REGULARIZATION STRATEGY FOR INVERSE PROBLEM FOR 1+1 DIMENSIONAL WAVE EQUATION

JUSSI KORPELA, MATTI LASSAS AND LAURI OKSANEN

ABSTRACT. An inverse boundary value problem for a 1+1 dimensional wave equation with wave speed $c(x)$ is considered. We give a regularisation strategy for inverting the map $\mathcal{A} : c \mapsto \Lambda$, where Λ is the hyperbolic Neumann-to-Dirichlet map corresponding to the wave speed c. That is, we consider the case when we are given a perturbation of the Neumann-to-Dirichlet map $\widetilde{\Lambda} = \Lambda + \mathcal{E}$, where $\mathcal{E}$ corresponds to the measurement errors, and reconstruct an approximative wave speed $\widetilde{c}$. We emphasize that $\widetilde{\Lambda}$ may not be in the range of the map $\mathcal{A}$. We show that the reconstructed wave speed $\widetilde{c}$ satisfies $\|\widetilde{c} - c\| \leq C\|\mathcal{E}\|^{1/54}$. Our regularisation strategy is based on a new formula to compute c from Λ.
Keywords: Inverse problem, regularization theory, wave equation.

1. INTRODUCTION

We consider an inverse boundary value problem for the wave equation

$$(\frac{\partial^2}{\partial t^2} - c(x)^2 \frac{\partial^2}{\partial x^2})u(t,x) = 0,$$

and introduce a regularization strategy to recover the sound speed $c(x)$ by using the knowledge of perturbed Neumann-to-Dirichlet map $\widetilde{\Lambda}$. Our approach is based on the Boundary Control method [2, 6, 54].

A variant of the Boundary Control method, called the iterative time-reversal control method, was introduced in [9]. The method was later modified in [15] to focus the energy of a wave at a fixed time, and in [47] to solve an inverse obstacle problem for the wave equation. Here we introduce yet another modification of the iterative time-reversal control method that is tailored for the 1+1 dimensional wave equation.

Classical regularization theory is explained in [16]. Iterative regularization of both linear and nonlinear inverse problems and convergence rates are discussed in Hilbert space setting in [10, 18, 20, 41, 43] and in Banach space setting in [19, 23, 24, 30, 48, 49, 50]. Our new results

Date: Feb. 25, 2016.

give a direct regularization method for the nonlinear inverse problem for the wave equation. The result contains an explicit (but not necessarily optimal) convergence rate.

By direct methods for non-linear problems we mean explicit construction of a non-linear map solving the problem without resorting to a local optimisation method. In our case the map is given by (63) below. The advantage of direct approaches is that they do not suffer from the possibility that the algorithm converges to a local minimum. In particular, they do not require a priori knowledge that the solution is in a small neighbourhood of a given function. There are currently only few regularized direct methods for non-linear inverse problems. An example is a regularisation algorithm for the inverse problem for the conductivity equation in [31]. Also, a direct regularized inversion for blind deconvolution is presented in [21].

1.1. **Statement of the results.** We define

$$\|c\|_{C^k(M)} = \sum_{p=0}^{k} \sup_{x\in(0,\infty)} |\frac{\partial^p c}{\partial x^p}(x)|, \tag{1}$$

where we denote by M the half axis $M = [0,\infty) \subset \mathbb{R}$. We denote the set of bounded $C^k(M)$-functions by

$$C_b^k(M) = \{c \in C^k(M); \|c\|_{C^k(M)} < \infty\}.$$

Let $C_0, C_1, L_0, L_1, m > 0$ and define the space of k times differentiable velocity functions

$$\begin{aligned} \mathcal{V}^k =&\{c \in C^k(M); C_0 \le c(x) \le C_1, \\ &\|c\|_{C^k(M)} \le m,\ c - 1 \in C_0^k([L_0, L_1])\}. \end{aligned} \tag{2}$$

Here $C_0^k([L_0, L_1])$ is the subspace of functions in $C_b^k(M)$ that are supported on $[L_0, L_1]$. Let

$$T > \frac{L_1}{C_0}. \tag{3}$$

For $c \in \mathcal{V}^2$ and $f \in L^2(0, 2T)$, the boundary value problem

$$\begin{aligned} &(\frac{\partial^2}{\partial t^2} - c(x)^2\frac{\partial^2}{\partial x^2})u(x,t) = 0 \quad \text{in } M \times (0, 2T), \\ &\partial_x u(0,t) = f(t), \\ &u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \end{aligned} \tag{4}$$

has a unique solution $u = u^f \in H^1(M \times (0, 2T))$. Using this solution we define the Neumann-to-Dirichlet operator $\Lambda = \Lambda_c$,

$$\Lambda : L^2(0, 2T) \to L^2(0, 2T), \quad \Lambda f = u^f|_{x=0}. \tag{5}$$

We define for a Banach space E

$$\mathcal{L}(E) := \{A : E \to E; A \text{ is linear and continuous}\}.$$

Let $X = L^\infty(M)$, $Z = C_b^2(M)$ and $Y = \mathcal{L}(L^2(0, 2T))$. We define $\mathcal{D}(\mathcal{A}) = \mathcal{V}^2$ and the direct map

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to \mathcal{R}(\mathcal{A}) \subset Y, \quad \mathcal{A}(c) = \Lambda. \tag{6}$$

The notation in (6) means that the range $\mathcal{R}(\mathcal{A}) = \mathcal{A}(\mathcal{V}^2)$ and the domain $\mathcal{D}(\mathcal{A})$ are equipped with the topologies of Y and Z, respectively. We show in Appendix A, Theorem 5, that the maps (5) and (6) are continuous.

We consider the inverse problem to recover the velocity function c by using the boundary measurements Λ. In our case, it is well-known that $\mathcal{A}$ is invertible, see e.g. [12, 13]. Let us record the following:

Theorem 1. $\mathcal{A}$ *is invertible, that is, there exist a map*

$$\mathcal{A}^{-1} : \mathcal{A}(\mathcal{V}^2) \subset Y \to \mathcal{V}^2 \subset Z, \quad \mathcal{A}^{-1}(\Lambda) = c.$$

For the convenience of the reader we give a proof of Theorem 1 in Section 2, where we also give a new formula to compute c from Λ. When we restrict $\mathcal{A}$ to the set $\mathcal{V}^3 \subset \mathcal{V}^2$, the map $\mathcal{A}|_{\mathcal{V}^3} : \mathcal{V}^3 \subset Z \to \mathcal{A}(\mathcal{V}^3)$ has a continuous inverse operator in the following sense:

Theorem 2. *The inverse map*

$$\mathcal{A}^{-1} : \mathcal{A}(\mathcal{V}^3) \subset Y \to \mathcal{V}^3 \subset Z, \quad \mathcal{A}^{-1}(\Lambda) = c,$$

is continuous.

Below, we will prove a result for the continuity modulus of $\mathcal{A}^{-1}$. The continuity of $\mathcal{A}^{-1}$ in Theorem 2 is abstract in the sense that it does not contain quantitative estimates. For the convenience of the reader we give a proof of Theorem 2 in Appendix B.

Our main result concerns perturbations of the Neumann-to-Dirichlet operator of the form

$$\widetilde{\Lambda} = \Lambda + \mathcal{E}, \tag{7}$$

where $\mathcal{E} \in Y$ models the measurement error. We assume that $\|\mathcal{E}\|_Y \leq \epsilon$, where $\epsilon > 0$ is known. In this situation we can not use the map $\mathcal{A}^{-1}$ to calculate function c since $\widetilde{\Lambda}$ may not be in the range $\mathcal{R}(\mathcal{A})$. We recall the definition of a regularization strategy, see e.g. [16] and [30].

Definition 1. *Let Z, Y be Banach spaces and $\Omega \subset Z$. Let $\mathcal{A} : \Omega \subset Z \to Y$ be a continuous mapping. Let $\alpha_0 \in (0, \infty]$. A family of continuous maps $\mathcal{R}_\alpha : Y \to Z$ parametrized by $0 < \alpha < \alpha_0$ is called a regularization strategy for $\mathcal{A} : \Omega \to Y$ if*

$$\lim_{\alpha \to 0} \mathcal{R}_\alpha(\mathcal{A}(c)) = c$$

for every $c \in \Omega$. A regularization strategy is called admissible, if the parameter α is chosen as a function of $\epsilon > 0$ so that $\lim_{\epsilon \to 0} \alpha(\epsilon) = 0$ and for every $c \in \Omega$

$$\lim_{\epsilon \to 0} \sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \widetilde{\Lambda} - c \right\|_Z : \widetilde{\Lambda} \in Y, \left\| \widetilde{\Lambda} - \mathcal{A}(c) \right\|_Y \leq \epsilon \right\} = 0.$$

Below we will use Definition 1 for $\mathcal{A}$ given in (6) with $\Omega = \mathcal{V}^3$. Figure 1 gives a schematic illustration of regularization.

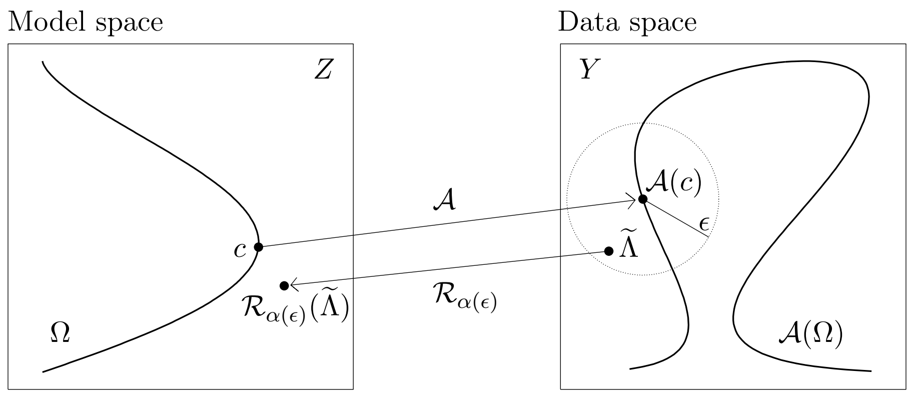

FIGURE 1. The idea of regularization is to construct a family $\mathcal{R}_{\alpha(\epsilon)}$ of continuous maps from the data space Y to the model space Z in such a way that c can be approximately recovered from noisy data $\widetilde{\Lambda}$. For a smaller noise level ϵ the approximation $\mathcal{R}_{\alpha(\epsilon)}(\widetilde{\Lambda})$ is closer to c. More details can be found in [44, Fig. 11.5].

We are now ready to formulate our main result:

Theorem 3. *Let $\beta = \frac{1}{54}$. For operator $\mathcal{A} : V^3 \subset Z \to Y$, there exists an admissible regularization strategy $\mathcal{R}_\alpha$ with the choice of parameter*

$$\alpha(\epsilon) = 2^{\frac{13}{9}} T^{\frac{4}{9}} \epsilon^{\frac{4}{9}}$$

that satisfies the following: For every $c \in \mathcal{V}^3$ there are ϵ_0 and $C > 0$ such that

$$\sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \widetilde{\Lambda} - c \right\|_Z : \widetilde{\Lambda} \in Y, \left\| \mathcal{A}(c) - \widetilde{\Lambda} \right\|_Y \leq \epsilon \right\} \leq C \epsilon^\beta,$$

for all $\epsilon \in (0, \epsilon_0)$.

