A LIPSCHITZ STABLE RECONSTRUCTION FORMULA FOR THE INVERSE PROBLEM FOR THE WAVE EQUATION

SHITAO LIU AND LAURI OKSANEN

Abstract. We consider the problem to reconstruct a wave speed $c \in C^\infty(M)$ in a domain $M \subset \mathbb{R}^n$ from acoustic boundary measurements modelled by the hyperbolic Dirichlet-to-Neumann map Λ. We introduce a reconstruction formula for c that is based on the Boundary Control method and incorporates features also from the complex geometric optics solutions approach. Moreover, we show that the reconstruction formula is locally Lipschitz stable for a low frequency component of c^{-2} under the assumption that the Riemannian manifold $(M, c^{-2}dx^2)$ has a strictly convex function with no critical points. That is, we show that for all bounded C^2 neighborhoods U of c, there is a C^1 neighborhood V of c and constants $C, R > 0$ such that

$$|\mathcal{F}\left(\widetilde{c}^{-2} - c^{-2}\right)(\xi)| \le Ce^{2R|\xi|} \left\|\widetilde{\Lambda} - \Lambda\right\|_*, \quad \xi \in \mathbb{R}^n,$$

for all $\widetilde{c} \in U \cap V$, where $\widetilde{\Lambda}$ is the Dirichlet-to-Neumann map corresponding to the wave speed $\widetilde{c}$ and $\|\cdot\|_*$ is a norm capturing certain regularity properties of the Dirichlet-to-Neumann maps.

1. Introduction

Let $M \subset \mathbb{R}^n$ be a compact set with nonempty interior and a smooth boundary ∂M and let $c \in C^\infty(M)$ be strictly positive. We consider the wave equation on M,

$$\begin{aligned}
&\partial_t^2 u(t,x) - c(x)^2 \Delta u(t,x) = 0, && (t,x) \in (0,\infty) \times M, \\
&u(t,x) = f(t,x), && (t,x) \in (0,\infty) \times \partial M, \\
&u|_{t=0}(x) = 0, \quad \partial_t u|_{t=0}(x) = 0, && x \in M.
\end{aligned} \tag{1}$$

Let us denote the solution of (1) by $u^f(t,x) = u(t,x)$ and let $T > 0$. We define the operator

$$\Lambda_{c,T} : f \mapsto \partial_\nu u^f|_{(0,T)\times\partial M}, \quad f \in C_0^\infty((0,T) \times \partial M),$$

1991 *Mathematics Subject Classification.* Primary: 35R30.

Key words and phrases. Inverse problems, wave equation, Lipschitz stability.

where ∂_ν is the Euclidean normal derivative on ∂M. Often we write $\Lambda_T := \Lambda_{c,T}$. The operator Λ_T models acoustic boundary measurements and is called the Dirichlet-to-Neumann operator. Let us denote by $C^\infty_+(M)$ the set of the strictly positive functions in $C^\infty(M)$. Then the inverse problem for the wave equation can be formulated as follows:

(IP) Reconstruct $c \in C^\infty_+(M)$ given the operator $\Lambda_{c,T}$.

The finite speed of propagation for the wave equation (1) gives a necessary condition for T in order to (IP) to have a unique solution. Indeed, if there is $x_0 \in M$ such that $T < 2d(x_0, \partial M)$, where d is the distance function of the Riemannian manifold $(M, c^{-2}dx^2)$, then the measurements Λ_T can not contain any information about $c(x_0)$. Conversely, the problem (IP) is known to be uniquely solvable for T strictly greater than the maximum of $2d(x, \partial M)$ for $x \in M$. The global uniqueness can be proven either by using the Boundary Control (BC) method originated from [7] or by using the complex geometric optics (CGO) solutions originated from [40]. However, a typical application of the BC method depends on Tataru's unique continuation theorem [41], whence only logarithmic type stability is expected for such an application. The CGO solutions based approach is also typically limited to logarithmic type stability [30].

Here we will introduce a global reconstruction method and prove that it is locally Lipschitz stable in the sense that we will describe below. The method is a modification of the BC method and employs also the harmonic exponential functions of the form

$$e^{i(\xi+i\eta)\cdot x/2}, \quad \xi, \eta \in \mathbb{R}^n,\ |\xi| = |\eta|,\ \xi \perp \eta,$$

that are CGO solutions for the Euclidean Laplacian.

Hölder stability with an exponent strictly better than $1/2$ allows an inverse problem to be solved locally by the nonlinear Landweber iteration [15]. Moreover, the convergence rate of the iteration is linear if and only if the problem is Lipschitz stable. Hence Lipschitz stability for (IP) would be important even without our explicit reconstruction method.

Hölder type stability results for (IP) were first obtained in [36, 37], and the best Hölder exponent available in the literature is $1/2$, see [9]. However, the Hölder exponent $1/2$ does not allow the convergence result [15] to be applied in a straightforward manner. Moreover, the technique in [9] does not give a global reconstruction method since it employs the geometric optics solutions corresponding to a fixed wave speed c_0, and requires c_0 to be known a priori.

The stability result in [9] depends on the assumption that the Riemannian manifold $(M, c_0^{-2}dx^2)$ is simple. Similarly, our result depends on an assumption that we call *stable observability* (see Definition 1 below) and that is also of geometrical nature. Let us also point out that Stefanov and Uhlmann [38] have considered the linear inverse problem to recover the initial value of a solution u to the wave equation in $\mathbb{R}^n$ given its trace on $\partial\Omega \times (0,T)$ where $\Omega \subset \mathbb{R}^n$. They prove that reconstruction of $u(0)$ supported in Ω can not be Lipschitz stable if there is a geodesic γ such that $\gamma(0) \in \Omega$ and $\gamma([-T,T]) \cap \partial\Omega = \emptyset$. Thus it would be unexpected if the non-linear inverse problem (IP) was locally Lipschitz stable without additional assumptions on the geometry $(M, c^{-2}dx^2)$.

1.1. **Statement of the main results.** We recall that the wave equation (1) is said to be continuously observable from open $\Gamma \subset \partial M$ in time $T > 0$ if there is $C_{obs} > 0$ such that

$$\|u(T)\|_{H_0^1(M)} + \|\partial_t u(T)\|_{L^2(M)} \le C_{obs} \|\partial_\nu u\|_{L^2((0,T)\times\Gamma)}, \tag{2}$$

where u is a solution of the wave equation

$$\begin{aligned} &\partial_t^2 u - c^2 \Delta u = 0, && \text{in } (0,T) \times M, \\ &u|_{(0,T)\times\partial M} = 0, && \text{in } (0,T) \times \partial M. \end{aligned} \tag{3}$$

The condition by Bardos, Lebeau and Rauch gives a geometric characterization of the continuous observability [6, 14]. In particular, if $\Gamma = \partial M$ and $(M, c^{-2}dx^2)$ is non-trapping then the continuous observability (2) is valid. This is analogous with the condition in the above mentioned [38]. We refer to [6] for the precise formulation of the geometric condition.

Definition 1. *Let $U \subset C_+^\infty(M)$. We say that the wave equations (1) are* stably observable *for $c \in U$, from open $\Gamma \subset \partial M$ in time $T > 0$, if there is $C_{obs} > 0$ satisfying the following: for all $c \in U$ the solutions of the wave equation (3) satisfy the observability inequality (2).*

This stronger form of observability does not follow from the technique in [6] since the compactness-uniqueness argument there does not bound the constant C_{obs} in terms of the geometry $(M, c^{-2}dx^2)$. However, we will prove the following theorem.

Theorem 1. *Let $c \in C_+^\infty(M)$ and suppose that there is a strictly convex function $\ell \in C^3(M)$ with respect to the metric tensor $c^{-2}dx^2$, and that ℓ has no critical points. Let $U \subset C_+^\infty(M)$ be bounded in $C^2(M)$ and let $\Gamma \subset \partial M$ be a neighborhood of*

$$\{x \in \partial M;\ \nabla\ell(x) \cdot \nu \ge 0\}.$$

Then there is a neighborhood V of c in $C^1(M)$ and $T > 0$ such that the wave equations (1) are stably observable for the wave speeds in the set $U \cap V$, from Γ in time T.

