Stability of the unique continuation for the wave operator via Tataru inequality and applications

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Abstract

In this paper we study the stability of the unique continuation in the case of the wave equation with variable coefficients independent of time. We prove a logarithmic estimate in a arbitrary domain of $\mathbb{R}^{n+1}$, where all the parameters are calculated explicitly in terms of the C^1-norm of the coefficients and on the other geometric properties of the problem. We use the Carleman-type estimate proved by Tataru in 1995 and an iteration of the local stability. We apply the result to the case of a wave equation with data on a cylinder an we get a stable estimate for any positive time, also after the first conjugate point associated with the geodesics of the metric of the variable coefficients.

Keywords: wave equation, unique continuation property, stability, analysis on manifolds, optimal control time.

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1 Introduction

We consider the wave operator in $\mathbb{R}^{n+1}$,

$$P(y, D) = -D_0^2 + \sum_{j,k=1}^{n} g^{jk}(x) D_j D_k + \sum_{j=1}^{n} h^j(x) D_j + q(x), \tag{1.1}$$

where $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ are the time-space variables, $D_0 = -i\partial_t$, $D_j = -i\partial_{x_j}$. The coefficients $g^{jk} \in C^1(\mathbb{R}^n)$ are real and independent of time, and $[g^{jk}]$ is a symmetric positive-definite matrix. The coefficients $h^j, q \in C^0(\mathbb{R}^n)$ are complex valued and independent of time.
An operator $P(y, D)$ is said to have the unique continuation property if for any solution u to $Pu = 0$ in a connected open set $\Omega \subset \mathbb{R}^{n+1}$ and vanishing on an open subset $B \subset \Omega$, it follows that u vanishes in Ω.
In the paper [28] Tataru proved for the first time the unique continuation property for (1.1) across every non-characteristic C^2-hypersurface with no limitation to the normal

direction. The key point of these results is a Carleman-type estimate involving an exponential pseudo-differential operator.

Much is known about the consequences of the general unique continuation property for the corresponding Cauchy problem. Actually the unique continuation property has proved to be instructive in many areas of mathematics, e.g. in studying the uniqueness for linear and nonlinear PDEs together with their blow up or traveling wave solutions [11], in studying the Anderson localization [7], in control theory to get controllability results [30, 31], in inverse problems to obtain uniqueness and stability estimates [18]. In particular Tataru's result [28] is crucial for the development of the Boundary Control method (see [5] for pioneering work and [17] for detailed exposition of the further developments).

Concerning the continuous dependence of the unique continuation property, that is its stability, less results are available. The elliptic and the parabolic cases have been studied in several settings by using either Carleman estimates or some versions of the three ball theorem (see [1], for a review of the results).

To our knowledge the hyperbolic case like (1.1) is still open for arbitrary domains and arbitrary matrix valued coefficients $g^{jk}(x)$, while there exist results for particular coefficients or domains (see [24, 32]). This is maybe related to the difficulty of using the standard Carleman estimates for hyperbolic operators in order to prove the unique continuation close to the characteristic directions, that is the reason why Tataru's work was so important in this field.

The aim of the present work is then to prove a global stability estimate for the unique continuation of the operator $P(y, D)$.

In a previous work [6] we proved this property for the local case. Namely, given $S = \{y \in \Omega; \psi(y) = 0\}$ a $C^{2,\rho}$-smooth oriented hypersurface, which is non-characteristic in Ω, for some fixed $\rho \in (0, 1)$, we assume that $u \in H^1(\Omega)$ is supported in $\{y; \psi(y) \leq 0\} \cap \Omega$, and $P(y, D)u \in L^2(\Omega)$. Then, for each $y_0 \in S$, with $\psi'(y_0) \neq 0$, we find R, r with $R \geq 2r > 0$ such that the following stability estimate holds:

$$\|u\|_{L^2(B(y_0,r))} \leq c_{111} \frac{\|u\|_{H^1(B(y_0,2R))}}{\ln\left(1 + \frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\right)}.$$

Here $B(y_0, r)$ is a ball in $\mathbb{R}^{n+1}$ of radius $r > 0$ centered in y_0 and $B(y_0, r) \subset B(y_0, 2R) \subset \Omega$. The radii r and R and the coefficient c_{111} have been explicitly calculated with dependency on the geometric parameters and on the function ψ in [6].

In this work we use the previous local stability inequality to prove a similar logarithmic estimate for quite general domains of $\mathbb{R}^{n+1}$.

Moreover we propose a procedure to calculate all the constants involved, dependent on the norms of ψ, the coefficients in (1.1), the properties of the domains and the smooth localizers. The procedure is described in Proposition 2.4 and Appendix A.

Concerning the proof, in the unpublished manuscript [27], Tataru suggested the possibility of obtaining a log-stability result, by splitting the estimate for high and low temporal frequencies and by using Gevrey-class localizers to improve the estimates of u for low temporal frequencies. Here and in [6] we have advanced that idea, by employing tools of subharmonic functions and proper choice of the localizers in the

iterating procedure, together with the explicit computations of the uniform radii r, R and the time frequencies used in the iteration. Of fundamental importance is the calculation of the positive lower bound of the radius r, without which the iterative procedure could stop before covering the desired subdomain of Ω.
The technique used consists in iterating the local stability result, but considering the low temporal frequencies separately from the high temporal frequencies. The advantage is that one can avoid the usual $(\ln\ln ...|\ln\|Pu\|_{L^2}|)^{-\theta}$ iterated estimate (for $\|u\|_{H^1} = 1$ and $\|Pu\|_{L^2} << 1$, $\theta \in (0,1)$) and get a $(|\ln\|Pu\|_{L^2}|)^{-\theta}$ results. As a consequence one obtains a stable control of the solution u inside Ω, for any positive time. Moreover, we can come as close as we wish to the optimal time of the control T_{opt}, i.e. the time to reach the uniqueness in Ω (see Corollary 3.4, as example of computation). The importance of this issue has also been underlined in [24], who worked with FBI transform technique to get a log-stability estimate for large times. Hoermander in [13] proved an upper bound of the type $\sqrt{27/23}T_{opt}$. The issue of reaching T_{opt} for (1.1) has been solved in [28], see also [14, 25]. Here we can derive the stable determination of it.
Like in the elliptic case, many possible applications can be derived out of it. In particular we plan to use the inequalities in Theorem 1.1-1.2 to obtain an explicit modulus of continuity for the inverse problem for the wave operator on manifolds. This would improve the existing inverse stability results for Riemannian manifolds, which are currently based either on compactness-type arguments, see [3, 21], or on very strong geometrical conditions for the coefficients, e.g. in [10, 19, 20]. Here is important to be able to relate the explicit estimates with some geometric invariant of the manifold (Ricci curvature, sectional curvature, diameter, etc.).
As application, in section 3 we apply Theorem 1.2 to the case of an arbitrary domain of influence in $\mathbb{R}^{n+1}$. This is a special case of manifold, once one considers g^{jk} as the inverse of the metric tensor. We start with a time-cylinder where the wave solution vanishes (or has small data) and we get the stability in any compact subset of the associated domain of influence at time T. The control of solution in a stable way in the domain of influence can have numerous important applications in inverse problems and in in control theory. Here we consider also the case in which the ray field has also singularities, i.e. behind the corresponding cut-locus. This means that in principle we are able to deal with manifolds that possesses conjugate points, trapped rays and other singularities of geodesics. Thus, we remove the usual non-trapping conditions used in the Carleman estimates.
The paper is organized as follows: in Section 2 we prove Theorems 1.1 and 1.2, in Section 3 we present the application to the case of a domain of influence of the wave solution vanishing in a small cylinder. In Appendix A we present the table with the estimates for the parameters used in Sec. 2 and we study the uniform estimates for the distance function d_g and the related function ψ defined in Sec. 3.

We first introduce some assumptions.

Assumption A1 Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator (1.1), with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in C^0(\Omega)$. We assume that $u \in H^1(\Omega)$

and that $P(y, D)u \in L^2(\Omega)$. Assume that there is a function $\psi \in C^{2,\rho}(\Omega)$, for some $\rho \in (0,1)$, such that in a domain $\Omega_0 \subseteq \Omega$ one has $p(y, \psi'(y)) \neq 0$ and $\psi'(y) \neq 0$, where $p(y,\xi) = -\xi_0^2 + g^{jk}(x)\xi_j\xi_k$ is the principal symbol of P.
Assume that there exist values $\psi_{min} < \psi_{max}$ and a connected nonempty set $\Upsilon \subset \Omega_0$ such that: $\mathrm{supp}\,(u) \cap \Upsilon = \emptyset$; and $\emptyset \neq \{y \in \Omega_0; \psi(y) > \psi_{max}\} \subset \Upsilon$ (which implies that Ω_0 contains a subset Υ where u vanishes, and that the value ψ_{max} is obtained for points inside the domain Ω_0).
Assume that ψ_{min} is such that the open set $\Omega_a = \{y \in \Omega_0 \backslash \overline{\Upsilon} : \psi_{min} < \psi(y) < \psi_{max}\}$ is nonempty, connected and satisfies $\mathrm{dist}(\partial\Omega_0, \Omega_a) > 0$.

See remark 2.8 for comments about the construction.

Assumption A2 We define $A(D_0)$ to be a pseudo-differential operator with symbol $a(\xi_0)$, $0 \le a \le 1$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \le 2$, equal to one in $|\xi_0| \le 1$. Furthermore let $a \in G_0^{1/\alpha}(\mathbb{R})$ for a fixed $\alpha \in (0,1)$. Here $G_0^{1/\alpha}$ is the set of Gevrey functions of class $1/\alpha$ with compact support, defined in [15, 24]. We also define the smooth localizer $b(y)$, supported in $|y| \le 2$, $0 \le b \le 1$ and equal to one in $|y| \le 1$.

The main results of the paper are the following Theorems 1.1 and 1.2, together with their application in Section 3 Theorem 3.3.

Theorem 1.1. *Under the conditions of Assumption A1-A2, define the open set* $\Omega_1 = \Omega_0 \backslash \overline{\Upsilon}$ *containing* Ω_a. *Then for every* $0 < \theta < 1$ *we have*

$$\|u\|_{L^2(\Omega_a)} \le c_{160} \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)\right)^{\theta}}.$$

Moreover, for any $m \in (0,1]$ *we get*

$$\|u\|_{H^{1-m}(\Omega_a)} \le c_{160}^m \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln \left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)\right)^{m\theta}}.$$

The constant c_{160} *is calculated in the proof.*

The dependency of the constant c_{160} from the geometric parameters of the problems and from ψ and θ is described in Proposition 2.4.

Assumption A3 Let Ω be a connected open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator (1.1), with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in C^0(\Omega)$. Let $u \in H^1(\Omega)$ and $P(y, D)u \in L^2(\Omega)$.
In Ω we assume the existence of open connected subsets Λ_k, $\Omega_{0,k}$, a connected set Υ and functions ψ_k for $k = 1, 2, \ldots, K$ defined in this way:
1. $\psi_k \in C^{2,\rho}(\Omega)$ for some $\rho \in (0,1)$, such that $p(y, \psi_k'(y)) \neq 0$ and $\psi_k'(y) \neq 0$ in $\Omega_{0,k}$,

where $p(y,\xi) = -\xi_0^2 + g^{jk}(x)\xi_j\xi_k$ is the principal symbol of P.
2. Assume that: $\text{supp}\,(u) \cap \Upsilon = \emptyset$. Define $\Upsilon_1 = \Omega_{0,1} \cap \Upsilon$ and for $k \geq 2$ set $\Upsilon_k = \Omega_{0,k} \cap \big(\bigcup_{j<k} \Lambda_j \cup \Upsilon\big)$.
Assume that there exist values $\psi_{min,k} < \psi_{max,k}$ such that: $\big((\text{supp}\,(u) \cap \Omega_{0,k}) \setminus \bigcup_{j<k} \Lambda_j\big) \subset \{y; \psi_k(y) \leq \psi_{max,k}\}$; and $\emptyset \neq \{y \in \Omega_{0,k}; \psi_k(y) > \psi_{max,k}\} \subset \Upsilon_k$.
3. Assume that $\psi_{min,k}$ is such that $\Lambda_k = \{y \in \big(\Omega_{0,k} \setminus \overline{\Upsilon_k}\big);\ \psi_{min,k} < \psi_k(y) < \psi_{max,k}\}$ is nonempty, connected and satisfies $\text{dist}(\partial\Omega_{0,k}, \Lambda_k) > 0$.
4. Assume that $\Lambda = \cup_{k=1}^K \Lambda_k$ is a connected set.

Theorem 1.2. *Under the conditions of Assumptions A2-A3, define the open set* $\Omega_1 = \bigcup_{k=1}^K \Omega_{0,k} \backslash \overline{\Upsilon}$ *containing* Λ*. Then for every* $0 < \theta < 1$ *we have*

$$\|u\|_{L^2(\Lambda)} \leq c_{161} \frac{\|u\|_{H^1(\Omega_1)}}{\big(\ln\big(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\big)\big)^{\theta}}.$$

Moreover, for any $m \in (0,1]$ *we get*

$$\|u\|_{H^{1-m}(\Lambda)} \leq c_{161}^m \frac{\|u\|_{H^1(\Omega_1)}}{\big(\ln\big(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\big)\big)^{m\theta}}.$$

The constant c_{161} *is calculated in the proof.*

The dependency of the constant c_{161} from the geometric parameters of the problems and from ψ and θ is described in Proposition 2.4.

2 Global Stability

Notations. We start by introducing some notations and definitions used in the rest of the article: first we consider $y = (t,x) \in \mathbb{R} \times \mathbb{R}^n$ the time-space variable and call $\xi = (\xi_0, \widetilde{\xi})$ its Fourier dual variable. We remind that the exponential pseudodifferential operator in Theorem 2.3 is defined as $e^{-\epsilon|D_0|^2/2\tau} v = \mathcal{F}^{-1}_{\xi_0 \to t} e^{-\epsilon\xi_0^2/2\tau} \mathcal{F}_{t' \to \xi_0} v$, with $\mathcal{F}$ and $\mathcal{F}^{-1}$ representing respectively the Fourier transform and its inverse. Then $e^{-\epsilon|D_0|^2/2\tau}$ is an integral operator with kernel $(\frac{\tau}{2\pi\epsilon})^{1/2} e^{-\tau|t'-t|^2/2\epsilon}$. We consider a pseudo-differential operator $A(D_0)$ with symbol $a(\xi_0) \in C_0^\infty(\mathbb{R})$, $0 \leq a \leq 1$, supported in $|\xi_0| \leq 2$ and equal to one in $|\xi_0| \leq 1$. Hence we can write $A(\beta|D_0|/\omega)v = \mathcal{F}^{-1}_{\xi_0 \to t} a(\beta|\xi_0|/\omega) \mathcal{F}_{t' \to \xi_0} v$ and the integral kernel is $(\frac{\omega}{2\pi\beta})^{1/2} \widehat{a}(\frac{\omega|t'-t|}{\beta})$. We will often work under the Assumption A2, where the symbol a is of Gevrey class. The smooth localizer $b(y)$ is always supported in $|y| \leq 2$ and equal to one in $|y| \leq 1$.
The norm of the Sobolev space H^s_τ is defined as $\|u\|_{s,\tau} = \|(|\xi|^2 + \tau^2)^{s/2} \mathcal{F}_{y \to \xi} u\|_{L^2}$, and the space H^s corresponds to the case $\tau = 1$.
According to our notations the positive coefficients denoted by c_x with $x \geq 100$ are defined just once, independently on the variables μ, τ, and they are calculated explicitly in terms of the coefficients of the operator (1.1) and the geometric parameters. This is essential to finally recover the value of c_{160} and the radii R, r in Table 4.3.

A first step is the following lemma, proven in [6], introducing a property often used in this section.

Lemma 2.1. *Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0)$, where $a \in C_0^\infty(\mathbb{R})$ is a smooth localizer supported in $|\xi_0| \le 2$ and equal one in $|\xi_0| \le 1$. Assume that $f(y) \in C_0^\infty(\mathbb{R}^n_x, G_0^{1/\alpha}(\mathbb{R}^1_t))$, where $0 < \alpha < 1$. Then, for every $\mu > 0$, $\beta_1 > 2$, $v \in L^2_{loc}(\mathbb{R}^{n+1})$ there are two constants c_{106}, c_{107} independent of μ such that*

$$a)\quad \|A(\beta_1 D_0/\mu) f(y)(1 - A(D_0/\mu))v\|_0 \le c_{107} e^{-c_{106}\mu^\alpha} \|v\|_0 \,.$$

Moreover, if $v \in L^2(\mathbb{R}^{n+1})$ and $h \in C_0^\infty(\mathbb{R}^{n+1})$ is a function such that $h \equiv 1$ on supp(f), then

$$b)\quad \|A(\beta_1 D_0/\mu) f h v\|_0 \le \|f\|_\infty \|A(D_0/\mu)h(y)v\|_0 + c_{107} e^{-c_{106}\mu^\alpha} \|hv\|_0 \,.$$

If $v \in H^m_{loc}(\mathbb{R}^{n+1})$, $m \ge 1$, then the estimate above holds also in $H^m(\mathbb{R}^{n+1})$,

$$c)\quad \|A(\beta_1 D_0/\mu) f (1 - A(D_0/\mu))v\|_m \le c_{108} e^{-c_{106}\mu^\alpha} \|v\|_m \,.$$

Proof. See [6] for the entire proof. Here we remind how to obtain the coefficients.
$a)$ On the set supp$[(1 - a(\xi_0/\mu))a(\beta_1\xi_0^1/\mu)]$ one obtains $|\xi_0^1 - \xi_0|^\alpha \ge (\mu - 2\mu/\beta_1)^\alpha$ and the assumption $f(t,.) \in G_0^{1/\alpha}(\mathbb{R}_t)$ implies, uniformly in x on a compact set $K \subset \mathbb{R}^n$ and for some $c_3 = c_3(\alpha, K)$, $c_{117} = c_{117}(\alpha, K)$ and $c_{106} = c_{117}(1 - 2/\beta_1)^\alpha/4$,

$$|\mathcal{F}_{t' \to (\xi_0^1 - \xi_0)}[f(t', x)]| \le c_3 e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha} \le c_3 e^{-2c_{106}\mu^\alpha} e^{-c_{117}|\xi_0^1 - \xi_0|^\alpha/2}. \tag{2.1}$$

The coefficient $c_3 = c_3(\alpha, K)$ is proportional to $c_{1,f}$, the Gevrey parameter of f, that is [15, 26]

$$|D^\kappa(f(s))| \le c_{1,f}^{|\kappa|+1}(|\kappa| + 1)^{|\kappa|/\alpha}, \ s \in \text{supp}(f).$$

We have $c_3 = c_{1,f}\text{Vol}(\text{supp}\,(f))$ and $c_{117} = 1/(ec_3)^\alpha$. We then estimate in the Fourier space the operator $A(\beta_1 D_0/\mu) f(\cdot)(1 - A(D_0/\mu))$, with $c_{107} = (c_3 \frac{8}{\beta_1}\Gamma\big(\frac{1}{\alpha}\big)\frac{1}{\alpha(c_{117})^{1/\alpha}}\frac{1}{(\alpha c_{106})^{\frac{1}{\alpha-1}}}))^{1/2}$.

$$\|a(\beta_1\xi_0^1/\mu)\mathcal{F}_{t' \to \xi_0^1}\big(f(t', x)(\mathcal{F}^{-1}_{\xi_0 \to t'}(1 - a(\xi_0/\mu))\mathcal{F}_{t \to \xi_0}[v])\big)\|_0^2 \le c_{107}^2 e^{-2c_{106}\mu^\alpha}\|v\|_0^2 \,.$$

□

According to the splitting $y = (t, x)$, the conormal bundle in $\mathbb{R}^{n+1}$ with respect to the foliation $x =$const is defined as:
$N^*F := \{(y, \xi) \in T^*\mathbb{R}^{n+1};$ with $\xi = (\xi_0, \widetilde{\xi})$ and $\xi_0 = 0\}$.
Its reduction to a subset $K \subset \mathbb{R}^{n+1}$ is $\Gamma_K := \{(y, \xi) \in T^*K, \, \xi_0 = 0\}$,
and its fibre in y_0 is $\Gamma_{y_0} := \{(y_0, \xi) \in N^*F\}$.

We then recall the concept of conormally strongly pseudoconvex function, alias strongly pseudoconvex function with respect to P on Γ_{y_0} ([28, 29]).