We will give explicit choices of $\mathcal{R}_\alpha$ and ϵ_0, in formulas (62), (63), and (64) below. For the convenience of the reader we give a short summary on the regularization strategy. Assume that we are given $\widetilde{\Lambda} \in Y$, that is, the Neumann-to-Dirichlet map for the unknown wave speed $c(x)$ with measurements errors. Then the regularization strategy is obtainded by doing the following steps:

(1) Using (8) and (24) we calculate the operator $\widetilde{H}_r = P_r(R\widetilde{\Lambda}RJ - J\widetilde{\Lambda})P_r$ for $r \in [0, T]$. This operator determines approximately the inner products of the waves by $\langle u^{f_1}(T), u^{f_2}(T)\rangle_{L^2(M)} \approx \langle \widetilde{H}_r f_1, f_2\rangle_{L^2}$ for all boundary sources $f_1, f_2 \in L^2(T-r, T)$.
(2) Using operator $\widetilde{H}_r$ we construct in (34) a source $\widetilde{f}_{\alpha,r}$ that approximates the solution $f_{\alpha,r}$ of the minimization problem (13). Here, the source $f_{\alpha,r}$ produces a wave such that $u^{f_{\alpha,r}}(t,x)|_{t=T}$ is close to the indicator function $1_{M(r)}(x)$ of the domain of influence $M(r)$, see (11) and Figure 2.
(3) Using sources $\widetilde{f}_{\alpha,r}$ we compute approximately the volumes $V(r) = \mathrm{Vol}_c(M(r))$ of the domains of influences, see (20).
(4) Using finite differeces we compute approximate values of the derivative of the volume of the domain influences $\partial_r V(r)$, see (44).
(5) We interpolate the obtained values of $\partial_r V(r)$. This determines the approximate values of the wave speed $v(r)$ in the travel time coordinates, see (21).
(6) Finally, we change coordinates from the travel time coordinates to the Euclidean coordiantes to obtain the approximate values of the wave speed $c(x)$ for $x \in M$.

1.2. **Previous literature.** From the point of view of uniqueness questions, the inverse problem for the 1+1 dimensional wave equation is equivalent with the one dimensional inverse boundary spectral problem. The latter problem was thoroughly studied in 1950s [17, 32, 42] and we refer to [22, pp. 65-67] for a historical overview. In 1960s Blagoveščenskiĭ [12, 13] developed an approach to solve the inverse problem for the 1+1 dimensional wave equation without reducing the problem to the inverse boundary spectral problem. This and later dynamical methods have the advantage over spectral methods that they require data only on a finite time interval. Applications of 1-dimensional inverse probems have beed discussed widely in [11, 22, 26]

The method in the present paper is a variant of the Boundary Control method that was pioneered by M. Belishev [2] and developed by M.

Belishev and Y. Kurylev [5, 6] in late 80s and early 90s. Of crucial importance for the method was the result of D. Tataru [54] concerning a Holmgren-type uniqueness theorem for non-analytic coefficients. The Boundary Control method for multidimensional inverse problems has been summarized in [3, 26], and considered for 1+1 dimensional scalar problems in [4, 7] and for multidimensional scalar problems in [25, 28, 33, 36, 37]. For systems it has been considered in [34, 35]. Stability results for the method have been considered in [1] and [29].

The inverse problem for the wave equation can be solved also by using complex geometrical optics solutions. These solutions were developed in the context of elliptic inverse boundary value problems [53], and in [45] they were employed to solve an inverse boundary spectral problem. Local stability results can be proven using (real) geometrical optics solutions [8, 51, 52], and in [40] a stability result was proved by using ideas from the Boundary Control method together with complex geometrical optics solutions. Finally we mention the important method based on Carleman estimates [14] that can be used to show stability results when the initial data for the wave equation is non-vanishing.

2. Modification of the iterative time-reversal control method

In this section we prove Theorem 1 in such a way that we can utilize the proof to construct a regularization strategy as in Theorem 3. Let Λ be as defined in (5). Let $r \in [0, T]$. We define linear operators in Y by

$$\begin{aligned} &Jf(t) = \frac{1}{2}\int_0^{2T} 1_\triangle(t,s) f(s)\,ds, \\ &Rf(t) = f(2T - t), \quad K = R\Lambda R J - J\Lambda, \\ &Bf(t) = 1_{(0,T)}(t)\int_t^T f(s)\,ds, \quad P_r f(t) = 1_{(T-r,T)}(t) f(t), \end{aligned} \tag{8}$$

where

$$1_\triangle(t,s) = \begin{cases} 1, & t + s \le 2T \text{ and } s > t > 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$1_{(T-r,T)}(t) = \begin{cases} 1, & t \in (T - r, T), \\ 0, & \text{otherwise.} \end{cases}$$

Let $f \in L^2(0,2T)$. Using the solution $u^f \in H^1(M \times (0,2T))$ of (4) we define

$$U_T : L^2(0,2T) \mapsto H^1(M), \quad U_T f = u^f|_{t=T}. \tag{9}$$

We show in Appendix A, Theorem 5, that the map (9) is continuous. Let us denote $dV = c^{-2}dx$ and $u^f(T) = u^f|_{t=T}$. Let us recall the Blagovestchenskii identities

$$\begin{aligned} \langle u^f(T), 1\rangle_{L^2(M;dV)} &= \langle f, B1\rangle_{L^2(0,2T)}, \\ \langle u^f(T), u^h(T)\rangle_{L^2(M;dV)} &= \langle f, Kh\rangle_{L^2(0,2T)}. \end{aligned} \tag{10}$$

The identities (10) originate from [11], and their proofs can be found e.g. in [9]. We define the domain of influence

$$M(r) = \{x \in M; d(x,0) \le r\}, \tag{11}$$

where $d(x,0) = \int_0^x \frac{1}{c(t)}dt$ is the travel time of the waves from 0 to the point x. See Figure 2 for a visualization of $M(r)$.

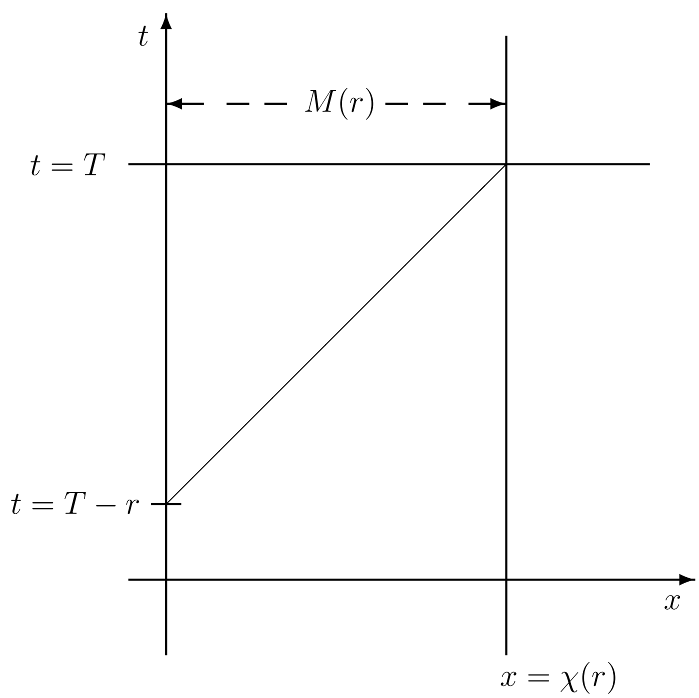

FIGURE 2. When the boundary source f satisfies, $\mathrm{supp}(f) \subset [T-r, T]$, the solution $u^f(t,x)|_{t=T}$ at time T is supported in the domain of influence $M(r)$.

We use the following result that is closely related to [9], [46].

Theorem 4. *Let $r \in [0,T]$ and $\alpha > 0$. Let K, B, and P_r be as defined in (8). Let us define*

$$S_r = \{f \in L^2(0,2T) \,:\, \mathrm{supp}(f) \subset [T-r,T]\}. \tag{12}$$

Then the regularized minimization problem

$$\min_{f\in S_r} \left(\langle f, Kf\rangle_{L^2(0,2T)} - 2\langle f, B1\rangle_{L^2(0,2T)} + \alpha \|f\|^2_{L^2(0,2T)} \right), \tag{13}$$

has unique minimizer

$$f_{\alpha,r} = (P_r K P_r + \alpha)^{-1} P_r B1 \tag{14}$$

and the map $r \mapsto f_{\alpha,r}$ is continuous $[0,T] \to L^2(0,2T)$. Moreover $u^{f_{\alpha,r}}(T)$ converges to the indicator function of the domain of influence,

$$\lim_{\alpha\to 0} \left\| u^{f_{\alpha,r}}(T) - 1_{M(r)} \right\|_{L^2(M;dV)} = 0. \tag{15}$$

For the convenience of the reader we give a proof.

Proof of Theorem 4. Let $\alpha > 0$ and let $f \in S_r$. We define the energy function

$$E(f) := \langle f, Kf\rangle_{L^2(0,2T)} - 2\langle f, B1\rangle_{L^2(0,2T)} + \alpha \|f\|^2_{L^2(0,2T)}. \tag{16}$$

The finite speed of wave propagation implies $\mathrm{supp}(u^f(T)) \subset M(r)$. Using (10) we can write

(17)
$$E(f) = \left\| u^f(T) - 1_{M(r)} \right\|^2_{L^2(M;dV)} - \left\| 1_{M(r)} \right\|^2_{L^2(M;dV)} + \alpha \|f\|^2_{L^2(0,2T))}.$$

Let $(f_j)_{j=1}^\infty \subset S_r$ be such that

$$\lim_{j\to\infty} E(f_j) = \inf_{f\in S_r} E(f).$$

Then

$$\alpha \|f_j\|_{L^2(0,2T))} \leq E(f_j) + \left\| 1_{M(r)} \right\|^2_{L^2(M;dV)},$$

and we see that $(f_j)_{j=1}^\infty$ is bounded in S_r. As S_r is a Hilbert space, there is a subsequence of $(f_j)_{j=1}^\infty$ converging weakly in S_r. Let us denote the limit by $f_\infty \in S_r$ and the subsequence still by $(f_j)_{j=1}^\infty$.

The map $U_T : L^2(0,2T) \to H^1(M)$, as defined in (9), is bounded. The embedding $I : H^1(M) \hookrightarrow L^2(M)$ is compact and thus $U_T : f \mapsto u^f(T)$ is a compact operator

$$U_T : L^2(0,2T) \to L^2(M).$$

Hence we have a subsequence $(f_j)_{j=1}^\infty$ for which $u^{f_j}(T) \to u^{f_\infty}(T)$ in $L^2(M)$ as $j \to \infty$. Moreover, the weak convergence implies

$$\|f_\infty\|_{L^2(0,2T))} \leq \liminf_{j\to\infty} \|f_j\|_{L^2(0,2T)}.$$

Hence

$$E(f_\infty) = \lim_{j\to\infty} \left\|u^{f_j}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} - \left\|1_{M(r)}\right\|^2_{L^2(M;dV)} + \alpha \left\|f_\infty\right\|^2_{L^2(0,2T)}$$
$$\leq \lim_{j\to\infty} \left\|u^{f_j}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} - \left\|1_{M(r)}\right\|^2_{L^2(M;dV)} + \alpha \liminf_{j\to\infty} \left\|f_j\right\|^2_{L^2(0,2T)}$$
$$= \liminf_{j\to\infty} E(f_j) = \inf_{f\in S_r} E(f),$$

and thus $f_\infty \in S_r$ is a minimizer for (16). We denote by D_h the Fréchet derivative to direction h. Note that $\inf_{f\in S_r} E(f) = \inf_{f\in L^2(0,2T)} E(P_r f)$.

