Bianchi permutability for the anti-self-dual Yang-Mills equations

G. Benincasa and R. G. Halburd

Department of Mathematics, University College London
Gower Street, London WC1E 6BT

Abstract

The anti-self-dual Yang-Mills equations are known to have reductions to many integrable differential equations. A general Bäcklund transformation (BT) for the ASDYM equations generated by a Darboux matrix with an affine dependence on the spectral parameter is obtained, together with its Bianchi permutability equation. We give examples in which we obtain BTs of symmetry reductions of the ASDYM equations by reducing this ASDYM BT. Some discrete integrable systems are obtained directly from reductions of the ASDYM Bianchi system.

Dedicated to Mark Ablowitz on his 70th birthday

1 Introduction

It is well known that many discrete integrable equations arise from the permutability of Bäcklund transformations of continuous integrable systems. In turn, many continuous integrable systems are known to be symmetry reductions of the anti-self-dual Yang-Mills (ASDYM) equations [19, 5, 2, 10, 1]. In this paper we will derive a form of the Bäcklund transformations for the ASDYM equations that is well-suited for obtaining Bäcklund transformations of reductions. Furthermore, we will use the Bianchi permutability of these Bäcklund transformations to obtain discrete integrable equations.

Let $A := A_z \, \mathrm{d}z + A_w \, \mathrm{d}w + A_{\tilde{z}} \, \mathrm{d}\tilde{z} + A_{\tilde{w}} \, \mathrm{d}\tilde{w}$ be a one-form with components in some Lie algebra $\mathfrak{g}$. In all of our examples, $\mathfrak{g}$ will be $\mathfrak{sl}(2;\mathbb{C})$. In terms of the components of this one-form, the ASDYM equations with respect to the metric $\mathrm{d}s^2 = 2(\mathrm{d}z\,\mathrm{d}\tilde{z} - \mathrm{d}w\,\mathrm{d}\tilde{w})$ are

$$\begin{aligned} &\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0, \\ &\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0, \\ &\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0. \end{aligned} \tag{1}$$

This is a system of three equations in four $\mathfrak{g}$-valued functions A_z, A_w, $A_{\tilde{z}}$ and $A_{\tilde{w}}$ of the four variables z, $\tilde{z}$, w and $\tilde{w}$. We will allow these independent variables to be complex. The system (1) is the compatibility condition for the Lax pair

$$\begin{aligned} (\partial_z - \zeta\partial_{\tilde{w}})\Psi &= -(A_z - \zeta A_{\tilde{w}})\Psi, \\ (\partial_w - \zeta\partial_{\tilde{z}})\Psi &= -(A_w - \zeta A_{\tilde{z}})\Psi. \end{aligned} \tag{2}$$

This Lax pair is equivalent to the statement that the differential operators $L = D_w - \zeta D_{\tilde{z}}$ and $M = D_z - \zeta D_{\tilde{w}}$ commute, where $D_z = \partial_z + A_z$, $D_w = \partial_w + A_w$, $D_{\tilde{z}} = \partial_{\tilde{z}} + A_{\tilde{z}}$, and $D_{\tilde{w}} = \partial_{\tilde{w}} + A_{\tilde{w}}$.

The conformal symmetries of $\mathbb{R}^{2,2}$ (translations, rotations/boosts, dilations, inversions) induce (Lie point) symmetries of the ASDYM equations. This large group of symmetries gives rise to a rich collection of inequivalent reductions. The ASDYM equations are also invariant under the gauge transformation

$$A_\mu \mapsto g^{-1}\partial_\mu g + g^{-1}A_\mu g,$$

for any G-valued function g, where G is the Lie group of $\mathfrak{g}$. The gauge freedom can be used to remove arbitrary functions in reductions.

The ASDYM equations also admit a variety of Bäcklund transformations, which in general are non-point symmetries in that they depend on derivatives of solutions and not just values of the solutions. Bäcklund transformations for the (A)SDYM equations have been introduced by several authors including Corrigan, Fairlie, Yates and Goddard [7], Prasad, Sinha and Chau Wang [17], Bruschi, Levi and Ragnisco [3], Papachristou and Kent Harrison [15], Mason, Chakravarty and Newman [8], and Tafel [18].

Despite the fact that many integrable systems are known to be reductions of the ASDYM equations, there are very few instances where BTs for the reduced equations have been obtained as reductions of an appropriate BT for ASDYM equations. Masuda obtained the affine Weyl group symmetry of P_{II}, P_{III} and P_{IV} from the ASDYM Bäcklund transformation [11, 12].