Definition 2.2. A C^2 real valued function ϕ is *conormally strongly pseudoconvex* with respect to P at y_0 if

$$
\begin{aligned}
&Re\{\overline{p}, \{p, \phi\}\}(y_0, \xi) > 0 && (2.2)\\
&\quad \text{on} \quad p(y_0, \xi) = 0, \quad 0 \neq \xi \in \Gamma_{y_0};\\
&\{\overline{p(y, \xi + i\tau\phi'(y))}, p(y, \xi + i\tau\phi'(y))\}/(2i\tau) > 0 && (2.3)\\
&\quad \text{for } y = y_0, \quad 0 \neq \xi \in \Gamma_{y_0}, \text{such that } p(y_0, \xi + i\tau\phi'(y_0)) = 0,\ \tau > 0.
\end{aligned}
$$

In particular, for the wave operator (1.1) the conditions are void for non-characteristic surfaces $\phi = const$. As consequence one can state the following Theorem (Theorem 2.1 in [6]), where the Carleman-type estimate by Tataru is recalled.

Theorem 2.3. *Let Ω be an open subset of $\mathbb{R} \times \mathbb{R}^n$. Let $P(y, D)$ be the wave operator* (1.1)*, with $g^{jk}(x) \in C^1(\Omega)$, $h^j, q \in C^0(\Omega)$. Let $y_0 \in \Omega$ and $\psi \in C^{2,\rho}(\Omega)$ be real valued, for some fixed $\rho \in (0, 1)$, such that $\psi'(y_0) \neq 0$ and $S = \{y; \psi(y) = 0\}$ being an oriented hypersurface non-characteristic in y_0.*
Consequently there is $\lambda > 1$ such that $\phi(y) = \exp(\lambda\psi)$ is a conormally strongly pseudoconvex function with respect to P at y_0.
Then there is a real valued quadratic polynomial f defined in (4.1) *with proper $\sigma > 0$, and a ball $B_{R_2}(y_0)$ such that $f(y) < \phi(y)$ when $y \in B_{R_2} - \{y_0\}$ and $f(y_0) = \phi(y_0)$; and f being a conormally strongly pseudoconvex function with respect to P in B_{R_2}. This implies that there exist ϵ_0, τ_0, $c_{1,T}$, $c_{2,T}, R$, such that, for each small enough $\epsilon < \epsilon_0$ and large enough $\tau > \tau_0$, we have*

$$
\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau f} u\|_{1,\tau} \leq c_{1,T}\, \tau^{-1/2} \|e^{-\epsilon|D_0|^2/2\tau} e^{\tau f} P(y, D)u\|_0 + c_{2,T} e^{-\tau R_2^2/4\epsilon} \|e^{\tau f} u\|_{1,\tau}.
$$

Here $u \in H^1_{loc}(\Omega)$, with $P(y, D)u \in L^2(\Omega)$ and $supp(u) \subset B_R(y_0)$.

This last estimate was used in [6] to prove local stability of the unique continuation with explicit coefficients. We recall these results in the following proposition.

Proposition 2.4. *Let P the wave operator* (1.1)*. Then, under the Assumption A1-A2, and using the result of Theorem 2.3, there exist two positive radii R and r such that the local stability results (i.e. Lemma 2.6 and Theorems 1.2 in [6]) hold true in every point of Ω_0, with the same parameters.*
Moreover, starting by the Assumptions A1, we are able to calculate all the constants involved in the local stability in a uniform way over Ω_0. The geometric parameters are constructed in Table (4.3) *while the derived constants are in the proof of Theorems 1.1 and 1.2 of [6].*
All the constants depend on :
- the coefficients in (1.1) *and their bounds:*

$$
|g^{jk}|_{C^1(\Omega_0)}, |h|_{C^0(\Omega_0)}, |q|_{C^0(\Omega_0)}, a_1, b_1,
$$

- the assumptions on the domains:

$$
dist_{\mathbb{R}^{n+1}}\{\partial\Omega_0, \Omega_a\} > 0,
$$

- the non-characteristic condition and the non-vanishing condition upon ψ':

$$p_1 = \min_{y\in\overline{\Omega}_0} p(y,\psi'),\; C_l = \min_{y\in\overline{\Omega}_0} |\psi'(y)|,$$

- the norms of ψ (see (4.2) for notations)

$$|\psi'|^2_{C^0(\Omega_0)}, |\psi''|^2_{C^0(\Omega_0)}, |\lambda\psi|_{max,\Omega_0},$$

- the norms of the smooth localizers, in time-space and frequency, together with their Gevrey parameters:

$$|\chi_1(s)|_{C^2(\mathbb{R})}, |b(s)|_{C^2(\mathbb{R})}, |a(\xi_0)|_{C^0(\mathbb{R})}.$$

We then need to reformulate Lemma 2.6 of [6] in the case of more general assumptions.

Lemma 2.5. *Under the assumption A1, let $y_0 \in \Omega_0$ and φ be the quadratic polynomial $\varphi(y) = f(y) - f(y_0)$, with f defined in (4.1). Let $0 < \alpha < 1$ and $\chi(s) \in G_0^{1/\alpha}(\mathbb{R})$ be a localizer supported in $[-8\delta, \delta]$ and equal 1 in $[-7\delta, \delta/2]$. Let $\mu > 0$, $\delta > 0$, be given constants, $b \in C_0^\infty(\mathbb{R}^{n+1})$ and $a \in C_0^\infty(\mathbb{R})$. Let $A(D_0)$ be a pseudodifferential operator with symbol a. If*

$$u \in H^1(B_{2R}), \quad Pu \in L^2(B_{2R}), \quad \|A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \le c_A e^{-\mu^\alpha},$$

then for each $\tau \ge 0$, there are constants c_{110}, c_{109} such that

$$\|e^{-\epsilon|D_0|^2/2\tau} e^{\tau\varphi}\chi(\varphi)b(\frac{y-y_0}{R})P(y,D)u\|_0 \le \max\{c_{110}, c_A\}e^{2\tau\delta - c_{109}\mu^\alpha} \max\{1, \|Pu\|_{L^2(B_{2R})}\}.$$

Using Lemma 2.5 we now reformulate Theorem 1.1. of [6] with more general assumptions.

Lemma 2.6. *Under the assumptions A1-A2, let $y_0 \in S = \{y; \psi(y) = 0\}$ be an $C^{2,\rho}$-oriented hypersurface, which is non-characteristic in y_0 and with $\psi'(y_0) \neq 0$.*
We also assume that $u \in H^1(\Omega)$ is such that $supp\,(u) \cap B_{2R}(y_0) \subset \{y; \psi(y) \le 0\}$.
Let $b \in G_0^{1/\alpha}(\mathbb{R}^{n+1})$ be Gevrey functions with compact support, with $0 < \alpha < 1$. Then, for $\mu \ge 1$, if for some positive coefficients c_U, c_P, c_A

$$\|u\|_{H^1(B_{2R(y_0)})} \le c_U, \quad \|Pu\|_{L^2(B_{2R}(y_0))} < c_P, \quad \|A(D_0/\mu)b((y-y_0)/R)Pu\|_0 \le c_A e^{-\mu^\alpha},$$

then, there are constants $c_{150}, c_{131}, c_{132}$ independent of μ, such that

$$\|A(D_0/\omega)b((y-y_0)/r)u\|_{H^1} \le c_{150} e^{-c_{132}\mu^{\alpha^2}}, \quad \forall\, \omega \le \mu^\alpha/(3c_{131}).$$

Moreover c_{131} and c_{132} are independent of c_U, c_P, c_A, while c_{150} depends on them. The dependency of all the constants is as described in Proposition 2.4.

Proof. The proof is identical to the one of Theorem 1.1. in [6]. Th. 2.3 is used for the function $\chi(\varphi)b(\frac{y-y_0}{R})u$ that is supported in $B_R(y_0) \cap \{y; \phi(y) \le \phi(y_0)\} \cap \{y; -8\delta < \varphi(y) < \delta\}$. Moreover $[P, \chi(\varphi)b(\frac{y-y_0}{R})]u = [P, \chi(\varphi))]b(\frac{y-y_0}{R})u$ (since $D(\chi b)u = bD(\chi)u$), while in $B_R(y_0)$ one has $\chi(\varphi)b(\frac{y-y_0}{R})u = \chi(\varphi)u$.

Here we have just to recalculate the related coefficients, distinguishing the ones dependent upon the parameters c_U, c_P, c_A from the ones independent of them.
We first list of coefficients independent of c_U, c_P, c_A, but dependent on the Gevrey parameters of the localizers and from the geometric constants r, R, δ (see Table (4.3)):

$$c_{1X} = c_{1X}(1/\alpha),\ c_{2X} = 1/(eNc_{1X})^{\alpha},\ c_{101} = c_{101}(\alpha),\ c_{102} = c_{102}(\alpha, c_{101}),$$
$$c_{119} = \delta c_{1X}(\alpha),\ B = \delta^{\alpha} c_{1X}(\alpha),\ |\partial_t^k \varphi(y)|_{C^0(B_R)} \leq c_{118}(R)(> 1),$$
$$c_{114} = c_{1,T}^2 |g|_{C^1}^2 |\chi_1|_{C^2}^2 (1 + |\varphi'|_{C^0}^4/\delta^4 + |\varphi''|_{C^0}^2/\delta^2),$$
$$c_{115} = c_{2,T}^2 (|\varphi'|_{C^0}^2 + 1)(3^3 e^{-3}/\delta^3)(1 + |\chi_1'|_{C^0}^2/\delta^2),$$
$$c_{120} = \Big(\frac{(1-\alpha)}{\alpha \ln B}\Big)^{\frac{(1-\alpha)}{\alpha}} B^{-\frac{(1-\alpha)}{\alpha \ln B}+1},\ c_{121} = \frac{2\pi c_{119}}{\alpha}\Gamma(2)\Big(\frac{2}{\alpha}\Big)^{\alpha} c_{120},$$
$$c_{122} = \max(1, 4c_{118}(R)c_{121}, c_{1X}/R) \geq 1,\ c_{123} = (ec_{122})^{\alpha} < 1,\ c_{128} = \frac{1}{3^{\alpha}2} c_{123} < 1,$$
$$c_{110} = (c_{122}(8/3)\Gamma(1/\alpha)/[\alpha c_{123}^{1/\alpha}(\alpha c_{128})^{1/(\alpha-1)}])^{1/2},$$
$$c_{109} = \min(\sqrt{\epsilon\,\delta/36}, c_{128}/2, 1) \leq 1,\ c_{130} = \frac{3c_{109}}{4\delta}\Big(\frac{1}{16}\Big)^5,$$
$$c_{131} = \max\{\sqrt{2}(16)^6,\ (\sqrt{2}(16)^6 3^{(\alpha-1)}\sqrt{\epsilon_0\delta})/c_{123},\ ((16)^6\sqrt{\epsilon_0\delta})/(3\sqrt{2})\} > 1,$$
$$c_{135} = r^{\alpha} c_{2X} \frac{1}{23^{\alpha^1}},\ c_{132} = \min(c_{135}(r), c_{137}),$$
$$c_{137} = \min\Big(\frac{1}{2}\Big(c_{102}\delta^{\alpha}\frac{(c_{130})^{\alpha}}{(\sqrt{2})^{\alpha}}, \delta\frac{c_{130}}{2\sqrt{2}}\Big), \frac{1}{2}c_{102}\delta^{\alpha}\Big(\frac{1}{2\sqrt{2}}c_{130}\Big)^{\alpha}\Big).$$

Then the coefficients dependent on c_U, c_P, c_A are:

$$c_{116} = 3\max(c_{1,T}^2 \max\{c_{110}, c_A\}^2 \max\{1,\ c_P\}^2, c_{114}c_U^2, c_{115}c_U^2),$$
$$c_{113} = \max(c_{116}, c_U^2 c_{112}(1+\tau_0^3)(1 + |\chi_1'|_{C^0}^2/\delta^2)e^{12\delta\tau_0}),$$
$$c_{134} = c_U\Big((rc_{1X})\frac{8}{3}\Gamma\Big(\frac{1}{\alpha}\Big)\frac{1}{\alpha(r^{\alpha}c_{2X})^{1/\alpha}(\alpha c_{135})^{1/(\alpha-1)}}\Big)^{1/2}$$
$$c_{136} = 2c_{101}\sqrt{c_{113}}\int_{\mathbb{R}} \sqrt{(s/\delta)^2+1}\, e^{-c_{102}s^{\alpha}/2} ds +$$
$$+ c_{101}\Big(\frac{2c_{113}(1+c_{130}^2)}{\min\{1,\ c_{130}^2/2\}}\Big)^{\frac{1}{2}}\Big(2\int_0^{+\infty} e^{-y'/2}dy' + \int_{\mathbb{R}} e^{-c_{102}|x'|^{\alpha}}dx'\Big)$$
$$c_{129} = \max(c_{134}, c_{136}).$$

We rename c_{129} with c_{150} to underline its new dependencies. □

We now introduce the main assumptions to prove the global stability result.
We recall that the support condition in Lemma 2.6 is not fulfilled everywhere for $u \in H^1(\Omega)$. The idea is that at each step one applies the local stability result of Lemma 2.6 in a ball centred in the point y_j and then one removes from supp (u) (in a smooth way) a part of the ball $B_r(y_j)$ already calculated, for example by subtracting by $b(2(y-y_j)/r)u_j$, which is supported in $B_r(y_j)$. Then u_{j+1} fulfills the support condition in Lemma 2.6 in the ball $B_{2R}(y_{j+1})$, also due to our Assumption A1 or A3.

Assumption A4 Let Ω_0 and ψ be as in Assumption A1. Then consider r and R the uniform radii defined in Proposition 2.4.
We define the set of points $\mathcal{E} = \{y_j \in \Omega_0,\ j = 1, .., N\}$, such that $\overline{\Omega_a} \subset \bigcup_{j=1}^N B_r(y_j) \subset \Omega_0$, in the following way:
1. Let $y_1 \in \Omega_0$ be the maximum point for ψ in $\overline{\Omega}_a$.
Set $u_1 = u$ and $u_2(y) = \big(1 - b(\frac{2(y-y_1)}{r})\big)u_1$,
2. Let $y_2 \in \Omega_0$ be the maximum for ψ in $\overline{\Omega}_a \backslash B_{r/2}(y_1)$,
3. In general, let $y_j \in \Omega_0$ be the maximum for ψ in $\overline{\Omega}_a \backslash \bigcup_{k=1}^{j-1} B_{r/2}(y_k)$, i.e. $y_j \in \operatorname{argmax}\{\psi(y),\ y \in (\overline{\Omega}_a \backslash \bigcup_{k=1}^{j-1} B_{r/2}(y_k))\}$.
Then we define:

$$u_j = \prod_{k=1..j-1} (1 - b_k)u, \quad b_k := b\big(\frac{2(y-y_k)}{r}\big). \tag{2.4}$$

Each y_j lies on the surface $S_j = \{y;\ \psi(y) = \psi(y_j)\}$. Notice that, since $|y_j - y_k| \geq r/2$ for $j \neq k$,

$$N \leq c_{170} = \frac{Vol(\Omega_0)}{\omega_{n+1}(\frac{r}{4\sqrt{\max(b_1,1)}})^{n+1}}, \tag{2.5}$$

where ω_{n+1} is the volume of the ball of radius one in $\mathbb{R}^{n+1}$, where we consider the following bound for the coefficients $a_1\delta^{jk} \leq g^{jk}(x) \leq b_1\delta^{jk}$.
We finally define $l(y) \in C_0^\infty(\mathbb{R}^{n+1})$ a localizer such that $l = 1$ on $\mathcal{L} = \{y \in \Omega_0;\ \mathrm{dist}(y, \partial\Omega_0) \geq \frac{R_1}{4}\}$, $0 \leq l \leq 1$ and $\mathrm{supp}(l) \subset \Omega_0$. Observe that $\cup_{k=1}^N B_{2R}(y_k) \subseteq \mathcal{L}$.

We now can formulate a stability estimate of inverse exponential type for the low temporal frequencies of u_j.

Theorem 2.7. *Under the Assumptions A1-A2-A4, let $y_k \in \mathcal{E}$ and let $b \in G_0^{1/\alpha}(\mathbb{R}^{n+1})$ be a Gevrey functions of class $1/\alpha$ with compact support, such that $0 < \alpha < 1$. Then, there exist constants R, r with $R \geq 2r > 0$, and $c_{159} > 1$ such that for $\mu > c_{159}$ there are coefficients $c_{151}, c_{152}, c_{154}, c_{155}, c_{156}, \beta, N$ for which, if*

$$\|u\|_{H^1(\Omega_1)} = 1, \quad \|Pu\|_{L^2(\Omega_1)} < 1, \quad \|A(\frac{D_0}{\beta\mu})l(y)Pu\|_{L^2} \leq \exp(-\mu^\alpha), \tag{2.6}$$

then calling $\mu_1 = \mu$ and $\mu_j = c_{156}\mu_{j-1}^\alpha$ for $2 \leq j \leq N$, we have $\mu_j \geq 1$ and

$$\|u_j\|_{H^1(B_{2R}(y_j))} \leq c_{152}, \quad \|Pu_j\|_{L^2(B_{2R}(y_j))} \leq c_{153}, \tag{2.7}$$

$$\|A(D_0/\mu_j)b((y-y_j)/R)Pu_j\|_0 \leq c_{154,j}\exp(-\mu_j^\alpha), \tag{2.8}$$

and consequently

$$\|A(D_0/\omega)b((y-y_j)/r)u_j\|_{H^1} \leq c_{155,j}\exp(-c_{132}\mu_j^{\alpha^2}), \quad \forall\, \omega \leq \mu_j^\alpha/(3c_{131}). \tag{2.9}$$

The radii r and R are defined in Table (4.3), while the coefficients c_k are calculated in the proof of the Theorem.