We will prove Theorem 1 in Section 4 also for anisotropic wave speeds by first deriving a geometric Carleman estimate for the wave equation. The main feature of the estimate is the absence of lower order terms, and we will follow the tradition of this type of estimates, see [25, 27, 42, 43], where the continuous observability was studied but the dependence of the constant C_{obs} on the coefficients of the equation was not considered; and [16], where the dependence of C_{obs} on the zeroth order coefficient of the equation was studied.

There are non-simple Riemannian manifolds (M, g) that admit strictly convex functions with no critical points (a trivial example being a non-convex subset of the Euclidean space). See [19] and [31] for further discussion on the relations between simplicity, the existence of a strictly convex function and the characterization by Bardos, Lebeau and Rauch.

To formulate our main result, let us recall that the Dirichlet-to-Neumann operator is continuous,

$$\Lambda_T : H^1_{cc}((0,T) \times \partial M) \to L^2((0,T) \times \partial M),$$

where $H^1_{cc}((0,T) \times \partial M) := \{f \in H^1((0,T) \times \partial M);\ f(0,x) = 0\}$, see [26]. Moreover, in Section 2 we show that Λ_{2T} has the additional regularity-symmetry property,

$$K(\Lambda_{2T}) : L^2((0,T) \times \partial M) \to L^2((0,T) \times \partial M), \tag{4}$$

where $K(\Lambda_{2T}) := R\Lambda_T RJ\Theta - J\Lambda_{2T}\Theta$, R is the time reversal on $(0,T)$, that is $Rf(t) := f(T-t)$, Θ is the extension by zero from $(0,T)$ to $(0,2T)$ and

$$Jf(t) := \frac{1}{2}\int_t^{2T-t} f(s)ds, \quad f \in L^2(0,2T),\ t \in (0,T).$$

The additional regularity can be understood by noticing that $K(\Lambda)$ corresponds formally to the operator $(\Lambda^* - \Lambda)J + [\Lambda, J]$, where $\Lambda = \Lambda_{2T}$ and J are operators of orders -1 and zero, respectively. We denote $\Upsilon := (0,T) \times \partial M$ and define

$$\|\Lambda_{2T}\|_* := \|K(\Lambda_{2T})\|_{L^2(\Upsilon)\to L^2(\Upsilon)} + \|\Lambda_T\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)}\,.$$

We will show that $\|\cdot\|_*$ is a norm in the appendix below. Our main result is the following.

Theorem 2. *Suppose that the wave equations (1) are stably observable for the wave speeds in a set $U \subset C_+^\infty(M)$, from ∂M in time $T > 0$. Suppose, furthermore, that there is $\epsilon_U > 0$ such that*

$$c(x) \geq \epsilon_U, \quad \text{for all } x \in M \text{ and } c \in U.$$

Let $R > 0$ satisfy $M \subset B(0,R)$ and let $c \in U$. Then there is $C > 0$ depending on M, T, c, ϵ_U and C_{obs} such that for all $\widetilde{c} \in U$

$$|\mathcal{F}\left(\widetilde{c}^{-2} - c^{-2}\right)(\xi)| \leq Ce^{2R|\xi|}\left\|\Lambda_{\widetilde{c},2T} - \Lambda_{c,2T}\right\|_*, \quad \xi \in \mathbb{R}^n, \tag{5}$$

where $\mathcal{F}(\rho)$, $\rho \in C^\infty(M)$, denotes the Fourier transform of the extension by zero of ρ onto $\mathbb{R}^n$.

The estimate (5) gives Lipschitz stability for low frequencies. However, because of the exponential constant in the estimate, high frequencies are reconstructed only with logarithmic stability. We have the following corollary that appears to cover some geometrical cases where no previous logarithmic estimates were available.

Corollary 1. *Let $U \subset C_+^\infty(M)$ satisfy the assumptions of Theorem 2 and suppose moreover that $c = \widetilde{c}$ near ∂M for all $c, \widetilde{c} \in U$ and that U is bounded in $C^1(M)$. Then for all $c \in U$ there is $C > 0$ such that*

$$\left\|\widetilde{c}^{-2} - c^{-2}\right\|_{L^2(M)} \leq C|\log(\|\Lambda_{\widetilde{c},2T} - \Lambda_{c,2T}\|_*)|^{-2}, \quad \widetilde{c} \in U.$$

1.2. **Previous literature.** We refer to the monograph [24] and to the review article [8] for literature concerning the BC method, and to the review article [44] concerning the CGO solutions. A stability result without a modulus of continuity for the former approach was proved in [4] and the first logarithmic type stability result for the latter in [1]. Hölder type stability results for (IP) are proved in the above mentioned articles [36, 37, 9].

There has been recent interest in results showing that the ill-posedness of the inverse problem for the Helmholtz equation decreases when the frequency increases, see [22, 32] and the references therein. Moreover, in a recent preprint [3], Lipschitz stability for determining the low frequency component of a potential in the inverse scattering problem was established. This result is similar in spirit with the present one but it is based on different techniques.

As for recovering the potential in a wave equation, Hölder type stability result was first established in [39]. This was improved later to an almost Lipschitz stability result with the Hölder exponent being $1-\epsilon$ in [5]. Lipschitz type stability can be obtained if the potential is assumed to be parametrized in a finite dimensional space [33]. See also [2, 34] for a Lipschitz stability result with finite number of parameters.

The inverse problem in the present paper is formulated with many boundary measurements. On the other hand, for a different formulation of the inverse problem with a single measurement, Lipschitz type stability results can be achieved. For example, [35] proved Lipschitz stability for recovering the sound speed from a single measurement in the context of multi-wave imaging. However, such formulation typically requires non-vanishing initial data which is not in favor of practical applications if only acoustic waves are used for imaging. The main methodology used in the inverse problems for the wave equation with a single measurement is based on Carleman type estimates. The technique was originated in [13] and has been developed tremendously since then. In particular, the continuous observability inequality (2) may be used to derive the Lipschitz type stability. For more details about the single measurement formulation, we refer to [20, 21, 29] and the references therein.

Let us also point out that the continuous observability inequality (2) is equivalent to the exact controllability of the wave equation, i.e. the surjectivity of the control to solution map. This well-known link to the control theory has been well studied since 1980s and we refer to the review articles [28, 18] and the references therein for more details on the exact controllability of wave equations.

2. A modification of the Boundary Control method

Let $\rho \in C^\infty(M)$ and let us extend ρ by zero to $\mathbb{R}^n$. We denote by $\mathcal{F}(\rho)$ the Fourier transform of the extension. Moreover, let us define the operator

$$B(\Lambda_{c,T}) := R\Lambda_{c,T} R I \mathcal{T}_0 - I\mathcal{T}_1,$$

where R is again the time reversal operator $Rf(t) = f(T-t)$, $\mathcal{T}_j$, $j = 0, 1$, are the first two traces on ∂M, that is, $\mathcal{T}_0 \phi = \phi|_{\partial M}$ and $\mathcal{T}_1 \phi = \partial_\nu \phi|_{\partial M}$, and

$$If(t) := \int_t^T f(s) ds, \quad f \in L^2(0,T),\ t \in (0,T).$$

In this section we will prove the following reconstruction formula.

Theorem 3. *Let $c \in C_+^\infty(M)$ and suppose that the wave equation (1) is continuously observable from ∂M in time $T > 0$. Let $\xi, \eta \in \mathbb{R}^n$ satisfy $|\xi| = |\eta|$ and $\xi \perp \eta$, and define the functions*

$$\phi_{\xi,\eta}(x) := e^{i(\xi + i\eta)\cdot x/2}, \quad \psi_{\xi,\eta}(x) := e^{i(\xi - i\eta)\cdot x/2}.$$

Then

$$\mathcal{F}\left(c^{-2}\right)(\xi) = (K(\Lambda_{c,2T})^{\dagger} B(\Lambda_{c,T})\phi_{\xi,\eta}, B(\Lambda_{c,T})\psi_{\xi,\eta})_{L^2((0,T)\times\partial M)}, \tag{6}$$

where $K(\Lambda_{c,2T})^{\dagger}$ *is the pseudoinverse of* $K(\Lambda_{c,2T})$.

