BURMAN†, PETER HANSBO‡, AND MATS G. LARSON§

Abstract. We design and analyse a Nitsche method for contact problems. Compared to the seminal work of Chouly and Hild [*SIAM J. Numer. Anal.*, 51 (2013), pp. 1295–1307], our method is constructed by expressing the contact conditions in a nonlinear function for the displacement variable instead of the lateral forces. The contact condition is then imposed using the nonsymmetric variant of Nitsche's method that does not require a penalty term for stability. Nonconforming piecewise affine elements are considered for the bulk discretization. We prove optimal error estimates in the energy norm.

Key words. finite element, Nitsche's method, contact, Signorini problem

AMS subject classifications. 65N12, 65N30, 74M15

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1. Introduction.

We consider the Signorini problem. Find u such that

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma_D \\ \partial_n u &= 0 \text{ on } \Gamma_N \\ u \leq 0,\ \partial_n u \leq 0,\ u\partial_n u &= 0 \text{ on } \Gamma_C, \end{aligned} \tag{1.1}$$

where $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a convex polygonal (polyhedral) domain with boundary $\partial\Omega$, and $\Gamma_D \cup \Gamma_N \cup \Gamma_C = \partial\Omega$. We assume that Γ_C coincides with one of the sides of the polygon. We write $\partial_n u := n \cdot \nabla u$, where n denotes the outwards-pointing normal of $\partial\Omega$. We present the problem here with homogeneous boundary data for simplicity, but the arguments below can be extended to the nonhomogeneous case after minor modifications.

It is well known that this problem admits a unique solution $u \in H^1(\Omega)$. This follows from the theory of Stampacchia applied to the corresponding variational inequality (see, e.g., [19]). We will also assume the additional regularity $u \in H^{\frac{3}{2}+\nu}(\Omega)$ $0 < \nu \leq \frac{1}{2}$. There exists a large body of literature treating finite element methods for contact problems. In general, however, it has proven difficult to prove optimal error estimates without making assumptions on the regularity of the exact solution on the contact zone. In the pioneering work of Scarpini and Vivaldi [28], $O(h^{\frac{3}{4}})$ convergence was proved in the energy norm for solutions in $H^2(\Omega)$. Brezzi, Hager,

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†Department of Mathematics, University College London, London WC1E 6BT, UK (e.burman@ucl.ac.uk).
‡Department of Mechanical Engineering, Jönköping University, SE-551 11 Jönköping, Sweden (peter.hansbo@ju.se).
§Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87 Umeå, Sweden (mats.larson@math.umu.se).

and Raviart [9] then proved $O(h)$ convergence under the additional condition that the solution was in $W^{1,\infty}(\Omega)$ or that the number of points where the contact condition changes from binding to nonbinding is finite. These initial works were followed by a series of papers where the scope was widened and sharper estimates obtained [26, 18, 5, 4, 6, 31, 22, 12]. Discretization of (1.1) is usually performed on the variational inequality or using a penalty method. The first case, however, leads to some nontrivial choices in the construction of the discretization spaces in order to satisfy the nonpenetration condition, and only recently has optimal error estimates been derived [16]. The latter case leads to the usual consistency and conditioning issues of penalty methods. A detailed analysis of the penalty method was recently performed by Chouly and Hild [12]. Another approach proposed by Hild and Renard [21] is to use a stabilized Lagrange multiplier in the spirit of Barbosa and Hughes [3], using the reformulation of the contact condition

$$\partial_n u = -\gamma^{-1}[u - \gamma\partial_n u]_+, \quad \gamma > 0, \tag{1.2}$$

where $[x]_\pm = \pm\max(0, \pm x)$ proposed by Alart and Curnier [2] in an augmented Lagrangian framework. Using the close relationship between the Barbosa and Hughes method and Nitsche's method [27] as discussed by Stenberg [29], this method was then further developed in the elegant Nitsche-type formulation introduced by Chouly, Hild, and Renard [11, 13]. In these works, optimal error estimates for solutions in $H^{\frac{3}{2}+\nu}(\Omega)$ to the above model problem were obtained for the first time. Their method was proposed in a nonsymmetric and a symmetric version similar to Nitsche's method for the imposition of boundary conditions; it has, however, been observed that in their framework, there was no equivalent to the penalty-free nonsymmetric Nitsche method proposed in [10]. Our aim in this work is to fill this gap, rather adding a piece to the puzzle than pretending to propose a method superior to the previous variants.

2. Nitsche's method for the Signorini problem. We will first discuss an alternative route to Nitsche-type methods for the contact problem that allows us to formulate the nonsymmetric penalty-free Nitsche method that will be the main concern of this work. The penalty-free Nitsche method can be interpreted as a Lagrange multiplier method where the multiplier and the corresponding test function have been replaced by the normal flux of the solution variable and of its test function, respectively. To design this method for contact problems, we take a slightly different approach than in [11]. Instead of working on the formulation (1.2) for the lateral forces, we use a similar relation on the displacement:

$$u = -\gamma[\partial_n u - \gamma^{-1}u]_+, \quad \gamma > 0. \tag{2.1}$$

Setting $P_\gamma(u) = \gamma\partial_n u - u$, we may write this relation as

$$u = -[P_\gamma(u)]_+. \tag{2.2}$$

It is straightforward to show that this is equivalent to the contact condition of equation (1.1). First assume that (2.1) holds. Then by construction $u \leq 0$. Assume that $\partial_n u > 0$. Then $u = -\gamma(\partial_n u - \gamma^{-1}u) < u$, which is a contradiction; hence, $\partial_n u \leq 0$. Assuming now $u < 0$ and $\partial_n u < 0$ implies that $u = -\gamma[\partial_n u - \gamma^{-1}u]_+ > u$, also a contradiction, and it follows that $u\partial_n u = 0$. On the other hand, if $u\partial_n u = 0$, $u \leq 0$, and $\partial_n u \leq 0$, we see that where $u < 0$, (2.1) becomes $u = -\gamma[-\gamma^{-1}u]_+$, and where $\partial_n u < 0$, (2.1) becomes $[\partial_n u]_+ = 0$, both which are true.

We multiply the first equation of (1.1) by a function v with zero trace on Γ_D and apply Green's formula to obtain

$$a(u,v) - \langle \partial_n u, v\rangle_{\Gamma_C} = (f,v)_\Omega,$$

where $(\cdot,\cdot)_\Omega$ and $\langle \cdot,\cdot\rangle_{\Gamma_C}$ denote the L^2-scalar product on Ω and Γ_C, respectively, and $a(u,v) := (\nabla u, \nabla v)_\Omega$. We then add a term imposing (2.1) on the form

$$\left\langle u + \gamma[\partial_n u - \gamma^{-1} u]_+, \theta_1 \partial_n v + \theta_2 \gamma^{-1} v\right\rangle_{\Gamma_C}, \tag{2.3}$$

resulting in family of Nitsche formulations defined by two parameters θ_1 and θ_2:

$$\begin{aligned} &a(u,v) - \langle \partial_n u, v\rangle_{\Gamma_C} + \theta_1 \langle \partial_n v, u\rangle_{\Gamma_C} + \theta_2 \gamma^{-1} \langle u, v\rangle_{\Gamma_C} \\ &\quad + \left\langle \gamma[\partial_n u - \gamma^{-1} u]_+, \theta_1 \partial_n v + \theta_2 \gamma^{-1} v\right\rangle_{\Gamma_C} = (f,v)_\Omega. \end{aligned}$$

Taking $\theta_1 \in \{-1,0,1\}$ and $\theta_2 = 1$ results in Nitsche methods on the form

$$\begin{aligned} &(\nabla u, \nabla v)_\Omega - \langle \partial_n u, v\rangle_{\Gamma_C} + \theta_1 \langle u, \partial_n v\rangle_{\Gamma_C} + \left\langle \gamma^{-1} u, v\right\rangle_{\Gamma_C} \\ &\quad + \left\langle [P_\gamma(u)]_+, \gamma^{-1} v + \theta_1 \partial_n v\right\rangle_{\Gamma_C} = (f,v)_\Omega. \end{aligned} \tag{2.4}$$

From this expression, we deduce that the linear part of the formulation coincides with the classical version of Nitsche's method, and the constraint is then imposed exactly as a nonhomogeneous Dirichlet condition in Nitsche's method. We will now show that the formulation (2.4) is equivalent with the formulation proposed in [13] that is derived using the condition (1.2) and may be written formally: find u such that

$$(\nabla u, \nabla v)_\Omega + \gamma^{-1} \langle [u - \gamma \partial_n u]_+, v + \theta_1 \gamma \partial_n v\rangle_{\Gamma_C} + \theta_1 \gamma \langle \partial_n u, \partial_n v\rangle_{\Gamma_C} = (f,v)_\Omega. \tag{2.5}$$

Before proceeding, we prove equivalence of (2.4) and (2.5).