Proof. Let $b \in G_0^{1/\alpha}(\mathbb{R}^{n+1})$ be a localizer with support as in Assumption 2.
Observe that according to our definitions we have :

$$B_r(y_j) \subset \operatorname{supp} b((y-y_j)/r) \subseteq B_{2r}(y_j) \subset B_R(y_j) \subset \operatorname{supp} b((y-y_j)/R) \subseteq B_{2R}(y_j)\,,$$

We now proceed step by step.
Step 1. We consider $y_1 \in \mathcal{E}$ defined in Assumption A4. From the hypotheses (2.6) the following inequalities hold true for $u_1 = u$:

$$\|u\|_{H^1(B_{2R}(y_1))} \leq 1, \quad \|Pu\|_{L^2(B_{2R}(y_1))} \leq 1.$$

From the definition of l in Assumption A4 and applying Lemma 2.1.(b) with $f = b(\frac{y-y_1}{R})$, $h = l$, $\beta_1 = \beta$, $\mu = \beta\mu$ we get

$$\begin{aligned} \|A(\frac{D_0}{\mu})b\big(\frac{y-y_1}{R}\big)l(y)Pu\|_{L^2} &\leq \|A(\frac{D_0}{\beta\mu})l(y)Pu\|_{L^2} + \widetilde{c}_{107}\exp(-\widetilde{c}_{106}\beta^\alpha\mu^\alpha)\|l(y)Pu\|_{L^2} \\ &\leq \exp(-\mu^\alpha) + \widetilde{c}_{107}\exp(-\mu^\alpha) \leq c_{154,1}\exp(-\mu^\alpha) \quad (2.10) \end{aligned}$$

with $c_{154,1} = 1 + \widetilde{c}_{107}$ and where $\beta > 2$ is a parameter chosen as:

$$\beta = 2 + (\frac{4}{\widetilde{c}_{117}})^{1/\alpha} \tag{2.11}$$

in order to have $\widetilde{c}_{106}\beta^\alpha = 1$. Indeed, applying Lemma 2.1.(b), one gets $\widetilde{c}_{106} = \frac{\widetilde{c}_{117}}{4}(1-\frac{2}{\beta})^\alpha$ where $\widetilde{c}_{117} = 1/(2\widetilde{c}_3)^\alpha$ and $\widetilde{c}_3 = c_{1,b}(R)\cdot \mathrm{Vol}(\operatorname{supp}(b(\frac{y-y_1}{R})))$, with $c_{1,b}(R)$ the Gevrey parameter associated with $b(\frac{y}{R})$. For the calculation of $\widetilde{c}_{107}$ see lemma 2.1. Notice that $\widetilde{c}_{106}$ and $\widetilde{c}_{107}$ are independent of y_1, since the calculation is invariant up to translations.
Calling $\widetilde{\psi}(y) = \psi(y) - \psi(y_1)$ we notice that u fulfils $\operatorname{supp}(u) \cap B_{2R}(y) \subset \{y; \psi(y) \leq \psi(y_1)\}$ We are then allowed to apply Lemma 2.6, with $y_0 = y_1$, $\psi = \widetilde{\psi}$, $c_U = 1, c_P = 1, c_A = c_{154,1}$ and calling $c_{155,1} = c_{150}$,

$$\|A(\frac{D_0}{\omega})b(\frac{y-y_1}{r})u\|_{H^1} \leq c_{155,1}\exp(-c_{132}\mu^{\alpha^2}), \quad \forall \omega \leq \frac{\mu^\alpha}{3c_{131}}.$$

Step $j > 1$.
Here we consider $y_j \in \mathcal{E}$ and u_j defined in (2.4) and notice that $\operatorname{supp}(u_j) \subseteq \operatorname{supp}(u)\backslash \cup_{k=1}^{j-1} B_{r/2}(y_k)$ and that $u_j = u$ on $\operatorname{supp}(u)\backslash \cup_{k=1}^{j-1} B_r(y_k)$.
Calling $\widetilde{\psi}(y) = \psi(y) - \psi(y_j)$ we notice that by construction u_j is such that $\operatorname{supp}(u_j) \cap B_{2R}(y_j) \subset \{y; \psi(y) \leq \psi(y_j)\}$. We then will apply Lemma 2.6, with $\psi = \widetilde{\psi}$ and $y_0 = y_j$.
We start by calculating the first estimate in (2.7):

$$\begin{aligned} \|u_j\|_{H^1(B_{2R}(y_j))} &\leq \|u\|_{H^1(B_{2R}(y_j))} + |\nabla \prod_{k=0..j-1}\big(1 - b(\frac{2(y-y_k)}{r})\big)|_{C^0}\|u\|_{L^2(B_{2R}(y_j))} \quad (2.12) \\ &\leq 2(1 + j\frac{|b'|_{C^0}}{r})\|u\|_{H^1(B_{2R}(y_j))}. \end{aligned}$$

Since $j \le N$ we get a uniform bound for all j

$$\|u_j\|_{H^1(B_{2R}(y_j))} \le c_{152}, \quad c_{152} = 2(1 + N\frac{|b'|_{C^0}}{r}). \tag{2.13}$$

Then we consider the second estimate in (2.7)

$$\|Pu_j\|_{L^2(B_{2R}(y_j))} \le \|Pu\|_{L^2(B_{2R}(y_j))} + \|[P, \prod_{k=0..j-1}(1 - b(\frac{2(y-y_k)}{r}))]u\|_{L^2(B_{2R}(y_j))} \tag{2.14}$$

$$\le 1 + 2j(1 + n^2|g^{kr}|_{C^0} + |h^s|_{C^0})(\frac{|b'|_{C^0}}{r} + \frac{|b''|_{C^0}}{r^2} + (j-1)\frac{|b'|^2_{C^0}}{r^2})\|u\|_{H^1(B_{2R}(y_j))} \le c_{153},$$

where the commutator is, for $b_k = b(2(y-y_k)/r)$:

$$[b_k, P]u = (-P_2 b_k)u + 2D_0 b_k D_0 u - 2g^{hr}(x)D_{x_h} b_k D_{x_r} u + ih^s(x) D_{x_s} b_k u$$

with $P_2 = -D_0^2 + g^{hr}(x)D_h D_r$ and, for all $j \le N$,

$$c_{153} = 1 + 2N(1 + n^2|g^{kr}|_{C^0} + |h^s|_{C^0})(\frac{|b'|_{C^0}}{r} + \frac{|b''|_{C^0}}{r^2} + (N-1)\frac{|b'|^2_{C^0}}{r^2}). \tag{2.15}$$

The third estimate (2.8) requires information of Step $j-1$.
Like in (2.10), from the definition of l and applying Lemma 2.1.(b) with $f = b(\frac{y-y_j}{R})$, $h = l$, $\beta_1 = \beta$, $\mu = \beta\mu_j$ we get

$$\|A(\frac{D_0}{\mu_j})b(\frac{y-y_j}{R})l(y)Pu_j\|_{L^2} \le \|A(\frac{D_0}{\beta\mu_j})l(y)Pu_j\|_{L^2} + c_{153}\widetilde{c}_{107}\exp(-\mu_j^\alpha). \tag{2.16}$$

where $\beta > 2$ is the parameter (2.11).
The first term on the right hand side of (2.16) becomes

$$\|A(\frac{D_0}{\beta\mu_j})l(y)Pu_j\|_0 \le \|A(\frac{D_0}{\beta\mu_j})l(y)Pu_{j-1}\|_0 + \|A(\frac{D_0}{\beta\mu_j})l(y)b_{j-1}Pu_{j-1}\|_0$$
$$+\|A(\frac{D_0}{\beta\mu_j})[b_{j-1}, P]u_{j-1}\|_0. \tag{2.17}$$

One can find recursively the estimate above for $j = 1$ by using (2.6) with $c_{162,1} = 1$, and stating for $j-1$

$$\|A(\frac{D_0}{\beta\mu_{j-1}})l(y)Pu_{j-1}\|_0 \le c_{162,j-1}e^{-\mu_{j-1}^\alpha}, \tag{2.18}$$

with $c_{162,j-1}$ a positive parameter.
By the inductive hypothesis and in analogy with (2.10),

$$\|A(\frac{D_0}{\mu_{j-1}})b(\frac{y-y_{j-1}}{R})l(y)Pu_{j-1}\|_0 \le c_{154,j-1}e^{-\mu_{j-1}^\alpha}, \tag{2.19}$$

where $c_{154,j-1} = c_{162,j-1} + c_{153}\widetilde{c}_{107}$.
The first term on the right hand side of (2.17) becomes, for $\mu_j \le \mu_{j-1}/2$,

$$\|A(\frac{D_0}{\beta\mu_j})l(y)Pu_{j-1}\|_0 \le \|A(\frac{D_0}{\beta\mu_{j-1}})l(y)Pu_{j-1}\|_0.$$

The second term on the right hand side of (2.17) becomes, for $2\mu_j \le \mu_{j-1}/3$,

$$\|A(\frac{D_0}{\beta\mu_j})l(y)b_{j-1}Pu_{j-1}\|_0 \le \|A(\frac{D_0}{\beta\mu_{j-1}})l(y)Pu_{j-1}\|_0 + c_{153}c_{164}\exp(-c_{165}\mu_{j-1}^{\alpha})$$

where by lemma 2.1.b) with $f = b_{j-1}$, $h = l$, $\beta_1 = 3$, $\mu = \beta\mu_{j-1}$, we have $c_{164} = c_{107}$, $c_{165} = c_{106}\beta^{\alpha} = c_{117}\beta^{\alpha}/(3^{\alpha}4)$. Notice that c_{165} and c_{164} are independent of y_j, since the calculation is invariant up to translations.
The term with the commutator in (2.17) can be split in the following way:

$$\begin{aligned} I_1 + I_2 + I_3 &= \|A(\frac{D_0}{\beta\mu_j})\big((-P_2 b_{j-1}) + ih^s(x)D_{x_s}b_{j-1}\big)u_{j-1}\|_{L^2} \\ &\quad + \|A(\frac{D_0}{\beta\mu_j})\big(2D_0 b_{j-1}D_0 u_{j-1}\big)\|_{L^2} \\ &\quad + \|A(\frac{D_0}{\beta\mu_j})\big(2g^{kr}(x)D_{x_k}b_{j-1}D_{x_r}u_{j-1}\big)\|_{L^2}. \end{aligned}$$

We notice that the localizer $b\big(\frac{(y-y_{j-1})}{r}\big) = 1$ on $\mathrm{supp}[b_{j-1}, P]u$, then we multiply u_{j-1} in I_1 with it to keep its support in $B_{2r}(y_{j-1})$ in order to use the estimates of Step $j-1$. For $\nu \le \frac{\mu_{j-1}^{\alpha}}{3c_{131}}$ a positive parameter, one has

$$\begin{aligned} I_1 &\le \|A(\frac{D_0}{\beta\mu_j})\big((-P_2 b_{j-1}) + ih^s(x)D_{x_s}b_{j-1}\big)A(\frac{D_0}{\nu})b(\frac{(y-y_{j-1})}{r})u_{j-1}\|_{L^2} \\ &+ \|A(\frac{D_0}{\beta\mu_j})\big((-P_2 b_{j-1}) + ih^s(x)D_{x_s}b_{j-1}\big)(1 - A(\frac{D_0}{\nu}))b(\frac{(y-y_{j-1})}{r})u_{j-1}\|_{L^2} \\ &\le c_{155,j-1}|-P_2 b_{j-1} + h^s(x)D_{x_s}b_{j-1}|_{C^0}\exp(-c_{132}\mu_{j-1}^{\alpha^2}) \\ &\quad + c_{107}c_{152}(1 + n^2|g^{kr}|_{C^0} + |h^s|_{C^0})\exp(-c_{106}\nu^{\alpha}) \end{aligned}$$

Notice that the first estimate on the right hand side is done by using the inductive hypothesis and by applying to the term $\|A(\frac{D_0}{\nu})b(\frac{(y-y_{j-1})}{r})u_{j-1}\|_{L^2}$ Lemma 2.6 with coefficients $c_U = c_{152}$, $c_P = c_{153}$, $c_A = c_{154,j-1}$ defined in (2.19) and then calling $c_{155,j-1}$ the resulting coefficient $c_{150} = c_{150}(c_{152}, c_{153}, c_{154,j-1})$.
For the second term on the right hand side we assume that $2\beta\mu_j \le \nu/3$ in order to write, both with $s = 0$ (i.e. L^2) and $s = 1$ (i.e H^1):

$$\|A(\frac{D_0}{\beta\mu_j})v\|_s \le \|A(\frac{3D_0}{\nu})v\|_s. \tag{2.20}$$

Then we apply Lemma 2.1.(a) with $\beta_1 = 3$, $\mu = \nu$ and f of this form (after moving out of the norm g^{kh}, h^s and the complex variable)

$$f_1(t,x) = \partial_t^2 b_{j-1} + \partial_{x_r}\partial_{x_h}b_{j-1} + \partial_{x_s}b_{j-1}, \tag{2.21}$$

involving just derivatives of smooth functions in $C_0^{\infty}(\mathbb{R}^n, G_0^{1/\alpha}(\mathbb{R}_t))$.
To recover an expressions for the coefficients we recall that the κ_2-derivative of

$h \in G_0^{1/\alpha}$ (with Gevrey constant $c_{1,h}$) is:

$$
\begin{aligned}
|D^{\kappa_1}(D^{\kappa_2}h(s))| &\leq c_{1,h}^{|\kappa_1|+|\kappa_2|+1}(|\kappa_1|+|\kappa_2|+1)^{(|\kappa_1|+|\kappa_2|)/\alpha} \\
&\leq c_{1,h}^{|\kappa_2|+|\kappa_1|+1}2^{|\kappa_2|(|\kappa_1|+|\kappa_2|)/\alpha}e^{|\kappa_2||\kappa_1|/\alpha}(|\kappa_1|+1)^{|\kappa_1|/\alpha}, \quad s \in \mathrm{supp}\,(h).
\end{aligned} \tag{2.22}
$$

In our case we must just consider time derivatives, in order to estimate (2.1). Since the translations play no role for the Fourier transform, we can calculate coefficients independently upon j. Call $c_{1,b}, c_{1,b'}, c_{1,b''}$ the Gevrey coefficients of the functions $b(y)$, $D_x b(y)$, $D_x^2 b(y)$. Then, define

$$
c_{f_1} = 2^{2/\alpha}e^{2/\alpha}c_{1,b}^2(r) + c_{1,b''}(r) + c_{1,b'}(r).
$$

Analogously we can get the values for the functions associated with I_1, I_2 (see below for definition of f_2, f_3):

$$
c_{f_2} = 2^{1/\alpha+1}e^{1/\alpha}c_{1,b}(r) + 2^{2/\alpha+1}e^{2/\alpha}c_{1,b}^2(r), \quad c_{f_3} = 4c_{1,b'}(r) + 2c_{1,b''}(r).
$$

In analogy to the computations above we can calculate c_{Df_2}, c_{Df_3} (the Gevrey parameters of $Df_2 = \partial_t(2\partial_t b_{j-1}) + \nabla_x(2\partial_t b_{j-1})$, $Df_3 = \partial_t(2\partial_{x_k}b_{j-1})) + \nabla_x(2\partial_{x_k}b_{j-1}))$, in order to apply Lemma 2.1.(c) with H^1-norms.
Now call $c_{comm} = c_{f_1}+c_{f_2}+c_{f_3}+c_{Df_2}+c_{Df_3}$ the biggest Gevrey parameter, common to all the functions f_1, f_2, f_3 inside the commutator, set $c_3 = c_{comm} \cdot \max_i \mathrm{Vol}(\mathrm{supp}\,(f_i))$, and $c_{117} = 1/(ec_3)^\alpha$. Then, define the following coefficients in Lemma 2.1, that are independent of the center point y_j:

$$
c_{106} = \frac{1}{(3^\alpha 4)(ec_3)^\alpha}, \quad c_{107} = c_{108} = (c_3\frac{8}{3}\Gamma(\frac{1}{\alpha})\frac{1}{\alpha(c_{117})^{1/\alpha}}\frac{1}{(\alpha c_{106})^{\frac{1}{\alpha-1}}})^{1/2}. \tag{2.23}
$$

Next we estimate I_2 moving the derivative D_0 of u_{j-1} in front of the integrand, then multiplying u_{j-1} with $b(\frac{y-y_{j-1}}{r})$, and finally adding and subtracting operators $A(D_0/\nu)$ with $\nu \leq \frac{\mu_{j-1}^\alpha}{3c_{132}}$,

$$
\begin{aligned}
I_2 &\leq \|A(\frac{D_0}{\beta\mu_j})2D_0 b_{j-1}\big[A(\frac{D_0}{\nu}) + (1-A(\frac{D_0}{\nu}))\big]b(\frac{(y-y_{j-1})}{r})u_{j-1}\|_{H^1} \\
&\quad +\|A(\frac{D_0}{\beta\mu_j})D_0(2D_0 b_{j-1})\big[A(\frac{D_0}{\nu}) + (1-A(\frac{D_0}{\nu}))\big]b(\frac{(y-y_{j-1})}{r})\big]u_{j-1}\|_{L^2} \\
&\leq c_{155,j-1}|2D_0 b_{j-1}|_{C^1}\exp(-c_{132}\mu_{j-1}^{\alpha^2}) + c_{152}c_{108}\exp(-c_{106}\nu^\alpha) \\
&\quad +c_{155,j-1}|D_0(2D_0 b_{j-1})|_{C^0}\exp(-c_{132}\mu_{j-1}^{\alpha^2}) + c_{152}c_{107}\exp(-c_{106}\nu^\alpha)
\end{aligned}
$$

To get the estimate above we apply twice Lemma 2.6 with the same parameters as in I_1. Next using (2.20) we estimate the terms $\|A(\frac{3D_0}{\nu})f(1-A(\frac{D_0}{\nu}))v\|_s$ using Lemma 2.1 c) and a).
Proceeding like with I_1, we have to calculate the time-Fourier transform of:

$$
f_2(y) = 2\partial_t b_{j-1} + 2\partial_t^2 b_{j-1},
$$

and the associated coefficients are (2.23). Finally, moving the derivative D_{x_r} of u_{j-1} in front of the integrand, multiplying u_{j-1} with $b(\frac{y-y_{j-1}}{r})$, adding and subtracting operators $A(D_0/\nu)$ with $\nu \le \frac{\mu_{j-1}^\alpha}{3c_{132}}$,

$$\begin{aligned} I_3 &\le \|A(\frac{D_0}{\beta\mu_j})2g^{kr}(x)D_{x_k}b_{j-1}\big[A(\frac{D_0}{\nu})+(1-A(\frac{D_0}{\nu}))\big]b(\frac{(y-y_{j-1})}{r})u_{j-1}\|_{H^1} \\ &\quad +\|A(\frac{D_0}{\beta\mu_j})D_{x_r}(2g^{kr}(x)D_{x_k}b_{j-1})\big[A(\frac{D_0}{\nu})+(1-A(\frac{D_0}{\nu}))\big]b(\frac{(y-y_{j-1})}{r})\big]u_{j-1}\|_{L^2} \\ &\le c_{155,j-1}|2ng^{kr}D_kb_{j-1}|_{C^1}\exp(-c_{132}\mu_{j-1}^{\alpha^2})+c_{152}c_{108}n^2|g^{kr}|_{C^1}\exp(-c_{106}\nu^\alpha) \\ &\quad +c_{155,j-1}|D_r(2g^{kr}D_kb_{j-1})|_{C^0}\exp(-c_{132}\mu_{j-1}^{\alpha^2})+c_{107}c_{152}n^2|g^{kr}|_{C^1}\exp(-c_{106}\nu^\alpha). \end{aligned}$$

Proceeding like with I_1 we have to calculate the time-Fourier transform of

$$f_3(y)=2\partial_{x_k}b_{j-1}+2\partial_{x_k}b_{j-1}+2\partial_{x_k}^2b_{j-1}.$$

and the associated coefficients are (2.23). By collecting all the terms of the estimate for (2.17), the bound for (2.16) becomes

$$\begin{aligned} &\|A(\frac{D_0}{\mu_j})b(\frac{y-y_j}{R})Pu_j\|_0 \le \|A(\frac{D_0}{\beta\mu_j})l(y)Pu_j\|_{L^2}+c_{153}\widetilde{c}_{107}\exp(-\mu_j^\alpha) \\ &\le c_{162,j}\big(\max\big(\exp(-\mu_{j-1}^\alpha),\exp(-c_{165}\mu_{j-1}^\alpha),\exp(-c_{132}\mu_{j-1}^{\alpha^2}),\exp(-c_{106}\nu^\alpha)\big)\big) \\ &\quad +c_{153}\widetilde{c}_{107}\exp(-\mu_j^\alpha) \end{aligned} \tag{2.24}$$

where, for all $j\ge 2$,

$$\begin{aligned} c_{162,j} &= 2c_{162,j-1}+c_{153}c_{164}+c_{155,j-1}|-P_2b_{j-1}+h^s(x)D_{x_s}b_{j-1}|_{C^0} \\ &\quad +c_{107}c_{152}(1+n^2|g^{kr}|_{C^0}+|h^s|_{C^0}) \\ &\quad +c_{155,j-1}|2D_0b_{j-1}|_{C^1}+c_{152}c_{108} \\ &\quad +c_{155,j-1}|D_0(2D_0b_{j-1})|_{C^0}+c_{152}c_{107} \\ &\quad +c_{155,j-1}|2ng^{kr}D_kb_{j-1}|_{C^1}+c_{152}c_{108}n^2|g^{kr}|_{C^1} \\ &\quad +c_{155,j-1}|D_r(2g^{kr}D_kb_{j-1})|_{C^0}+c_{107}c_{152}n^2|g^{kr}|_{C^1}. \end{aligned}$$

In order to write (2.24) in the form

$$\|A(\frac{D_0}{\mu_j})b(\frac{y-y_j}{R})Pu_j\|_0 \le c_{154,j}e^{-\mu_j^\alpha},$$

we set in (2.8)

$$c_{154,j} = c_{162,j}+c_{153}\widetilde{c}_{107}$$

and we look for μ_j of the form $\mu_j = c_{156,j}\mu_{j-1}^\alpha$ such that, for β as in (2.11), and collecting all the constraints on μ_j used in the proof,

$$\mu_j \le \frac{1}{6\beta}\nu = \frac{1}{6\beta}\frac{\mu_{j-1}^\alpha}{3c_{131}}, \quad \text{and consider}$$

$$c_{156}=c_{156,j}=\min\big(\frac{1}{18\beta c_{131}},c_{132}^{1/\alpha},c_{165}^{1/\alpha},\frac{c_{106}^{1/\alpha}}{3c_{131}}\big). \tag{2.25}$$

The right hand side of (2.25) is independent upon j due to the definition of c_{165}, c_{106} in (2.23), and the fact that c_{132} and c_{131} do not change during the iteration (see proof of Lemma 2.6). Therefore we define a parameter $c_{156}(< 1)$ independent of j by the formula (2.25). We then estimate μ_j from below

$$\mu_j = c_{156}\mu_{j-1}^{\alpha} = c_{156}(c_{156}\mu_{j-2}^{\alpha})^{\alpha} = c_{156}(c_{156}(c_{156}\mu_{j-3}^{\alpha})^{\alpha})^{\alpha}$$
$$= c_{156}^{1+\alpha+\alpha^2+\ldots+\alpha^{j-2}}\mu^{\alpha^{j-1}} \geq c_{156}^{1/(1-\alpha)}\mu^{\alpha^{j-1}}$$

where we apply $1 < \sum_{m=0}^{j-2}\alpha^m \leq 1/(1-\alpha)$. Finally, in order to obtain the requested condition $\mu_j \geq 1$, we set

$$c_{156}^{1/(1-\alpha)}\mu^{\alpha^{j-1}} \geq 1,\ \forall j \in [1, N],\ \text{which implies } \mu \geq c_{156}^{-\frac{1}{\alpha^{N-1}(1-\alpha)}}$$

and we finally find $c_{159} = c_{156}^{-\frac{1}{\alpha^{N-1}(1-\alpha)}}$ in the line before (2.6).
By applying Lemma 2.6 with $c_U = c_{152}$, $c_P = c_{153}$, $c_A = c_{154,j}$,
$c_{155,j} = c_{150}(c_{152}, c_{153}, c_{154,j})$, one obtains the last inequality (2.9) of the Theorem. □

Remark 2.8. 1. In order to work with the pseudodifferential operators $e^{-\epsilon|D_0|^2/2\tau}$ and $A(\beta|D_0|/\omega)$ one needs smooth functions in the time variable. Hence one should first operate a proper regularization in the time variable. We proceed in the same way as done in [10] or [17]. Observe that the functions u, Pu and u_j, Pu_j are always multiplied by a smooth localizer when $A(D_0)$ and e^{-D_0} are applied to them.