We refer to [17] and [10] for the definition and general theory of pseudoinverse operators.

By inspecting the proof of Theorem 3 we see that if $\Gamma \subset \partial M$ is an open set such that the wave equation (1) is continuously observable from Γ in time $T > 0$, and we consider the restriction

$$\Lambda_{c,2T,\Gamma} : f \mapsto \partial_\nu u^f|_{(0,2T)\times\Gamma}, \quad f \in C_0^\infty((0,2T)\times\Gamma),$$

then the pseudoinverse of $K(\Lambda_{c,2T,\Gamma})$ is a bounded operator on the space $L^2((0,T)\times\Gamma)$. However, the formula (6) does not generalize straightforwardly to the partial data case where only the restriction $\Lambda_{c,2T,\Gamma}$ is known, since the harmonic exponentials $\phi_{\xi,\eta}$ and $\psi_{\xi,\eta}$ have traces that are supported on the whole boundary.

2.1. Blagoveščenskiĭ type identities.

The BC method is based on the following identity that originates from [12],

$$(u^f(T), u^h(T))_{L^2(M;c^{-2}(x)dx)} = (f, K(\Lambda_{c,2T})h)_{L^2((0,T)\times\partial M)}, \tag{7}$$

where $f, h \in C_0^\infty((0,T)\times\partial M)$, $T > 0$ and u^f denotes the solution of (1). The formulation of the identity (7) by using the operator $K(\Lambda_{c,2T})$ is from [11]. Let us also mention that the iterative time-reversal control method introduced there can be adapted to give an efficient implementation of the reconstruction formula (6).

We define the map $W_{c,T} f := u^f(T)$. Then it is well-known, see e.g. [26], that this control to solution map

$$W_{c,T} : L^2((0,T)\times\partial M) \to L^2(M).$$

is continuous. To simplify the notation, we often write $K := K(\Lambda_{c,2T})$, $B := B(\Lambda_{c,T})$ and $W := W_{c,T}$. Then (7) yields $K = W^* W$, and K extends as a continuous operator on $L^2((0,T)\times\partial M)$ by the L^2 continuity of W.

Lemma 1. *Let* $f \in C_0^\infty((0,T)\times\partial M)$ *and let* $\phi \in C^\infty(M)$ *be harmonic. Then*

$$(u^f(T), \phi)_{L^2(M;c^{-2}(x)dx)} = (f, B\phi)_{L^2((0,T)\times\partial M)}. \tag{8}$$

In particular, B *is the restriction of* W^* *on harmonic functions.*

Proof. Let $t \in (0,T)$. Then integration by parts gives

$$\begin{aligned}&\partial_t^2 (u^f(t), \phi)_{L^2(M;c^{-2}(x)dx)} \\ &\quad = (\Delta u^f(t), \phi)_{L^2(M;dx)} - (u^f(t), \Delta\phi)_{L^2(M;dx)} \\ &\quad = (\Lambda_T f(t), \phi)_{L^2(\partial M)} - (f(t), \partial_\nu \phi)_{L^2(\partial M)}.\end{aligned}$$

By solving this differential equation with vanishing initial conditions at $t = 0$ we get

$$\begin{aligned}&(u^f(T), \phi)_{L^2(M;c^{-2}(x)dx)} \\ &\quad = \int_0^T \int_0^s (\Lambda_T f(t), \phi)_{L^2(\partial M)} - (f(t), \partial_\nu \phi)_{L^2(\partial M)} dt ds \\ &\quad = (\mathcal{I}\Lambda_T f, \phi)_{L^2((0,T)\times\partial M)} - (\mathcal{I} f, \partial_\nu \phi)_{L^2((0,T)\times\partial M)},\end{aligned}$$

where $\mathcal{I} f(s) := \int_0^s f(t) dt$. The equation (8) follows since $\Lambda_T^* = R\Lambda_T R$ and $\mathcal{I}^* = I$. $\square$

2.2. **Computation of boundary controls.** Let $\phi \in L^2(M)$ and consider the control equation

$$W f = \phi, \quad \text{for } f \in L^2((0,T) \times \partial M). \tag{9}$$

Typically W is not injective and we can hope to solve (9) only in the sense of the pseudoinverse $f = W^\dagger \phi$. It is well-known that the pseudoinverse $W^\dagger$ is a bounded operator if and only if W has closed range $R(W)$. If $T > 0$ is large enough then $R(W)$ is dense in $L^2(M)$ by Tataru's unique continuation [41]. Hence the pseudoinverse $W^\dagger$ is a bounded operator if and only if W is surjective. It is well-known, see e.g. [6], that the map

$$f \mapsto (u^f(T), \partial_t u(T)) : L^2((0,T) \times \partial M) \to L^2(M) \times H^{-1}(M)$$

is surjective if and only if the continuous observability (2) holds with $\Gamma = \partial M$. Let us now assume that (2) holds. Then

$$W^\dagger : L^2(M) \to L^2((0,T) \times \partial M)$$

is continuous and $WW^\dagger$ is the identity operator.

The pseudoinverse can be written as $W^\dagger = (W^*W)^\dagger W^*$. Moreover, $R(K) = R(W^*W) = R(W^*)$ is closed since $R(W)$ is closed. Hence $K^\dagger$ is continuous on $L^2((0,T)\times\partial M)$ and $KK^\dagger$ is the orthogonal projection onto $R(K)$. In particular, if $\phi \in C^\infty(M)$ is harmonic, then $W^\dagger \phi = K^\dagger B \phi$ can be computed from the boundary data Λ_{2T}.

Let $\phi, \psi \in C^\infty(M)$ be harmonic. Then the identity (8) yields

$$(\phi,\psi)_{L^2(M;c^{-2}(x)dx)} = (WW^\dagger\phi,\psi)_{L^2(M;c^{-2}(x)dx)}$$
$$= (W^\dagger\phi, B\psi)_{L^2((0,T)\times\partial M)} = (K^\dagger B\phi, B\psi)_{L^2((0,T)\times\partial M)}.$$

Notice that the functions $\phi := \phi_{\xi,\eta}$ and $\psi := \psi_{\xi,\eta}$ defined in Theorem 3 are harmonic and $\phi(x)\psi(x) = e^{i\xi\cdot x}$. Thus

$$\mathcal{F}\left(c^{-2}\right)(\xi) = (\phi,\psi)_{L^2(M;c^{-2}(x)dx)} = (K^\dagger B\phi, B\psi)_{L^2((0,T)\times\partial M)}.$$

This proves Theorem 3.

3. Stability of the reconstruction

Let us assume that $c \in C^\infty_+(M)$, $T > 0$ and $\Gamma = \partial M$ satisfy (2). We denote $\Upsilon := (0,T)\times\partial M$. Notice that $W^*\phi = \partial_\nu u|_\Upsilon$, where u is the solution of the wave equation (3) with $(u, \partial_t u) = (0, \phi)$ as the initial data at $t = T$. Hence we have

$$\|\phi\|_{L^2(M)} \le C_{obs}\,\|W^*\phi\|_{L^2(\Upsilon)}\,.$$

In particular, W^* is an injection and $(W^*)^{-1} : R(W^*) \to L^2(M)$ satisfies $\|(W^*)^{-1}\|_{R(W^*)\to L^2(M)} \le C_{obs}$. Notice that

$$(W^\dagger)^* = (W^*)^\dagger = (W^*)^{-1}P_{R(W^*)},$$

where $P_{R(W^*)}$ is the orthogonal projection onto $R(W^*)$. Hence

$$\left\|W^\dagger\right\|_{L^2(M)\to L^2(\Upsilon)} = \left\|(W^*)^\dagger\right\|_{L^2(\Upsilon)\to L^2(M)} \le C_{obs}.$$

Moreover, $(W^*W)^\dagger = W^\dagger(W^*)^\dagger$, which implies

$$\left\|K^\dagger\right\|_{L^2(\Upsilon)\to L^2(\Upsilon)} = \left\|(W^*W)^\dagger\right\|_{L^2(\Upsilon)\to L^2(\Upsilon)} \le C_{obs}^2.$$

We are now ready to prove Theorem 2 formulated in the introduction.