PROPOSITION 2.1. *The formulations* (2.4) *and* (2.5) *are equivalent.*

Proof. We will show that the left-hand side of the expression (2.4) is equivalent to that of (2.5). To this end, first observe that

$$[P_\gamma(u)]_+ + u - \gamma \partial_n u = [-(u - \gamma \partial_n u)]_+ + u - \gamma \partial_n u = [u - \gamma \partial_n u]_+ \tag{2.6}$$

and that

$$\begin{aligned} &- \langle \partial_n u, v\rangle_{\Gamma_C} + \theta_1 \langle \partial_n v, u\rangle_{\Gamma_C} + \gamma^{-1} \langle u, v\rangle_{\Gamma_C} - \left\langle u - \gamma \partial_n u, \gamma^{-1} v + \theta_1 \partial_n v\right\rangle_{\Gamma_C} \\ &\quad = \theta_1 \gamma \langle \partial_n u, \partial_n v\rangle_{\Gamma_C}. \end{aligned} \tag{2.7}$$

Then add and subtract $u - \gamma \partial_n u$ in the last term in the left-hand side of (2.4), resulting in

$$\begin{aligned} &\left\langle [P_\gamma(u)]_+ + (u - \gamma \partial_n u) - (u - \gamma \partial_n u), \gamma^{-1} v + \theta_1 \partial_n v\right\rangle_{\Gamma_C} \\ &\quad = \gamma^{-1} \langle [u - \gamma \partial_n u]_+, v + \theta_1 \gamma \partial_n v\rangle_{\Gamma_C} - \gamma^{-1} \langle u - \gamma \partial_n u, v + \theta_1 \gamma \partial_n v\rangle_{\Gamma_C}, \end{aligned}$$

where we used the relation (2.6). The claim follows by applying (2.7). $\square$

A similar approach to the analysis that we propose below in the case of the penalty-free Nitsche method can also be applied to (2.4), using standard conforming finite elements, and in that case it offers an alternative analysis of the method (2.4), (2.5), resulting in the same bounds as those proven in [13].

Herein we will consider the method obtained when $\theta_1 = 1$ and $\theta_2 = 0$, in which case the term imposing the contact condition reduces to

$$\left\langle u + \gamma[\partial_n u - \gamma^{-1} u]_+, \partial_n v \right\rangle_{\Gamma_C}.$$

Observe that the two terms differ only by the exclusion of the last term, which corresponds to a penalty, and in that sense the latter variant is penalty free.

It follows that the penalty-free version leads to the following formal restatement of (1.1) for smooth u:

$$(\nabla u, \nabla v)_\Omega - \langle \partial_n u, v\rangle_{\Gamma_C} + \langle u, \partial_n v\rangle_{\Gamma_C} + \langle [P_\gamma(u)]_+, \partial_n v\rangle_{\Gamma_C} = (f, v)_\Omega. \tag{2.8}$$

Observe that the linear part of the system is equivalent to that proposed in [10] for Dirichlet boundary conditions but that here this is used to enforce the condition (2.2) on u.

For the discretization of (2.8), we will use the Crouzeix–Raviart nonconforming piecewise affine element with midpoint continuity on element edges (or continuity of averages over faces in three dimensions). As we shall see below, this element is advantageous for the formulation proposed since the necessary stability results are relatively straightforward to prove. The nonconforming finite element space has been analyzed for the Signorini problem by Hua and Wang [24]. They prove optimal convergence up to a logarithmic factor for $H^2(\Omega)$ solutions under the assumption that the number of points where the constraint changes from binding to nonbinding is finite. In this work, we prove the same optimal results for solutions in $H^{\frac{3}{2}+\nu}(\Omega)$, $\nu > 0$ as those obtained in [11, 13].

To handle the nonconformity error, we need to make an additional mild assumption on the source term: the trace of f must be well defined in the vicinity of the contact boundary Γ_C. To make this precise, we let $\|\cdot\|_X$ denote the L^2-norm over the domain X, which may be a subset of either $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$, and define

$$\Omega_{t_C} := \{x \in \bar{\Omega} : x = y - n_y t, \text{ where } y \in \Gamma_C \text{ and } 0 \le t \le t_C\},$$

where n_y denotes the outward-pointing normal on Γ_C at the point y. For a fixed t, we then define

$$\partial_t \Omega := \{x \in \bar{\Omega} : x = y - n_y t, \text{ where } y \in \Gamma_C\}.$$

Observe that for any function $v \in H^s(\Omega_{t_C})$ with $s > \frac{1}{2}$, there holds

$$\|v\|_{\Omega_{t_C}} \lesssim t_C^{\frac{1}{2}} \sup_{0 \le t \le t_C} \|v\|_{\partial_t \Omega}. \tag{2.9}$$

We introduce the norm $\|v\|_{L^2_\infty(\Omega)} := \|v\|_{L^2(\Omega)} + \sup_{0 \le t \le t_C} \|v\|_{\partial_t \Omega}$ and assume that

$$\exists\ t_C > 0 \text{ such that } \|f\|_{L^2_\infty(\Omega)} < \infty. \tag{2.10}$$

3. The nonconforming finite element method.

To simplify the analysis below, we will work with the nonconforming finite element space proposed by Crouzeix and Raviart in [14]. Let $\{\mathcal{T}_h\}_h$ denote a family of shape regular and quasi-uniform tessellations of Ω into nonoverlapping simplices such that for any two different simplices κ, $\kappa' \in \mathcal{T}_h$, $\kappa \cap \kappa'$ consists of either the empty set, a common face or edge, or a common vertex. The diameter of a simplex κ will be denoted h_κ and the outward-pointing normal n_κ. The family $\{\mathcal{T}_h\}_h$ is indexed by the maximum element size of $\mathcal{T}_h$, $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$. We denote the set of element faces in $\mathcal{T}_h$ by $\mathcal{F}$ and let $\mathcal{F}_i$ denote the set of interior faces and $\mathcal{F}_\Gamma$ the set of faces in some $\Gamma \subset \partial\Omega$. We will assume that the mesh is fitted to the subsets of $\partial\Omega$ representing the boundary conditions Γ_D, Γ_N, and Γ_C so that the boundaries of these subsets coincide with the boundaries of element faces. To each face F, we associate a unit normal vector, n_F. For interior faces, its orientation is arbitrary but fixed. On the boundary $\partial\Omega$, we identify n_F with the outward-pointing normal of $\partial\Omega$. The subscript on the normal is dropped in cases where it follows from the context.

We define the jump over interior faces $F \in \mathcal{F}_i$ by

$$[\![v]\!]|_F := \lim_{\epsilon \to 0^+} (v(x|_F - \epsilon n_F) - v(x|_F + \epsilon n_F)),$$

and for faces on the boundary, $F \in \partial\Omega$, we let $[\![v]\!]|_F := v|_F$. Similarly, we define the average of a function over an interior face F by

$$\{v\}|_F := \tfrac{1}{2} \lim_{\epsilon \to 0^+} (v(x|_F - \epsilon n_F) + v(x|_F + \epsilon n_F)),$$

and for F on the boundary, we define $\{v\}|_F := v|_F$. The classical nonconforming space of piecewise affine finite element functions (see [14]) then reads

$$V_h := \{v_h \in L^2(\Omega) : \int_F [\![v_h]\!] \, \mathrm{d}s = 0, \forall F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_D} \text{ and } v_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(\kappa)$ denotes the set of polynomials of degree less than or equal to one restricted to the element κ.

