

# Explicit Model Predictive Control of Hybrid Systems and Multiparametric Mixed Integer Polynomial Programming

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DOI 10.1002/aic.15396

Published online July 27, 2016 in Wiley Online Library (wileyonlinelibrary.com)

*Hybrid systems are dynamical systems characterized by the simultaneous presence of discrete and continuous variables. Model-based control of such systems is computationally demanding. To this effect, explicit controllers which provide control inputs as a set of functions of the state variables have been derived, using multiparametric programming mainly for the linear systems. Hybrid polynomial systems are considered resulting in a Mixed Integer Polynomial Programming problem. Treating the initial state of the system as a set of bounded parameters, the problem is reformulated as a multiparametric Mixed Integer Polynomial optimization (mp-MIPOPT) problem. A novel algorithm for mp-MIPOPT problems is proposed and the exact explicit control law for polynomial hybrid systems is computed. The key idea is the computation of the analytical solution of the optimality conditions while the binary variables are treated as relaxed parameters. Finally, using symbolic calculations exact nonconvex critical regions are computed. © 2016 The Authors AIChE Journal published by Wiley Periodicals, Inc. on behalf of American Institute of Chemical Engineers AIChE J, 62: 3441–3460, 2016*  
 Keywords: hybrid systems, multiparametric programming, mixed integer polynomial programming, explicit model predictive control

## Introduction

This article studies the optimal control of hybrid systems for the case that the dynamics are expressed by discrete time polynomial equations. Hybrid systems have attracted the interest of the research community throughout the years but their online optimal control remains a challenging task that gets exacerbated when nonlinear dynamics are considered. To this effect, we propose a novel multiparametric algorithm for the offline solution of the optimal control problem of hybrid polynomial systems and thus alleviating the computational effort required.

Changes in thermodynamic equilibria, reaction kinetics that change with respect to the temperature of the reactor, flow regime changes from laminar to turbulent and vice versa, process safety, and so forth are typical examples of hybrid process systems.<sup>1–3</sup> Batch processes exhibit a hybrid nature which is more related to the control system than the process itself; as the procedures in a batch process are specific and in certain order, one could identify the distinct regimes of the process and the logical constraints that indicate the changes in their sequence. In general, propositional logic has been a way of integrating qualitative knowledge in the optimization scheme.<sup>4–6</sup> Modeling of hybrid systems has gained the attention of researchers over the years. An algorithm for modeling and dynamic optimization of hybrid

systems in continuous time domain, using differential/algebraic equality and inequality constraints was presented in Avraam et al.<sup>7</sup> The mixed logical dynamical framework was introduced in Bemporad and Morari,<sup>8</sup> in which the logical components and the propositional logic expressions are transformed into linear constraints, resulting in a Mixed Integer Linear Programming (MILP) problem. Because of their importance in process systems engineering, a vast amount of research work on the optimal control of hybrid systems has been reported in the literature<sup>9–13</sup> with model predictive control (MPC) being one of them.

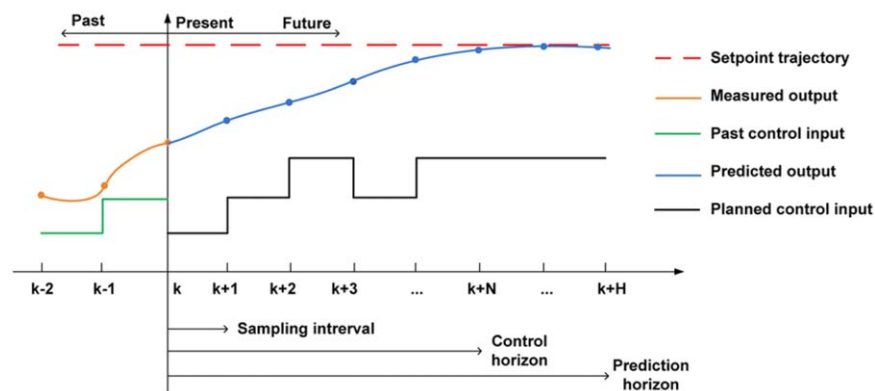
MPC has been developed and applied extensively in the industry and academia and refers to a wide class of control schemes that utilizes explicitly the model of the process so as to compute the manipulated variable trajectory and optimize a desirable performance index, subject to the system's constraints over a prediction horizon.<sup>14,15</sup> Typically, the performance index to be minimized is a weighted quadratic error function between the actual and the desirable state of the controlled variables of the system. After the optimization problem is solved, the first optimal control action is implemented on the system and when the next measurements are available the procedure is repeated in a receding horizon fashion as shown in Figure 1. The main disadvantage of the MPC, especially when one has to control processes with fast dynamics, is the computational burden associated with the required repetitive solution of an optimization problem at regular intervals.<sup>16</sup>

During the last 15 years, a significant amount of research work has been presented on how to shift the computational effort offline while preserving the benefits that MPC offers. In Bemporad et al.,<sup>17</sup> the authors proposed an algorithm for offline computation of the explicit solution to the MPC of

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**Figure 1. The receding horizon approach in the MPC framework.<sup>16</sup>**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

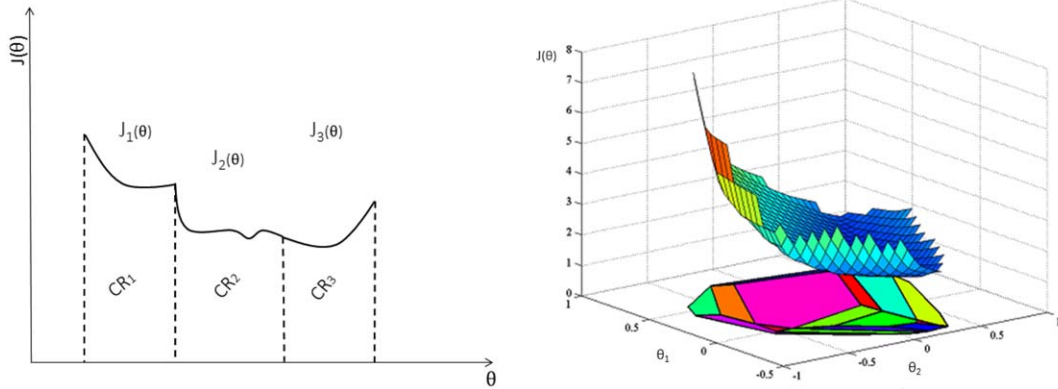
systems with linear constraints using multiparametric quadratic programming. Discrete time hybrid systems with linear dynamics and performance index based on 1-norm or  $\infty$ -norm were studied in Bemporad et al.,<sup>18</sup> where the authors solved a multiparametric (mp)-MILP problem<sup>19</sup> to compute the explicit optimal control law. In Sakizlis et al.,<sup>20</sup> the use of parametric programming was outlined, to compute the explicit control law of linear hybrid systems for problems that also include quadratic cost. All the aforementioned approaches indicate the emergence of a new systematic way to solve offline the computationally demanding closed loop MPC problem that result in a “look-up table” with all the optimal future control moves, which can be then implemented online. The general idea is to solve the optimization problem offline and compute the vector of optimization variables, that is, optimal future control inputs, as explicit functions of the state of the system along with the corresponding regions, known as critical regions (CRs), where these functions are valid.<sup>17,21–26</sup> Recently, an algorithm for explicit MPC of hybrid systems exploiting the concepts of constrained dynamical programming and multiparametric programming was proposed.<sup>27</sup> The authors focused on hybrid systems with linear cost function and piece-wise affine dynamics that resulted in the solution of a mp-MILP dynamic programming problem. Another algorithm for the explicit MPC of linear hybrid systems, in which a preprocessing step is included in order to avoid storing more than one solution in every CR was presented in Oberdieck and Pistikopoulos.<sup>28</sup>

As mentioned above, the optimal control of linear hybrid systems has been studied extensively, with emphasis given on the ones with piece-wise affine dynamics. To accomplish higher degree of controllability and avoid instabilities, nonlinearities that inherently exist in process systems, should be considered. MPC of nonlinear hybrid systems results in the solution of a Mixed Integer Nonlinear Programming (MINLP) problem, which can be NP-hard and computationally prohibitive for online implementation. The safety case of two aircraft collision was presented in Lygeros et al.<sup>9</sup> and the authors proposed an algorithm for controller synthesis of nonlinear hybrid systems. Their approach was based on the grounds of “hybrid automata,”<sup>29</sup> but its efficiency was strongly dependent on the ability to solve the Hamilton-Jacobi equations that the problem involved. Multiple linearized models and an optimization framework with fixed structure for the MPC of nonlinear hybrid systems, was presented in Nandola and Bhartiya.<sup>30</sup> Patil et al.<sup>31</sup> suggested a global optimization algorithm for the

optimal control of nonlinear hybrid systems using Bernstein polynomial form and a branch and bound procedure. Polynomial parametric optimization techniques using algebraic geometry techniques were presented in Fotiou et al.,<sup>32</sup> in which cylindrical algebraic decomposition and Gröbner bases theory are employed to solve the system of polynomial equations that arise in the MPC scheme.

In the present work, hybrid systems with polynomial dynamics are considered. Recently, in Fotiou et al.<sup>33</sup> the authors dealt with the optimal control of piece-wise polynomial hybrid systems and in order to efficiently compute the optimal solution online they proposed an algorithm where cylindrical algebraic decomposition to extract structural information and then utilized it to speed up the online calculations. Mixed Integer Polynomial Programming (MIPP) problems can be a promising class of MINLPs because of the ability to solve the system of polynomial equations analytically and compute a closed form solution that entails a number of solutions.<sup>34</sup> This can be achieved using, for example, the theory of Gröbner Bases, which transforms the system of equations into a triangular format.<sup>34,35</sup> The computational burden grows exponentially with the number of variables<sup>36</sup>; however, current developments such as parallel computing are expected to improve the computational performance. The main contribution of this work is a novel multiparametric programming algorithm for computation of the explicit control law of such systems. A salient feature of the proposed algorithm is the fact that, in the context of multiparametric programming theory, nonconvex CRs are computed exactly and no convex-approximation is employed.

The remainder of the article is organized as follows: in the next section, the reader is given some preliminary information about multiparametric programming and then the concept of Mixed Integer Polynomial Programming is reviewed. Later, we present the multiparametric Mixed Integer Polynomial Programming; the proposed algorithm is outlined and the explicit MPC of hybrid polynomial systems is illustrated. Then, in section “Illustrative examples,” the proposed algorithm is first applied on a numerical example and then the explicit control of a hybrid two tanks system is examined. The results from the application of the proposed algorithm for the design of an explicit hybrid controller along with specific computational issues are discussed in section “Results and Discussion.” Finally, concluding remarks are drawn along with proposals for future research directions.



(a) Parametric profile of optimal objective value in different CRs (b) 3D graph of a value function in the parametric space  $(\theta_1, \theta_2)$  with the corresponding CRs on the floor of the graph

**Figure 2. Illustrations of arbitrary (multi)parametric programming solutions.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

## Problem Background

In this section, an overview of the main components of the present work is given. First, the reader is introduced to the notion of multiparametric programming; next, the general formulation for MIPPs is presented along with a solution approach for MIPPs as shown in Dua.<sup>37</sup> Then, we introduce a novel algorithm for multiparametric MIPPs where the general principles and steps are given in detail and finally we show how the proposed algorithm can be used for the explicit MPC of hybrid polynomial systems.

### Multiparametric programming

Generally, a multiparametric programming problem is of the following form<sup>19,38–44</sup>

$$\begin{aligned} J(\theta) &= \min_{\mathbf{x}} f(\mathbf{x}, \theta) \\ \text{subject to : } & \mathbf{h}(\mathbf{x}, \theta) = 0 \\ & \mathbf{g}(\mathbf{x}, \theta) \leq 0 \\ & \mathbf{x} \in X \subseteq \mathbb{R}^{n_x}, \quad \theta \in \Theta \subseteq \mathbb{R}^{n_\theta} \end{aligned} \quad (1)$$

where  $\mathbf{x}$  is the vector of optimization variables and belongs to the bounded set  $X \in \mathbb{R}^{n_x}$ ,  $\theta$  is the vector of uncertain parameters and belongs to the bounded set  $\Theta \in \mathbb{R}^{n_\theta}$ . Solving the system described in (1) results in a solution of the following general structure

$$\mathbf{x}(\theta) = \begin{cases} x_1(\theta) & \text{if } \theta \in \text{CR}_1 \\ x_2(\theta) & \text{if } \theta \in \text{CR}_2 \\ \vdots & \\ x_n(\theta) & \text{if } \theta \in \text{CR}_n \end{cases} \quad (2)$$

where  $\mathbf{x}(\theta)$ , optimal vector of optimization variables, is computed as explicit function of the problem's parameters along with a number of CRs in which each solution is optimal. Graphically, this is shown in Figure 2a, where three CRs have been computed and solution  $J_1(\theta)$  is valid in  $\text{CR}_1$ ,  $J_2(\theta)$  is valid in  $\text{CR}_2$ , and  $J_3(\theta)$  is valid in  $\text{CR}_3$  while another graph in a two-parameters' space is given in Figure 2b. Note, in Figure 2 the illustrations of explicit solutions in the parametric space, along with the CRs, are

schematic representations for the general case of arbitrary nonconvex multiparametric programs.