One of the technical difficulties in reducing the BT for the ASDYM equations is the need to change gauge. In this paper we will derive the general form of a Bäcklund transformation generated by a Darboux matrix with an affine dependence on the spectral parameter. This form of the Bäcklund transformation makes it much easier to deal with issues related to gauge.

The first two of the equations in (1) guarantee the existence of G-valued functions H and K respectively such that $\partial_z H = -A_z H$, $\partial_w H = -A_w H$, $\partial_{\tilde{z}} K = -A_{\tilde{z}} K$, and $\partial_{\tilde{w}} K = -A_{\tilde{w}} K$. The final equation,

$$\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0,$$

then takes the compact form

$$\partial_w (J^{-1} \partial_{\tilde{w}} J) - \partial_z (J^{-1} \partial_{\tilde{z}} J) = 0, \tag{3}$$

where $J = K^{-1} H$. This is known as *Yang's equation* and was first written in [20]. Equation (3) has the obvious symmetry

$$J(z, w, \tilde{z}, \tilde{w}) \mapsto M(z, w) J(z, w, \tilde{z}, \tilde{w}) \widetilde{M}(\tilde{z}, \tilde{w}).$$

Apart from reductions to the Ernst equation and the chiral fields models, this form of ASDYM is not usually considered in relation to symmetry reductions. One reason for this is that Yang's equation has lost some of the Lie symmetries of the original system (the rotations and boosts). Another reason is that the A_μ, in the original formalism appear directly in the Lax pair of ASDYM, which makes it easier to identify reductions by looking at the reduced Lax pair.

In section 2 we will derive a Bäcklund transformation for the ASDYMEs. This system is most naturally expressed in terms of Yang's equation (3). The equations describing these transformations depend on two matrix-valued functions, $C(z, w)$ and $\widetilde{C}(\tilde{z}, \tilde{w})$. These matrices contain important gauge information. This form of the BT is particularly useful when considering reductions. The case in which C and $\widetilde{C}$ are constants was derived previously by Bruschi, Levi and Ragnisco [3]. In section 3 we consider the known reductions of the ASDYMEs to the sine-Gordon equation and the sixth Painlevé equation. We show how these reductions can be extended to the BTs derived in section 2 to obtain the Bäcklund transformations for the reductions. Finally, in section 4, we derive an equation describing the Bianchi permutability of the BTs for the ASDYMEs derived in section 2. We show how reductions of this system lead to discrete integrable equations.

2 Bäcklund transformations for ASDYM equations

Starting from the ASDYM Lax pair (2), we perform a ζ-dependent gauge transformation

$$\Psi \mapsto \hat{\Psi} = (S + \zeta T)\Psi, \tag{4}$$

such that the resulting system has the same form:

$$\begin{aligned}(\partial_z - \zeta\partial_{\tilde{w}})\hat{\Psi} &= -(\hat{A}_z - \zeta\hat{A}_{\tilde{w}})\hat{\Psi},\\ (\partial_w - \zeta\partial_{\tilde{z}})\hat{\Psi} &= -(\hat{A}_w - \zeta\hat{A}_{\tilde{z}})\hat{\Psi}.\end{aligned}$$

This gives

$$\begin{aligned}&\big\{(S_w - SA_w + \hat{A}_w S) + \zeta(T_w - S_{\tilde{z}} + SA_{\tilde{z}} - TA_w + \hat{A}_w T - \hat{A}_{\tilde{z}} S)\\ &\qquad\qquad + \zeta^2(-T_{\tilde{z}} + TA_{\tilde{z}} - \hat{A}_{\tilde{z}} T)\big\}\Psi = 0,\\ &\big\{(S_z - SA_z + \hat{A}_z S) + \zeta(T_z - S_{\tilde{w}} + SA_{\tilde{w}} - TA_z + \hat{A}_z T - \hat{A}_{\tilde{w}} S)\\ &\qquad\qquad + \zeta^2(-T_{\tilde{w}} + TA_{\tilde{w}} - \hat{A}_{\tilde{w}} T)\big\}\Psi = 0.\end{aligned}$$

Equating the coefficients of the various powers of ζ yields

$$\begin{aligned}S_w = SA_w - \hat{A}_w S, &\qquad S_z = SA_z - \hat{A}_z S,\\ S_{\tilde{z}} - T_w &= SA_{\tilde{z}} - \hat{A}_{\tilde{z}} S - TA_w + \hat{A}_w T,\\ S_{\tilde{w}} - T_z &= SA_{\tilde{w}} - \hat{A}_{\tilde{w}} S - TA_z + \hat{A}_z T,\\ T_{\tilde{z}} = TA_{\tilde{z}} - \hat{A}_{\tilde{z}} T, &\qquad T_{\tilde{w}} = TA_{\tilde{w}} - \hat{A}_{\tilde{w}} T.\end{aligned} \tag{5}$$

