Stackings and the W-cycles conjecture

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Abstract

We prove Wise's W-cycles conjecture: Consider a compact graph Γ' immersing into another graph Γ. For any immersed cycle $\Lambda : S^1 \to \Gamma$, we consider the map Λ' from the circular components $\mathbb{S}$ of the pullback to Γ'. Unless Λ' is reducible, the degree of the covering map $\mathbb{S} \to S^1$ is bounded above by minus the Euler characteristic of Γ'. As a corollary, any finitely generated subgroup of a one-relator group has finitely generated Schur multiplier.

1 Introduction

As part of his work on the coherence of one-relator groups, Wise made a conjecture about the number of lifts of a cycle in a free group along an immersion, which we will call the *W-cycles conjecture*. If $f_1 : \Gamma_1 \looparrowright \Gamma$ and $f_2 : \Gamma_2 \looparrowright \Gamma$ are immersions of graphs, then the fibre product

$$\Gamma_1 \times_\Gamma \Gamma_2 = \{(x, y) \in \Gamma_1 \times \Gamma_2 \mid f_1(x) = f_2(y)\}$$

immerses into Γ_1 and Γ_2, and is the pullback of f_1 and f_2. An immersed loop $\Lambda : S^1 \looparrowright \Gamma$ is *primitive* if it does not factor properly through any other immersion $S^1 \looparrowright \Gamma$.

With this definition, the W-cycles conjecture can be stated as follows.

Conjecture 1 (Wise [Wis05]). *Let $\rho : \Gamma' \to \Gamma$ be an immersion of finite connected core graphs and let $\Lambda : S^1 \to \Gamma$ be a primitive immersed loop. Let $\mathbb{S}$ be the union of the circular components of $\Gamma' \times_\Gamma S^1$. Then the number of components of $\mathbb{S}$ is at most the rank of Γ'.*

The purpose of this note is to prove Wise's conjecture; indeed, we prove a stronger statement. As usual, if π is a covering map then $\deg \pi$ denotes its degree, the number of preimages of a point. An immersion of a union of circles $\Lambda : \mathbb{S} \to \Gamma$ is called *reducible* if there is an edge of Γ which is traversed at most once by Λ.

Theorem 2. *Let $\rho : \Gamma' \looparrowright \Gamma$ be an immersion of finite connected core graphs and let $\Lambda : S^1 \to \Gamma$ be a primitive immersed loop. Suppose that $\mathbb{S}$, the union of* *the circular components of* $\Gamma' \times_\Gamma S^1$*, is non-empty, so there is a natural covering map* $\sigma : \mathbb{S} \looparrowright S^1$*. Then either*

$$\deg \sigma \leq -\chi(\Gamma')$$

or the pullback immersion $\Lambda' : \mathbb{S} \to \Gamma'$ *is reducible.*

The statement of the conjecture is a corollary of this theorem. Indeed, the inequality in the theorem is strictly stronger than the inequality in the conjecture; alternatively, in the reducible case, we may remove an edge and proceed by induction.

Wise's notion of *nonpositive immersions* provides a connection with a famous question of Baumslag [Bau74]: is every one-relator group coherent? (Recall that a group is *coherent* if every finitely generated subgroup is finitely presented.) As in the case of graphs, an immersion of cell complexes is a locally injective cellular map.

Definition 3 (Wise)**.** A cell complex X has *nonpositive immersions*, or *NPI* if, for every immersion of compact, connected complexes $Y \looparrowright X$, either $\chi(Y) \leq 0$ or Y has trivial fundamental group.

Presentation complexes of one-relator groups with torsion do not have nonpositive immersions. Let C_k be the presentation complex of $\mathbb{Z}/k\mathbb{Z}$ associated to the presentation $\langle a \mid a^k \rangle$, and for $l \mid k$, let $C_{k,l}$ be the l–fold cover of C_k.