Typically,  $\theta$  corresponds to uncertain parameters such as production demand, reaction temperature, and machine availability. For an extended discussion on this topic, the interested reader is referred to the book of Pistikopoulos et al.<sup>23</sup>

### Mixed integer polynomial programming

In the general case, a Mixed Integer Polynomial Programming problem can be formulated as follows<sup>37</sup>

*Mixed Integer Polynomial Programming:*

$$\begin{aligned} J_1 &= \min_{\mathbf{x}, \mathbf{y}} f(\mathbf{x}, \mathbf{y}) \\ \text{subject to : } & \mathbf{h}(\mathbf{x}, \mathbf{y}) = 0 \\ & \mathbf{g}(\mathbf{x}, \mathbf{y}) \leq 0 \\ & \mathbf{x} \in \mathbb{R} \\ & \mathbf{y} \in \{0, 1\}^{n_y} \end{aligned} \quad (3)$$

where  $\mathbf{x}$  is the vector of continuous variables with dimensionality  $n_x$ ,  $\mathbf{y}$  is the vector of integer variables with dimensionality  $n_y$ ,  $\mathbf{h}$  is the vector of equality constraints and is  $n_h$  dimensional,  $\mathbf{g}$  is the vector of inequality constraints and is  $n_g$  dimensional, while  $f$  is a scalar objective function. The integer variables of the MIPP are relaxed as continuous variables and then by treating them as parameters a multiparametric programming problem arises as follows:

*Multiparametric polynomial programming:*

$$\begin{aligned} J_2(\mathbf{y}) &= \min_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) \\ \text{subject to : } & \mathbf{h}(\mathbf{x}, \mathbf{y}) = 0 \\ & \mathbf{g}(\mathbf{x}, \mathbf{y}) \leq 0 \\ & \mathbf{x} \in \mathbb{R}^{n_x} \\ & \mathbf{y} \in [0, 1]^{n_y} \end{aligned} \quad (4)$$

Problem (4) is now parametric in the relaxed integer variables, that is,  $\mathbf{y}$ . The first-order Karush-Kuhn-Tucker (KKT) conditions of Problem (4) are given by the equality constraints (5)–(7) and the inequality constraints (8) and (9).

$$\mathbf{u}^*(\mathbf{x}) = \beta(\mathbf{x}) \tag{11}$$

$$\nabla_{\mathbf{x}}L(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0 \tag{5}$$

$$\lambda_j g_j(\mathbf{x}, \mathbf{y}) = 0, \quad \forall j=1, \dots, n_g \tag{6}$$

$$\mathbf{h}(\mathbf{x}, \mathbf{y}) = 0 \tag{7}$$

$$\mathbf{g}(\mathbf{x}, \mathbf{y}) \leq 0 \tag{8}$$

$$\lambda_j \geq 0, \quad \forall j=1, \dots, n_g \tag{9}$$

where  $L(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}, \mathbf{y}) + \sum_{i=1}^{n_h} \mu_i h_i(\mathbf{x}, \mathbf{y}) + \sum_{j=1}^{n_g} \lambda_j g_j(\mathbf{x}, \mathbf{y})$  is the Lagrange function of the system and it is parametric in  $\mathbf{y}$ . Notice that in the KKT system, the equality constraints are parametric in  $\mathbf{y}$  and polynomial in  $[\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}]$ . Solving the system of equality constraints analytically, with  $\mathbf{y}$  being relaxed, the optimal solution and the Lagrange multipliers are computed as functions of  $\mathbf{y}$ , that is,  $\mathbf{x}(\mathbf{y}), \boldsymbol{\lambda}(\mathbf{y}), \boldsymbol{\mu}(\mathbf{y})$ . The final solutions of the MIPOPT problem are then computed by imposing the integrality condition on  $\mathbf{y}$  and then qualifying with the inequality constraints, that is, (8) and (9), of the KKT conditions along with Linear Independence Constraint Qualification (LICQ).

**Remark 1.** Note that Gueddar and Dua,<sup>45</sup> proposed a Branch and Bound (B&B) algorithm for general MINLPs using an approximate parametric solution, whereas KKT conditions can be solved analytically for the case when nonlinearities are due to the presence of polynomial terms only.<sup>37</sup>

### Using multiparametric programming in the context of MPC

Consider the following discrete time MPC formulation where the initial condition of the system is available at time  $t = t_k$

$$\mathcal{F}(\mathbf{x}(t_k)) = \min_{\mathbf{x}} \sum_{t=0}^{N-1} \mathcal{L}(\mathbf{x}_t, \mathbf{u}_t) + E(\mathbf{x}_N)$$

subject to

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{x}(t_k) \\ \mathbf{x}_{t+1} &= \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t) \quad t = 0, 1, \dots, N-1 \\ \mathbf{z}_t &= \mathbf{h}(\mathbf{x}_t, \mathbf{u}_t) \quad t = 1, \dots, N-1 \\ A\mathbf{x}_t &\leq \mathbf{a} \quad t = 0, 1, \dots, N-1 \\ B\mathbf{z}_t &\leq \mathbf{b} \quad t = 0, 1, \dots, N-1 \\ C\mathbf{u}_t &\leq \mathbf{c} \quad t = 0, 1, \dots, N-1 \end{aligned} \tag{10}$$

where  $\mathbf{x}_t, \mathbf{u}_t, \mathbf{z}_t$  are the state, control input, and system output vectors, respectively, at every sampling instance,  $t$ , and are  $n_x, n_u, n_y$  dimensional.  $A, B, C$  are matrices of appropriate dimensions and  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  vectors of appropriate dimensions which represent inequality constraints for the state, output, and control inputs while  $\mathcal{L} : \mathbb{R}^{n_x+n_u} \rightarrow \mathbb{R}$  is a stage cost and  $E : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$  is a terminal cost function.  $N$  is the prediction horizon which for the sake of notation simplicity is taken as equal to the control horizon. The repetitive solution of problem (10) provides the optimal cost  $\mathcal{F}(\mathbf{x}(t_k))$  and the optimization vector, which in this case is the sequence of optimal control inputs  $\mathbf{u}^* = [\mathbf{u}_1^*, \mathbf{u}_2^*, \dots, \mathbf{u}_{N-1}^*]$  over the finite prediction horizon  $N$ . In contrast with the conventional receding horizon policy, in which the optimization problem has to be solved at every sampling instance, according to the explicit MPC strategy, an explicit control law is derived<sup>25,46</sup>

where  $\beta : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u}$ . The way this is achieved is based on the principles of multiparametric programming, where the initial condition of the system at every sampling instance is considered as a vector of varying parameters, that is,  $\boldsymbol{\theta} := \mathbf{x}_0$ . The solution of this multiparametric program results in the explicit control law, that is, the optimal control sequence as an explicit function of the initial state of the system at every sampling instance. This way, the computational effort associated with the online computations, which for systems with nonlinear dynamics and discrete decisions can be prohibitive, is shifted to an offline multiparametric step. In following section, an algorithm for multiparametric Mixed Integer Polynomial optimization (mp-MIPOPT) problems is presented and the explicit MPC of hybrid polynomial systems is illustrated using the proposed mp-MIPOPT algorithm.

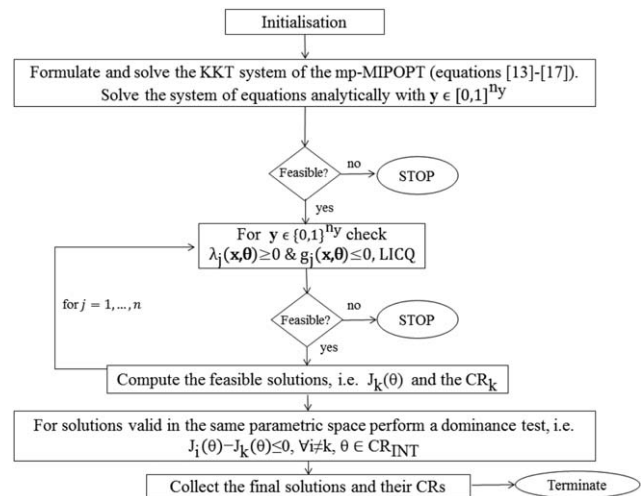
### Multiparametric Mixed Integer Polynomial Programming

#### An algorithm for mp-MIPOPT problems

Consider the multiparametric polynomial programming (mp-PP) problem presented in (4). In addition to the already existing parameters of that problem, that is, the integer variables, a vector of parameters  $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^{n_\theta}$  is introduced. Now the resulting relaxed mp-MIPOPT problem is of the following form

$$\begin{aligned} J_3(\mathbf{y}, \boldsymbol{\theta}) &= \min_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) \\ \text{subject to : } &\mathbf{h}(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) = \mathbf{0} \\ &\mathbf{g}(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) \leq \mathbf{0} \\ &\mathbf{x} \in X \subseteq \mathbb{R}^{n_x} \\ &\mathbf{y} \in [0, 1]^{n_y} \\ &\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^{n_\theta} \end{aligned} \tag{12}$$

The problem presented in (12) is a multiparametric programming problem where  $\boldsymbol{\theta}$  together with the relaxed  $\mathbf{y}$  are considered as parameters. An outline of the proposed algorithm for mp-MIPOPT problems is presented in Algorithm 1. The main idea of the proposed algorithm is to exploit the ability of solving analytically the system of equality constraints that arise from



**Figure 3. Multiparametric Mixed Integer Polynomial Programming algorithm outline.**



the first-order KKT conditions of problem (12). The equality constraints of the KKT system are given by Eqs. (13)–(15).

*KKT system of mp-MIPOPT*

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}) = 0 \quad (13)$$

$$\lambda_j(\boldsymbol{\theta}) g_j(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) = 0, \quad \forall j = 1, \dots, n_g \quad (14)$$

$$\mathbf{h}(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) = 0 \quad (15)$$

$$\mathbf{g}(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) \leq 0 \quad (16)$$

$$\lambda_j(\boldsymbol{\theta}) \geq 0, \quad \forall j = 1, \dots, n_g \quad (17)$$

where  $L(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\theta}) = f(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) + \sum_{i=1}^{n_h} \mu_i(\boldsymbol{\theta}) h_i(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta}) + \sum_{j=1}^{n_g} \lambda_j(\boldsymbol{\theta}) g_j(\mathbf{x}, \mathbf{y}, \boldsymbol{\theta})$  is the Lagrange function of the system and it is parametric in  $\mathbf{y}$  and  $\boldsymbol{\theta}$ . Again, the system of equality constraints is polynomial in  $[\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}]$  and parametric in  $\mathbf{y}$  and  $\boldsymbol{\theta}$ . Solving analytically the system of Eqs. 13–15 using a symbolic manipulation software, for example, Mathematica®, we compute the vector of optimization variables, that is,  $\mathbf{x}(\mathbf{y}, \boldsymbol{\theta})$  and the Lagrange multipliers, that is,  $\boldsymbol{\lambda}(\mathbf{y}, \boldsymbol{\theta})$ ,  $\boldsymbol{\mu}(\mathbf{y}, \boldsymbol{\theta})$ , as functions of the problem's parameters, that is,  $\mathbf{y}$  and  $\boldsymbol{\theta}$ . At the next step, the integrality condition on the integer variables is imposed and the inequality constraints from the KKT system, that is, (16) and (17), are evaluated. For the candidate solutions that the integrality condition is satisfied (integer feasible solutions), the vector of optimization variables and the Lagrange multipliers are left as parametric in  $\boldsymbol{\theta}$ , that is,  $\mathbf{x}(\boldsymbol{\theta})$ ,  $\boldsymbol{\lambda}(\boldsymbol{\theta})$ ,  $\boldsymbol{\mu}(\boldsymbol{\theta})$ . At this step, for the integer feasible solutions (now parametric only in  $\boldsymbol{\theta}$ ), the CRs are defined by the following set of parametric inequality constraints:  $\mathbf{g}(\boldsymbol{\theta}) \leq 0$  and  $\lambda_j(\boldsymbol{\theta}) \geq 0$ ,  $\forall j = 1, \dots, n_g$  along with certain Constraint Qualifications (CQ). The final solutions of the mp-MIPOPT problem are those for which both the integrality condition and the parametric inequality constraints are simultaneously satisfied. After the computation of the final explicit solutions, it is possible that more than one solution is feasible within the same parametric space; for those solutions, a comparison procedure is followed (see Appendix) so as to keep only one optimal solution per CR.

Notice that in the proposed algorithm, after computing the continuous variables as functions of the uncertain parameters, that is,  $\mathbf{x}(\boldsymbol{\theta})$ , the explicit form of the objective function, that is,  $J(\boldsymbol{\theta})$ , is also computed by substituting  $\mathbf{x}(\boldsymbol{\theta})$ , for a given  $\mathbf{y}$ , in  $f(\mathbf{x}, \mathbf{y})$ . The LICQ at Step 5 of Algorithm, is applied to evaluate the integer feasible solutions. The gradients of the active constraints for the each solution are calculated and tested to prove their linear independence.<sup>47</sup> Particularly, in symbolic

manipulation software there exist built-in functions that automatically perform this task; however, in our case a small code was written to perform the qualification. In Figure 3, the general outline of the proposed algorithm is presented. A procedure for the dominance criterion (test) used in the proposed algorithm is presented in the Appendix, which is along the lines of the mp-MILP algorithms.<sup>39</sup>

### Explicit MPC of hybrid systems using mp-MIPOPT

MPC of nonlinear hybrid systems is computationally demanding as one has to solve the corresponding MINLP problem on-line. Algorithms for explicit solutions of multiparametric MINLPs are presented in Dua and Pistikopoulos.<sup>41</sup> The problem of NMPC of hybrid systems, for the case that the nonlinearities are in polynomial form, is considered in the present work. In this article, hybrid systems will be considered in the following form

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\delta}(t)) \quad (18)$$

$$\mathbf{z}(t) = \boldsymbol{\xi}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\delta}(t)) \quad (19)$$

subject to

$$\mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\delta}(t)) \leq 0 \quad (20)$$

where  $\mathbf{x}(t)$  represents the state vector,  $\mathbf{z}(t)$  the output vector,  $\mathbf{u}(t)$  the control inputs, and  $\boldsymbol{\delta}(t)$  the auxiliary variables that are used to model the hybrid nature of the system and might be either continuous or discrete.<sup>48</sup> The vector  $\mathbf{h}$  stands for the equality and inequality constraints used to model logic-based decisions as well as physical constraints of the system and can be either linear or polynomial,  $\mathbf{f}$  is a vector of polynomial functions. Considering the initial state of the system as the vector of parameters, that is,  $\mathbf{x}_0$ , the corresponding mp-MIPOPT problem that arises from the MPC of the hybrid polynomial system is as follows

$$J^*(\mathbf{x}_0) = \min_{u_0, \dots, u_{N-1}, z_0, \dots, z_{N-1}, \delta_0, \dots, \delta_{N-1}} \sum_{t=0}^{N-1} \mathcal{L}(\mathbf{x}_t, \mathbf{z}_t, \boldsymbol{\delta}_t) + E(\mathbf{x}_N)$$

subject to

$$\mathbf{x}_0 = \mathbf{x}_{t=0} \quad (21)$$

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t), \mathbf{z}(t), \boldsymbol{\delta}(t)), \quad t = 0, 1, \dots, N-1$$

$$\mathbf{z}(t+1) = \boldsymbol{\xi}(\mathbf{x}(t), \mathbf{z}(t), \boldsymbol{\delta}(t)), \quad t = 0, 1, \dots, N-1$$

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#### Algorithm 1 mp-MIPOPT

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**Step 1.** Reformulate the MIPOPT problem as an mp-PP, by relaxing the integer variables, i.e.,  $\mathbf{y}$ , and treating them as parameters together with the uncertain parameters, i.e.,  $\boldsymbol{\theta}$ .