The finite element method takes the following form: find $u_h \in V_h$ such that

$$A_h(u_h; v_h) = L(v_h) \quad \forall v_h \in V_h, \tag{3.1}$$

where $L(v_h) := (f, v_h)_\Omega$ and

$$A_h(u_h; v_h) := a_h(u_h, v_h) + \langle u_h + [P_\gamma(u_h)]_+, \partial_n v_h \rangle_{\Gamma_C} \tag{3.2}$$

with $P_\gamma(u_h) = \gamma \partial_n u_h - u_h$ and $\gamma > 0$ a parameter to determine. The linear form $a_h(\cdot, \cdot)$ coincides with the consistent part of Nitsche's method,

$$a_h(u_h, v_h) := a(u_h, v_h) - \langle \partial_n u_h, v_h \rangle_{\Gamma_C},$$

where we have redefined $a(u, v) := \sum_{\kappa \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K$. To see the effect of the nonlinear term, let Γ_C^+ denote the part of the contact zone where $\gamma[\partial_n u - \gamma^{-1} u]_+ > 0$ and $\Gamma_C^0 = \Gamma_C \setminus \Gamma_C^+$. We may then write the form $A(\cdot, \cdot)$

$$a(u_h, v_h) - \langle \partial_n u, v_h \rangle_{\Gamma_C} + \langle \partial_n v, u_h \rangle_{\Gamma_C^0} + \langle \gamma \partial_n u, \partial_n v \rangle_{\Gamma_C^+}.$$

This corroborates the naive idea that the method should impose a Dirichlet condition on Γ_C^0, here using the penalty-free Nitsche method, and a Neumann condition on Γ_C^+, here in the form of a penalty term. Observe that the continuity of the form that is obvious in the formulation (3.2) (by the continuity of $[\cdot]_+$; see more details below) is no longer clear in this latter expression.

For comparison, in the method of Chouly, Hild, and Renard [13], the form (2.4) $A(\cdot,\cdot)$ takes the form

$$a(u_h, v_h) - \langle \partial_n u, v_h \rangle_{\Gamma_C^0} + \theta_1 \langle \partial v_n, u_h \rangle_{\Gamma_C^0} + \left\langle u_h, \gamma^{-1} v_h \right\rangle_{\Gamma_C^0} + \theta_1 \langle \gamma \partial_n u, \partial_n v \rangle_{\Gamma_C^+},$$

where θ takes the values -1 or 1 for the symmetric and nonsymmetric versions, respectively. Clearly in this case, the Dirichlet condition on Γ_C^0 is imposed using the classical Nitsche method and the Neumann condition on Γ_C^+ is imposed either weakly or with an additional penalty term (in the symmetric case, this term has the wrong sign and does not stabilize the boundary condition).

3.1. Preliminary results. For the analysis below, we will use some elementary tools that we collect here. We will use the notation $a \lesssim b$ for $a \le Cb$, where C is a constant independent of h.

The following norm and semi-norm on $H^{\frac{3}{2}+\nu}(\Omega)+V_h$ will be used below to simplify the notation

$$\|v\|_h := \left(\sum_{\kappa \in \mathcal{T}_h} \|v\|_\kappa^2 \right)^{\frac{1}{2}}, \quad \|v\|_{h,\Gamma_C} := \left(\sum_{F \in \mathcal{F}_{\Gamma_C}} \|v\|_F^2 \right)^{\frac{1}{2}}.$$

We also define the broken H^1-norms:

$$\|v\|_{1,h} := \|\nabla v\|_h + \|v\|_h$$

and

$$\|v\|_{1,C} := \|v\|_{1,h} + \gamma^{\frac{1}{2}} \|\partial_n v\|_{h,\Gamma_C} + \gamma^{-\frac{1}{2}} \|v\|_{h,\Gamma_C}.$$

We recall, for future reference, the following inequalities:

- Poincaré inequality (see [8, Theorems 6.2 and 7.2]). There exists $\alpha > 0$ such that

$$\alpha \|v\|_{1,h}^2 \le \|\nabla v\|_h^2 \quad \forall v \in V_h + H^1(\Omega). \tag{3.3}$$

- Inverse inequality (see [15, section 1.4.3]):

$$|v|_{H^1(\kappa)} \le C_I h_\kappa^{-1} \|v\|_{L^2(\kappa)} \quad \forall v \in \mathbb{P}_1(\kappa). \tag{3.4}$$

- Trace inequalities (see [15, section 1.4.3]):

$$\|v\|_{L^2(\partial\kappa)} \le C_T \left(h_\kappa^{-\frac{1}{2}} \|v\|_{L^2(\kappa)} + h_\kappa^{\frac{1}{2}} |v|_{H^1(\kappa)} \right) \quad \forall v \in H^1(\kappa) \tag{3.5}$$

  and

$$\|v\|_{L^2(\partial\kappa)} \le C_t h_\kappa^{-\frac{1}{2}} \|v\|_{L^2(\kappa)} \quad \forall v \in \mathbb{P}_1(\kappa). \tag{3.6}$$

For the analysis below, we also need a quasi-interpolation operator that maps piecewise linear nonconforming functions into the space of piecewise linear conforming functions. Let $I_{\mathrm{cf}} : V_h \mapsto V_h \cap H^1(\Omega)$ denote a quasi-interpolation operator [23, 1, 25] such that

$$\|I_{\mathrm{cf}} v_h - v_h\|_\Omega + h \|\nabla (I_{\mathrm{cf}} v_h - v_h)\|_h \lesssim \|h^{\frac{1}{2}} [\![v_h]\!]\|_{\mathcal{F}_i} \lesssim h \|\nabla v_h\|_h. \tag{3.7}$$

Stability is based on the fact that we can construct a function that is zero in the bulk of the domain and with a certain value of the flux on the boundary. We make this precise in the following lemma.

LEMMA 3.1. *Let $r : \Gamma_C \mapsto \mathbb{R}$ be a face-wise constant function such that $r|_F = r_F \in \mathbb{R}$ for all $F \in \mathcal{F}_{\Gamma_C}$. There exists $v_h \in V_h$ such that*

$$\partial_n v_h|_F = r_F \quad \textit{for } F \in \mathcal{F}_{\Gamma_C}, \tag{3.8}$$

$$\int_F \{v_h\}\ ds = 0 \quad \textit{for } F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_D} \cup \mathcal{F}_{\Gamma_N}, \tag{3.9}$$

and

$$\|v_h\|_\Omega \lesssim h^{\frac{3}{2}} \|r\|_{\Gamma_C}. \tag{3.10}$$

Proof. For a given simplex κ with one face in $\mathcal{F}_{\Gamma_C}$, assume that $x_1, \dots, x_d$ are the vertices in Γ_C and x_0 is the vertex in the bulk. Define $v_\kappa \in \mathbb{P}_1(\kappa)$ by $v_\kappa(x_i) = 1$, $i = 1, \dots, d$, and $v_\kappa(x_0) = 1 - d$. Then it follows that for $F \subset \partial\kappa \cap \Omega$,

$$\int_F v_\kappa \ \mathrm{d}x = 0$$

and $\nabla v_\kappa := |\nabla v_\kappa| n_{\partial\Omega}$, where $n_{\partial\Omega}$ is the normal to Ω on $\partial\kappa \cap \partial\Omega$ and $|\nabla v_h| = c_\kappa h_\kappa^{-1}$, where c_κ is a positive constant that depends only on the shape regularity of κ. For the remaining simplices κ that do not have a face in Γ_C, we define $v_\kappa \equiv 0$. It follows that

$$v_h := \sum_{\kappa \in \mathcal{T}_h} v_\kappa \in V_h.$$

We conclude by multiplying v_h in each element with $h_\kappa c_\kappa^{-1} r_F$. Then, by construction, (3.8) and (3.9) are satisfied. The stability (3.10) is a consequence of an inverse trace inequality. For κ such that $v_\kappa \neq 0$, there holds $\|v_\kappa\|_\kappa \leq h_\kappa^{\frac{1}{2}} \|v_\kappa\|_{\partial\kappa\cap\Gamma_C}$. This follows by mapping to the reference element, norm equivalence, and a scaling argument. As a consequence, we have

$$\|v_h\|_\Omega \lesssim \left(\sum_{F \in \mathcal{F}_{\Gamma_C}} h_\kappa \|h_\kappa c_\kappa^{-1} r_F\|_F^2 \right)^{\frac{1}{2}} \lesssim h^{\frac{3}{2}} \|r\|_{\Gamma_C}. \qquad \square$$

The nonlinearity satisfies the following monotonicity and continuity properties.

