Adaptive output feedback finite time control for a class of second order nonlinear systems

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***Abstract*—This paper develops a finite time output feedback based control scheme for a class of nonlinear second order systems. The system representation includes both model uncertainty and uncertain parameters. A finite time parameter estimator is developed. This facilitates the design of a finite time observer based on the well-established step-by-step sliding mode observer design approach. A terminal sliding mode control scheme is then developed using the corresponding state estimates. The methodology is applied to a continuous stirred tank reactor system to validate the effectiveness of the proposed approach.**

***Keywords*—Finite time stability; parameter estimation; adaptive state observer; terminal sliding mode control; continuous stirred tank reactor**

I. INTRODUCTION

Finite time stabilization is an active area of research in control theory which has been motivated by increasing robustness requirements coupled with demands for enhanced tracking performance [1]. Early contributions on continuous finite time controllers for the double integrator system appear in [2], [3] and subsequently many other results have been produced exploring both the theory and application of the finite time control paradigm. For the case of a known system representation without uncertainty, an output feedback based finite time controller can be found in [4] and finite time stability of a class of time invariant continuous systems can be found in [5]. More recent work on the finite time stabilisation of a double integrator system can be found in [6] where finite time output feedback is studied without considering robustness to disturbances. A Lyapunov function for the perturbed double integrator is proposed in [7] but the robustness claims are presented without proof. An augmented continuous sliding mode controller, which assumes full state feedback, is shown to be robust to persisting disturbances but with the trade-off that the derivative of the disturbance is required to be bounded in [8]. Very recently, the problem of finite-time output stabilisation of the double integrator has been addressed applying a homogeneity approach. A homogeneous controller and a homogeneous observer are designed ensuring finite-time stabilization and the robustness of the resulting scheme is analysed [9]. Finite time control via output regulation for a chain of integrators subject to both matched and unmatched disturbances is presented in [10]. A finite time control methodology is developed using a finite time disturbance observer.

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This paper considers finite time stabilization by output feedback control in the case of an uncertain system where the class of second order systems considered is broader than the double integrator system which has been the topic of many previous contributions. As not all system states can be measured, an observer based output feedback control will be employed. A number of candidate asymptotic observers have been presented for second order nonlinear systems using a range of design paradigms including a high gain nonlinear state observer [11], a sliding mode observer [12], an adaptive nonlinear observer [13] and a robust observer designed using Lyapunov techniques and LMIs [14]. By combining such observers with existing state feedback control methodologies, various output feedback control strategies have been designed [15-17]. It should be noted however that many of these existing approaches prescribe asymptotic stability of the output error. The price of increased speed of convergence and tighter tracking accuracy in these approaches will frequently be a high gain system which can prove unacceptable in practice.

To achieve finite time stability of the closed-loop system, it is necessary to use a finite time observer to estimate the states for use by the control law. There are several candidate finite time stable state observers described in the literature, including the super twisting second order sliding mode observer [18], terminal sliding mode observer [19] and finite time Lyapunov function based observer [20]. Here the step by step observer framework is selected because of its straightforward and intuitive design approach [21]. This method requires computation of the equivalent injection in order to provide an estimate of the state estimation error. This yields problems when the system contains disturbances and/or uncertainty in the dynamics of the measured state and existing contributions applying this method for second order nonlinear systems assume the corresponding disturbance signals and/or uncertainty are known or measurable [22,23]. This is clearly a limiting assumption. In this paper, a finite time parameter estimator is developed to estimate the corresponding uncertainty in the measureable state which can be used in the equivalent injection computations within the step by step observer design. A novel adaptive finite time observer results and an integral terminal sliding mode controller is then designed. The parameter estimator, observer and controller are all shown to be finite time stable. Finally, the Continuous Stirred Tank Reactor (CSTR), a typical chemical engineering process, is used to validate the proposed approach.