**Step 2.** Formulate the first-order Karush-Kuhn-Tucker (KKT) conditions for problem (12) resulting in the corresponding system of equations and inequalities (13)–(17).

**Step 3.** Solve the resulting system of Eqs. (13)–(15) and compute the continuous variables and the Lagrange multipliers as explicit functions of the  $\mathbf{y}$  and  $\boldsymbol{\theta}$ , i.e., obtain  $\mathbf{x}(\mathbf{y}, \boldsymbol{\theta})$  and  $\boldsymbol{\lambda}(\mathbf{y}, \boldsymbol{\theta})$ .

**Step 4.** Fix  $\mathbf{y}$  to all possible combinations of integer values and evaluate  $\mathbf{x}(\mathbf{y}, \boldsymbol{\theta})$  and  $\boldsymbol{\lambda}(\mathbf{y}, \boldsymbol{\theta})$  to obtain  $\mathbf{x}(\boldsymbol{\theta})$  and  $\boldsymbol{\lambda}(\boldsymbol{\theta})$ .

**Step 5.** Substitute  $[\mathbf{x}(\boldsymbol{\theta}), \boldsymbol{\lambda}(\boldsymbol{\theta})]$  in to (16) and (17) to obtain a set of inequality constraints in terms of  $\boldsymbol{\theta}$ , along with CQ, such as Linear Independence CQ (LICQ). The regions formed by the corresponding sets of inequality constraints may no be polyhedrons.

**Step 6.** Perform a feasibility test, on the set of inequality constraints obtained in Step 5. Set(s) of solutions with infeasible constraints are rejected.

**Step 7.** Collect all the feasible sets of solutions corresponding to the same binary node in order to obtain a compact representation of the CRs in its parametric space.

**Step 8.** For different integer solutions which are valid in the same parametric space, perform a dominance test (See Appendix) in order to keep only the ones that lead to the optimal value of the objective function in each CR and hence obtain a compact representation of all the CRs in one parametric space.

**Table 1. Parametric Solutions of Equality Constraints in KKT Conditions (Step 3 of Algorithm 1)**

| $x_1$                        | $x_2$                               | $\lambda_1$                          | $\lambda_2$                | $\lambda_3$           | $\lambda_4$                      | $\lambda_5$                                       | $\lambda_6$                 |
|------------------------------|-------------------------------------|--------------------------------------|----------------------------|-----------------------|----------------------------------|---|-----------------------------|
| 0                            | 0                                   | 0                                    | 0                          | $\frac{14}{5}$        | $\frac{8}{5}$                    | 0   | 0                           |
| $\frac{4}{5}$                | $\frac{8}{5}$                       | 0                                    | 0                          | $\frac{16}{5}$        | 0                                | 0   | 0                           |
| 2                            | 1                                   | 0                                    | 0                          | 0                     | 0                                | 0   | 0                           |
| 2                            | $6y_2$                              | 0                                    | 0                          | 0                     | 0                                | 0   | $-2+10y_2$                  |
| $\frac{21}{10}$              | $\frac{7}{10}$                      | 0                                    | 0                          | 0                     | $\frac{1}{5}$                    | 0   | 0                           |
| $6y_1$                       | 1                                   | 0                                    | 0                          | 0                     | 0                                | $-4+12y_1$  | 0                           |
| $6y_1$                       | $2y_1$                              | 0                                    | 0                          | 0                     | $\frac{2}{3}-\frac{4y_1}{3}$     | $-\frac{14}{3}+\frac{40y_1}{3}$                   | 0                           |
| $6y_1$                       | $12y_1$                             | 0                                    | 0                          | $-2+24y_1$            | 0                                | $-8+60y_1$  | 0                           |
| $6y_1$                       | $5y_2$                              | 0                                    | 0                          | 0                     | 0                                | $-4+12y_1$  | $-2+10y_2$                  |
| $6y_1$                       | $-5+12y_1-3\theta$                  | 0                                    | $12-24y_2+6\theta$         | 0                     | 0                                | $-28+60y_1-12\theta$                              | 0                           |
| $6y_1$                       | $\frac{5}{3}+2y_1-\frac{\theta}{3}$ | $\frac{4-2\theta}{9}+\frac{4y_1}{3}$ | 0                          | 0                     | 0                                | $-\frac{32}{9}+\frac{40y_1}{3}-\frac{2\theta}{9}$ | 0                           |
| $\frac{5y_2}{2}$             | $5y_2$                              | 0                                    | 0                          | $2-\frac{5y_2}{2}$    | 0                                | 0   | $-4+\frac{25y_2}{2}$        |
| $15y_2$                      | $5y_2$                              | 0                                    | 0                          | 0                     | $-4+30y_2$                       | 0   | $-14+100y_2$                |
| $\frac{5-\theta}{5}$         | $2-\frac{2\theta}{5}$               | $\frac{2}{5}-\frac{2\theta}{5}$      | 0                          | $\frac{4+2\theta}{5}$ | 0                                | 0   | 0                           |
| $\frac{16-\theta}{10}$       | $\frac{11}{5}-\frac{3\theta}{10}$   | $\frac{4}{5}-\frac{\theta}{5}$       | 0                          | 0                     | 0                                | 0   | 0                           |
| $-5+15y_2+\theta$            | $5y_2$                              | $14+30y_2-2\theta$                   | 0                          | 0                     | 0                                | 0   | $-44+100y_2+6\theta$        |
| $\frac{4(5+2\theta)}{5}$     | $3+\frac{\theta}{5}$                | $\frac{12}{5}+\frac{4\theta}{5}$     | $\frac{16}{5}+2\theta$     | 0                     | 0                                | 0   | 0                           |
| $\frac{3(5+3\theta)}{5}$     | $1+\frac{3\theta}{5}$               | 0                                    | $\frac{6+12\theta}{5}$     | 0                     | $-\frac{2}{5}-\frac{6\theta}{5}$ | 0   | 0                           |
| $\frac{2(7+3\theta)}{5}$     | $\frac{3}{5}-\frac{3\theta}{5}$     | 0                                    | $\frac{4+6\theta}{5}$      | 0                     | 0                                | 0   | 0                           |
| $\frac{(5+5y_2+3\theta)}{2}$ | $5y_2$                              | 0                                    | $\frac{1+5y_2+3\theta}{2}$ | 0                     | 0                                | 0   | $\frac{3+25y_2+3\theta}{2}$ |

$$\mathbf{h}(\mathbf{x}(t), \mathbf{z}(t), \delta(t)) \leq 0, \quad t = 0, 1, \dots, N-1$$

where  $N$  is the prediction horizon of the constrained finite time optimal control problem. Solving the mp-MIPOPT problem in (21) using the algorithm described in section, the optimal control inputs, that is,  $\mathbf{u}(\mathbf{x}_{N-1})$ , along with discrete variables for the prediction horizon, that is,  $\mathbf{y}(\mathbf{x}_{N-1})$  are computed as explicit functions of the initial state of the system, that is,  $\mathbf{x}_0$ , at each sampling instance.

In the next section, we present two examples to illustrate the merits of the proposed algorithm. First, a numerical example is solved and the solution procedure is illustrated and then the mp-MIPOPT algorithm is used for the explicit control of a two tanks system.

### Illustrative Examples

#### Numerical example with uncertain parameter

Consider the following modified example from Dua<sup>37</sup> to demonstrate the basic steps of the mp-MIPOPT algorithm when only one uncertain parameter is considered

$$\min_{x,y} J = 2y_1 + 2y_2 + 4x_1 - x_1^2 - x_2^2 + 2x_2 + 4$$

subject to

$$g_1 = -x_1 + 3x_2 + \theta - 5 \leq 0$$

$$g_2 = 2x_1 - 3x_2 - 3\theta - 5 \leq 0$$

$$g_3 = -2x_1 + x_2 \leq 0$$

$$g_4 = x_1 - 3x_2 \leq 0$$

**Table 2. Set of Feasible Parametric Solutions and CRs Satisfying the Inequality Constraints in KKT Conditions**

| No. of Feasible Solution | $y_1$ | $y_2$ | $x_1(\theta)$            | $x_2(\theta)$           | $J(\theta)$                                 | CR   |
|--------------------------|-------|-------|--------------------------|-------------------------|---|--|
| 1                        | 0     | 0     | 0                        | 0                       | 4   | $-\frac{5}{3} \leq \theta \leq 5$            |
| 2                        | 1     | 0     | 0                        | 0                       | 6   | $-\frac{5}{3} \leq \theta \leq 5$            |
| 3                        | 0     | 1     | 0                        | 0                       | 5   | $-\frac{5}{3} \leq \theta \leq 5$            |
| 4                        | 0     | 0     | $\frac{4}{5}$            | $\frac{8}{5}$           | $\frac{36}{5}$                              | $-\frac{5}{3} \leq \theta \leq 1$            |
| 5                        | 1     | 1     | 2                        | 1                       | 12  | $-\frac{2}{3} \leq \theta \leq 4$            |
| 6                        | 1     | 1     | $\frac{21}{10}$          | $\frac{7}{10}$          | 11.9  | $-\frac{1}{2} \leq \theta \leq 4$            |
| 7                        | 1     | 1     | 6                        | $\frac{5-\theta}{3}+2$  | $\frac{(-100+16\theta-\theta^2)}{9}$        | $\frac{5}{4} \leq \theta \leq 5$             |
| 8                        | 1     | 1     | $\frac{5-\theta}{5}$     | $\frac{10-2\theta}{5}$  | $\frac{(50+2\theta-\theta^2)}{5}$           | $-\frac{5}{3} \leq \theta \leq 1$            |
| 9                        | 1     | 1     | $\frac{16+\theta}{10}$   | $\frac{22-3\theta}{10}$ | $\frac{(104+8\theta-\theta^2)}{10}$         | $-\frac{8}{5} \leq \theta \leq 5$            |
| 10                       | 1     | 1     | $\frac{4(5+2\theta)}{5}$ | $3+\frac{\theta}{5}$    | $4-\frac{36\theta}{5}-\frac{13\theta^2}{5}$ | $-\frac{5}{3} \leq \theta \leq \frac{5}{4}$  |
| 11                       | 1     | 1     | $\frac{3(5+3\theta)}{5}$ | $1+\frac{3\theta}{5}$   | $\frac{(55-18\theta-18\theta^2)}{5}$        | $-\frac{1}{2} \leq \theta \leq -\frac{1}{3}$ |
| 12                       | 1     | 1     | $\frac{2(7+3\theta)}{5}$ | $\frac{3-3\theta}{5}$   | $\frac{(55-12\theta-9\theta^2)}{5}$         | $-\frac{2}{3} \leq \theta \leq -\frac{1}{3}$ |

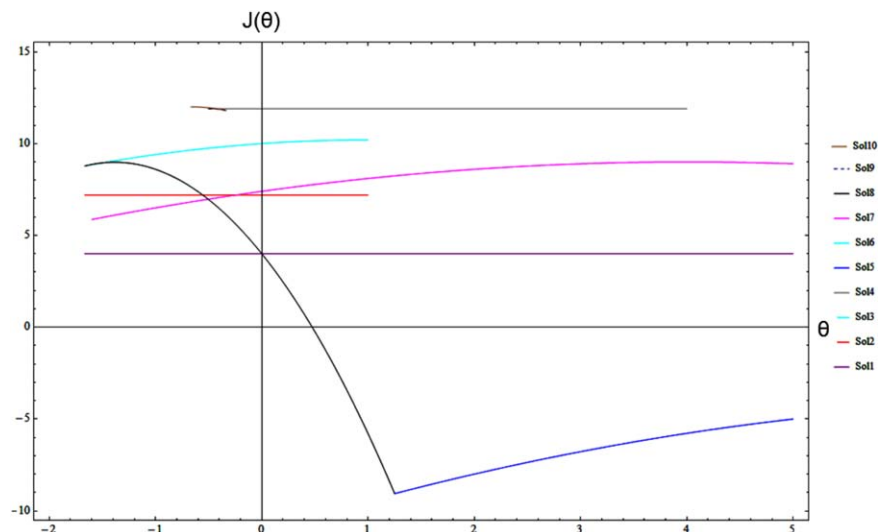


Figure 4. Graphical representation of feasible parametric solutions in the parametric space.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

$$g_5 = -6y_1 + x_1 \leq 0$$

$$g_6 = -5y_2 + x_2 \leq 0$$

$$\theta \in [-2, 5]$$

where  $x_1, x_2$  are continuous variables,  $y_1, y_2$  are 0-1 binary variables, and  $\theta$  is an uncertain parameter. Following Algorithm 1, after the relaxation of binary variables as continuous and treating them as parameters along with the uncertain parameter  $\theta$ , the Lagrange function, is given by

$$L = 4 + 4x_1 - x_1^2 + 2x_2 - x_2^2 + 2y_1 + y_2 + (-5 - x_1 + 3x_2 + \theta)\lambda_1 \\ + (-5 + 2x_1 - x_2 - 3\theta)\lambda_2 \\ + (-2x_1 + x_2)\lambda_3 + (x_1 - 3x_2)\lambda_4 + (x_1 - 6y_1)\lambda_5 + (x_2 - 5y_2)\lambda_6$$