LEMMA 3.2. *Let $a, b \in \mathbb{R}$. Then there holds*

$$\begin{aligned} ([a]_+ - [b]_+)^2 &\leq ([a]_+ - [b]_+)(a - b), \\ |[a]_+ - [b]_+| &\leq |a - b|. \end{aligned}$$

Proof. Developing the left-hand side of the expression, we have

$$[a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \leq [a]_+ a + [b]_+ b - a[b]_+ - [a]_+ b = ([a]_+ - [b]_+)(a - b).$$

The second claim follows from the first by taking absolute values on the factors in the right-hand side and dividing by $|[a]_+ - [b]_+|$. $\square$

LEMMA 3.3 (Continuity of A_h). *Let $v_1, v_2 \in H^{\frac{3}{2}+\nu} + V_h$ and $w_h \in V_h$. Then there holds*

$$|A_h(v_1; w_h) - A_h(v_2; w_h)| \lesssim \|v_1 - v_2\|_{1,C} \|w_h\|_{1,C} \lesssim \Theta(h, \gamma)^2 \|v_1 - v_2\|_\Omega \|w_h\|_\Omega.$$

Proof. By the Cauchy–Schwarz inequality, we have

$$|a_h(v_1 - v_2, w_h)| \leq \|v_1 - v_2\|_{1,C}\|w_h\|_{1,C}.$$

For the nonlinear term, the following bound holds as a consequence of the third inequality of Lemma 3.2 and the inequalities (3.4)–(3.6):

$$\begin{aligned}
&\left\langle\gamma([\partial_n v_1 - \gamma^{-1}v_1]_+ - [\partial_n v_2 - \gamma^{-1}v_2]_+, \partial_n w_h\right\rangle_{\Gamma_C} \\
&\quad\leq \left\langle|\gamma^{\frac{1}{2}}\partial_n(v_1 - v_2) - \gamma^{-\frac{1}{2}}(v_1 - v_2)|, \gamma^{\frac{1}{2}}|\partial_n w_h|\right\rangle_{\Gamma_C} \\
&\quad\lesssim \|v_1 - v_2\|_{1,C}\|w_h\|_{1,C} \\
&\quad\lesssim \Theta(h,\gamma)^2\|v_1 - v_2\|_\Omega\|w_h\|_\Omega
\end{aligned}$$

with $\Theta(h,\gamma) := 1 + h^{-1}(C_I + C_t C_I \gamma^{\frac{1}{2}} h^{-\frac{1}{2}} + C_t \gamma^{-\frac{1}{2}} h^{\frac{1}{2}})$. ∎

4. Existence and uniqueness of discrete solutions. In this section, we will prove that the finite dimensional nonlinear system (3.1) admits a unique solution under suitable assumptions on the parameter γ. First, with $N_V := \dim V_h$, define the mapping $G : \mathbb{R}^{N_V} \mapsto \mathbb{R}^{N_V}$ by

$$(G(U), V)_{\mathbb{R}^{N_V}} := A_h(u_h; v_h) - L(v_h), \tag{4.1}$$

where $U = \{u_i\}$ with u_i denoting the degrees of freedom of V_h associated with the Crouzeix–Raviart basis functions $\{\varphi_i\}_{i=1}^{N_V}$ and similarly $V = \{v_i\}$ denotes the vector of degrees of freedom associated with the test function v_h. The nonlinear system associated to (3.1) may then be written: find $U \in \mathbb{R}^{N_V}$ such that $G(U) = 0$.

Let us next prove a positivity result for the formulation (3.1) that will be useful when proving existence and uniqueness.

PROPOSITION 4.1. *Assume that $\gamma = \gamma_0 h$ with γ_0 large enough. Then for $u_1, u_2 \in V_h$, there exists $v_h \in V_h$ such that*

$$\begin{aligned}
&\frac{\alpha}{2}\|u_1 - u_2\|_{1,h}^2 + \gamma^{-1}\|u_1 - u_2 + [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+\|_{\Gamma_C}^2 \\
&\qquad\lesssim A_h(u_1; v_h) - A_h(u_2; v_h),
\end{aligned} \tag{4.2}$$

where α is the constant from (3.3). *Moreover, for γ_0 large enough, there exists $B \in \mathbb{R}^{N_V \times N_V}$ such that for X with $|X|_{\mathbb{R}^{N_V}}$ large enough,*

$$(G(X), BX)_{\mathbb{R}^{N_V}} > 0,$$

and there exists $b_1, b_2 > 0$ associated to B such that

$$b_1|X|_{\mathbb{R}^{N_V}} \leq |BX|_{\mathbb{R}^{N_V}} \leq b_2|X|_{\mathbb{R}^{N_V}}.$$

Proof. Let $w_h := u_1 - u_2$. Observe that by Lemma 3.1, we can choose $\xi_h(w_h) \in V_h$ such that

$$\partial_n \xi_h|_F = \gamma^{-1}|F|^{-1}\int_F w_h \, \mathrm{d}s =: \gamma^{-1}\bar{w}|_F, \qquad \text{for } F \in \mathcal{F}_{\Gamma_C} \tag{4.3}$$

and

$$\int_F \{\xi_h\} \, \mathrm{d}s = 0 \text{ for } F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_D} \cup \mathcal{F}_{\Gamma_N}. \tag{4.4}$$

It follows using integration by parts that for all $y_h \in V_h$, there holds

$$(\nabla y_h, \nabla \xi_h) - \langle \partial_n y_h, \xi_h \rangle_{\Gamma_C} = 0.$$

Now taking $v_h = w_h + \xi_h$ leads to

$$\begin{aligned} A_h(u_1; v_h) - A_h(u_2; v_h) = \|\nabla w_h\|_h^2 + (\nabla w_h, \nabla \xi_h)_h - \langle \partial_n w_h, \xi_h \rangle_{\Gamma_C} + \left\langle \gamma^{-1} \bar{w}, w_h \right\rangle_{\Gamma_C} \\ + \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \partial_n w_h + \gamma^{-1} \bar{w}. \right\rangle_{\Gamma_C}. \end{aligned}$$

Then by adding and subtracting $\gamma^{-1} w_h$ in the right slot of the last term in the right-hand side, we obtain

$$\begin{aligned} A_h(u_1; v_h) - A_h(u_2; v_h) &= \|\nabla w_h\|_\Omega^2 + \left\langle \gamma^{-1} \bar{w}, w_h \right\rangle_{\Gamma_C} \\ &\quad + \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \partial_n w_h - \gamma^{-1} w_h \right\rangle_{\Gamma_C} \\ &\quad + \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \gamma^{-1}(w_h + \bar{w}) \right\rangle_{\Gamma_C} \\ &= \|\nabla w_h\|_\Omega^2 + \left\langle \gamma^{-1} w_h, w_h \right\rangle_{\Gamma_C} \\ &\quad + \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \partial_n w_h - \gamma^{-1} w_h \right\rangle_{\Gamma_C} \\ &\quad + 2 \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \gamma^{-1} w_h \right\rangle_{\Gamma_C} \\ &\quad + \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+ + w_h, \gamma^{-1}(\bar{w} - w_h) \right\rangle_{\Gamma_C}. \end{aligned}$$