Proof of Theorem 2. Let $c, \widetilde{c} \in U$ and denote $\Lambda_T = \Lambda_{c,T}$, and $\widetilde{\Lambda}_T = \Lambda_{\widetilde{c},T}$ and let us define W, $\widetilde{W}$, K, $\widetilde{K}$, B and $\widetilde{B}$ analogously. From now on we will omit writing $L^2(\Upsilon)$ as a subscript. We have, see e.g. [23],

$$\left\|\widetilde{K}^\dagger - K^\dagger\right\| \le 3\max(\left\|\widetilde{K}^\dagger\right\|^2, \left\|K^\dagger\right\|^2)\left\|\widetilde{K} - K\right\| \le 3C_{obs}^4\left\|\widetilde{K} - K\right\|.$$

Notice that for $\phi \in C^\infty(M)$ the function

$$(RI\mathcal{T}_0\phi)(t,x) = t\phi(x), \quad t \in [0,T],\ x \in \partial M,$$

is in $H^1_{cc}(\Upsilon)$. Thus there is $C_0 > 0$ depending only on T and M such that

$$\begin{aligned}\left\|(\widetilde{B} - B)\phi\right\| &= \left\|(\widetilde{\Lambda}_T - \Lambda_T)RI\mathcal{T}_0\phi\right\| \\ &\le C_0 \left\|\widetilde{\Lambda}_T - \Lambda_T\right\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)} \|\phi\|_{C^1(\partial M)}.\end{aligned}$$

Moreover, if $\phi \in C^\infty(M)$ is harmonic then

$$\|B\phi\| = \|W^*\phi\| \le C_c \|\phi\|_{L^2(M;c^{-2}dx)},$$

where we have denoted $C_c := \|W^*\|_{L^2(M)\to L^2(\Upsilon)}$.

Let $\phi, \psi \in C^\infty(M)$ be harmonic. Then

$$\begin{aligned}|(\phi,\psi)_{L^2(M;\widetilde{c}^{-2}dx)} - (\phi,\psi)_{L^2(M;c^{-2}dx)}| &\le |((\widetilde{K}^\dagger - K^\dagger)B\phi, B\psi)| \\ &+ |(\widetilde{K}^\dagger B\phi, (\widetilde{B} - B)\psi)| + |(\widetilde{K}^\dagger(\widetilde{B} - B)\phi, \widetilde{B}\psi)|.\end{aligned}$$

We have

$$\begin{aligned}&|((\widetilde{K}^\dagger - K^\dagger)B\phi, B\psi)| \\ &\quad \le 3C_{obs}^4 C_c^2 \left\|\widetilde{K} - K\right\| \|\phi\|_{L^2(M;c^{-2}dx)} \|\psi\|_{L^2(M;c^{-2}dx)}, \\ &|(\widetilde{K}^\dagger B\phi, (\widetilde{B} - B)\psi)| \\ &\quad \le C_{obs}^2 C_c C_0 \left\|\widetilde{\Lambda}_T - \Lambda_T\right\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)} \|\phi\|_{L^2(M;c^{-2}dx)} \|\psi\|_{C^1(\partial M)}.\end{aligned}$$

Notice that $(\widetilde{K}^\dagger)^* = \widetilde{K}^\dagger$ since $\widetilde{K}^* = \widetilde{K}$. Moreover, as $\widetilde{K}^\dagger \widetilde{B}\psi = \widetilde{W}^\dagger \phi$, we have

$$\begin{aligned}|(\widetilde{K}^\dagger(\widetilde{B} - B)\phi, \widetilde{B}\psi)| &= |((\widetilde{B} - B)\phi, \widetilde{W}^\dagger\psi)| \\ &\le C_0 C_{obs} \left\|\widetilde{\Lambda}_T - \Lambda_T\right\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)} \|\phi\|_{C^1(\partial M)} \|\psi\|_{L^2(M;\widetilde{c}^{-2}dx)}.\end{aligned}$$

Hence there is a constant $C > 0$ depending on M, T, c, ϵ_U and C_{obs} such that for all $\widetilde{c} \in U$ and harmonic $\phi, \psi \in C^1(M)$

(10)

$$\begin{aligned}&|(\phi,\psi)_{L^2(M;\widetilde{c}^{-2}dx)} - (\phi,\psi)_{L^2(M;c^{-2}dx)}| \\ &\quad \le C\left(\left\|\widetilde{K} - K\right\| + \left\|\widetilde{\Lambda}_T - \Lambda_T\right\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)}\right) \|\phi\|_{C^1(M)} \|\psi\|_{C^1(M)}.\end{aligned}$$

Let $\xi, \eta \in \mathbb{R}^n$ satisfy $|\xi| = |\eta|$ and define $\phi(x) := e^{i(\xi+i\eta)\cdot x/2}$. Then $|\phi(x)| \le e^{R|\xi|/2}$ and

$$|\partial_{x^j}\phi(x)| \le (|\xi_j| + |\eta_j|)e^{R|\eta|/2} \le 2|\xi|e^{R|\xi|/2} \le C_R e^{R|\xi|},$$

where R is the radius of a ball $B(0,R)$ that contains M and $C_R > 0$ is a constant depending only on R. Hence (10) implies that there is $C > 0$ such that

$$|\mathcal{F}\left(\widetilde{c}^{-2} - c^{-2}\right)(\xi)| \le Ce^{2R|\xi|} \left\|\widetilde{\Lambda}_{2T} - \Lambda_{2T}\right\|_*, \quad \xi \in \mathbb{R}^n,\ \widetilde{c} \in U.$$

□

4. Stable observability

In this section we will prove Theorem 1 formulated in the introduction. As the proof is of geometric nature, we will consider the wave equation

$$\begin{aligned} &\partial_t^2 u - \Delta_{g,\mu} u = 0, && \text{in } (0,T)\times M,\\ &u|_{(0,T)\times\partial M} = 0, && \text{in } (0,T)\times\partial M. \end{aligned}$$

on a smooth compact Riemannian manifold (M,g) with boundary. Here $\Delta_{g,\mu}$ is the weighted Laplace-Beltrami operator,

$$\Delta_{g,\mu} u = \mu^{-1}\operatorname{div}_g(\mu\nabla_g u),$$

where $\mu \in C^\infty(M)$ is strictly positive and div_g and ∇_g denote the divergence and the gradient with respect to the metric tensor g. To prove Theorem 1 we will apply the results proven in this section to $g = c(x)^{-2}dx^2$ and $\mu(x) = c(x)^{n-2}$. Then $\Delta_{g,\mu} = c(x)^2\Delta$, where Δ is the Euclidean Laplacian.

We denote by $|\cdot|_g$, $(\cdot,\cdot)_g$, dV_g, dS_g, ν_g and D_g^2 the norm, the inner product, the volume and the surface measures, the exterior unit normal vector and the Hessian with respect to g, and will omit writing the subscript g when considering a fixed Riemannian metric tensor. We recall that the Hessian satisfies

$$D^2w(X,Y) = (D_X\nabla w, Y), \quad D^2w(X,Y) = D^2w(Y,X),$$

where $w \in C^2(M)$, $X, Y \in TM$ and D_X is the covariant derivative with respect to g. We denote $\operatorname{div}_\mu X := \mu^{-1}\operatorname{div}(\mu X)$ and have the formula

$$\operatorname{div}_\mu(wX) = (\nabla w, X) + w\operatorname{div}_\mu X.$$

We will obtain a stable observability inequality from a Carleman estimate similar to that in [43].