The first-order KKT conditions are given by:

1. Equality Constraints

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda_1 + 2\lambda_2 - 2\lambda_3 + \lambda_4 + \lambda_5$$

Table 3. Summary of Final Solutions ( $i$ ) for Example 1

| $i$ | $CR_i$                            | $y_i$ | $J_i(\theta)$                                   |
|-----|-----------------------------------|-------|---|
| 1   | $-\frac{5}{3} \leq \theta \leq 0$ | [0 0] | 4   |
| 2   | $0 \leq \theta \leq \frac{5}{4}$  | [1 1] | $4 - \frac{36\theta}{5} - \frac{13\theta^2}{5}$ |
| 3   | $\frac{5}{4} \leq \theta \leq 5$  | [1 1] | $\frac{(-100 + 16\theta - \theta^2)}{9}$        |

$$\frac{\partial L}{\partial x_2} = 2 - 2x_2 + 3\lambda_1 - \lambda_2 + \lambda_3 - 3\lambda_4 + \lambda_6$$

$$(-5 - x_1 + 3x_2 + \theta)\lambda_1 = 0$$

$$(-5 + 2x_1 - x_2 - 3\theta)\lambda_2 = 0$$

$$(-2x_1 + x_2)\lambda_3 = 0$$

$$(x_1 - 3x_2)\lambda_4 = 0$$

$$(x_1 - 6y_1)\lambda_5 = 0$$

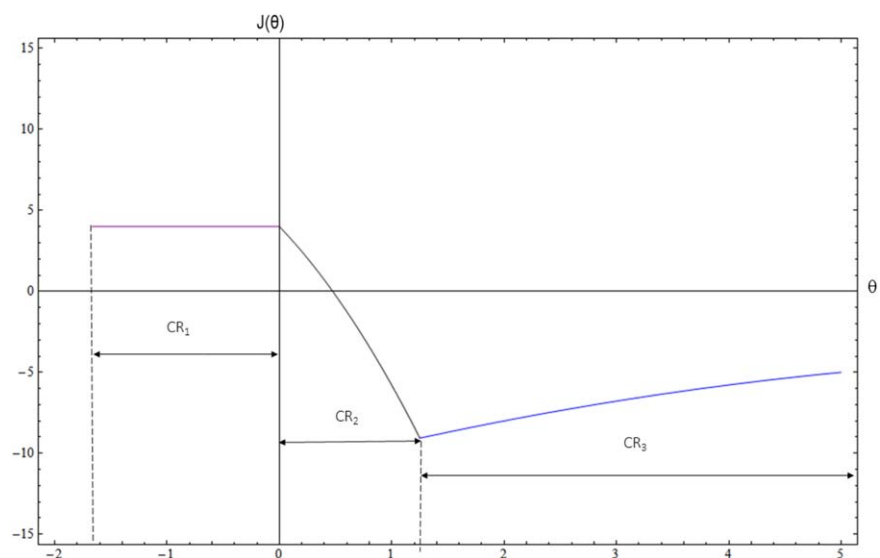
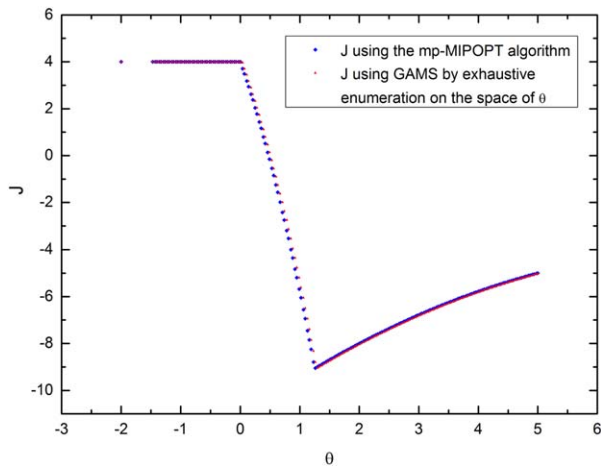


Figure 5. Optimal value of the objective function in the parametric space.

[Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]



**Figure 6. Evaluation of parametric solution of p-MIPOPT using GAMS.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

$$(x_2 - 5y_2)\lambda_6 = 0$$

### 2. Inequality Constraints

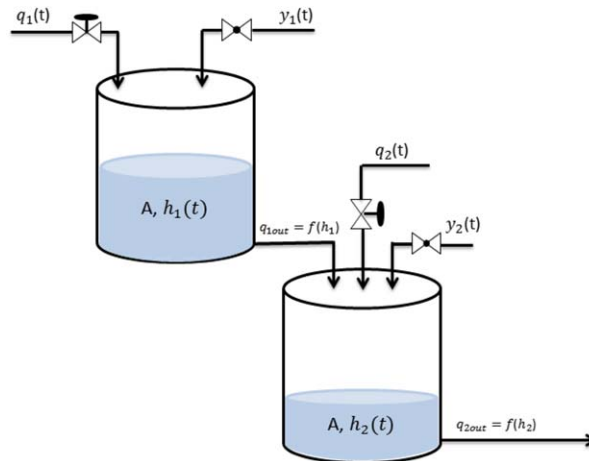
$$g_1 = -x_1 + 3x_2 + \theta - 5 \leq 0$$

$$g_2 = 2x_1 - 3x_2 - 3\theta - 5 \leq 0$$

$$g_3 = -2x_1 + x_2 \leq 0$$

$$g_4 = x_1 - 3x_2 \leq 0$$

$$g_5 = -6y_1 + x_1 \leq 0$$



**Figure 7. Two tanks polynomial hybrid system.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

$$g_6 = -5y_2 + x_2 \leq 0$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0$$

Solving the system of equality constraints of the KKT system the solutions  $[\mathbf{x}(\mathbf{y}, \theta), \boldsymbol{\lambda}(\mathbf{y}, \theta)]$  computed are given in Table 1.

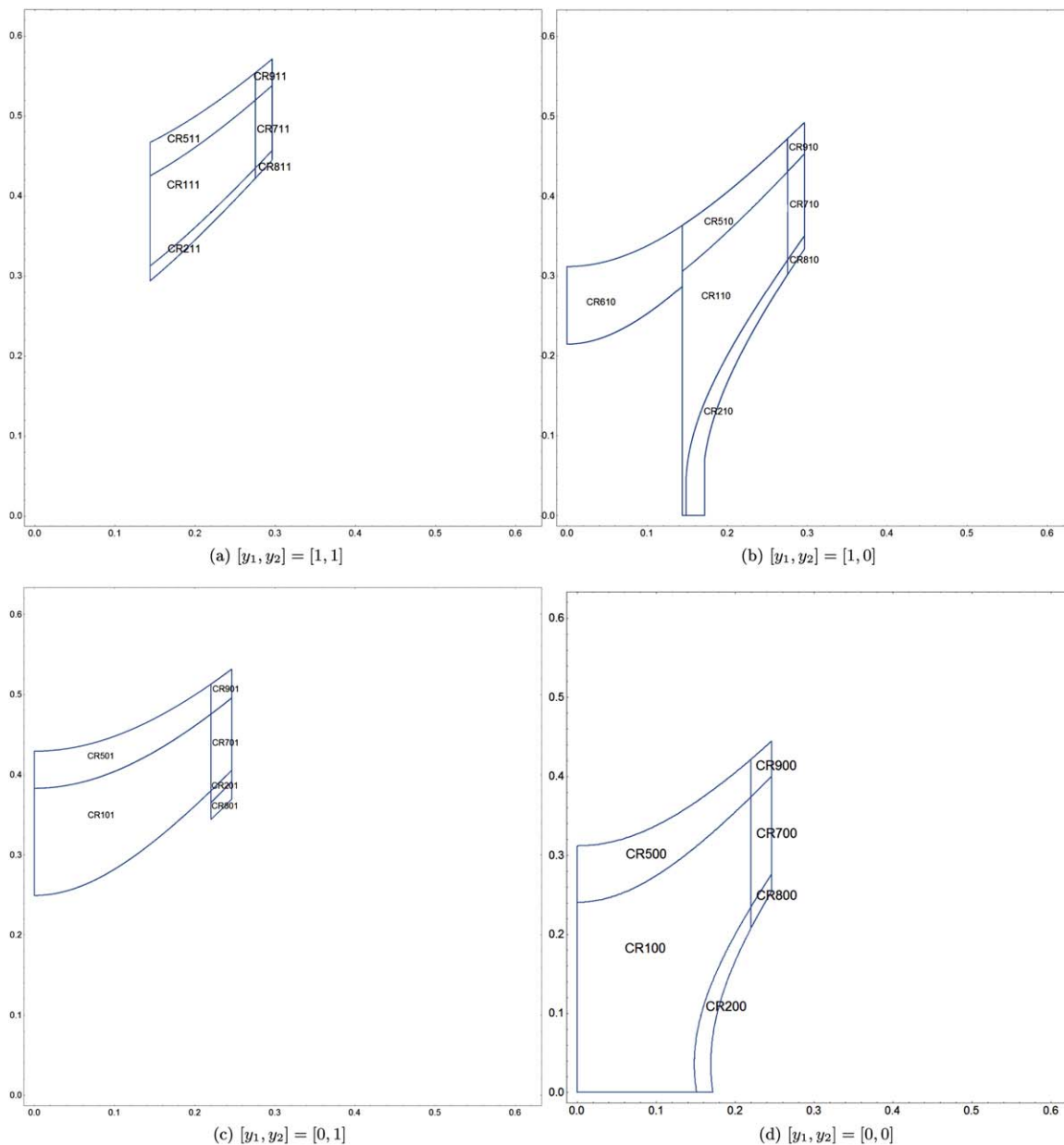
The solutions  $[\mathbf{x}(\mathbf{y}, \theta), \boldsymbol{\lambda}(\mathbf{y}, \theta)]$  are then evaluated by fixing  $\mathbf{y}$  at all possible combinations of the integer values. The evaluations result in  $[\mathbf{x}(\theta), \boldsymbol{\lambda}(\theta)]$ . Substituting  $[\mathbf{x}(\theta), \boldsymbol{\lambda}(\theta)]$  into the inequality constraints ( $g_j(\theta) \leq 0, \lambda_j(\theta) \geq 0$ ) results in a set of inequality constraints in terms of  $\theta$ . Solutions that satisfy this step together with LIQC are stored while the rest are

T1

**Table 4. Feasible Solutions for Hybrid Two Tanks System**

| $\lambda_1$   | $\lambda_2$   | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$   |
|---|---|-------------|-------------|-------------|---|
| 0   | 0   | 0           | 0           | 0           | 0   |
| 0   | 0   | 0           | 0           | 0           | 0   |
| 0   | 0   | 0           | 0           | 0           | 0   |
| 0   | 0   | 0           | 0           | 0           | 0   |
| 0   | 0   | 0           | 0           | 0           | $-0.795733 - 1.95608y_2 - 29.3426\theta_1^2 - 1.3986\theta_2 + 19.5608\theta_2^2$ |
| 0   | 0   | 0           | 0           | 0           | $-0.795733 - 1.95608y_2 - 29.3426\theta_1^2 - 1.3986\theta_2 + 19.5608\theta_2^2$ |
| $-1.14538 - 1.95608y_1 - 1.3986\theta_1 + 29.3426\theta_1^2$  | 0   | 0           | 0           | 0           | 0   |
| $-1.14538 - 1.95608y_1 - 1.3986\theta_1 + 29.3426\theta_1^2$  | 0   | 0           | 0           | 0           | 0   |
| $-1.14538 - 1.95608y_1 - 1.3986\theta_1 + 29.3426\theta_1^2$  | 0   | 0           | 0           | 0           | $-0.795733 - 1.95608y_2 - 29.3426\theta_1^2 - 1.3986\theta_2 + 19.5608\theta_2^2$ |
| $\lambda_7$   | $\lambda_8$   |             |             |             |   |
| 0   | 0   |             |             |             |   |
| 0   | $-0.671328 + 1.95608y_2 + 29.3426\theta_1^2 + 1.3986\theta_2 - 19.5608\theta_2^2$ |             |             |             |   |
| $-0.321678 + 1.95608y_1 + 1.3986\theta_1 - 29.3426\theta_1^2$ | 0   |             |             |             |   |
| $-0.321678 + 1.95608y_1 + 1.3986\theta_1 - 29.3426\theta_1^2$ | $-0.671328 + 1.95608y_2 + 29.3426\theta_1^2 + 1.3986\theta_2 - 19.5608\theta_2^2$ |             |             |             |   |
| 0   | 0   |             |             |             |   |
| $-0.321678 + 1.95608y_1 + 1.3986\theta_1 - 29.3426\theta_1^2$ | 0   |             |             |             |   |
| 0   | 0   |             |             |             |   |
| 0   | $-0.671328 + 1.95608y_2 + 29.3426\theta_1^2 + 1.3986\theta_2 - 19.5608\theta_2^2$ |             |             |             |   |
| 0   | 0   |             |             |             |   |
| $q_1$   | $q_2$   |             |             |             |   |
| $0.3289 - 2y_1 - 1.43\theta_1 + 30.0014\theta_1^2$            | $0.686401 - 2y_2 - 30.0014\theta_1^2 - 1.43\theta_2 + 20\theta_2^2$               |             |             |             |   |
| $0.3289 - 2y_1 - 1.43\theta_1 + 30.0014\theta_1^2$            | 0   |             |             |             |   |
| 0   | $0.686401 - 2y_2 - 30.0014\theta_1^2 - 1.43\theta_2 + 20\theta_2^2$               |             |             |             |   |
| 0   | 0   |             |             |             |   |
| $0.3289 - 2y_1 - 1.43\theta_1 + 30.0014\theta_1^2$            | 1.5   |             |             |             |   |
| 0   | 1.5   |             |             |             |   |
| 0   | $0.686401 - 2y_2 - 30.0014\theta_1^2 - 1.43\theta_2 + 20\theta_2^2$               |             |             |             |   |
| 1.5   | 0   |             |             |             |   |
| 1.5   | 1.5   |             |             |             |   |





**Figure 8. Critical Regions for different binary values for two tanks system.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

discarded. After imposing the integrality conditions on the feasible solutions, only those which are integer feasible are kept and the rest are discarded while in Table 2 for the sake of brevity the best integer feasible solution from each set of solutions is reported.