Recall that two of the three ASDYM equations guarantee the existence of the G-valued potential functions H and K such that $\partial_z H = -A_z H$, $\partial_w H = -A_w H$, $\partial_{\tilde{z}} K = -A_{\tilde{z}} K$, and $\partial_{\tilde{w}} K = -A_{\tilde{w}} K$. Similarly there are functions $\hat{H}$ and $\hat{K}$ such that $\partial_z \hat{H} = -\hat{A}_z \hat{H}$, $\partial_{\tilde{z}} \hat{K} = -\hat{A}_{\tilde{z}} \hat{K}$, etc. Now define $C := \hat{K}^{-1} T K$ and $\tilde{C} := \hat{H}^{-1} S H$. Then the first two equations in (5) are equivalent to $\widetilde{C} \equiv \widetilde{C}(\tilde{z}, \tilde{w})$ and the last two equations in (5) are equivalent to $C \equiv C(z, w)$. In terms of $J = K^{-1} H$, the remaining two equations in (5) become

$$\begin{aligned}\hat{J}\left(\hat{J}^{-1} C J\right)_z &= \left(\hat{J}\widetilde{C} J^{-1}\right)_{\tilde{w}} J,\\ \hat{J}\left(\hat{J}^{-1} C J\right)_w &= \left(\hat{J}\widetilde{C} J^{-1}\right)_{\tilde{z}} J.\end{aligned} \tag{6}$$

In the case when C and $\widetilde{C}$ are constant multiples of the identity, this system was first derived by Prasad, Sinha and Chau Wang [17]. The case in which in which C and $\widetilde{C}$ constant was later derived by Bruschi, Levi and Ragnisco [3]. The more general dependence $C \equiv C(z, w)$ and $\widetilde{C} \equiv \widetilde{C}(\tilde{z}, \tilde{w})$ is needed to ensure that $\hat{J}$ remains in G (e.g., if $G = \mathrm{SL}(2;\mathbb{C})$ then we need this freedom C and $\widetilde{C}$ to ensure that $\det(\hat{J}) = 1$) and plays an important role in some reductions.

3 Examples of reductions

We review some standard reductions of the ASDYM equations in the form (1) and extend these reductions to include Bäcklund transformations.

3.1 Reduction to the sine-Gordon equation

Following Chakravarty and Ablowitz [4], we look for solutions of the ASDYM equations such that the A_μ's depend on z and $\tilde{z}$ only. We choose a gauge such that $A_{\tilde{z}} = 0$. The field equations (1) become

$$\partial_z A_w + [A_z, A_w] = 0, \quad \partial_{\tilde{z}} A_{\tilde{w}} = 0, \quad \text{and} \quad \partial_{\tilde{z}} A_z + [A_w, A_{\tilde{w}}] = 0.$$

Generically, we can use the remaining gauge freedom to put $A_{\tilde{w}}$ in the form

$$k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For the remaining matrices, we take

$$A_w = \begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix} \quad \text{and} \quad A_z = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}.$$

This leads to the equations

$$a_z = 2ibc,\ b_z = -2iac \quad \text{and} \quad c_{\tilde{z}} = 2ikb.$$

The first two equations give $a^2 + b^2 = \lambda^2$, where λ is a constant. When $\lambda \neq 0$, we introduce the parametrization $a = \lambda \cos\theta$ and $b = \lambda \sin\theta$, we find that $c = \frac{i}{2}\theta_z$ and $\theta_{z\tilde{z}} = 4k\lambda \sin\theta$. Finally, by rescaling z and $\tilde{z}$, without loss of generality we take $k = \lambda = 1/2$. So the sine-Gordon reduction is

$$A_z = \frac{i\theta_z}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_w = \frac{1}{2} \begin{pmatrix} 0 & \exp(-i\theta) \\ \exp(i\theta) & 0 \end{pmatrix},$$

$$A_{\tilde{z}} = 0, \quad A_{\tilde{w}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\theta \equiv \theta(z, \tilde{z})$ solves the sine-Gordon equation

$$\theta_{z\tilde{z}} = \sin\theta. \tag{7}$$

In order to construct the Bäcklund transformation, we first construct J, and hence H and K. From $\partial_w H = -A_w H$ and $\partial_z H = -A_z H$, we have

$$H = \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cosh(w/2) & -\sinh(w/2) \\ -\sinh(w/2) & \cosh(w/2) \end{pmatrix} \widetilde{M}(\tilde{z}, \tilde{w})$$

and from $\partial_{\tilde{z}} K = -A_{\tilde{z}} K$ and $\partial_{\tilde{w}} K = -A_{\tilde{w}} K$, we have

$$K = \exp\left\{ -\frac{\tilde{w}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} M(z, w)^{-1},$$

where $M(z, w)$ and $\widetilde{M}(\tilde{z}, \tilde{w})$ are in $SL(2; \mathbb{C})$.

