Research Article

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The hyperbolic lattice point problem in conjugacy classes

Abstract: For Γ a cocompact or cofinite Fuchsian group, we study the hyperbolic lattice point problem in conjugacy classes, which is a modification of the classical hyperbolic lattice point problem. We use large sieve inequalities for the Riemann surfaces $\Gamma \backslash \mathbb{H}$ to obtain average results for the error term, which are conjecturally optimal. We give a new proof of the error bound $O(X^{2/3})$, due to Good. For $\mathrm{SL}_2(\mathbb{Z})$ we interpret our results in terms of indefinite quadratic forms.

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1 Introduction

Let $\mathbb{H}$ be the hyperbolic plane, z, w be two fixed points in $\mathbb{H}$ and $\rho(z, w)$ denote the hyperbolic distance. Let also $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a cocompact or cofinite Fuchsian group. The classical hyperbolic lattice point problem asks to estimate the number of points in the orbit Γz that belong in a disk of radius R and center w, i.e., to give an asymptotic formula for

$$\#\{\gamma \in \Gamma : \rho(\gamma z, w) \le R\}.$$

Let $\cosh \rho(z, w) = 2u(z, w) + 1$, where $u(z, w)$ is the standard point-pair invariant function

$$u(z, w) = \frac{|z - w|^2}{4 \mathfrak{I}(z) \mathfrak{I}(w)},$$

and define

$$H(X; z, w) = \#\{\gamma \in \Gamma : 4u(z, \gamma w) + 2 \le X\}.$$

Selberg [22] proved that

$$H(X; z, w) = \sum_{1/2 < s_j \le 1} \pi^{1/2} \frac{\Gamma(s_j - 1/2)}{\Gamma(s_j + 1)} u_j(z) \overline{u_j(w)} X^{s_j} + E(X; z, w) \tag{1.1}$$

with

$$E(X; z, w) = O(X^{2/3}),$$

where $\{u_j\}_{j=1}^{\infty}$ is an orthonormal system of eigenfunctions for the discrete spectrum of the hyperbolic Laplacian and the sum is over the small eigenvalues $\lambda_j = s_j(1 - s_j) < 1/4$ of the surface $\Gamma \backslash \mathbb{H}$. For earlier results and extensions see [12, 19].

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Now, let $\mathcal{H} \subset \Gamma$ be a hyperbolic conjugacy class of Γ. Write $\mathcal{H}$ as $\mathcal{H} = \mathcal{P}^\nu$, where $\mathcal{P}$ is a primitive conjugacy class, i.e., $\mathcal{H} = \{ag^\nu a^{-1}, a \in \Gamma\}$, where g is a primitive hyperbolic element of Γ. For $\gamma \in \Gamma$ define

$$\mu(\gamma) = \inf_{z \in \mathbb{H}} \rho(z, \gamma z).$$

Notice that $\mu(\gamma)$ is constant in conjugacy classes, hence we can define $\mu := \mu(\mathcal{H}) = \mu(g^\nu)$, which is the length of the closed geodesic corresponding to the hyperbolic class $\mathcal{H}$.

Let z be a fixed point in $\mathbb{H}$, and define the quantity

$$N_z(t) = \#\{\gamma \in \mathcal{H} : \rho(z, \gamma z) \leq t\}.$$

For Γ cocompact, Huber [14] was the first one who posed and studied the problem of estimating the asymptotic behavior of $N_z(t)$ as $t \to \infty$. Huber proved that the asymptotic behavior of $N_z(t)$ as $t \to \infty$ is

$$N_z(t) \sim \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \frac{\mu}{\nu} X, \tag{1.2}$$

where

$$X = \frac{\sinh(t/2)}{\sinh(\mu/2)} \tag{1.3}$$

and $\text{vol}(\Gamma \backslash \mathbb{H})$ is the area of $\Gamma \backslash \mathbb{H}$.

There is a nice geometric interpretation of this problem, explained in [14] and [15]. Let ℓ be the invariant closed geodesic of g. Then, $N_z(t)$ counts the number of $\gamma \in \Gamma / \langle g \rangle$ such that $\rho(\gamma z, \ell) \leq t$. This is the number of geodesic segments on $\Gamma \backslash \mathbb{H}$ from z perpendicular to ℓ of length less than or equal to t. After conjugation, one can assume that ℓ lies on $\{yi, y > 0\}$. Huber's interpretation shows that $N_z(t)$ actually counts γ in $\Gamma / \langle g \rangle$ such that $\cos v \geq X^{-1}$, where v is the angle defined by the ray from 0 to γz and the geodesic $\{yi, y > 0\}$.

For Γ cocompact or cofinite, Good [10] proved a general sum formula that covers many cases of decompositions of the group $G = \text{SL}_2(\mathbb{R})$. One of these cases corresponds to Huber's hyperbolic lattice point problem in conjugacy classes. In Good's notation the hyperbolic lattice point problem in conjugacy classes corresponds to the $_\eta G_\zeta$ case, whereas the classical one corresponds to the $_\zeta G_\zeta$ case [10, equation (3.12)]. His method is based on defining certain Poincaré series $P_\xi(z, s, m)$ [10, equation (7.1)] as sums over cosets of a hyperbolic subgroup of Γ of his basic eigenfunctions $V_\xi(z, s, \lambda)$ [10, equation (4.8)]. These Poincaré series generalise the Eisenstein series and the resolvent kernel. He then expands a modification of $P_\xi(z, s, m)$ into automorphic eigenfunctions, and computes the Fourier expansion around ζ. This involves generalizations of Kloosterman sums, leading to a local trace formula [10, Theorem 1]. The end result is Good's general formula [10, Theorem 4]. After matching notation for $m = n = 0$ this formula implies

$$N_z(t) = \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \frac{\mu}{\nu} X + 2\lambda_{\mathcal{H}} \lambda_z \sum_{1/2 < s_j < 1} a_j(\mathcal{H}, z) X^{s_j} + E_z(t),$$

where

$$E_z(t) = O(X^{2/3}),$$

$\lambda_{\mathcal{H}}, \lambda_z$ are specific constants and $a_j(\mathcal{H}, z)$ are functions depending on $s_j, u_j, \mathcal{H}, z$ and special functions.

Later Huber [15] proved for Γ cocompact that

$$\left| N_z(t) - \frac{2}{\text{vol}(\Gamma \backslash \mathbb{H})} \frac{\mu}{\nu} X \right| \leq c_1 X^\tau + c_2 X^{3/4} + c_3 X^{1/2},$$

where τ is an parameter related to the eigenvalue λ_1 and c_1, c_2, c_3 are specific constants.

As the natural parametrization is given by (1.3), we denote

$$N(\mathcal{H}, X; z) = N_z(t),$$

and work with $N(\mathcal{H}, X; z)$ for the rest of this paper. Clearly,

$$N(\mathcal{H}, X; z) = \#\left\{\gamma \in \mathcal{H} : \frac{\sinh(\rho(z, \gamma z)/2)}{\sinh(\mu/2)} \leq X\right\}.$$

We also define

$$M(\mathcal{H}, X; z) = \sum_{1/2 < s_j \leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j}, \tag{1.4}$$

where

$$A(s) = 2^{s-1}(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)})\frac{\Gamma(\frac{s+1}{2})\Gamma(1 - \frac{s}{2})\Gamma(s - \frac{1}{2})}{\pi\Gamma(s+1)}, \tag{1.5}$$

$$\hat{u}_j = \int_\sigma u_j \, ds \tag{1.6}$$

is the period integral of u_j across a segment σ of the invariant geodesic of g^v with length $\int_\sigma ds = \mu/v$ (see Lemma 2.1) and the sum in (1.4) is over the small eigenvalues of the hyperbolic Laplacian of $\Gamma\backslash\mathbb{H}$. We denote by $E(\mathcal{H}, X; z)$ the error term

$$E(\mathcal{H}, X; z) = N(\mathcal{H}, X; z) - M(\mathcal{H}, X; z).$$

In Section 2 we refine the machinery of Huber in [15]. We compute his special functions $\xi_\lambda(v)$ (see [15, equations (10) and (11)]) in terms of the Legendre functions $P_{s-1}^0(i \tan v)$. This allows to show the oscillatory behaviour of the Huber transform $d(f^\pm, t)$, see Proposition 2.4. In Sections 3 and 4 we give a new proof of the following theorem.

Theorem 1.1 ([10, Theorem 4]). *Let Γ be a cocompact or cofinite Fuchsian group and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then,*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

We combine the results and techniques of Section 2 with the large sieve inequalities obtained by Chamizo in [4] to prove average results for the error term $E(\mathcal{H}, X; z)$, similar to those in [3] for the error term of the classical hyperbolic lattice point problem. In Section 5 we prove the main results of this paper.

Theorem 1.2. *Let Γ be a cocompact or cofinite Fuchsian group and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then,*

$$\frac{1}{X}\int_X^{2X} |E(\mathcal{H}, x; z)|^2 \, dx \ll X \log^2 X,$$

where the constant implied in '$\ll$' depends on Γ, $\mathcal{H}$ and z.

Theorem 1.3. *Let Γ be a cocompact Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then, for $n = 1, 2$,*

$$\int_{\Gamma\backslash\mathbb{H}} |E(\mathcal{H}, X; z)|^{2n} \, d\mu(z) \ll X^n \log^{2n} X,$$

where the constant implied in '$\ll$' depends on Γ and $\mathcal{H}$.

Finally, in Section 6 we use Huber's geometric interpretation to study some arithmetic consequences of our results. More specifically, for $\mathcal{H}$ a hyperbolic class of $SL_2(\mathbb{Z})$, we interpret the quantity $N(\mathcal{H}, X; z)$ in terms of the number of solutions of indefinite quadratic forms in four variables with restrictions.