2. About the construction.
a) Assumption A1 (and analogously A3) implies that: $(\text{supp}\,(u) \cap \Omega_0) \subset \{y; \psi(y) \leq \psi_{max}\}$; and that the level sets $\{y \in \Omega_0; \psi(y) = c\}$, with $c \in [\psi_{min}, \psi_{max}]$, are contained in $\overline{\Upsilon \cup \Omega_a}$. An example of this construction is in section 3.
b) Assumption A1 and A3 can be relaxed in this way. Instead of defining ψ_{min}, Ω_a (or Λ_k), we just observe that the assumptions on ψ, Ω_0 together with $(\text{supp}\,(u) \cap \Omega_0) \subset \{y; \psi(y) \leq \psi_{max}\}$ imply the existence of a non empty set $\Omega_a \subset \Omega_0$ for which Theorem 1.1 holds. Ω_a can be defined as $\cup_j B_r(y_j)$, with $y_j \in \mathcal{E}$ (see Assumption 4) such that the support condition $\text{supp}\,(u_j) \cap B_{2R}(y_j) \subset \{y \in \Omega_0; \psi \leq \psi(y_j)\}$ is fulfilled for every j. This construction requires to follow step by step the local iteration and sometimes this is difficult. That is why the a-priori knowledge that the level set $\{y \in \Omega_0 \setminus \overline{\Upsilon}; \psi(y) = \psi_{min}\}$ is strictly contained in Ω_0 is useful, even if it excludes for example the case where the level sets of ψ are parallel hyperplanes and $\text{supp}\,(u)$ is on one side of one of them.

3. In Theorem 2.7 we have worked under the assumptions

$$\|u\|_{H^1(\Omega_1)} = 1, \quad \|Pu\|_{L^2(\Omega_1)} < 1, \quad \|A(D_0/(\beta\mu))l(y)Pu\|_{L^2} \leq \exp(-\mu^{\alpha}),$$

in order to apply Theorem 1.1 easily. One can generalize the assumptions by setting

$$\|u\|_{H^1(\Omega_1)} \leq c_{U,g}, \quad \|Pu\|_{L^2(\Omega_1)} \leq c_{P,g}, \quad \|A(D_0/(\beta\mu))l(y)Pu\|_{L^2} \leq c_{A,g}\exp(-\mu^{\alpha}),$$

and by changing the coefficients $c_{152}, c_{153}, c_{154,j}, c_{155,j}$ accordingly.
This gives a statement of global stability of the unique continuation for low temporal frequencies.

4. Notice that Lemma 2.5, Lemma 2.6, and Theorems 2.4, 1.1,1.2 can be reformulated for localizers supported on cylinders (instead of on balls), defined on cylinders

$$C_s(y_0) = \{(t,x) : |t-t_0| \le s,\ |x-x_0| \le s\},$$

by observing that:

$$C_{r/\sqrt{2}}(y_0) \subset B_r(y_0) \subset B_R(y_0) \subset C_{\sqrt{2}R}(y_0).$$

The advantage is to be able to reduce the assumptions on the regularity of the $x-$localizers. Namely, one can replace $b(y)$ in $G_0^{1/\alpha}(\mathbb{R}^{n+1})$ with the product $b_{ti}(t)b_{sp}(x)$, where $b_{ti} \in G_0^{1/\alpha}(\mathbb{R}_t)$ and $b_{sp} \in C_0^2(\mathbb{R}^n)$ are supported in B_2, equal to one in B_1 and $0 \le b_{ti}, b_{sp} \le 1$.
One can also replace the global localizer l in the proof of Theorems 2.4 with $l(y) = \sum_{m=1}^{M} l_{ti,m}(t) l_{sp,m}(x)$, with $l = 1$ in $\{y \in \Omega_0; \operatorname{dist}(y, \partial\Omega_0) \ge \frac{R_1}{4}\}$, which contains $\cup_{k=1}^{N} C_{\sqrt{2}R}(y_k)$, and $\operatorname{supp}(l) \subset \Omega_0$, and $l_{ti,m} \in C_0^\infty(\mathbb{R})$ and $l_{sp,m} \in C_0^2(\mathbb{R}^n)$.

2.1 Proof of Theorem 1.1

Proof. We consider two cases:
Case A. Assume $\|Pu\|_{L^2(\Omega_1)} \ge \exp(-c_{159})\|u\|_{H^1(\Omega_1)}$, where $c_{159} := c_{156}^{-\frac{1}{(1-\alpha)}\frac{1}{\alpha^{N-1}}} > 1$ has been defined in (2.25). Then the estimate is trivial

$$\|u\|_{L^2(\Omega_a)} \le \|u\|_{H^1(\Omega_1)} \le (\ln(1+\exp(c_{159})))^\theta \frac{\|u\|_{H^1(\Omega_1)}}{\ln\left(1+\frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)^\theta}.$$

Case B. Assume now $\|Pu\|_{L^2(\Omega_1)} < \exp(-c_{159})\|u\|_{H^1(\Omega_1)}$ and without restriction of generality take $\|u\|_{H^1(\Omega_1)} = 1$. Our aim is to consider separately estimates for low and high temporal frequencies. Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0) \in G_0^{1/\alpha}$, defined in Assumption A2. Let $b \in G_0^{1/\alpha}(\mathbb{R}^{n+1})$ be another localizer with support like in Assumption A2.
The parameter $\alpha \in (0,1)$ is then common to all the localizers in time, space and temporal frequency. Let $y_j \in \mathcal{E}$ be the set of the points defined in Assumption A4 and consider the balls $B_r(y_j)$ centred in those points. Observe that according to our definitions we have :

$$B_r(y_j) \subset \operatorname{supp} b((y-y_j)/r) \subseteq B_{2r}(y_j) \subset B_R(y_j) \subset \operatorname{supp} b((y-y_j)/R) \subseteq B_{2R}(y_j),$$

Recall that $b(\frac{y-y_j}{r}) = 1$ in $B_r(y_j)$ and observe that $u_j = u$ in $B_r(y_j)\backslash\cup_{s=1}^{j-1}B_r(y_s)$, with u_j defined in (2.4).
We then cover Ω_a by the disjoint sets $B_r(y_j)\backslash\cup_{s=1}^{j-1}B_r(y_s)$ and operate the initial

estimate as follows:

$$
\begin{aligned}
\|u\|_{L^2(\Omega_a)} \le \|u\|_{L^2(B_r(y_1))} + \|u\|_{L^2(B_r(y_2)\setminus B_r(y_1))} + \|u\|_{L^2(B_r(y_3)\setminus \cup_{s=1}^{2} B_r(y_s))} \\
+ ... + \|u\|_{L^2(B_r(y_N)\setminus \cup_{s=1}^{N-1} B_r(y_s))} \quad (2.26)\\
= \|b(\frac{y-y_1}{r})u\|_{L^2(B_r(y_1))} + \|b(\frac{y-y_2}{r})u_2\|_{L^2(B_r(y_2)\setminus B_r(y_1))} \\
+ ... + \|b(\frac{y-y_N}{r})u_N\|_{L^2(B_r(y_N)\setminus \cup_{s=1}^{N-1} B_r(y_s))} \\
\le \sum_{j=1}^{N} \|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} + \sum_{j=1}^{N} \|\big(1-A(\frac{D_0}{\omega})\big)b(\frac{y-y_j}{r})u_j\|_{L^2} := H_1 + H_2.
\end{aligned}
$$

In the last estimate we have chosen $\omega > 0$ and split all the terms in their low and high temporal component, i.e.

$$
\|b(\frac{y-y_j}{r})u_j\|_{L^2(B_r(y_j))} \le \|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} + \|\big(1-A(\frac{D_0}{\omega})\big)b(\frac{y-y_j}{r})u_j\|_{L^2}.
$$

To estimate H_2 we have

$$
\begin{aligned}
\|\big(1-A(\frac{D_0}{\omega})\big)b(\frac{y-y_j}{r})u_j\|_{L^2} \le \|(1-a(\xi_0/\omega))\mathcal{F}_{t\to\xi_0}(b((y-y_j)/r)u_j(y))\|^2_{L^2} \quad (2.27)\\
\le \frac{1}{\omega^2}\int_{|\xi_0|>\omega}\int_{\mathbb{R}^n} |\xi_0 \mathcal{F}_{t\to\xi_0}(b((y-y_j)/r)u_j(t,x))|^2 dx d\xi_0 \le \frac{1}{\omega^2}\|b((y-y_j)/r)u_j(y)\|^2_{H^1}.
\end{aligned}
$$

To estimate H_1 we first consider $\mu > c_{159}$ and we set $\|Pu\|_{L^2(\Omega_1)} = e^{-\mu}$, that implies $\|A(\frac{D_0}{\zeta\mu})l(y)Pu\|_0 \le e^{-\mu^\alpha}$, for all $\zeta > 0$. Then we choose $\omega = \mu_N^\alpha/(3c_{131})$ and apply Theorem 2.7 to each term of the sum:

$$
\|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} \le c_{155,N}\exp(-c_{132}\mu_N^{\alpha^2}). \qquad (2.28)
$$

This is possible since $\mu_N > 1$ is the smallest time frequency of the set μ_j, while $c_{155,N}$ is the biggest coefficient $c_{155,j}$, j=1,..,N.
Collecting the two bounds and reminding that $\mu_N \ge c_{156}^{1/(1-\alpha)}\mu^{\alpha^{N-1}} > 1$, we have:

$$
\begin{aligned}
\|u\|_{L^2(\Omega_a)} &\le Nc_{155,N}\exp(-c_{132}\mu_N^{\alpha^2}) + \frac{3c_{131}}{\mu_N^\alpha}\sum_{j=1}^{N}\|b(\frac{y-y_j}{r})u_j\|_{H^1(\Omega_1)} \\
&\le Nc_{155,N}\exp(-c_{132}(c_{156}^{1/(1-\alpha)}\mu^{\alpha^{N-1}})^{\alpha^2}) \\
&\quad + \frac{3Nc_{131}}{(c_{156}^{1/(1-\alpha)}\mu^{\alpha^{N-1}})^\alpha}\big(1+\frac{|b'|_{C^0}}{r}\big)c_{152} \\
&\le \frac{c_{158}}{\mu^{\alpha^N}} = \frac{c_{158}}{(-\ln(\|Pu\|_{L^2(\Omega_1)}))^{\alpha^N}} \le \frac{2^{\alpha^N}c_{158}\|u\|_{H^1(\Omega_1)}}{\big(\ln\big(1+\frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\big)\big)^{\alpha^N}},
\end{aligned}
$$

where c_{156} is defined in (2.25) and

$$
c_{158} = Nc_{155,N} + 3Nc_{131}c_{152}\big(1+\frac{|b'|_{C^0}}{r}\big)c_{156}^{-\alpha/(1-\alpha)}.
$$

In the last step we have applied $\ln(y) \geq \ln(1+y)/2$ for $y = \|u\|_{H^1(\Omega_1)}/\|Pu\|_{L^2(\Omega_1)} > e$, and then we have returned to the original notation.
Now we choose α such that $\alpha = (\theta)^{1/N}$ and which belongs to $(0,1)$ so that, defining $c_{160} = \left(\ln(1+e^{c_{159}(\theta)})\right)^{\theta} + 2^{\theta} c_{158}(\theta)$, we obtain the result. □

In the previous theorem the dependency of c_{160} upon θ is very bad.
For some applications it is better to keep α and N independent and formulate the following consequence:

Corollary 2.9. *Consider the assumptions of Theorem 1.1. Then for every* $0 < \alpha < 1$ *we have*

$$\|u\|_{L^2(\Omega_a)} \leq c_{160} \frac{\|u\|_{H^1(\Omega_1)}}{\left(\ln\left(1 + \frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\right)\right)^{\alpha^N}}$$

with $c_{160} = \left(\ln(1+e^{c_{159}})\right)^{\alpha^N} + 2^{\alpha^N} c_{158}$. *Here* $N \leq c_{170}$ *given by* (2.5).

2.2 Proof of Theorem 1.2

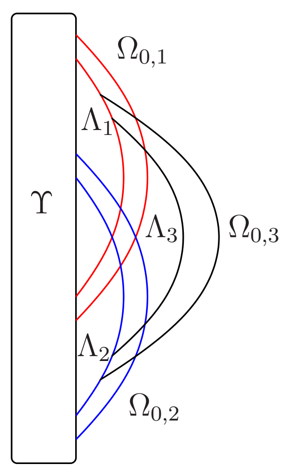

FIGURE 1. A possible construction of the domains $\Upsilon, \Omega_{0,k}, \Lambda_k$

Proof. Initialization of the radii and the localizers: Let $A(D_0)$ be a pseudo-differential operator with symbol $a(\xi_0) \in G_0^{1/\alpha}(\mathbb{R})$, defined in Assumption A2. We define the localizer $b(y) \in G_0^{1/\alpha}(\mathbb{R}^{n+1})$ with support like in Assumption A2.
Using Assumption A3, in each domain $\Omega_{0,k}$ we can calculate a table like (4.3), with $\Omega_{0,k}$ in place of Ω_0, and Λ_k in place of Ω_a, and where all the constants dependency is described in Proposition 2.4. By comparing the tables of the several $\Omega_{0,k}$ we can consider $R_2 = \min_k R_{2,k}$ and find $R = \frac{1}{4}\left(16 + \frac{1}{16}\right)^{-1/2} R_2$ the common radius for the local stability in Λ_k. After fixing R, we reduce also the values r_k so that $r = \min_k r_k$ is the common radius of the ball where the L^2 local estimate can be performed in Λ_k.
Construction of the set E *and the functions* u_j: For $\Omega_0 = \Omega_{0,1}$, $\Omega_a = \Lambda_1$ and $\psi = \psi_1$, we define $y_j \in \mathcal{E}_1$ the set of the maximal points for ψ_1, according to the procedure in Assumption A4. Call N_1 the number of points of the covering of Λ_1, i.e. $\Lambda_1 \subset$

$\cup_{j=1}^{N_1} B_r(y_j)$. Then we remove Λ_1 from Ω and we restart the procedure with the set Λ_2.
Namely, for $\Omega_0 = \Omega_{0,2}\backslash\Lambda_1$, $\psi = \psi_2$ and $\Omega_a = \Lambda_2$, we define $y_j \in \mathcal{E}_2$ according to the procedure of Assumption A4, where we use the indexing $j = N_1 + 1, .., N_2$.
When also Λ_2 is covered, one skips to Λ_3 and so on.
At the end one can define the set of points $\mathcal{E} = \cup_{k=1}^{K} \mathcal{E}_k$ of cardinality $N = \sum_{k=1}^{K} N_k$.
Consider the balls $B_r(y_j)$ centred in those points and define u_j as in (2.4).
Observe that according to our definitions we have :

$$B_r(y_j) \subset \operatorname{supp} b((y-y_j)/r) \subseteq B_{2r}(y_j) \subset B_R(y_j) \subset \operatorname{supp} b((y-y_j)/R) \subseteq B_{2R}(y_j)\,,$$

Moreover $b(\frac{y-y_j}{r}) = 1$ in $B_r(y_j)$ and observe that $u_j = u$ in $B_r(y_j)\backslash \cup_{s=1}^{j-1} B_r(y_s)$, with u_j defined in (2.4).
Uniform parameters: With the previous assumptions Lemma 2.6 can then be applied in each ball $B_{2R}(y_j)$, with $y_j \in \mathcal{E}_k$ in place of y_0, and we call $c_{131,k}, c_{132,k}, c_{165,k}, c_{150,k}$ the related parameters, that are constant for every ball in the region $\Omega_{0,k}$, but in principle change from region to region.
Therefore the following uniform constants are introduced:

$$c_{131,*} = \max_{k=1..K} c_{131,k},\ c_{132,*} = \min_{k=1..K} c_{132,k},\ c_{165,*} = \min_{k=1..K} c_{165,k},$$
$$c_{156,*} = \min\Big(\frac{1}{18\beta c_{131,*}}, c_{132,*}^{1/\alpha}, c_{165,*}^{1/\alpha}, \frac{c_{106}^{1/\alpha}}{3c_{131,*}}\Big). \tag{2.29}$$

We define $l(y) \in C_0^\infty(\mathbb{R}^{n+1})$ a localizer such that $0 \le l \le 1$, $l = 1$ on the set $\{y \in \bigcup_{k=1}^{K} \Omega_{0,k};\ \mathrm{dist}(y, \partial\Omega_0) \ge \frac{R_1}{4}\}$, which contains $\cup_{j=1}^{N} B_{2R}(y_j)$, and $\mathrm{supp}(l) \subset \Omega_0$.
In particular we consider β as in (2.11), and the related $\widetilde{c}_{106}$.
Construction:
We consider two cases:
Case A. Assume $\|Pu\|_{L^2(\Omega_1)} \ge \exp(-c_{159,*})\|u\|_{H^1(\Omega_1)}$, where $c_{159,*} := c_{156,*}^{-\frac{1}{(1-\alpha)}\frac{1}{\alpha^{N-1}}} > 1$.
Then the estimate is trivial

$$\|u\|_{L^2(\Lambda)} \le \|u\|_{H^1(\Omega_1)} \le \big(\ln(1+\exp(c_{159}))\big)^\theta \frac{\|u\|_{H^1(\Omega_1)}}{\ln\big(1+\frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\big)^\theta}\,.$$

Case B. Assume now $\|Pu\|_{L^2(\Omega_1)} < \exp(-c_{159,*})\|u\|_{H^1(\Omega_1)}$ and without restriction of generality take $\|u\|_{H^1(\Omega_1)} = 1$. Our aim is to consider separately estimates for low and high temporal frequencies. We cover Λ by the disjoint sets $B_r(y_j)\backslash\cup_{s=1}^{j-1} B_r(y_s)$ and operate the initial estimate as follows:

$$\begin{aligned}\|u\|_{L^2(\Lambda)} &\le \|u\|_{L^2(B_r(y_1))} + \|u\|_{L^2(B_r(y_2)\backslash B_r(y_1))} + ... + \|u\|_{L^2(B_r(y_N)\backslash\cup_{s=1}^{N-1} B_r(y_s))}\\ &= \|b(\frac{y-y_1}{r})u\|_{L^2(B_r(y_1))} + ... + \|b(\frac{y-y_N}{r})u_N\|_{L^2(B_r(y_N)\backslash\cup_{s=1}^{N-1} B_r(y_s))}\\ &\le \sum_{j=1}^{N} \|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} + \sum_{j=1}^{N} \|\big(1-A(\frac{D_0}{\omega})\big)b(\frac{y-y_j}{r})u_j\|_{L^2} := H_1 + H_2.\end{aligned}$$

In the last estimate we took $\omega > 0$ and split all the terms in their low and high temporal component, i.e.

$$\|b(\frac{y-y_j}{r})u_j\|_{L^2(B_r(y_j))} \le \|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} + \|\big(1-A(\frac{D_0}{\omega})\big)b(\frac{y-y_j}{r})u_j\|_{L^2}.$$