Substituting

$$J = F(\tilde{w}) \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{pmatrix} F(w)^{-1},$$

where

$$F(x) = \exp\left\{ \frac{x}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} = \begin{pmatrix} \cosh(x/2) & \sinh(x/2) \\ \sinh(x/2) & \cosh(x/2) \end{pmatrix}, \tag{8}$$

into equations (6) with

$$C = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad \widetilde{C} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix},$$

gives

$$\tilde{a}\partial_{\tilde{z}}(\hat{\theta} - \theta) = 2b \sin\left(\frac{\hat{\theta} + \theta}{2}\right), \quad \tilde{b}\partial_{\tilde{z}}(\hat{\theta} + \theta) = 2a \sin\left(\frac{\hat{\theta} - \theta}{2}\right),$$

$$a\partial_z(\hat{\theta} - \theta) = 2\tilde{b} \sin\left(\frac{\hat{\theta} + \theta}{2}\right), \quad b\partial_z(\hat{\theta} + \theta) = 2\tilde{a} \sin\left(\frac{\hat{\theta} - \theta}{2}\right).$$

Compatibility implies that either $a = \tilde{b} = 0$ or $b = \tilde{a} = 0$. These are the standard Bäcklund transformations for the sine-Gordon equation.

3.2 Reduction to the sixth Painlevé equation

Each of the six Painlevé equations,

$$u'' = 6u^2 + z, \tag{9}$$

$$u'' = 2u^3 + zu + \alpha, \tag{10}$$

$$u'' = \frac{1}{u}u'^2 - \frac{1}{z}u' + \frac{1}{z}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}, \tag{11}$$

$$u'' = \frac{1}{2u}u'^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - \alpha)u + \frac{\beta}{u}, \tag{12}$$

$$\begin{aligned} u'' &= \left\{\frac{1}{2u} + \frac{1}{u-1}\right\} u'^2 - \frac{1}{z}u' \\ &\quad + \frac{(u-1)^2}{z^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{z} + \frac{\delta u(u+1)}{u-1}, \end{aligned} \tag{13}$$

$$\begin{aligned} u'' &= \frac{1}{2}\left\{\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right\} u'^2 - \left\{\frac{1}{z} + \frac{1}{u-1} + \frac{1}{u-z}\right\} u' \\ &\quad + \frac{u(u-1)(u-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{u^2} + \frac{\gamma(z-1)}{(u-1)^2} + \frac{\delta z(z-1)}{(u-z)^2}\right\}, \end{aligned} \tag{14}$$

where α, β, γ and δ are constants, is known to be a reduction of the ASDYM equations with Lie algebra $\mathfrak{sl}(2;\mathbb{C})$ (Mason and Woodhouse [9, 10]). Here we will describe the reduction to the sixth Painlevé equation, $\mathrm{P_{VI}}$ (equation 14).

In terms of the variables $p = -\log w$, $q = -\log\tilde{z}$, $r = \log(\tilde{w}/\tilde{z})$, and $t = (z\tilde{z})/(w\tilde{w})$, we consider the reduction in which the connection one-form takes the form $A = P(t)\mathrm{d}p + Q(t)\mathrm{d}q + R(t)\mathrm{d}r$, where $P(t)$, $Q(t)$ and $R(t)$ are functions of t only. We have

$$\begin{aligned} \mathbf{A} &= A_z\,\mathrm{d}z + A_w\,\mathrm{d}w + A_{\tilde{z}}\,\mathrm{d}\tilde{z} + A_{\tilde{w}}\,\mathrm{d}\tilde{w} \\ &= P\mathrm{d}p + Q\mathrm{d}q + R\mathrm{d}r \\ &= -\frac{1}{w}P\mathrm{d}w - \frac{1}{\tilde{z}}Q\mathrm{d}\tilde{z} + R\left(\frac{\mathrm{d}\tilde{w}}{\tilde{w}} - \frac{\mathrm{d}\tilde{z}}{\tilde{z}}\right). \end{aligned}$$

Hence $zA_z = 0$, $wA_w = -P$, $\tilde{z}A_{\tilde{z}} = -(Q+R)$ and $\tilde{w}A_{\tilde{w}} = R$.