Remark 1.4. Our use of piecewise linear functions $f^\pm$ to define the smooth automorphic functions $A(f^\pm)(z)$ is much simpler than the construction and spectral expansion of Poincaré series in Good [10]. Moreover, the oscillatory behaviour that is crucial in the application of the large sieve seems difficult to identify in the local trace formula in [10]. Even matching Good's expansion [10, Theorem 4] with $M(\mathcal{H}, X; z)$ seems to be a complicated task needing extensive calculations. Only the leading term of $M(\mathcal{H}, X; z)$ is easy to match.

Remark 1.5. Eskin and McMullen used ergodic methods to study the asymptotics of various counting problems on Lie groups, one of which is the conjugacy class problem [7, §III.2]. Duke, Rudnick and Sarnak [5] gave another proof of the main term [5, Example 1.5]. For the classical hyperbolic lattice point problem one deals with the locally symmetric space $\Gamma\backslash SL_2(\mathbb{R})/SO(2)$. Our case involves the space $\Gamma\backslash SL_2(\mathbb{R})/A$ which is not even Hausdorff, where A is the group of diagonal matrices.

Remark 1.6. Hill and Parnovski [13] studied the asymptotic behavior of the variance of the hyperbolic lattice point counting function for the classical hyperbolic lattice point problem. When Γ has no small eigenvalues, in Theorem 1.3 we provide an upper bound for the variance of the hyperbolic lattice point function in our situation.

Remark 1.7. Recently Parkkonen and Paulin [18] studied the hyperbolic lattice point problem in conjugacy classes for higher dimensional negatively curved manifolds. In special cases, i.e., for compact manifolds or arithmetic group of isometries, they obtain bounds for the error term. It would be interesting to prove error bounds analogous to Theorem 1.1 and average results analogous to Theorems 1.2, 1.3.

Remark 1.8. An interesting application of the hyperbolic lattice point problem in conjugacy classes and its geometric interpretation concerns degenerating Riemann surfaces and the appearance of Eisenstein series, see [9].

2 Certain automorphic functions and their spectral expansion

2.1 Plan of proof and comparison with the classical problem

Let $K(z, w)$ be the automorphic kernel defined as

$$K(z, w) = \sum_{\gamma \in \Gamma} k(u(\gamma z, w))$$

for a test function $k(u)$. If $k(u)$ is the characteristic function of the interval $[0, (X - 2)/4]$, then we have $K(z, w) = H(X; z, w)$, and the asymptotics of $H(X; z, w)$ can be studied using the pre-trace formula for the kernel $K(z, w)$. In practice one needs approximations $k_\pm(u)$ of $k(u)$ and estimates of the corresponding Selberg–Harish-Chandra transforms $h_\pm(t)$ of $k_\pm(u)$.

When we restrict the summation to the conjugacy class $\mathcal{H} \subset \Gamma$, we do not get an automorphic kernel $g(u)$ in the place of $k(u)$. Therefore, Selberg theory does not apply in this case. Huber, however, defined an automorphic function $A(f)$ that plays the role of $K(z, w)$ for a suitable test function f when $z = w$, see (2.1). The spectral expansion of $A(f)$ provides the asymptotic behavior of $N(\mathcal{H}, X; z)$.

Assume that Γ is cocompact. Let $C_0^*[1, \infty)$ be the space of real functions of compact support that are bounded in $[1, \infty)$ and have at most finitely many discontinuities. For an f in $C_0^*[1, \infty)$, define the Γ-automorphic function

$$A(f)(z) = \sum_{\gamma \in \mathcal{H}} f\left(\frac{\cosh \rho(z, \gamma z) - 1}{\cosh \mu(\gamma) - 1} \right). \tag{2.1}$$

Since Γ is cocompact and f has compact support, the sum in (2.1) is finite.

Let Δ be the hyperbolic Laplace operator on $\Gamma \backslash \mathbb{H}$ and $\{u_j\}_{j=0}^{\infty}$ be an orthonormal system of (real-valued) automorphic eigenfunctions of $-\Delta$, with corresponding eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$. Then, $A(f)$ has an L^2-expansion:

$$A(f) = \sum_j c(f, t_j) u_j(z),$$

where $\lambda_j = 1/4 + t_j^2$ and

$$c(f, t_j) = \int_{\Gamma \backslash \mathbb{H}} A(f)(z) u_j(z) \, d\mu(z)$$

is the j-th Fourier coefficient of $A(f)$. We have the following lemma.

Lemma 2.1 ([15]). *We have*

$$c(f, t_j) = 2\hat{u}_j d(f, t_j),$$

where $\hat{u}_j$ is the integral

$$\hat{u}_j = \int_\sigma u_j \, ds \tag{2.2}$$

across a segment σ of the invariant geodesic of g with length $\int_\sigma ds = \mu/\nu$,

$$d(f, t) = \int_0^{\pi/2} f\left(\frac{1}{\cos^2 v}\right) \frac{\xi_\lambda(v)}{\cos^2 v}\, dv \tag{2.3}$$

with $\lambda = 1/4 + t^2$, and ξ_λ is the solution of the differential equation

$$\xi_\lambda''(v) + \frac{\lambda}{\cos^2 v}\xi_\lambda(v) = 0, \quad v \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{2.4}$$

with $\xi_\lambda(0) = 1$ and $\xi_\lambda'(0) = 0$.

The coefficient $d(f, t)$, which we call the Huber transform of f, now plays the role of the Selberg–Harish-Chandra transform. For our choice of test functions f we can use properties of special functions to estimate the Fourier coefficients $d(f, t_j)$, see Proposition 2.4. Table 1 summarizes the analogies between the two problems.

Classical problem	For conjugacy classes
z, w	$z, \mathcal{H}$
$k(u)$	$f\left(\frac{\cosh \rho - 1}{\cosh \mu - 1}\right)$
$K(z, w)$	$A(f)(z)$
$h(t)$	$d(f, t)$
$u_j(w)$	$\hat{u}_j$

Table 1. Corresponding quantities for the two problems.

In order to bound $E(\mathcal{H}, X; z)$, we will need the following bound (weak type of Weyl's law) for the period integrals of $\hat{u}_j$'s defined in Lemma 2.1

Lemma 2.2 ([15, equation (63)]). *For the sequence of the period integrals $\{\hat{u}_j\}_{j=0}^\infty$, the following estimate holds:*

$$\sum_{t_j \leq T} |\hat{u}_j|^2 \ll T.$$

The exact asymptotic behavior was first proved by Good [10, Theorem 2], see also [17], and in greater generality by Tsuzuki [23, Theorem 1].

2.2 Special functions and test functions

For the proof of Theorem 1.1 it is crucial to identify the special function $\xi_\lambda(v)$ and its relevant properties. Using [8, equation (87)], [6, equations (10) and (12)] and [11, equation (9.132.2)], we see that the general solution of equation (2.4) can be written in the form

$$\xi_\lambda(v) = a(s)F\left(s, 1-s, 1; \frac{1 - i\tan(v)}{2}\right) + b(s)F\left(s, 1-s, 1; \frac{1 + i\tan(v)}{2}\right),$$

where $F(a, b, c; z)$ is the Gauss hypergeometric function. The initial conditions of Lemma 2.1 imply that

$$a(s) = b(s) = (2 \cdot F(s, 1-s, 1; 1/2))^{-1}.$$

Using [11, equation (8.702)] and [6, equation (50)], we can write $\xi_\lambda(v)$ as

$$\xi_\lambda(v) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)(P_{s-1}^0(i\tan v) + P_{s-1}^0(-i\tan v)),$$

where $P_\nu^\mu(z)$ is the associated Legendre function of the first kind. Using the change of variable $x = \tan(v)$, we get

$$d(f, t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1 - \frac{s}{2}\right)\int_0^\infty f(x^2 + 1)(P_{s-1}^0(ix) + P_{s-1}^0(-ix))\, dx.$$

Huber's interpretation shows that we are counting $\gamma \in \mathcal{H}$ such that $(\cos v)^{-1} \le X$, i.e., $x^2 + 1 \le X^2$. Hence, choosing

$$f(x^2 + 1) = \begin{cases} 1 & \text{for } x \le \sqrt{X^2 - 1}, \\ 0 & \text{for } x > \sqrt{X^2 - 1}, \end{cases}$$

we get

$$A(f)(z) = N(\mathcal{H}, X; z).$$

Let us set

$$U = \sqrt{X^2 - 1}. \tag{2.5}$$

Motivated by [1, p. 269] we define the following test functions for $x > 0$ and $0 < U/2 < T < U < V < 2U$:

$$f^+(x^2 + 1) = \begin{cases} 1 & \text{for } x \le U, \\ \dfrac{V - x}{V - U} & \text{for } U \le x \le V, \\ 0 & \text{for } V \le x, \end{cases} \tag{2.6}$$

$$f^-(x^2 + 1) = \begin{cases} 1 & \text{for } x \le T, \\ \dfrac{U - x}{U - T} & \text{for } T \le x \le U, \\ 0 & \text{for } U \le x. \end{cases} \tag{2.7}$$

Denote $Y = V - U$. Notice that

$$f(x^2 + 1) = \begin{cases} 1 & \text{for } x \le U, \\ 0 & \text{for } U < x, \end{cases}$$

hence $f^- \le f \le f^+$. This gives

$$A(f^-)(z) \le N(\mathcal{H}, X; z) \le A(f^+)(z).$$

Since $U = X + O(X^{-1})$ as $U, X \to \infty$, we can translate estimates involving X to ones with U and vice versa. We compute $d(f^+, t)$ and $d(f^-, t)$. The analysis for $d(f^-, t)$ is similar to the one for $d(f^+, t)$ with U and T instead of V and U. Therefore, we consider only $d(f^+, t)$.

