SMALL SUBSET SUMS

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Abstract. Let $\|.\|$ be a norm in $\mathbb{R}^d$ whose unit ball is B. Assume that $V \subset B$ is a finite set of cardinality n, with $\sum_{v\in V} v = 0$. We show that for every integer k with $0 \le k \le n$, there exists a subset U of V consisting of k elements such that $\|\sum_{v\in U} v\| \le \lceil d/2\rceil$. We also prove that this bound is sharp in general. We improve the estimate to $O(\sqrt{d})$ for the Euclidean and the max norms. An application on vector sums in the plane is also given.

1. Definitions, notation, results

We consider the real d-dimensional vector space $\mathbb{R}^d$ with a norm $\|.\|$ whose unit ball is B. For a finite set $U \subset \mathbb{R}^d$, $|U|$ stands for the cardinality of U, and $s(U)$ for the sum of the elements of U, so $s(U) = \sum_{u\in U} u$, and $s(\emptyset) = 0$ of course.

In 1914 Steinitz [12] proved that, in the case of the Euclidean norm, for every finite set $V \subset B$ with $|V| = n$ and $s(V) = 0$, there exists an ordering $v_1, \dots, v_n$ of the vectors in V such that *all partial sums* have norm at most $2d$, that is

$$\max_{k=1,\dots,n} \Big\| \sum_1^k v_i \Big\| \leqslant 2d.$$

It is important here that the bound $2d$ does not depend on n, the size of V. Steinitz's result implies that for every norm and every finite $V \subset B$ with $s(V) = 0$ there is an ordering along which all partial sums are bounded by a constant that depends only on B. Let $S(B)$ denote the smallest such constant for a given norm with unit ball B, and set $S(d) = \sup S(B)$ where the supremum is taken over all norms in R^d. The best known bounds on $S(d)$ are: $S(B) \le d$, proved by Sevastyanov [9], and by Grinberg and Sevastyanov [7], and $S(d) \ge \frac{d+1}{2}$, which is shown by an example coming from the ℓ_1 norm [7]. For specific norms, stronger results may hold. In particular, for ℓ_2 and ℓ_∞, it is conjectured that the right order of magnitude of $S(B)$ is $\sqrt{d}$ – although not even $o(d)$ is known.

Steinitz's result immediately implies that for every finite set $V \subset B$ with $s(V) = 0$ and every integer k, $0 \le k \le |V|$, there is a subset $U \subset V$ such that $|U| = k$ and $\|s(U)\|$ is not greater than a constant depending only on d, B, k, for instance $S(B)$ is such a constant. Let $T(B, k)$ be the smallest constant with this property, set $T(B) = \sup_k T(B, k)$, and $T(d) = \sup T(B)$ where the supremum is taken over all norms in $\mathbb{R}^d$. It is evident that $T(B, k) \le k$.

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In this paper we investigate $T(B,k), T(B)$ and $T(d)$. Here come our main results. First, the estimate for general norms.

Theorem 1. *Let B be the unit ball of an arbitrary norm on $\mathbb{R}^d$. For any finite set $V \subset B$ with $s(V) = 0$, and for any $k \leqslant |V|$, there exists a subset $U \subset V$ with k elements, so that*

$$\|s(U)\| \leqslant \left\lceil \frac{d}{2} \right\rceil .$$

In other words, $T(d) \leqslant \lceil \frac{d}{2} \rceil$.

Theorem 2. *For every $d \geqslant 1$, there exists a norm in $\mathbb{R}^d$ with unit ball B, so that $T(B,k) = \lceil \frac{d}{2} \rceil$ for infinitely many values of k. Also, $T(B,k) = k$ for all $k \leq \lfloor \frac{d}{2} \rfloor$.*

Theorems 1 and 2 imply that $T(d) = \lceil \frac{d}{2} \rceil$ for all integers $d \geq 1$.

One expects that for specific norms better estimates are valid. We have proved this in some cases. The unit ball of the norm ℓ_p^d will be denoted by B_p^d. We have the following results in the cases $p = 1, 2, \infty$.

Theorem 3. $\frac{d}{2} \leq T(B_1^d) \leq \lceil \frac{d}{2} \rceil$.

Theorem 4. $\frac{1}{2}\sqrt{d+2} \leq T(B_2^d) \leq \frac{1+\sqrt{5}}{2}\sqrt{d}$

Theorem 5. $\frac{1}{3}\sqrt{d} \leq T(B_\infty^d) \leq O(\sqrt{d})$

We mention that in Theorems 4 and 5 the order of magnitude is the same as the conjectured value of the Steinitz constant.

Remark 1. Note that there is a "complementary" symmetry here. Namely, for every $U \subset V$, $s(U) = -s(V \setminus U)$, hence $\|s(U)\| = \|s(V \setminus U)\|$, and the cases k and $n-k$ are symmetric. Hence, we may assume $k \leq n/2$.