In Figure 4, a graph of the feasible parametric solutions within their CRs is shown. As it is obvious, there exist feasible solutions within the same parametric range for which a dominance test needs to be performed, that is,  $J_i(\theta) - J_k(\theta) \leq 0$ ,  $i \neq k$ , in order to identify  $\min(J_i(\theta), J_k(\theta))$ .

After the dominance test, the final parametric solutions and the corresponding CRs are collected. In Figure 5, the final CRs are depicted and the optimal value of the objective function in the parametric space is shown.

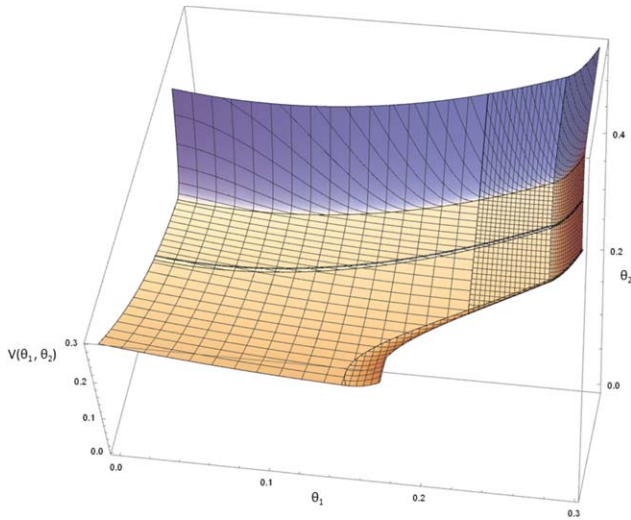
In Table 3, the summary of the final parametric solutions for Example 1 is given.

To compare the parametric solution computed with the proposed mp-MIPOPT, the same problem is solved using GAMS and in Figure 6, a comparison of the value of the objective function using the mp-MIPOPT algorithm and that computed by GAMS is shown. The corresponding MINLP problem was solved in GAMS using BARON,<sup>49</sup> with  $\theta$  being a varying parameter.

As it can be seen in Figure 6, the parametric solution and the solution obtained by exhaustive enumeration of  $\theta$  are exactly the same.

### Two tanks system

Consider the following polynomial hybrid system as shown in Figure 7. The present system involves two flow control valves that allow continuous flow of the liquid, that is,  $q_1(t)$ ,  $q_2(t)$ , two valves that can be either on or off, that is,



**Figure 9.** Three-dimensional graph of the objective function in terms of the initial conditions of the system, that is,  $\theta_1$  and  $\theta_2$ .

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

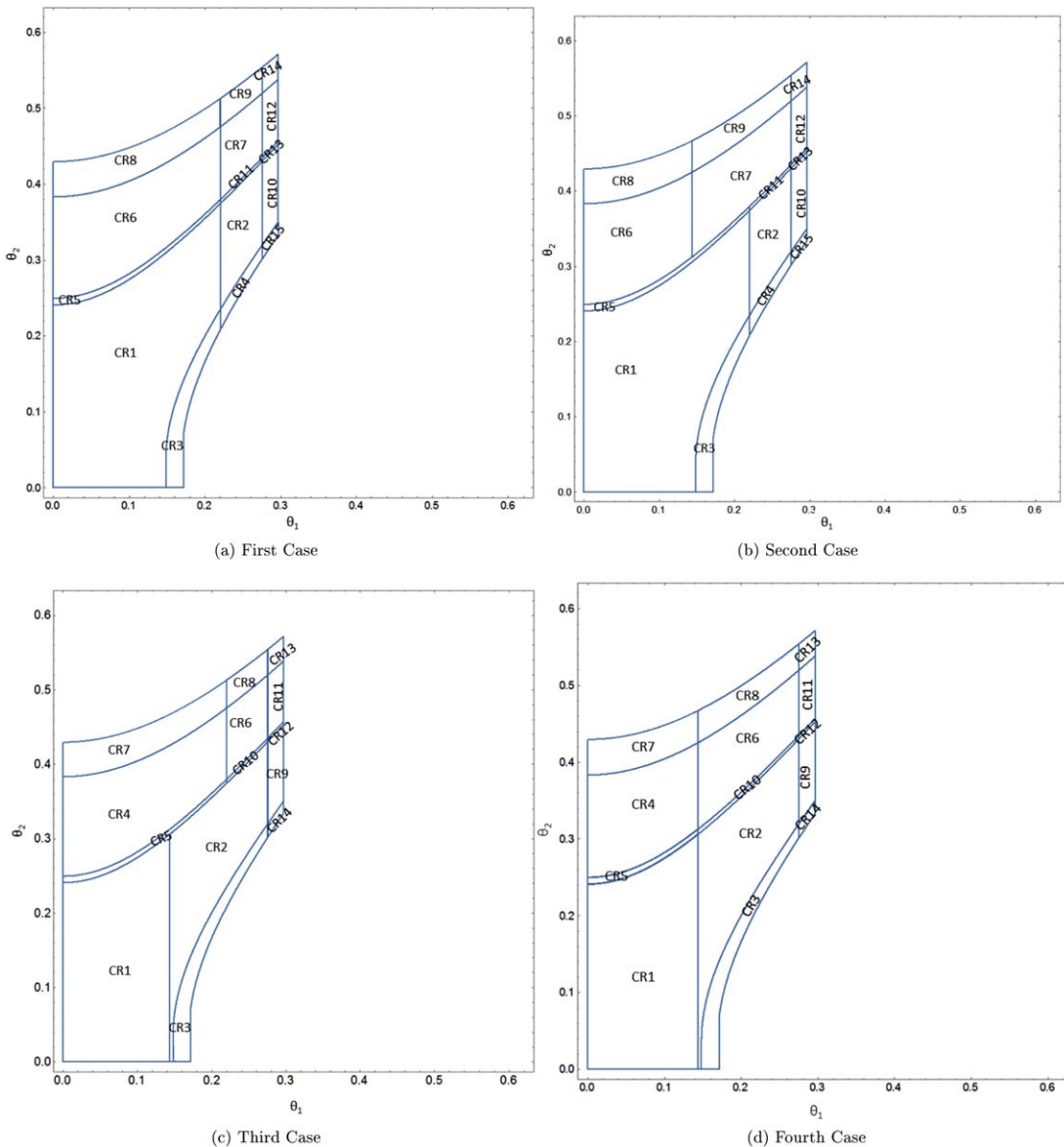
$y_1(t), y_2(t)$ . The flow resistance from each tank is assumed to be a polynomial function of the level of the liquid in each tank for illustration purposes.

The aim is to design an explicit controller which is able to control the level of both the first and the second tank at specific set-points, that is,  $h_1^{sp} = 0.23$  m and  $h_2^{sp} = 0.48$  m. Both of the tanks have a surface of  $0.0143$  m<sup>2</sup> and the height of the tanks is  $0.62$  m. The restriction on the maximum flow from the continuous valves is given by  $q_i(t) \leq 1.5$  for,  $i = 1, 2$ . The dynamic behavior of the system is given by Eqs. 22 and 23

$$A\dot{h}_1 = 0.002q_1(t) + 0.0015y_1(t) - 0.06h_1^2(t) \quad (22)$$

$$A\dot{h}_2 = 0.06h_1(t) + 0.00315y_2(t) + 0.002q_2(t) - 0.04h_2^2(t) \quad (23)$$

To compute the discrete time system, Euler integration was employed and the prediction and control horizon is set to  $N=1$ . The use of prediction horizon of unity leads to less computationally intensive optimization problem but issues with stability may be encountered. Considering the initial conditions of the system at each sampling instance as the vector of



**Figure 10.** Critical regions of hybrid two tanks system.

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

**Table 5. Critical Regions of the Two Tanks System**

|                    |  |  |
|--------------------|--|--|
| CR <sub>1</sub> =  | $0 \leq \theta_1 \leq 0.148344$        | $0 \leq \theta_2 \leq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2}$  |
|                    | $0.148344 < \theta_1 \leq 0.151186$    | $0 \leq \theta_2 \leq 0.03575 - 0.0000452206 \sqrt{-1.61582 * 10^7 + 7.34265 * 10^8 \theta_1^2}$ |
|                    |  | $0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2$                 |
| CR <sub>2</sub> =  | $0.151186 < \theta_1 \leq 0.220133$    | $0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2} \geq \theta_2$         |
|                    |  | $0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2$                 |
|                    | $0.220133 \leq \theta_1 \leq 0.275414$ | $0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2$                 |
| CR <sub>3</sub> =  | $0.148344 \leq \theta_1 \leq 0.171825$ | $0 \leq \theta_2 \leq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2}$  |
|                    | $0.171825 < \theta_1 \leq 0.220133$    | $0.03575 + 0.000452206 \sqrt{-210533 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2$                 |
|                    |  | $0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \geq \theta_2$                 |
| CR <sub>4</sub> =  | $0.220133 \leq \theta_1 \leq 0.275414$ | $0.03575 + 0.000452206 \sqrt{-210533 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2$                 |
|                    | $0.275414 < \theta_1 \leq 0.324286$    | $0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \geq \theta_2$                 |
|                    |  | $0 \leq \theta_1 \leq 0.220133$  |
| CR <sub>5</sub> =  | $0.324286 < \theta_1 \leq 0.373158$    | $\theta_2 \leq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2}$                  |
|                    | $0.373158 < \theta_1 \leq 0.422030$    | $\theta_2 \geq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2}$                  |
|                    |  | $0.220133 \leq \theta_1 \leq 0.275414$   |
| CR <sub>6</sub> =  | $0.422030 < \theta_1 \leq 0.470902$    | $\theta_2 \geq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2}$                  |
|                    | $0.470902 < \theta_1 \leq 0.519774$    | $\theta_2 \leq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2}$         |
|                    |  | $0.220133 \leq \theta_1 \leq 0.275414$   |
| CR <sub>7</sub> =  | $0.519774 < \theta_1 \leq 0.568646$    | $\theta_2 \leq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    | $0.568646 < \theta_1 \leq 0.617518$    | $\theta_2 \geq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    |  | $0.220133 \leq \theta_1 \leq 0.275414$   |
| CR <sub>8</sub> =  | $0.617518 < \theta_1 \leq 0.666390$    | $\theta_2 \geq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    | $0.666390 < \theta_1 \leq 0.715262$    | $\theta_2 \leq 0.03575 + 0.0000226103 \sqrt{3.03052 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    |  | $0.220133 \leq \theta_1 \leq 0.275414$   |
| CR <sub>9</sub> =  | $0.715262 < \theta_1 \leq 0.764134$    | $\theta_2 \leq 0.03575 + 0.0000226103 \sqrt{3.03052 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    | $0.764134 < \theta_1 \leq 0.813006$    | $\theta_2 \geq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2}$         |
|                    |  | $0.275414 \leq \theta_1 \leq 0.296312$   |
| CR <sub>10</sub> = | $0.813006 < \theta_1 \leq 0.861878$    | $\theta_2 \leq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2}$         |
|                    | $0.861878 < \theta_1 \leq 0.910750$    | $0.220133 \leq \theta_1 \leq 0.275414$   |
|                    |  | $\theta_2 \geq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2}$         |
| CR <sub>11</sub> = | $0.910750 < \theta_1 \leq 0.959622$    | $\theta_2 \leq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2}$                  |
|                    | $0.959622 < \theta_1 \leq 1.008494$    | $\theta_2 \geq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2}$                  |

$$\begin{aligned}
\text{CR}_{12} &= \begin{cases} 0.275313 \leq \theta_1 \leq 0.296312 \\ \theta_2 \geq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2} \\ \theta_2 \leq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2} \\ 0.242774 \leq \theta_1 \leq 0.261042 \end{cases} \\
\text{CR}_{13} &= \begin{cases} \theta_2 \geq 0.03575 + 0.0000226103 \sqrt{8.20733 * 10^7 + 2.93426 * 10^9 \theta_1^2} \\ \theta_2 \leq 0.03575 + 0.000452206 \sqrt{223033 + 7.34265 * 10^6 \theta_1^2} \\ 0.275313 \leq \theta_1 \leq 0.296312 \end{cases} \\
\text{CR}_{14} &= \begin{cases} \theta_2 \geq 0.03575 + 0.0000226103 \sqrt{2.35919 * 10^8 + 2.93706 * 10^9 \theta_1^2} \\ \theta_2 \leq 0.03575 + 0.0000226103 \sqrt{3.0352 * 10^8 + 2.93706 * 10^9 \theta_1^2} \\ 0.275313 \leq \theta_1 \leq 0.296312 \end{cases} \\
\text{CR}_{15} &= \begin{cases} 0.03575 + 0.000452206 \sqrt{-210533 + 7.34265 * 10^6 \theta_1^2} \leq \theta_2 \\ 0.03575 + 0.000452206 \sqrt{-161582 + 7.34265 * 10^6 \theta_1^2} \geq \theta_2 \end{cases}
\end{aligned}$$