II. Problem formulation

Consider the following second order nonlinear system

$$\begin{aligned}\dot{x}_1 &= -\beta x_1 + x_2 + f_1(x_1,x_2) + d_1(x_1,x_2)\\ \dot{x}_2 &= g(x_2)u + f_2(x_1,x_2) + d_2(x_2)\theta \\ y &= x_2\end{aligned} \tag{1}$$

Here $x_1, x_2 \in R$ are the system states, $g(x_2) \in R$ is known and $g^{-1}(x_2)$ exists, with both being bounded, $u \in R$ is the control input, $f_1(x_1,x_2) \in R$, $f_2(x_1,x_2) \in R$ are known and bounded, $d_1(x_1,x_2) \in R$ is uncertain, $d_2(x_2) \in R^{1\times n}$ is known, $\theta \in R^{n\times 1}$ represents an unknown parameter vector, $\beta > 0$ is a constant and $y \in R$ is the system output. The following assumptions are made on the system (1):

Assumption 1: $\left|d_1(x_1,x_2)\right| \le \rho_1$, $\rho_1 > 0$ is known.

Assumption 2: $\left\|d_2(x_2)\theta\right\| \le \rho_2$, $\rho_2 > 0$ is known.

Remark 1: Equation (1) can represent a chemical system such as the CSTR (23) . If $\beta = 0$, $f_1(x_1,x_2) = 0$ and $d_1(x_1,x_2) = 0$ it represents mechanical systems [26].

A finite time output feedback control for the system (1) which contains uncertainty in the parameters and the dynamics is sought. To achieve this objective, an adaptive finite time parameter estimation approach is first developed in order to facilitate the design of a sliding mode, step by step observer. Then, a finite time output feedback sliding mode control can be designed by combining the finite time observer with an integral terminal sliding mode control strategy.

III. A finite time stable observer

In this section, a finite time stable observer will be designed using a step-by-step observer design combined with an adaptive finite time parameter estimator.

A. Finite time parameter estimator

Initially assume x_1 and x_2 are measurable. A finite time parameter estimator can be designed by using the filter method [24]. The dynamic equation of x_2 in (1) is first rewritten as:

$$\dot{x}_2 = \varphi(x_1,x_2,u) + \Phi(x_2)\theta \tag{2}$$

where $\varphi(x_1,x_2,u) = g(x_2)u + f_2(x_1,x_2)$ and $\Phi(x_2) = d_2(x_2)$. Define the following filters:

$$\begin{aligned} &k\dot{x}_{2f} + x_{2f} = x_2, \quad x_{2f}(0) = 0\\ &k\dot{\Phi}_f(x_2) + \Phi_f(x_2) = \Phi(x_2), \quad \Phi_f(0) = 0\\ &k\dot{\varphi}_f(x_1,x_2,u) + \varphi_f(x_1,x_2,u) = \hat{\varphi}(x_1,x_2,u), \quad \varphi_f(0,0,0) = 0\end{aligned} \tag{3}$$

where $k > 0$ is the filter parameter.

$$\dot{x}_{2f} = \frac{x_2 - x_{2f}}{k} = \hat{\varphi}_f + \Phi_f\theta \tag{4}$$

Define the following auxiliary filtered regressor matrices:

$$\begin{aligned}\dot{P} &= -lP + \Phi_f^T\Phi_f\\ \dot{Q} &= -lQ + \Phi_f^T\left(\frac{x_2 - x_{2f}}{k} - \varphi_f\right)\end{aligned} \tag{5}$$

where $l > 0$. The solution to (5) is given as:

$$\begin{aligned}P(t) &= \int_0^t e^{-l(t-r)}\Phi_f^T(r)\Phi_f(r)dr\\ Q(t) &= \int_0^t e^{-l(t-r)}\Phi_f^T(r)\left[\left(x_2(r) - x_{2f}(r)\right)/k - \varphi_f(r)\right]dr\end{aligned} \tag{6}$$

It is obvious that:

$$\theta = P^{-1}(t)Q(t) \tag{7}$$

Now, define another auxiliary variable as:

$$W(t) = P(t)\hat{\theta}(t) - Q(t) \tag{8}$$

where $\hat{\theta}$ is the estimate of θ. The finite time adaptive parameter estimator is then defined by:

$$\dot{\hat{\theta}} = -\Gamma P^T(t)\operatorname{sgn}(W(t)) \tag{9}$$

where $\operatorname{sgn}(W(t)) = [\operatorname{sgn}(W_1(t)),\cdots,\operatorname{sgn}(W_n(t))]^T$.

Lemma 1 [25]: A vector or matrix function $\phi(x)$ is persistently excited (PE) if there exist $T > 0$ and $\varepsilon > 0$ such that $\int_t^{t+\Delta t}\phi(r)\phi^T(r)dr \ge \varepsilon I$, $\forall t \ge 0$.