When establishing Helly-type theorems for sums of vectors in a normed plane, Bárány and Jerónimo-Castro proved the following result [3, Lemma 5], which matches our scheme: *Given* 6 *vectors in the unit ball of a normed plane whose sum is 0, there always exist* 3 *among them, whose sum has norm at most 1.* In fact, this statement served as the starting point for our current research. An application of Theorem 1 implies an extension of one of the Helly-type results [3, Theorem 3], which we formulate slightly differently and prove in the last section.

Theorem 6. *Let $k \geqslant 2$ be a positive integer, and $n = m(k-1)+1$ for some $m \geqslant 1$. Assume B is the unit ball of a norm in $\mathbb{R}^2$, $V \subset B$ is of size n and $\|s(V)\| \leq 1$. Then V contains a subset W of size k such that $\|s(W)\| \leq 1$.*

2. Proof of Theorem 1

We are to consider linear combinations $\sum_{v \in V} \alpha(v)v$ of the vectors in V. The coefficients $\alpha(v)$ form a vector $\alpha \in \mathbb{R}^V$. Define the convex polytope

$$P(V,k) = \Big\{ \alpha \in \mathbb{R}^V : \sum_{v \in V} \alpha(v)v = 0,\ \sum_{v \in V} \alpha(v) = k,\ 0 \leq \alpha(v) \leq 1 \ (\forall v \in V) \Big\}.$$

$P(V,k)$ is non-empty as $\alpha(v) \equiv k/n$ lies in it (here $n = |V|$). From now on let α denote a fixed vertex of $P(V,k)$. The basic idea is to choose U to be the set of vectors from V that have the k largest coefficients $\alpha(v)$. This works directly when d is odd, and some extra care is needed for even d.

We note first that $P(V,k)$ is determined by $d+1$ linear equations and $2n$ inequalities for the coefficients $\alpha(v)$, so at a vertex at most $d+1$ coefficients are strictly between 0 and 1. Define $U_1 = \{v \in V : \alpha(v) = 1\}$ and $Q = \{v \in V : 0 < \alpha(v) < 1\}$. Set $q = \sum_{v\in Q} \alpha(v)$, q is an integer since $q + |U_1| = k$. Split now Q into two parts, E and F, so that $|E| = q$ and E contains the vectors with the q largest coefficients in Q, and F the rest (ties broken arbitrarily). Then $U = U_1 \cup E$ has exactly k elements and

$$\begin{aligned} s(U) &= \sum_{v\in U_1} v + \sum_{v\in E} v \\ &= \sum_{v\in V} \alpha(v)v + \sum_{v\in E}(1-\alpha(v))v - \sum_{v\in F}\alpha(v)v. \end{aligned}$$

Here $\sum_{v\in V} \alpha(v)v = 0$, so by the triangle inequality

$$||s(U)|| \le \sum_{v\in E}(1-\alpha(v)) + \sum_{v\in F}\alpha(v).$$

The average of the coefficients in Q is $a := q/|Q|$. Thus, the average of the coefficients is at least a in E, and it is at most a in F. Consequently, the last sum is maximal when $\alpha(v) = a$ for all $v \in Q$:

$$||s(U)|| \le q(1-a) + (|Q|-q)a = \frac{2}{|Q|}\, q\,(|Q|-q) \le \frac{|Q|}{2}\,.$$

This finishes the proof when d is odd as $|Q| \le d+1$, and also when d is even and $|Q| \le d$.

We are left with the case when d is even and $|Q| = d+1$. The vectors in Q are linearly dependent, so there is a non-zero $\beta \in \mathbb{R}^V$ with $\beta(v) = 0$ when $v \notin Q$ such that $\sum_{v\in Q} \beta(v)v = 0$. We can assume that $\sum_{v\in Q} \beta(v) \le 0$. Then $\sum_{v\in V}(\alpha(v) + t\beta(v))v = 0$ for every $t \in \mathbb{R}$. Choose $t > 0$ maximal so that $0 \le \gamma(v) = \alpha(v) + t\beta(v)) \le 1$ for every $v \in V$. This means that, for some $v^* \in Q$, $\gamma(v^*) = 0$ or 1.

Assume for the time being that $q \le (d+1)/2$.

Suppose first that $\gamma(v^*) = 0$. This time we split $Q^* := Q \setminus v^*$ again into E and F so that $|E| = q$ and E contains the vectors from Q^* with the q largest coefficients. Note that $\sum_{v\in Q^*} \gamma(v) \le \sum_{v\in Q} \alpha(v) = q$ and that $|Q^*| = d$, so the average a^* of $\gamma(v)$ over Q^* is at most q/d. We use again $U = U_1 \cup E$ and we have, the same way as before,

$$||s(U)|| \le \sum_{v\in E}(1-\gamma(v)) + \sum_{v\in F}\gamma(v).$$

The right hand side is maximal again if every $\gamma(v)$ equals their average a^*, hence

$$||s(U)|| \le q(1-a^*) + (d-q)a^* = q + (d-2q)a^* \le q + (d-2q)\frac{q}{d} \le \frac{d}{2},$$

because d is even so $q \le (d+1)/2$ implies $2q \le d$. Thus, $||s(U)|| \le d/2$.