To estimate H_2 we have

$$\|(1-A(D_0/\omega))b((y-y_j)/r)u_j(y)\|^2_{L^2} \le \frac{1}{\omega^2}\|b((y-y_j)/r)u_j(y)\|^2_{H^1}.$$

To estimate H_1 we first observe that $\mathrm{supp}(b(\frac{y-y_j}{r})u_j)$ is in $\Omega_{0,1}$ for $j = 1, .., N_1$, where ψ_1 is defined. Then $\mathrm{supp}(b(\frac{y-y_j}{r})u_j)$ is in $\Omega_{0,2}$ for $j = N_1+1, .., N_2$ where ψ_2 is defined, and so on.
Consider $\mu > c_{159,*}$ and we set $\|Pu\|_{L^2(\Omega_1)} = e^{-\mu}$, that implies $\|A(\frac{D_0}{\zeta\mu})l(y)Pu\|_0 \le e^{-\mu^\alpha}$, for all $\zeta > 0$.
We also set $c_{152} = 2(1+N\frac{|b'|_{C^0}}{r})$, $c_{153} = 1+2N(1+n^2|g^{kr}|_{C^0}+|h^s|_{C^0})(\frac{|b'|_{C^0}}{r}+\frac{|b''|_{C^0}}{r^2}+(N-1)\frac{|b'|^2_{C^0}}{r^2})$, and $\mu_j = c_{156,*}\mu^\alpha_{j-1}$. Then we choose $\omega = \mu_N^\alpha/(3c_{131,*})$ and apply Theorem 2.7 to each term of the sum:

$$\|A(\frac{D_0}{\omega})b(\frac{y-y_j}{r})u_j\|_{L^2} \le c_{155,N}\exp(-c_{132,*}\mu_N^{\alpha^2}).$$

This is possible since $\mu_N > 1$ is the smallest time frequency of the set μ_j, while $c_{155,N}$ is the biggest coefficient $c_{155,j}$, j=1,..,N.
Collecting the two bounds and reminding that $\mu_N \ge c_{156,*}^{1/(1-\alpha)}\mu^{\alpha^{N-1}} > 1$, we have:

$$\begin{aligned}\|u\|_{L^2(\Lambda)} &\le Nc_{155,N}\exp(-c_{132,*}\mu_N^{\alpha^2}) + \frac{3c_{131,*}}{\mu_N^\alpha}\sum_{j=1}^N \|b(\frac{y-y_j}{r})u_j\|_{H^1(\Omega_1)}\\ &\le \frac{c_{158,*}}{\mu^{\alpha^N}} = \frac{c_{158,*}}{(-\ln(\|Pu\|_{L^2(\Omega_1)}))^{\alpha^N}} \le \frac{2^{\alpha^N}c_{158,*}\|u\|_{H^1(\Omega_1)}}{\big(\ln\big(1+\frac{\|u\|_{H^1(\Omega_1)}}{\|Pu\|_{L^2(\Omega_1)}}\big)\big)^{\alpha^N}},\end{aligned}$$

where

$$c_{158,*} = Nc_{155,N} + 3Nc_{131,*}c_{152}\big(1+\frac{|b'|_{C^0}}{r}\big)c_{156,*}^{-\alpha/(1-\alpha)}.$$

Now we choose α such that $\alpha = (\theta)^{1/N}$ and which belongs to $(0,1)$ so that, defining $c_{161} = \big(\ln(1+e^{c_{159,*}(\theta)})\big)^\theta + 2^\theta c_{158,*}(\theta)$, we obtain the result. $\square$

3 Applications

Assumption A5 Assume that $M = \mathbb{R}^n$ and on M we have a metric tensor g satisfying

$$a_0 I \le [g_{jk}(x)]^n_{j,k=1} \le b_0\, I, \quad \text{and} \quad \|g_{jk}\|_{C^4(M)} \le b_3. \tag{3.1}$$

Note that then

$$
\begin{aligned}
&b_0^{-1}I \le [g^{jk}(x)]_{j,k=1}^n \le a_0^{-1}I, \qquad (3.2)\\
&\|g^{jk}\|_{C^1(M)} \le b_3 a_0^{-2},\\
&\sqrt{a_0}d_{\mathbb{R}^n}(x_1,x_2) \le d_g(x_1,x_2) \le \sqrt{b_0}d_{\mathbb{R}^n}(x_1,x_2),\\
&\frac{1}{\sqrt{b_0}}\|\xi\|_{\mathbb{R}^n} \le \|\xi\|_g \le \frac{1}{\sqrt{a_0}}\|\xi\|_{\mathbb{R}^n}, \quad \xi \in T_x^*M,\\
&\sqrt{a_0}\|\eta\|_{\mathbb{R}^n} \le \|\eta\|_g \le \sqrt{b_0}\|\eta\|_{\mathbb{R}^n}, \quad \eta \in T_xM.
\end{aligned}
$$

Here we assume that $a_0 < 1 < b_0$. In Appendix A we call $a_1 = b_0^{-1}$ and $b_1 = a_0^{-1}$. Note that (3.1) implies that

$$|\mathrm{Sec}| < \Lambda_M = \Lambda_M(a_0, b_0, b_3),$$

where Sec is the sectional curvature of (M, g). Also assume that the injectivity radius of (M, g) satisfies $\mathrm{inj}(M, g) > i_0$ with $0 < i_0 < \min(1, \pi/\Lambda_M^{1/2})$.
Consider the wave operator (1.1). Assume that the lower order coefficients are such that

$$\|h^j\|_{C^0(M)} + \|q\|_{C^0(M)} \le b_3.$$

We fix the three positive parameters ℓ, T, γ as follows:

$$\ell \le i_0/4, \quad T > \ell, \quad 0 < \gamma \le T - \ell.$$

In this section, we use the following definitions.

Definition 3.1. $B_g(z, r_1) \subset M = \mathbb{R}^n$ is the ball with center z and radius r_1, defined using the Riemannian metric g. Also, $B_{\mathbb{R}^n}(x, r_1)$ is the Euclidean ball in $\mathbb{R}^n$. For $y = (t, x) \in \mathbb{R} \times M$, let

$$\mathcal{C}_g(y, r_1) = (t - r_1, t + r_1) \times B_g(x, r_1) \qquad (3.3)$$

and $\mathcal{C}(y, r_1) = \mathcal{C}_{\mathbb{R}^{n+1}}(y, r_1) = (t - r_1, t + r_1) \times B_{\mathbb{R}^n}(x, r_1)$. Also, denote

$$d_{\mathbb{R}\times(\mathbb{R}^n,g)}((t_1, x_1), (t_2, x_2)) = \max(|t_1 - t_2|, d_g(x_1, x_2))$$

and

$$d_{\mathbb{R}\times(\mathbb{R}^n,e)}((t_1, x_1), (t_2, x_2)) = \max(|t_1 - t_2|, d_{\mathbb{R}^n}(x_1, x_2)).$$

Let $z \in M$, and define the *hyperbolic function* as and

$$\psi(t, x; T, z) = (T - d_g(x, z))^2 - t^2, \quad y = (t, x) \in \mathbb{R} \times \mathbb{R}^n. \qquad (3.4)$$

Note that as $\mathrm{inj}(M) > i_0$, $(t, x) \mapsto \psi(t, x; T, z)$ is smooth in $\mathbb{R} \times (B_g(z, i_0) \setminus \{z\})$.
Define

$$
\begin{aligned}
S_{\ell,\gamma} &= S(z, \ell, T, \gamma) \qquad (3.5)\\
&:= \{(t, x) \in [-T + \ell, T - \ell] \times \mathbb{R}^n;\ \psi(t, x; T, z) \ge \gamma^2,\ d_g(x, z) \le T\}.
\end{aligned}
$$

Also let $z \in M$ and

$$\Sigma(z,\ell,T) = \{(t,x) \in \mathbb{R}\times\mathbb{R}^n;\ |t| \le T-\ell,\ |t| \le T - d_g(x,z)\} \tag{3.6}$$

be the *domain of influence* of the cylinder

$$W(z,\ell,T) = (-T+\ell, T-\ell)\times B_g(z,\ell). \tag{3.7}$$

3.0.1 Some geometric estimates for domains of influences

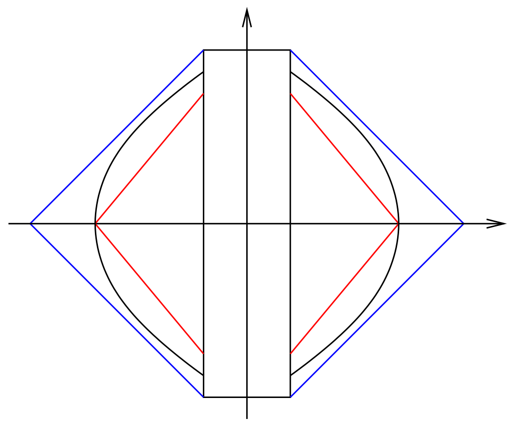

FIGURE 2. The hyperbolic surface between two domains of influence

Lemma 3.2. *Let T, ℓ, γ be as in Assumption A5. Denote $T_{\ell,\gamma} = ((T-\ell)^2 - \gamma^2)^{\frac{1}{2}} + \ell$, Then*

$$\Sigma(z,\ell,T-\gamma) \subset S(z,\ell,T,\gamma) \subset \Sigma(z,\ell,T_{\ell,\gamma}) \cup W(z,\ell,T) \subset \Sigma(z,\ell,T).$$

Proof. Assume that $x \in \mathbb{R}^n$ is such that $s = d_g(x,z) \in [\ell, T-\gamma]$ and that $|t| \le T-\gamma-s$. Then

$$\begin{aligned} \psi(t,x;T,z) &= (T-s)^2 - t^2 \\ &\ge\ (T-s)^2 - (T-\gamma-s)^2 \\ &\ge\ \Big((T-s)-(T-\gamma-s)\Big)\Big((T-s)+(T-\gamma-s)\Big) \\ &\ge\ \gamma(2(T-s)-\gamma) \\ &\ge\ \gamma^2. \end{aligned}$$

We see that

$$S(z,\ell,T,\gamma)\ \supset\ \{(t,x)\in[-T+\ell,T-\ell]\times\mathbb{R}^n;\ d_g(x,z)\ge\ell,\ |T-\gamma-d_g(x,z)|\ge|t|\}.$$

This proves

$$\Sigma(z,\ell,T-\gamma) \subset S(z,\ell,T,\gamma). \tag{3.8}$$

Assume next that $s = d_g(x,z) \in [\ell, T_{\ell,\gamma}]$ and $|t| > T_{\ell,\gamma} - s$. Then $T_{\ell,\gamma} - T < 0$ implies that

$$\begin{aligned}
\psi(t,x;T,z) &= (T-s)^2 - t^2 \\
&< (T-\ell-(s-\ell))^2 - ((T_{\ell,\gamma}-\ell)-(s-\ell))^2 \\
&\leq ((T-\ell)^2 - (T_{\ell,\gamma}-\ell)^2) - 2(T-\ell)(s-\ell) + 2(T_{\ell,\gamma}-\ell)(s-\ell) \\
&\leq \gamma^2.
\end{aligned}$$

Thus, we see that the complement of $S_{\ell,\gamma}$ satisfies

$$\{(t,x) \in [-T+\ell, T-\ell] \times \mathbb{R}^n;\ |T_{\ell,\gamma} - d_g(x,z)| < |t|\} \subset S^c_{\ell,\gamma} \cup W(z,\ell,T).$$

Hence we see that

$$S_{\ell,\gamma} \setminus W(z,\ell,T) \subset \Sigma(z,\ell,T_{\ell,\gamma}) \setminus W(z,\ell,T).$$

This and (3.8) yield the claim.

□

3.1 Applications: Stability on the domain of influence of a cylinder

Here we consider the case when the solution of the wave equation (1.1) vanishes in the cylinder $W(z_0,\ell,T)$ and T may be so large that we have to consider also singular points for d_g. We refer to Definition 3.1 for the definition of sets used.
Our aim is to prove the following:

Theorem 3.3. *Under the conditions of Assumption A5, let $z_0 \in \mathbb{R}^n$, and define*

$$\Omega = (-T,T) \times M,\ \Upsilon = W(z_0,T,\ell),\ \Lambda = S(z_0,\ell,T,\gamma) \setminus \Upsilon,$$
$$\Omega_0 = S(z_0,\ell,T,\frac{\gamma}{\sqrt{2}}) \setminus \{(t,x) : t \in \mathbb{R}, d_g(z_0,x) \leq \frac{\ell}{4}\},\ \Omega_1 = \Omega_0 \setminus \Upsilon.$$

Assume that $u \in H^1(\Omega)$ satisfies

$$P(x,D)u(y) = f(y), \quad \textit{for}\ y \in \Omega$$

and

$$u|_{W(z_0,\ell,T)} = 0. \tag{3.9}$$

Then for every $0 < \theta < 1$ we have

$$\|u\|_{L^2(\Lambda)} \leq c_{163} \frac{\|u\|_{H^1(\Omega_1)}}{\Big(\ln\big(e + \frac{\|u\|_{H^1(\Omega_1)}}{\|f\|_{L^2(\Omega_1)}}\big)\Big)^{\theta}}.$$

Here, c_{163} depends only on $a_0, b_0, b_3, T, \gamma, \ell,\ i_0,$ and θ.

Corollary 3.4. *By Lemma 3.2 we observe that, after a reparametrization of the time, $\Sigma(z_0,\ell,T) \subset S(z_0,\ell,T+\gamma,\gamma)$. Consider the wave equation formulated in Theorem 3.3. Hence for each γ such that $0 < \gamma < T-\ell$, the optimal time of the control $T-\ell$ (with $T-\ell = \max_{x,y\in\Sigma(z_0,\ell,T)\setminus W(z_0,\ell,T)} d_g(y,x)$) can be approximated from above by $T-\ell+\gamma$, using a result of stability of the unique continuation.*

3.1.1 Local stability estimate

Below, we say that the cut-off function corresponding to a center point $\widehat{y} = (\widehat{t}, \widehat{x}) \in \mathbb{R} \times \mathbb{R}^n$ and a radius $\widehat{r}$ is the product of a "time-variable cut-off function" and "space-variable cut-off function", given by

$$b_{\widehat{y},\widehat{r}}(t,x) = b^{(ti)}_{\widehat{t},\widehat{r}}(t) b^{(sp)}_{\widehat{x},\widehat{r}}(x), \tag{3.10}$$
$$b^{(ti)}_{\widehat{t},\widehat{r}}(t) = f^{(ti)}(\frac{t-\widehat{t}}{\widehat{r}}), \quad f^{(sp)}_{\widehat{x},\widehat{r}}(x) = f^{(1)}(\frac{x-\widehat{x}}{\widehat{r}}),$$

where $f^{(sp)} \in C^2_0(\mathbb{R}^n)$ and $f^{(ti)} \in G^{1/\alpha}_0(\mathbb{R}^1)$. We assume that $0 \le f^{(sp)} \le 1$ and $0 \le f^{(ti)} \le 1$. We assume that

$$\text{supp}\,(f^{(ti)}) \subset B_{\mathbb{R}}(0,\sqrt{2}), \quad f^{(ti)}|_{B_{\mathbb{R}}(0,1)} = 1,$$
$$\text{supp}\,(f^{(sp)}) \subset B_{\mathbb{R}^n}(0,\sqrt{2}), \quad f^{(sp)}|_{B_{\mathbb{R}^n}(0,1)} = 1.$$

Then we have for $\widehat{y} \in \mathbb{R}^{n+1}$ and $\widehat{r} > 0$

$$\text{supp}\,(b_{\widehat{y},\widehat{r}}) \subset \mathcal{C}(\widehat{y}, 2\widehat{r}), \quad b_{\widehat{y},\widehat{r}}|_{\mathcal{C}(\widehat{y},\widehat{r})} = 1. \tag{3.11}$$

Note that by (3.1) and (3.3), $\mathcal{C}(\widehat{y}, 2\widehat{r}) \subset \mathcal{C}_g(\widehat{y}, 2\sqrt{b_0}\widehat{r})$ and $\mathcal{C}_g(\widehat{y}, \sqrt{a_0}\widehat{r}) \subset \mathcal{C}(\widehat{y}, \widehat{r})$.

For the proof of the global stability we must define the following points $\widehat{y}$ and functions $\psi_{\widehat{z},\widehat{T}}$.

Definition 3.5. (see Figure 2 below, where $\widehat{y} = y_j$.) Let $\widehat{y} = (\widehat{x}, \widehat{t}) \in S(z_0, \ell, T, \gamma) \setminus \{y;\ t \in \mathbb{R}; d_g(z_0, \widehat{x}) < \ell\}$:
a) If $\ell \le d_g(z_0, \widehat{x}) \le \frac{7}{8} i_0$, then define $\widehat{z} = z_0$, $\widehat{T} = T$, $\psi_{\widehat{z},\widehat{T}}(y) = \psi(y; z_0, T)$,
b) If $\frac{7}{8} i_0 < d_g(z_0, \widehat{x})$, then let $\psi_{\widehat{z},\widehat{T}}(y) = \psi(y; \widehat{z}, \widehat{T})$. Calling $\gamma_{z_0,\widehat{\xi}}$ a distance minimizing, unit speed geodesic from z_0 to $\widehat{x}$ in M, we define $\widehat{z}, \widehat{T}$ as follows:

$$\widehat{L} := d_g(z_0, \widehat{x}) - \frac{i_0}{4} > \frac{5i_0}{8}, \qquad \widehat{T} := T - \widehat{L}, \qquad \widehat{z} := \gamma_{z_0,\widehat{\xi}}(\widehat{L}),$$

Note that the choice of the point $\widehat{z}$ is not unique as there may be several distance minimizing geodesics from z_0 to $\widehat{x}$.
Observe that for $\ell < i_0/4$ and $T > \frac{7i_0}{8}$, we have

$$\gamma + \ell < \gamma + \frac{i_0}{4} \le \widehat{T} = T - d_g(z_0, \widehat{x}) + \frac{i_0}{4} < T - \frac{5i_0}{8} < T - \ell - \frac{3i_0}{8}. \tag{3.12}$$

We then introduce the sets:

$$\Omega_2(\widehat{z}, \widehat{T}, \ell, \gamma) = S(\widehat{z}, \widehat{T}, \ell, \gamma) \cap \{(t,x) : t \in \mathbb{R}, \ell \le d_g(\widehat{z}, x) \le \frac{5}{8} i_0\}, \tag{3.13}$$
$$\Omega_3(\widehat{z}, \widehat{T}, \ell, \gamma) = S(\widehat{z}, \widehat{T}, \ell, \frac{\gamma}{\sqrt{2}}) \cap \{(t,x) : t \in \mathbb{R}, \frac{1}{4}\ell \le d_g(\widehat{z}, x) \le \frac{7}{8} i_0\}.$$

Lemma 3.6. *Under the Assumptions A2, A5, let* $\widehat{y}$ *and* $\psi_{\widehat{z},\widehat{T}}$ *be as in Definition 3.5. Then there exist* $r, R, c_U, c_P, c_A,$ *and* $c_{150}, c_{131}, c_{132}$*, dependent only on the parameters* $a_0, b_0, b_3, T, \gamma, \ell,$ *and* i_0 *such that the following property holds:*
If $v \in H_0^1(\Omega)$ *and* $\mu \geq 1$ *is such that*

$$supp\,(v) \cap \mathcal{C}(\widehat{y}, 2R) \subset \{y \in \mathcal{C}(\widehat{y}, 2R);\ \psi_{\widehat{z},\widehat{T}}(y) \leq \psi_{\widehat{z},\widehat{T}}(\widehat{y})\} \tag{3.14}$$

and

$$\|v\|_{H^1(\mathcal{C}(\widehat{y},2R))} \leq c_U, \quad \|Pv\|_{L^2(\mathcal{C}(\widehat{y},2R))} < c_P, \quad \|A(D_0/\mu)(b_{\widehat{y},R}Pv)\|_{L^2(\mathbb{R}^{n+1})} \leq c_A e^{-\mu^\alpha},$$

then,

$$\|A(D_0/\omega)(b_{\widehat{y},r}v)\|_{H^1(\mathbb{R}^{n+1})} \leq c_{150}\exp(-c_{132}\mu^{\alpha^2}), \quad \textit{for all } \omega \leq \mu^\alpha/(3c_{131}).$$