If

$$0 = D_h E(f) = 2\langle h, P_r K P_r f\rangle_{L^2(0,2T)} - 2\langle h, P_r B1\rangle_{L^2(0,2T)} + 2\alpha\langle h, f\rangle_{L^2(0,2T)},$$

for all $h \in S_r \subset L^2(0, 2T)$, then

$$(P_r K P_r + \alpha) f = P_r B1.$$

Using (10) we have

$$\langle (P_r K P_r + \alpha) f, f\rangle_{L^2(0,2T)} = \langle u^{P_r f}(T), u^{P_r f}(T)\rangle_{L^2(M;dV)} + \langle \alpha f, f\rangle_{L^2(0,2T)}.$$

Operator $P_r K P_r + \alpha$ is coercive when $\alpha > 0$. The Lax-Milgram Theorem implies that it is invertible, and we have an expression for minimizer

$$f_{\alpha,r} := f_\infty = (P_r K P_r + \alpha)^{-1} P_r B1.$$

According to [54], see also [27], we know that

$$\{u^f(T) \in L^2(M(r));\ f \in S_r\}$$

is dense in $L^2(M(r))$. Let $\delta > 0$. For $\epsilon = \frac{\delta^2}{2}$, let us choose $f_\epsilon \in S_r$, $f_\epsilon \neq 0$ such that

$$\left\|u^{f_\epsilon}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} \leq \epsilon. \tag{18}$$

Using (17) we have

$$\left\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} \leq E(f_{\alpha,r}) + \left\|1_{M(r)}\right\|^2_{L^2(M;dV)}.$$

Because $E(f_{\alpha,r}) \leq E(f_\epsilon)$ we have

$$\left\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} \leq \left\|u^{f_\epsilon}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} + \alpha \left\|f_\epsilon\right\|^2.$$
$$\leq \epsilon + \alpha \left\|f_\epsilon\right\|^2.$$

When $0 < \alpha < \alpha_r = \frac{\delta^2}{2\|f_\epsilon\|^2}$, we see that

$$\left\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\right\|_{L^2(M;dV)} \leq (\epsilon + \alpha \left\|f_\epsilon\right\|^2)^{\frac{1}{2}} = \delta.$$

Thus

$$\lim_{\alpha\to 0} \left\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\right\|_{L^2(M;dV)} = 0.$$

□

We define the travel time coordinates for $x \in M$ by

$$\tau : [0,\infty) \to [0,\infty), \quad \tau(x) = d(x,0).$$

The function τ is strictly increasing and we denote its inverse by

$$\chi = \tau^{-1} : [0,\infty) \to [0,\infty).$$

We have

$$\chi(0) = 0, \quad \chi'(t) = \frac{1}{\tau'(\chi(t))} = c(\chi(t)). \tag{19}$$

Thus denoting $v(t) = c(\chi(t))$ and using $V(r)$ to denote the volume of $M(r)$ with respect to the measure dV we have

$$V(r) = \left\|1_{M(r)}\right\|^2_{L^2(M;dV)} = \int_0^{\chi(r)} \frac{dx}{c(x)^2} = \int_0^r \frac{\chi'(t)dt}{v(t)^2} = \int_0^r \frac{dt}{v(t)}. \tag{20}$$

Note that $M(r) = [0, \chi(r)]$. In particular, $V(r)$ determines the wave speed in the travel time coordinates,

$$v(r) = \frac{1}{\partial_r V(r)}, \tag{21}$$

and also in the original coordinates since

$$c(x) = v(\chi^{-1}(x)), \quad \chi(t) = \int_0^t v(t')dt'. \tag{22}$$

Using Theorem 4 and (10) we have a method to compute the volumes of the domains of influence

$$V(r) = \left\|1_{M(r)}\right\|^2_{L^2(M;dV)} = \lim_{\alpha\to 0}\langle f_{\alpha,r}, B1\rangle_{L^2(0,2T)}, \tag{23}$$

where $r \in [0,T]$. We are ready to prove Theorem 1.

Proof of Theorem 1. For a given measurement Λ, Theorem 4 and equations (21), (22), (23) give us a way to calculate for all $x \in (0,L)$ the value of the velocity function

$$c(x) = v(\chi^{-1}(x)) = \mathcal{A}^{-1}(\Lambda)(x).$$

As we assumed that outside of the interval $(0,L)$ the function c is identically one, the proof for the existence of inverse map $\mathcal{A}^{-1}$ is complete. □

3. Stability of regularized problem

In this section we prove Theorem 3. We will construct the operator $\mathcal{R}_{\alpha(\epsilon)}$ as a composition of several operators. The construction is motivated by the proof of Theorem 1. We define for a Banach space E

$$\mathcal{K}(E) = \{A \in \mathcal{L}(E); A \text{ is compact}\}.$$

Let J, R be as defined in (8). Using (8) we see that $J \in \mathcal{K}(L^2(0,2T))$. We define

$$\boldsymbol{K} : Y \to \mathcal{K}(L^2(0,2T)), \quad \boldsymbol{K}\widetilde{\Lambda} = R\widetilde{\Lambda}RJ - J\widetilde{\Lambda}. \tag{24}$$
$$\boldsymbol{H} : Y \mapsto C([0,T],Y), \quad \boldsymbol{H}\widetilde{\Lambda} = r \mapsto P_r(\boldsymbol{K}\widetilde{\Lambda})P_r.$$

Proposition 1. *We have* $\|\boldsymbol{H}\|_{Y\to C([0,T],Y)} \le T$.

Proof. Let $r \in [0,T]$. We have estimates $\|P_r\|_Y \le 1$, $\|R\|_Y \le 1$, $\|J\|_Y \le \frac{T}{2}$, and

$$\left\|\boldsymbol{H}\widetilde{\Lambda}(r)\right\|_{\mathcal{L}(L^2(0,2T))} \le 2\,\|J\|_{\mathcal{L}(L^2(0,2T))} \left\|\widetilde{\Lambda}\right\|_{\mathcal{L}(L^2(0,2T))} \le T \left\|\widetilde{\Lambda}\right\|_{\mathcal{L}(L^2(0,2T))}.$$

Thus

$$\|\boldsymbol{H}\|_{Y\to L^\infty([0,T],Y)} \le T.$$

It remains to show that $r \mapsto \boldsymbol{H}\widetilde{\Lambda}(r)$ is continuous. Let us denote $\widetilde{K} = \boldsymbol{K}\widetilde{\Lambda}$. Let $r,s \in [0,T]$. We use the singular value decomposition for the compact operator $\widetilde{K}$. There are orthonormal bases $\{\phi_n\}_{n=1}^\infty \in L^2(0,2T)$ and $\{\psi_n\}_{n=1}^\infty \in L^2(0,2T)$ such that

$$\widetilde{K}f = \sum_{n=1}^{\infty} \mu_n \langle f, \phi_n\rangle_{L^2(0,2T)} \psi_n, \tag{25}$$

for all $f \in L^2(0,2T)$, where $\mu_n \in \mathbb{R}$ are the singular values of $\widetilde{K}$. We define the family $\{\widetilde{K}^m\}_{m=1}^\infty$ of finite rank operators by the formula

$$\widetilde{K}^m f = \sum_{n=1}^{m} \mu_n \langle f, \phi_n\rangle_{L^2(0,2T)} \psi_n. \tag{26}$$

Then

(27)

$$\begin{aligned}&\left\|P_r\widetilde{K}P_rf - P_s\widetilde{K}P_sf\right\|_{L^2(0,2T)}\\ &\le \left\|P_r\widetilde{K}P_rf - P_r\widetilde{K}^mP_rf\right\|_{L^2(0,2T)} + \left\|P_r\widetilde{K}^mP_rf - P_s\widetilde{K}^mP_rf\right\|_{L^2(0,2T)} +\\ &\quad \left\|P_s\widetilde{K}^mP_rf - P_s\widetilde{K}^mP_sf\right\|_{L^2(0,2T)} + \left\|P_s\widetilde{K}^mP_sf - P_s\widetilde{K}P_sf\right\|_{L^2(0,2T)}.\end{aligned}$$

Let $\epsilon > 0$ and let $\|f\|_{L^2(0,2T)} \leq 1$. By choosing m large enough we have

$$\left\|P_r\widetilde{K}P_rf - P_r\widetilde{K}^mP_rf\right\|_{L^2(0,2T)} + \left\|P_s\widetilde{K}^mP_sf - P_s\widetilde{K}P_sf\right\|_{L^2(0,2T)} \leq \frac{\epsilon}{2}.$$

Applying projections to (26) we see that

$$P_s\widetilde{K}^mP_rf = \sum_{n=1}^{m} \mu_n\langle f, P_r\phi_n\rangle P_s\psi_n.$$

For the second term in the sum (27) we have an estimate

$$\left\|P_r\widetilde{K}^mP_rf - P_s\widetilde{K}^mP_rf\right\|_{L^2(0,2T)} = \left\|\sum_{n=1}^{m} \mu_n\langle f, P_r\phi_n\rangle(P_r - P_s)\psi_n\right\|_{L^2(0,2T)}$$

$$\leq \sum_{n=1}^{m} |\mu_n|\, \|(P_r - P_s)\psi_n\|_{L^2(0,2T)} \leq C(m)|r-s|^{\frac{1}{2}}.$$

For the third term in the sum we have an analogous estimate

$$\left\|P_s\widetilde{K}^mP_rf - P_s\widetilde{K}^mP_sf\right\|_{L^2(0,2T)} \leq C(m)|r-s|^{\frac{1}{2}}.$$

Putting these estimates together and choosing $|r-s| \leq \delta(\epsilon) = \frac{\epsilon^2}{4C(m)^2}$, we see that

$$\left\|P_r\widetilde{K}P_r - P_s\widetilde{K}P_s\right\|_Y \leq \epsilon. \qquad \square$$

Let us define

$$M_1 = \sup\{\|\mathcal{A}(c)\|_{\mathcal{L}(L^2(0,2T))}\,; c \in \mathcal{V}^2\}. \tag{28}$$

Using the continuity of $\mathcal{A}$, see Theorem 5 below, we see that $M_1 < \infty$. We define $M_2 = 2TM_1$. Let $c \in \mathcal{V}^2$ and denote $\Lambda = \mathcal{A}(c)$. We use again the notations $H = \boldsymbol{H}\Lambda$, $\widetilde{H} = \boldsymbol{H}\widetilde{\Lambda}$ and $\widetilde{H}_r = \boldsymbol{H}\widetilde{\Lambda}(r)$. Using Proposition 1 we have

$$\|H\|_{C([0,T],Y)} \leq M_2. \tag{29}$$

We define $M_3 = M_2 + 3$ and a family $\{\Psi_\alpha^Z\}_{\alpha\in(0,2]} \in C(\mathbb{R})$ by

$$\Psi_\alpha^Z(s) = \begin{cases} 0, & \text{if } \ s > M_3 - \frac{\alpha}{4}, \\ -\frac{4}{\alpha}s + \frac{4M_3}{\alpha} - 1, & \text{if } \ s \in (M_3 - \frac{\alpha}{2}, M_3 - \frac{\alpha}{4}], \\ 1, & \text{if } \ s \leq M_3 - \frac{\alpha}{2}. \end{cases}$$

For $\alpha \in (0,2]$ we define

$$\boldsymbol{Z}_\alpha : C([0,T],Y) \to C([0,T],Y), \tag{30}$$

$$\boldsymbol{Z}_\alpha(\widetilde{H}) = r \mapsto \Psi_\alpha^Z\Big(\left\|M_3 - (\widetilde{H} + \alpha)\right\|_{C([0,T],Y)}\Big)\Big(\widetilde{H}_r + \alpha\Big)^{-1}.$$