The case when $\gamma(v^*) = 1$ is similar: this time v^* is added to U_1, $Q^* = Q \setminus v^*$ is split into E and F with $|E| = q-1$ so that E contains the vectors with the largest $q-1$ coefficients. Now $\sum_{v\in Q^*} \gamma(v) \le \sum_{v \in Q} \alpha(v) - 1 = q-1$, and thus the average a^* of $\gamma(v)$ over Q^* is at most $(q-1)/d$. As above, we are led to the inequality

$$||s(U)|| \le (q-1)(1-a^*) + (d-(q-1))a^* = (q-1) + (d-2(q-1))a^*.$$

Using that $d - 2(q-1) \ge 0$ and $a^* \le (q-1)/d$, we conclude that $||s(U)|| \le d/2 - 2/d < d/2$.

Finally we consider the case $q > (d+1)/2$. By complementary symmetry $s(U) = -s(V \setminus U)$. For $q > (d+1)/2$, we consider the complementary problem of finding $U \subset V$ with $n-k$ elements so that $||s(U)|| \le \lceil d/2 \rceil$. It is easy to see that $1 - \alpha(.) \in \mathbb{R}^V$ is a vertex of $P(V, n-k)$, for which $\sum_{v \in Q}(1 - \alpha(v)) < (d+1)/2$. □

The same proof yields a stronger statement.

Theorem 7. *Let $W \subset B$ finite. Then for every $k \leqslant |W|$ and for every vector $w_0 \in \operatorname{conv} W$, there is a subset $U \subset W$ of cardinality k, so that*

$$\|s(U) - kw_0\| \leqslant \left\lceil \frac{d}{2} \right\rceil.$$

The proof is the same as above, except that instead of the convex polytope $P(V,k)$, we consider the coefficient vectors $\alpha : W \to [0,1]$ satisfying

$$\sum_{w \in W} \alpha(w) w = kw_0 \text{ and } \sum_{w\in W} \alpha(w) = k.$$

The condition $w_0 \in \operatorname{conv} W$ ensures that this set is a non-empty convex polytope. The rest of the argument is unchanged.

Remark 2. For later reference we record the fact that the linear dependence α defines the sets U_1 and Q, and if $|Q| = d+1$, then the new linear dependence γ defines $v^* \in Q$ and Q^*. Note that this works for even and odd d, we only need $|Q| = d+1$. For later use we define

$$A = \{v \in V : \gamma(v) = 1\} \text{ and } C = \{v \in V : 0 < \gamma(v) < 1\}. \tag{1}$$

3. Proof of Theorem 2

We are going to use the following fact. If the unit ball of a norm $\|.\|$ is the convex hull of the vectors $v_1, \dots, v_m, -v_1, \dots, -v_m \in \mathbb{R}^d$, then for every vector $x \in \mathbb{R}^d$,

$$\|x\| = \min\Big\{ \sum_1^m |a_i| \ : \sum_1^m a_i v_i = x \Big\}.$$

Let $e_1, \dots, e_d$ be the standard basis vectors of $\mathbb{R}^d$, and set $e_0 = -\sum_1^d e_i$. We define V to be s copies of $\{e_0, e_1, \dots, e_d\}$, where $s \geqslant 1$ is an integer. The unit ball is set to be $B = \operatorname{conv}\{V, -V\}$. Let $k < n = s(d+1)$ be a positive integer congruent to $\lceil \frac{d}{2} \rceil$ mod $(d+1)$. We claim that for every k-element subset U of V, $\|s(U)\| \ge \lceil \frac{d}{2} \rceil$.

Assume that U contains b_i copies of e_i for every i, so $k = \sum_0^d b_i$. We have to estimate the norm of the vector $v = \sum_0^d b_i e_i$. Assume that

$$v = \sum_0^d a_i e_i$$

for some $a_i \in \mathbb{R}$. Then $\sum_0^d (b_i - a_i)e_i = 0$. Since the only linear dependence of the vectors $e_0, \dots, e_d$ is $x \sum_0^d e_i = 0$ for some constant $x \in \mathbb{R}$, we obtain that $a_i = b_i - x$ for every i. Set

$$f(x) := \sum_0^d |b_i - x|,$$

Then $\|v\| = \min f(x)$ by the fact from the beginning of this section. We are going to estimate $f(x)$. Since $b_i \in \mathbb{Z}$ for every i, the function $f(x)$ is piecewise linear on $\mathbb{R}$ (it is affine on all intervals $(q, q+1)$ for $q \in \mathbb{Z}$). Therefore, there exists $c \in \mathbb{Z}$ so that the minimum of $f(x)$ is attained at c.