**Table 6. Parametric Solutions of the Two Tanks System (I)**

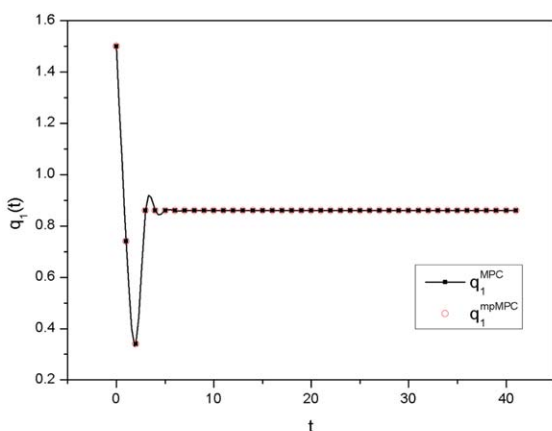
|  |   |   |  |
|--|---|---|--|
| if $[\theta_1, \theta_2] \in \text{CR}_1$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 0.686 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 0 \\ y_2 = 0 \end{cases} $   | if $[\theta_1, \theta_2] \in \text{CR}_8$ then    | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 1.5 \\ y_1 = 0 \\ y_2 = 1 \end{cases} $   |
| if $[\theta_1, \theta_2] \in \text{CR}_2$ then | $ \begin{cases} q_1 = -0.423 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 0.686 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 1 \\ y_2 = 0 \end{cases} $  | if $[\theta_1, \theta_2] \in \text{CR}_9$ then    | $ \begin{cases} q_1 = -0.423 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 1.5 \\ y_1 = 1 \\ y_2 = 1 \end{cases} $  |
| if $[\theta_1, \theta_2] \in \text{CR}_3$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 0 \\ y_1 = 0 \\ y_2 = 0 \end{cases} $  | if $[\theta_1, \theta_2] \in \text{CR}_{10}$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 0.686 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 1 \\ y_2 = 0 \end{cases} $  |
| if $[\theta_1, \theta_2] \in \text{CR}_4$ then | $ \begin{cases} q_1 = -0.423 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 0 \\ y_1 = 1 \\ y_2 = 0 \end{cases} $   | if $[\theta_1, \theta_2] \in \text{CR}_{11}$ then | $ \begin{cases} q_1 = -0.423 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 1.5 \\ y_1 = 1 \\ y_2 = 0 \end{cases} $  |
| if $[\theta_1, \theta_2] \in \text{CR}_5$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = 1.5 \\ y_1 = 0 \\ y_2 = 0 \end{cases} $  | if $[\theta_1, \theta_2] \in \text{CR}_{12}$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = -0.884 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 1 \\ y_2 = 1 \end{cases} $ |
| if $[\theta_1, \theta_2] \in \text{CR}_6$ then | $ \begin{cases} q_1 = 0.336 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = -0.884 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 0 \\ y_2 = 1 \end{cases} $  | if $[\theta_1, \theta_2] \in \text{CR}_{13}$ then | $ \begin{cases} q_1 = 1.5 \\ q_2 = 0 \\ y_1 = 1 \\ y_2 = 1 \end{cases} $   |
| if $[\theta_1, \theta_2] \in \text{CR}_7$ then | $ \begin{cases} q_1 = -0.423 - 1.46\theta_1 + 30.7\theta_1^2 \\ q_2 = -0.884 - 30\theta_1^2 - 1.43\theta_2 + 20\theta_2^2 \\ y_1 = 1 \\ y_2 = 1 \end{cases} $ | if $[\theta_1, \theta_2] \in \text{CR}_{14}$ then | $ \begin{cases} q_1 = 1.5 \\ q_2 = 1.5 \\ y_1 = 1 \\ y_2 = 1 \end{cases} $   |
|  |   | if $[\theta_1, \theta_2] \in \text{CR}_{15}$ then | $ \begin{cases} q_1 = 1.5 \\ q_2 = 0 \\ y_1 = 1 \\ y_2 = 0 \end{cases} $   |

**Table 7. Parametric Solutions of the Two Tanks System (II)**

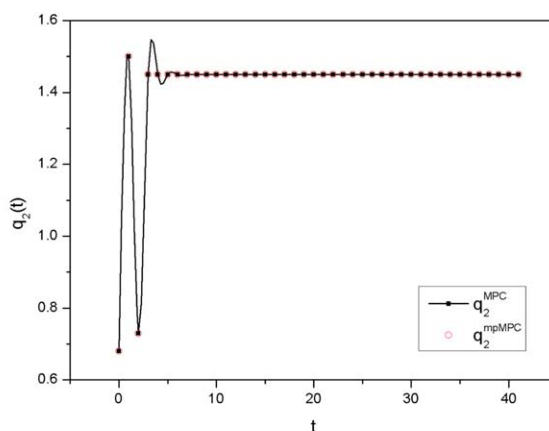
|  |   |   |  |
|--|---|---|--|
| if $[\theta_1, \theta_2] \in \text{CR}_1$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 0.48 \end{cases}$  | if $[\theta_1, \theta_2] \in \text{CR}_8$ then    | $\begin{cases} x_1(t) = 0.52 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 2.15 + 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$ |
| if $[\theta_1, \theta_2] \in \text{CR}_2$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 0.48 \end{cases}$  | if $[\theta_1, \theta_2] \in \text{CR}_9$ then    | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 2.15 + 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$                           |
| if $[\theta_1, \theta_2] \in \text{CR}_3$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 21\theta_1^2 + \theta_2 - 14\theta_2^2 \end{cases}$        | if $[\theta_1, \theta_2] \in \text{CR}_{10}$ then | $\begin{cases} x_1(t) = 1.55 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 0.48 \end{cases}$   |
| if $[\theta_1, \theta_2] \in \text{CR}_4$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 21\theta_1^2 + \theta_2 - 14\theta_2^2 \end{cases}$        | if $[\theta_1, \theta_2] \in \text{CR}_{11}$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 0.48 \end{cases}$   |
| if $[\theta_1, \theta_2] \in \text{CR}_5$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 1.05 + 21\theta_1^2 + \theta_2 - 14\theta_2^2 \end{cases}$ | if $[\theta_1, \theta_2] \in \text{CR}_{12}$ then | $\begin{cases} x_1(t) = 1.55 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 1.1 + 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$  |
| if $[\theta_1, \theta_2] \in \text{CR}_6$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 0.48 \end{cases}$  | if $[\theta_1, \theta_2] \in \text{CR}_{13}$ then | $\begin{cases} x_1(t) = 1.55 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 1.1 + 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$  |
| if $[\theta_1, \theta_2] \in \text{CR}_7$ then | $\begin{cases} x_1(t) = 0.23 \\ x_2(t) = 0.48 \end{cases}$  | if $[\theta_1, \theta_2] \in \text{CR}_{14}$ then | $\begin{cases} x_1(t) = 1.55 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 2.15 + 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$ |
|  |   | if $[\theta_1, \theta_2] \in \text{CR}_{15}$ then | $\begin{cases} x_1(t) = 1.55 + \theta_1 - 21\theta_1^2 \\ x_2(t) = 21\theta_1^2 + \theta_2 - 13.98\theta_2^2 \end{cases}$        |

parameters of the mp-MIPOPT problem, the resulting multi-parametric program consists of 12 constraints, 2 parameters, 2 binary variables, and 4 optimization variables whereas the

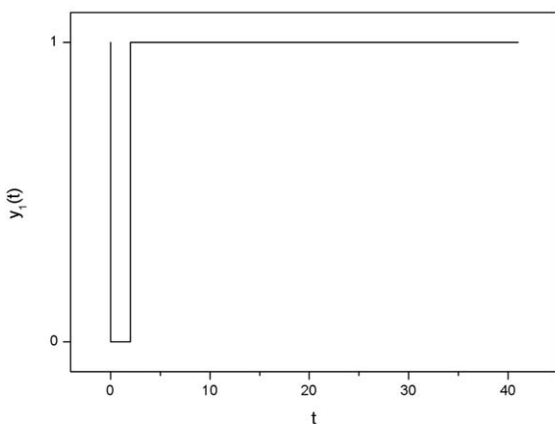
objective function is considered to be the quadratic error function between the desired set-point and the predicted value of the tank's liquid level as shown in Eq. 24



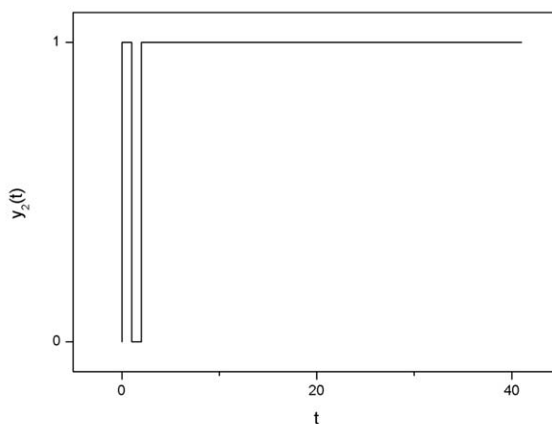
(a)  $q_1 = f(t)$



(b)  $q_2 = f(t)$



(c)  $y_1 = f(t)$

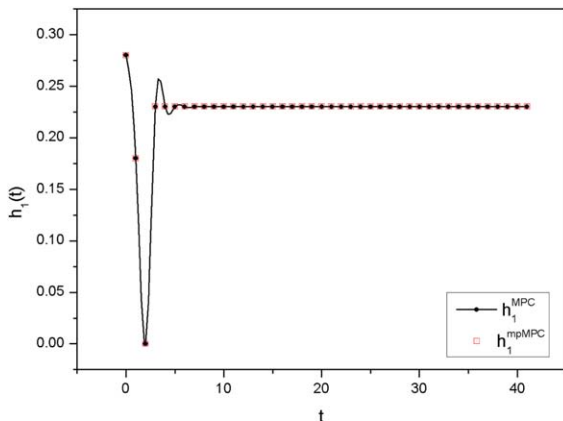


(d)  $y_2 = f(t)$

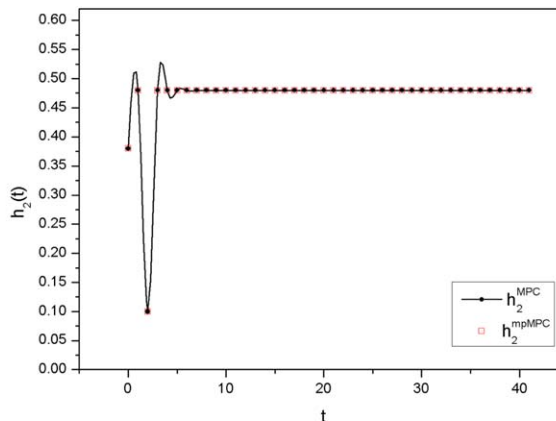
**Figure 11. Validation of the control inputs of the parametric controller for the two tanks system.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]





(a)  $h_1 = f(t)$



(b)  $h_2 = f(t)$

**Figure 12. Validation of the explicit controller for the two tanks system.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

$$V(\theta, \mathbf{y}) = \sum_{i=1}^2 \sum_{t=0}^N (x_{i,t}^{\text{calc}}(\theta, \mathbf{y}) - x_{i,t}^{\text{sp}})^2 \quad (24)$$

For simplicity, no terminal set constraints were imposed in the present example in order to enhance stability as well as path constraints and velocity constraints were also not employed. Solving the resulting mp-MIPOPT problem in Mathematica 10©, the KKT system of the problem was first formulated and solved resulting in 25 candidate solutions. Following the proposed algorithm, these candidate solutions are qualified for the non-negativity of the Lagrange multipliers and after this step, 9 are the feasible sets of solutions and are shown in Table 4.

As shown in Table 4, both the Lagrange multipliers and the control inputs are parametric in  $\mathbf{y}, \theta$ . Next, the binary variables that were until this step relaxed, as continuous variables  $\mathbf{y} \in [0, 1]$ , are fixed to all their possible values, that is, 1-0, and from this procedure 36 candidate integer solutions are

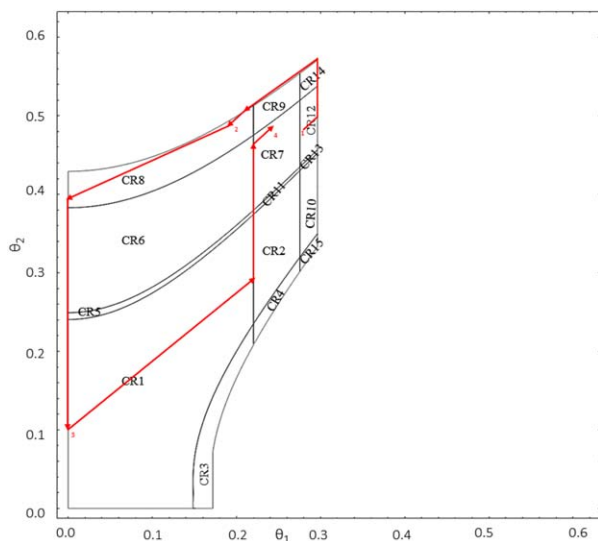
computed for which by applying the feasibility and optimality conditions, that is,  $\lambda_j(\mathbf{y}, \theta) \geq 0$  and  $g_j(\mathbf{y}, \theta) \leq 0$ , for  $j=1, \dots, 9$  the integer feasible solutions are 36. For the final 36 integer feasible solutions, the optimal value of the objective function along with the corresponding CRs are computed. After Step 8 of the mp-MIPOPT algorithm, the final number of CRs (and the corresponding optimal parametric solutions) was 17. Within the 17 CRs, there were 10 CRs where parametric solutions corresponding to two different integer solution were overlapping. In Figure 8, the corresponding CRs for the four integer feasible solution are presented.

For the overlapping CRs, the dominance test is performed after which only the solution with the lowest objective value is kept and the other is discarded. Because of the nonconvex nature of the underlying optimization problem as we will discuss in section “Overlapping Critical Regions,” for the present example some overlapping CRs had solutions with the same objective value and as a result no dominant solution could be proven. Figure 9 is the three-dimensional graph of the objective function as a function of  $\theta_1$  and  $\theta_2$ . Considering that the problem involves polynomial terms, it is not surprising that the objective function profile is nonconvex.

The overlapping CRs corresponded to different integer nodes namely,  $[y_1, y_2] = [1, 0]$  or  $[0, 0]$  and  $[y_1, y_2] = [1, 1]$  or  $[0, 1]$ . The CRs corresponding to the four possible case are shown in Figure 10 and the parametric solution for the case  $\mathbf{y} = [1, 1]$  is given in Table 5. Since for some regions in the space of  $\theta$  one may obtain same parametric solution for different integer solutions, the control engineers may prefer an integer solution over another integer solution. Our methodology provides all the optimal parametric profiles, allowing the control engineers to make an informed decision.