Definition 4. A cell complex X has *not too positive immersions*, or *NTPI* if, for every immersion of compact, connected complexes $Y \looparrowright X$, Y is homotopy equivalent to a wedge of subcomplexes of $C_{k,l}$s and a compact 2-complex $Y' \subset Y$ with $\chi(Y') \leq 0$.

For $k = 1$ this reduces to NPI, since $C_{1,l}$ is a disk. Our main theorem implies that presentation complexes associated to one-relator groups have NTPI; in particular, in the torsion-free case, they have NPI.

Corollary 5. *Let* X *be compact 2-complex with one 2-cell* e^2 *and suppose that the attaching map* $\Lambda \colon S^1 \to X^{(1)}$ *of* e^2 *is an immersion. Then* X *has NTPI.*

Proof. Suppose that $\rho \colon Y \looparrowright X$ is an immersion of a compact 2-complex Y into X. Let $\Gamma = X^{(1)}$, $\Gamma' = Y^{(1)}$, and $\Lambda' \colon \mathbb{S} \to \Gamma'$ be the pullback immersion, in the notation of Theorem 2. Let $\mathbb{S}'$ be the union of the components $S_1, \ldots, S_m$ of $\mathbb{S}$ that are realized by boundaries of 2-cells of Y. If $\chi(Y) > 0$ then $\deg(\sigma) > -\chi(\Gamma')$ and so, by Theorem 2, Λ' is reducible. That is, there is some edge e of Γ' traversed by at most one component S of $\mathbb{S}$.

If S isn't contained in $\mathbb{S}'$, we may remove the edge e and proceed by induction on the size of the one-skeleton of Y.

We may therefore suppose that S is a component of $\mathbb{S}'$. Suppose that Λ is realized (up to conjugacy) by a kth power w^k in $\pi_1\Gamma$, and that the covering map $S \to S^1$ has degree l. Then l divides k, and Y is homotopy equivalent to a wedge $D_{k,l} \vee Y'$, where $D_{k,l}$ is a subcomplex of $C_{k,l}$ and Y' is the subcomplex of Y with the edge e and all 2-cells attached to S removed. We now proceed by induction on the number of 2-cells of Y. □

Wise has conjectured that, if a 2-complex X has nonpositive immersions, then its fundamental group is coherent. Although Baumslag's conjecture remains open, we do obtain a weaker statement: every finitely generated subgroup of a one-relator group has finitely generated Schur multiplier.

Corollary 6. *Let G be a one-relator group. If $H < G$ is finitely generated then*

$$\text{rank}(H_2(H, \mathbb{Z}))) \leq b_1(H) - 1$$

In his proof that three-manifold groups are coherent [Sco73], Scott introduces the notion of *indecomposable covers*: If G is a finitely generated freely indecomposable group then $K \twoheadrightarrow G$ is an indecomposable cover if it doesn't factor (surjectively) through a free product. The next lemma is a straightforward consequence of the existence of indecomposable covers.

Lemma 7. *Let $G = G_1 * \cdots * G_n * \mathbb{F}_k$ be the Grushko decomposition of a finitely generated group G, with G_i freely indecomposable. There is a finitely presented group $H = H_1 * \cdots * H_n * \mathbb{F}_k$ and a surjective homomorphism $\varphi\colon H \twoheadrightarrow G$ such that $\varphi|_{H_i}\colon H_i \twoheadrightarrow G_i$ is an indecomposable cover.*

Let X be the presentation complex of a one-relator group G, and let $Y \looparrowright X$ be a covering map corresponding to a finitely generated subgroup H. By a trivial generalization of Stallings' folding technique [Sta83], there is a sequence of immersions of finite complexes obtained by first immersing a graph Y_1 in X and repeatedly adding relations and folding

$$Y_1 \looparrowright Y_2 \looparrowright \ldots \looparrowright Y_n \looparrowright \ldots \looparrowright Y$$

with the property that each immersion $Y_i \looparrowright Y_{i+1}$ induces a surjection on fundamental groups and such that $Y = \varinjlim Y_i$. If H is one-ended, by Lemma 7, we may assume that each Y_i has one-ended fundamental group and, by Corollary 5, that $\chi(Y_i) \leq 0$.