The facts $k = \sum_0^d b_i \equiv \lceil d/2 \rceil \bmod (d+1)$ and $c \in \mathbb{Z}$ imply that $\sum_0^d (b_i - c) \equiv \lceil d/2 \rceil \bmod (d+1)$. Thus,

$$\left\lceil \frac{d}{2} \right\rceil \leqslant \Big| \sum_0^d (b_i - c) \Big| \leqslant \sum_0^d |b_i - c|,$$

hence, $\|v\| \geqslant \lceil d/2 \rceil$.

We show next that $T(B, k) = k$ when $1 \leq k < \lceil d/2 \rceil$. The unit ball B is the same as above and $V = \{e_0, \dots, e_d\}$. Assume $U \subset V$ with $|U| = k$ and $\|s(U)\| < k$. Add $\lceil d/2 \rceil - k$ vectors from $V \setminus U$ to U to obtain a subset W of $\lceil d/2 \rceil$ elements. Every addition increases the norm of the sum by at most one (because of the triangle inequality), so we get $\|s(W)\| \leq \|s(U)\| + \lceil d/2 \rceil - k < \lceil d/2 \rceil$, contrary to what was established above. Thus $T(B, k) \geq k$, while $T(B, k) \leq k$ follows from the triangle inequality. □

Further examples showing $T(B, k) = \lceil d/2 \rceil$ will be given in the next section.

Remark 3. We mention that for large enough n, there is no vector set that works simultaneously for all k with $d/2 \leqslant k \leqslant n - d/2$. This follows from Steinitz's theorem: let $v_1, \dots, v_n$ be the ordering where all partial sums lie in dB. Then necessarily two partial sums, with at least $d/2$ summands whose cardinalities differ by at least $d/2$, are close to each other: a standard volume estimate shows that their distance is bounded above by $4dn^{-1/d}$. Then their difference, which is a k-sum with some $d/2 \leqslant k \leqslant n - d/2$, must be small.

4. The ℓ_1 norm, proof of Theorem 3

The upper bound follows from Theorem 1. For the lower bound let V consist of $e_1, \dots, e_d$ and d copies of $\frac{1}{d}e_0$ (with the same notation as in the previous section). Assume $U \subset V$ has exactly d elements. If U contains p vectors out of $e_1, \dots, e_d$, then $s(U)$ has p coordinates equal to $\frac{p}{d}$ and $d - p$ coordinates equal to $\frac{p}{d} - 1$. Thus $\|s(U)\|_1 = \frac{1}{d}(p^2 + (d-p)^2)$. The last expression is minimal when $p = \lfloor \frac{d}{2} \rfloor$. The minimum equals $\frac{d}{2}$ when d is even and $\frac{d}{2} + \frac{1}{2d}$ when d is odd. This is slightly better (for d odd) than the stated lower bound. $\square$

This example shows that $T(B_1^d) = T(B_1^d, d) = d/2$ for even d. A small modification gives further examples implying $T(B_1^d, k) = d/2$ for even d and for all $k \geq d$. Namely, given $d \geq 1$ and $k \geq d$, let V consist of the vectors $e_1, \ldots, e_d$, and $2k - d$ copies of $\frac{1}{2k-d} e_0$. Then $V \subset B_1^d$ and $s(V) = 0$. It is not hard to check that this shows $T(B_1^d, k) = d/2$ for every $k \geqslant d$ (d is even).

5. The ℓ_2 norm, proof of Theorem 4

In this section, $\|.\|$ stands for the Euclidean norm. For the upper bound we will need two lemmas. The first is Lemma 2.2 in Beck's paper [4]. A similar result is given in [1, Theorem 4.1]. The second is a Steinitz type statement.

Lemma 1. *Let $Q \subset B_2^d$ be finite, and $\alpha : Q \to [0,1]$. Then there exists $\varepsilon : Q \to \{0,1\}$ such that $\| \sum_{v \in Q} (\varepsilon(v) - \alpha(v)) v \| \leq \sqrt{d}/2$.*

Lemma 2. *Assume that $V \subset B_2^d$ is a finite set and $\|s(V)\| = \sigma$. Then there exists an ordering $v_1, \ldots, v_n$ of the elements of V, such that, for all $h \leqslant n$,*

$$\Big\| \sum_1^h v_i \Big\| \leqslant \sqrt{\sigma^2 + h}.$$

Proof. Choose $v_1 \in V$ arbitrarily. For $h \geqslant 2$, we select v_h inductively. We set $S_h = \sum_1^h v_i$. Assume that $\|S_{h-1}\| \leqslant \sqrt{\sigma^2 + h - 1}$, and set $W = V \setminus \{v_1, \ldots, v_{h-1}\}$. We consider three cases.

Case 1. If $\|S_{h-1}\| \leqslant \sigma - 1$, then choose $v_h \in W$ arbitrary: $\|S_h\| \leqslant \sigma$ holds by the triangle inequality.