Let E be a Banach space and let $H \in E$. Let $\epsilon > 0$. We denote

$$\mathcal{B}_E(H,\epsilon) := \{\widetilde{H} \in E : \left\| H - \widetilde{H} \right\|_E < \epsilon\}. \tag{31}$$

Proposition 2. *Let $\epsilon \in (0,1)$ and let $p \in (0, \frac{1}{2})$. Let $\alpha = 2\epsilon^p$ and let $\|H\|_{C([0,T],Y)} \le M_2$. Let $H_r \in Y$ be positive semidefinite. Let us assume that $\widetilde{H} \in \mathcal{B}_{C([0,T],Y)}(H,\epsilon)$. Then*

$$\left\| \boldsymbol{Z}_\alpha\big(H\big) - \boldsymbol{Z}_\alpha\big(\widetilde{H}\big) \right\|_{C([0,T],Y)} \le 2^{-1}\epsilon^{1-2p}.$$

Proof. By the definition (30) of Ψ^Z_α, we see that if

$$\Psi^Z_\alpha\Big(\left\| M_3 - (\widetilde{H} + \alpha) \right\|_{C([0,T],Y)} \Big) \neq 0,$$

then

$$\left\| M_3 - (\widetilde{H} + \alpha) \right\|_{C([0,T],Y)} \le M_3 - \frac{\alpha}{4} < M_3$$

and $\left(\widetilde{H}_r + \alpha\right)^{-1}$ is defined by the formula

$$\left(\widetilde{H}_r + \alpha\right)^{-1} = \frac{1}{M_3}\Big(I - \frac{M_3 - (\widetilde{H}_r + \alpha)}{M_3}\Big)^{-1} = \frac{1}{M_3}\sum_{l=1}^{\infty}\Big(\frac{M_3 - (\widetilde{H}_r + \alpha)}{M_3}\Big)^l.$$

This gives that $\boldsymbol{Z}_\alpha\big(\widetilde{H}\big)(r) \in Y$, when $r \in [0,T]$. Proposition 1 gives continuity for the map $r \mapsto \widetilde{H}_r$. As $\left(\widetilde{H}_r + \alpha\right) \mapsto \left(\widetilde{H}_r + \alpha\right)^{-1}$ is continuous operation we see that $\boldsymbol{Z}_\alpha\big(\widetilde{H}\big) \in C([0,T],Y)$. It remains to show that the norm estimate holds. By assumption, $H_r = H(r)$ is positive semidefinite, that is, $H_r : L^2(0,2T) \to L^2(0,2T)$ is selfadjoint and $H_r \ge 0$. Also, $\|H_r\|_Y \le M_2$. Thus $0 \le H_r \le M_2$ and as $M_3 = M_2 + 3$ and $0 \le \alpha \le 2$, we have $0 \le M_2 - H_r \le M_2$. Thus

$$I \le M_3 - \alpha I - H_r \le M_3 - \alpha I.$$

Hence $\|(M_3 - \alpha)I - H\|_{C([0,T],Y)} \le M_3 - \alpha$. As $\|H - \widetilde{H}\|_{C([0,T],Y)} \le \epsilon \le \frac{\alpha}{2}$, we have

$$\left\| M_3 - (\widetilde{H} + \alpha) \right\|_{C([0,T],Y)} \le M_3 - \frac{\alpha}{2}.$$

Thus $\Psi^Z_\alpha(\|M_3 - (\widetilde{H} + \alpha)\|_{C([0,T],Y)}) = 1$ and $\boldsymbol{Z}_\alpha\big(\widetilde{H}\big)$ is the map

$$r \mapsto \left(\widetilde{H}_r + \alpha\right)^{-1}.$$

Let $r \in [0,T]$. We denote

$$H_{\alpha,r} = (H_r + \alpha), \quad \widetilde{H}_{\alpha,r} = (\widetilde{H_r + \alpha}), \quad E = \widetilde{H}_{\alpha,r} - H_{\alpha,r}.$$

As H_r is positive semidefinite we have

$$\left\| H_{\alpha,r}^{-1} \right\|_Y \leq \alpha^{-1}. \tag{32}$$

Moreover

$$\widetilde{H}_{\alpha,r}^{-1} - H_{\alpha,r}^{-1} = \left(\left[I + H_{\alpha,r}^{-1} E \right]^{-1} - I \right) H_{\alpha,r}^{-1}.$$

Thus

$$\left\| (\widetilde{H}_{\alpha,r})^{-1} - (H_{\alpha,r})^{-1} \right\|_Y \leq \frac{\|(H_{\alpha,r})^{-1} E\|_Y}{1 - \|(H_{\alpha,r})^{-1} E\|_Y} \left\| (H_{\alpha,r})^{-1} \right\|_Y. \tag{33}$$

We have $\frac{1}{2} \geq \frac{\epsilon}{\alpha}$. Using (32) and (33) we have

$$\left\| (\widetilde{H}_{\alpha,r})^{-1} - (H_{\alpha,r})^{-1} \right\|_Y \leq \frac{\epsilon \alpha^{-1}}{1 - \frac{1}{2}} \left\| (H_{\alpha,r})^{-1} \right\|_Y \leq 2 \frac{\epsilon}{\alpha^2} = 2^{-1} \epsilon^{1-2p}. \quad \square$$

Let P_r and B be as defined in (8). We define

$$\begin{aligned} &\boldsymbol{S} : C([0,T], Y) \to C([0,T]), \\ &\boldsymbol{S}\big(\widetilde{Z}_\alpha\big)(r) = \langle \widetilde{Z}_\alpha(r) P_r B1, B1 \rangle_{L^2(0,2T)}, \\ &\widetilde{f}_{\alpha,r} = \widetilde{Z}_\alpha(r) P_r B1. \end{aligned} \tag{34}$$

Proposition 3. *We have* $\|\boldsymbol{S}\|_{C([0,T])} \leq \frac{T^3}{3}$.

Proof. As the maps $r \mapsto P_r B1$ and $r \mapsto Z_\alpha(r)$ are continuous, we have that $\boldsymbol{S}\big(\widetilde{Z}_\alpha\big) \in C([0,T])$. Let $r \in [0,T]$. We have

$$\|P_r\|_Y \leq 1, \quad \|B1\|_{L^2(0,2T)}^2 = \frac{T^3}{3},$$

and therefore

$$|\boldsymbol{S}\big(\widetilde{Z}_\alpha\big)(r)| = |\langle \widetilde{Z}_\alpha(r) P_r B1, B1 \rangle_{L^2(0,2T)}| \leq \frac{T^3}{3} \left\| \widetilde{Z}_\alpha(r) \right\|_Y. \qquad \square$$

Lemma 1. *Let $c \in \mathcal{V}^2$. There is $C > 0$ such that for all $r > 0$ and $p \in H^1(M)$ satisfying supp$(p) \subset M(r)$ there is $f \in S_r$ such that $u^f(x,T) = p(x)$ and*

$$\|f\|_{L^2(0,2T)} \leq C \|p\|_{H^1(M)}. \tag{35}$$

We recall that $M(r)$ is defined by (11) and S_r is defined by (12). We note that in the study of multidimensional inverse problem, estimate (35) need to be replaced by the Tataru inequality [54], [**?**] (see also [**?**], [**?**]), that is significantly weaker than (35). This is one of the key differences between one and multidimensional case.

Proof. Let us consider the wave equation with time and space having the exchanged roles

$$\begin{aligned}(36)\qquad &(\partial_x^2 - c(x)^{-2}\partial_t^2)\widetilde{u}(x,t) = 0, && (x,t) \in (0,\chi(T)) \times (0,T),\\ &\widetilde{u}(x,T) = p(x), && x \in [0,\chi(T)],\\ &\widetilde{u}(\chi(T),t) = \partial_x \widetilde{u}(\chi(T),t) = 0, && t \in (0,T).\end{aligned}$$

By [38] and [39] the solution of (36) satisfies

$$\|\widetilde{u}(0,\cdot)\|_{H^1(0,T)} \leqslant C\,\|p\|_{H^1(M(T))}\,. \tag{37}$$

If $supp(p) \subset M(r)$ then $supp(\widetilde{u}(0,\cdot)) \subset [T-r,T]$ and $\widetilde{u}(x,0) = \partial_t \widetilde{u}(x,0) = 0$, when $x \in [0,\chi(T)]$, by finite speed of propagation. We choose $f(t) = \widetilde{u}(0,t)$. □

Let $f_{\alpha,r}$ be as in (14) and define

$$s_\alpha \in C([0,T]), \quad s_\alpha(r) := \langle f_{\alpha,r}, B1\rangle_{L^2(0,2T)}. \tag{38}$$

Lemma 2. *Let $\alpha \in (0, \min(1, \frac{1}{\chi(T)^2}))$. Let V be as defined in (20). Then there is $C > 0$, independent α, such that*

$$\|s_\alpha - V\|_{C([0,T])} \leq C\alpha^{\frac{1}{4}}.$$

Proof. Let $r \in [0,T]$ and $\delta > 0$. Let us define $w_\delta \in H^1(M)$

$$w_\delta(x) = \begin{cases} 1, & \text{if} \quad x \in (0,\chi(r)),\\ 1 - \frac{x-\chi(r)}{\delta}, & \text{if} \quad x \in [\chi(r),\chi(r)+\delta],\\ 0, & \text{if} \quad x \in (\chi(r)+\delta,\infty).\end{cases}$$

Using $c(x) > C_0$ we have

$$\left\|w_\delta - 1_{M(r)}\right\|^2_{L^2(M;dV)} \leq \frac{\delta}{3C_0^2}. \tag{39}$$

When $\delta \in (0, \min(1, \frac{1}{\chi(T)}))$ we have

$$\|w_\delta\|^2_{H^1(M)} \leq \chi(T) + \frac{\delta}{3} + \frac{1}{\delta} \leq \frac{3}{\delta}. \tag{40}$$

Below $C > 0$ denotes a constant that may grow between inequalities, and that depends only on m, C_0, C_1, L_1. Lemma 1 gives us f_δ for which $u^{f_\delta}(x,T) = w_\delta(x)$. Thus (40) implies

$$\|f_\delta\|_{L^2(0,2T)} \leq C\,\|w_\delta\|_{H^1(M)} \leq \frac{C}{\delta^{\frac{1}{2}}}. \tag{41}$$

Let $f \in S_r$. We define

$$G_{\alpha,r}(f) = \left\|u^f(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} + \alpha\,\|f\|^2_{L^2(0,2T)}\,. \tag{42}$$

Using (39) and (41) we have

$$G_{\alpha,r}(f_\delta) = \left\|w_\delta - 1_{M(r)}\right\|^2_{L^2(M;dV)} + \alpha \left\|f_\delta\right\|^2_{L^2(0,2T)} \leq \frac{\delta}{C} + \alpha\frac{C}{\delta}. \tag{43}$$

Functional (42) and the functional defined in Theorem 4 have the same minimizer $f_{\alpha,r}$. Using (10), (23), and (38) we have

$$\begin{aligned}
\|s_\alpha - V\|^2_{C([0,T])} &= \sup_{r\in[0,T]} \left|\langle f_{\alpha,r}, B1\rangle_{L^2([0,2T])} - V(r)\right|^2. \\
&= \sup_{r\in[0,T]} \left|\langle u^{f_{\alpha,r}}(T), 1\rangle_{L^2(M;dV)} - \langle 1_{M(r)}, 1\rangle_{L^2(M;dV)}\right|^2 \\
&\leq C \sup_{r\in[0,T]} \left\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\right\|^2_{L^2(M;dV)} \leq C \sup_{r\in[0,T]} G_{\alpha,r}(f_{\alpha,r}) \\
&\leq C \sup_{r\in[0,T]} G_{\alpha,r}(f_\delta)
\end{aligned}$$