To obtain the last equality we once again added and subtracted $\gamma^{-1} w_h$, this time in the term $\left\langle \gamma^{-1} \bar{w}, w_h \right\rangle_{\Gamma_C}$. Applying the monotonicity

$$\gamma^{-1} \|[P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+\|_{\Gamma_C}^2 \leq \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \partial_n w_h - \gamma^{-1} w_h \right\rangle_{\Gamma_C},$$

we see that

$$\begin{aligned} \gamma^{-1} \|[P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+ + w_h\|_{\Gamma_C}^2 \leq \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \partial_n w_h - \gamma^{-1} w_h \right\rangle_{\Gamma_C} \\ + \left\langle \gamma^{-1} w_h, w_h \right\rangle_{\Gamma_C} + 2 \left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+, \gamma^{-1} w_h \right\rangle_{\Gamma_C}. \end{aligned}$$

Then using the arithmetic-geometric inequality together with the approximation properties of the piecewise constant approximation $\bar{w}$ and an element-wise trace inequality to get the bound

$$\begin{aligned} &\left\langle [P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+ + w_h, \gamma^{-1}(\bar{w} - w_h) \right\rangle_{\Gamma_C} \\ &\quad \leq \frac{1}{2} \gamma^{-1} \|[P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+ + w_h\|_{\Gamma_C}^2 + \frac{1}{2} \gamma^{-1} C h \|\nabla w_h\|_\Omega^2, \end{aligned}$$

we finally obtain

$$\begin{aligned} &\left(1 - \frac{1}{2} \gamma^{-1} C h\right) \|\nabla w_h\|_h^2 + \frac{1}{2} \gamma^{-1} \|[P_\gamma(u_1)]_+ - [P_\gamma(u_2)]_+ + w_h\|_{\Gamma_C}^2 \\ &\quad \leq A_h(u_1; v_h) - A_h(u_2; v_h). \end{aligned}$$

We conclude by choosing $\gamma > Ch$ and applying (3.3).

For the second claim, first consider equation (4.2) with $u_1 = u_h$, $u_2 = 0$:

$$\frac{\alpha}{2} \|u_h\|_{1,h}^2 + \gamma^{-1} \|u_h - [P_\gamma(u_h)]_+\|_{\Gamma_C}^2 \lesssim A_h(u_h; u_h + \xi_h(u_h)). \tag{4.5}$$

Let the positive constants c_h and C_h denote the square roots of the smallest and the largest eigenvalues, respectively, of the matrix given by $(\varphi_i, \varphi_j)_\Omega$, $1 \le i, j \le N_V$ such that

$$c_h |U|_{\mathbb{R}^{N_V}} \le \|u_h\|_\Omega \le C_h |U|_{\mathbb{R}^{N_V}}.$$

Let B denote the transformation matrix such that the finite element function corresponding to the vector BU is the function $u_h + \xi_h(u_h)$. First, we show that for γ sufficiently large, there are constants b_1 and b_2 such that $b_1 |U|_{\mathbb{R}^{N_V}} \le |BU|_{\mathbb{R}^{N_V}} \le b_2 |U|_{\mathbb{R}^{N_V}}$. This can be seen by observing that

$$\|u_h\|_\Omega \le \|u_h + \xi_h\|_\Omega + \|\xi_h\|_\Omega \le C_h |BU|_{\mathbb{R}^{N_V}} + C\gamma^{-1} h \|u_h\|_\Omega$$

so that

$$c_h (1 - C\gamma^{-1} h) |U|_{\mathbb{R}^{N_V}} \le (1 - C\gamma^{-1} h) \|u_h\|_\Omega \le C_h |BU|_{\mathbb{R}^{N_V}}.$$

Similarly, we may prove the upper bound using $c_h |BU|_{\mathbb{R}^{N_V}} \le \|u_h + \xi_h\|_\Omega$ so that

$$c_h |BU|_{\mathbb{R}^{N_V}} \le \|u_h\|_\Omega + \|\xi_h\|_\Omega \le \|u_h\|_\Omega + C\gamma^{-1} h \|u_h\|_\Omega \le C_h (1 + C\gamma^{-1} h) |U|_{\mathbb{R}^{N_V}}.$$

Then there holds, using (4.5),

$$\begin{aligned}
(G(U), BU)_{\mathbb{R}^{N_V}} &= A_h(u_h; u_h + \xi_h(u_h)) - L(u_h + \xi_h(u_h)) \\
&\ge \frac{\alpha}{4} \|u_h\|^2_{1,h} - \frac{C_*^2}{\alpha} \|f\|^2_\Omega \ge \frac{\alpha}{4} \lambda_1 |U|^2_{\mathbb{R}^{N_V}} - \frac{C_*^2}{\alpha} \|f\|^2_\Omega,
\end{aligned}$$

where C_* is the constant such that $L(u_h + \xi_h(u_h)) \le C_* \|f\|_\Omega \|u_h\|_{1,h}$ and λ_1 is the smallest eigenvalue of the matrix defined by $(\nabla \varphi_i, \nabla \varphi_j)_h + (\varphi_i, \varphi_j)_\Omega$, $1 \le i, j \le N_V$. We conclude that for

$$|U|_{\mathbb{R}^{N_V}} > \frac{2C_*}{\alpha \lambda_1^{\frac{1}{2}}} \|f\|_\Omega,$$

there holds

$$(G(U), BU)_{\mathbb{R}^{N_V}} > 0. \qquad \square$$

PROPOSITION 4.2. *The formulation* (3.1) *admits a unique solution for $\gamma = \gamma_0 h$, with γ_0 large enough.*

Proof. The proof of existence uses Brouwer's fixed point theorem in a classical way (see, e.g., [30, Lemma 1.4, Chapter 2].) Fix $h > 0$. Observe that G defined by (4.1) is continuous since by Lemma 3.3,

$$\begin{aligned}
|G(U_1) - G(U_2)|_{\mathbb{R}^{N_V}} &= \sup_{W \in \mathbb{R}^{N_V} : |W| = 1} (G(U_1) - G(U_2), W)_{\mathbb{R}^{N_V}} \\
&= \sup_{w \in V_h} (A_h(u_1; w_h) - A_h(u_2; w_h)) \\
&\lesssim \Theta(h)^2 \|u_1 - u_2\|_\Omega \|w_h\|_\Omega \le \Theta(h)^2 C_h^2 |U_1 - U_2|_{\mathbb{R}^{N_V}}.
\end{aligned}$$

By the second claim of Proposition 4.1, we may fix $q \in \mathbb{R}_+$ such that for $X \in \mathbb{R}^{N_V}$ with $|X| \ge q$, there holds

$$(G(X), BX)_{\mathbb{R}^{N_V}} > 0. \tag{4.6}$$

Assume that there exists no $X \in \mathbb{R}^{N_V}$ such that $G(X) = 0$ and define the function $\phi(X) = -q/b_1 B^{\mathrm{T}} G(X)/|G(X)|_{\mathbb{R}^{N_V}}$. Since $G(X) \ne 0$ and by the continuity of $G(X)$,

$\phi(\cdot)$ is well defined and continuous. The transpose of B satisfies the same bounds as B, and therefore ϕ maps the ball of radius qb_2/b_1 in $\mathbb{R}^{N_V}$ into itself. It then follows by Brouwer's fixed point theorem that ϕ admits a fixed point: there exists $Z \in \mathbb{R}^{N_V}$ such that

$$Z = \phi(Z) = -q/b_1 B^{\mathrm{T}} G(Z)/|G(Z)|_{\mathbb{R}^{N_V}},$$

and, since $|B^{\mathrm{T}}G(Z)|/|G(Z)|_{\mathbb{R}^{N_V}} \geq b_1$, it follows that $|Z| \geq q$. By definition then, $|Z|^2_{\mathbb{R}^{N_V}} = -q/b_1 (G(Z), BZ)_{\mathbb{R}^{N_V}}/|G(Z)|_{\mathbb{R}^{N_V}}$, which contradicts the assumption (4.6). It follows that there exists at least one $U \in \mathbb{R}^{N_V}$ such that $G(U) = 0$.

Uniqueness of the discrete solution is an immediate consequence of Proposition 4.1. Indeed, assume that both u_1 and u_2 are solutions to (3.1); then for v_h chosen as in the proposition,

$$\alpha \|u_1 - u_2\|^2_{1,h} \lesssim A_h(u_1; v_h) - A_h(u_2; v_h) = (f, v_h)_\Omega - (f, v_h)_\Omega = 0. \qquad \square$$

5. A priori error estimates. A priori error estimates may now be derived by combining the techniques of the uniqueness argument above with the Galerkin perturbation arguments.