Lemma 2. *Let $\ell \in C^2(M)$, $\psi \in C^1(M)$ and let $w \in C^2(\mathbb{R}\times M)$. We denote $\phi := \Delta_\mu \ell - \psi$ and $\widetilde{q} := \phi - |\nabla\ell|^2$ and define*

$$v := e^\ell w, \quad \vartheta := \psi v + 2(\nabla v, \nabla\ell), \quad Y := ((\partial_t v)^2 + |\nabla v|^2 - \widetilde{q}v^2)\nabla\ell.$$

Then

$$
\begin{aligned}
(11)\qquad & e^{2\ell}(\partial_t^2 w - \Delta_\mu w)^2/2 - \partial_t(\vartheta \partial_t v) + \operatorname{div}_\mu(\vartheta \nabla v) + \operatorname{div}_\mu Y \\
&\quad = (\partial_t^2 v - \Delta_\mu v + \widetilde{q} v)^2/2 + \vartheta^2/2 \\
&\qquad + \phi(\partial_t v)^2 - \phi|\nabla v|^2 + 2D^2\ell(\nabla v, \nabla v) \\
&\qquad + v(\nabla\psi, \nabla v) + (\widetilde{q}\psi - \operatorname{div}_\mu(\widetilde{q}\nabla\ell))v^2.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\widetilde{q}\psi - \operatorname{div}_\mu(\widetilde{q}\nabla\ell) &= \phi|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell) \\
&\quad + \phi\psi - \operatorname{div}_\mu(\phi\nabla\ell).
\end{aligned}
$$

Proof. Notice that

$$
\begin{aligned}
\Delta_\mu v &= (\nabla v, \nabla\ell) + v\Delta_\mu \ell + (\nabla e^\ell, \nabla w) + e^\ell \Delta_\mu w, \\
&= 2(\nabla v, \nabla \ell) + (\Delta_\mu \ell - |\nabla\ell|^2)v + e^\ell \Delta_\mu w.
\end{aligned}
$$

We expand the square

$$
\begin{aligned}
(12)\qquad e^{2\ell}(\partial_t^2 w - \Delta_\mu w)^2 &= (\partial_t^2 v - \Delta_\mu v + \widetilde{q} v + \vartheta)^2 \\
&= (\partial_t^2 v - \Delta_\mu v + \widetilde{q} v)^2 + \vartheta^2 + 2(\partial_t^2 v - \Delta_\mu v + \widetilde{q} v)\vartheta,
\end{aligned}
$$

and study the cross terms. We have

(13)

$$
\begin{aligned}
\vartheta\Delta_\mu v &= \operatorname{div}_\mu(\vartheta\nabla v) - (\nabla\vartheta, \nabla v) \\
&= \operatorname{div}_\mu(\vartheta\nabla v) - v(\nabla\psi, \nabla v) - \psi|\nabla v|^2 - 2(\nabla(\nabla v, \nabla\ell), \nabla v), \\
&= \operatorname{div}_\mu(\vartheta\nabla v - |\nabla v|^2\nabla\ell) - v(\nabla\psi, \nabla v) + (\Delta_\mu\ell - \psi)|\nabla v|^2 \\
&\quad - 2D^2\ell(\nabla v, \nabla v),
\end{aligned}
$$

since

$$
\begin{aligned}
2(\nabla(\nabla v, \nabla\ell), \nabla v) &= 2(D_{\nabla v}\nabla v, \nabla\ell) + 2(\nabla v, D_{\nabla v}\nabla\ell) \\
&= 2D^2 v(\nabla v, \nabla\ell) + 2D^2\ell(\nabla v, \nabla v),
\end{aligned}
$$

and $2D^2 v(\nabla v, \nabla\ell) = \operatorname{div}_\mu(|\nabla v|^2\nabla\ell) - |\nabla v|^2\Delta_\mu\ell$. The two remaining cross terms can be manipulated similarly, and we have

$$
\vartheta\partial_t^2 v = \partial_t(\vartheta\partial_t v) - \operatorname{div}_\mu((\partial_t v)^2\nabla\ell) + (\Delta_\mu\ell - \psi)(\partial_t v)^2, \tag{14}
$$

$$
\vartheta\widetilde{q} v = (\widetilde{q}\psi - \operatorname{div}_\mu(\widetilde{q}\nabla\ell))v^2 + \operatorname{div}_\mu(v^2\widetilde{q}\nabla\ell). \tag{15}
$$

The first claim follows by inserting (13), (14) and (15) into (12).

For the second claim notice that

$$
\begin{aligned}
\widetilde{q}\psi - \operatorname{div}_\mu(\widetilde{q}\nabla\ell) &= (\phi - |\nabla\ell|^2)\psi - \operatorname{div}_\mu((\phi - |\nabla\ell|^2)\nabla\ell) \\
&= (\Delta_\mu \ell - \psi)|\nabla\ell|^2 + 2D^2\ell(\nabla\ell, \nabla\ell) \\
&\quad + \phi\psi - \operatorname{div}_\mu(\phi\nabla\ell).
\end{aligned}
$$

□

Corollary 2 (Pointwise Carleman inequality)**.** *Let $\ell \in C^3(M)$ and $\rho > 0$ satisfy*

$$
D^2\ell(X,X) \geq \rho|X|^2, \quad X \in T_x M,\ x \in M. \tag{16}
$$

Let $\tau > 0$ and $w \in C^2(\mathbb{R} \times M)$. We define

$$
\begin{aligned}
&v := e^{\tau\ell} w, \quad \vartheta := \tau((\Delta_\mu \ell - \rho)v + 2(\nabla v, \nabla\ell)), \\
&Y := \tau((\partial_t v)^2 + |\nabla v|^2 - (\tau\rho - \tau^2|\nabla\ell|^2)v^2)\nabla\ell.
\end{aligned}
$$

Then

$$
\begin{aligned}
&e^{2\tau\ell}(\partial_t^2 w - \Delta_\mu w)^2/2 - \partial_t(\vartheta\partial_t v) + \operatorname{div}_\mu(\vartheta\nabla v) + \operatorname{div}_\mu Y \\
&\quad \geq e^{2\tau\ell}(\rho\tau - 1)((\partial_t w)^2 + |\nabla w|^2)/2 + e^{2\tau\ell}(2\rho|\nabla\ell|^2\tau - C_1)\tau^2 w^2,
\end{aligned}
$$

where $C_1 = \rho^2 + \max_{x\in M} |\nabla(\Delta_\mu \ell(x))|^2$.

Proof. We invoke Lemma 2 with ℓ replaced by $\tau\ell$ and $\psi = \tau(\Delta_\mu\ell - \rho)$. Then $\phi = \tau\rho$. Notice that the two first terms on the right-hand side of (11) are positive. We employ (16) for $X = \nabla v$ and for $X = \nabla\ell$ to get

$$
\begin{aligned}
&e^{2\tau\ell}(\partial_t^2 w - \Delta_\mu w)^2/2 - \partial_t(\vartheta\partial_t v) + \operatorname{div}_\mu(\vartheta\nabla v) + \operatorname{div}_\mu Y \\
&\quad \geq \rho\tau(\partial_t v)^2 + \rho\tau|\nabla v|^2 + v(\tau\nabla\Delta_\mu\ell, \nabla v) + 3\rho|\nabla\ell|^2\tau^3 v^2 - \rho^2\tau^2 v^2 \\
&\quad \geq \rho\tau(\partial_t v)^2 + (\rho\tau - 1)|\nabla v|^2 + 3\rho|\nabla\ell|^2\tau^3 v^2 - (\rho^2 + |\nabla\Delta_\mu\ell|^2)\tau^2 v^2,
\end{aligned}
$$

where we have applied Cauchy-Schwartz inequality to the third term in going from the first inequality to second inequality. The claim then follows by noticing that

$$
e^{-2\tau\ell}|\nabla v|^2 = |\tau w\nabla\ell + \nabla w|^2 \geq \frac{1}{2}|\nabla w|^2 - |\nabla\ell|^2\tau^2 w^2.
$$