Proof. In the Appendix we have calculated uniform estimates for the function $\psi(y; z_0, T)$ defined in Definition 3.5 a); see (4.6), (4.7) (with $\gamma_I = \gamma/\sqrt{2}$), (4.8), (4.12), (4.13). Analogously, one can estimate the functions $\psi(y; \widehat{z}, \widehat{T})$ defined in Definition 3.5 b): calling $\Omega_2 = \Omega_2(\widehat{z}, \widehat{T}, \ell, \gamma)$ and $\Omega_3 = \Omega_3(\widehat{z}, \widehat{T}, \ell, \gamma)$, we have

$$\begin{aligned}
&\|\psi'_{\widehat{z},\widehat{T}}\|_{C^0(\Omega_3)} \leq b_4(T+1),\\
&\|\psi''_{\widehat{z},\widehat{T}}\|_{C^0(\Omega_3)} \leq b_4(T+1)((\ell/4)^{-1}+1),\\
&\|\psi'''_{\widehat{z},\widehat{T}}\|_{C^0(\Omega_3)} \leq b_4(T+1)((\ell/4)^{-2}+(\ell/4)^{-1}),\\
&\min_{y\in\Omega_3}|d\psi_{\widehat{z},\widehat{T}}(y)| \geq \sqrt{2}\gamma b_0^{-1/2}, \quad \min_{y\in\Omega_3}|p(y, d\psi_{\widehat{z},\widehat{T}}(y))| \geq 2\gamma^2,\\
&d_{\mathbb{R}\times(\mathbb{R}^n,e)}(\Omega_2, \partial\Omega_3) \geq \frac{1}{\sqrt{b_0}}\min\{\frac{i_0}{4}, \frac{\gamma^2}{8(T-\ell)}, \frac{3\ell}{4}\}.
\end{aligned} \tag{3.15}$$

Next define $R_0 = (2\sqrt{b_0})^{-1}\min\{i_0/4,\ 3\ell/4,\ \gamma^2/(8(T-\ell))\}$ a uniform radius that let the ball $B_{2R_0}(\widehat{y})$ stay inside the injectivity radius (in order to assure the regularity of $\psi_{\widehat{z},\widehat{T}}$) and inside the set Ω_3 (in order to assure that $\psi_{\widehat{z},\widehat{T}}$ is non-characteristic in the ball), according to Lemma 4.3. Moreover, $\mathcal{C}(\widehat{y}, 2R_0) \subset \Omega_3$. We then consider the procedure of Appendix A to determine the radii r, R related to the function $\psi = \psi_{\widehat{z},\widehat{T}}(y) - \psi_{\widehat{z},\widehat{T}}(\widehat{y})$. We set $R_1 = \min\{1, R_0, (\lambda|d\psi|_{C^0(\Omega_3)})^{-1}\}$ in the Table (4.3), and we observe that, using the estimates (3.15) for the derivatives of ψ and Assumption A5, we can choose radii r, R, R_2 that are the same for each $\widehat{y}$, and consequently also the derived parameters.
As seen in section 2, all the parameters in Lemma 2.6, $r, R, c_U, c_P, c_A, c_{150}, c_{131}, c_{132}$, depend on the uniform estimates for the quantities listed in Proposition 2.4. As we saw above, these estimates depend on the parameters a_0, b_0, b_3, T, γ, ℓ, and i_0. Then, for each $\widehat{y}$, the claim follows from Lemma 2.6 for v in place of u, with the function $\psi = \psi_{\widehat{z},\widehat{T}}(y) - \psi_{\widehat{z},\widehat{T}}(\widehat{y})$. □

3.1.2 Global stability estimate

Rule of choosing the center points of small balls:
We are going to apply the local stability estimate for the solution u of the wave

equation. Let r, R be the radii defined in Lemma 3.6 and consider the cylinders $\mathcal{C}(y_j, r/2)$ having center points at y_j chosen iteratively below, see (3.3). For each point y_j we define a localizer $b_j(y)$ associated with $y_j = (t_j, x_j)$ (see (3.11)) by

$$b_j(t,x) = b_{y_j, r/2}(t,x).$$

We proceed in analogy with Assumption A4, with Λ in place of Ω_a, with Λ and Ω_0 defined in the statement of Theorem 3.3. Next set $\psi(y) = \psi(y; T, z_0)$, as in (3.4). The main difference is that here ψ is not everywhere a $C^{2,\rho}$ function, as explained below.

We define the set $\mathcal{E} = \{y_j \in \overline{\Lambda}, j = 1, ..., J_0\}$ and the functions $u_j(y)$, iteratively as follows:

1) For $j = 1$ we define $u_1(t,x) = u(t,x)$ and consider $\overline{\Lambda} \subset \Omega_0$

$$y_1 \in \operatorname{argmax} \{\psi(y; T, z_0)\,;\, y = (t,x) \in \overline{\Lambda}\}. \tag{3.16}$$

2) For $j \geq 2$, after $y_1, y_2, \ldots, y_{j-1} \in \mathcal{E}$ have been chosen and the function $u_j(t,x)$ has been constructed we proceed as follows: If $\operatorname{supp}(u_j) \cap \overline{\Lambda} \neq \emptyset$, we choose y_j to be a point that satisfies

$$y_j \in \operatorname{argmax} \{\psi(y; T, z_0)\,;\, y = (t,x) \in \overline{\Lambda} \setminus \cup_{k=1}^{j-1} \mathcal{C}(y_k, r/2)\} \tag{3.17}$$

and define

$$u_j(y) = (1 - b_{j-1}(y)) u_{j-1}(y). \tag{3.18}$$

We notice that by construction $\operatorname{supp}(u_j) \cap \mathcal{C}(y_j, 2R) \subset \{y; \psi(y; T, z_0) \leq \psi(y_j; T, z_0)\}$. When $\operatorname{supp}(u_j) \cap \overline{\Lambda} = \emptyset$, we end the iteration and we set J_0 equal to j.

Next we estimate the number of iteration steps J_0. By construction, the points y_j in steps 1 and 2 satisfy $d_{\mathbb{R}\times(\mathbb{R}^n, e)}(y_j, y_k) \geq r/2, \quad j \neq k$. Moreover,

$$\overline{\Lambda} \subset \mathcal{C}(y_I, \rho_0), \quad \text{where } \rho_0 = \frac{2}{\sqrt{a_1}}(T+1), \quad y_I = (0, z_0),$$

see (3.3). Thus the maximal number J_0 of steps is smaller or equal to the maximal number of points in a $\widetilde{r}$ net in the set $\overline{\Lambda}$ that is bounded by

$$J_0 \leq \frac{\operatorname{vol}_{\mathbb{R}\times\mathbb{R}^n}(\mathcal{C}(0, \rho_0 + r))}{\operatorname{vol}_{\mathbb{R}\times\mathbb{R}^n}(\mathcal{C}(0, \frac{r}{4}))} \leq C_1 \frac{(T+2)^{n+1}}{r^{n+1}} \tag{3.19}$$

where C_1 is a uniform constant that can be estimated in an explicit way.

Note that above we have always chosen $y_j = (t_j, x_j)$ as maximal points for the hyperbolic function $\psi(y; T, z_0)$ associated with the "original" center point z_0 and time T. The motivation for this choice is that the level sets of $\psi(y; T, z_0)$ are the best approximation of the domain of influence $\Sigma(z_0, \ell, T)$ that we want to approach.

When the distance of x_j to the point z_0 is larger than the injectivity radius, the function $y \mapsto \psi(y; T, z_0)$ is only Lipschitz-smooth but it may happen that it is not C^2-smooth. To apply Lemma 3.6 in this case, we choose a different hyperbolic function ψ_{z_j, T_j} that changes at each step of the iteration and depends on the point y_j.

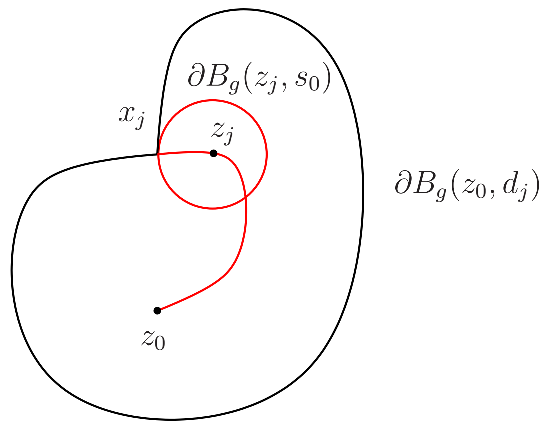

FIGURE 3. Preparations to do unique continuation near $y_j = (t_j, x_j) \in \mathbb{R} \times \mathbb{R}^n$. When $d_j = d_g(z_0, x_j)$ is larger than the injectivity radius, the boundary of the ball $B_g(z_0, d_j)$ may be non-smooth (the black external contour in the figure). For $x_j \in B_g(z_0, d_j)$ we choose some distance minimizing geodesic $\gamma_{z_0,\xi_j}([0, d_j])$ that connects z_0 to x_j. In the figure, this geodesic is the red curve from z_0 to x_j. On this geodesic we choose a point $z_j = \gamma_{z_0,\xi_j}(d_j - s_0)$. The boundary of the ball $B_g(z_j, s_0)$ (the red circle in the figure) is smooth and contains the point x_j. We do unique continuation near the point y_j using the hyperbolic function ψ_{z_j,T_j}, associated with the center z_j, that is smooth near y_j.

We then distinguish two cases as in Definition 3.5:
a) If $\ell \le d_g(z_0, x_j) \le \frac{7i_0}{8}$, then we consider $\psi(y; T_j, z_j)$ with $z_j = z_0$ and $T_j = T$.
b) Next, assume that $d_j := d_g(z_0, x_j) > \frac{7i_0}{8}$. Then, we define $\widehat{y} = y_j$ and as in Definition 3.5:

$$L_j = \widehat{L}, \quad T_j = \widehat{T}, \quad z_j = \widehat{z}, \quad \psi_{z_j,T_j}(y) = \psi(y; T_j, z_j).$$

Note that the choice of the point z_j is not unique as there may be several distance minimizing geodesics from z_0 to x_j.

Lemma 3.7. *For the points $y_j = (t_j, x_j) \in \overline{\Lambda}$, $z_j \in \mathbb{R}^n$, the time $T_j > 0$, and the function u_j chosen above, the support condition (3.14) is valid in the cylinder $\mathcal{C}(y_j, 2R)$ for the function $\psi_{z_j,T_j}(y)$, that is,*

$$\mathit{supp}\,(u_j) \cap \mathcal{C}(y_j, 2R) \subset \{y \in \mathbb{R}^{n+1};\ \psi_{z_j,T_j}(y) \le \psi_{z_j,T_j}(y_j)\}. \tag{3.20}$$

Moreover, we have $\psi(y_j; T, z_0) = \psi(y_j; T_j, z_j)$ and

$$y_j \in \partial S(z_j, T_j, \ell, \gamma_j) \cap \mathcal{C}(y_j, 2R), \quad \textit{where } \gamma_j := \sqrt{\psi(y_j; T, z_0)} \ge \gamma.$$

Proof. If $\ell \le d_g(z_0, x_j) \le \frac{7i_0}{8}$, then the property is trivial because of $\psi(y; T_j, z_j) = \psi(y; T, z_0)$ and the definition of $y_j \in \mathcal{E} \subset \overline{\Lambda}$.
Assume now that $d_g(z_0, x_j) > \frac{7i_0}{8}$. Recall that by definition of $R < R_0$ in the proof of Lemma 3.6, we have $2R < b_0^{-1/2} \frac{\gamma^2}{T-\ell} \le b_0^{-1/2}\gamma$. Let us consider $(t, x) \in \mathcal{C}(y_j, 2R)$. By the triangle inequality and the definition of L_j we have

$$d_g(x, z_j) + L_j \ge d_g(x, z_0), \quad d_g(x_j, z_j) + L_j = d_g(x_j, z_0).$$

This yields

$$T-(d_g(x,z_j)+L_j)\le T-d_g(x,z_0). \tag{3.21}$$

Since

$$\begin{aligned} d_g(x,z_j)+L_j &\le d_g(x,x_j)+d_g(x_j,z_j)+L_j \le d_g(x,x_j)+d_g(x_j,z_0)\\ &\le 2Rb_0^{1/2}+T-\gamma \le T, \end{aligned}$$

we have $T-(d_g(x,z_j)+L_j)\ge 0$, and (3.21) implies

$$\psi(t,x;T_j,z_j)=(T-L_j-d_g(x,z_j))^2-t^2\le (T-d_g(x,z_0))^2-t^2=\psi(t,x;T,z_0). \tag{3.22}$$

Hence,

$$\begin{aligned} &\operatorname{supp}(u_j)\cap\mathcal{C}(y_j,2R)\subset\\ &\{(t,x)\in\mathcal{C}(y_j,2R);\ \psi(t,x;T,z_0)\le\gamma_j^2\}\subset\{(t,x)\in\mathcal{C}(y_j,2R);\ \psi(t,x;T_j,z_j)\le\gamma_j^2\} \end{aligned}$$

Moreover, when $\widetilde{x}\in\gamma_{z_0,\xi_j}([L_j,d_j])$, where $\gamma_{z_0,\xi_j}([0,d_j])$ is a length minimizing minimizing geodesic connecting z_0 to x_j, and $\widetilde{t}\in\mathbb{R}$, we have $d_g(\widetilde{x},z_j)+L_j=d_g(\widetilde{x},z_0)$ and

$$\begin{aligned} \psi(\widetilde{t},\widetilde{x};T_j,z_j) &= (T-L_j-d_g(\widetilde{x},z_j))^2-\widetilde{t}^{\,2}\\ &= (T-d_g(\widetilde{x},z_0))^2-\widetilde{t}^{\,2} = \psi(\widetilde{t},x;T,z_0). \end{aligned} \tag{3.23}$$

In particular, when $(\widetilde{t},\widetilde{x})$ is equal to $y_j=(t_j,x_j)$, we see that $\psi(t_j,x_j;T_j,z_j)=\psi(t_j,x_j;T,z_0)$.
The above implies that

$$\mathcal{C}(y_j,2R)\cap S(z_j,T_j,\ell,\gamma_j)\subset\mathcal{C}(y_j,2R)\cap S(z_0,T,\ell,\gamma_j). \tag{3.24}$$

Note that the boundary of $S(z_0,T,\ell,\gamma_j)$ may be non-smooth in the ball $\mathcal{C}(y_j,2R)$, while the boundary of $S(z_j,T_j,\ell,\gamma_j)$ is smooth. That is why we have introduced the new function ψ_{z_j,T_j}. We also recall that $\mathcal{C}(y_j,2R)\subset S(z_0,T,\ell,\frac{\gamma}{\sqrt{2}})$ and $\mathcal{C}(y_j,2R)\subset S(z_j,T_j,\ell,\frac{\gamma_j}{\sqrt{2}})$.
By the construction of u_j and its support and the inclusion (3.24) we deduce that

$$u_j=0,\quad \text{for } y\in\mathcal{C}(y_j,2R)\cap S(z_j,T_j,\ell,\gamma_j). \tag{3.25}$$

Moreover, since $\psi(t_j,x_j;T_j,z_j)=\psi(t_j,x_j;T,z_0)=\gamma_j^2$, we have $y_j\in\partial S(z_j,T_j,\ell,\gamma_j)\cap\mathcal{C}(y_j,2R)$. □

Proof of Theorem 3.3. We apply Theorem 1.2 in a special way. As mentioned before, here Lemma 3.6 replaces Lemma 2.6.
Set y_j like in (3.17), and u_j like in (3.18).
Step 1. *Within the injectivity radius.* Let $d_g(z_0,x_j)\le 7i_0/8$. Define like in (3.13) and Lemma 3.6.

$$\begin{aligned} \Omega_{0,1} &= \Omega_3(z_0,T,\ell,\gamma),\\ \psi_1(y) &= \psi(y;T,z_0),\\ \Lambda_1 &= \Omega_2(z_0,T,\ell,\gamma_1). \quad \gamma_1=T-\frac{5i_0}{8}\ge\gamma. \end{aligned}$$

Here $\psi_{max,1} = (T-\ell)^2$, $\psi_{min,1} = \gamma_1^2$. Observe that $\Lambda_1 \subseteq \Omega_2(z_0, T, \ell, \gamma)$, the set used in Lemma 3.6 to compute the uniform radius R. By construction every $y \in \mathcal{C}(y_j, 2R)$ is such that $\frac{\ell}{4} \leq d_g(z_0, x_j) \leq \frac{7i_0}{8}$, hence $x \to d_g(z_0, x)$ is C^3-smooth in $\mathcal{C}(y_j, 2R)$. Moreover ψ_1 is $C^3(\Omega_{0,1})$ and hence regular enough to apply the local stability result of Lemma 3.6. Here we are in the case where $\widehat{y} = y_j$ is like Definition 3.5 a). The condition (3.14) is fulfilled by u_j due to the initial assumption that $u = 0$ in $W(z_0, \ell, T)$ and the construction of u_j step by step. Call N_1 the number of points y_j used to cover Λ_1. If $T \leq 7i_0/8$, then the procedure stops here. If also $T \leq 5i_0/8$, then it is enough to use a fraction of i_0 above to define Ω_2 and Ω_3.
Step $j > N_1$. Case $T > 7i_0/8$.
Here we change $\Omega_{0,j}$, $\psi_j(y)$ and Λ_j at each step. We have 2 cases:
a) If $y_j \in \Omega_{0,1} \setminus \Lambda_1$, then we simply consider $\Omega_{0,j} = \Omega_{0,1}$ and $\psi_j = \psi_1$ (that is C^3 regular since $d_g(x_j, z_0) \leq 7i_0/8$). Here $\Lambda_j = \Omega_2(z_0, T, \ell, \gamma_j)$, with $\gamma_j = \sqrt{\psi(y_j; T, z_0)}$, but $\mathcal{E}_j = \{y_j\}$, in the sense that we apply the local unique continuation step just once, in a cylinder $\mathcal{C}(y_j, 2R)$ centred in $y_j \in \{y; \psi_1(y) = \psi_1(y_j)\}$. Observe that $\Lambda_j \subseteq \Omega_2(z_0, T, \ell, \gamma)$. Again for $\widehat{y} = y_j$ holds Definition 3.5 a) and the condition (3.14) is fulfilled by construction. By Remark 2.8 2.b) there is no need of defining $\psi_{min,j}$ here.
b) If $y_j \notin \Omega_{0,1}$. Then, $d_g(z_0, x_j) > 7i_0/8$.
Here we are outside of the domain where ψ_1 is certainly smooth, since the function $x \to d_g(z_0, x)$ can fail to be C^2-smooth in $\mathcal{C}(y_j, 2R)$. So even if $y_j \in \{y; \psi_1(y) = \psi_1(y_j)\}$, to apply the local stability we choose another function ψ_j passing through y_j and having the good properties outlined in Lemma 3.7. Calling $\widehat{y} = y_j$ and defining z_j, T_j, ψ_j as in Definition 3.5 b), we can consider

$$
\begin{aligned}
\Omega_{0,j} &= \Omega_3(z_j, T_j, \ell, \gamma), \\
\psi_j(y) &= \psi(y; T_j, z_j), \\
\Lambda_j &= \Omega_2(z_j, T_j, \ell, \gamma_j), \quad \gamma_j = \sqrt{\psi(y_j; T, z_0)} \geq \gamma.
\end{aligned}
$$

Observe that $\Lambda_j \subseteq \Omega_2(z_j, T_j, \ell, \gamma)$. Again we have $\mathcal{E}_j = \{y_j\}$, in the sense that we apply unique continuation just in a cylinder $\mathcal{C}(y_j, 2R)$ centred in $y_j \in \{y; \psi_j(y) = \psi_j(y_j)\}$. The condition $\mathrm{supp}\,(u_j) \cap \mathcal{C}(y_j, 2R) \subset \{y; \psi_j(y) \leq \psi_j(y_j)\}$ is fulfilled due to Lemma 3.7. By Remark 2.8 2.b) there is no need of defining $\psi_{min,j}$ here.
Notice that, due to our uniform estimates, the radii R and r remain unchanged for every y_j and the other constants of the Table (4.3) are chosen uniformly. This implies that $c_{156,*} = c_{156,1}$. We also recall that $\mathcal{C}(y_j, 2R) \subset S(z_0, T, \ell, \frac{\gamma}{\sqrt{2}})$ for every j, by the construction of the points y_j and the choice of R. Here $l(y) \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies $l = 1$ on $\cup_{k=1}^{J_0} \mathcal{C}(y_k, 2R)$, $0 \leq l \leq 1$ and $\mathrm{supp}(l) \subset S(z_0, T, \ell, \frac{\gamma}{\sqrt{2}})$, see Remark 2.8-4. The coefficient c_{163} is computed like c_{161}. □

Remark 3.8. We remark that an alternative proof of Th. 3.3 is possible by applying Th. 1.2 in the following way. Define a net of center points (t_k, z_k) for the translated hyperbolic functions:

$$\psi(y; T_k, z_k, t_k) = (T_k - d_g(x, z_k))^2 - (t - t_k)^2.$$

Choose $\Upsilon = W(z_0, , T, \ell)$, $\Omega_{0,k} \subset \{y; y \in [-T_k + t_k, T_k + t_k] \times \mathbb{R}^n;\ \psi(y; T_k, z_k, t_k) \geq \gamma_k^2/2,\ T_k \geq d_g(x, z_k)\}$ and $\Lambda_k \subset \{y; y \in [-T_k + t_k, T_k + t_k] \times \mathbb{R}^n;\ \psi(y; T_k, z_k, t_k) \geq \gamma_k^2,\ T_k \geq d_g(x, z_k)\}$. The construction is similar to the one in Figure 1 of section 2. In this case one does not need to introduce the points $\widehat{y}$ of Definition 3.5. The parameters $(t_k, z_k, T_k, \gamma_k)$ should be chosen such that $\Omega_{0,k}$ is contained in the domain $0 < d_g(z_k, x) \leq \frac{7}{8} i_0$ (to guarantee the regularity of $\psi(y; T_k, z_k, t_k)$). Moreover $\Lambda_k \subset \Sigma(z_0, \ell, T)$ and their union should cover a subset of the domain of influence $\Sigma(z_0, \ell, T)$.