Lemma 2 [24]: The matrix $P(t)$ is positive definite and satisfies $\lambda_{\min}(P(t)) > \sigma$ for $t > T$ and $\sigma > 0$, $T > 0$, if the regressor matrix $\Phi(x_2)$ is PE.

Lemma 3 [26]: If $a_1, a_2, \cdots, a_n$ are all positive numbers, and $0 < p < 2$, then the following inequality holds:

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^p \le \left(a_1^p + a_2^p + \cdots a_n^p\right)^2$$

The following theorem is now ready to be presented.

Theorem 1: For system (1), define the estimation error $\tilde{\theta} = \theta - \hat{\theta}$. If the parameter estimation law is designed as (8), and $\Phi(x_2)$ is PE, then $\tilde{\theta} = 0$ in finite time.

Proof: Selecting a Lyapunov function as:

$$V_1 = \frac{1}{2\Gamma}\tilde{\theta}^T\tilde{\theta} \tag{10}$$

By using (7-9), it follows that:

$$\dot{V}_1 = \tilde{\theta}^T P^T(t)\operatorname{sgn}\left(-P(t)\tilde{\theta}\right) \tag{11}$$

$$\dot{V}_1 = \tilde{\theta}^T P^T(t)\left[\operatorname{sgn}\left(-\left(P(t)\tilde{\theta}\right)_1\right),\cdots,\operatorname{sgn}\left(-\left(P(t)\tilde{\theta}\right)_n\right)\right]^T \tag{12}$$

By using Lemmas 2 and 3, the following equation holds:

$$\dot{V}_1 = -\sum_{i=1}^n\left(P(t)\tilde{\theta}\right)_i \le -\left\|P(t)\tilde{\theta}\right\| \tag{13}$$

$$\dot{V}_1 \le -\mu_1\sqrt{V_1} \tag{14}$$

where $\mu_1 = \sigma\sqrt{2/\lambda_{\max}(\Gamma^{-1})}$, and $\tilde{\theta} = 0$ as $t \geq t_{s1}$, $t_{s1} \leq 2\sqrt{V_1(0)}/\mu_1$.

Remark 2: The parameter estimator (9) requires that the system states are measurable as in [24]. Frequently, however, states such as acceleration and velocity in mechanical systems and concentrations in chemical systems cannot be measured. To overcome this constraint, the parameter estimator will be combined with a step-by-step sliding mode state observer in order to develop a novel, adaptive finite time observer and parameter estimator.

B. Finite time state observer

To design a finite time observer using the step by step observer approach, the equivalent injection approach is used in the x_2 error dynamics to derive the corresponding estimation error for the x_1 subsystem. If there are parameter uncertainties in the dynamics of x_2 , the corresponding estimation error for x_1 cannot be obtained. It is thus first necessary to estimate the unknown parameters for use in the equivalent injection computation within the observer design.

If x_1 is not measurable, the filters in (3)-(6) are redesigned as:

$$
\begin{aligned}
&k\dot{x}_{2f} + x_{2f} = x_2, \quad x_{2f}(0) = 0 \\
&k\dot{\Phi}_f(x_2) + \Phi_f(x_2) = \Phi(x_2), \quad \Phi_f(0) = 0 \\
&k\dot{\hat{\varphi}}_f(\hat{x}_1, x_2, u) + \hat{\varphi}_f(\hat{x}_1, x_2, u) = \hat{\varphi}(\hat{x}_1, x_2, u), \quad \varphi_f(0,0,0) = 0
\end{aligned} \tag{15}
$$

$$
\dot{x}_{2f} = \frac{x_2 - x_{2f}}{k} = \hat{\varphi}_f + \Phi_f\theta - \Phi_f\Delta(t) \tag{16}
$$

$$
\begin{aligned}
&\dot{P} = -lP + \Phi_f^T\Phi_f \\
&\dot{Q} = -lQ + \Phi_f^T\left(\frac{x_2 - x_{2f}}{k} - \hat{\varphi}_f\right)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&P(t) = \int_0^t e^{-l(t-r)}\Phi_f^T(r)\Phi_f(r)dr \\
&Q(t) = \int_0^t e^{-l(t-r)}\Phi_f^T(r)\left[\left(x_2(r) - x_{2f}(r)\right)/k - \hat{\varphi}_f(r)\right]dr
\end{aligned} \tag{18}
$$

$$
\theta = P^{-1}(t)Q(t) + \Delta(t) \tag{19}
$$

where $\Delta(t) \in R^{n\times 1}$ is caused by the error between the actual state x_1 and the corresponding estimate.