Proof of Corollary 6. Let Y and Y_i be the spaces constructed in the previous paragraph. By [Lyn50], both $H_2(G, \mathbb{Z})$ and $H_2(H, \mathbb{Z})$ are torsion-free, so it suffices to show that $b_2(Y) \leq b_1(H) - 1$. Combining Corollary 5 with Lemma 7 we may assume that H is one-ended and that $\chi(Y_i) \leq 0$. No Y_i is simply connected and so, since X has NTPI and H is one-ended, $\chi(Y_i) \leq 0$ for all i. Since homology commutes with direct limits, it follows that $\text{rank}(H_2(Y, \mathbb{Z})) \leq b_1(H) - 1$ as claimed. □

Our proof of Theorem 2 was inspired by the proof of the following theorem of Duncan and Howie. In particular, the punch line in Lemma 13 is essentially their proof of [DH91, Lemma 3.1].

The *genus* of an element w in a free group F is the minimal number g so that $w = \Pi_{i=1}^{g} [x_i, y_i]$ has a solution in F, or equivalently, the minimal genus of a once-holed surface mapping into a graph representing F with boundary w.

Theorem ([DH91, Corollary 5.2])**.** *Let w be an indivisible element in a free group F. Then the genus of w^m is at least $m/2$.*

While this work was in preparation, we learned that Helfer and Wise have also proved the W-cycles conjecture [HW14] and its generalization to staggered presentations (See Remark 18).

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2 Stackings

2.1 Computing the characteristic of a free group

By a *circle*, we mean a graph homeomorphic to S^1.

Definition 8. Let Γ be a finite graph, let $\mathbb{S}$ be a disjoint union of finitely many circles, and let $\Lambda\colon \mathbb{S} \looparrowright \Gamma$ be a map of graphs. Consider the trivial $\mathbb{R}$-bundle $\pi\colon \Gamma\times\mathbb{R}\to\Gamma$. A *stacking* is an embedding $\hat{\Lambda}\colon \mathbb{S}\hookrightarrow\Gamma\times\mathbb{R}$ such that $\pi\hat{\Lambda}=\Lambda$.

Although this definition is very simple, it leads to a natural way of estimating the Euler characteristic of a graph.

Let π and ι be the projections of $\Gamma\times\mathbb{R}$ to Γ and $\mathbb{R}$, respectively. Let

$$\mathcal{A}_{\hat{\Lambda}} = \{x\in\mathbb{S} \mid \forall y\neq x\ (\Lambda(x)=\Lambda(y)\Rightarrow \iota(\hat{\Lambda}(x))>\iota(\hat{\Lambda}(y)))\}$$

and

$$\mathcal{B}_{\hat{\Lambda}} = \{x\in\mathbb{S} \mid \forall y\neq x\ (\Lambda(x)=\Lambda(y)\Rightarrow \iota(\hat{\Lambda}(x))<\iota(\hat{\Lambda}(y)))\}$$

Intuitively, $\mathcal{A}_{\hat{\Lambda}}$ is the set of points of $\hat{\Lambda}(\mathbb{S})$ that one sees if one looks at $\hat{\Lambda}(\mathbb{S})$ from above, and likewise $\mathcal{B}_{\hat{\Lambda}}$ is the set of points of $\hat{\Lambda}(\mathbb{S})$ that one sees from below.

Henceforth, assume that $\Lambda:\mathbb{S}\to\Gamma$ is an immersion. The stacking $\hat{\Lambda}$ is called *good* if $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$ each meet every connected component of $\mathbb{S}$. For brevity, we will call a subset $s\subseteq\mathbb{S}$ an *open arc* if it is connected, simply connected, open, and a union of vertices and interiors of edges.