Case 2. If $\|S_{h-1}\| \geqslant \sigma$, then by the assumption $\|S\| = \sigma$, there exists a vector $v_h \in W$, for which $\langle S_{h-1}, v_h \rangle \leqslant 0$. Therefore,

$$\|S_h\|^2 = \|S_{h-1} + v_h\|^2 \leqslant \|S_{h-1}\|^2 + \|v_h\|^2 \leqslant (\sigma^2 + h - 1) + 1 = \sigma^2 + h.$$

Case 3. If $\sigma - 1 < \|S_{h-1}\| < \sigma$, define $\varepsilon = \sigma - \|S_{h-1}\|$, so $0 < \varepsilon < 1$ and $\varepsilon \leq \sigma$. Then

$$\sum_{v \in W} \langle v, S_{h-1} \rangle = \langle S_h - S_{h-1}, S_{h-1} \rangle \leqslant \sigma(\sigma - \varepsilon) - (\sigma - \varepsilon)^2 = \varepsilon(\sigma - \varepsilon).$$

Thus, there exists $v_h \in W$, for which $\langle v_h, S_{h-1} \rangle \leqslant \varepsilon(\sigma - \varepsilon)$. Then

$$\begin{aligned} \|S_h\|^2 = \|S_{h-1} + v_h\|^2 &\leqslant (\sigma - \varepsilon)^2 + 2\,\varepsilon(\sigma - \varepsilon) + 1 \\ &= \sigma^2 + 1 - \varepsilon^2 < \sigma^2 + h. \end{aligned}$$

$\square$

Proof of Theorem 4. For the lower bound let V be the set of vertices of a regular simplex inscribed in B_2^d. Then $s(V) = 0$. Let $U \subset V$ have $\lceil \frac{d}{2} \rceil$ elements. A routine computation shows that $\|s(U)\|$ equals $\frac{\sqrt{d+2}}{2}$ when d is even and $\frac{d+1}{2\sqrt{d}} > \frac{\sqrt{d+2}}{2}$ when d is odd. This implies the lower bound $T(B_2^d) \geq \frac{\sqrt{d+2}}{2}$.

For the upper bound we have to prove the existence of $U \subset V$ with $|U| = k$ and $\|s(U)\| \le \frac{1+\sqrt{5}}{2}\sqrt{d}$. From the proof of Theorem 1 recall the definition of $P(V,k)$ and its vertex $\alpha \in \mathbb{R}^V$ and $U_1 = \{v \in V : \alpha(v) = 1\}$ and $Q = \{v \in V : 0 < \alpha(v) < 1\}$. Here $|Q| \le d+1$.

If $|Q| = 0$, then $|U_1| = k$ and $s(U_1) = 0$, so we can set $U = U_1$. The case $|Q| = 1$ is impossible because the sum of all $\alpha(v)$ is an integer. From now on we assume that $2 \le |Q|$ implying $|U_1| + 1 \le k \le |U_1| + |Q| - 1$. Using Lemma 1 for α restricted to Q we find $\varepsilon : Q \to \{0,1\}$ such that $\|\sum_{v\in Q}(\varepsilon(v) - \alpha(v))v\| \le \sqrt{d}/2$.

Define $W = U_1 \cup \{v \in Q : \varepsilon(v) = 1\}$, then W has the properties that $\|s(W)\| \le \sqrt{d}/2$ and $||W| - k| \le d$. Because of the complementary symmetry, we can assume that $k \le |W| \le k + d$. Set $h = |W| - k$. Then Lemma 2 applies to W: writing $\sigma = \|s(W)\|$ we have $\sigma \le \sqrt{d}/2$ and so the elements of W can be ordered as $w_1, w_2, \dots$ so that $\|\sum_1^m w_i\| \le \sqrt{\sigma^2 + m}$ for every m. In particular, with $m = h \le d$, $\|\sum_1^h w_i\| \le \sqrt{\sigma^2 + h} \le \sqrt{d/4 + d}$. Then for $U = W \setminus \{w_1, ...w_h\}$, we have $|U| = k$ and $\|s(U)\| \le \frac{1+\sqrt{5}}{2}\sqrt{d}$. □

6. The ℓ_∞ norm, proof of Theorem 5

Here, $\|.\|$ denotes the maximum norm. We need two lemmas again, the first is similar to Lemma 1.

Lemma 3. *If $C \subset B_\infty^d$ consists of d linearly independent vectors, then for every point z of the parallelotope $P = \sum_{v\in C}[0, v]$, there is a vertex u of P with $\|z - u\|_\infty = O(\sqrt{d})$.*

This is a result of Spencer [10, Corollary 8], and also of Gluskin [6] whose work relies on that of Kashin [8]. Spencer's proof gives the estimate $\|z-u\| \le 6\sqrt{d}$. The linear independence condition is only needed to ensure that P is a parallelotope, and so its vertices are of the form $s(D) = \sum_{v\in D} v$ for some subset $D \subset C$.

The next statement is the (weaker) analogue of Lemma 2 for the l_∞ norm. Note that we require the set W to contain only a few vectors. The proof is longer and it uses Chobanyan's transference theorem (for the ℓ_∞ norm) so we postpone it to Section 7.