THEOREM 5.1. *Assume that $u \in H^{\frac{3}{2}+\nu}(\Omega)$, with $0 < \nu \leq \frac{1}{2}$ is the solution of the problem* (1.1). *Assume that u_h denotes the solution of* (3.1)–(3.2), *where $\gamma = \gamma_0 h$. If γ_0 is chosen sufficiently large and $h \leq t_C$, where t_C is the constant of assumption* (2.10), *then there holds, with $e := u - u_h$,*

$$\alpha^{\frac{1}{2}} \|e\|_{1,h} + \gamma^{-\frac{1}{2}} \|[P_\gamma u_h]_+ + u_h\|_{\Gamma_C} \lesssim \inf_{v_h \in V_h} (\|u - v_h\|_{1,C} + h^{\frac{1}{2}} \|\partial_n (u - v_h)\|_{\mathcal{F}_i}) + h\|f\|_{L^2_\infty(\Omega)}.$$

Proof. Using the definition of the form $a(\cdot,\cdot)$, we have

$$\|\nabla e\|^2_h \leq a(e,e) = a(e, u - v_h) + a(e, v_h - u_h). \tag{5.1}$$

For the first term, we have

$$a(e, u - v_h) \leq \frac{1}{2}\|\nabla e\|^2_h + \frac{1}{2}\|\nabla(u - v_h)\|^2_h. \tag{5.2}$$

Considering the second term, we see that by integrating by parts in the term $a(u, v_h - u_h)$ to obtain the conformity error term $\langle \{\partial_n u\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}\setminus\mathcal{F}_{\Gamma_C}}$, we have the consistency relation

$$\begin{aligned} a(e, v_h - u_h) = {} & \langle \{\partial_n u\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}\setminus\mathcal{F}_{\Gamma_C}} + \langle \partial_n e, v_h - u_h\rangle_{\Gamma_C} \\ & - \langle \partial_n (v_h - u_h), e\rangle_{\Gamma_C} \\ & - \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u_h)\rangle_{\Gamma_C}. \end{aligned} \tag{5.3}$$

Using that

$$\langle \partial_n e, v_h - u_h\rangle_{\Gamma_C} - \langle \partial_n(v_h - u_h), e\rangle_{\Gamma_C} = \langle \partial_n e, v_h - u\rangle_{\Gamma_C} - \langle \partial_n (v_h - u), e\rangle_{\Gamma_C}$$

and

$$\begin{aligned} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u_h)\rangle_{\Gamma_C} = {} & \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u)\rangle_{\Gamma_C} \\ & + \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n e\rangle_{\Gamma_C} \\ = {} & \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \partial_n(v_h - u)\rangle_{\Gamma_C} \\ & + \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, P_\gamma e\rangle_{\Gamma_C} \\ & + \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, e\rangle_{\Gamma_C}, \end{aligned}$$

we arrive at the identity

$$
\begin{aligned}
a(e, v_h - u_h) = {} & \langle \{\partial_n u\}, [\![v_h - u_h]\!] \rangle_{\mathcal{F}\setminus\mathcal{F}_{\Gamma_C}} + \langle \partial_n e, v_h - u \rangle_{\Gamma_C} \\
& - \langle \partial_n (v_h - u), e + ([P_\gamma u]_+ - [P_\gamma u_h]_+) \rangle_{\Gamma_C} \\
& - \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, P_\gamma (u - u_h) \rangle_{\Gamma_C} \\
& - \gamma^{-1} \langle e, [P_\gamma u]_+ - [P_\gamma u_h]_+ \rangle_{\Gamma_C} .
\end{aligned}
\tag{5.4}
$$

Observe now that the following relation holds using monotonicity and the elementary relation $a^2 + b^2 + 2ab = (a+b)^2$, with $a = \gamma^{-1/2}(u - u_h)$ and $b = \gamma^{-1/2}([P_\gamma u]_+ - [P_\gamma u_h]_+)$:

$$
\begin{aligned}
& - \gamma^{-1} \|e\|^2_{\Gamma_C} - \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, P_\gamma e \rangle_{\Gamma_C} - \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, 2e \rangle_{\Gamma_C} \\
& \quad \le -\gamma^{-1} \|e\|^2_{\Gamma_C} - \gamma^{-1} \|[P_\gamma u]_+ - [P_\gamma u_h]_+\|^2_{\Gamma_C} - \gamma^{-1} \langle [P_\gamma u]_+ - [P_\gamma u_h]_+, 2e \rangle_{\Gamma_C} \\
& \quad \le - \|\gamma^{-\frac{1}{2}} ([P_\gamma u_h]_+ + u_h)\|^2_{\Gamma_C}.
\end{aligned}
$$

We deduce the following bound:

$$
\begin{aligned}
a(e, v_h - u_h) \le {} & \langle \{\partial_n u\}, [\![v_h - u_h]\!] \rangle_{\mathcal{F}\setminus\mathcal{F}_{\Gamma_C}} + \langle \partial_n e, v_h - u \rangle_{\Gamma_C} \\
& - \langle \partial_n (v_h - u), e + ([P_\gamma u]_+ - [P_\gamma u_h]_+) \rangle_{\Gamma_C} \\
& - \|\gamma^{-\frac{1}{2}} ([P_\gamma u_h]_+ + u_h)\|^2_{\Gamma_C} + \gamma^{-1} \langle e, [P_\gamma u]_+ - [P_\gamma u_h]_+ \rangle_{\Gamma_C} \\
& + \gamma^{-1} \|e\|^2_{\Gamma_C}.
\end{aligned}
\tag{5.5}
$$

Choosing now $\xi_h(u - u_h) \in V_h$ constructed in the same fashion as the special function v_h of Lemma 3.1 but with $\partial_n \xi_h|_F = \gamma^{-1}(\bar{u} - \bar{u}_h)|_F = \gamma^{-1}\bar{e}|_F$ on faces $F \subset \Gamma_C$ as test function in (3.1), we obtain

$$
a_h(e, \xi_h) - \langle \{\partial_n u\}, [\![\xi_h]\!] \rangle_{\mathcal{F}} + \gamma^{-1} \|\bar{e}\|^2_{\Gamma_C} + \left\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \gamma^{-1} \bar{e} \right\rangle_{\Gamma_C} = 0.
$$

Here arguments similar to those of (5.3) were used together with the property that $\partial_n \xi_h|_F = \gamma^{-1}\bar{e}|_F$. Note that using orthogonality on the faces of $e - \bar{e}$, we have

$$
\gamma^{-1} \|\bar{e}\|^2_{\Gamma_C} = \gamma^{-1} \|e\|^2_{\Gamma_C} - \gamma^{-1} \|\bar{e} - e\|^2_{\Gamma_C},
$$

and once again using orthogonality and also the contact condition,

$$
\begin{aligned}
\left\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \gamma^{-1} \bar{e} \right\rangle_{\Gamma_C} = {} & \left\langle [P_\gamma u]_+ - [P_\gamma u_h]_+ + e, \gamma^{-1} e \right\rangle_{\Gamma_C} \\
& - \left\langle [P_\gamma u_h]_+ + u_h, \gamma^{-1} \bar{e} - \gamma^{-1} e \right\rangle_{\Gamma_C} \\
& - \gamma^{-1} \|\bar{e} - e\|^2_{\Gamma_C}.
\end{aligned}
$$

For the last term in the right-hand side, we may add and subtract $v_h - \bar{v}_h$ and use the triangle inequality followed by the interpolation properties of the projection onto piecewise constants and a trace inequality to obtain

$$
\begin{aligned}
\gamma^{-1} \|\bar{e} - e\|^2_{\Gamma_C} & \le C(\gamma^{-1} \|u - v_h\|^2_{\Gamma_C} + \gamma^{-1} h^{-1} h^2 \|\nabla (v_h - u_h)\|^2_h) \\
& \le C(\|u - v_h\|^2_{1,C} + \gamma^{-1} h^{-1} h^2 \|\nabla e\|^2_h).
\end{aligned}
\tag{5.6}
$$