An adaptive parameter estimator is designed as:

$$
\dot{\hat{\theta}} = -\Gamma\left(P^T(t)\mathrm{sgn}\left(W(t)\right) - \tilde{x}_2 d_2^T(\hat{x}_2)\right) \tag{20}
$$

Define the following corresponding finite time observer:

$$
\begin{aligned}
&\dot{\hat{x}}_1 = -\beta\hat{x}_1 + \hat{x}_2 + f_1(\hat{x}_1, \hat{x}_2) + \alpha_1\,\mathrm{sgn}(\bar{x}_1 - \hat{x}_1) \\
&\dot{\hat{x}}_2 = g(\hat{x}_2)u + f_2(\hat{x}_1, \hat{x}_2) + d_2(\hat{x}_2)\hat{\theta} + \alpha_2\,\mathrm{sgn}(y - \hat{x}_2)
\end{aligned} \tag{21}
$$

where $\alpha_1, \alpha_2 > 0$, $\bar{x}_1$ are to be defined. Define the estimation error between the plant and observer as:

$$
\begin{aligned}
&\tilde{x}_1 = x_1 - \hat{x}_1 \\
&\tilde{x}_2 = x_2 - \hat{x}_2
\end{aligned} \tag{22}
$$

Let:

$\Delta f_1(x_1, t) = f_1(x_1, x_2) - f_1(\hat{x}_1, \hat{x}_2)$,

$\Delta g(x_2) = g(x_2) - g(\hat{x}_2)$,

$\Delta d_2(x_2) = d_2(x_2) - d_2(\hat{x}_2)$,

$\Delta f_2(x_1, x_2) = f_2(x_1, x_2) - f_2(\hat{x}_1, \hat{x}_2)$

The observer error system may then be expressed as:

$$
\begin{aligned}
&\dot{\tilde{x}}_1 = -\beta\tilde{x}_1 + \tilde{x}_2 + \Delta f_1(x_1, x_2) + d_1(x_1, x_2) - \alpha_1\,\mathrm{sgn}(\bar{x}_1 - \hat{x}_1) \\
&\dot{\tilde{x}}_2 = \Delta g(x_2)u + \Delta f_2(x_1, x_2) + d(\hat{x}_2)\tilde{\theta} + \Delta d_2(x_2)\theta \\
&\quad -\alpha_2\,\mathrm{sgn}(y - \hat{x}_2)
\end{aligned} \tag{23}
$$

In order to prove that the observer estimation error converges to zero in finite time, it is necessary to impose the following assumptions on the terms relating to the plant and observer mismatch and the boundedness of the applied control signal.

Assumption 3: $|\Delta f_1(x_1, x_2)| \leq \rho_4$, $|\Delta g(x_2)| \leq \rho_5$ and $|\Delta f_2(x_1, x_2)| \leq \rho_6$, $|\Delta d_2(x_2)\theta| \leq \rho_7$, $\rho_4, \rho_5, \rho_6, \rho_7 > 0$.

Assumptions 4: The control input u is essentially bounded.

Assumption 5: $\left|\left(P(t)\Delta(t)\right)_i\right| < \left|\left(P(t)\tilde{\theta}(t)\right)_i\right|$, where $(\square)_i$ denotes the *i*th element of a vector, $i = 1, \cdots, n$.

Assumption 6: $\tilde{x}_1$ is a real root of the equation $f_2(x_1, \hat{x}_2) - f_2(\hat{x}_1, \hat{x}_2) = c$, c is a constant.

The following result may now be presented.

Theorem 2: Under Assumptions 1-6, the adaptive state observer defined in equations (20) and (21) is finite time stable and $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{\theta}$ converge to zero in finite time.

Proof:

Step 1: To prove $\tilde{x}_2$ will be zero in finite time.