Lemma 9. *If Λ is an immersion then each connected component of $\mathcal{A}_{\hat{\Lambda}}$ or $\mathcal{B}_{\hat{\Lambda}}$ is either a connected component of $\mathbb{S}$ or an open arc in $\mathbb{S}$.*

Proof. It suffices to prove the lemma for $\mathcal{A}_{\hat{\Lambda}}$. Let $s\subseteq\mathbb{S}$ be a connected component of $\mathcal{A}_{\hat{\Lambda}}$. It follows from the definition that s is open. Note also that if one point p in the interior of an edge e is contained in $\mathcal{A}_{\hat{\Lambda}}$ then the whole interior of e is contained in $\mathcal{A}_{\hat{\Lambda}}$. This completes the proof. □

The next lemma characterizes reducible maps in terms of a stacking; in particular, reducibility is reduced to non-disjointness of $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$.

Lemma 10. *If $\hat{\Lambda}$ is a stacking of an immersion $\Lambda : \mathbb{S} \to \Gamma$, then $\mathcal{A}_{\hat{\Lambda}} \cap \mathcal{B}_{\hat{\Lambda}}$ contains the interior of an edge if and only if Λ is reducible. If $\hat{\Lambda}$ is a good stacking and $\mathcal{A}_{\hat{\Lambda}}$ or $\mathcal{B}_{\hat{\Lambda}}$ contains a circle then $\hat{\Lambda}$ is reducible.*

Proof. To first assertion is immediate from the definitions. It suffices to prove the second assertion for $\mathcal{A}_{\hat{\Lambda}}$. Let S be a component of $\mathbb{S}$ contained in $\mathcal{A}_{\hat{\Lambda}}$. Since $\mathbb{S}$ is good, there is an edge e of S contained in $\mathcal{B}_{\hat{\Lambda}}$. Therefore, e is contained in both $\mathcal{A}_{\hat{\Lambda}}$ and $\mathcal{B}_{\hat{\Lambda}}$. It follows that e is traversed exactly once $\hat{\Lambda}$, so $\hat{\Lambda}$ is reducible. □

The final lemma of this section is completely elementary, but is the key observation in the proof. It asserts that number of open arcs in $\mathcal{A}_{\Lambda}$ or $\mathcal{B}_{\Lambda}$ computes the Euler characteristic of the image of Λ.

Lemma 11. *Let $\hat{\Lambda} : \mathbb{S} \to \Gamma \times \mathbb{R}$ be a stacking of a surjective immersion $\Lambda : \mathbb{S} \to \Gamma$. The number of open arcs in $\mathcal{A}_{\hat{\Lambda}}$ or $\mathcal{B}_{\hat{\Lambda}}$ is equal to $-\chi(\Gamma)$.*

Proof. As usual, it suffices to prove the lemma for $\mathcal{A}_{\hat{\Lambda}}$. Let x be a vertex of Γ of valence $v(x)$. Because Λ is surjective, exactly $v - 2$ edges incident at x are covered by open arcs of $\mathcal{A}_{\hat{\Lambda}}$ that end at x. Therefore, the number of open arcs is

$$\frac{1}{2} \sum_{x \in V(\Gamma)} (v(x) - 2)$$

which is easily seen to be $-\chi(\Gamma)$. □

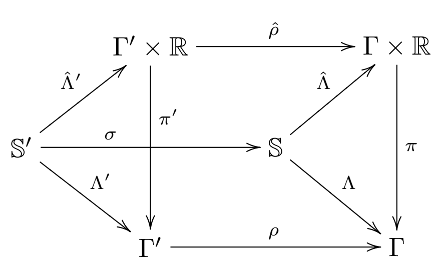
2.2 Computing the characteristic of a subgroup

As in the previous section, Γ is a finite graph, $\Lambda : \mathbb{S} \looparrowright \Gamma$ is an immersion and $\hat{\Lambda} : \mathbb{S} \hookrightarrow \Gamma \times \mathbb{R}$ is a stacking. Consider now an immersion of finite graphs $\rho : \Gamma' \to \Gamma$, and let $\mathbb{S}'$ be the circular components of the fibre product $\mathbb{S} \times_{\Gamma} \Gamma'$, which is equipped with a map $\sigma : \mathbb{S}' \to \mathbb{S}$ and an immersion $\Lambda' : \mathbb{S}' \to \Gamma'$. Note that if $\mathbb{S}'$ is non-empty then σ is a covering map. In order to prove Theorem 2, we would like to estimate the characteristic of Γ' in terms of $\hat{\Lambda}$.