Using (43) and choosing $\delta = \alpha^{\frac{1}{2}}$ we have

$$\|s_\alpha - V\|^2_{C([0,T])} \leq C\alpha^{\frac{1}{2}}.$$

□

Lemma 3. *There is $\widetilde{m} > 0$ such that following holds: When $c \in \mathcal{V}^2$, the functions v and V, defined in (21) and (20), satisfy*

$$\|v\|_{C^2([0,T])} \leq \widetilde{m} \quad \text{and} \quad \|V\|_{C^3([0,T])} \leq \widetilde{m}.$$

Proof. Equations (19), (20), (21), and (22) with the chain rule and the formula for the derivatives of inverse functions give us the result. □

For small $h > 0$ we consider the partition

$$(0, T) = (0, h) \cup [h, 2h) \cup [2h, 3h) \cup ... \cup [Nh - h, Nh) \cup [Nh, T),$$

where $N \in \mathbb{N}$ satisfies $T - h \leq Nh < T$. We define a discretized and regularized approximation of the derivative operator ∂_r by

$$D_h : C([0, T]) \to L^\infty(0, T), \tag{44}$$

$$D_h(\widetilde{s}_\alpha)(r) = \begin{cases} \frac{\widetilde{s}_\alpha(h)}{h}, & \text{if} \quad r \in (0, h), \\ \frac{\widetilde{s}_\alpha(jh+h)-\widetilde{s}_\alpha(jh)}{h}, & \text{if} \quad r \in [jh, jh + h), \\ \frac{\widetilde{s}_\alpha(T)-\widetilde{s}_\alpha(Nh)}{h}, & \text{if} \quad r \in [Nh, T). \end{cases}$$

Proposition 4. *Let $\beta > 0$ and $\epsilon \in (0, \min(\frac{1}{\beta^{\frac{1}{4}}}, \frac{1}{\beta^{\frac{1}{4}}\chi(T)^{\frac{1}{2}}})$. Let $\alpha = \beta\epsilon^4$, $h = \epsilon^{\frac{1}{2}}$, V be as defined in (20) and let s_α be as defined in (38). Let us assume that $\widetilde{s}_\alpha \in \mathcal{B}_{C([0,T])}(s_\alpha, \epsilon)$. Then*

$$\|D_h(\widetilde{s}_\alpha) - \partial_r V\|_{L^\infty(0,T)} \leq C\epsilon^{\frac{1}{2}},$$

where C is independent of α and $\widetilde{s}_\alpha$.

Proof. Let $r \in [jh, jh+h)$. Using the definition of $D_h(\widetilde{s}_\alpha)$ (44) we have

$$\begin{aligned}\Big|D_h(\widetilde{s}_\alpha)(r) - \partial_r V(r)\Big| &= \Big|\frac{\widetilde{s}_\alpha(jh+h) - \widetilde{s}_\alpha(jh)}{h} - \partial_r V(r)\Big| \\ \leq \Big|\frac{\widetilde{s}_\alpha(jh+h) - s_\alpha(jh+h)}{h}\Big| &+ \Big|\frac{s_\alpha(jh) - \widetilde{s}_\alpha(jh)}{h}\Big| \\ + \Big|\frac{s_\alpha(jh+h) - V(jh+h)}{h}\Big| &+ \Big|\frac{V(jh) - s_\alpha(jh)}{h}\Big| \\ + \Big|\frac{V(jh+h) - V(jh)}{h} &- \partial_r V(r)\Big|.\end{aligned}$$

Lemma 3 gives us $\|V\|_{C^3([0,T])} \leq \widetilde{m}$. When $r \in [jh, jh+h)$ there is $\xi \in (jh, jh+h)$ such that

$$\Big|\frac{V(jh+h) - V(jh)}{h} - \partial_r V(r)\Big| = \Big|\partial_r V(\xi) - \partial_r V(r)\Big| \leq h\widetilde{m}. \tag{45}$$

Using (45) and Lemma 2 with assumption we get

$$\Big|D_h(\widetilde{s}_\alpha)(r) - \partial_r V(r)\Big| \leq \frac{2\epsilon}{h} + \frac{2C\alpha^{\frac{1}{4}}}{h} + h\widetilde{m}.$$

Let us choose $h = \epsilon^{\frac{1}{2}}$ and $\alpha = \beta\epsilon^4$. Then

$$\Big|D_h(\widetilde{s}_\alpha)(r) - \partial_r V(r)\Big| \leq C\epsilon^{\frac{1}{2}}. \tag{46}$$

The proof is almost identical when $r \in (0, h)$ or $r \in [Nh, T)$. Note that the right hand side of (46) is independent of r.

□

Let C_0 and C_1 be as in (2). Let $\widetilde{k}_\alpha \in L^\infty(0,T)$ and we define

$$\Psi^W(\widetilde{k}_\alpha)(r) = \begin{cases} \frac{1}{C_1}, & \text{if} \quad \widetilde{k}_\alpha(r) < C_1^{-1}, \\ \frac{1}{\widetilde{k}_\alpha(r)}, & \text{if} \quad C_1^{-1} \leq \widetilde{k}_\alpha(r) \leq C_0^{-1}, \\ \frac{1}{C_0}, & \text{if} \quad \widetilde{k}_\alpha(r) > C_0^{-1}. \end{cases}$$

We define

$$W : L^\infty(0,T) \to L^\infty(M), \quad W(\widetilde{k}_\alpha)(r) = \begin{cases} \Psi^W(\widetilde{k}_\alpha)(r), & \text{if } r \in (0,T), \\ 1, & \text{if } r \in [T, \infty). \end{cases} \tag{47}$$

Proposition 5. *Let V be as defined in (20) and v be as defined in (21). Let us assume that $\widetilde{k}_\alpha \in \mathcal{B}_{L^\infty(0,T)}(\partial_r V, \epsilon)$. Then*

$$\Big\|W(\widetilde{k}_\alpha) - v\Big\|_{L^\infty(M)} \leq C_1^2\epsilon.$$

Proof. For all $x \in M$, we have $0 < C_0 \le c(x) \le C_1$. Let $r \in (0,T)$ and assume that $C_1^{-1} \le \widetilde{k}_\alpha(r) \le C_0^{-1}$. Using (21) and (22) we have $0 < \frac{1}{C_1} \le \partial_r V(r) \le \frac{1}{C_0}$. Then

$$\Big|\frac{1}{\widetilde{k}_\alpha(r)} - \frac{1}{\partial_r V(r)}\Big| = \Big|\frac{\widetilde{k}_\alpha(r) - \partial_r V(r)}{\widetilde{k}_\alpha(r)\partial_r V(r)}\Big| \le C_1^2 \epsilon. \tag{48}$$

In the case when $r \in (0,T)$ and $\widetilde{k}_\alpha(r) < C_1^{-1}$ or $\widetilde{k}_\alpha(r) > C_0^{-1}$ we obtain similar estimates. Note that the right hand side of (48) is independent of r. When $r \ge T$ the left hand side is identically zero. □

For $\widetilde{w}_\alpha \in L^\infty(M)$ we define two operators
(49)

$$\Psi^\Phi : L^\infty(M) \to L^\infty(M), \quad \Psi^\Phi(\widetilde{w}_\alpha)(r) := \begin{cases} C_0, & \text{if} \quad \widetilde{w}_\alpha(r) < C_0, \\ \widetilde{w}_\alpha(r), & \text{if} \quad C_0 \le \widetilde{w}_\alpha(r) \le C_1, \\ C_1, & \text{if} \quad w_\alpha(r) > C_1. \end{cases}$$

and

$$\Upsilon : L^\infty(M) \to C(M), \quad \Upsilon(\widetilde{w}_\alpha)(t) = \int_0^t \widetilde{w}_\alpha(t')dt'. \tag{50}$$

Using (49) and (50) we see that $\Upsilon \circ \Psi^\Phi(\widetilde{w}_\alpha) : M \to M$ is bijective as a function of t. Let us denote $\widetilde{\chi} = \Upsilon \circ \Psi^\Phi(\widetilde{w}_\alpha)$ and $\widetilde{\chi}^{-1} = (\Upsilon \circ \Psi^\Phi(\widetilde{w}_\alpha))^{-1}$. We define an operator
(51)

$$\Phi : L^\infty(M) \to L^\infty(\mathbb{R}), \quad \Phi(\widetilde{w}_\alpha) = \begin{cases} 1, & \text{if} \quad x \in (-\infty, 0), \\ \widetilde{w}_\alpha \circ \widetilde{\chi}^{-1}, & \text{if} \quad x \in [0, L_1), \\ 1, & \text{if} \quad x \in [L_1, \infty). \end{cases}$$

Let us define $\eta \in C^\infty(\mathbb{R})$ by

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{x^2-1}\right), & \text{if} \quad x \in (-1,1), \\ 0, & \text{if} \quad |x| \ge 1, \end{cases} \tag{52}$$

where the constant $C > 0$ selected so that $\int_{\mathbb{R}} \eta(x) = 1$. For $\nu > 0$ we define

$$\eta_\nu(x) = \frac{1}{\nu}\eta\Big(\frac{x}{\nu}\Big). \tag{53}$$

By using convolution we define a smooth approximation to a given function $\Phi(\widetilde{w}_\alpha) \in L^\infty(\mathbb{R})$ by setting

$$\Gamma_\nu : L^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad \Gamma_\nu(\Phi(\widetilde{w}_\alpha)) = \eta_\nu * \Phi(\widetilde{w}_\alpha). \tag{54}$$

Let us denote $\widetilde{c}_\nu = (\Gamma_\nu \circ \Phi)(\widetilde{w}_\alpha) = \eta_\nu * \Phi(\widetilde{w}_\alpha)$.