Lemma 4. *Assume $W \subset B_\infty^d$, $|W| = m \le 5d$, and $\|s(W)\|_\infty = O(\sqrt{d})$. Then there is an ordering $w_1, \dots, w_m$ of the vectors in W such that*

$$\max_{h=1,\dots,m} \Big\| \sum_1^h w_i \Big\|_\infty = O(\sqrt{d}).$$

Proof of Theorem 5. The lower bound uses Hadamard matrices and is given in [1].

For the upper bound we assume, rather for convenience than necessity, that the set $V \subset \mathbb{R}^d$ is in general position, for instance, no d vectors from V are linearly dependent. The general case follows from this by a limit argument. We assume further that $|V| = n > 5d$ since for $n \le 5d$ the result is a consequence of Lemma 4. Set $m = \lfloor n/(2d) \rfloor$.

We are going to define linear dependencies γ_i, for $i = 1, 2, \dots, m-1$ so that the sets

$$A_i = \{v \in V : \gamma_i(v) = 1\},\ C_i = \{v \in V : 0 < \gamma_i(v) < 1\}$$

satisfy the conditions

$$A_i \subset A_{i+1},\ (2i-1)d \le |A_i| < h_i := \sum_{v\in V} \gamma_i(v) \le 2di,\ |C_i| = d.$$

The construction is recursive and is similar to how α and $\gamma \in \mathbb{R}^V$ were constructed. For $i = 1$ we take an arbitrary vertex α of the convex polytope $P(V, 2d)$, then $|Q| = d+1$ (because of the general position assumption) and $d \le |U_1| < 2d$ follows. We construct γ as specified in Remark 2 and (1). Then define $\gamma_1 = \gamma$, set $A_1 = \{v \in V : \gamma_1(v) = 1\}$, $C_1 = \{v \in V : 0 < \gamma_1(v) < 1\}$. General position implies that $|C_1| = d$ and then $d \le |A_1| < h_1 = \sum_{v\in V} \gamma_1(v) \le 2d$.

Assume next that $\gamma_1, \dots, \gamma_i$ have been constructed $(1 < i < m-1)$, and the sets A_j, C_j for $j \le i$ satisfy the required conditions. Define the convex polytope

$$P_{i+1} = \{\alpha \in P(V, 2d(i+1)) : \alpha(v) = 1\ (\forall v \in A_i)\}$$

We check that P_{i+1} is non-empty. As $|A_i| < h_i \le 2di$, the linear dependence $\alpha = \gamma_i + t(\mathbf{1} - \gamma_i)$ lies in P_{i+1} for a suitable t, we only have to check that $0 < t < 1$ as this implies $0 \le \alpha(v) = \gamma_i(v) + t(1 - \gamma_i(v)) \le 1$. To fulfill the condition $\sum_{v\in V} \alpha(v) = 2d(i+1)$, we must set

$$t = \frac{2d(i+1) - h_i}{n - h_i} = 1 - \frac{n - 2d(i+1)}{n - h_i}.$$

Thus $0 < t < 1$ indeed as $h_i \le 2di$.

Next, let α_{i+1} be a fixed vertex of P_{i+1}. The method recorded in Remark 2 gives another linear dependence γ_{i+1} with $|C_{i+1}| = d$. $A_i \subset A_{i+1}$ by the construction. All $v \in V$ with $\alpha_{i+1}(v) = 1$ are in A_{i+1}, and there are at least $2d(i+1) - d$ of them. Thus $(2i+1)d \le |A_{i+1}|$. Further $|A_{i+1}| < h_{i+1}$ follows since $\gamma_{i+1}(v) = 1$ for every $v \in A_{i+1}$ and $h_{i+1} \le 2d(i+1)$ because $h_{i+1} = \sum_{v\in V} \gamma_{i+1}(v) \le \sum_{v\in V} \alpha_{i+1}(v) = 2d(i+1)$.

The construction is almost finished, as a last step we define $A_0 = C_0 = \emptyset$.

We use Lemma 3 next. The parallelotope $P := \sum_{v\in C_i} [0, v]$ contains the point $-s(A_i)$, since $0 = s(A_i) + \sum_{v\in C_i} \gamma_i(v)v$. A vertex of P is of the form $s(D) = \sum_{v\in D} v$, where D is a subset of C_i. By Lemma 3, there is a $D_i \subset C_i$ such that the vertex $s(D_i)$ is at distance $O(\sqrt{d})$ from $-s(A_i)$. Thus the vector $z_i = s(A_i \cup D_i)$ is short, namely, $\|z_i\| = O(\sqrt{d})$. Note that by setting $D_0 = \emptyset$, we have $z_0 = 0$ which is again of norm $O(\sqrt{d})$.