Selecting a Lyapunov function as:

$$
V_2 = \frac{1}{2}\tilde{x}_2^2 + \frac{1}{2\Gamma}\tilde{\theta}^T\tilde{\theta} \tag{24}
$$

$$
\begin{aligned}
\dot{V}_2 = {} & \tilde{x}_2\left[\Delta g(x_2)u + \Delta f_2(x_1, x_2) + d(\hat{x}_2)\tilde{\theta} + \Delta d_2(x_2)\theta\right. \\
& \left. -\alpha_2\,\mathrm{sgn}(y - \hat{x}_2)\right] + \tilde{\theta}^T P^T(t)\mathrm{sgn}\left(-P(t)\tilde{\theta} + P\Delta(t)\right) \\
& -\tilde{x}_2\tilde{\theta}^T d_2^T(\hat{x}_2)
\end{aligned} \tag{25}
$$

In light of Assumptions 3-6, it follows that:

$$
\begin{aligned}
\dot{V}_2 \leq {} & -\alpha_2|\tilde{x}_2| + |\tilde{x}_2|\left|\Delta g(x_2)u + \Delta f_2(x_1, x_2) + \Delta d_2(x_2)\theta\right| \\
& +\tilde{\theta}^T P^T(t)\left[\mathrm{sgn}\left(-\left(P(t)\tilde{\theta}\right)_1\right), \cdots, \mathrm{sgn}\left(-\left(P(t)\tilde{\theta}\right)_n\right)\right]^T
\end{aligned} \tag{26}
$$

$$\dot{V}_2 \le -|\tilde{x}_2|\left(\alpha_2 - \left|\Delta g(x_2)u + \Delta f_2(x_1,x_2) + \Delta d_2(x_2)\theta\right|\right) - \left\|P(t)\tilde{\theta}\right\| \tag{27}$$

If α_2 is large enough, $\alpha_2 - \left|\Delta g(x_2)u + \Delta f_2(x_1,x_2) + \Delta d_2(x_2)\theta\right| = \eta_1 > 0$, (27) becomes:

$$\dot{V}_2 \le -\eta_1|\tilde{x}_2| - \left\|P(t)\tilde{\theta}\right\| \tag{28}$$

$$\dot{V}_2 \le -\mu_2\sqrt{\frac{1}{2}\tilde{x}_2^2} - \mu_1\sqrt{\frac{1}{2}\tilde{\theta}^T\tilde{\theta}} \tag{29}$$

where $\mu_2 = \sqrt{2}\eta_1$. Let $\mu = \min\{\mu_1,\mu_2\}$ and use Lemma 3 again:

$$\dot{V}_2 \le -\mu\left(\frac{1}{2}\tilde{x}_2^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta}\right)^{\frac{1}{2}} = -\mu V^{\frac{1}{2}} \tag{30}$$

According to the finite time stability principle, $\tilde{x}_2$ and $\tilde{\theta}$ will be zero after $t \ge t_{s2}$, $t_{s2} = \frac{V^{0.5}(0)}{2\mu}$.

As $t \ge t_{s2}$, the following equation holds:

$$\Delta g(x_2)u + \Delta f_2(x_1,x_2) + d(\hat{x}_2)\tilde{\theta} + \Delta d_2(x_2)\theta - \alpha_2\,\mathrm{sgn}(y - \hat{x}_2) = 0 \tag{31}$$

Note that, as $t \ge t_{s2}$, $\tilde{\theta} = 0$, $\Delta g(x_2) = 0$, $\Delta d_2(x_2) = 0$, then

$$\Delta f_2(x_1,x_2) = \left[\alpha_1\,\mathrm{sgn}(y - \hat{x}_1)\right]_{eq} \tag{32}$$

After $t \ge t_{s2}$, $\Delta f_2(x_1,x_2) = f_2(x_1,\hat{x}_2) - f_2(\hat{x}_1,\hat{x}_2)$, consider Assumption 6 and let:

$$\tilde{x}_1 = F\left(\left[\alpha_1\,\mathrm{sgn}(y - \hat{x}_1)\right]_{eq}\right) \tag{33}$$

where $F(\cdot)$ is a smooth function.

Step 2: To prove $\tilde{x}_1$ will be zero in finite time.