The ASDYM equations (1) reduce to the system of three matrix-valued ODEs

$$P' = 0, \quad tQ' = [R, Q] \quad \text{and} \quad t(1-t)R' = [tP + Q, R], \tag{15}$$

where prime denotes differentiation with respect to t. It follows from these equations that the traces of P^2, Q^2, R^2 and $(P+Q+R)^2$ are all constants. These are related to the constants α, β, γ and δ appearing in $\mathrm{P_{VI}}$. Furthermore, on introducing the scaled spectral parameter $\lambda = -\tilde{z}/(w\zeta)$ and taking $\Psi(z, w, \tilde{z}, \tilde{w}; \zeta) = \Phi(t; \lambda)$, we can extend this reduction to the Lax pair (2), giving

$$\begin{aligned}\partial_t \Phi &= -\left(\frac{R}{\lambda - t}\right)\Phi,\\ \partial_\lambda \Phi &= \left(\frac{Q}{\lambda} - \frac{P+Q+R}{\lambda - 1} + \frac{R}{\lambda - t}\right)\Phi.\end{aligned}$$

The second equation in (15) shows that there is an $SL(2;\mathbb{C})$-valued function of t, $G(t)$, and a constant $Q_0 \in \mathfrak{sl}(2;\mathbb{C})$, such that

$$Q(t) = G(t)^{-1} Q_0 G(t) \quad \text{and} \quad R(t) = -tG(t)^{-1}G'(t).$$

The form of J is then

$$J = \tilde{z}^{-Q_0} G(t) w^P.$$

From the last equation in (15), we see that the ASDYM equations in this reduction become

$$(1-t)(tG^{-1}G')' = [tP + G^{-1}Q_0 G, G^{-1}G'].$$

In the general case, we can take the constant matrices P and Q_0 to have the form

$$P = \frac{\theta_\infty}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad Q_0 = \frac{\theta_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So $J = U(\tilde{z})^{-1} G(t) V(w)$, where $t = \dfrac{z\tilde{z}}{w\tilde{w}}$ and

$$U(\tilde{z}) = \tilde{z}^{Q_0} = \begin{pmatrix} \tilde{z}^{\theta_0/2} & 0 \\ 0 & \tilde{z}^{-\theta_0/2} \end{pmatrix}, \qquad V(w) = w^P \begin{pmatrix} w^{\theta_\infty/2} & 0 \\ 0 & w^{-\theta_\infty/2} \end{pmatrix}.$$

The parameters in $\mathrm{P_{VI}}$ (equation 14) are given by $\alpha = \frac{1}{2}(\theta_\infty - 1)^2$, $\beta = -\frac{1}{2}\theta_0^2$, $\gamma = \mathrm{tr}\{(P+Q+R)^2\}$ and $\delta = \mathrm{tr}(R^2) + \frac{3}{2}$.

Muğan and Sakka [13] derived 12 Schlesinger transformations for P_{VI}. All of these Schlesinger transformations follow from the ASDYM Bäcklund transformation (6) with simple choices of C and $\widetilde{C}$. For example, the first

Schlesinger transformation in [13], which leaves the parameters γ and δ unchanged and acts on α and β via $\theta \mapsto \hat{\theta}_\infty = \theta_\infty + 1$ and $\theta \mapsto \hat{\theta}_0 = \theta_0 + 1$, corresponds to the choice

$$C_1 = w^{1/2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}_1 = \tilde{z}^{1/2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

A self-contained derivation of the Schlesinger transformations for the Painlevé equations will be published separately.

4 Bianchi permutability for the ASDYM equations

In terms of C and $\widetilde{C}$, the transformation (4) takes the form

$$\Psi \mapsto \hat{\Psi} = (S + \zeta T)\Psi = (\hat{H}\widetilde{C}H^{-1} + \zeta \hat{K} C K^{-1})\Psi.$$

Set $\Phi = K^{-1}\Psi$. Then, since $J = K^{-1}H$, we have

$$\Phi \mapsto \hat{\Phi} = \left(\hat{J}\widetilde{C}J^{-1} + \zeta C \right) \Phi.$$