For the next step of the proof we first check that the size of the symmetric difference $(A_{i+1} \cup D_{i+1}) \triangle (A_i \cup D_i)$ is at most $5d$. This holds for $i = 0$. For larger i, D_{i+1} and A_{i+1} are disjoint, and A_{i+1} contains A_i, so the symmetric difference is the same a $X \triangle D_i$, where $X = (A_{i+1} \setminus A_i) \cup D_{i+1}$. Here $|A_{i+1} \setminus A_i| < 3d$, and both D_i and D_{i+1} have at most d elements, which gives the upper bound $5d$.

Now $z_i - s(D_i) + s(X) = z_{i+1}$. Thus, adding at most $5d$ vectors from B^d_∞ to z_i one arrives at z_{i+1}, and both z_i, z_{i+1} are short. Define

$$W = \{-u : u \in D_i \setminus X\} \bigcup (X \setminus D_i).$$

Then W is a subset of B^d_∞, of at most $5d$ elements, such that $s(W) = \sum_{w \in W} w = z_{i+1} - z_i$. Thus $\|s(W)\| = O(\sqrt{d})$. By applying Lemma 4 to W we get an ordering $w_1, \dots, w_m$ such that every partial sum along this ordering is $O(\sqrt{d})$. Then for every $h = 1, \dots, m$.

$$\|z_i + \sum_1^h w_j\| \le \|z_i\| + \|\sum_1^h w_j\| = O(\sqrt{d}).$$

In the original problem we have to show that for every $k \le n$ there is a set $U \subset V$ of size k with $\|s(U)\| = O(\sqrt{d})$. This is clear when k equals the size of some $A_i \cup D_i$, but what is to be done for the other k? Well, such a k lies between $|A_i \cup D_i|$ and $|A_{i+1} \cup D_{i+1}|$ for some i. Note that $z_i = s(A_i \cup D_i)$. Moreover, each sum $z_i + w_1 + \ldots + w_h$ is the sum of vectors in a subset of V. This can be seen by induction on h. The case $h = 0$ is clear. The induction step $h - 1 \to h$ is clear again when w_h does not come from D_i, simply one more term appears in the sum. If however w_h comes from D_i, then it cancels the previous $-w_h$ that is a unique term in $s(A_i \cup D_i)$. So each partial sum is a subset-sum. The number of elements in these subsets increases or decreases by one when the next w_h is added. Then for every k between $|A_i \cup D_i|$ and $|A_{i+1} \cup D_{i+1}|$ there is a partial sum containing exactly k terms. □

Remark 4. The above proof yields a slightly stronger statement: we construct a chain of subsets of V, each with sum of order of magnitude $O(\sqrt{d})$, so that the cardinality of two consecutive subsets differ by one, and the chain traverses from the empty set to V. We have hoped to give a better value for the Steinitz constant $S(B^d_2)$ or $S(B^d_\infty)$ by a suitable modification of the argument (we would need an *increasing* chain of subsets with the previous properties), but our efforts have failed so far.

Remark 5. A simpler proof may be given if one only aims for the existence a k-element subset with small sum. We may assume that $k \leqslant n - d$. Starting from a vertex of $P(v, k-d)$ and using Lemma 3, similarly to the proof of Theorem 4, we can construct a set W so that $\|s(W)\| \leqslant 6\sqrt{d}$, and $k - 2d \leqslant |W| \leqslant k$. Let α be the characteristic function of W, i. e. $\alpha(v) = 1$ if $v \in W$, and 0 otherwise. Let $l = |W|$, and set t so that $l + t(n - l) = k + d$. Then $t \leqslant 1$.

Next, consider the set P of the linear dependencies $\beta : V \to [0, 1]$ with

$$\sum_{v \in V} \beta(v)v = (1-t)s(W), \; \sum_{v \in V} \beta(v) = k + d, \; \beta(v) = 1 (\forall v \in W).$$

Then P is a non-empty convex polytope, since $\alpha + t(\mathbf{1} - \alpha)$ satisfies all the above conditions. Take an arbitrary a vertex of P. As before, invoking Lemma 3, we find a set Y so that $\|s(Y) - (1-t)s(W)\|_\infty = O(\sqrt{d})$, and

$||Y| - (k+d)| \leqslant d$. Furthermore, the construction implies that $W \subset Y$. Hence,

$$k - 2d \leqslant |W| \leqslant k \leqslant |Y| \leqslant k + 2d,$$

and $\|s(W)\| = O(\sqrt{d})$ as well as $\|s(Y)\| = O(\sqrt{d})$. We finish the proof by applying Lemma 4 to the set $Y \setminus W$.

Remark 6. The above proofs translate for arbitrary norms as long as the analogues of Lemmas 1 and 2 (or Lemmas 3 and 4) may be established.

7. Proof of Lemma 4

For this lemma it is natural to use Chobanyan's transference theorem [5] (see also [1]), which connects Steinitz's theorem with sign assignments to vectors in a sequence.