Defining $\bar{x}_1 = F\left(\left[\alpha_1\,\mathrm{sgn}(y - \hat{x}_1)\right]_{eq}\right) + \hat{x}_1$, and considering (23), the following Lyapunov function can be selected as:

$$V_3 = \frac{1}{2}\tilde{x}_1^2 \tag{34}$$

$$\dot{V}_3 = \tilde{x}_1\left(-\beta\tilde{x}_1 + \tilde{x}_2 + \Delta f_1(x_1,x_2) + d_1(x_1,x_2) - \alpha_1\,\mathrm{sgn}(\bar{x}_1 - \hat{x}_1)\right) \tag{35}$$

$$\dot{V}_3 \le -\alpha_1|\tilde{x}_1| - \beta|\tilde{x}_1| + |\tilde{x}_1|\left|\tilde{x}_2 + \Delta f_1(x_1,x_2) + d_1(x_1,x_2)\right| \tag{36}$$

$$\dot{V}_3 \le -|\tilde{x}_1|\left(\alpha_1 + \beta - \left|\tilde{x}_2 + \Delta f_1(x_1,x_2) + d_1(x_1,x_2)\right|\right) \tag{37}$$

If α_1 is large enough, there must be a sufficiently large η_2 such that:

$$\alpha_1 + \beta - \left|\tilde{x}_2 + \Delta f_1(x_1,x_2) + d_1(x_1,x_2)\right| > \eta_2 > 0$$

Then the following inequality is satisfied:

$$\dot{V}_3 \le -\eta_2|\tilde{x}_1| \tag{38}$$

It follows that $\tilde{x}_1 = 0$ for $t \ge t_{s3}$, $t_{s3} \le |\tilde{x}_1(0)|/\eta_2$. In light of the above, $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{\theta}$ converges to zero in finite time and the Theorem is proved.

IV. Finite time output feedback sliding mode control

In this section an output feedback Terminal Sliding Mode Control (TSMC) will be designed for (1) using the state estimates from the observer developed in the previous section. The following additional assumption is required on the boundedness of the desired trajectory.

Assumption 7: The desired trajectory $y_r, \dot{y}_r \in R$ is bounded.

By using estimated $\hat{x}_2$, the tracking error is defined as:

$$\hat{e}_2 = \hat{x}_2 - y_r \tag{39}$$

A corresponding sign integral terminal sliding mode is defined by:

$$\hat{s} = \hat{e}_2 + \lambda\hat{e}_{2I}$$
$$\dot{\hat{e}}_{2I} = \mathrm{sgn}(\hat{e}_2), \quad \hat{e}_{2I}(0) = -\frac{\hat{e}_2(0)}{\lambda} \tag{40}$$

where $\lambda > 0$, as $t_{s4} \ge \frac{|\hat{e}_2(0)|}{\lambda}$, $e_2 = e_{2I} = 0$ [27]. Using (21) it follows that:

$$\dot{\hat{s}} = g(\hat{x}_2)u + f_2(\hat{x}_1,\hat{x}_2) + d_2(\hat{x}_2)\hat{\theta} + \alpha_2\,\mathrm{sgn}(y - \hat{x}_2) - \dot{y}_r + \lambda\,\mathrm{sgn}(\hat{e}_2) \tag{41}$$

In terms of (36), the terminal sliding mode control can be defined as:

$$u_{eq} = g^{-1}(\hat{x}_2)\left[-f_2(\hat{x}_1,\hat{x}_2) - d_2(\hat{x}_2)\hat{\theta} - \alpha_2\,\mathrm{sgn}(y - \hat{x}_2) + \dot{y}_r - \lambda\,\mathrm{sgn}(\hat{e}_2)\right]$$
$$u_s = -g^{-1}(\hat{x}_2)K\,\mathrm{sgn}(\hat{s}) \tag{42}$$
$$u = u_{eq} + u_s$$

Theorem 3: If Assumptions 1-7 hold, the tracking error $e_2 = x_2 - y_r$ will be finite time stable and x_1 will be Lyapunov stable if the control law is designed as in (42).