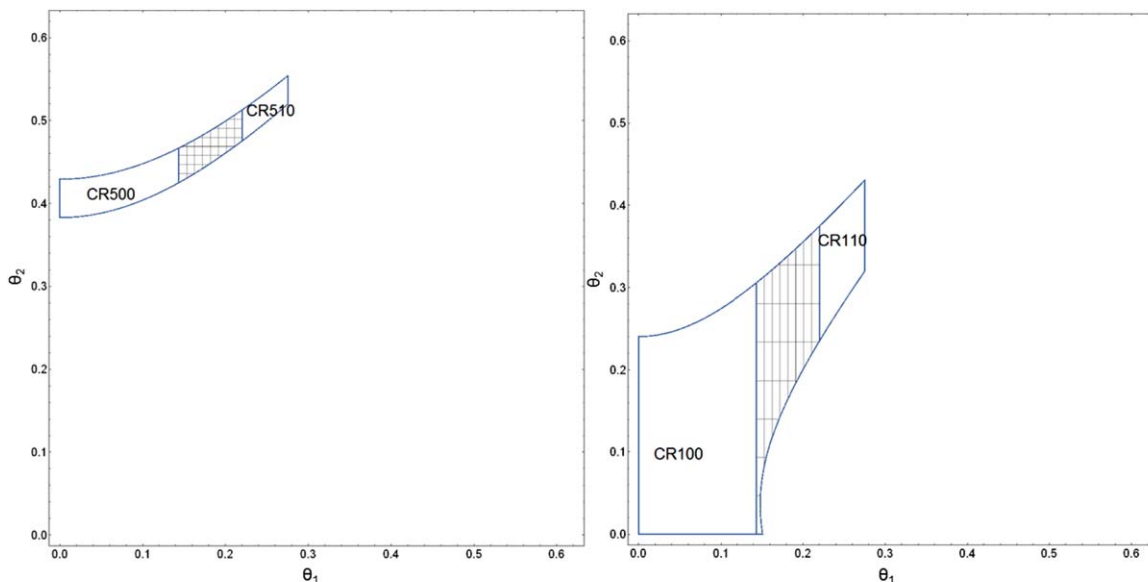
The parametric solutions of the control inputs of the hybrid control system example are shown in Table 6 while the explicit functions for the states of the system are given in Table 7.

Therefore, the parametric controller that was designed using mp-MIPOPT involves two parameters, 15 CRs and the optimization variables, that is, the control inputs of the system computed as explicit functions of the initial conditions of the system at each sampling instance. Figures 11 and 12 provide a comparison between the explicit controller and an online controller solving a MINLP at each iteration. As shown, the



**Figure 13. Optimal control policy in the parametric space  $\theta_1, \theta_2$ .**

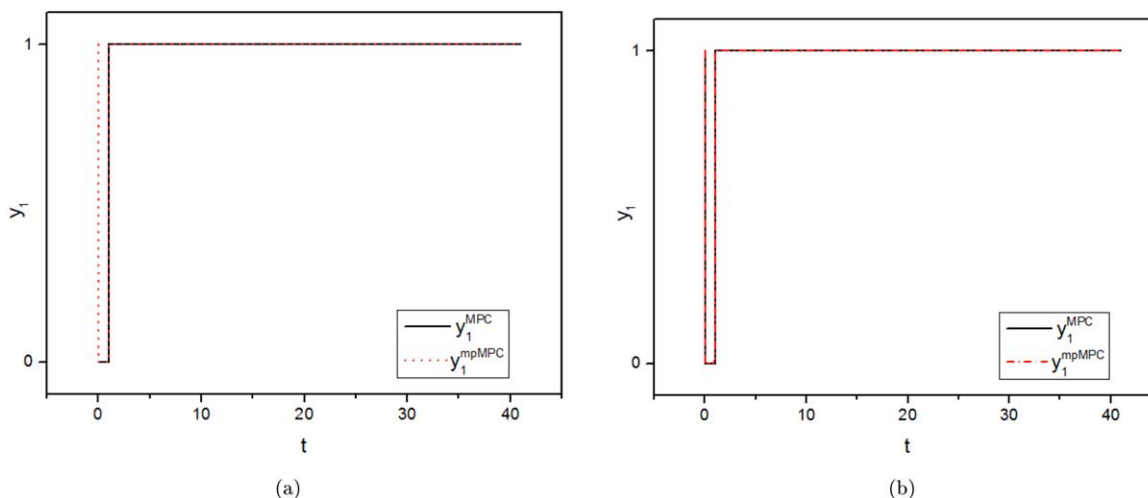
[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]



(a) CR500 overlapping with CR510 in the meshed parametric region (b) CR100 overlapping with CR110 in the meshed parametric region

**Figure 14. Overlapping CRs in the parametric space.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]



**Figure 15. Third case validation.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

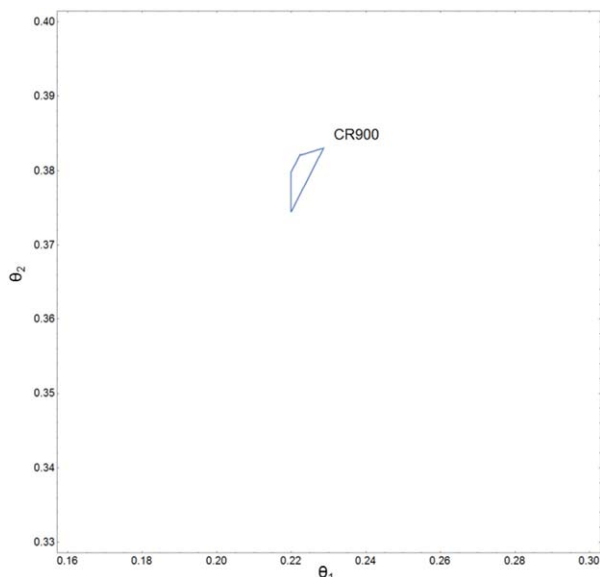
explicit controller and the MINLP solver provide the same solution. The online optimization, that is, the corresponding MINLP problem was solved using ANTIGONE<sup>50</sup> in GAMS. The computations were conducted on a single threaded machine with an Intel Core i7-5600 CPU at 2.60 GHz and 16 GB of RAM. The dynamic evolution of the system in the parametric space is depicted in Figure 13.

In Figures 11 and 12, the superscripts MPC and mpMPC refer to the online solution of the problem using GAMS/ANTIGONE and the mp-MIPOPT algorithm that is proposed in the present work.

**Remark 2.** In the present example, prediction horizon of unity was employed for the MPC scheme to alleviate the computational complexity. Concerning the offline computations, it takes 0.2 s to compute the complete map of candidate solutions while the algorithm returns the final solutions after 30 s and to get a better idea about the offline computational complexity when the prediction horizon

is set to 2, Mathematica needs 3204 s to return 763 candidate feasible solutions. Concerning the benefits of applying the proposed algorithm online for the explicit MPC of hybrid polynomial systems it takes 0.002 s to compute the optimal solution while when using ANTIGONE the optimal solution is found in 0.125 s. Because of the polynomial nature of the problem the time needed to compute the optimal solution using online optimization grows rapidly as ANTIGONE, needs 0.9 s for  $N = 2$ , 50 s for  $N = 3$ , and 690 s for  $N = 4$ .

**Remark 3.** In the context of the decomposition algorithms proposed in the literature for solving multiparametric mixed integer programming problems,<sup>19,39,41,42</sup> the main advantage of the algorithm proposed here is that an approximation of the nonlinear function (where applicable) is not required. A current limitation of the proposed algorithm is the computational implementation for symbolic solution of polynomial equations.



**Figure 16. Final CR900.**

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

## Results and Discussion

As shown in Figures 11 and 12, the proposed algorithm provides the exact explicit solution for optimal control of hybrid polynomial systems. Exploiting the ability to solve the optimization problem analytically, by formulating the KKT conditions, nonpolyhedral CRs are computed instead of the approximate polyhedral ones that have been presented in the literature for mp-MINLPs. It would be of interest to discuss two topics concerning the results of the case study: (i) The overlapping CRs with equal objective values and (ii) the numerical accuracy vs. the computational complexity of the parametric controller.

### Overlapping CRs

In the literature of multiparametric programming, there have been reports of overlapping CRs especially in nonconvex optimization problems. A CR is the region in the parametric space where a solution remains feasible and optimal; it is a common case that two different solutions share a part of the same CR in which case the solution with the lowest objective value is kept, the other one is rejected and the CR is partitioned based on the dominance criterion as explained in the Appendix. This may not always be possible to achieve because some regions in the parametric space may have different integer solutions but same objective function values.

Having two cost-wise same alternatives under the same condition makes the decision making required in the optimal control of hybrid systems even more challenging. These alternative solutions may have different dynamic impact on the system. From a practical point of view if in the previous time interval, a pressure safety valve is closed and in the next state the valve can be open or closed without any impact on the optimal operation of the system then probably one would choose to keep to valve closed. In the example of the two tanks, as can be seen in Figures 14a, b two CRs of the same optimal cost share a common region in the parametric space.

Overlaps like these can be avoided with the addition of a penalty term in the objective function that would account for the

values of the continuous control inputs, but that would also increase the computational complexity of the optimization problem. In Figure 15, we present the results from the simulation that validated the third case study in which the  $y = [1, 1]$  and  $[1, 0]$  solutions are preferred. As shown in Figure 15a, when no explicit constraint is implemented in GAMS for the overlaps the results are slightly different in terms of binary variables; this is overcome as shown in Figure 15b when we specify our preference which does not affect the value of the objective function.

### Computation of the CRs

In the hybrid two tanks system, the “CR900” which can be seen in Figure 8d, after the comparison of its solution with other solutions that were valid in part of the same parametric space, was found to be a quite small CR as show in Figure 16. The graph was rescaled in order to have a better picture of the size of this CR; note that in the same parametric region another solution is also feasible but suboptimal. The importance of neglecting the aforementioned CR was evaluated by numerical simulations. In the final explicit solution, of the case study presented, this CR was not included.

So, while the explicit solutions that we obtain by analytically solving the KKT conditions are exact, the comparison procedure requires numerical techniques and hence can result in some numerical issues.

## Concluding Remarks

In this work, MPC of nonlinear hybrid systems, where nonlinearities are due to polynomial terms only, is considered. This results in a mixed integer polynomial programming (MIPP) problem. The online solution of the MIPP can be computationally very intensive, especially because MIPPs in general are nonconvex. To circumvent this issue, the problem is reformulated as a mp-MIPOPT problem—this is achieved by treating initial conditions of the state variables as parameters bounded between certain lower and upper bounds. An algorithm for the solution of mp-MIPOPT problems is presented—to the best of our knowledge this is the first ever reported algorithm for mp-MIPOPT problems that provides the exact solution. The solution is given by optimal value of the control variables as a set of explicit functions of the state variables, reducing online optimization to simple function evaluations. The main ideas of the algorithm are demonstrated through two illustrative examples. The focus of the ongoing work is on implementing the algorithm for large scale problems and testing it on wider range of control applications. For that purpose, we will explore the structure of the underlying mathematical problem while an implementation of the proposed algorithm in a custom made environment would offer higher degree of freedom so as to speed up the calculations needed for the solution of the system of polynomial equations.

## Acknowledgment

The authors gratefully acknowledge financial support from EPSRC grants EP/M027856/1 and EP/M028240/1.

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## Appendix : Comparison Procedure for the Dominance Criterion

Defining redundant constraints and computing the new CRs within the comparison procedure is a nontrivial task, especially for non-convex problems. A comparison procedure for parametric solutions valid in the same parametric space can be found in Acevedo and Pistikopoulos,<sup>39</sup> so as to keep only the one that provides the better solution (dominance criterion). For the case of convex CRs, that is, when the CRs are defined as a set of linear inequality constraints. In general, while solving a mp-MIPOPT problem it can happen that two different parametric solutions, that is,  $J_1(\theta)$  and  $J_2(\theta)$  to be



**Table A1. Possible Outcomes in the Definition of  $CR_{INT}$**

|        |   |
|--------|---|
| Case 1 | $CR_1 \subseteq CR_2$ which means that all constraints of $CR_2$ are redundant and $CR_{INT} = CR_1$                                |
| Case 2 | $CR_1 \supseteq CR_2$ which means that all constraints of $CR_1$ are redundant and $CR_{INT} = CR_2$                                |
| Case 3 | The $CR_{INT}$ is defined by a set of active constraints from both $CR_1$ and $CR_2$ as both CRs have some nonredundant constraints |

feasible in the same parametric space. The comparison procedure aims to identify the regions were

$$J_1(\theta) - J_2(\theta) \leq 0 \tag{A1}$$

and

$$J_2(\theta) - J_1(\theta) \leq 0 \tag{A2}$$

given that  $J_1(\theta)$  is valid in  $CR_1$  and  $J_2(\theta)$  is valid in  $CR_2$ . The first step is to compute  $CR_{INT} = CR_1 \cap CR_2$ .