Assume $v_1, \dots, v_n$ is a sequence of vectors from the unit ball B of an arbitrary norm on $\mathbb{R}^d$. It is proved in [2] that there are signs $\varepsilon_1, \dots, \varepsilon_n = \pm 1$ such that

$$\max_{k=1,\dots,n} \Big\| \sum_{i=1}^{k} \varepsilon_i v_i \Big\| \leq 2d - 1. \tag{2}$$

This is a general bound that does not depend on n and the norm. But better estimates are valid for specific norms and some (small) values of n. For fixed B and n let $F(B,n)$, the *sign sequence constant of* B, be defined as the smallest number that one can write on the right hand side of (2), and let $F(B) = \sup_n F(B,n)$. It is quite easy to see for instance that $F(B_2^d, n) \leq \sqrt{n}$ for all n (but we don't need this). What we need is a result of Spencer [11, Theorem 1.4]:

Fact 1. $F(B_\infty^d, d) \leq K\sqrt{d}$ where K is a universal constant.

Chobanyan's transference theorem [5] says that, for every norm with unit ball B, $S(B) \leq F(B)$, that is, the Steinitz constant is at most as large as the sign sequence constant. We need a slightly stronger variant, so we define $S(B,n)$ as the smallest number R such that for every set $V \subset B$ with $s(V) = 0$ and $|V| = n$ there is an ordering $v_1, \dots, v_n$ of the elements in V such that

$$\max_{k=1,\dots,n} \Big\| \sum_{i=1}^{k} v_i \Big\| \leq R.$$

Of course, $S(B) = \sup_n S(B,n)$. Here comes the stronger version of Chobanyan's theorem, and comes without proof as the proof is identical with the original one.

Theorem 8. *For every norm in $\mathbb{R}^d$ with unit ball B, $S(B,n) \leq F(B,n)$.*

Theorem 8 and Fact 1 imply the following.

Fact 2. Given $V \subset B_\infty^d$ with $|V| = m$ where $m \leq 5d$ and $s(V) = 0$, there is an ordering $v_1, \dots, v_m$ of V such that $\max_{h=1,\dots,m} \| \sum_1^h v_i \|_\infty \leq K_1 \sqrt{d}$, where K_1 is a universal constant.

Proof. We note first that for $m \le d$ this follows directly from Fact 1 and Theorem 8 with $K_1 = K$. For $m \ge d$, take the natural embedding of $\mathbb{R}^d$ into $\mathbb{R}^m$, the set V lies in the ℓ_∞ unit ball of $\mathbb{R}^m$. Apply Fact 1 and Theorem 8 there, and you get an ordering of V in $\mathbb{R}^d$ along which all partial sums have norm at most $K\sqrt{m} \le K\sqrt{5d}$. Thus Fact 2 holds with $K_1 = \sqrt{5}K$. □

Proof of Lemma 4. We need a concrete bound on $\|s(W)\|_\infty$, so suppose that $\|s(W)\|_\infty \le K_2\sqrt{d}$. For $w \in W$ define $w^* = w - \frac{1}{m}s(W)$. Then $\|w^*\|_\infty \le \|w\|_\infty + \frac{1}{m}\|s(W)\|_\infty \le 2$ as $s(W)$, being the sum of m vectors from B^d_∞, has norm at most m. Further, $\sum_{w\in W} w^* = 0$ and $W \subset 2B^d_\infty$. By Fact 2 there is an ordering $w_1, \dots, w_m$ of the vectors in W such that for every h

$$\Big\| \sum_1^h w_i^* \Big\|_\infty \le 2K_1\sqrt{d}.$$

We check that $\sum_1^h w_i = \sum_1^h w_i^* + \frac{h}{m}s(W)$ and so for every h

$$\Big\| \sum_1^h w_i \Big\|_\infty \le \Big\| \sum_1^h w_i^* \Big\|_\infty + \Big\| s(W) \Big\|_\infty \le 2K_1\sqrt{d} + K_2\sqrt{d} = O(\sqrt{d}). \quad \square$$

8. An application: proof of Theorem 6

We proceed by induction on m. For $m = 1$, the assertion is clearly true. For the induction step $(m-1) \to m$ let $V \subset B$ with $|V| = (k-1)m + 1$ and $\|s(V)\| \le 1$. Set $v_0 = -s(V)$ so $\|v_0\| \le 1$. Define $V_0 = V \cup \{v_0\}$. Then $V_0 \subset B$ and $s(V_0) = 0$. So by Theorem 1, there exists a subset U of V_0 of size k, with $\|s(U)\| \leqslant 1$. We are done if $v_0 \notin U$. So suppose that $v_0 \in U$. Then $v_0 \notin W := V \setminus U$, and $\|s(W)\| \le 1$ because

$$s(U) = -s(W).$$

Here W is of size $(m-1)(k-1) + 1$, so the induction hypothesis implies that W contains a subset U of size k with $\|s(U)\| \le 1$. □

We mention finally that Theorem 6 is equivalent to the following Helly type statement. If $V \subset B$ and $|V| = (k-1)m+1$, and $\|s(U)\| > 1$ for every set $U \subset V$ of size k, then $\|s(V)\| > 1$.

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