Proof:

Selecting a Lyapunov function as:

$$V_4 = \frac{1}{2}\hat{s}^2 \tag{43}$$

$$\dot{V}_4 = \hat{s}\left[g(\hat{x}_2)u + f_2(\hat{x}_1,\hat{x}_2) + d_2(\hat{x}_2)\hat{\theta} + \alpha_2\,\mathrm{sgn}(y - \hat{x}_2) - \dot{y}_r + \lambda\,\mathrm{sgn}(\hat{e}_2)\right] \tag{44}$$

Substituting (42) into (41):

$$\dot{V}_4 = -K|\hat{s}| \tag{45}$$

From (45), $\hat{e}_2$ will reach $\hat{s}$ in finite time $t_{s5} \ge \frac{|\hat{s}(0)|}{K}$, then $\hat{e}_2$ will converge to zero in finite time along $\hat{s}$ after time $t = t_{s4} + t_{s5}$. It should be noted that, if $\hat{x}_1$ and $\hat{x}_2$ converge to x_1 and x_2, the control law should be:

$$\bar{u}_{eq} = g^{-1}(x_2)\left[-f_2(x_1,x_2) - d_2(x_2)\theta + \dot{y}_r - \lambda \operatorname{sgn}(e_2)\right]$$
$$\bar{u}_s = -g^{-1}(x_2) K \operatorname{sgn}(s) \tag{46}$$
$$\bar{u} = \bar{u}_{eq} + \bar{u}_s$$

where

$$e_2 = x_2 - y_r$$
$$s = e_2 + \lambda e_{2I}$$
$$\dot{e}_{2I} = \operatorname{sgn}(e_2), \quad e_{2I}(0) = -\frac{e_2(0)}{\lambda}$$

Substituting (46) into (1):

$$\dot{e}_2 = -\operatorname{sgn}(e_2) \tag{47}$$

It is obvious that e_2 will be zero in finite time. If x_2 converges to y_r in finite time, the dynamics of x_1 will be:

$$\dot{x}_1 = -\beta x_1 + y_r + f_1(x_1, y_r) + d_1(x_1, y_r) \tag{48}$$

Selecting a Lyapunov function candidate for (40) as:

$$V_5 = \frac{1}{2} x_1^2 \tag{49}$$
$$\dot{V}_5 = x_1 \dot{x}_1 = -\beta x_1^2 + x_1 y_r + x_1 f_1(x_1, y_r) + x_1 d_1(x_1, x_2) \tag{50}$$
$$\dot{V}_5 \le -\beta x_1^2 + |x_1||y_r| + |x_1||f_1(x_1, y_r)| + |x_1||d_1(x_1, x_2)| \tag{51}$$

Note that $|y_r|$, $|f_1(x_1,y_r)|$ and $|d_1(x_1,x_2)|$ are all bounded, let that $\left(|y_r| + |f_1(x_1,y_r)| + |d_1(x_1,x_2)|\right) \le \eta_3$, $\eta_3 > 0$ is a constant.

$$\dot{V}_5 \le -|x_1|\left[\beta |x_1| - \eta_3\right] \tag{52}$$

Then, x_1 will converge to a residual set $\Omega = \left\{ x_1 \middle| |x_1| \le \frac{\eta_3}{\beta} \right\}$ and x_1 is Lyapunov stable.

V. CASE STUDY: CONTINUOUSLY STIRRED TANK REACTOR

In this section, a CSTR system is used to illustrate the proposed approach. The dimensionless dynamic equations of the CSTR are taken from [28]:

$$\dot{x}_1 = -x_1 + D_a(1 - x_1)e^{x_2/(x_2/\gamma+1)} - d_1$$
$$\dot{x}_2 = -x_2 + BD_a(1 - x_1)e^{x_2/(x_2/\gamma+1)} - \beta(x_2 - x_{2c}) + \beta u + \theta x_2$$
$$y = x_2$$

where $x_1, x_2 \in R$ are the states, $y \in R$ is the system output which represents the dimensionless temperature, $d_1 \in R$ is the external disturbance, $d_2 = \theta x_2$ in which θ_2 is the heat transfer coefficient. The parameters are set as: $B = 8$, $\beta = 0.3$, $\gamma = 20$, $D_a = 0.078$, $x_{2c} = 0$, $d_1 = 0.01$, $\theta = 0.04$. The desired trajectory of the system output is assumed to be:

$$y_r = x_{2s}\left(1 - k_1 e^{-k_2 t}\right)$$

where $x_{2s} = 2.7517$, $k_1 = 1$ and $k_2 = 1$. The controller parameters are chosen as: $k = 100$, $l = 1$, $\Gamma = 5$, $\alpha_1 = 0.5$, $\alpha_2 = 0.5$, $\lambda = 0.2$, $K = 0.2$.