Now suppose that we have two classes of BTs given by the pairs of functions $C^{(1)}$, $\widetilde{C}^{(1)}$ and $C^{(2)}$, $\widetilde{C}^{(2)}$. Now we impose the condition that these two Bäcklund transformations commute. To this end, let $J_{m,n}$ be a solution of equation (3), with corresponding eigenfunction $\Phi_{m,n}$. Suppose that under a type 1 Bäcklund transformation, the pair $(J_{m,n}, \Phi_{m,n})$ is mapped to $(J_{m+1,n}, \Phi_{m+1,n})$ and under a type 2 Bäcklund transformationit is mapped to $(J_{m,n+1}, \Phi_{m,n+1})$. Requiring that this can be done consistently for all integer m and n (i.e., requiring that a type 2 BT applied to $(J_{m+1,n}, \Phi_{m+1,n})$ results in the same solution as a type 1 BT applied to $(J_{m,n+1}, \Phi_{m,n+1})$, namely $(J_{m+1,n+1}, \Phi_{m+1,n+1})$) demands the compatibility of the system

$$\begin{aligned} \Phi_{m+1,n} &= \left(J_{m+1,n}\widetilde{C}^{(1)} J^{-1}_{m,n} + \zeta C^{(1)} \right) \Phi_{m,n}, \\ \Phi_{m,n+1} &= \left(J_{m,n+1}\widetilde{C}^{(2)} J^{-1}_{m,n} + \zeta C^{(2)} \right) \Phi_{m,n}. \end{aligned} \tag{16}$$

Compatibility gives

$$\begin{aligned} & J^{-1}_{m+1,n+1} \left(C^{(2)} J_{m+1,n} \widetilde{C}^{(1)} - C^{(1)} J_{m,n+1} \widetilde{C}^{(2)} \right) \\ & + \left(\widetilde{C}^{(2)} J^{-1}_{m+1,n} C^{(1)} - \widetilde{C}^{(1)} J^{-1}_{m,n+1} C^{(2)} \right) J_{m,n} = 0, \end{aligned} \tag{17}$$

together with $[C^{(1)}, C^{(2)}] = [\widetilde{C}^{(1)}, \widetilde{C}^{(2)}] = 0$. Equations (16) form a Lax pair for the Bianchi permutability equation (17).

Chau and Chinea [6] previously considered the Bianchi permutatbility for the special Bäcklund transformations derived in Prasad, Sinha and Chau Wang [17].

4.1 Sine-Gordon permutability as a reduction

Substituting

$$J_{m,n} = F(\tilde{w}) \begin{pmatrix} e^{i\theta_{m,n}/2} & 0 \\ 0 & e^{-i\theta_{m,n}/2} \end{pmatrix} F(w)^{-1},$$

where F is given by (8), into the permutability equation (17) with

$$C^{(j)} = b_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{C}^{(j)} = \tilde{a}_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives

$$\sin\left(\frac{\theta_{m+1,n} + \theta_{m,n}}{2}\right) - \sin\left(\frac{\theta_{m,n+1} + \theta_{m+1,n+1}}{2}\right) + \kappa\left[\sin\left(\frac{\theta_{m+1,n} + \theta_{m+1,n+1}}{2}\right) - \sin\left(\frac{\theta_{m,n+1} + \theta_{m,n}}{2}\right)\right] = 0.$$

This is equivalent to the standard form

$$\tan\left(\frac{\theta_{m+1,n+1} - \theta_{m,n}}{4}\right) = \frac{\kappa+1}{\kappa-1} \tan\left(\frac{\theta_{m,n+1} - \theta_{m+1,n}}{4}\right),$$

of the Bianchi permutability theorem for the sine-Gordon equation.

4.2 Non-autonomous ASDYM Bianchi system

A non-autonomous version of the Bianchi system can be derived by allowing the matrices $C^{(j)}$ and $\widetilde{C}^{(j)}$, $j = 1, 2$, to depend on m and n. To this end, we replace the system (16) with the system

$$\Phi_{m+1,n} = \left(J_{m+1,n}\widetilde{C}^{(1)}_{m,n}J^{-1}_{m,n} + \zeta C^{(1)}_{m,n}\right)\Phi_{m,n},$$
$$\Phi_{m,n+1} = \left(J_{m,n+1}\widetilde{C}^{(2)}_{m,n}J^{-1}_{m,n} + \zeta C^{(2)}_{m,n}\right)\Phi_{m,n}.$$

Compatibility gives

$$\begin{aligned} & J_{m+1,n+1}^{-1}\left(C_{m+1,n}^{(2)} J_{m+1,n} \widetilde{C}_{m,n}^{(1)} - C_{m,n+1}^{(1)} J_{m,n+1} \widetilde{C}_{m,n}^{(2)}\right) \\ & + \left(\widetilde{C}_{m+1,n}^{(2)} J_{m+1,n}^{-1} C_{m,n}^{(1)} - \widetilde{C}_{m,n+1}^{(1)} J_{m,n+1}^{-1} C_{m,n}^{(2)}\right) J_{m,n} = 0, \end{aligned} \tag{18}$$

where

$$C_{m,n+1}^{(1)} C_{m,n}^{(2)} = C_{m+1,n}^{(2)} C_{m,n}^{(1)} \tag{19}$$

and

$$\widetilde{C}_{m,n+1}^{(1)} \widetilde{C}_{m,n}^{(2)} = \widetilde{C}_{m+1,n}^{(2)} \widetilde{C}_{m,n}^{(1)}. \tag{20}$$

This is a nonautonomous version of the Bianchi system (17).