For an $A > 0$, we define $I(A)$ and $J(A)$ by

$$I(A) = \int_0^A (P_{s-1}^0(ix) + P_{s-1}^0(-ix))(A - x)\,dx,$$

$$J(A) = (A^2 + 1)(P_{s-1}^{-2}(iA) + P_{s-1}^{-2}(-iA)).$$

Then, it is easy to see that

$$d(f^+, t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) \cdot \frac{I(V) - I(U)}{V - U}.$$

Lemma 2.3. *The functions $I(A)$ and $J(A)$ satisfy the relation*

$$I(A) = J(A) - 2P_{s-1}^{-2}(0).$$

Proof. Using integration by parts, [11, equation (8.752.3)] and the fact that the function $(z^2 - 1)^{1/2} P_{s-1}^{-1}(z)$ is single-valued in the disk with center $(1, 0)$ and radius 2, we get

$$I(A) = -i \int_0^A (-x^2 - 1)^{1/2}(P_{s-1}^{-1}(ix) - P_{s-1}^{-1}(-ix))\,dx.$$

Using again twice [11, equation (8.752.3)] for $m = 1, 2$, we get

$$I(A) = (x^2 + 1)(P_{s-1}^{-2}(ix) + P_{s-1}^{-2}(-ix))\Big|_0^A.$$

The result is immediate. □

2.3 Estimates for the Huber transform

Lemma 2.3 implies that

$$d(f^+, t) = (2\sqrt{\pi})^{-1}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) \cdot \frac{J(V) - J(U)}{V - U}.$$ (2.8)

Equation (8.776.1) in [11] implies

$$d(f^+, t) = B(s) \cdot \frac{(V^2 + 1)V^{s-1} - (U^2 + 1)U^{s-1}}{V - U} \cdot (1 + O(U^{-2}))$$

$$+ D(s) \cdot \frac{(V^2 + 1)V^{-s} - (U^2 + 1)U^{-s}}{V - U} \cdot (1 + O(U^{-2})),$$ (2.9)

where

$$B(s) = 2^{s-2}(e^{i\frac{\pi}{2}(s-1)} + e^{-i\frac{\pi}{2}(s-1)})\frac{\Gamma(\frac{s+1}{2})\Gamma(1 - \frac{s}{2})\Gamma(s - \frac{1}{2})}{\pi\Gamma(s + 2)},$$ (2.10)

$$D(s) = (e^{i\frac{\pi}{2}(-s)} + e^{-i\frac{\pi}{2}(-s)})\frac{\Gamma(\frac{s+1}{2})\Gamma(1 - \frac{s}{2})\Gamma(\frac{1}{2} - s)}{\pi\Gamma(3 - s)2^{s+1}}.$$ (2.11)

Proposition 2.4. (a) *For any $s = 1/2 + it$ we have*

$$d(f^+, t) = B(1/2 + it)(3/2 + it)X^{1/2+it} + D(1/2 + it)(3/2 - it)X^{1/2-it}$$

$$+ O(B(1/2 + it)|t|^2 X^{-1/2+it}Y + D(1/2 + it)|t|^2 X^{-1/2-it}Y).$$

(b) *Let $t \in \mathbb{R}$ (i.e., $\mathbb{R}(s) = 1/2$) and $t \neq 0$. Then, $d(f^+, t)$ can be written in the form*

$$d(f^+, t) = a(t, Y/X)X^{1/2+it} + b(t, Y/X)X^{1/2-it},$$

where the coefficients $a(t, Y/X)$ and $b(t, Y/X)$ satisfy the bound

$$a(t, Y/X), b(t, Y/X) = O(t^{-2}\min\{t, XY^{-1}\}).$$

Hence,

$$d(f^+, t) = O(t^{-2}\min\{t, XY^{-1}\}X^{1/2}).$$

(c) *Let $t \notin \mathbb{R}$, i.e., $s \in (1/2, 1]$. Then,*

$$d(f^+, t) = B(s)(s + 1)X^s + D(s)(2 - s)X^{1-s} + O(\Gamma(s - 1/2)Y + \Gamma(1/2 - s)X^{1/2}).$$

(d) *For $t = 0$, we get*

$$d(f^+, 0) = O(X^{1/2}\log X).$$

Proof. (a) First, apply the mean value theorem to the function $f(x) = x^{s+1} + x^{s-1}$ to get

$$\frac{(V^2 + 1)V^{s-1} - (U^2 + 1)U^{s-1}}{V - U} = (s + 1)X^s + O(s(s + 1)X^{s-1}Y + (s - 2)X^{-1}).$$

Applying it again to the function $g(x) = x^{2-s} + x^{-s}$, we have

$$\frac{(V^2 + 1)V^{-s} - (U^2 + 1)U^{-s}}{V - U} = (2 - s)X^{1-s} + O((2 - s)(1 - s)X^{-s}Y + (-s)X^{-3/2}).$$

Plugging $s = 1/2 + it$ in (2.9) and using that $O(U^{-2}) = O(X^{-2})$ and the above estimates, we get the result.

(b) First, consider the function $f(x)$ as above. We know from part (a) that the terms containing $X^{1/2+it}$ come from the terms containing $f(x)$. The mean value theorem implies

$$\frac{(V^2 + 1)V^{s-1} - (U^2 + 1)U^{s-1}}{V - U} \ll |t| \cdot X^{1/2},$$

whereas, trivial estimates imply

$$\frac{(V^2 + 1)V^{s-1} - (U^2 + 1)U^{s-1}}{V - U} \ll X^{3/2}Y^{-1}.$$

Hence, if we set

$$a(t, Y/X) = (1 + O(U^{-2}))B(s) \cdot \frac{(V^2 + 1)V^{s-1} - (U^2 + 1)U^{s-1}}{V - U} X^{-(1/2+it)},$$

and use Stirling's formula for the Γ function, we get the bound

$$a(t, Y/X) = O(t^{-2} \min\{t, XY^{-1}\}).$$

Doing the same for $g(x)$ as above and the coefficient $b(t, Y/X)$ defined as

$$b(t, Y/X) = (1 + O(U^{-2}))D(s) \cdot \frac{(V^2 + 1)V^{-s} - (U^2 + 1)U^{-s}}{V - U} X^{-(1/2-it)},$$

we get (b).

(c) It follows from (a). We estimate three of the Γ-factors in $B(s)$, $D(s)$ (equations (2.10)–(2.11)) and keep the factors $\Gamma(s - 1/2)$ and $\Gamma(1/2 - s)$ accordingly.

(d) Putting $t = 0$ in (2.8) we get

$$d(f^+, 0) = (2\sqrt{\pi})^{-1}\Gamma^2(3/4)\frac{H(V) - H(U)}{V - U},$$

where $H(z) = (z^2 + 1)(P_{-1/2}^{-2}(iz) + P_{-1/2}^{-2}(-iz))$. Thus, applying once again the mean value theorem, there exists a $\xi \in [U, V]$ such that

$$d(f^+, 0) = (2\sqrt{\pi})^{-1}\Gamma^2(3/4)H'(\xi).$$

For $H'(z)$ we have

$$H'(z) = 2z\big(P_{-1/2}^{-2}(iz) + P_{-1/2}^{-2}(-iz)\big) + (z^2 + 1)\frac{d}{dz}\big(P_{-1/2}^{-2}(iz) + P_{-1/2}^{-2}(-iz)\big).$$

Equation (8.731.1) in [11] implies

$$H'(z) = \frac{3z}{2}\big(P_{-1/2}^{-2}(iz) - P_{-1/2}^{-2}(-iz)\big) - \frac{5i}{2}\big(P_{1/2}^{-2}(iz) - P_{1/2}^{-2}(-iz)\big). \tag{2.12}$$

Consider the first bracket. Using [11, equation (8.713.2)], we get

$$P_{-1/2}^{-2}(i\xi) - P_{-1/2}^{-2}(-i\xi) \ll (\xi^2 + 1)\int_0^\infty (\cosh^2 t + \xi^2)^{-5/4}\, dt \ll \xi^{-1/2}\int_0^\infty \left(\left(\frac{\cosh t}{\xi}\right)^2 + 1\right)^{-5/4}\, dt.$$

Setting $x = \cosh t/\xi$, we get

$$\int_0^\infty \left(\left(\frac{\cosh t}{\xi}\right)^2 + 1\right)^{-5/4}\, dt = \int_{1/\xi}^\infty (x^2 + 1)^{-5/4}\frac{\xi}{(\xi^2 x^2 - 1)^{1/2}}\, dx$$

$$= \int_{1/\xi}^1 (x^2 + 1)^{-5/4}\frac{\xi}{(\xi^2 x^2 - 1)^{1/2}}\, dx + \int_1^\infty (x^2 + 1)^{-5/4}\frac{\xi}{(\xi^2 x^2 - 1)^{1/2}}\, dx.$$

Since $U, V \to \infty$, we can assume that $\xi \geq 2$. We see that

$$\int_1^\infty (x^2 + 1)^{-5/4}\frac{\xi}{(\xi^2 x^2 - 1)^{1/2}}\, dx \ll \int_1^\infty (x^2 + 1)^{-5/4}\, dx = O(1)$$

and, after setting $u = x\xi$,

$$\int_{1/\xi}^1 (x^2 + 1)^{-5/4}\frac{\xi}{(\xi^2 x^2 - 1)^{1/2}}\, dx = \int_1^\xi \left(\frac{\xi^2}{u^2 + \xi^2}\right)^{5/4}\frac{\xi}{(u^2 - 1)^{1/2}}\frac{du}{\xi} \leq \int_1^\xi \frac{1}{\sqrt{u^2 - 1}}\, du \ll \log \xi.$$

Combining these estimates we get

$$P_{-1/2}^{-2}(i\xi) + P_{-1/2}^{-2}(-i\xi) \ll \xi^{-1/2} \log \xi.$$

For the second bracket, using once again [11, equation (8.713.2)], we get

$$P_{1/2}^{-2}(i\xi) - P_{1/2}^{-2}(-i\xi) \ll (\xi^2 + 1) \int_0^\infty \cosh t (\cosh^2 t + \xi^2)^{-5/4} \, dt \ll \xi^{1/2} \int_0^\infty \frac{\cosh t}{\xi} \left(\left(\frac{\cosh t}{\xi} \right)^2 + 1 \right)^{-5/4} \, dt.$$

As above, set $x = \cosh t / \xi$ and split the integral into two integrals:

$$\int_{1/\xi}^1 (x^2 + 1)^{-5/4} \frac{\xi x}{(\xi^2 x^2 - 1)^{1/2}} \, dx + \int_1^\infty (x^2 + 1)^{-5/4} \frac{\xi x}{(\xi^2 x^2 - 1)^{1/2}} \, dx.$$

As above, assuming $\xi \geq 2$, the second integral is easily seen to converge, whereas the first one, setting $u = x\xi$ is again bounded by $\int_1^\xi (u^2 - 1)^{-1/2} \, du$. Finally, combining all the above estimates, we get

$$d(f^+, 0) \ll H'(\xi) \ll \xi^{1/2} \log \xi \ll V^{1/2} \log V,$$

which implies the desired bound, since $X \sim U$ and $V < 2U$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 The cocompact case

We can now prove Theorem 1.1 when Γ is cocompact.