□

Lemma 3. *Let $T > 0$ and let $w \in C^2([0,T] \times M)$ satisfy $w(t,x) = 0$ for $(t,x) \in [0,T] \times \partial M$. Let $\tau, \rho > 0$ and let $\ell \in C^3(M)$. We define v, ϑ and Y as in Corollary 2. Moreover, we denote $dm := \mu dV$ and*

$$
\Gamma := \{x \in \partial M;\ (\nabla\ell, \nu) > 0\}. \tag{17}
$$

Then

$$\int_0^T \int_M \operatorname{div}_\mu(\vartheta \nabla v) + \operatorname{div}_\mu Y \, dm dt \le C_2 e^{B_\ell \tau} \tau \int_0^T \int_\Gamma (\partial_\nu w)^2 \mu dS dt,$$

where $B_\ell = 2\max_{x\in M} \ell(x)$ *and* $C_2 = 3\max_{x \in M} |\nabla \ell(x)|$. *Moreover,*

$$\int_M |\vartheta \partial_t v| dm \le (C_3 \tau + C_4) e^{B_\ell \tau} \tau \int_M (\partial_t w)^2 + |\nabla w|^2 dm,$$

where

$$C_3 = C_F \max_{x\in M} |\nabla \ell(x)|^2, \quad C_4 = \max_{x \in M} |\nabla \ell| + C_F \max_{x\in M} |\Delta_\mu \ell - \rho|/2$$

and $C_F \ge 1$ *is a constant satisfying the Friedrich inequality*

$$\int_M \phi^2 dm \le C_F \int_M |\nabla \phi|^2 dm, \quad \phi \in C_0^\infty(M). \tag{18}$$

Proof. Notice that on $[0,T] \times \partial M$ we have $v = \partial_t v = 0$ and

$$\nabla v = e^{\tau \ell} \nabla w + \tau v \nabla \ell = e^{\tau\ell} \nabla w = e^{\tau \ell} (\nabla w, \nu) \nu,$$

since w vanishes there. Thus

$$\begin{aligned}\vartheta(\nabla v, \nu) + (Y, \nu) &= 2\tau (e^{\tau\ell}(\nabla w, \nu)\nu, \nabla \ell)(e^{\tau \ell} \nabla w, \nu) \\ &\quad + \tau |e^{\tau\ell}(\nabla w, \nu)\nu|^2 (\nabla \ell, \nu) \\ &= 3 e^{2\tau\ell} \tau (\nabla w, \nu)^2 (\nabla \ell, \nu).\end{aligned}$$

By the divergence theorem,

$$\int_M \operatorname{div}(\mu \vartheta \nabla v + \mu Y) dV = \int_{\partial M} \mu(\vartheta \nabla v + Y, \nu) dS,$$

and the first claim follows.

For the second claimed inequality, notice that

$$\begin{aligned}&\tau^{-1} e^{-2\tau\ell} |\vartheta \partial_t v| = |((\Delta_\mu \ell - \rho) w + 2(\nabla w + \tau w \nabla \ell, \nabla \ell)) \partial_t w| \\ &\quad \le |(\Delta_\mu \ell - \rho) + 2\tau |\nabla \ell|^2| |w \partial_t w| + 2|\nabla \ell| |\nabla w| |\partial_t w| \\ &\quad \le |(\Delta_\mu \ell - \rho)/2 + \tau |\nabla \ell|^2| (w^2 + (\partial_t w)^2) + |\nabla \ell| (|\nabla w|^2 + (\partial_t w)^2).\end{aligned}$$

Hence

$$\begin{aligned}&e^{-B_\ell \tau} \tau^{-1} \int_M |\vartheta \partial_t v| dm \\ &\quad \le (\max |\Delta_\mu \ell - \rho|/2 + \tau \max |\nabla \ell|^2) \left(\int_M w^2 dm + \int_M (\partial_t w)^2 dm \right) \\ &\qquad + \max |\nabla \ell| \int_M |\nabla w|^2 + (\partial_t w)^2 dm,\end{aligned}$$

and the second claimed inequality follows from (18) with $\phi = w$. $\square$

Remark 1. *Let $u \in C^2([0,\infty) \times M)$ be a solution of*

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta_\mu u(t,x) = 0, && (t,x) \in (0,\infty) \times M, \\ &u(t,x) = 0, && (t,x) \in (0,\infty) \times \partial M, \end{aligned} \tag{19}$$

Then the energy,

$$E(t) := \int_M (\partial_t u(t))^2 + |\nabla u(t)|^2 dm,$$

is constant for $t \in [0,\infty)$.

We recall that the constant C_1 is defined in Corollary 2 and the constants C_2, C_3, C_4 and B_ℓ are defined in Lemma 3. Moreover, we define the constant $\beta_\ell = 2 \min_{x \in M} \ell(x)$.

Theorem 4 (Observability inequality)**.** *Suppose that there is a strictly convex function $\ell \in C^3(M)$ with no critical points. Let $\rho, r > 0$ satisfy*

$$D^2\ell(X,X) \geq \rho |X|^2, \quad |\nabla \ell(x)| \geq r,$$

for all $X \in T_x M$ and $x \in M$, and let $\Gamma \subset \partial M$ contain the set (17). Suppose that

$$T > 2(C_3\tau + C_4)e^{(B_\ell - \beta_\ell)\tau}\tau, \quad \textit{where } \tau = \max\left(\frac{3}{\rho}, \frac{C_1}{2\rho r^2}\right). \tag{20}$$

Let $u \in C^2([0,T] \times M)$ be a solution of (19). Then

$$E(0) \leq C(T) \int_0^T \int_\Gamma (\partial_\nu u)^2 \mu dS dt,$$

where

$$C(T) = \frac{C_2 e^{(B_\ell - \beta_\ell)\tau}\tau}{T - 2(C_3\tau + C_4)e^{(B_\ell - \beta_\ell)\tau}\tau}.$$

Proof. We will integrate the inequality of Corollary 2 with $w = u$. Notice that

$$\rho\tau - 1 \geq 2 \quad \text{and} \quad 2\rho|\nabla \ell|^2\tau - C_1 \geq 0.$$

By combining these observations with Lemma 3, we get

$$\begin{aligned} &e^{\beta_\ell \tau} \int_0^T \int_M (\partial_t u)^2 + |\nabla u|^2 dm dt \\ &\quad \leq \int_0^T \int_M \operatorname{div}_\mu(\vartheta \nabla v) + \operatorname{div}_\mu Y \, dm dt - \left[\int_M (\vartheta \partial_t v) dm \right] \Big|_{t=0}^{t=T} \\ &\quad \leq C_2 e^{B_\ell \tau}\tau \int_0^T \int_\Gamma (\partial_\nu u)^2 \mu dS dt + 2(C_3\tau + C_4)e^{B_\ell \tau}\tau E(0), \end{aligned}$$

where we have also used the fact $E(0) = E(T)$.

By Lemma 3 and Remark 1

$$\begin{aligned}&e^{\beta_\ell \tau} \int_0^T \int_M (\partial_t u)^2 + |\nabla u|^2 dm dt \\ &\quad \le 2(C_3\tau + C_4) e^{B_\ell \tau} \tau E(0) + C_2 e^{B_\ell \tau} \tau \int_0^T \int_\Gamma (\partial_\nu u)^2 \mu dS dt.\end{aligned}$$