Proposition 6. *Let $\epsilon > 0$ and $\nu = \epsilon^{\frac{1}{3}}$. Let $m > 0$ as in (2). Let v as in (21). Let $c \in \mathcal{V}^3$. Let us assume that $\widetilde{w}_\alpha \in \mathcal{B}_{L^\infty(M)}(v, \epsilon)$. Thus we have*

(*i*) $\quad \|\Phi(\widetilde{w}_\alpha) - c\|_{L^\infty(M)} \leq C\epsilon,$

(*ii*) $\quad \|\widetilde{c}_\nu - c\|_{C^2(M)} \leq C\epsilon^{\frac{1}{3}}.$

Proof. Let $x \in [0, L_1)$. Let us denote $t = \chi^{-1}(x)$ and $\widetilde{t} = \widetilde{\chi}^{-1}(x)$. Having χ as in (22) and $\widetilde{\chi}$ as in (51) we see that

$$|\Phi(\widetilde{w}_\alpha)(x) - c(x)| = |\widetilde{w}_\alpha(\widetilde{t}) - v(t)| \leq |\widetilde{w}_\alpha(\widetilde{t}) - v(\widetilde{t})| + |v(\widetilde{t}) - v(t)|.$$

Lemma 3 gives us $\|v\|_{C^2(0,T)} \leq \widetilde{m}$ and we have

$$|v(\widetilde{t}) - v(t)| \leq \widetilde{m}|\widetilde{t} - t|. \tag{55}$$

Using (2) and (22) we see that $0 < C_0 \leq v(t) \leq C_1$ and hence

$$C_0|\widetilde{t} - t| \leq |\int_t^{\widetilde{t}} v(t')dt'| = |\chi(\widetilde{t}) - \chi(t)|. \tag{56}$$

Having $\widetilde{\chi}(\widetilde{t}) = x = \chi(t)$ and using (22) and (50) we see that

$$|\chi(\widetilde{t}) - \chi(t)| = |\chi(\widetilde{t}) - \widetilde{\chi}(\widetilde{t})| = |\int_0^{\widetilde{t}} \Big(v(t') - \Psi^\Phi(\widetilde{w}_\alpha)(t')\Big)dt'|. \tag{57}$$

Using (22) and (50) we see that $|v(t') - \Psi^\Phi(\widetilde{w}_\alpha)(t')| \leq |v(t') - \widetilde{w}_\alpha(t')|$ for all $t' \in M$. Hence

$$|\chi(\widetilde{t}) - \chi(t)| = |\int_0^{\widetilde{t}} (v(t') - \widetilde{w}_\alpha(t'))dt'| \leq \widetilde{\chi}^{-1}(L_1)\epsilon. \tag{58}$$

Using (55),(56), and (58) we have

$$|\Phi(\widetilde{w}_\alpha)(x) - c(x)| \leq \Big(1 + \frac{\widetilde{m}\widetilde{\chi}^{-1}(L_1)}{C_0}\Big)\epsilon. \tag{59}$$

Note that the right hand side in (59) does not depend on x. When $x \in [L_1, \infty)$ the left hand side in identically zero and we have inequality in case (i).
(ii) Let us define that $c(x) = 1$, for $x \in (-\infty, 0)$. Using (1) we have

$$\|\eta_\nu * \Phi(\widetilde{w}_\alpha) - \eta_\nu * c)\|_{C^2(\mathbb{R})} \leq \|\eta_\nu\|_{W^{2,1}(\mathbb{R})} \|\Phi(\widetilde{w}_\alpha) - c\|_{L^\infty(\mathbb{R})}.$$

Let $\nu \in (0, 1)$. Using inequality (i) from Proposition 6 and definitions (51), (53) and (54) we have

$$\|\eta_\nu * \Phi(\widetilde{w}_\alpha) - \eta_\nu * c)\|_{C^2(\mathbb{R})} \leq C\nu^{-2}\epsilon. \tag{60}$$

Using (52) and (53) we see that $\mathrm{supp}(\eta_\nu) \subset [-\nu, \nu]$. Combining that with assumption that $c \in \mathcal{V}^3$ we have

$$\|\eta_\nu * c - c\|_{C^2(\mathbb{R})} \le 2\nu \, \|c\|_{C^3(\mathbb{R})} \le 2\nu m. \tag{61}$$

By $\nu = \epsilon^{\frac{1}{3}}$ and using (60) and (61) we have

$$\|\widetilde{c}_\nu - c\|_{C^2(\mathbb{R})} \le C\nu^{-2}\epsilon + \nu 2m \le (C + 2m)\epsilon^{\frac{1}{3}}.$$

□

Proof of Theorem 3. Let

$$\epsilon_0 = min\{1, \frac{1}{2T}, \frac{1}{2^{\frac{13}{4}}T}, \frac{1}{2^{\frac{13}{4}}T\chi(T)^{\frac{9}{2}}}, \frac{3^9 2^8}{C^{18}C_1^{36}T^{28}}\}. \tag{62}$$

Suppose that $\widetilde{\Lambda} \in \mathcal{B}_Y(\Lambda, \epsilon)$ and $\epsilon \in (0, \epsilon_0)$. We denote $H = \boldsymbol{H}\Lambda$ and $\widetilde{H} = \boldsymbol{H}\widetilde{\Lambda}$. Using Proposition 1 we get

$$\left\|H - \widetilde{H}\right\|_{C([0,T],Y)} \le 2T\epsilon.$$

We denote $Z_\alpha = \boldsymbol{Z}_\alpha\big(H\big)$ and $\widetilde{Z}_\alpha = \boldsymbol{Z}_\alpha\big(\widetilde{H}\big)$. We have $\widetilde{H} \in \mathcal{B}_{C([0,T],Y)}(H, 2T\epsilon)$ and $\epsilon \in \big(0, \min(1, \frac{1}{2T})\big)$. Proposition 2 with $p = \frac{4}{9}$ gives us

$$\left\|Z_\alpha - \widetilde{Z}_\alpha\right\|_{C([0,T],Y)} \le 2^{-\frac{8}{9}}T^{\frac{1}{9}}\epsilon^{\frac{1}{9}} =: \epsilon_1,$$

since $\alpha = 2^{p+1}T^p\epsilon^p = 2^{\frac{13}{9}}T^{\frac{4}{9}}\epsilon^{\frac{4}{9}}$.
We denote $s_\alpha = \boldsymbol{S}Z_\alpha$ and $\widetilde{s}_\alpha = \boldsymbol{S}\widetilde{Z}_\alpha$. We have $\widetilde{Z}_\alpha \in \mathcal{B}_{C([0,T],Y)}(Z_\alpha, \epsilon_1)$. Proposition 3 gives us

$$\|s_\alpha - \widetilde{s}_\alpha\|_{C([0,T])} \le 3^{-1} \cdot 2^{-\frac{8}{9}}T^{\frac{28}{9}}\epsilon^{\frac{1}{9}} =: \epsilon_2.$$

We denote $\widetilde{k}_\alpha = D_h(\widetilde{s}_\alpha)$. We have $\widetilde{s}_\alpha \in \mathcal{B}_{C([0,T])}(s_\alpha, \epsilon_2)$ and $\epsilon_2 \in \Big(0, \min(\frac{1}{\beta^{\frac{1}{4}}}, \frac{1}{\beta^{\frac{1}{4}}\chi(T)^{\frac{1}{2}}})\Big)$. Proposition 4 with $\beta = 3^4 2^5 T^{-12}$ gives us

$$\left\|\widetilde{k}_\alpha - \partial_r V\right\|_{L^\infty(0,T)} \le C3^{-\frac{1}{2}} \cdot 2^{-\frac{4}{9}}T^{\frac{14}{9}}\epsilon^{\frac{1}{18}} =: \epsilon_3,$$

where $\alpha = \beta(3^{-1} \cdot 2^{-\frac{8}{9}}T^{\frac{28}{9}}\epsilon^{\frac{1}{9}})^4 = \beta(3^{-4} \cdot 2^{-\frac{32}{9}}T^{\frac{112}{9}}\epsilon^{\frac{4}{9}}) = 2^{\frac{13}{9}}T^{\frac{4}{9}}\epsilon^{\frac{4}{9}}$.
We denote $\widetilde{w}_\alpha = W(\widetilde{k}_\alpha)$. We have $\widetilde{k}_\alpha \in \mathcal{B}_{L^\infty(0,T)}(\partial_r V, \epsilon_3)$. Proposition 5 gives us

$$\|\widetilde{w}_\alpha - v\|_{L^\infty(M)} \le C_1^2 C3^{-\frac{1}{2}} \cdot 2^{-\frac{4}{9}}T^{\frac{14}{9}}\epsilon^{\frac{1}{18}} =: \epsilon_4.$$

Let $\epsilon_4 \in (0,1)$ and $\nu = \epsilon_4^{\frac{1}{3}}$. We denote $\widetilde{c}_\nu = \eta_\nu * \Phi(\widetilde{w}_\alpha)$. We have $\widetilde{w}_\alpha \in \mathcal{B}_{L^\infty(M)}(v, \epsilon_4)$. Let $c \in \mathcal{V}^3$ and Proposition 6 gives us

$$\|\widetilde{c}_\nu - c\|_{C^2(M)} \le C\epsilon^{\frac{1}{54}},$$

where $\epsilon \in (0, \epsilon_0)$. Using (24),(30),(34),(44),(47), (51) and (54) we define

$$\mathcal{R}_{\alpha(\epsilon)} : Y \to Z, \qquad (63)$$
$$\mathcal{R}_{\alpha(\epsilon)} = \Gamma_\nu \circ \Phi \circ W \circ D_h \circ \boldsymbol{S} \circ \boldsymbol{Z}_\alpha \circ \boldsymbol{H},$$

and we have an estimate

$$\left\| \mathcal{R}_{\alpha(\epsilon)}(\widetilde{\Lambda}) - c \right\|_Z \leq C \epsilon^{\frac{1}{54}}. \qquad (64)$$

□

Appendix A: The direct problem

Theorem 5. *Let $c \in \mathcal{V}^2$ and $f \in L^2(0, 2T)$. Then the boundary value problem (4) has a unique solution $u^f \in H^1((0, 2T) \times M)$. The operators Λ and U_T, defined in (5) and (9), are bounded, and the direct map $\mathcal{A} : \mathcal{V}^2 \subset Z \to Y$, defined in (6), is continuous, and moreover*

$$M_1 = \sup\{\|\mathcal{A}(c)\|_{\mathcal{L}(L^2(0,2T))}\,; c \in \mathcal{V}^2\} < \infty.$$

Proof. Let us consider the wave equation (4). When $c = 1$ on M we denote the solution by u_0^f and have

$$u_0^f(t, x) = h(t - x), \quad h(s) = \begin{cases} -\int_0^s f(t)dt, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Notice that $f \mapsto u_0^f$ is continuous from $L^2(0, 2T)$ to $C^1([0, 2T]; L^2(M)) \cap C([0, 2T]; H^1(M))$.

Let us consider the wave equation

$$\begin{aligned} &(\partial_t^2 - c(x)^2 \partial_x^2) w(t, x) = F(t, x) \quad \text{in } (0, 2T) \times M, \\ &\partial_x w(t, 0) = 0, \\ &w|_{t=0} = \partial_t w|_{t=0} = 0. \end{aligned} \qquad (65)$$

For the wave equation (65) the existence and uniqueness of the solutions and the continuity of the map $W : F \mapsto w$, given by

$$W : L^2((0, 2T) \times M) \to C([0, 2T]; H^1(M)) \cap C^1([0, 2T]; L^2(M)),$$

follow from the results of [38, Ch. 3, Theorems 8.1 and 8.2], and [39, p. 93].

Let $\psi \in C^\infty(M)$ be such that $\psi = 1$ near $x = 0$ and $\psi = 0$ when $x > \frac{L_0}{2}$. Note that $c = 1$ in the support of ψ. The commutator $A = [\partial_x^2, \psi]$ is a first order differential operator, whence $Au_0^f \in L^2((0, 2T) \times M)$ for $f \in L^2(0, 2T)$. Let us choose $F(t, x) = Au_0^f(t, x)$ in (65) and define $u^f = \psi u_0^f + w$. Then

$$(\partial_t^2 - c^2 \partial_x^2) u^f = \psi(\partial_t^2 - \partial_x^2) u_0^f + Au_0^f - Au_0^f = 0,$$

where $u^f := \psi u_0^f + w \in C^1([0,2T]; L^2(M)) \cap C([0,2T]; H^1(M))$ is the solution of (4). As $\psi = 1$ near $x = 0$, we see that u satisfies also the boundary conditions in (4). In particular, $u^f \in H^1((0,2T) \times M)$. The above shows that $f \mapsto u^f$ is continuous operator from $L^2(0,2T)$ to $u^f \in C^1([0,2T]; L^2(M)) \cap C([0,2T]; H^1(M))$, and hence $U_T : L^2(0,2T) \to H^1(M)$ is continuous. Using Trace theorem, we see that the map Λ is continuous from $L^2(0,2T)$ to $H^{\frac{1}{2}}(0,2T)$.