Theorem 3.1. *Let Γ be a cocompact Fuchsian group and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then, the error term $E(\mathcal{H}, X; z)$ satisfies the bound*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

Proof. We begin with the spectral expansion of $A(f^+)$:

$$A(f^+)(z) = \sum_j c(f^+, t_j) u_j(z) = \sum_j 2d(f^+, t_j) \hat{u}_j u_j(z).$$

Using Proposition 2.4, we write it in the form

$$A(f^+)(z) = \sum_{1/2 < s_j \leq 1} 2B(s_j)(s_j + 1) \hat{u}_j u_j(z) X^{s_j} + \sum_{1/2 < s_j \leq 1} 2D(s_j)(2 - s_j) X^{1 - s_j}$$
$$+ O\left(\sum_{1/2 < s_j \leq 1} \Gamma(s_j - 1/2) \hat{u}_j u_j(z) Y + \sum_{1/2 < s_j \leq 1} \Gamma(1/2 - s_j) \hat{u}_j u_j(z) X^{1/2} \right)$$
$$+ \sum_{0 \neq t_j \in \mathbb{R}} 2d(f^+, t_j) \hat{u}_j u_j(z) + O(X^{1/2} \log X).$$

Since the spectrum is discrete, for s_j corresponding to a small eigenvalue, $s_j - 1/2$ is bounded away from zero. As the number of small eigenvalues is finite, we get

$$\sum_{1/2 < s_j \leq 1} \Gamma(s_j - 1/2) \hat{u}_j u_j(z) Y + \sum_{1/2 < s_j \leq 1} \Gamma(1/2 - s_j) \hat{u}_j u_j(z) X^{1/2} = O(Y + X^{1/2}).$$

By the same argument,

$$\sum_{1/2 < s_j \leq 1} 2D(s_j)(2 - s_j) X^{1 - s_j} = O(X^{1/2}).$$

Let $A(s)$ be the function defined in equation (1.5). Then,

$$A(s) = 2B(s)(s + 1),$$

and after setting

$$G(f^+, z) = \sum_{0 \neq t_j \in \mathbb{R}} 2d(f^+, t_j)\hat{u}_j u_j(z),$$

we can rewrite the spectral expansion of $A(f^+)(z)$ as

$$A(f^+)(z) = \sum_{1/2 < s_j \leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j} + G(f^+, z) + O(Y + X^{1/2} \log X). \tag{3.1}$$

Using again Proposition 2.4 and the discreteness of the spectrum, we get

$$G(f^+, z) = \sum_{|t_j| \geq 1} 2d(f^+, t_j)\hat{u}_j u_j(z) + \sum_{|t_j| < 1} 2d(f^+, t_j)\hat{u}_j u_j(z)$$

$$= \sum_{|t_j| \geq 1} 2d(f^+, t_j)\hat{u}_j u_j(z) + O(X^{1/2}).$$

Since $d(f, t)$ is an even function of t, after using dyadic decomposition, we get the bound

$$\sum_{|t_j| \geq 1} d(f^+, t_j)\hat{u}_j u_j(z) \ll \sum_{t_j \geq 1} d(f^+, t_j)\hat{u}_j u_j(z)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{2^n \leq t_j < 2^{n+1}} d(f^+, t_j)\hat{u}_j u_j(z) \right)$$

$$\ll \sum_{n=0}^{\infty} \sup_{2^n \leq t_j < 2^{n+1}} d(f^+, t_j) \left(\sum_{2^n \leq t_j < 2^{n+1}} \hat{u}_j u_j(z) \right).$$

Using [16, Proposition 7.2], Proposition 2.4 and Lemma 2.2, we get

$$G(f^+, z) \ll \sum_{n=0}^{\infty} 2^{-2n} \min\{2^n, XY^{-1}\} X^{1/2} \left(\sum_{t_j < 2^{n+1}} |\hat{u}_j|^2 \right)^{1/2} \left(\sum_{t_j < 2^{n+1}} |u_j(z)|^2 \right)^{1/2} + X^{1/2}$$

$$\ll X^{1/2} \left(\sum_{n=0}^{\infty} 2^{-n/2} \min\{2^n, XY^{-1}\} \right) + X^{1/2}.$$

We split the sum according to $n < \log_2(X/Y)$ and $n > \log_2(X/Y)$. We get

$$G(f^+, z) \ll X^{1/2} \left(\sum_{n < \log_2(X/Y)} 2^{-n/2} \min\{2^n, XY^{-1}\} \right) + X^{1/2} \left(\sum_{n \geq \log_2(X/Y)} 2^{-n/2} \min\{2^n, XY^{-1}\} \right) + X^{1/2},$$

which is bounded by

$$X^{1/2} \sum_{n < \log_2(X/Y)} 2^{n/2} + X^{3/2} Y^{-1} \sum_{n \geq \log_2(X/Y)} 2^{-n/2} + X^{1/2} \ll XY^{-1/2} + X^{1/2}.$$

By (3.1), we finally get

$$A(f^+)(z) = \sum_{1/2 < s_j \leq 1} A(s_j)\hat{u}_j u_j(z) X^{s_j} + O(XY^{-1/2} + Y + X^{1/2} \log X). \tag{3.2}$$

We work similarly for $A(f^-)$ and we use

$$A(f^-)(z) \leq N(\mathcal{H}, X; z) \leq A(f^+)(z)$$

to obtain

$$E(\mathcal{H}, X; z) = O(XY^{-1/2} + Y + X^{1/2} \log X).$$

The optimal error arises for $Y = XY^{-1/2}$, i.e., $Y = X^{2/3}$, which yields

$$E(\mathcal{H}, X; z) = O(X^{2/3}). \qquad \square$$

Remark 3.2. For $\lambda_0 = 0$, i.e., $s_0 = 1$, the contribution to $M(\mathcal{H}, X; z)$ is $2\hat{u}_0 u_0(z) X$ with

$$u_0(z) = \frac{1}{\sqrt{\text{vol}(\Gamma\backslash\mathbb{H})}}.$$

It is immediate to see that

$$\hat{u}_0 = \frac{1}{\sqrt{\text{vol}(\Gamma\backslash\mathbb{H})}} \frac{\mu}{\nu},$$

hence we get Huber's main term (1.2).

4 The cofinite case

Now, let Γ be a cofinite Fuchsian group, and define $A(f)$ as in equation (2.1). The first obstacle we face is to examine whether $A(f)$ is in $L^2(\Gamma\backslash\mathbb{H})$. To see this, suppose that f is compactly supported in $[1, K]$ with $K > 0$ fixed, and consider the counting function

$$\tilde{N}(z, \delta) = \#\{\gamma \in \mathcal{H} : u(\gamma z, z) \leq \delta\}.$$

An element γ contributes to the summation in $A(f)$ exactly when $\gamma \in \tilde{N}(z, \delta)$ with $\delta = K(\cosh(\mu) - 1)$. To prove that $A(f)$ is in $L^2(\Gamma\backslash\mathbb{H})$, it suffices to prove that $\tilde{N}(z, \delta)$ in uniformly bounded (i.e., independently of z).

Lemma 4.1. *The counting function $\tilde{N}(z, \delta)$ is uniformly bounded, hence $A(f) \in L^2(\Gamma\backslash\mathbb{H})$.*

Proof. For simplicity, assume that γ has only one cusp at $\mathfrak{a}$. Conjugating, we can assume that $\mathfrak{a} = \infty$. Then, for $Y > 0$, consider the set

$$A(Y) = \{\gamma \in \Gamma_\infty\backslash\Gamma : \Im(\gamma z) > Y\}.$$

Lemma 2.10 in [16] shows that

$$\#A(Y) < 1 + \frac{10}{c_\infty Y},$$

where c_∞ is a constant depending only on the cusp ∞. This means that there exists a Y_0 such that for every $Y > Y_0$,

$$\#A(Y) \leq 1,$$

i.e., for Y large enough $A(Y)$ contains at most one class $\Gamma_\infty\gamma$. Since the fundamental domain contains points of deformation ≤ 1, we have $\Im(z) \geq \Im(\gamma z) > Y$. Since the trivial class leaves the $\Im(\gamma z) = \Im(z)$, we have $\gamma \in \Gamma_\infty$. Hence, if $\gamma \in \mathcal{H}$, then, for all z, $\Im(\gamma z) < Y_0$, where Y_0 depends only on the cusp ∞.