The stacking $\hat{\Lambda}$ of Λ naturally pulls back to a stacking $\hat{\Lambda}'$ of Λ'. More precisely, there is a natural isomorphism

$$(\Gamma \times \mathbb{R}) \times_{\Gamma} \Gamma' \cong \Gamma' \times \mathbb{R}$$

and the universal property of the fibre bundle defines a map $\hat{\Lambda}' : \mathbb{S}' \to \Gamma \times \mathbb{R}$, so we have the following commutative diagram.



Lemma 12. *If $\hat{\Lambda}$ is a stacking then $\hat{\Lambda}'$ is also a stacking. Furthermore, if $\hat{\Lambda}$ is good then $\hat{\Lambda}'$ is also good.*

Proof. The proof of the first assertion is a diagram chase, which we leave as an exercise to the reader. The second assertion follows immediately from the observation that $\sigma^{-1}(\mathcal{A}_{\hat{\Lambda}}) \subseteq \mathcal{A}_{\hat{\Lambda}'}$ and $\sigma^{-1}(\mathcal{B}_{\hat{\Lambda}}) \subseteq \mathcal{B}_{\hat{\Lambda}'}$. □

The final lemma in this section estimates the Euler characteristic of Γ' using a stacking of the pullback immersion Λ'. Since all finitely generated subgroups of free groups can be realized by immersions of finite graphs, this can be thought of as an estimate for the rank of a subgroup of a free group; this point of view motivates the title of this subsection.

Lemma 13. *If $\hat{\Lambda}$ is a good stacking then either $\Lambda' : \mathbb{S}' \to \Gamma'$ is reducible or*

$$-\chi(\Lambda'(\mathbb{S}')) \geq \deg \sigma$$

Proof. Suppose Λ' is not reducible; in particular, Λ' is surjective.

Let e be an edge in $\mathcal{A}_{\hat{\Lambda}}$ and consider its $\deg \sigma$ preimages $\{e'_j\}$. Since Λ' is not reducible, no component of $\mathcal{A}_{\hat{\Lambda}'}$ is a circle, by Lemma 10, and so every e'_j is contained in an open arc of $\mathcal{A}_{\hat{\Lambda}'}$.

If $-\chi(\Gamma') < \deg \sigma$ then, by Lemma 11 and the pigeonhole principle, two distinct preimages e'_i and e'_j are contained in the same open arc A. But then, for any f an edge of $\mathbb{S}$ contained in $\mathcal{B}_{\hat{\Lambda}}$ (which again exists because $\hat{\Lambda}$ is good), A also contains an edge f' that maps to f. Therefore, $\mathcal{A}_{\hat{\Lambda}'} \cap \mathcal{B}_{\hat{\Lambda}'}$ contains f', and so Λ' is reducible by Lemma 10. See Figure 1. □