3.1.3 The case of solutions with small values in a cylinder

Our purpose is to reformulate Theorem 3.3 for a wave equation with vanishing source term and a solution u that is no longer vanishing but it is small inside a cylindrical set.

Corollary 3.9. *Under the conditions of Assumption A5, let $z_0 \in \mathbb{R}^n$. Also, let $\Omega = (-T, T) \times B_g(z_0, T + \ell)$, $\Omega_1 = S(z_0, \ell, T, \frac{\gamma}{\sqrt{2}}) \setminus \{(t, x) : t \in \mathbb{R}, d_g(z_0, x) \leq \ell/4\}$, $\Omega_2 = S(z_0, \ell, T, \gamma)$. Assume that $u \in H^1(\Omega)$ satisfies*

$$P(x, D)w(t, x) = 0, \quad \textit{for } (t, x) \in \Omega$$

and define

$$W_1 = (-T + \ell, T - \ell) \times B_g(z_0, \ell + \gamma). \tag{3.26}$$

Then for every $0 < \theta < 1$ we have

$$\|w\|_{L^2(\Omega_2 \setminus W_1)} \leq c_{166} \frac{\|w\|_{H^1(\Omega_1 \setminus W_0)}}{\left(\ln\left(e + \frac{\|w\|_{H^1(\Omega_1 \setminus W_0)}}{C' \|w|_{W_1}\|_{H^1(W_1)}}\right)\right)^{\theta}}\ . \tag{3.27}$$

Here, c_{166} depends only on $a_0, b_0, b_3, T, \gamma, \ell$, i_0, and θ.

Proof. Let $B_0 = B_g(z_0, \ell)$, $B_1 = B_g(z_0, \ell + \gamma)$, and $W_0 = W(z_0, \ell, T) = (-T + \ell, T - \ell) \times B_0 \subset W_1$. We use a cut-off function $\eta(x) \in C_0^2(B_1)$, $0 \leq \eta \leq 1$, that is equal one in B_0 and satisfies $\|\eta\|_{C^2(\mathbb{R}^n)} \leq c_0 \gamma^{-2}$, where c_0 is a uniform constant. Then $\widetilde{w}(x, t) = (1 - \eta(x))w(x, t)$ vanishes in W_0 and we have

$$\begin{aligned} &P(y, D)\widetilde{w}(t, x) = F(t, x), \quad \text{in } \Omega, \\ &F(t, x) = -g^{jk}(x)(D_j \eta D_k \eta)\widetilde{w} - g^{jk}(x)(D_j \eta D_k \widetilde{w}) - h^j(x)(D_j \eta)\widetilde{w}\ \in L^2(\Omega), \end{aligned}$$

and since η is supported in B_1,

$$\|F\|_{L^2(\Omega_1)} \leq \|F\|_{L^2(W_1)} \leq c_1 \|w|_{W_1}\|_{H^1(W_1)}, \tag{3.28}$$

where c_1 is a uniform constant. Also, since $w = \widetilde{w}$ in $\Omega \setminus W_1$, we have

$$\begin{aligned} \|w - \widetilde{w}\|_{H^1(\Omega_1)} &\leq c_2 \|w|_{W_1}\|_{H^1(W_1)}, & (3.29) \\ \|\widetilde{w}\|_{H^1(\Omega_1)} &\leq c_2 \|w\|_{H^1(\Omega_1)} & (3.30) \end{aligned}$$

where c_2 is a uniform constant. Summarizing, above we have seen that

$$\begin{aligned} P(y,D)\widetilde{w}(t,x) &= F(t,x), \quad \text{in } (-T+\ell, T-\ell)\times B_g(z_0, T+\ell), \\ \widetilde{w}|_{W_0} &= 0. \end{aligned}$$

Moreover F in $(-T+\ell, T-\ell)\times B_g(z_0, T+\ell)$ vanishes outside of W_1 and F is small if $\|w|_{W_1}\|_{H^1(W_1)}$ is small. Then, applying Theorem 3.3 to $\widetilde{w}$ we get

$$\|\widetilde{w}\|_{L^2(\Omega_2\setminus W_0)} \le c_{163}\frac{\|\widetilde{w}\|_{H^1(\Omega_1\setminus W_0)}}{\left(\ln\left(e+\frac{\|\widetilde{w}\|_{H^1(\Omega_1\setminus W_0)}}{\|F\|_{L^2(\Omega_1\setminus W_0)}}\right)\right)^\theta}.$$

As $\|w\|_{L^2(\Omega_2\setminus W_1)} \le \|\widetilde{w}\|_{L^2(\Omega_2\setminus W_0)}$, and by (3.30),

$$\|\widetilde{w}\|_{H^1(\Omega_1\setminus W_0)} \le c_2\|w\|_{H^1(\Omega_1\setminus W_0)},$$

and since the function $t\mapsto \frac{t}{(\ln(e+t))^\theta}$ is increasing for $t\ge 0$, we get

$$\|w\|_{L^2(\Omega_2\setminus W_1)} \le c_{163}\frac{c_2\|w\|_{H^1(\Omega_1\setminus W_0)}}{\left(\ln\left(e+\frac{c_2\|w\|_{H^1(\Omega_1\setminus W_0)}}{c_1\|w|_{W_1}\|_{H^1(W_1))}}\right)\right)^\theta}.$$

This proves the claim with $c_{166} = \max(c_2/c_1, c_2 c_{163})$. □

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When finalizing this article, it came to our attention that another group, Camille Laurent and Matthieu Léautaud has been working independently on issues related to this paper.

4 Appendix

4.1 A: Geometric constants

We write the table of the constants used in the article. This is a special version of Table (5.1) in [6], since now all the coefficients are calculated independently of the center point y_k and of the local information.
In order to get the uniform coefficients we use the same notations as in Section 3.1 of [6]:

a) By Assumption A1, we consider the case of the wave operator (1.1) with principal symbol $p(y,\xi) = -\xi_0^2 + \sum_{jk=1}^n g^{jk}(x)\xi_j\xi_k$, with $0 < a_1\,\delta^{jk} \le g^{jk}(x) \le b_1\,\delta^{jk}$, $a_1,\, b_1 > 0$.
   Call $\xi = (\xi_0, \widetilde{\xi}) \in \mathbb{R}\times\mathbb{R}^n$, where $|\widetilde{\xi}|^2 = \sum_{j=1}^n \xi_j^2$.

b) We consider the function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$, for some $\rho \in (0,1)$, such that $p(y, \psi'(y)) \neq 0$ and $\psi'(y) \neq 0$ in a domain $\Omega_0 \subseteq \Omega$. Let $y_0 \in \Omega_0$ be a general point lying on the level set $S = \{y;\, \psi(y) = 0\}$. Call $p_1 = \min_{y \in \overline{\Omega}_0} p(y, \psi') > 0$, $C_l = \min_{y \in \overline{\Omega}_0} |\psi'(y)| > 0$.

Moreover we use Einstein's convention for the repeated indexes.

We recall the three Steps - procedure to calculate the geometric parameters in [6].

Step 1 . Given a function $\psi \in C^{2,\rho}(\mathbb{R}^{n+1})$ fulfilling the assumptions above in a domain Ω_0, we find positive constants M_2, M_1, M_P such that the following inequality holds true

$$M_2\xi_0^2 + M_1\big(\frac{|p(y, \xi + i\tau\psi'(y))|^2}{\tau^2 + |\xi|^2} + |\langle p'_\xi(y, \xi + i\tau\psi'(y), \psi'(y)\rangle|^2\big)$$
$$+\frac{\{\overline{p(y, \xi + i\tau\psi'(y))}, p(y, \xi + i\tau\psi'(y))\}}{2i\tau} \geq M_P(\tau^2 + |\xi|^2)$$

for every $\xi \in \mathbb{R} \times \mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the hypersurface $S = \{y;\, \psi(y) = 0\}$ is conormally strongly pseudoconvex w.r.t. P in Ω_0.

Step 2 . For $\phi = e^{\lambda\psi}$, with y_0 on the level set $\phi(y) = 1$, we find $\lambda > 0$ such that the following inequality holds true

$$M_2\xi_0^2 + \frac{M_1}{\min\{1, \lambda^2\phi^2(y)\}}\frac{|p(y, \xi + i\tau\phi'(y))|^2}{\tau^2 + |\xi|^2}$$
$$+\frac{1}{\lambda\phi(y)}\frac{\{\overline{p(y, \xi + i\tau\phi'(y))}, p(y, \xi + i\tau\phi'(y))\}}{2i\tau} \geq M_P \min\{1, \lambda^2\phi^2(y)\}(\tau^2 + |\xi|^2)$$

for every $\xi \in \mathbb{R} \times \mathbb{R}^n$, $\xi \neq 0$, $\tau \in \mathbb{R}$. The previous inequality proves that the function ϕ is conormally strongly pseudoconvex w.r.t. P in Ω_0.

Step 3 . We consider a perturbation of ϕ by the shifted 2nd order polynomial centred in the point y_0,

$$f(y) = \sum_{|v| \leq 2} (\partial^v\phi)(y_0)\,(y - y_0)^v/v! - \sigma|y - y_0|^2. \tag{4.1}$$

In a ball $B(y_0, R_1) \subset \Omega_0$ where $f' \neq 0$ we define

$$\phi_0 = \min_{y \in B(y_0, R_1)} \phi(y), \qquad \phi_M = \max_{y \in B(y_0, R_1)} \phi(y).$$

We find σ and $R_2 > 0$ small enough such that in the ball $B(y_0, R_2)$ the following inequalities hold true: $\quad f(y) < \phi(y) \quad$ in $B(y_0,\, R_2)\backslash\{y_0\}$, $\quad$ and

$$M_2\xi_0^2 + 2M_1\frac{|p(y, \xi + i\tau f'(y))|^2}{\tau^2 + |\xi|^2} + \frac{\{\overline{p(y, \xi + i\tau f'(y))}, p(y, \xi + i\tau f'(y))\}}{(\lambda\phi_0)2i\tau}$$
$$\geq \frac{1}{2}(\tau^2 + |\xi|^2)\,.$$

The previous inequality proves that the function f is conormally strongly pseudoconvex w.r.t. P in $B(y_0, R_2)$.

How to obtain the uniform Table (4.3)**, starting from the table in [6]** .
We need to recalculate R_2 invariantly. Hence we take:

$$\begin{aligned} \sigma &= 2c_T R_2^\rho,\ c_T = n|\lambda\psi|_{max,\Omega_0} \\ |\lambda\psi|_{max,\Omega_0} &= \phi_M \max(\lambda|\psi''|_{C^{0,\rho}(\Omega_0)},\ \lambda^2|\psi|_{C^{0,1}(\Omega_0)}|\psi''|_{C^0(\Omega_0)},\ \lambda^3|\psi|_{C^{0,1}(\Omega_0)}|\psi'|^2_{C^0(\Omega_0)}) \end{aligned} \tag{4.2}$$

and after a first estimate in $B_{R_1}(y_0)$ we estimate M_R calculating the norms in Ω_0 instead of in $B_{R_1}(y_0)$:

$$M_R \geq 1 - c_{100}\big[|\lambda\psi|^2_{max,\Omega_0} R_2^{2(1+\rho)} M_1(1+\lambda^2\phi_M^2|\psi'|^2_{C^0(\Omega_0)})$$
$$+|\lambda\psi|_{max,\Omega_0} R_2^\rho \frac{1}{\lambda\phi_0}(1+\lambda^2\phi_M^2|\psi'|^2_{C^0(\Omega_0)} + \lambda^2\phi_M^2(|\psi'|_{C^0(\Omega_0)}|\psi''|_{C^0(\Omega_0)} + \lambda|\psi'|^3_{C^0(\Omega_0)}))\big] .$$

Recall also that

$$\begin{aligned} |f'|_{C^0(B_{R_2}(y_0))} &\leq \lambda\phi_M|\psi'|_{C^0(\Omega_0)} + 5n|\lambda\psi|_{max,\Omega_0} R_2^{1+\rho}, \\ |f''|_{C^0(B_{R_2}(y_0))} &\leq \lambda\phi_M(|\psi''|_{C^0} + \lambda|\psi'|^2_{C^0}) + 4n|\lambda\psi|_{max,\Omega_0} R_2^\rho, \\ |\phi''(y)| &= |\phi\lambda(\psi'' + \lambda\psi'\times\psi')| \geq e^{-1}\lambda^2 C_l^2/2. \end{aligned}$$

In the table we anticipate the definition of ϵ_0 in order to embody in R_2 the condition $R_2 \leq \frac{1}{8}\frac{\epsilon_0}{\sqrt{2M_2}}\left(16+\frac{1}{16}\right)^{1/2}$, that is essential to define the inequality in Th. 2.3. The estimate for the minimum r was done in [6] and we refine it calculating the norms in Ω_0.

Consider now the points $y_k \in \mathcal{E}$ defined in Assumption A4. For each k, via the translation $\psi(y) - \psi(y_k)$, we have $\psi(y_k) = 0$ and we can replace y_0 with y_k in the definitions above. After the previous translation, i.e. $\phi(y) = \exp(\lambda(\psi(y) - \psi(y_k)))$, we still have $e^{-1} = \phi_0 \leq \phi(y) \leq \phi_M = e$ in each ball $B_{R_1}(y_k)$ and we can consider $\phi(y_k) = 1$. All these translated curves share the same parameters as in Table (4.3).

Remark 4.1. In [6] we assumed that $\mathrm{dist}_{\mathbb{R}^{n+1}}\{\partial\Omega_0, \partial\Omega\} > 0$.
Here we have the condition $\mathrm{dist}_{\mathbb{R}^{n+1}}\{\partial\Omega_0, \Omega_a\} > 0$, and then we assume that $\Omega_a \subset \Omega_1$. In Section 3 we apply Table 4.3 to a C^3-function ψ. Instead of calculating the $C^{2,\rho}-$norm, it is more practical to use the C^3-norm, but this requires the following modifications. Set $\rho = 1$ and c_T in place of $n|\lambda\psi|_{max,\Omega_0}$, where

$$\begin{aligned} |\phi''|_{C^0(B_{R_2})} &\leq c_{T1} := \lambda\phi_M(|\psi''|_{C^0(\Omega_0)} + \lambda|\psi'|^2_{C^0(\Omega_0)}), \\ |\phi'''|_{C^0(B_{R_2})} &\leq c_{T2} := \lambda\phi_M(3\lambda|\psi'|_{C^0(\Omega_0)}|\psi''|_{C^0(\Omega_0)} + \lambda^2|\psi'|^3_{C^0(\Omega_0)} + |\psi'''|_{C^0(\Omega_0)}), \\ \sigma = 2c_T R_2, &\quad c_T = c_{T1} + c_{T2},\ \delta = c_T q^2\frac{R_2^3}{8},\ q = \frac{1}{4}(16+\frac{1}{16})^{-1/2}, \\ f - \phi \leq -c_T q^2 R_2^2,\ |\phi' - f'|_{C^0(B_{R_2})} &\leq 5c_T R_2^2,\ |\phi'' - f''|_{C^0(B_{R_2})} \leq 5c_T R_2. \end{aligned}$$

The bound for r remains unchanged since $|\phi''|_{C^0(B_{R_2})} q^2 R_2^3/8 \leq \delta$ (see [6]).

Table for the constants calculated as in [6] (4.3)

Name		Limit Value
C_3	$\geq$	$20(1+n^2\|g^{jk}\|^2_{C^1(\Omega_0)})\|\psi'\|_{C^1(\Omega_0)}(1+\|\psi'\|^2_{C^0(\Omega_0)})$
M_P	$\leq$	1
M_1	$\geq$	$(M_P+C_3)\max\big\{\frac{2}{a_1^2},\ \frac{1}{2(p_1)^2}\big\}$
M_2	$\geq$	$\frac{2}{\min\{1,a_1\}}(M_P+C_3)+\frac{(b_1+a_1)}{2}M_1$
λ	$\geq$	$\max\{M_1, e, \frac{2\|\psi''\|_{C^0(\Omega_0)}}{C_l^2}\}$
ϕ_0	$\geq$	e^{-1}
ϕ_M	$\leq$	e
R_1	$\leq$	$\min\{1, \mathrm{dist}_{\mathbb{R}^{n+1}}\{\partial\Omega_0, \Omega_a\}, \frac{1}{\lambda\|\psi'\|_{C^0(\Omega_0)}}\}$
c_{100}	$\geq$	$10(1+n^4\|g^{jk}\|^2_{C^1(\Omega_0)})$
ϵ_0	$\leq$	$\frac{1}{2n(\lambda\phi_M(\|\psi''\|_{C^0(\Omega_0)}+\lambda\|\psi'\|^2_{C^0(\Omega_0)})+4n\|\lambda\psi\|_{max,\Omega_0})}$
R_2	$\leq$	$\min\big\{R_1,\ \big(\frac{C_l}{2\phi_M(\|\psi''\|_{C^0(\Omega_0)}+\lambda\|\psi'\|^2_{C^0(\Omega_0)}+4n\|\lambda\psi\|_{max,\Omega_0}/\lambda)}\big),\ \big(\frac{\lambda^2\phi_M C_l^2}{4n\|\lambda\psi\|_{max,\Omega_0}}\big)^{\frac{1}{\rho}},$ $\big(\frac{1}{4c_{100}\|\lambda\psi\|^2_{max,\Omega_0}M_1(1+\lambda^2\phi_M^2\|\psi'\|^2_{C^0(\Omega_0)})}\big)^{\frac{1}{2+2\rho}},\ \frac{1}{8}\frac{\epsilon_0}{\sqrt{2M_2}}\big(16+\frac{1}{16}\big)^{1/2},$ $\big(\frac{\lambda\phi_0}{4c_{100}\|\lambda\psi\|_{max,\Omega_0}\big(1+\lambda^2\phi_M^2\|\psi'\|^2_{C^0(\Omega_0)}+\lambda^2\phi_M^2(\|\psi'\|_{C^0(\Omega_0)}\|\psi''\|_{C^0(\Omega_0)}+\lambda\|\psi'\|^3_{C^0(\Omega_0)})\big)}\big)^{\frac{1}{\rho}}\big\}$
σ	$\geq$	$2n\|\lambda\psi\|_{max,\Omega_0}R_2^\rho$
τ_0	$\geq$	$\max\{1,\ 64\big(4M_1+\frac{1}{4\lambda\phi_0}\big)\big((\lambda\phi_M(\|\psi''\|_{C^0}+\lambda\|\psi'\|^2_{C^0})+$ $4n\|\lambda\psi\|_{max,\Omega_0}R_2^\rho)^2(1+n^2\|g^{jk}\|_{C^0(\Omega_0)})^2+n\|h\|^2_{C^0(\Omega_0)}(2+$ $2(\lambda\phi_M\|\psi'\|_{C^0(\Omega_0)}+5n\|\lambda\psi\|_{max,\Omega_0}R_2^{1+\rho})^2+2\|q\|^2_{C^0(\Omega_0)})\big)\}$
R	$\leq$	$\frac{1}{4}\big(16+\frac{1}{16}\big)^{-1/2}R_2$
δ	$\leq$	$n\frac{1}{32}\big(16+\frac{1}{16}\big)^{-1}\|\lambda\psi\|_{max,\Omega_0}R_2^{2+\rho}$
r	$\leq$	$\frac{n\lambda^2C_l^2\frac{1}{4}\big(16+\frac{1}{16}\big)^{-1}R_2^{2+\rho}}{2e\big(\|\phi'\|_{C^0(\Omega_0)}+5n\|\phi''\|_{C^{0,\rho}(\Omega_0)}\big)}$
$c_{1,T}$	$\geq$	$\sqrt{4\big(\frac{4M_1}{\tau_0}+\frac{1}{4(\lambda\phi_0)}\big)}$
$c_{2,T}$	$\geq$	$(\frac{1}{2}+\sqrt{2M_2})(1+\frac{2\|\chi_1'\|_{C^0(\Omega_0)}}{\tau_0 4R})+\frac{c_{1,T}}{\sqrt{\tau_0}}c_{133}$
c_{133}	$\geq$	$2(1+n^2\|g^{jk}\|_{C^0(\Omega_0)})\big(\frac{\|\chi_1''\|_{C^0(\Omega_0)}}{\tau_0(4R)^2}+\frac{\|\chi_1'\|_{C^0(\Omega_0)}}{4R}(1+\lambda\phi_M\|\psi'\|_{C^0(\Omega_0)}+$ $5n\|\lambda\psi\|_{max,\Omega_0}R_2^{1+\rho}+\frac{\|h\|_{L^\infty(\Omega_0)}}{\tau_0}))$

4.1.1 Uniform estimates for the hyperbolic function ψ

Uniform regularity estimates for the distance function d_g. It is a well known fact, see [9], that if a metric is C^m-smooth, then the Riemannian normal coordinates are C^{m-1}-smooth, and the metric tensor in these coordinates is C^{m-2}-smooth. In particular, the distance function $x \mapsto d_g(x,z)$ is C^{m-1}-smooth. In the following we consider how to obtain uniform bounds for the distance function under suitable assumptions.