Now, let γ be an element of $\mathcal{H}$ such that $u(\gamma z, z) \leq \delta$. Using the formula

$$u(z, w) = \frac{|z - w|^2}{4\Im(z)\Im(w)},$$

we obtain

$$\Im(z) - \Im(\gamma z) \leq |z - \gamma z| \leq \sqrt{4\delta\Im(z)\Im(\gamma z)} \leq \sqrt{4\delta Y_0\Im(z)},$$

hence

$$\Im(z) \leq \sqrt{4\delta Y_0\Im(z)} + Y_0.$$

This inequality implies an upper bound $\Im(z) \leq M$, where M depends only on Y_0 and δ. We also get

$$\Re(\gamma z) - \Re(z) \leq |z - \gamma z| \leq \sqrt{4\delta\Im(z)\Im(\gamma z)} \leq \sqrt{4\delta Y_0 M},$$

and since, for Γ cofinite, we have a uniform bound $|\Re(z)| < M$, we also get a uniform bound for $\Re(\gamma z)$. This means, for all $\gamma \in \mathcal{H}$ satisfying $u(\gamma z, z) \leq \delta$, γz lies in a compact set, which does not depend of z but only on δ. This proves $\tilde{N}(z, \delta)$ in uniformly bounded, and thus $A(f) \in L^2(\Gamma\backslash\mathbb{H})$. $\square$

Lemma 4.1 allows us to write a spectral expansion for $A(f)$. For Γ cofinite but not cocompact the continuous spectrum of $-\Delta$ covers the segment $[1/4, \infty)$ uniformly with multiplicity the number of cusps; the eigenfunction that corresponds to the eigenvalue $\lambda = 1/4 + t^2 \geq 1/4$ is the Eisenstein series $E_{\mathfrak{a}}(z, 1/2 + it)$. The spectral expansion of $A(f)$ becomes

$$A(f) = \sum_j c(f, t_j) u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} c_{\mathfrak{a}}(f, t) E_{\mathfrak{a}}(z, 1/2 + it)\, dt. \tag{4.1}$$

The rest of the proof for the cofinite case is the same as for the cocompact case: the estimates of $d(f, t)$ that we wrote in Section 3 (Proposition 2.4) do not depend on the compactness of the group Γ, but only on the spectral parameter t. To complete the proof of Theorem 1.1, we need the analogues of Lemmas 2.1 and 2.2 for Eisenstein series.

Examining the proof of Lemma 2.1 in [15], we notice that, along the same lines, we can prove the following version for Eisenstein series.

Lemma 4.2. *We have*

$$c_{\mathfrak{a}}(f, t) = 2\hat{E}_{\mathfrak{a}}(1/2 + it) d(f, t),$$

where $\hat{E}_{\mathfrak{a}}(1/2 + it)$ is the integral

$$\hat{E}_{\mathfrak{a}}(1/2 + it) = \int_{\sigma} E_{\mathfrak{a}}(z, 1/2 + it)\, ds$$

across a segment σ of the invariant geodesic of γ with length $\int_{\sigma} ds = \mu/\nu$, $d(f, t)$ given by (2.3).

The analogue of Lemma 2.2 for Eisenstein series is the following lemma.

Lemma 4.3. *We have the bound*

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2\, dt \ll T.$$

Proof. For $T > 0$, define the angle $v_T \in (0, \frac{\pi}{2})$ by the relation

$$\tan(v_T) = \frac{\sqrt{2}}{T},$$

and the function f as

$$f(u) = \begin{cases} 1 & \text{for } 1 \leq u \leq \cos^{-2}(v_T), \\ 0 & \text{for } \cos^{-2}(v_T) < u. \end{cases}$$

Thus, for

$$X = \sqrt{1 + \frac{2}{T^2}},$$

we have $A(f)(z) = N(\mathcal{H}, X; z)$. By Lemma 4.1, we get that $A(f)$ is in $L^2(\Gamma \backslash \mathbb{H})$. Moreover, Lemma 4.1 shows that

$$M_X := \sup_{z \in \mathbb{H}} N(\mathcal{H}, X; z) < \infty.$$

Since we are interested about the estimate as $T \to \infty$, X remains bounded and hence M_X can be chosen uniformly bounded by some M. Then, we have the trivial bound

$$\int_{\Gamma \backslash \mathbb{H}} (A(f)(z))^2\, d\mu(z) \leq M \int_{\Gamma \backslash \mathbb{H}} A(f)(z)\, d\mu(z),$$

and, by [15, equation (60)], we get the bound

$$\int_{\Gamma \backslash \mathbb{H}} A(f)(z)\, d\mu(z) \ll T^{-1}.$$

By Lemma 4.2, for $\lambda = 1/4 + t^2$ we get

$$c_{\mathfrak{a}}(f, t) = 2\hat{E}_{\mathfrak{a}}(1/2 + it) \int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} \, dv.$$

On the other hand, from Parseval's identity we get

$$\int_F (A(f)(z))^2 \, d\mu(z) = \sum_j |c(f, t_j)|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{+\infty} |c_{\mathfrak{a}}(f, t)|^2 \, dt$$

$$\geq \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} |c_{\mathfrak{a}}(f, t)|^2 \, dt$$

$$\geq \sum_{\mathfrak{a}} \frac{1}{\pi} \int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \left(\int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} \, dv \right)^2 \, dt.$$

We use [15, Appendix, equation (5)], as in the proof of Lemma 2.2 in [15, p. 24], to get

$$\int_0^{v_T} \frac{\xi_\lambda(v)}{\cos^2(v)} \, dv \gg T^{-1},$$

hence

$$\int_{-T}^{T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2 \ll T. \qquad \square$$

We can now finish the proof of Theorem 1.1 as follows.

Theorem 4.4. *Let Γ be a cofinite Fuchsian group, and $\mathcal{H}$ a hyperbolic conjugacy class of Γ. Then,*

$$E(\mathcal{H}, X; z) = O(X^{2/3}).$$

Proof. The part that corresponds to the Maaß cusp forms can be handled exactly as in the cocompact case. For the contribution of Eisenstein series in the spectral expansion of $A(f)(z)$ we need the estimate

$$\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} d(f, t) \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z, 1/2 + it) \, dt = O(XY^{-1/2} + X^{1/2}).$$

We use Proposition 2.4, Lemmas 4.2 and 4.3 exactly the same way as in the proof of Theorem 3.1. $\qquad \square$

5 Averaging results

Let Γ be a cofinite Fuchsian group. We now apply the large sieve results of [4] for the Riemann surfaces $\Gamma \backslash \mathbb{H}$ to obtain averaging results for $E(\mathcal{H}, X; z)$. To be precise, let a_j be a sequence of complex numbers and, for each cusp $\mathfrak{a}$, let $a_{\mathfrak{a}}(t)$ be a continuous function of t. We have the following results.

Theorem 5.1 ([4]). *Given $z \in \Gamma \backslash \mathbb{H}$, $T, X > 1$ and $x_1, x_2, \ldots, x_R \in [X, 2X]$, if $|x_k - x_\ell| > \delta > 0$ for $k \neq \ell$, then*

$$\sum_{m=1}^{R} \left| \sum_{|t_j| \leq T} a_j x_m^{it_j} u_j(z) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} a_{\mathfrak{a}}(t) x_m^{it} E_{\mathfrak{a}}(z, 1/2 + it) \, dt \right|^2 \ll (T^2 + XT\delta^{-1}) \|a\|_*^2,$$

where

$$\|a\|_* = \left(\sum_{|t_j| \leq T} |a_j|^2 + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} |a_{\mathfrak{a}}(t)|^2 \, dt \right)^{1/2},$$

and the constant implied in '$\ll$' depends on Γ and $y_\Gamma(z)$.

Theorem 5.2 ([4]). *Given $T > 1$ and $z_1, z_2, \ldots, z_R \in \Gamma \backslash \mathbb{H}$, if $\rho(z_k, z_\ell) > \delta > 0$ for $k \neq \ell$, then*

$$\sum_{m=1}^{R} \left| \sum_{|t_j| \leq T} a_j u_j(z_m) + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-T}^{T} a_{\mathfrak{a}}(t) E_{\mathfrak{a}}(z_m, 1/2 + it) \, dt \right|^2 \ll (T^2 + \delta^{-2}) \|a\|_*^2,$$

where $\|a\|_*$ is defined as above and the constant implied in '$\ll$' depends on Γ and $\max y_\Gamma(z_m)$.

We will use Theorem 5.1 to prove the following result for the radial averaging of $E(\mathcal{H}, X; z)$.

Proposition 5.3. *Let $X > 2$ and $X_1, X_2, \ldots, X_R \in [X, 2X]$, satisfying the condition $|X_i - X_j| > \delta$ for some $\delta > 0$, when $i \neq j$. Then, we have*

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X,$$

where the constant implied in '$\ll$' depends on Γ, $\mathcal{H}$ and z.

Theorem 5.4. *If $R\delta \gg X$ and $R > X^{1/2}$, then*

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll X \log^2 X. \tag{5.1}$$

Letting R go to infinity, we get

$$\frac{1}{X} \int_{X}^{2X} |E(\mathcal{H}, x; z)|^2 \, dx \ll X \log^2 X. \tag{5.2}$$

For the spatial average, we use Theorem 5.2 to prove the following.

Proposition 5.5. *Let $X > 2$ and $z_1, z_2, \ldots, z_R$ be points in $\Gamma \backslash \mathbb{H}$ away from the cusps, satisfying the condition $\rho(z_i, z_j) > \delta$ for some $\delta > 0$, when $i \neq j$. Then, we have*

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll \delta^{-2} X + R^{1/3} X^{4/3} \log^2 X, \tag{5.3}$$

and

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^4 \ll \delta^{-2} X^2 \log^4 X + R^{1/3} X^{8/3} \log^3 X, \tag{5.4}$$

where the constant implied in '$\ll$' depends on Γ, $\mathcal{H}$ and z.

Theorem 5.6. *If $R\delta^2 \gg 1$ and $R > X^{1/2}$, then, for $n = 1, 2$,*

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^{2n} \ll X^n \log^{2n} X,$$

Letting R go to infinity, if Γ is cocompact, we get

$$\int_{\Gamma \backslash \mathbb{H}} |E(\mathcal{H}, X; z)|^{2n} \, d\mu(z) \ll X^n \log^{2n} X.$$

Before giving the proof of the above results, we need to fix the following notation. For a function $f \in C_0^*[1, \infty)$, denote by $E_f(\mathcal{H}, X; z)$ the difference

$$E_f(\mathcal{H}, X; z) = A(f)(z) - \sum_{1/2 \leq s_j \leq 1} 2d(f, t_j) \hat{u}_j u_j(z).$$

In the proofs of Theorems 3.1 and 4.4, we proved that for Γ cocompact or cofinite we have

$$E_{f^+}(\mathcal{H}, X; z) = O(XY^{-1/2} + X^{1/2}),$$
$$E_{f^-}(\mathcal{H}, X; z) = O(XY^{-1/2} + X^{1/2}),$$
$$E_{f^-}(\mathcal{H}, X; z) < E(\mathcal{H}, X; z) + O(Y + X^{1/2} \log X) < E_{f^+}(\mathcal{H}, X; z).$$

We begin with the proof of Proposition 5.3.