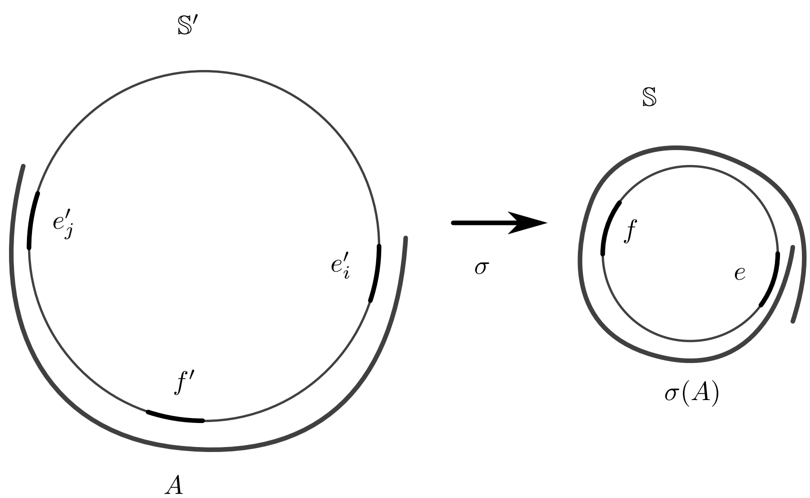


Figure 1: If $-\chi(\Gamma')$ is smaller than the sum of the degrees then Λ' is reducible.

3 A tower argument

In order to apply Lemma 13 to prove Theorem 2, we need to prove that stackings exist. The proof here employs a *cyclic tower argument* of the kind used by Brodskiĭ and Howie to prove that one-relator groups are right-orderable and locally indicable [Bro80, How82].

Definition 14. Let X be a complex. A *(cyclic) tower* is the composition of a finite sequence of maps

$$X_0 \looparrowright X_1 \looparrowright \ldots \looparrowright X_n = X$$

such that each map $X_i \looparrowright X_{i+1}$ is either an inclusion of a subcomplex or a covering map (resp. a normal covering map with infinite cyclic deck group).

One can argue by induction with towers because of the following lemma of Howie (building on ideas of Papakyriakopoulos and Stallings) [How81].

Lemma 15. *Let $Y \to X$ be cellular map of compact complexes. Then there exists a maximal (cyclic) tower map $X' \looparrowright X$ such that $Y \to X$ lifts to a map $Y \to X'$.*

As in the previous sections let Γ be a graph. To apply a cyclic tower argument, one needs to know that the phenomene of interest are preserved by cyclic coverings. In our case, that control is provided by the following lemma.

Lemma 16. *Consider an infinite cyclic cover of a graph Γ. Then there is an embedding $\tilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ such that the diagram*

$$\begin{CD} \tilde{\Gamma} \times \mathbb{R} @>\tilde{\pi}>> \tilde{\Gamma} \\ @VVV @VVV \\ \Gamma \times \mathbb{R} @>\pi>> \Gamma \end{CD}$$

commutes where, as usual π and $\tilde{\pi}$ denote coordinate projections onto Γ and $\tilde{\Gamma}$ respectively. (Note that the embedding $\tilde{\Gamma} \times \mathbb{R} \hookrightarrow \Gamma \times \mathbb{R}$ is usually not *natural with respect to the coordinate projections onto $\mathbb{R}$.)*

Proof. Elements g of the group $\pi_1\Gamma$ act by deck transformations $x \mapsto gx$ on the covering space $\tilde{\Gamma}$. The infinite cyclic covering $\tilde{\Gamma} \to \Gamma$ also defines a homomorphism $\pi_1\Gamma \to \mathbb{Z}$, which in turn allows elements g of $\pi_1\Gamma$ to act by translation on $\mathbb{R}$.

Consider the diagonal action of $\pi_1\Gamma$ on $\tilde{\Gamma}\times\mathbb{R}$. The quotient is homeomorphic to $\Gamma\times\mathbb{R}$. Let $X = \tilde{\Gamma}\times(-1/2, 1/2) \subset \tilde{\Gamma}\times\mathbb{R}$. Distinct translates of X are disjoint, and so the map $X \hookrightarrow \tilde{\Gamma}\times\mathbb{R}$ descends to an embedding $X \hookrightarrow \Gamma\times\mathbb{R}$. Any choice of homeomorphism $(-1/2, 1/2) \cong \mathbb{R}$ identifies X with $\tilde{\Gamma} \times \mathbb{R}$. It is straightforward to check that the claimed diagram commutes. □

We are now ready to prove that stackings exist. A very simple example of a stacking is illustrated in Figure 2.