Let $m \geq 2$ be an integer, and $a, r_0 > 0$, $Q_0 > 0$ be fixed parameters.

Assume that on M are local coordinates (U_k, Ψ_k), where $U_k \subset M$ are open and $\Psi_k : U_k \to \mathbb{R}^n$ such that:
(i) For any $x \in M$ there is k such that the metric balls $B_g(x, r_0) \subset U_k$, Let $W_k = \Psi_k(U_k) \subset \mathbb{R}^n$.
(ii) The metric tensor satisfies in local coordinates

$$\frac{1}{4} I \leq (\Psi_k)_* g \leq 4\, I, \tag{4.4}$$

(iii) We have $\|(\Psi_k)_* g\|_{C^m(\overline{W}_k)} \leq Q_0$,
(iv) The transition functions satisfy: $\|\Psi_k \circ \Psi_j^{-1}\|_{C^{m+1}(\Psi_j(U_j \cap U_k))} \leq Q_0$,
(v) The injectivity radius satisfies: $\mathrm{inj}(M, g) \geq 2r_0 = i_0$, where $0 < i_0 < \frac{\pi}{2\sqrt{\Lambda_M}}$ and Λ_M is an upper bound for the sectional curvature of M.

Under the previous assumptions and using the notation $\langle v \rangle = (1 + |v|^2)^{1/2}$, one can obtain the estimate for the derivatives of d_g when $0 < d_g(x_0, x) < i_0$:

$$|D_x^\alpha d_g(x_0, x)| \leq e^{c_{m,n}^\alpha \langle Q_0 \rangle^4 \langle d_g(x_0,x) \rangle^2}\, d_g(x_0, x)^{(1-|\alpha|)}, \quad |\alpha| \geq 0. \tag{4.5}$$

Here $c_{m,n}^\alpha \geq 1$ are coefficients which depend only on m, n; their value may be explicitly found from combinatorics.
Consequently, we consider Assumption A5. Then for all $z \in \mathbb{R}^n$ and $A_1 = \{x \in \mathbb{R}^n;\ \frac{1}{4}\ell \leq d_g(x, z) \leq \frac{7}{8} i_0\}$ we have (see (4.5) and [9] for details)

$$\|d_g(\,\cdot\,, z)\|_{C^1(A_1)} \leq b_2, \quad \|d_g(\,\cdot\,, z)\|_{C^3(A_1)} \leq b_2(\ell/4)^{-2}, \tag{4.6}$$

where b_2 depends on a_0, b_0, b_3, and i_0 in an explicit way.
Let $z \in M$, $T > 0$ and recall the 'hyperbolic function' introduced in Definition 3.1

$$\psi_{z,T}(t, x) := \psi(t, x; T, z) = (T - d_g(x, z))^2 - t^2.$$

In the following we consider properties of this function in order to construct the related Table (4.3).
We recall that the principal symbol of the wave operator P, at $y = (t, x)$, $\xi = (\xi_0, \xi_1, \ldots, \xi_n) \in T_y^*(\mathbb{R} \times \mathbb{R}^n)$ by $p(y, \xi) = -\xi_0^2 + \sum_{j,k=1}^n g^{jk}(x) \xi_j \xi_k$.

Lemma 4.2. *Let $z \in \mathbb{R}^n$, $A_1 = \{x \in \mathbb{R}^n;\ \frac{1}{4}\ell \leq d_g(x, z) \leq \frac{7}{8} i_0\}$ and $\mathcal{A} = [-T, T] \times A_1$. Let $y = (t, x) \in \mathcal{A}$ be such that $|t| < T - d(x, z)$. Also, assume that $\psi_{z,T}(y) \geq \gamma_I^2$, with $0 < \gamma_I < T$. Then $d\psi_{z,T}(y)$ is well defined and satisfies*

$$\begin{aligned} &p(y, d\psi_{z,T}(y)) \geq 4\gamma_I^2, \\ &\min_{y \in \mathcal{A}} |d\psi_{z,T}(y)| \geq 2\gamma_I b_0^{-1/2}. \end{aligned} \tag{4.7}$$

Moreover, we have

$$\begin{aligned} &\|\psi'_{z,T}\|_{C^0(\mathcal{A})} \leq b_4(T + 1), \\ &\|\psi''_{z,T}\|_{C^0(\mathcal{A})} \leq b_4(T + 1)((\ell/4)^{-1} + 1), \\ &\|\psi'''_{z,T}\|_{C^0(\mathcal{A})} \leq b_4(T + 1)((\ell/4)^{-2} + (\ell/4)^{-1}), \end{aligned} \tag{4.8}$$

where $b_4 = b_4(a_0, b_0, b_3)$ is a uniform constant.

Proof. Consider the co-normal of the level set of $\psi_{z,T}$,

$$\nu = d\psi_{z,T}(y) = (-2t\,,\ -2(T-d_g(x,z))\partial_x d_g(x,z)) \in T^*_y(\mathbb{R}\times\mathbb{R}^n).$$

Using the two following facts :

$$\sum_{j,k=1}^{n} g^{jk}(x)\partial_{x_j} d_g(x,z)\partial_{x_k} d_g(x,z) = \|\nabla d_g(\,\cdot\,,z)\|_g^2 = 1,$$

$$\Big| -\xi_0^2 + \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k\Big| \le \xi_0^2 + \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k =: \|\xi\|^2_{dt^2+g},$$

we obtain

$$\begin{aligned} &p(y, d\psi_{z,T}(y)) = 4(-t^2 + (T-d_g(x,z))^2) \ge 4\gamma_I^2 > 0, \qquad (4.9)\\ &|d\psi_{z,T}(y)|_{dt^2+g} \ge \sqrt{4\gamma_I^2} = 2\gamma_I \text{ implying } |d\psi_{z,T}(y)| \ge 2\gamma_I b_0^{-1/2}. \end{aligned}$$

Let us now analyze derivatives of $\psi_{z,T}$. We have

$$\begin{aligned} \nabla\psi_{z,T}|_{(t,x)} &= (-2t, -2(T-d_g(x,z))\nabla_x d_g(x,z)),\\ \nabla^2\psi_{z,T}|_{(t,x)} &= (-2, -2(T-d_g(x,z))\nabla_x^2 d_g(x,z) + 2\nabla_x d_g(x,z)\otimes\nabla_x d_g(x,z)), \end{aligned}$$

$$\begin{aligned} \nabla^3\psi_{z,T}|_{(t,x)} &= (0, V),\\ V &= -2(T-d_g(x,z))\nabla_x^3 d_g(x,z) + 4\nabla_x d_g(x,z)\otimes\nabla_x^2 d_g(x,z)\\ &\quad +2\nabla_x^2 d_g(x,z)\otimes\nabla_x d_g(x,z). \end{aligned}$$

Thus, by calling $\psi' = d\psi_{z,T}$, $\psi'' = \nabla^2\psi_{z,T}$, $\psi''' = \nabla^3\psi_{z,T}$, we get

$$\begin{aligned} &\|\psi'_{z,T}\|_{C^0(\mathcal{A})} \le 4(T+1)\|d_g(\,\cdot\,,z)\|_{C^1(A_1)},\\ &\|\psi''_{z,T}\|_{C^0(\mathcal{A})} \le 4(T+1)(\|d_g(\,\cdot\,,z)\|_{C^2(A_1)} + \|d_g(\,\cdot\,,z)\|^2_{C^1(A_1)}),\\ &\|\psi'''_{z,T}\|_{C^0(\mathcal{A})} \le 6(1+T)(\|d_g(\,\cdot\,,z)\|_{C^3(A_1)} + \|d_g(\,\cdot\,,z)\|_{C^1(A_1)}\|d_g(\,\cdot\,,z)\|_{C^2(A_1)}),\qquad (4.10) \end{aligned}$$

and then one can use (4.6) at the right hand side, where b_4 is a uniform constant. ∎

Next we estimate the distance between two level sets of $\psi_{z,T}$, or $\psi_{\widehat{z},\widehat{T}}$, outside of a cylinder of radius ℓ (see Definition 3.1) and its consequences.

Lemma 4.3. *Under the Assumption A5, we have,*
a) For $i=1,2$ define $L_i = \{y; y\in\partial S(z,T,\ell,\gamma_i), d_g(x,z) > \ell\}$, with $\gamma_1 = \gamma/\sqrt{2}$ and $\gamma_2 = \gamma$. Hence,

$$dist_{dt^2+g}(L_1, L_2) \ge \frac{\gamma^2}{8(T-\ell)} := c_{180}. \qquad (4.11)$$

Consequently, defining z_0, Λ and Ω_0 as in Theorem 3.3 we get

$$dist_{\mathbb{R}^{n+1}}(\Lambda, \partial\Omega_0) \ge \frac{1}{\sqrt{b_0}}\min\{c_{180}, \frac{3\ell}{4}\}. \qquad (4.12)$$

b) Let $\widehat{y} = (\widehat{x}, \widehat{t})$ be a point defined in Definition 3.5, and let $\psi(y; \widehat{T}, \widehat{z})$ be the associated hyperbolic function. For $i = 3, 4$ define $L_i = \{y; y \in \partial S(\widehat{z}, \widehat{T}, \ell, \gamma_i), d_g(x, \widehat{z}) > \ell\}$. Then, for $\gamma_3 = \gamma/\sqrt{2}$ and $\gamma_4 = \gamma$,

$$dist_{dt^2+g}(L_3, L_4) \geq c_{180}. \tag{4.13}$$

Proof. a) Let $y_0 = (t_0, x_0) \in L_2$ and $y_1 = (t_1, x_1) \in L_1$. Our aim is to get a positive lower bound for $\mathrm{dist}_{dt^2+g}(y_1, y_0)$. The two points lay on two level sets of $\psi(y; T, z)$, and we consider their difference:

$$\psi(y_0; T, z) - \psi(y_1; T, z) = (T - d_g(z, x_0))^2 - t_0^2 - (T - d_g(z, x_1))^2 + t_1^2 = \frac{\gamma^2}{2}$$

$$(d_g(z, x_1) - d_g(z, x_0))(2T - d_g(z, x_0) - d_g(z, x_1)) + t_1^2 - t_0^2 = \frac{\gamma^2}{2}. \tag{4.14}$$

By definition, we know that

$$(2T - d_g(z, x_0) - d_g(z, x_1)) \geq 0, \quad \ell \leq d_g(z, x_0) \leq T - \gamma, \quad \ell \leq d_g(z, x_1) \leq T - \frac{\gamma}{\sqrt{2}} < T.$$

Assume w.r.o.g. that $t_0, t_1 \geq 0$.
Case 1. $t_1^2 - t_0^2 = m\frac{\gamma^2}{2}$, $m \in [0, 1)$,

$$d_g(x_1, x_0) \geq (d_g(z, x_1) - d_g(z, x_0)) = \frac{(1-m)\gamma^2}{2(2T - d_g(z, x_0) - d_g(z, x_1))} \geq \frac{(1-m)\gamma^2}{2(2T - 2\ell)}.$$

Case 2. $t_1^2 - t_0^2 = \frac{\gamma^2}{2}$, that implies $d_g(z, x_0) = d_g(z, x_1)$ and $d_g(x_1, x_0) \geq 0$.
Case 3. $t_1^2 - t_0^2 = q\frac{\gamma^2}{2}$, $q > 1$

$$(d_g(z, x_1) - d_g(z, x_0))(2T - d_g(z, x_0) - d_g(z, x_1)) = -(q-1)\frac{\gamma^2}{2}$$

here one can reverse the signs and prove as in case 1:

$$d_g(x_1, x_0) \geq (d_g(z, x_0) - d_g(z, x_1)) = \frac{(q-1)\gamma^2}{2(2T - d_g(z, x_0) - d_g(z, x_1))} \geq \frac{(q-1)\gamma^2}{2(2T - 2\ell)}.$$

Case 4. $t_1^2 - t_0^2 = -p\frac{\gamma^2}{2} < 0$, for $p > 0$

$$(d_g(z, x_1) - d_g(z, x_0))(2T - d_g(z, x_0) - d_g(z, x_1)) = \frac{(1+p)\gamma^2}{2}.$$

Case 1. is then the worse case. Hence for $t_1 = \sqrt{m\frac{\gamma^2}{2} + t_0^2} \leq \sqrt{\frac{\gamma^2}{2} + (T-\ell)^2}$ and since $\gamma \leq (T - \ell)$, we have

$$\mathrm{dist}_g(y_1, y_0) = \max\{|t_1 - t_0| = \frac{(t_1^2 - t_0^2)}{(t_1 + t_0)}, d_g(x_1, x_2)\} \geq$$

$$\geq \max\{m\frac{\gamma^2}{(\sqrt{\frac{(T-\ell)^2}{2} + (T-\ell)^2} + T - \ell)}, (1-m)\frac{\gamma^2}{2(2T - 2\ell)}\} \geq \frac{\gamma^2}{8(T-\ell)}.$$

b) We then consider y_0, y_1 belonging to two level sets of the function $\psi(y; \widehat{z}, \widehat{T})$ to calculate the left hand site of (4.13). We repeat the same computation as above with the new values. We recall (3.12). By triangular inequality, since $\widehat{T} = T - d_g(z_0, \widehat{z})$,

$$(2\widehat{T} - d_g(\widehat{z}, x_0) - d_g(\widehat{z}, x_1)) \leq 2T - d_g(z_0, x_0) - d_g(z_0, x_1) \leq 2(T - \ell).$$

And again $t_1 = \sqrt{m\frac{\gamma^2}{2} + t_0^2} \leq \sqrt{\frac{(\widehat{T}-\ell)^2}{2} + (\widehat{T} - \ell)^2} \leq 2(\widehat{T} - \ell) \leq 2(T - \ell)$.
Hence the new estimate is bounded from below by c_{180}. $\square$

References

[1] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, *The stability for the Cauchy problem for elliptic equations*, Inverse Problems 25 (12), (2009).

[2] H. Amann, Ordinary differential equations. An introduction to nonlinear analysis. Translated from the German by Gerhard Metzen. de Gruyter Studies in Mathematics, 13. Walter de Gruyter, Berlin, 1990. xiv+458 pp.

[3] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor, *Boundary regularity for the Ricci equation, Geometric Convergence, and Gel'fand's Inverse Boundary Problem*, Inventiones Mathematicae, 158, 261-321, (2004).

[4] G. Birkhoff, G.-C. Rota-Ordinary differential equations-Wiley (1989).

[5] M. Belishev, *An approach to multidimensional inverse problems for the wave equation.* (Russian) Dokl. Akad. Nauk SSSR 297 (1987), no. 3, 524–527.

[6] R. Bosi, Y. Kurylev, M. Lassas, *Stability of the unique continuation for the wave operator via Tataru inequality: the local case*, to apper in Journal d'Analyse Mathematique. Preprint at http://www.rni.helsinki.fi/~mjl/publications_time.html

[7] J. Bourgain, E. C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. 161, no. 2, 389426, (2005).

[8] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970), 61–75.

[9] D. DeTurck, J. Kazdan, Some regularity theorems in Riemannian geometry. Ann. Sci. Ecole Norm. Sup. (4) 14 (1981), no. 3, 249–260.

[10] M. Eller, V. Isakov, G. Nakamura, D. Tataru, *Uniqueness and stability in the Cauchy problem for Maxwell and elasticity systems*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV, 329349, Stud. Math. Appl., 31, North-Holland, Amsterdam, (2002).

[11] L. Escauriaza, C.E. Kenig, G. Ponce, L. Vega, *Unique continuation for Schrödinger evolutions, with applications to profiles of concentration and traveling waves.* Comm. Math. Phys. 305, no. 2, 487512, (2011).

[12] S. Gallot, D. Hulin, J. Lafontaine, Riemannian geometry. Second edition. Universitext. Springer-Verlag, Berlin, 1990. xiv+284 pp.

[13] L . Hoermander, *A uniqueness theorem for second order hyperbolic differential equations.*, Comm. Partial Differential Equations 17 (1992), no. 5-6, 699714.

[14] L. Hoermander, *On the uniqueness of the Cauchy problem under partial analiticity assumptions*, in "Geometrical optics and related topics", Birkaeuser, (1997).

[15] L. Hoermander, The analysis of linear partial differential equations I, Springer, (1990).

[16] L. Hoermander, The analysis of linear partial differential equations II, Springer, (1990).

[17] A. Katchalov, Y. Kurylev, M. Lassas, Inverse boundary spectral problems, Chapman & Hall/CRC, (2001).

[18] V. Isakov, Inverse Problems for Partial Differential Equations, Springer, (2005).

[19] O. Imanuvilov, V. Isakov, M. Yamamoto, *An inverse problem for the dynamical Lame system with two sets of boundary data*, Comm. Pure Appl. Math., 56, 1366-1382, (2003).

[20] O. Imanuvilov, V. Isakov, M. Yamamoto, *New realization of the pseudoconvexity and its application to an inverse problem*, Appl. Anal. 88, no. 5, 637652, (2009).

[21] Y. Kurylev, M. Lassas, T. Yamaguchi, *Uniqueness and Stability in Inverse Spectral Problems for Collapsing Manifolds*, preprint.

[22] B. Ya. Levin, Lectures on entire functions, American Mathematical Society, (1996).

[23] P. Petersen, Riemannian Geometry, Springer, 1st edition, New York (1998).

[24] L. Robbiano, *Fonction de coût et contrôle des solutions des équations hyperboliques*, Asym. Anal. 10, 95-115, (1995).

[25] L. Robbiano, C. Zuily, *Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients,* Invent. Math. 131, no. 3, (1998).

[26] L. Rodino, *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific, (1993).

[27] D. Tataru, *Carleman estimates, unique continuation and applications*, unpublished manuscript, see *http : //math.berkeley.edu/ ∼tataru/ucp.html*.

[28] D. Tataru, *Unique continuation for solutions to PDE's; Between Hörmander's theorems and Holmgren's theorem*, Comm. PDE, 20 (5& 6), 855-884 (1995).

[29] D. Tataru, *Unique continuation operators with partially analytic coefficients*, J. Math. Pures Appl., 78, 505-521 (1999).

[30] D. Tataru, *Carleman estimates, unique continuation and controllability for anizotropic PDEs*. Optimization methods in partial differential equations, 267279, Contemp. Math., 209, Amer. Math. Soc., Providence, RI, (1997).

[31] D. Tataru, *Boundary controllability for conservative PDEs*. Appl. Math. Optim., 31, 257-295 (1995).

[32] S. Vessella, *Quantitative estimates of strong unique continuation for anisotropic wave equations*, http://arxiv.org/pdf/1406.7798.pdf.

[33] W. Walter, R. Thompson, *Ordinary differential equations*. Graduate texts in mathematics Readings in mathematics 182 Springer, 1998.

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