As a consequence,

$$\begin{aligned}\gamma^{-1}\|e\|_{\Gamma_C}^2 + \left\langle [P_\gamma u]_+ - [P_\gamma u_h]_+, \gamma^{-1} e\right\rangle_{\Gamma_C} \leq \frac{1}{4}\|\gamma^{-\frac{1}{2}}([P_\gamma u_h]_+ + u_h)\|_{\Gamma_C}^2 \\ + C(\|u-v_h\|_{1,C}^2 + \gamma^{-1}h^{-1}h^2\|\nabla e\|_h^2) \\ - a_h(e,\xi_h) + \langle \{\partial_n u\}, [\![\xi_h]\!]\rangle_{\mathcal{F}}\,. \end{aligned} \tag{5.7}$$

Collecting the results of equations (5.1), (5.2), (5.5), and (5.7) and applying the Poincaré inequality (3.3) leads to

$$\begin{aligned}\alpha\left(\frac{1}{2} - C\frac{h}{\gamma}\right)\|e\|_{1,h}^2 + \frac{1}{2\gamma}\|([P_\gamma u_h]_+ + u_h)\|_{\Gamma_C}^2 \leq\; & - a(e,\xi_h) + \langle \partial_n e, v_h - u\rangle_{\Gamma_C} \\ & - \langle [P_\gamma u_h]_+ + u_h, \partial_n(v_h-u)\rangle_{\Gamma_C} \\ & + \langle\{\partial_n u\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}_i} \\ & + \langle\{\partial_n e\}, [\![\xi_h]\!]\rangle_{\mathcal{F}} \\ & + C\left(1+\frac{h}{\gamma}\right)\|u-v_h\|_{1,\Gamma_C}^2. \end{aligned} \tag{5.8}$$

Observe that $a(u_h,\xi_h) - \langle\{\partial_n u_h\}, [\![\xi_h]\!]\rangle_{\mathcal{F}} = 0$ using integration by parts and the construction of ξ_h. Then, once again by integration by parts, we have

$$a(e,\xi_h) - \langle\{\partial_n e\}, [\![\xi_h]\!]\rangle_{\mathcal{F}} = (-\Delta u, \xi_h)_{\Omega_C} \leq \|\Delta u\|_{\Omega_C}\|\xi_h\|_{\Omega_C},$$

where Ω_C is the set of elements with one face on Γ_C. Let $h_C > 0$ be the largest value such that $\partial_{h_C}\Omega \cap \Omega_C \neq \emptyset$ and assume that $h_C \leq t_C$. Observe that by the construction of ξ_h and adding and subtracting v_h, there holds

$$\begin{aligned}\|\xi_h\|_{\Omega_C} \lesssim h^{\frac{1}{2}} h\gamma^{-1}\|\bar{e}\|_{\Gamma_C} &\lesssim h^{\frac{1}{2}} h\gamma^{-1}\|e\|_{\Gamma_C} \\ &\lesssim h^{\frac{1}{2}}(h\gamma^{-1})(\|u-v_h\|_{\Gamma_C} + \|u_h - v_h\|_{\Gamma_C}).\end{aligned}$$

Let $w_h = u_h - v_h$; then, by adding and subtracting $I_{\mathrm{cf}} w_h$ and applying the local trace inequality (3.6) and the standard global trace inequality for functions in $H^1(\Omega)$, we obtain

$$\begin{aligned}\|w_h\|_{\Gamma_C} &\leq \|w_h - I_{\mathrm{cf}} w_h\|_{\Gamma_C} + \|I_{\mathrm{cf}} w_h\|_{\Gamma_C} \\ &\lesssim h^{-\frac{1}{2}}\|w_h - I_{\mathrm{cf}} w_h\|_h + \|w_h - I_{\mathrm{cf}} w_h\|_{1,h} + \|w_h\|_{1,h}.\end{aligned}$$

Applying the discrete interpolation estimate (3.7), we then have

$$\|w_h\|_{\Gamma_C} \lesssim \|w_h\|_{1,h},$$

from which it follows that

$$(h\gamma^{-1})\|u_h - v_h\|_{\Gamma_C} \lesssim (h\gamma^{-1})(\|e\|_{1,h} + \|u - v_h\|_{1,h}).$$

For the factor $\|\Delta u\|_{\Omega_C}$, we use (2.9) to obtain the bound

$$\|\Delta u\|_{\Omega_C} \lesssim h^{\frac{1}{2}} \sup_{0\leq t\leq h_C}\|\Delta u\|_{\partial_t\Omega} \leq h^{\frac{1}{2}}\|f\|_{L^2_\infty(\Omega)}.$$

It follows that

$$a_h(e,\xi_h) - \langle\{\partial_n e\}, [\![\xi_h]\!]\rangle_{\mathcal{F}} \lesssim h\|f\|_{L^2_\infty(\Omega)}(h\gamma^{-1})(\|e\|_{1,h} + \|u - v_h\|_{1,h}). \tag{5.9}$$

For the remaining terms of (5.8), we have, by first adding and subtracting v_h and using the mean value property of the space V_h and then applying the Cauchy–Schwarz inequality followed by the arithmetic–geometric inequality,

$$\begin{aligned}
&\langle \partial_n e, v_h - u\rangle_{\Gamma_C} - \langle [P_\gamma u_h]_+ + u_h, \partial_n(v_h - u)\rangle_{\Gamma_C} + \langle \{\partial_n u\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}} \\
&\quad = \langle \partial_n(u - v_h), v_h - u\rangle_{\Gamma_C} + \langle \partial_n(v_h - u_h), v_h - u\rangle_{\Gamma_C} \\
&\qquad - \langle [P_\gamma u_h]_+ + u_h, \partial_n(v_h - u)\rangle_{\Gamma_C} + \langle \{\partial_n(u - v_h)\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}_i} \\
&\quad \le C\varepsilon^{-1}\|u - v_h\|^2_{1,C} + \gamma\|\partial_n(u - v_h)\|^2_{\mathcal{F}_i} + \tfrac{1}{4}\gamma^{-1}\|[P_\gamma u_h]_+ + u_h\|^2_{\Gamma_C} \\
&\qquad + \varepsilon(\gamma\|\partial_n(v_h - u_h)\|^2_{\Gamma_C} + \gamma^{-1}\|[\![v_h - u_h]\!]\|^2_{\mathcal{F}_i}).
\end{aligned} \tag{5.10}$$

Using the zero average property of the nonconforming space, element-wise trace inequalities, and a triangular inequality, we obtain

$$\begin{aligned}
\gamma\|\partial_n(v_h - u_h)\|^2_{\Gamma_C} + \gamma^{-1}\|[\![v_h - u_h]\!]\|^2_{\mathcal{F}_i} &\le C_{\gamma_0}\|v_h - u_h\|^2_{1,h} \\
&\le 2C_{\gamma_0}(\|e\|^2_{1,h} + \|v_h - u\|^2_{1,h}).
\end{aligned}$$

Observe that C_{γ_0} is constant for $\gamma_0 = \gamma/h$ fixed, but it cannot be made small by choosing γ_0 large (or small). Instead, we choose $\varepsilon < \alpha/(16C_{\gamma_0})$ to obtain the bound

$$\begin{aligned}
&\langle \partial_n e, v_h - u\rangle_{\Gamma_C} - \langle [P_\gamma u_h]_+ + u_h, \partial_n(v_h - u)\rangle_{\Gamma_C} + \langle \{\partial_n u\}, [\![v_h - u_h]\!]\rangle_{\mathcal{F}} \\
&\quad \le C\|u - v_h\|^2_{1,C} + \gamma\|\partial_n(u - v_h)\|^2_{\mathcal{F}_i} + \frac{1}{4\gamma}\|([P_\gamma u_h]_+ + u_h)\|^2_{\Gamma_C} + \frac{\alpha}{8}\|e\|^2_{1,h}.
\end{aligned} \tag{5.11}$$