Figure 1 shows the evolution of the parameter estimates where $\hat{\theta}$ converges to θ in finite time. Because of the finite time parameter estimation, $\bar{x}_1$ can be obtained from the equivalent injection principle and the state observer stabilizes $\hat{x}_1$ and $\hat{x}_2$ to x_1 and x_2 in finite time as shown in Figures 2-3. To avoid chattering, a saturation function $y = x/(|x| + \delta)$ is used instead of the sign function $y = \operatorname{sgn}(x)$, where $\delta > 0$ is a small constant. By using the saturation function, the control input is smooth (Figure 4).

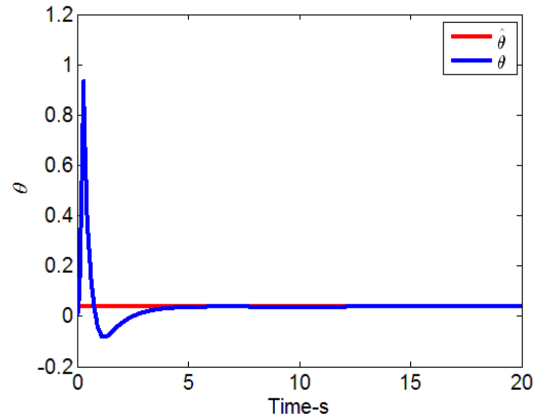

Figure 1 Parameter estimation

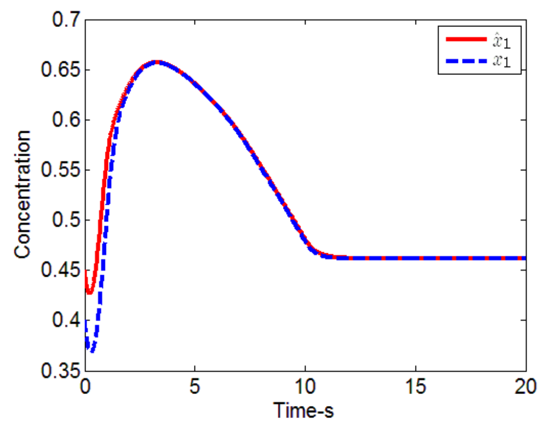

Figure 2 Concentration and its estimation

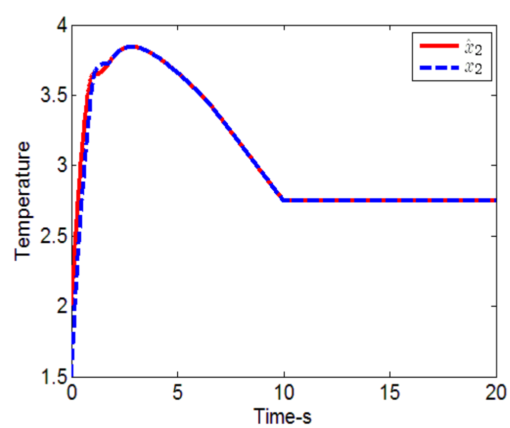

Figure 3 Temperature and its estimation

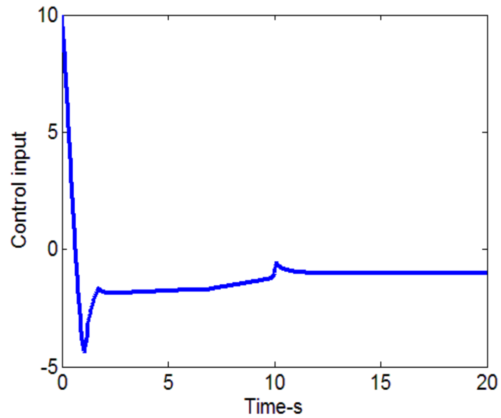

Figure 4 Control input

VI. Conclusion

The problem of finite time output feedback control for a class of second order nonlinear systems has been considered. The assumed system representation includes uncertainty in the parameters as well as model uncertainty. An adaptive finite time parameter estimator is first developed to estimate the unknown parameters. This is shown to facilitate finite time state observer design. Finally, a terminal sliding mode control is developed. The design procedure is straightforward and constructive. The proposed approach is validated by using simulation of CSTR system. Future work will involve practical implementation of the proposed control strategy.

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