Let us now suppose that $f \in C_0^\infty(0,2T)$. Let u^f be solution for the boundary value problem in (4) and $c(x)$ be as defined in (2). Let $x \in M$ and we define

$$k \in C^2(M), \quad k(x) = c(x)^{1/2} \tag{66}$$

and

(67)

$$G : C^2(M \times (0,2T)) \to C(M \times (0,2T)),$$

$$G(u) = k^{-1}\Big(\partial_t^2 - c^2\partial_x^2\Big)ku = \Big(\partial_t^2 - c^2\partial_x^2 - 2c^2k^{-1}(\partial_x k)\partial_x - c^2k^{-1}(\partial_x^2 k)\Big)u.$$

Let $x \in M$ and define

$$\phi(x) = \int_0^x c(x')^{-1}dx'. \tag{68}$$

Let us denote $\widetilde{x} = \phi(x)$ and define

$$\widetilde{u}^f(\widetilde{x},t) = \widetilde{u}^f(\phi(x),t) := \frac{u^f(x,t)}{k(x)}. \tag{69}$$

Using (3), (66), (67), (68), (69) and the property of finite speed of propagation we see that $\widetilde{u}^f(\widetilde{x},t)$ is a solution of the boundary value problem

$$\begin{aligned} &(\partial_t^2 - \partial_{\widetilde{x}}^2 + q(\widetilde{x}))\widetilde{u}^f(\widetilde{x},t) = 0, && (\widetilde{x},t) \in (0,2T) \times (0,2T), \\ &\partial_{\widetilde{x}}\widetilde{u}^f(0,t) = f(t), \quad \partial_x\widetilde{u}^f(2T,t) = 0, && t \in (0,2T), \\ &\widetilde{u}^f(x,0) = \partial_t\widetilde{u}^f(x,0) = 0, && \widetilde{x} \in [0,2T], \end{aligned} \tag{70}$$

where

$$q(\widetilde{x}) = -c^2(\phi^{-1}(\widetilde{x}))k^{-1}(\phi^{-1}(\widetilde{x}))\partial_x^2 k(\phi^{-1}(\widetilde{x})). \tag{71}$$

Let $\widetilde{x} \in [0,T]$. Using (2), (3), (66), (68), (71) we see that for every q that corresponds to some $c \in \mathcal{V}^2$ via formula (71) there is a constant $C_3 = C_3(C_0, C_1, L_1, m, T)$ for which

$$|q(x)| \le C_3. \tag{72}$$

Let $\widetilde{x} \ge T$. Using (2) we have $c(x) = 1$ and thus by using (71) we see that $q(\widetilde{x}) = 0$.

We define $\Lambda_q f = \widetilde{u}|_{\widetilde{x}=0}$. Let us consider two velocity functions c_1 and c_2, and let q_1 and q_2 be the potentials corresponding to c_1 and c_2 via formula (71). Using (69) and property that $C_0^\infty(0,2T) \subset L^2(0,2T)$ is dense we have

$$\|\mathcal{A}(c_1) - \mathcal{A}(c_2)\|_{\mathcal{L}(L^2(0,2T))} \le \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(L^2(0,2T))}. \tag{73}$$

Let us denote by $u^f_{q_1}$ and $u^f_{q_2}$ the two solutions with respect to potentials q_1 and q_2 for the problem (70). Let us define $w(\widetilde{x},t) = \widetilde{u}^f_{q_1}(\widetilde{x},t) - \widetilde{u}^f_{q_2}(\widetilde{x},t)$. Then w is the solution of

$$\begin{aligned}
&(\partial_t^2 - \partial_x^2 + q_1(\widetilde{x}))w(\widetilde{x},t) = F(\widetilde{x},t), \quad && (\widetilde{x},t) \in (0,2T)\times(0,2T),\\
&\partial_{\widetilde{x}} w(0,t) = 0, \quad \partial_{\widetilde{x}} w(2T,t) = 0, && t \in (0,2T),\\
&w(\widetilde{x},0) = \partial_t w(\widetilde{x},0) = 0, && x \in [0,2T],
\end{aligned} \tag{74}$$

where

$$F(\widetilde{x},t) = (q_1(\widetilde{x}) - q_2(\widetilde{x}))\widetilde{u}^f_{q_2}(\widetilde{x},t). \tag{75}$$

Using results of [38, Ch. 3], or alternatively, the same proof that is in [27], Lemma 1.9 for initial boundary value problem with Diriclet boundary condition, we see for (74), we see that there is a constant $C_4 = C_4(C_0, C_1, C_3, L_1, m, T)$ such that for all potentials q satisfying (72) the solution of the wave equation satisfies

$$\|w\|_{H^1((0,2T)\times(0,2T))} \le C_4 \|F\|_{L^2((0,2T)\times(0,2T))}. \tag{76}$$

Using (69) we see that $u^f_{q_2} \in H^1((0,2T)\times(0,2T))$. That with (72) and (75) imply

$$\|F\|_{L^2((0,2T)\times(0,2T))} \le \|q_1 - q_2\|_{L^\infty(0,2T)} \left\|u^f_{q_2}\right\|_{H^1((0,2T)\times(0,2T))}. \tag{77}$$

When $q = 0$, we can construct an explicit solution of (70), see [27], formula (1.34). Similarly to the estimate (76), we see using results of [38, Ch. 3] or [27] that there is a constant C_5 that depends only on C_0, C_1, L_1, m, T such that

$$\left\|\widetilde{u}^f_{q_2}\right\|_{H^1((0,2T)\times(0,2T))} \le C_5 \|f\|_{L^2(0,2T)}. \tag{78}$$

Using Trace Theorem we have

$$\|\Lambda_{q_1} f - \Lambda_{q_2} f\|_{L^2(0,2T)} \le C(T) \left\|u^f_{q_1} - u^f_{q_2}\right\|_{H^1((0,2T)\times(0,2T))}. \tag{79}$$

Havinq $q(\widetilde{x}) = 0$, when $\widetilde{x} \ge T$, and using (76), (77), (78), and (79) we have

$$\|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(L^2(0,2T))} \le C_7 \|q_1 - q_2\|_{L^\infty(0,2T)} = C_7 \|q_1 - q_2\|_{L^\infty(0,T)}, \tag{80}$$

where C_7 depends only on C_0, C_1, L_1, m, and T.

Let $c_1, c_2 \in \mathcal{V}^2$, where $c_1 \in \mathcal{V}^2$ is fixed. Let $\|c_1 - c_2\|_{C^2(M)} \le \epsilon$. Let $\widetilde{x} \in (0, T)$. Using (68) and (71) we have

$$q_1(\widetilde{x}) = -c_1^2(x)k_1^{-1}(x)\partial_x^2 k_1(x)|_{x=\phi_1^{-1}(\widetilde{x})},$$
$$q_2(\widetilde{x}) = -c_2^2(x)k_2^{-1}(x)\partial_x^2 k_2(x)|_{x=\phi_2^{-1}(\widetilde{x})}.$$

Let us denote $y = \phi_1^{-1}(\widetilde{x})$ and $x = \phi_2^{-1}(\widetilde{x})$. Note that for all $c_1, c_2 \in \mathcal{V}^2$ and $\widetilde{x} \in (0, T)$ we have $x, y \in [0, TC_1]$. Let $x \in M$ and define

$$h_1(x) = -c_1^2(x)k_1^{-1}(x)\partial_x^2 k_1(x), \quad h_2(x) = -c_2^2(x)k_2^{-1}(x)\partial_x^2 k_2(x).$$

We have

(81)
$$|q_1(\widetilde{x}) - q_2(\widetilde{x})| \le |h_1(\phi_1^{-1}(\widetilde{x})) - h_1(\phi_2^{-1}(\widetilde{x}))| + |h_1(\phi_2^{-1}(\widetilde{x})) - h_2(\phi_2^{-1}(\widetilde{x}))|$$

For the second term on the right hand side of (81) we have

$$\begin{aligned}
&|h_1(x) - h_2(x)| = |c_2^2(x)k_2^{-1}(x)\partial_x^2 k_2(x) - c_1^2(x)k_1^{-1}(x)\partial_x^2 k_1(x)| \\
&\le |c_1^2(x) - c_2^2(x)||k_1^{-1}(x)||\partial_x^2 k_1(x)| \\
&\quad + |k_1^{-1}(x) - k_2^{-1}(x)||c_2^2(x)||\partial_x^2 k_1(x)| \\
&\quad + |\partial_x^2 k_1(x) - \partial_x^2 k_2(x)||c_2^2(x)||k_2^{-1}(x)|,
\end{aligned}$$

where $x = \phi_2^{-1}(\widetilde{x})$. Having $x \in [0, TC_1]$ and using (2), (68), (66) we can bound each of these three terms and get

$$|h_1(x) - h_2(x)| \le C_8 \|c_1 - c_2\|_{C^2(M)}, \tag{82}$$

where C_8 depends only on C_0, C_1, L_1, m, T. As h_1 is continuous on M and zero on $[0, L_0) \cup (L_1, \infty)$, h_1 is uniformly continuous on M. Moreover, we have a function $\omega : M \to M$, the continuity modulus of h_1, for which

$$|h_1(x) - h_1(y)| \le \omega(\epsilon), \tag{83}$$

for $x, y \in M$ satisfying $|x - y| \le \epsilon$. Thus for the first term on the right hand side of (81) we have

$$|h_1(\phi_1^{-1}(\widetilde{x})) - h_1(\phi_2^{-1}(\widetilde{x}))| \le \omega(|\phi_1^{-1}(\widetilde{x}) - \phi_2^{-1}(\widetilde{x})|). \tag{84}$$

Having $x \in [0, TC_1]$ and using (2) and (68) we have

$$|\phi_1(x) - \phi_2(x)| \le \int_0^x \frac{|c_1(x') - c_2(x')|}{c_1(x')c_2(x')} dx \le \frac{TC_0 \|c_1 - c_2\|_{C^2(M)}}{C_0^2}. \tag{85}$$

As $\frac{1}{C_1} \le \frac{d\phi_1}{dx}(x) \le \frac{1}{C_0}$, we have

$$\frac{1}{C_1}|x - y| \le |\phi_1(x) - \phi_1(y)| \le \frac{1}{C_0}|x - y|. \tag{86}$$

Using (85) and (86) we have

$$\begin{aligned}(87)\quad |\phi_1^{-1}(\widetilde{x})-\phi_2^{-1}(\widetilde{x})| &\le C_1|\phi_1(\phi_1^{-1}(\widetilde{x}))-\phi_1(\phi_2^{-1}(\widetilde{x}))| \\ &\le C_1\Big(\phi_1(\phi_1^{-1}(\widetilde{x}))-\phi_2(\phi_2^{-1}(\widetilde{x}))|+|\phi_2(\phi_2^{-1}(\widetilde{x}))-\phi_1(\phi_2^{-1}(\widetilde{x}))|\Big) \\ &\le C_1\Big(0+\frac{TC_0\,\|c_1-c_2\|_{C^2(M)}}{C_0^2}\Big)\le C_0C_1T\,\|c_1-c_2\|_{C^2(M)}\,.\end{aligned}$$

Using (81), (82), (84), and (87) we have

$$(88)\qquad |q_1(\widetilde{x})-q_2(\widetilde{x})|\le \omega(C\,\|c_1-c_2\|_{C^2(M)})+C\,\|c_1-c_2\|_{C^2(M)}\,,$$

where $\widetilde{x}\in(0,T)$. As ω is continuous at zero, we see that when $c_2\to c_1$ in $\mathcal{V}^2\subset C_b^2(M)$ we get by using (88) that $q_2\to q_1$ in $L^\infty(0,T)$. Using this with (73) and (80) we obtain $\mathcal{A}(c_1)\to\mathcal{A}(c_2)$ in $\mathcal{L}(L^2(0,2T))$ when $c_2\to c_1$ in $\mathcal{V}^2\subset C_b^2(M)$.