Proof of Proposition 5.3. We choose Y such that $X^{1/2} \log X \ll Y \ll X$. We get

$$E_{f^-}(\mathcal{H}, X; z) < E(\mathcal{H}, X; z) + O(Y) < E_{f^+}(\mathcal{H}, X; z).$$

We choose f to be f^+ or f^- as in (2.6) and (2.7) with $X = X_m$ and U given by (2.5). We have

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll \sum_{m=1}^{R} |E_f(\mathcal{H}, X_m; z)|^2 + RY^2.$$

The estimates below are true for $f = f^+$ or f^-. We write

$$S(X, z, T) = 2 \sum_{T < |t_j| \le 2T} d(f, t_j) \hat{u}_j u_j(z) + \frac{1}{\pi} \sum_{\mathfrak{a}} \left(\int_{T}^{2T} + \int_{-2T}^{-T} \right) d(f, t) \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z, 1/2 + it) \, dt.$$

We now break the set of t_j, t in the following sets:

$$\begin{aligned}
A_1 &= \{t_j : 0 < |t_j| \le 1\}, & B_1 &= \{t : 0 < |t| \le 1\}, \\
A_2 &= \{t_j : 1 < |t_j|, \le X^2 Y^{-2}\}, & B_2 &= \{t : 1 < |t| \le X^2 Y^{-2}\}, \\
A_3 &= \{t_j : |t_j| > X^2 Y^{-2}\}, & B_3 &= \{t : |t| > X^2 Y^{-2}\}.
\end{aligned}$$

Using the notation

$$S_i(z) := 2 \sum_{t_j \in A_i} d(f, t_j) \hat{u}_j u_j(z) + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_{B_i} d(f, t) \hat{E}_{\mathfrak{a}}(1/2 + it) E_{\mathfrak{a}}(z, 1/2 + it) \, dt,$$

$E_f(\mathcal{H}, X; z)$ can be written as

$$E_f(\mathcal{H}, X; z) = S_1(z) + S_2(z) + S_3(z).$$

We first estimate $S_3(z)$. Using the estimates for $t_j \in \mathbb{R}$, we get the bound

$$\sum_{t_j \in A_3} 2d(f, t_j) \hat{u}_j u_j(z) \ll \sum_{|t_j| > X^2 Y^{-2}} |t_j|^{-2} \min\{t_j, X/Y\} X^{1/2} \hat{u}_j u_j(z)$$

$$\ll \sum_{t_j > X^2 Y^{-2}} t_j^{-2} X^{3/2} Y^{-1} \hat{u}_j u_j(z).$$

Using dyadic decomposition, this is bounded by

$$X^{3/2} Y^{-1} \sum_{n=0}^{\infty} \left(\sum_{2^n X^2 Y^{-2} < t_j \le 2^{n+1} X^2 Y^{-2}} t_j^{-2} \hat{u}_j u_j(z) \right)$$

and hence trivially bounded by

$$X^{3/2} Y^{-1} \sum_{n=0}^{\infty} 2^{-2n} X^{-4} Y^4 \left(\sum_{2^n X^2 Y^{-2} < t_j \le 2^{n+1} X^2 Y^{-2}} \hat{u}_j u_j(z) \right).$$

Using the Cauchy–Schwarz inequality, [16, Proposition 7.2] and Lemma 2.2, we get the bound

$$X^{-5/2} Y^3 \sum_{n=0}^{\infty} 2^{-2n} \left(\sum_{t_j \le 2^{n+1} X^2 Y^{-2}} |\hat{u}_j|^2 \right)^{1/2} \left(\sum_{t_j \le 2^{n+1} X^2 Y^{-2}} |u_j(z)|^2 \right)^{1/2}$$

$$\ll X^{-5/2} Y^3 \sum_{n=0}^{\infty} 2^{-2n} (2^{n/2} X Y^{-1})(2^n X^2 Y^{-2}) \ll X^{1/2} \ll Y.$$

Similarly we deal with the case of the Eisenstein series over B_3. We conclude $S_3(z) = O(Y)$.

We now consider $S_1(z)$. We get

$$\sum_{t_j \in A_1} d(f, t_j) \hat{u}_j u_j(z) \ll X^{1/2} \sum_{|t_j| < 1} t_j^{-2} \min\{t_j, X/Y\} \hat{u}_j u_j(z) \ll X^{1/2} \ll Y,$$

since there exist finitely many eigenvalues with spectral parameter $|t_j| \leq 1$. Similarly, we prove the $O(Y)$ bound for the Eisenstein series' contribution over B_1. We conclude that $S_1(z) = O(Y)$. Combining all the above, we get

$$E_f(\mathcal{H}, X; z) = 2 \sum_{t_j \in A_2} d(f, t_j)\hat{u}_j u_j(z) + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_{A_2} d(f, t)\hat{E}_{\mathfrak{a}}(1/2 + it)E_{\mathfrak{a}}(z, 1/2 + it)\, dt + O(Y).$$

Adding for $T = 2^k$, $k = 0, 1, \ldots, [\log_2(X^2 Y^{-2})]$, we get the bound

$$E_f(\mathcal{H}, X; z) \ll \sum_{1 \leq T < X^2 Y^{-2}} S(X, z, T) + O(Y),$$

and, adding for $X_1, \ldots, X_R$, we get

$$\sum_{m=1}^{R} |E_f(\mathcal{H}, X_m; z)|^2 \ll \sum_{m=1}^{R} \left| \sum_{1 \leq T < X^2 Y^{-2}} S(X_m, z, T) \right|^2 + RY^2. \tag{5.5}$$

The Cauchy–Schwarz inequality now yields

$$\left| \sum_{1 \leq T < X^2 Y^{-2}} S(X_m, z, T) \right|^2 \ll \log X \sum_{1 \leq T < X^2 Y^{-2}} |S(X_m, z, T)|^2, \tag{5.6}$$

which combined with the bound (5.5) gives

$$\sum_{m=1}^{R} |E_f(\mathcal{H}, X_m; z)|^2 \ll \log X \sum_{1 \leq T < X^2 Y^{-2}} \left(\sum_{m=1}^{R} |S(X_m, z, T)|^2 \right) + RY^2. \tag{5.7}$$

Using the estimates of Proposition 2.4, we can now write

$$d(f, t) = X^{1/2}\left(a(t, Y/X)X^{it} + b(t, Y/X)X^{-it}\right),$$

where $a(t, Y/X)$ and $b(t, Y/X)$ are functions satisfying

$$a(t, Y/X), b(t, Y/X) \ll |t|^{-2} \min\{|t|, XY^{-1}\}.$$

We apply Theorem 5.1, which implies that for $a_j = d(f, t_j)$ and $a(t) = d(f, t)$

$$\sum_{m=1}^{R} \left| \sum_{T < |t_j| \leq 2T} d(f, t_j)\hat{u}_j u_j(z) + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_{T}^{2T} d(f, t)\hat{E}_{\mathfrak{a}}(1/2 + it)E_{\mathfrak{a}}(z, 1/2 + it)\, dt \right|^2$$

is bounded by

$$(T^2 + XT\delta^{-1})\|a\|_*^2,$$

i.e.,

$$\sum_{m=1}^{R} |S(X_m, z, T)|^2 \ll (T^2 + XT\delta^{-1})\|a\|_*^2,$$

where

$$\|a\|_*^2 \ll \sum_{T < |t_j| \leq 2T} |t_j|^{-2} \min\{|t_j|, XY^{-1}\}X^{1/2}\hat{u}_j|^2 + \frac{1}{\pi} \sum_{\mathfrak{a}} \int_{T}^{2T} |t|^{-2} \min\{|t|, XY^{-1}\}X^{1/2}\hat{E}_{\mathfrak{a}}(1/2 + it)|^2.$$

The last expression can be bounded by

$$XT^{-4} \min\{T^2, X^2 Y^{-2}\}\left(\sum_{T \leq |t_j| \leq 2T} |\hat{u}_j|^2 + \sum_{\mathfrak{a}} \int_{T}^{2T} |\hat{E}_{\mathfrak{a}}(1/2 + it)|^2\, dt \right),$$

and using Lemma 2.2, we obtain

$$\|a\|_*^2 \ll XT^{-3} \min\{T^2, X^2 Y^{-2}\}. \tag{5.8}$$

Thus, we conclude

$$\sum_{m=1}^{R} |S(X_m, z, T)|^2 \ll (T^2 + XT\delta^{-1})(XT^{-3} \min\{T^2, X^2Y^{-2}\}),$$

hence

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll \log X \sum_{1 \le T < X^2 Y^{-2}} \left(\sum_{m=1}^{R} |S(X_m, z, T)|^2 \right) + RY^2$$

$$\ll \log X \sum_{1 \le T < X^2 Y^{-2}} (T^2 + XT\delta^{-1})(XT^{-3} \min\{T^2, X^2Y^{-2}\}) + RY^2.$$

We get the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll X \log X \left(\sum_{1 \le T < XY^{-1}} T \right) + X^2 \delta^{-1} \log X \left(\sum_{1 \le T < XY^{-1}} 1 \right)$$

$$+ X^3 Y^{-2} \log X \left(\sum_{XY^{-1} \le T < X^2 Y^{-2}} T^{-1} \right) + X^4 \delta^{-1} Y^{-2} \log X \left(\sum_{XY^{-1} \le T < X^2 Y^{-2}} T^{-2} \right) + RY^2.$$