To conclude notice that

$$\int_0^T \int_M (\partial_t u)^2 + |\nabla u|^2 dm dt = T E(0).$$

□

Corollary 3 (Stable observability). *Suppose that there is a strictly convex function $\ell \in C^3(M)$ with no critical points. Let $\rho, r > 0$ satisfy*

$$D^2 \ell(X, X) > \rho |X|^2, \quad |\nabla \ell(x)| > r,$$

for all $X \in T_x M$ and $x \in M$. Suppose that open $\Gamma \subset \partial M$ satisfies

$$\{x \in \partial M;\ (\nabla \ell, \nu) \ge 0\} \subset \Gamma.$$

Let U_0 be a bounded C^2 neighborhood of (g, μ). Then there is a C^1 neighborhood U of g and constants $C, T > 0$ satisfying the following: for all $(\widetilde{g}, \widetilde{\mu}) \in U_0$ such that $\widetilde{g} \in U$, the solutions

$$\widetilde{u} \in C([0,T]; H^1(M)) \cap C^1([0,T]; L^2(M)) \tag{21}$$

of the wave equation,

$$\begin{aligned}&\partial_t^2 \widetilde{u} - \Delta_{\widetilde{g},\widetilde{\mu}} \widetilde{u} = 0, && \text{on } (0,T) \times M, \\ &\widetilde{u} = 0, && \text{on } (0,T) \times \partial M,\end{aligned}$$

satisfy the observability inequality

$$\|\widetilde{u}(0)\|^2_{H_0^1(M)} + \|\partial_t \widetilde{u}(0)\|^2_{L^2(M)} \le C \|\partial_\nu \widetilde{u}\|^2_{L^2((0,T)\times\Gamma)}.$$

Proof. Let us choose a finite number of compact coordinate neighborhoods covering M and let K be one of them. A metric $\widetilde{g}$ is given in K by a smooth matrix valued function $\widetilde{g}_{ij}$. Let us denote by $\sigma(\widetilde{g}, x)$ the smallest eigenvalue of the matrix $\partial^2_{ij}\ell - \widetilde{\Gamma}^k_{ij}\partial_k \ell - \rho \widetilde{g}_{ij}$, where $\widetilde{\Gamma}^k_{ij}$ are the Christoffel symbols corresponding to $\widetilde{g}_{ij}$. Then $\sigma : C^1 \times K \to \mathbb{R}$ is continuous on the compact set $K_0 \times K$, where K_0 is the C^1 closure of the projection of U_0 on the metric tensors. In particular, there is a C^1 neighborhood U of g such that $\sigma(\widetilde{g}, x) > 0$ in $\overline{U} \cap K_0 \times K$. That is,

$$D^2_{\widetilde{g}} \ell(X, X) > \rho |X|^2_{\widetilde{g}}, \quad X \in TK,\ \widetilde{g} \in U \cap K_0.$$

The function $(\widetilde{g}, x) \mapsto |\nabla_{\widetilde{g}} \ell(x)|_{\widetilde{g}}$ is continuous on the compact set $K_0 \times K$, whence by making U smaller if necessary, we have

$$|\nabla_{\widetilde{g}} \ell(x)|_{\widetilde{g}} > r, \quad x \in K,\ \widetilde{g} \in U \cap K_0.$$

Let us suppose for a moment that K intersects the set $\partial M \setminus \Gamma$. We may assume that $\partial M \cap K$ is given by a defining function F and that $\widetilde{\nu} = \nabla_{\widetilde{g}} F / |\nabla_{\widetilde{g}} F|$. The function $(\widetilde{g}, x) \mapsto (\nabla_{\widetilde{g}} F(x), \nabla_{\widetilde{g}} \ell(x))_{\widetilde{g}}$ is continuous on the compact set $K_0 \times K$, whence by making U smaller if necessary, we have that

$$(\widetilde{\nu}(x), \nabla_{\widetilde{g}} \ell(x))_{\widetilde{g}} < 0, \quad x \in (K \cap \partial M) \setminus \Gamma,\ \widetilde{g} \in U \cap K_0.$$

By taking the intersection with respect to the finite cover chosen in the beginning of the proof, we see that there is a C^1 neighborhood U of g such that all $\widetilde{g} \in U \cap K_0$ satisfy the assumptions of Theorem 4 with the fixed ℓ, ρ, r and Γ.

Let us show next that the constants $C_j(\widetilde{g}, \widetilde{\mu})$, $j = 1, 2, 3, 4$, stay bounded in U_0. We may first work locally in a compact coordinate neighborhood K as above. Let us denote by $\lambda_0(\widetilde{g}, x)$ and $\lambda_n(\widetilde{g}, x)$ the smallest and the largest eigenvalue of $\widetilde{g}_{ij}(x)$. The functions λ_0 and λ_n are continuous $C^0 \times K \to \mathbb{R}$, $\widetilde{g}_{ij}(x)$ is positive definite and $\overline{U}_0 \times K$ is compact in $C^0 \times K$, where the closure is in C^0. In particular, we may choose $C > 0$ so that on $\overline{U}_0$

$$C^{-1} |X|_g \le |X|_{\widetilde{g}} \le C |X|_g, \quad X \in TK.$$

Moreover, we may choose $C > 0$ so that also the functions $|\widetilde{g}(x)|$ and $\widetilde{\mu}(x)$ are bounded below by C^{-1} and above by C on $\overline{U}_0 \times K$. Hence there is $C_F > 0$ such that the Friedrichs' inequality

$$\int_M \phi^2 d\widetilde{m} \le C_F \int_M |\nabla \phi|_{\widetilde{g}}^2 d\widetilde{m}, \quad \phi \in C_0^\infty(M),$$

holds for all $(\widetilde{g}, \widetilde{\mu}) \in \overline{U}_0$. Now it straightforward to see that $C_j(\widetilde{g}, \widetilde{\mu})$, $j = 1, 2, 3, 4$, are bounded on U_0 as they can be expressed in coordinates using the derivatives of $\widetilde{g}_{ij}$ and $\widetilde{\mu}$ up to the second order.

The map $(\widetilde{u}(0), \partial_t \widetilde{u}(0)) \mapsto \partial_\nu \widetilde{u}|_{(0,T)\times\Gamma}$ is continuous from $H_0^1(M) \times L^2(M)$ to $L^2((0,T) \times \Gamma)$ by [26]. Thus we may approximate the initial data by smooth compactly supported functions and get the observability also for solutions in the energy class (21). $\square$

Theorem 1 follows from Corollary 3 by choosing $g = c(x)^{-2} dx^2$ and $\mu(x) = c(x)^{n-2}$.

Appendix: A linear space for Dirichlet-to-Neumann operators

Let us consider the operator

$$A : H^1_{cc}((0,T) \times \partial M) \to L^2((0,2T) \times \partial M),$$

and define the map $\mathcal{K}(A) := RA_T RJ\Theta - JA$ and the restrictions

$$A_T f := (Af)|_{(0,T)\times\partial M}, \quad A_R f := (Af)|_{(T,2T)\times\partial M}.$$

Notice that $2J\Theta f(t) = \int_t^T f(s)ds$, whence $RJ\Theta f(0) = 0$ and $\mathcal{K}(A)$ is well-defined on $H^1_{cc}((0,T) \times \partial M)$. We define

$$\|A\|_* := \|\mathcal{K}(A)\|_{L^2(\Upsilon)\to L^2(\Upsilon)} + \|A_T\|_{H^1_{cc}(\Upsilon)\to L^2(\Upsilon)},$$

where we have denoted $\Upsilon := (0,T) \times \partial M$. Let us next show that $\|\cdot\|_*$ is a norm on

$$H^1_{DN} := \{A : H^1_{cc}(\Upsilon) \to L^2((0,2T) \times \partial M);\ \|A\|_* < \infty\}.$$

Clearly, $\|\cdot\|_*$ is homogeneous and subadditive. Suppose that $\|A\|_* = 0$ and let $f \in H^1_{cc}(\Upsilon)$. Then $A_T = 0$ and

$$0 = -2\mathcal{K}(A)f(t) = 2JAf(t) = \int_T^{2T-t} A_R f(s)ds, \quad t \in (0,T).$$

By differentiating, we see that $A_R f(2T-t) = 0$ for almost all $t \in (0,T)$, whence $A_R = 0$ and we have shown that $\|\cdot\|_*$ is a norm.