Choosing $c_2(x)=1$ for all $x\in M$, we see that $\mathcal{A}(c_2)f=0$ for all $f\in L^2(0,2T)$. Using this with (73) we have

$$(89)\quad \|\mathcal{A}(c_1)\|_{\mathcal{L}(L^2(0,2T))}=\|\mathcal{A}(c_1)-\mathcal{A}(c_2)\|_{\mathcal{L}(L^2(0,2T))}\le C_9\,\|q_1-q_2\|_{L^\infty(0,T)}\,,$$

where C_9 depends only on C_0,C_1,L_1,m,T. When $c_2(x)=1$ for all $x\in M$, q_2 is zero on M. This with (72) and (80) imply $\|q_1-q_2\|_{L^\infty(0,T)}\le C_3$, for all $c_1\in\mathcal{V}^2$. Using this and (89) we see that

$$M_1=\sup\{\|\mathcal{A}(c)\|_{\mathcal{L}(L^2(0,2T))}\,;c\in\mathcal{V}^2\}<\infty.$$

□

Appendix B: The proof of theorem (2)

Proof. Let $\mathcal{V}^k$ as defined in (2). By Theorem 5, the map

$$\mathcal{A}:\mathcal{V}^2\subset X\to Y,\quad \mathcal{A}(c)=\Lambda,$$

is continuous. Also by Theorem 1, $\mathcal{A}:\mathcal{V}^2\to\mathcal{A}(\mathcal{V}^2)$ is one-to-one. By Arzela-Ascoli Theorem $cl_{C^2(M)}(\mathcal{V}^3)=\overline{\mathcal{V}^3}$ is a compact subset of $C_b^2(M)$. Let $U\subset\overline{\mathcal{V}^3}$ is open. Thus $\overline{\mathcal{V}^3}\setminus U$ is closed and compact. Using continuity of $\mathcal{A}$ we see that $\mathcal{A}(\overline{\mathcal{V}^3}\setminus U)=\mathcal{A}(\overline{\mathcal{V}^3})\setminus\mathcal{A}(U)$ is compact. As Y is a Hausdorff space, $\mathcal{A}(\overline{\mathcal{V}^3})\setminus\mathcal{A}(U)$ is closed. Thus $\mathcal{A}(U)\subset A(\overline{\mathcal{V}^3})$ is open and

$$\mathcal{A}:\overline{\mathcal{V}^3}\to\mathcal{A}(\overline{\mathcal{V}^3}),\quad \mathcal{A}(c)=\Lambda,$$

is a homomorphism. Note that $\overline{\mathcal{V}^3}$ has the relative topology determined by the norm $\|\cdot\|_{C^k(M)}$ and $\mathcal{A}(\overline{\mathcal{V}^3})$ has the relative topology induced from Y. □

Acknowledgements. We thank Samuli Siltanen for inspiring discussions on the regularization on inverse problems.

L. Oksanen was partly supported by EPSRC. J. Korpela and M. Lassas were supported by Academy of Finland, projects 263235, 273979 and 284715.

References

[1] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, and M. Taylor. Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem. *Invent. Math.*, 158(2):261–321, 2004.

[2] M. I. Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk SSSR*, 297(3):524–527, 1987.

[3] M. I. Belishev. Boundary control in reconstruction of manifolds and metrics (the BC method). *Inverse Problems*, 13(5):R1–R45, 1997.

[4] M. I. Belishev and A. P. Kachalov. Methods in the theory of boundary control in an inverse spectral problem for an inhomogeneous string. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 179(Mat. Vopr. Teor. Rasprostr. Voln. 19):13, 14–22, 187, 1989.

[5] M. I. Belishev and Y. V. Kurylëv. A nonstationary inverse problem for the multidimensional wave equation "in the large". *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 165(Mat. Vopr. Teor. Rasprostr. Voln. 17):21–30, 189, 1987.

[6] M. I. Belishev and Y. V. Kurylev. To the reconstruction of a Riemannian manifold via its spectral data (BC-method). *Comm. Partial Differential Equations*, 17(5-6):767–804, 1992.

[7] M. I. Belishev, V. A. Ryzhov, and V. B. Filippov. A spectral variant of the VS-method: theory and numerical experiment. *Dokl. Akad. Nauk*, 337(2):172–176, 1994.

[8] M. Bellassoued and D. Dos Santos Ferreira. Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map. *Inverse Probl. Imaging*, 5(4):745–773, 2011.

[9] K. Bingham, Y. Kurylev, M. Lassas, and S. Siltanen. Iterative time-reversal control for inverse problems. *Inverse Probl. Imaging*, 2(1):63–81, 2008.

[10] N. Bissantz, T. Hohage, and A. Munk. Consistency and rates of convergence of nonlinear Tikhonov regularization with random noise. *Inverse Problems*, 20(6):1773–1789, 2004.

[11] A. S. Blagoveščenskiĭ. The inverse problem of the theory of seismic wave propagation. In *Problems of mathematical physics, No. 1: Spectral theory and wave processes (Russian)*, pages 68–81. (errata insert). Izdat. Leningrad. Univ., Leningrad, 1966.

[12] A. S. Blagoveščenskiĭ. A one-dimensional inverse boundary value problem for a second order hyperbolic equation. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 15:85–90, 1969.

[13] A. S. Blagoveščenskiĭ. The inverse boundary value problem of the theory of wave propagation in an anisotropic medium. *Trudy Mat. Inst. Steklov.*, 115:39–56. (errata insert), 1971.

[14] A. L. Bukhgeĭm and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.

[15] M. F. Dahl, A. Kirpichnikova, and M. Lassas. Focusing waves in unknown media by modified time reversal iteration. *SIAM J. Control Optim.*, 48(2):839–858, 2009.

[16] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[17] I. M. Gel′fand and B. M. Levitan. On the determination of a differential equation from its spectral function. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 15:309–360, 1951.

[18] M. Hanke. Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems. *Numer. Funct. Anal. Optim.*, 18(9-10):971–993, 1997.

[19] B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987–1010, 2007.

[20] T. Hohage and M. Pricop. Nonlinear Tikhonov regularization in Hilbert scales for inverse boundary value problems with random noise. *Inverse Probl. Imaging*, 2(2):271–290, 2008.

[21] L. Justen and R. Ramlau. A non-iterative regularization approach to blind deconvolution. *Inverse Problems*, 22(3):771–800, 2006.

[22] S. I. Kabanikhin, A. D. Satybaev, and M. A. Shishlenin. *Direct methods of solving multidimensional inverse hyperbolic problems*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 2005.

[23] B. Kaltenbacher and A. Neubauer. Convergence of projected iterative regularization methods for nonlinear problems with smooth solutions. *Inverse Problems*, 22(3):1105–1119, 2006.

[24] B. Kaltenbacher, A. Neubauer, and O. Scherzer. *Iterative regularization methods for nonlinear ill-posed problems*, volume 6 of *Radon Series on Computational and Applied Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[25] A. Katchalov and Y. Kurylev. Multidimensional inverse problem with incomplete boundary spectral data. *Comm. Partial Differential Equations*, 23(1-2):55–95, 1998.

[26] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[27] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[28] A. Katchalov, Y. Kurylev, M. Lassas, and N. Mandache. Equivalence of time-domain inverse problems and boundary spectral problems. *Inverse Problems*, 20(2):419–436, 2004.

[29] A. Katsuda, Y. Kurylev, and M. Lassas. Stability of boundary distance representation and reconstruction of Riemannian manifolds. *Inverse Probl. Imaging*, 1(1):135–157, 2007.

[30] A. Kirsch. *An Introduction to the Mathematical Theory of Inverse Problems.* Springer-Verlag New York, Inc., New York, NY, USA, 1996.

[31] K. Knudsen, M. Lassas, J. L. Mueller, and S. Siltanen. Regularized D-bar method for the inverse conductivity problem. *Inverse Probl. Imaging*, 3(4):599–624, 2009.

[32] M. G. Kreĭn. Solution of the inverse Sturm-Liouville problem. *Doklady Akad. Nauk SSSR (N.S.)*, 76:21–24, 1951.

[33] Y. Kurylev. An inverse boundary problem for the Schrödinger operator with magnetic field. *J. Math. Phys.*, 36(6):2761–2776, 1995.

[34] Y. Kurylev and M. Lassas. Inverse problems and index formulae for Dirac operators. *Adv. Math.*, 221(1):170–216, 2009.

[35] Y. Kurylev, M. Lassas, and E. Somersalo. Maxwell's equations with a polarization independent wave velocity: direct and inverse problems. *J. Math. Pures Appl. (9)*, 86(3):237–270, 2006.

[36] M. Lassas and L. Oksanen. Inverse problem for the Riemannian wave equation with Dirichlet data and Neumann data on disjoint sets. *Duke Math. J.*, 163(6):1071–1103, 2014.

[37] M. Lassas and L. Oksanen. Local reconstruction of a Riemannian manifold from a restriction of the hyperbolic Dirichlet-to-Neumann operator. In *Inverse problems and applications*, volume 615 of *Contemp. Math.*, pages 223–231. Amer. Math. Soc., Providence, RI, 2014.

[38] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[39] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II.* Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 182.

[40] S. Liu and L. Oksanen. A lipschitz stable reconstruction formula for the inverse problem for the wave equation. *Submitted. Preprint arXiv:1210.1094*, Oct. 2012.

[41] S. Lu, S. V. Pereverzev, and R. Ramlau. An analysis of Tikhonov regularization for nonlinear ill-posed problems under a general smoothness assumption. *Inverse Problems*, 23(1):217–230, 2007.

[42] V. A. Marčenko. Concerning the theory of a differential operator of the second order. *Doklady Akad. Nauk SSSR. (N.S.)*, 72:457–460, 1950.

[43] P. Mathé and B. Hofmann. How general are general source conditions? *Inverse Problems*, 24(1):015009, 5, 2008.

[44] J. L. Mueller and S. Siltanen. *Linear and nonlinear inverse problems with practical applications*, volume 10. Siam, 2012.

[45] A. Nachman, J. Sylvester, and G. Uhlmann. An n-dimensional Borg-Levinson theorem. *Comm. Math. Phys.*, 115(4):595–605, 1988.

[46] L. Oksanen. Solving an inverse problem for the wave equation by using a minimization algorithm and time-reversed measurements. *Inverse Probl. Imaging*, 5(3):731–744, 2011.

[47] L. Oksanen. Inverse obstacle problem for the non-stationary wave equation with an unknown background. *Comm. Partial Differential Equations*, 38(9):1492–1518, 2013.

[48] R. Ramlau. Regularization properties of Tikhonov regularization with sparsity constraints. *Electron. Trans. Numer. Anal.*, 30:54–74, 2008.

[49] R. Ramlau and G. Teschke. A Tikhonov-based projection iteration for nonlinear ill-posed problems with sparsity constraints. *Numer. Math.*, 104(2):177–203, 2006.

[50] E. Resmerita. Regularization of ill-posed problems in Banach spaces: convergence rates. *Inverse Problems*, 21(4):1303–1314, 2005.

[51] P. Stefanov and G. Uhlmann. Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media. *J. Funct. Anal.*, 154(2):330–358, 1998.

[52] P. Stefanov and G. Uhlmann. Recovery of a source term or a speed with one measurement and applications. Mar. 2011.

[53] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[54] D. Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.