4.3 Reduction of Bianchi system

Let us consider the system (18–20) independently of any connection with the ASDYM equations. Let

$$C_{m,n}^{(1)} = \frac{1}{\alpha_m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{m,n}^{(2)} = \frac{1}{\beta_n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}_{m,n}^{(1)} = \widetilde{C}_{m,n}^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$J_{m,n} = \begin{pmatrix} u_{m,n} & 0 \\ 0 & 1/u_{m,n} \end{pmatrix}.$$

Then the nonautonomous Bianchi system (18) reduces to

$$\begin{aligned} & \alpha_m(u_{m,n}u_{m+1,n} - u_{m,n+1}u_{m+1,n+1}) \\ & - \beta_n(u_{m,n}u_{m,n+1} - u_{m+1,n}u_{m+1,n+1}) = 0. \end{aligned} \tag{21}$$

Equation (21) is known as the nonautonomous lattice mKdV equation [16]. In order to try to interpret this in terms of Bäcklund transformations, note that symmetry reductions of the ASDYM equations in the original variables (A_μ) lead to reductions in Yang's form where J has the "dressed" form $J = AGB$, where G depends on the variables that will be the independent variables of the reduced equations and the "dressing matrices" A and B depend on some auxiliary combination of variables that do not appear in the reduced equation.

Substituting the form $J_{m,n} = AG_{m,n}B$ into the nonautonomous Bianchi permutability equation (18) gives

$$G_{m+1,n+1}^{-1}\left(D_{m+1,n}^{(2)} G_{m+1,n} \widetilde{D}_{m,n}^{(1)} - D_{m,n+1}^{(1)} G_{m,n+1} \widetilde{D}_{m,n}^{(2)}\right)$$

$$+\left(\widetilde{D}^{(2)}_{m+1,n}G^{-1}_{m+1,n}D^{(1)}_{m,n}-\widetilde{D}^{(1)}_{m,n+1}G^{-1}_{m,n+1}D^{(2)}_{m,n}\right)G_{m,n}=0,$$

where $D^{(j)}=A^{-1}C^{(j)}A$ and $\widetilde{D}^{(j)}=B\widetilde{C}^{(j)}B^{-1}$.

Furthermore, the conditions

$$C^{(1)}_{m,n+1}C^{(2)}_{m,n}=C^{(2)}_{m+1,n}C^{(1)}_{m,n}\quad\text{and}\quad \widetilde{C}^{(1)}_{m,n+1}\widetilde{C}^{(2)}_{m,n}=\widetilde{C}^{(2)}_{m+1,n}\widetilde{C}^{(1)}_{m,n}$$

become

$$D^{(1)}_{m,n+1}D^{(2)}_{m,n}=D^{(2)}_{m+1,n}D^{(1)}_{m,n}\quad\text{and}\quad \widetilde{D}^{(1)}_{m,n+1}\widetilde{D}^{(2)}_{m,n}=\widetilde{D}^{(2)}_{m+1,n}\widetilde{D}^{(1)}_{m,n}.$$

Now we return to the problem of interpreting the nonautonomous lattice mKdV (21) in terms of Bäcklund transformations. The ASDYM Bianchi system (18) with

$$J_{m,n}=F(\tilde{w})\begin{pmatrix}u_{m,n}(z,\tilde{z}) & 0\\ 0 & \frac{1}{u_{m,n}(z,\tilde{z})}\end{pmatrix}F(w)^{-1}$$

and

$$C^{(1)}_{m,n}=\frac{1}{\alpha_m}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad C^{(2)}_{m,n}=\frac{1}{\beta_n}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \widetilde{C}^{(1)}_{m,n}=\widetilde{C}^{(2)}_{m,n}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

again gives the nonautonomous lattice mKdV (21). However this is now a statement about BTs of a reduction of the ASDYM equations (specifically the sine-Gordon equation with $u_{m,n}(z,\tilde{z})=\mathrm{e}^{i\theta_{m,n}(z,\tilde{z})/2}$). Ormerod [14] has shown that dmKdV has a reduction to $\mathrm{qP_{VI}}$.