Trivial bounds for each term separately yield the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll X^2 Y^{-1} \log X + \delta^{-1} X^2 \log^2 X + RY^2.$$

The optimal choice for Y is $Y = R^{-1/3} X^{2/3}$, which implies the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X. \qquad \square$$

Proof of Theorem 5.4. Choosing $\delta^{-1} \ll RX^{-1}$ and $R > X^{1/2}$ in the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + \delta^{-1} X^2 \log^2 X,$$

we get

$$\sum_{m=1}^{R} |E(\mathcal{H}, X_m; z)|^2 \ll R^{1/3} X^{4/3} \log X + RX \log^2 X \ll RX \log^2 X,$$

and we get the bound (5.1). For the bound (5.2), we take the points X_i equally spaced in the interval $[X, 2X]$ with $\delta = R^{-1} X$. As $R \to \infty$,

$$\sum_{m=1}^{R} ||E(\mathcal{H}, X_m; z)|^2 \frac{X}{R} \to \int_{X}^{2X} |E(\mathcal{H}, x; z)|^2 \, dx,$$

hence

$$\frac{1}{X} \int_{X}^{2X} |E(\mathcal{H}, x; z)|^2 \, dx \ll X \log^2 X. \qquad \square$$

Proof of Proposition 5.5. For a sequence $\{a_k\}$, the Cauchy–Schwarz inequality implies

$$\left(\sum_{k=0}^{n} a_k \right)^2 \ll \sum_{k=0}^{n} (n+1-k)^2 a_k^2, \qquad \left(\sum_{k=0}^{n} a_k \right)^2 \ll \sum_{k=0}^{n} (k+1)^2 a_k^2.$$

The first inequality for $a_k = S(X, z_m, 2^k)$ implies the bound

$$\left| \sum_{1 \le T < X^2 Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2 Y^{-2}} |\log T^{-1} X^2 Y^{-2} + 1|^2 |S(X, z_m, T)|^2, \qquad (5.9)$$

whereas the second gives

$$\left| \sum_{1 \le T < X^2 Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2 Y^{-2}} (\log T + 1)^2 |S(X, z_m, T)|^2. \qquad (5.10)$$

The bounds (5.9) and (5.10) give

$$\left| \sum_{1 \le T < X^2 Y^{-2}} S(X, z_m, T) \right|^2 \ll \sum_{1 \le T < X^2 Y^{-2}} |c_T|^2 |S(X, z_m, T)|^2, \tag{5.11}$$

where $c_T = \min\{\log T^{-1} X^2 Y^{-2} + 1, \log T + 1\}$. Using Theorem 5.2, bound (5.8) and summing over $T = 2^k$, $k = 0, 1, \ldots, [\log_2(X^2 Y^{-2})]$, we get

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll \sum_{1 \le T < X^2 Y^{-2}} |c_T|^2 (T^2 + \delta^{-2})(X T^{-1} \min\{1, X^2 T^{-2} Y^{-2}\}) + R Y^2$$

$$\ll \delta^{-2} X + X^2 Y^{-1} \log^2 X + R Y^2,$$

where the last bound yields as in the proof of Proposition 5.3. The bound (5.3) is obtained for $Y = X^{2/3} R^{-1/3}$. For the fourth moment, we use Hölder's inequality to prove

$$\sum_{m=1}^{R} |E_f(\mathcal{H}, X; z_m)|^4 \ll \log^3 X \sum_{1 \le T < X^2 Y^{-2}} \left(\sum_{m=1}^{R} |S(X, z_m, T)|^4 \right) + R Y^4. \tag{5.12}$$

We can now finish the proof assuming the following large sieve inequality:

$$\sum_{m=1}^{R} \left| \sum_{|t_j| \le T} a_j u_j(z_m) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-T}^{T} a_{\mathfrak{a}}(t) E_{\mathfrak{a}}(z_m, 1/2 + it)\, dt \right|^4 \ll (T^4 + T^2 \delta^{-2}) \|a\|_*^4. \tag{5.13}$$

For the proof, see [2]. We can now derive the second part of the proposition by applying (5.13) to $a_j = d(f, t_j)$ and $a_{\mathfrak{a}}(t) = d(f, t)$. □

Proof of Theorem 5.6. Consider first the case $n = 1$. Choosing $\delta^{-2} \ll R$ and $R > X^{1/2}$ in the bound

$$\sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll \delta^{-2} X + R^{1/3} X^{4/3} \log^2 X,$$

we get

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \ll X + R^{-2/3} X^{4/3} \log^2 X \ll X \log^2 X,$$

and the first part follows. For the integral estimate we notice that as $R \to \infty$,

$$\frac{1}{R} \sum_{m=1}^{R} |E(\mathcal{H}, X; z_m)|^2 \to \int_{\Gamma \backslash \mathbb{H}} |E(\mathcal{H}, X; z)|^2\, d\mu(z).$$

Hence,

$$\int_{\Gamma \backslash \mathbb{H}} |E(\mathcal{H}, X; z)|^2\, d\mu(z) \ll X \log^2 X.$$

The case $n = 2$ follows in exactly the same way. □

For the error term $E(X; z, w)$ of the classical hyperbolic lattice point problem, the optimal bound is conjectured to be

$$E(X; z, w) = O(X^{1/2 + \epsilon})$$

for every $\epsilon > 0$. This is supported by the Ω-results of Phillips and Rudnick [20] and the averaging results in [3].

Theorems 5.4 and 5.6 lead us to formulate the analogous conjecture.

Conjecture 5.7. For Γ cocompact or cofinite and $\mathcal{H}$ a hyperbolic conjugacy class of Γ, the error term $E(\mathcal{H}, X; z)$ satisfies the bound

$$E(\mathcal{H}, X; z) = O(X^{1/2 + \epsilon})$$

for every $\epsilon > 0$.

6 Arithmetic applications

In this section we are interested in arithmetic corollaries of our results. We use the geometric interpretation to get an arithmetic interpretation of the quantity $N(\mathcal{H}, X; z)$. We restrict our attention to $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Fix a point z in $\mathbb{H}$. Huber's interpretation in [15] shows that $N(\mathcal{H}, X; z)$ counts γ in $\Gamma/\langle g \rangle$ such that $\cos v \geq X^{-1}$, where v is the angle defined by the ray from 0 to γz and the geodesic $\{yi, y > 0\}$. Denote by c the geodesic from γz perpendicular to $\{yi, y > 0\}$ and by $\ell(c)$ its length. Then,

$$\rho(\gamma z, \{iy, y > 0\}) = \ell(c) = \int_{\pi/2-v}^{\pi/2} \csc(t)\, dt = \log\left(\frac{1 + \sin v}{\cos v}\right). \tag{6.1}$$

On the other hand, the distance of γz to the imaginary axis is given by

$$\cosh \rho(\gamma z, i|\gamma z|) = \frac{|\gamma z|}{\Im(\gamma z)}. \tag{6.2}$$

For

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

we define

$$f_z(A, B, C, D) = \cosh \rho(\gamma z, i|\gamma z|) = \frac{|\gamma z|}{\Im(\gamma z)} = \frac{|Az + B||Cz + D|}{\Im(z)}. \tag{6.3}$$

Using $|Cz + D| = |C\bar{z} + D|$ and $AD - BC = 1$, f_z can be written in the form

$$f_z(A, B, C, D) = \frac{\left((AC|z|^2 + BD + AD\mathbb{R}(z) + BC\mathbb{R}(z))^2 + \Im(z)^2\right)^{1/2}}{\Im(z)}. \tag{6.4}$$

Using (6.1), the condition $\cos v \geq X^{-1}$ can now be written as

$$f_z(A, B, C, D) \leq \cosh\left(\log(X + \sqrt{X^2 - 1})\right) = X. \tag{6.5}$$

Let $z = \alpha + \beta i$. Inequality (6.5) takes the form

$$|(\alpha^2 + \beta^2)AC + BD + \alpha AD + \alpha BC| \leq \beta\sqrt{X^2 - 1} = \beta X + O(X^{-1}). \tag{6.6}$$

To get simple results, we take specific choices for z.

6.1 Quadratic forms

For the basics of indefinite quadratic forms we refer to [21]. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a primitive indefinite quadratic form in two variables, i.e., $(a, b, c) = 1$ and $b^2 - 4ac = d > 0$ is not a square. We denote Q by $[a, b, c]$. Two forms $[a, b, c]$, $[a', b', c']$ are called equivalent ($[a, b, c] \sim [a', b', c']$) if there is a $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma.$$

The automorphs of Q is the group $\mathrm{Aut}(Q) = \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ which fixes Q, under the action above. This group is infinite and cyclic with generator

$$M_{[a,b,c]} = \begin{pmatrix} \frac{t_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{t_0 + bu_0}{2} \end{pmatrix},$$

where $t_0, u_0 > 0$ is the fundamental solution of Pell's equation $x^2 - dy^2 = 4$. Since $t_0 > 2$, the matrix $M_{[a,b,c]}$ is hyperbolic. We denote by ε_d the quantity

$$\varepsilon_d = \frac{t_0 + \sqrt{d}u_0}{2}.$$

The quadratic form Q is associated with two real quadratic numbers

$$\theta_1 = \frac{-b + \sqrt{d}}{2a}, \quad \theta_2 = \frac{-b - \sqrt{d}}{2a},$$

the roots of the polynomial $a\theta^2 + b\theta + c$.

    The main reason we are interested in quadratic forms is that, according to the next proposition, they are in one-to-one correspondence with hyperbolic conjugacy classes of the modular group (a proof of this result can be found in [21, p. 232]).