**Computation of  $CR_{INT}$  and Redundant Constraints**

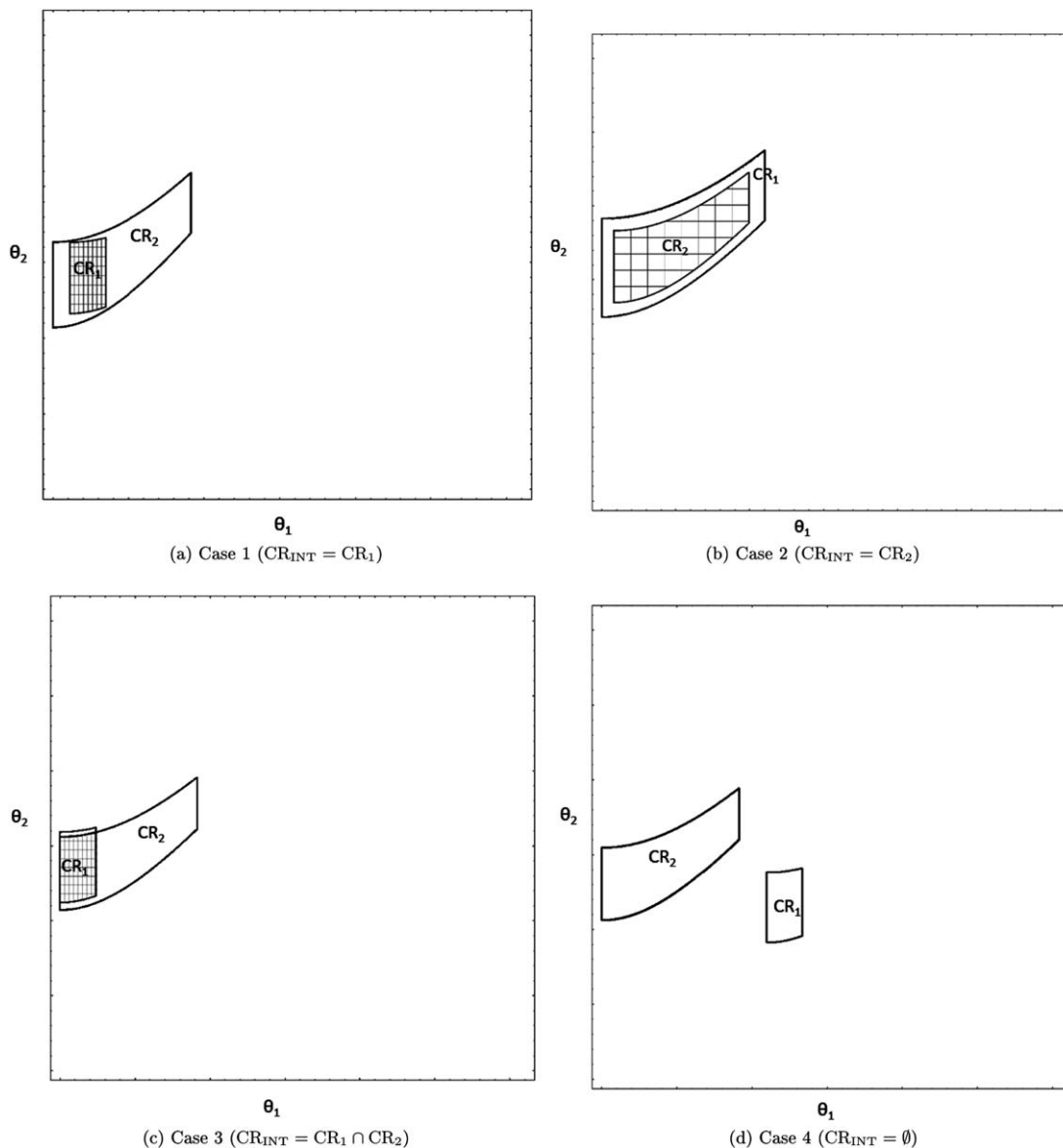
Excluding the case that  $CR_{INT} = \emptyset$  there are three possible outcomes in the definition of the  $CR_{INT}$  which are described in Table A1.

In Figure A1, the different cases for the definition of the  $CR_{INT}$  can be envisaged.

For illustration purposes assume that the following two randomly generated CRs, given by (A3) and (A4), are under examination. We have chosen to illustrate a case that two nonconvex CRs overlap and the  $CR_{INT}$  is nonconvex as well, in order to underline the salient feature of the proposed algorithm, i.e. computing exact nonconvex CRs.

$$CR_1 = \begin{cases} -8 \leq \theta_1 \leq 10 \\ \theta_2^3 > \theta_1 + \theta_1^2 + 3 \end{cases} \tag{A3}$$

$$CR_2 = \begin{cases} -10 \leq \theta_1 \leq 5 \\ \theta_2^2 + \theta_2 < 9 - \theta_1^3 + \theta_1^2 \end{cases} \tag{A4}$$



**Figure A1. Definition of  $CR_{INT}$ .**



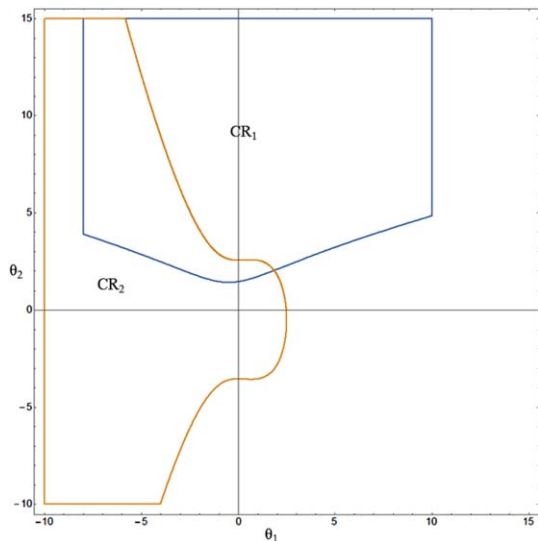


Figure A2. CR<sub>1</sub> and CR<sub>2</sub> in the parametric space.

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

Graphically, in the parametric space CR<sub>1</sub> and CR<sub>2</sub> are presented in Figure A2. These two CRs are nonpolyhedral and the inequalities defining the CRs are polynomial. As mentioned above, in the present work Mathematica was employed for the analytic solution of the mp-MIPOPT. Specifically, for the comparison procedure the command “Reduce” was employed. “Reduce” is a command in Mathematica that qualifies sets of conditional arguments within a given set of parameters and computes a new set within which these conditional statements are satisfied. For example in the definition of the intersection of two CRs (CR<sub>INT</sub>), “Reduce” identifies the redundant constraints of both CRs and computes the region of parametric space where both CRs exists; for the case that the CRs do not overlap the output of “Reduce” is a “False” statement equivalent to the argument CR<sub>INT</sub>=∅.

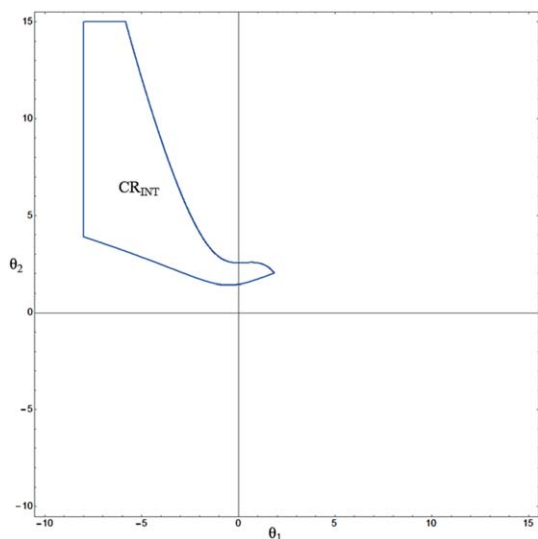


Figure A3. CR<sub>INT</sub> in the parametric space.

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

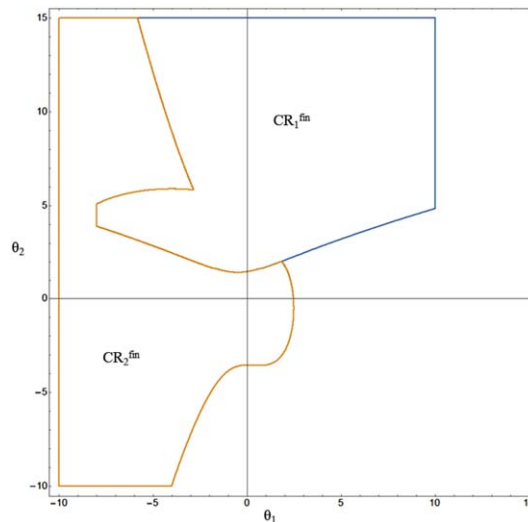


Figure A4. Final nonoverlapping CRs in the parametric space.

[Color figure can be viewed in the online issue, which is available at [wileyonlinelibrary.com](http://wileyonlinelibrary.com).]

In this illustrative case using “Reduce,” the CR<sub>INT</sub> is computed and the redundant inequalities from the two CRs are automatically removed from the set of inequalities forming the CR<sub>INT</sub>, which is given mathematically by (A5) and graphically presented in Figure A3; the CR<sub>INT</sub> as expected is non-polyhedral

$$CR_{INT} = \begin{cases} -8 \leq \theta_1 \leq 1.84742 \\ \theta_2 \geq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \end{cases} \quad (A5)$$

The redundant constraints from each CR can be computed as  $RC_{CR_i} = \{\theta | \theta \in (CR_i \wedge \neg CR_{INT})\}$ ,  $\forall i = 1, 2$  using Mathematica.

### Computation of CR<sub>REST</sub> and the Final Nonoverlapping CRs

After the definition of the CR<sub>INT</sub> the dominance criterion can be expressed by the conditional inequality (A6)

$$J_1(\theta) - J_2(\theta) \leq 0, \theta \in CR_{INT} \quad (A6)$$

As a next step, excluding the case that CR<sub>INT</sub>=∅, the comparison procedure is continued and a new set of conditional statements is qualified, given by (A6). The output of this step is used so as to define the CR<sub>REST<sub>i</sub></sub>, given by (A7) and (A8), and the two new CRs that satisfy the comparison procedure and no longer overlap

$$CR_{REST_1} = \{\theta | \theta \in (CR_{INT} \wedge (J_1(\theta) \leq J_2(\theta)))\} \quad (A7)$$

$$CR_{REST_2} = \{\theta | \theta \in (CR_{INT} \wedge (J_1(\theta) \geq J_2(\theta)))\} \quad (A8)$$

Following the comparison procedure for the previous illustrative case, assume that  $J_1(\theta) - J_2(\theta) = 0.03\theta_2^4 + \theta_1^2 + 8\theta_1 - 20$ . In order to compute the dominant solution for the illustrative case the “Reduce” command is again employed in order to qualify (A6). The output of the “Reduce” in the present case a new set of polynomial inequalities given by (A9), namely CR<sub>REST<sub>i</sub></sub>; this is the fraction of CR<sub>INT</sub> in which  $J_1(\theta) \leq J_2(\theta)$

$$\text{CR}_{\text{REST}_1} = \begin{cases} -8 \leq \theta_1 \leq -2.83459 \\ -2.83459 \leq \theta_1 \leq 2.49087 \end{cases} \begin{cases} \theta_2 \geq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \leq \sqrt{-2000 + 800\theta_1 + 100\theta_1^2 + \theta_1^4} \\ \theta_2 \geq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \end{cases} \quad (\text{A9})$$

Given the mathematical expression of  $\text{CR}_{\text{REST}_1}$ , the fraction of the  $\text{CR}_{\text{INT}}$  where  $J_1(\theta) \geq J_2(\theta)$  can be computed evaluating two equivalent conditional expressions given by (A10) and (A11). The output of "Reduce" for this evaluation is given mathematically, for the present illustrative case by (A12)

$$\text{CR}_{\text{REST}_2} = \{\theta \mid \theta \in (\text{CR}_{\text{INT}} \wedge (\neg \text{CR}_{\text{REST}_1}))\} \quad (\text{A10})$$

$$\text{CR}_{\text{REST}_2} = \{\theta \mid \theta \in (\text{CR}_{\text{INT}} \wedge (J_1(\theta) \geq J_2(\theta)))\} \quad (\text{A11})$$

$$\text{CR}_{\text{REST}_2} = \begin{cases} -8 \leq \theta_1 \leq -2.83459 \\ \theta_2 \geq \sqrt{-2000 + 800\theta_1 + 100\theta_1^2 + \theta_1^4} \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \end{cases} \quad (\text{A12})$$

After the  $\text{CR}_{\text{REST}}$  regions are computed the final CRs can be computed as follows:

$$\text{CR}_1^{\text{fin}} = \{\theta \mid \theta \in (\text{CR}_1 \wedge (\neg \text{CR}_{\text{REST}_2}))\} \quad (\text{A13})$$

$$\text{CR}_2^{\text{fin}} = \{\theta \mid \theta \in (\text{CR}_2 \wedge (\neg \text{CR}_{\text{REST}_1}))\} \quad (\text{A14})$$

Notice that in the previous step  $\text{CR}_{\text{REST}_2}$  could be implicitly computed employing the negative Boolean expressions and not explicitly as done here, alleviating thus the computational effort. Finally, the two CRs that no longer overlap are presented graphically in Figure A4, while their mathematical expression is given by (A15) and (A16). Notice that  $J_1(\theta)$  is optimal in  $\text{CR}_1^{\text{fin}}$  and  $J_2(\theta)$  is optimal in  $\text{CR}_2^{\text{fin}}$

$$\text{CR}_1^{\text{fin}} = \begin{cases} -8 \leq \theta_1 \leq -2.83459 \\ -2.83459 \leq \theta_1 \leq 10 \\ \theta_2 \geq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \end{cases} \begin{cases} \theta_2 \geq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \leq \sqrt{-2000 + 800\theta_1 + 100\theta_1^2 + \theta_1^4} \\ \theta_2 \geq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \end{cases} \quad (\text{A15})$$

$$\text{CR}_2^{\text{fin}} = \begin{cases} -10 \leq \theta_1 \leq -8 \\ -8 \leq \theta_1 \leq -2.83459 \\ -2.83459 \leq \theta_1 \leq 1.84742 \\ 1.84742 \leq \theta_1 \leq 2.49087 \end{cases} \begin{cases} \theta_2 \geq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \geq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \leq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \geq \sqrt{-2000 + 800\theta_1 + 100\theta_1^2 + \theta_1^4} \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \geq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \leq -\theta_1 - \theta_1^2 + \theta_1^3 - 3 \\ \theta_2 \geq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \\ \theta_2 \leq -0.5 + 0.5\sqrt{37 + 4\theta_1^2 - 4\theta_1^3} \end{cases} \quad (\text{A16})$$

Manuscript received Mar. 1, 2016, and revision received May 13, 2016.