5 Conclusion

Many integrable equations are known to be reductions of the ASDYM equations. We have derived a class of Bäcklund transformations (6) for these equations that depends on two functions $C(z,w)$ and $\widetilde{C}(\tilde{z},\tilde{w})$. This form is particularly useful when we consider reductions. In particular the Schlesinger transformations for the Painlevé equations follow from the standard reductions from the ASDYM equations together, where symmetry considerations restrict the possible choices for C and $\widetilde{C}$.

The Bianchi permutability of the Bäcklund transformations (6) results in the system (18). This is a discrete integrable system in its own right.

Through various reductions, this is a rich source of discrete systems. It also provides a way of identifying certain discrete equations as Bäcklund transformations of integrable differential systems. The richness of reductions of the ASDYM equations comes form the large (conformal) group of symmetries. The richness of discrete reductions of the ASDYM Bianchi system comes from the choices of $C(z, w)$, $\widetilde{C}(\tilde{z}, \tilde{w})$ as well as the form of J.

Acknowledgments

RH gratefully acknowledges support from the Engineering and Physical Science Research Council through grant EP/K041266/1.

References

[1] M. J. Ablowitz, S. Chakravarty and R. G. Halburd, Integrable systems and reductions of the self-dual Yang-Mills equations. *J. Math. Phys.* 44: 3147–3173 (2003).

[2] M. J. Ablowitz and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (LMS Lecture Note Series, 149. CUP, Cambridge, 1991)

[3] M. Bruschi, D. Levi and O. Ragnisco, Nonlinear partial differential equations and Bäcklund transformations related to the 4-dimensional self-dual Yang-Mills equations, *Lett. Nuovo Cimento* 33: 263–266 (1982).

[4] S. Chakravarty and M. J. Ablowitz, "On reduction of self-dual Yang-Mills equations, in Painlevé Transcendents, their Asymptotics and Applications, edited by D. Levi and P. Winternitz, Plenum, London, 1992.

[5] S. Chakravarty, M. J. Ablowitz and P. A. Clarkson, Reductions of self-dual Yang-Mills fields and classical systems, *Phys. Rev. Lett.* 65: 1085–1087 (1990).

[6] L.-L. Chau and F. J. Chinea, Permutability property for self-dual Yang-Mills fields. *Lett. Math. Phys.* 12: 189–192 (1986).

[7] E. F. Corrigan, D. B. Fairlie, R. G. Yates, and P. Goddard, The construction of self-dual solutions to SU(2) gauge theory, *Comm. Math. Phys.* 58: 223–240 (1978).

[8] L. Mason, S. Chakravarty and E. T. Newman, Bčklund transformations for the anti-self-dual Yang-Mills equations, *J. Math. Phys.* 29: 1005–1013 (1988).

[9] L. J. Mason and N. M. J. Woodhouse, Self-duality and the Painlevé transcendents, *Nonlinearity* 6: 569–581 (1993).

[10] L. J. Mason and N. M. J. Woodhouse, *Integrability, self-duality, and twistor theory* (London Mathematical Society Monographs. New Series, 15. OUP, New York, 1996).

[11] T. Masuda, The anti-self-dual Yang-Mills equation and classical transcendental solutions to the Painlevé II and IV equations, *J. Phys. A* 38: 6741–6757 (2005)

[12] T. Masuda, The anti-self-dual Yang-Mills equation and the Painlevé III equation, *J. Phys. A* 40: 14433–14445 (2007)

[13] U. Muğan and A. Sakka, Schlesinger transformations for Painlevé VI equation, *J. Math. Phys.* 36: 1284–1298 (1995).

[14] C. M. Ormerod, Reductions of lattice mKdV to q-P_{VI}. *Phys. Lett. A* 376: 2855–2859 (2012).

[15] C. J. Papachristou and B. Kent Harrison, Some aspects of the isogroup of the self-dual Yang-Mills system, *J. Math. Phys.* 28: 1261–1264 (1987).

[16] V. Papageorgiou, B. Grammaticos and A. Ramani, Integrable lattices and convergence acceleration algorithms, *Phys. Lett. A* 179: 111–115 (1993).

[17] M. K. Prasad, A. Sinha and L.-L. Chau Wang, Parametric Bäcklund transformation for self-dual SU(N) Yang-Mills fields, *Phys. Rev. Lett.* 43: 750–753 (1979).

[18] J. Tafel, A comparison of solution generating techniques for the self-dual Yang-Mills equations. II. *J. Math. Phys.* 31: 12341236 (1990).

[19] R. S. Ward, On self-dual gauge fields, *Phys. Lett. A* 61: 81–82 (1977).

[20] C. N. Yang, Condition of self-duality for SU(2) gauge fields on Euclidean four-dimensional space. *Phys. Rev. Lett.* 38: 1377–1379 (1977)