Proposition 6.1. *Define the map ϕ by*

$$\phi([a, b, c]) = M_{[a,b,c]}.$$

Then, the following hold:

(a) *ϕ is a bijective map of the set of primitive indefinite quadratic forms onto the set of primitive hyperbolic elements of $\mathrm{SL}_2(\mathbb{Z})$,*

(b) *ϕ commutes with the action of $\mathrm{SL}_2(\mathbb{Z})$: $[a, b, c] \sim [a', b', c']$ if and only if $M_{[a,b,c]}$ is conjugate to $M_{[a',b',c']}$.*

The matrix $M_{[a,b,c]}$ has eigenvalues:

$$\lambda_{1,2} = \frac{t_0 \pm \sqrt{d}u_0}{2} = \varepsilon_d^{\pm 1}. \tag{6.7}$$

Its diagonalisation is

$$M_{[a,b,c]} = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}, \quad T = \begin{pmatrix} \theta_1 & \theta_2 \\ 1 & 1 \end{pmatrix}. \tag{6.8}$$

Calculations imply that

$$M_{[a,b,c]}^n = \frac{1}{\theta_1 - \theta_2} \begin{pmatrix} \lambda_1^n \theta_1 - \lambda_2^n \theta_2 & (\lambda_2^n - \lambda_1^n)\theta_1\theta_2 \\ \lambda_1^n - \lambda_2^n & \lambda_2^n \theta_1 - \lambda_1^n \theta_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

    Let us first examine a simple case. Consider the case $b = 0$, i.e., we consider the form $Q(x, y) = ax^2 + cy^2$. Set $\theta = \theta_1 = -\theta_2 = \sqrt{d}/2a$. Thus, $\theta^2 = -c/a > 0$. In this case, $M_{[a,b,c]}^n$ takes the form

$$M_{[a,0,c]}^n = M^n = \frac{1}{2\theta} \begin{pmatrix} (\lambda_1^n + \lambda_2^n)\theta & (\lambda_1^n - \lambda_2^n)\theta^2 \\ \lambda_1^n - \lambda_2^n & (\lambda_1^n + \lambda_2^n)\theta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{6.9}$$

We now prove an application of Theorems 1.1 and 1.2.

Proposition 6.2. *Given $Q(x, y) = ax^2 + cy^2$ with $\theta^2 = -c/a > 0$ and $-ac$ not a square, let F_Q denote the indefinite quadratic form*

$$F_Q(\alpha, \beta, \gamma, \delta) = \alpha^2 - \frac{a}{c}\beta^2 + \frac{c}{a}\gamma^2 - \delta^2.$$

Let $P(X)$ be the number of solutions $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$ such that $\alpha\delta - \beta\gamma = 1$ and

$$|F_Q(\alpha, \beta, \gamma, \delta)| \le X,$$

under the equivalence: $(\alpha, \beta, \gamma, \delta) \sim (\alpha', \beta', \gamma', \delta')$ if there exists an integer n such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M^n \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Here M^n is given by (6.9). Then, the following hold:

(a) *$P(X)$ satisfies*

$$P(X) = \frac{6 \log \varepsilon_d}{\pi} X + E(X)$$

with ε_d given by (6.7) and $E(X) = O(X^{2/3})$.

(b) *$E(X)$ satisfies the average bound*

$$\frac{1}{X} \int_X^{2X} |E(x)|^2 \, dx \ll X \log^2 X,$$

where the constant implied in '$\ll$' depends on the quadratic form Q.

Proof. For (a), let ℓ be the invariant closed geodesic of M. Conjugating Γ with T, we bring ℓ on the imaginary axis. Let also $z = T^{-1}(i)$. In this case $N(\mathcal{H}, X; z)$ counts

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})/\langle M \rangle \simeq T^{-1}\mathrm{SL}_2(\mathbb{Z})T/G,$$

where

$$G = \left\langle \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

such that asymptotically

$$|AC + BD| \leq X.$$

For

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = T^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} T \in T^{-1}\mathrm{SL}_2(\mathbb{Z})T,$$

we get

$$2|AC + BD| = |F_Q(\alpha, \beta, \gamma, \delta)|.$$

Thus, $N(\mathcal{H}, X; z)$ counts points $\mathrm{SL}_2(\mathbb{Z})$ under the extra equivalence that comes from the quotient with $\langle M \rangle$ such that

$$|F_Q(\alpha, \beta, \gamma, \delta)| \leq 2X,$$

i.e., $P(X) = N(\mathcal{H}, X/2; z)$. By Theorem 4.4 and the fact that $\mathrm{SL}_2(\mathbb{Z})$ has no eigenvalues $\lambda \in (0, 1/4)$, we get

$$P(X) = \hat{u}_0 u_0(z) X + O(X^{2/3}).$$

For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, we have

$$u_0(z) = \sqrt{\frac{3}{\pi}}, \quad \hat{u}_0 = \sqrt{\frac{3}{\pi}} \frac{\mu}{\nu}.$$

Since M is primitive $\nu = 1$ and, for $\mathrm{SL}_2(\mathbb{Z})$, we know that $\mu = 2 \log \varepsilon_d$, which equals the length of the closed geodesic ℓ (see, for example, [21, Corollary 1.5]). Part (a) now follows. Part (b) follows immediately as $E(X) = E(\mathcal{H}, X/2; z)$. □

Remark 6.3. Note that, for $z = i$, by (6.3) we count solutions of

$$(A^2 + B^2)(C^2 + D^2) \leq X^2$$

with restrictions. The more general case $b \neq 0$ or $z \neq i$ leads to a more complicated quadratic form (see (6.6) and (6.8)).

Remark 6.4. The arithmetic corollaries of the classical hyperbolic lattice point problem (see [3]) differ from ours in the fact that the quadratic forms in [3] are positive definite, e.g., for $z = w = i$ one gets that $4u(\gamma i, i) + 2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$. The quadratic form $F_Q(\alpha, \beta, \gamma, \delta)$ is indefinite.

6.2 Hecke operators

Applying Hecke operators as in [3, 16] for the classical lattice point counting problem, we can count solutions of $|F(\alpha, \beta, \gamma, \delta)| \leq X$ lying in the hypersurface $\alpha\delta - \beta\gamma = n$ with $n > 1$. Let Γ_n be the set

$$\Gamma_n = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{Z}^{2 \times 2} : \alpha\delta - \beta\gamma = n \right\}.$$

For $n \in \mathbb{N}$, let $T_n \colon \mathcal{A}(\Gamma \backslash \mathbb{H}) \to \mathcal{A}(\Gamma \backslash \mathbb{H})$ be the n-th Hecke operator, see [16, Section 8.5 and Chapter 12], defined by

$$T_n(f)(z) = \frac{1}{\sqrt{n}} \sum_{\tau \in \Gamma \backslash \Gamma_n} f(\tau z).$$

As the Hecke operators commute with Δ, we choose a joint orthonormal basis u_j. We denote by $\lambda_j(n)$ the eigenvalue of T_n for $u_j(z)$, i.e.,

$$T_n u_j(z) = \lambda_j(n) u_j(z),$$

and $\eta_t(n)$ for the Eisenstein series, i.e.,

$$T_n E_\infty(z, 1/2 + it) = \eta_t(n) E_\infty(z, 1/2 + it),$$

where

$$\eta_t(n) = \sum_{ad=n} \left(\frac{a}{d}\right)^{it}.$$

Notice that counting solutions

$$|F_Q(\alpha, \beta, \gamma, \delta)| \leq X$$

with $\alpha\delta - \beta\gamma = n$ is equivalent to counting solutions

$$|f_i(A, B, C, D)| \leq nX$$

with $AD - BC = n$.

We apply T_n on both expressions of $A(f)(z)$. Applying T_n to the spectral expansion (4.1), we get

$$T_n A(f)(z) = \sum_j c(f, t_j) \lambda_j(n) u_j(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} c_\infty(f, t) \eta_t(n) E_\infty(z, 1/2 + it)\, dt. \tag{6.10}$$

On the geometric side, we have

$$T_n A(f)(z) = \frac{1}{\sqrt{n}} \sum_{\tau \in \Gamma \backslash \Gamma_n} \left(\sum_{\gamma \in \mathcal{H}} f\left(\frac{\cosh \rho(\tau^{-1}\gamma\tau z, z) - 1}{\cosh \mu(\gamma) - 1}\right) \right).$$

If $\mathcal{H}$ is the conjugacy class of the primitive hyperbolic matrix M, we define the set

$$\mathcal{H}_n = \{\gamma^{-1} M \gamma : \gamma \in \Gamma_n\}.$$

The set $\mathcal{H}_n$ is in one-to-one correspondence with the quotient set $\Gamma_n/\langle M \rangle$. Notice also that $\mu(\tau\gamma\tau^{-1}) = \mu(\gamma)$. Therefore,

$$T_n A(f)(z) = \frac{1}{\sqrt{n}} \sum_{\gamma \in \mathcal{H}_n} f\left(\frac{\cosh \rho(\gamma z, z) - 1}{\cosh \mu(\gamma) - 1}\right). \tag{6.11}$$

Using that $|\lambda_j(n)| \leq \lambda_0(n) = \sigma(n) n^{-1/2}$ and the uniform bound $|\eta_t(n)| \ll d(n) \ll \lambda_0(n)$ we conclude the following proposition.

Proposition 6.5. *Denote with $P_{Q,n}(X)$ the number of solutions $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}$ such that $\alpha\delta - \beta\gamma = n$ and*

$$|F_Q(\alpha, \beta, \gamma, \delta)| \leq X,$$

under the equivalence: $(\alpha, \beta, \gamma, \delta) \sim (\alpha', \beta', \gamma', \delta')$ if there exists an integer m such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = M_{[a,b,c]}^m \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Then, the following hold:

(a) *$P_{Q,n}(X)$ has the asymptotic behaviour*

$$P_{Q,n}(X) = \frac{6 \log \varepsilon_d}{\pi} \frac{\sigma(n)}{n} X + E_n(X)$$

with

$$E_n(X) = O\left(\frac{\sigma(n)}{n^{2/3}} X^{2/3}\right).$$

(b) *$E_n(X)$ satisfies the bound*

$$\frac{1}{X} \int_X^{2X} |E_n(x)|^2\, dx \ll \sigma^2(n) \frac{X}{n} \log^2\left(\frac{X}{n}\right),$$

where the constant implied in '$\ll$' depends on the quadratic form Q.

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