Let us now consider the Dirichlet-to-Neumann operator

$$\Lambda_{2T} : H^1_{cc}((0,2T) \times \partial M) \to L^2((0,2T) \times \partial M).$$

Let $E : H^1_{cc}(0,T) \to H^1_{cc}(0,2T)$ be an extension operator, that is, $Ef|_{[0,T]} = f$. Then

$$\Lambda_{2T} \circ E : H^1_{cc}((0,T) \times \partial M) \to L^2((0,2T) \times \partial M).$$

Moreover, causality of the wave equation (1) yields $(\Lambda_{2T} \circ E)_T = \Lambda_T$ and

$$(h, \mathcal{K}(\Lambda_{2T} \circ E)f) = (u^h(T), u^f(T))_{L^2(M;c^{-2}dx)} = (h, K(\Lambda_{2T})f),$$

for all $f, h \in C_0^\infty((0,T) \times \partial M)$. Hence the embedding $\Lambda_{2T} \mapsto \Lambda_{2T} \circ E$ of the Dirichlet-to-Neumann operators to H^1_{DN} does not depend on the choice of the extension operator E.

Acknowledgements. The research was supported by Finnish Centre of Excellence in Inverse Problems Research, Academy of Finland project COE 250215, Academy of Finland project 141075 and European Research Council advanced grant 400803.

References

[1] G. Alessandrini. Stable determination of conductivity by boundary measurements. *Appl. Anal.*, 27(1-3):153–172, 1988.
[2] G. Alessandrini and S. Vessella. Lipschitz stability for the inverse conductivity problem. *Adv. in Appl. Math.*, 35(2):207–241, 2005.
[3] H. Ammari, H. Bahouri, D. D. S. Ferreira, and I. Gallagher. Stability estimates for an inverse scattering problem at high frequencies. *Preprint, arXiv:1205.6563*, May 2012.
[4] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, and M. Taylor. Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem. *Invent. Math.*, 158(2):261–321, 2004.
[5] G. Bao and K. Yun. On the stability of an inverse problem for the wave equation. *Inverse Problems*, 25(4):7pp, 2009.
[6] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30(5):1024–1065, 1992.
[7] M. I. Belishev. An approach to multidimensional inverse problems for the wave equation. *Dokl. Akad. Nauk SSSR*, 297(3):524–527, 1987.
[8] M. I. Belishev. Recent progress in the boundary control method. *Inverse Problems*, 23(5):R1–R67, 2007.
[9] M. Bellassoued and D. Dos Santos Ferreira. Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map. *Inverse Probl. Imaging*, 5(4):745–773, 2011.
[10] A. Ben-Israel and T. N. E. Greville. *Generalized inverses*. CMS Books in Mathematics 15. Springer-Verlag, New York, 2003.
[11] K. Bingham, Y. Kurylev, M. Lassas, and S. Siltanen. Iterative time-reversal control for inverse problems. *Inverse Probl. Imaging*, 2(1):63–81, 2008.
[12] A. S. Blagoveščenskiĭ. The inverse problem of the theory of seismic wave propagation. In *Problems of mathematical physics, No. 1*, pages 68–81. Izdat. Leningrad. Univ., Leningrad, 1966.
[13] A. L. Bukhgeĭm and M. V. Klibanov. Uniqueness in the large of a class of multidimensional inverse problems. *Dokl. Akad. Nauk SSSR*, 260(2):269–272, 1981.
[14] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.
[15] M. de Hoop, L. Qiu, and O. Scherzer. Local analysis of inverse problems: Hölder stability and iterative reconstruction. *Inverse Problems*, 28:045001, 2012.
[16] T. Duyckaerts, X. Zhang, and E. Zuazua. On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(1):1–41, 2008.
[17] H. W. Engl, M. Hanke, and A. Neubauer. *Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.
[18] R. Gulliver, I. Lasiecka, W. Littman, and R. Triggiani. The case for differential geometry in the control of single and coupled pdes: the structure acoustic chamber. In *Geometric methods in inverse problems and PDE control*, volume 137 of *IMA Vol. Math. Appl.*, pages 73–181. Springer, New York, 2004.

[19] R. Gulliver and W. Littman. Chord uniqueness and controllability: the view from the boundary. In *Differential geometric methods in the control of partial differential equations*, volume 268 of *Contemp. Math.*, pages 145–175. Amer. Math. Soc., Providence, RI, 2000.

[20] O. Y. Imanuvilov and M. Yamamoto. Global uniqueness and stability in determining coefficients of wave equations. *Comm. Partial Differential Equations*, 26(7-8):1409–1425, 2001.

[21] V. Isakov. *Inverse problems for partial differential equations*, volume 127 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2006.

[22] V. Isakov. Increasing stability for the Schrödinger potential from the Dirichlet-to-Neumann map. *Discrete Contin. Dyn. Syst. Ser. S*, 4(3):631–640, 2011.

[23] S. Izumino. Convergence of generalized inverses and spline projectors. *J. Approx. Theory*, 38(3):269–278, 1983.

[24] A. Katchalov, Y. Kurylev, and M. Lassas. *Inverse boundary spectral problems*, volume 123 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[25] M. A. Kazemi and M. V. Klibanov. Stability estimates for ill-posed Cauchy problems involving hyperbolic equations and inequalities. *Appl. Anal.*, 50(1-2):93–102, 1993.

[26] I. Lasiecka, J.-L. Lions, and R. Triggiani. Nonhomogeneous boundary value problems for second order hyperbolic operators. *J. Math. Pures Appl. (9)*, 65(2):149–192, 1986.

[27] M. M. Lavrent′ev, V. G. Romanov, and S. P. Shishat·skiĭ. *Ill-posed problems of mathematical physics and analysis*, volume 64 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986.

[28] J.-L. Lions. Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, 30(1):1–68, 1988.

[29] S. Liu and R. Triggiani. Global uniqueness and stability in determining the damping coefficient of an inverse hyperbolic problem with nonhomogeneous neumann b.c. through an additional dirichlet boundary trace. *SIAM J. Math. Anal.*, 43(24):1631–1666, 2011.

[30] N. Mandache. Exponential instability in an inverse problem for the Schrödinger equation. *Inverse Problems*, 17(5):1435–1444, 2001.

[31] L. Michel. Escape function conditions for the observation, control, and stabilization of the wave equation. *SIAM J. Control Optim.*, 41(5):1554–1566, 2002.

[32] S. Nagayasu, G. Uhlmann, and J.-N. Wang. Increasing stability in an inverse problem for the acoustic equation. *Preprint, arXiv:1110.5145*, Oct. 2011.

[33] Rakesh. Reconstruction for an inverse problem for the wave equation with constant velocity. *Inverse Problems*, 6(1):91–98, 1990.

[34] L. Rondi. A remark on a paper by G. Alessandrini and S. Vessella: "Lipschitz stability for the inverse conductivity problem". *Adv. in Appl. Math.*, 36(1):67–69, 2006.

[35] P. Stefanov and G. Uhlmann. Recovery of a source term or a speed with one measurement and applications. To appear in *Trans. Amer. Math. Soc.*

[36] P. Stefanov and G. Uhlmann. Stability estimates for the hyperbolic Dirichlet-to-Neumann map in anisotropic media. *J. Funct. Anal.*, 154(2):330–358, 1998.

[37] P. Stefanov and G. Uhlmann. Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map. *Int. Math. Res. Not.*, (17):1047–1061, 2005.

[38] P. Stefanov and G. Uhlmann. Thermoacoustic tomography with variable sound speed. *Inverse Problems*, 25(7):075011, 16, 2009.

[39] Z. Sun. On continuous dependence for an inverse initial-boundary value problem for the wave equation. *J. Math. Anal. Appl*, 150(1):188–204, 1990.

[40] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[41] D. Tataru. Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem. *Comm. Partial Differential Equations*, 20(5-6):855–884, 1995.

[42] D. Tataru. Carleman estimates and unique continuation for solutions to boundary value problems. *J. Math. Pures Appl. (9)*, 75(4):367–408, 1996.

[43] R. Triggiani and P. F. Yao. Carleman estimates with no lower-order terms for general Riemann wave equations. Global uniqueness and observability in one shot. *Appl. Math. Optim.*, 46(2-3):331–375, 2002.

[44] G. Uhlmann. Electrical impedance tomography and calderon's problem. *Inverse Problems*, 25(12):123011, 2009.

Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 FI-00014

E-mail address: shitao.liu@helsinki.fi

E-mail address: lauri.oksanen@helsinki.fi