Lemma 17. *Any primitive immersion* $\Lambda\colon S^1 \to \Gamma$ *has a stacking*

$$\hat{\Lambda}\colon S^1 \to \Gamma \times \mathbb{R}$$

Proof. Let $\Gamma_0 \looparrowright \Gamma_1 \looparrowright \ldots \looparrowright \Gamma_m = \Gamma$ be a maximal cyclic tower lifting of Λ, and let $\Lambda_n : S^1 \to \Gamma_n$ be the lift of Λ to Γ_n. Note that Γ_0 is a circle and Λ_0 is a finite-to-one covering map. Since Λ is primitive, it follows that Λ_0 is a homeomorphism and hence trivially stackable.

Proceeding by induction on n, let $\hat{\Lambda}_{n-1} : S^1 \hookrightarrow \Gamma_{n-1} \times \mathbb{R}$ be a stacking of Λ_{n-1}. If $\Gamma_{n-1} \to \Gamma_n$ is an inclusion of subgraphs then it extends naturally to an inclusion $i : \Gamma_{n-1} \times \mathbb{R} \hookrightarrow \Gamma_n \times \mathbb{R}$, and so $\hat{\Lambda} = i \circ \hat{\Lambda}_{n-1}$ is a stacking.

Suppose therefore that $\Gamma_{n-1} \to \Gamma_n$ is an infinite cyclic covering map. Let $i : \Gamma_{n-1} \times \mathbb{R} \to \Gamma_n \times \mathbb{R}$ be the embedding provided by Lemma 16. Then $\hat{\Lambda}_n = i \circ \hat{\Lambda}_{n-1}$ is an embedding $S^1 \hookrightarrow \Gamma_n \times \mathbb{R}$, and a simple diagram chase confirms that $\hat{\Lambda}_n$ is a lift of Λ_n. This completes the proof. □

Remark 18. Note that any stacking of a map of a single circle is automatically good. Lemma 17 (also implicit in [HW14]) holds for graphs and immersions associated to staggered presentations.

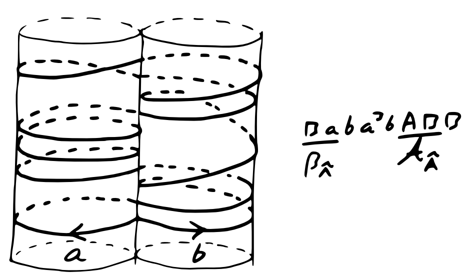


Figure 2: A stacking of the word $Baba^3bABB$.

Let $L = \langle x_1, \ldots, x_n \mid w \rangle$ be a one-relator group, where w is a cyclically reduced nonperiodic word $w = x_{i_1} \cdots x_{i_m}$ in the x_i. Duncan and Howie use right-orderability of L to assign heights to the (distinct, by [How82, Corollary 3.4]) elements $a_0 = 1$, $a_j = x_{i_1} \cdots x_{i_j}$, $j < m$, in L in the same way we use the embedding $\hat{\Lambda}$ to find open arcs which remain above ($\mathcal{A}$) or below ($\mathcal{B}$) every point of S^1 with the same image in Γ. Lemma 17 is equivalent to the existence of a right-invariant pre-order on L which distinguishes between the elements a_j. Lemma 17 is also closely related to the main theorem of [Far76].

Our main theorem is now a quick consequence of Lemmas 13 and 17.

Proof of Theorem 2. Let Γ, Γ', etc., be as in Theorem 2, and let $\hat{\Lambda}$ be the stacking provided by Lemma 17. Since S^1 is connected, the stacking $\hat{\Lambda}$ is automatically good. By hypothesis Λ' is not reducible, and therefore by Lemma 13, $-\chi(\Gamma') \geq -\chi(\Lambda'(\mathbb{S}')) \geq \deg \sigma$ as claimed. $\square$

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