Collecting the above bounds (5.8), (5.9), and (5.11), choosing $h\gamma^{-1}$ and ε small enough (i.e., γ_0 large enough), we conclude that for all $v_h \in V_h$,

$$\begin{aligned}
\alpha^{\frac{1}{2}}\|e\|_{1,h} + \gamma^{-\frac{1}{2}}\|[P_\gamma u_h]_+ + u_h\|_{\Gamma_C} &\lesssim (\|u - v_h\|_{1,C} + h^{\frac{1}{2}}\|\partial_n(u - v_h)\|_{\mathcal{F}_i}) \\
&\quad + \frac{h}{\alpha^{\frac{1}{2}}}\|f\|_{L^2_\infty(\Omega)}.
\end{aligned}$$ ∎

COROLLARY 5.2. *Under the assumptions of Theorem* 5.1*, there holds*

$$\alpha^{\frac{1}{2}}\|e\|_{1,h} + \gamma^{-\frac{1}{2}}\|[P_\gamma u_h]_+ + u_h\|_{\Gamma_C} \lesssim h^{\frac{1}{2}+\nu}\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)} + \frac{h}{\alpha^{\frac{1}{2}}}\|f\|_{L^2_\infty(\Omega)}. \tag{5.12}$$

Proof. This is immediate from the best approximation result of Theorem 5.1 and the existence of an optimal approximation of u in V_h. Since the Crouzeix–Raviart space contains the H^1-conforming space of piecewise affine functions, we may take the standard Lagrange interpolant $i_h u$ for which there holds (see [17, 13])

$$\|u - i_h u\|_{1,C} + h^{\frac{1}{2}}\|\partial_n(u - i_h u)\|_{\mathcal{F}_i} \lesssim h^{\frac{1}{2}+\nu}\|u\|_{H^{\frac{3}{2}+\nu}(\Omega)}.$$ ∎

6. Numerical example. Here we will consider two examples on the unit square, $\Omega = [0,1]^2$. We have used the package FreeFEM++ for the computations [20]. We let $\Gamma_D = [0,1] \times \{1\}$, $\Gamma_N = \{0\} \times [0,1] \cup \{1\} \times [0,1]$, and $\Gamma_C = [0,1] \times \{0\}$. Since the exact solution is not known, we solve the problem on a mesh with $h = 2\sqrt{2} \cdot 10^{-3}$ (a 500×500 mesh), using the nonsymmetric Nitsche method from [13] and piecewise quadratic conforming approxmation to obtain a reference solution. In all cases, we

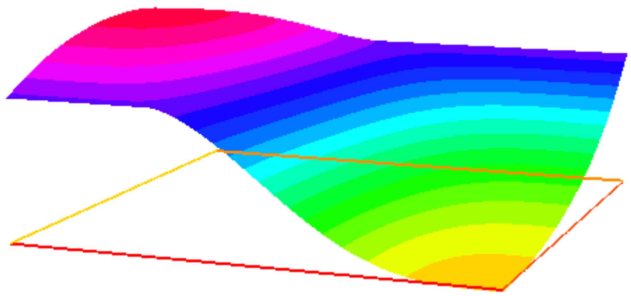

FIG. 1. *The fine mesh solution using* (6.1).

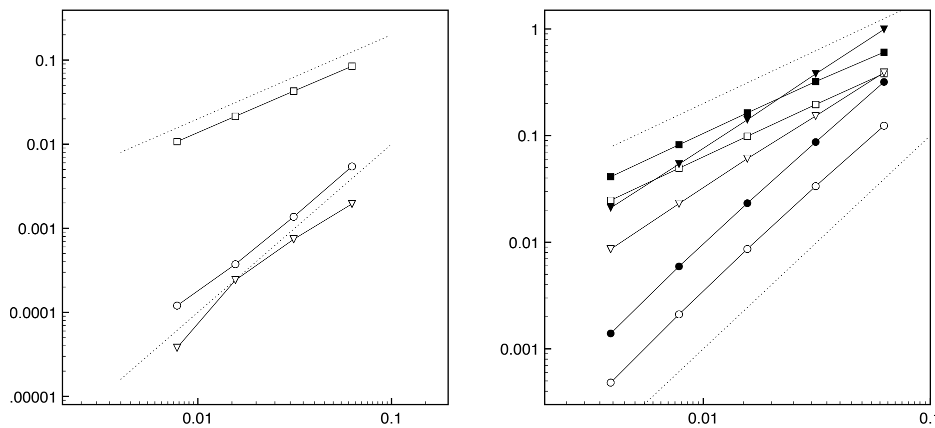

FIG. 2. *Convergence plots of the two numerical examples. Left: the problem from section* 6.1. *Right: the problem from section* 6.2. *Dotted lines are reference curves. Upper* $O(h)$, *lower* $O(h^2)$. *Square markers:* H^1*-error; circle markers:* L^2*-error; triangle markers: satisfaction of the contact condition,* $\|u_h + [P_\gamma(u_h)]_+\|_{\Gamma_C}$. *In the right plot, white markers indicate* $N = 3$, *and black markers indicate* $N = 5$.

use a fixed point iteration to compute the solution, and we iterate until the relative H^1-error of the increment is smaller than a tolerance `TOL`.

In the graphics below, the H^1-error is marked with squares, the L^2-error with circles, and finally the residual quantity $\|u_h + [P_\gamma(u_h)]_+\|_{\Gamma_C}$ by triangles. Dotted lines are reference lines for first- (upper) and second- (lower) order convergence.

6.1. Problem with one contact zone. We first consider an example from [5] that results in a solution where the contact condition is active in one-half of the boundary $y = 0$, and so the solution changes from contact to noncontact in one point. The right-hand side is chosen to be

$$f = -2\pi \sin(2\pi x). \tag{6.1}$$

We solve it on a sequence of Union Jack–style meshes with $h/\sqrt{2} \in \{2^{-(i+4)}\}_{i=0}^4$. An elevation of the solution is presented in Figure 1. The tolerance was set to `TOL`$=10^{-7}$. The fine-scale solution required 127 fixed point iteration to reach the required accuracy, and the solves using the penalty-free method required 79, 84, 66, 54, and 58 iterations, respectively. Convergences are reported in the left graphic of Figure 2.

As expected, we observe first-order convergence of the relative H^1-error and second-order convergence of the relative L^2-error, and the residual quantity measuring the satisfaction of the contact condition has a convergence close to $\mathcal{O}(h^{\frac{3}{2}})$.

6.2. Problem with several contact zones. Here we increase the number of contact zones by taking the right-hand side to be

$$f = (2\pi N)^2 \cos(2\pi N x), N \in \{3, 5\}. \tag{6.2}$$

As N increases, so does the number of contact zones on $y = 0$; the solution changes from contact to noncontact in $N + 1$ points. We solve the problem for $h/\sqrt{2} \in \{2^{-(i+4)}\}_{i=0}^{4}$ and compute the same quantities as in the previous case. The convergences are reported in the right graphic of Figure 2. The cases $N = 3$ and $N = 5$ are distinguished by the use of white and black markers, respectively; similar convergence orders were observed in both cases. First-order convergence is observed for the error in the H^1-norm and second-order convergence in the L^2–error. As before, the convergence of $\|u_h + [P_\gamma(u_h)]_+\|_{\Gamma_C}$ is approximately $\mathcal{O}(h^{\frac{3}{2}})$.

7. Conclusion. We have proved that the nonsymmetric Nitsche method of [10] may be applied in the framework of [11, 13] for the approximation of unilateral contact problems. An optimal error estimate for a method using a nonconforming finite element space was derived combining tools from the inf-sup analysis of [10] with the monotonicity argument of [11, 13]. The theoretical results were illustrated in two numerical examples. Herein, we considered only the simplified case of the Signorini problem based on Poisson's equation, but the extension to linearized elasticity may be feasible using the results from [7]. Another natural question is if the above analysis can be extended to the case of standard conforming elements. The difficulty here is to handle the nonlocal character of the function necessary for the stability argument, adding a layer of terms that must be estimated. Numerical experiments not reported here indicate that the conforming method also performs well.

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