NON-EXISTENCE OF COMPETITIVE EQUILIBRIA WITH DYNAMICALLY INCONSISTENT PREFERENCES*

Tommaso Gabrieli
Henley Business School, University of Reading

Sayantan Ghosal
Department of Economics, University of Warwick

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ABSTRACT. This paper shows the robust non existence of competitive equilibria even in a simple three period representative agent economy with dynamically inconsistent preferences. We distinguish between a sophisticated and naive representative agent. Even when underlying preferences are monotone and convex, at given prices we show by example that the induced preference of the sophisticated representative agent over choices in first period markets is both non convex and satiated. Even allowing for negative prices, the market clearing allocation is not contained in the convex hull of demand. Finally, with a naive representative agent, we show that perfect foresight is incompatible with market clearing and individual optimization at given prices.

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1. Introduction

Starting from Strotz (1956), choice problems with dynamically inconsistent preferences have been studied extensively[1]. There is a small but growing literature that studies the properties of competitive equilibrium models with dynamically inconsistent preferences[2]. The representative agent economy is a particularly simple (and widely used) model in macroeconomics and finance where both issues of optimization and market clearing arise[3]. This paper shows the robust non existence of competitive equilibria even in a simple deterministic three period representative agent economy with dynamically inconsistent preferences.

We distinguish between a naive and sophisticated representative agent. We formulate the decision problem of a sophisticated representative agent as an intra-personal game at given prices. In our simple exchange economy there is only one candidate market clearing allocation, namely one in which the representative agent consumes his endowments. We show, via a robust example, that there are no prices such that, at the solution of the intra-personal game, the representative agent consumes his endowments.

The models of Barro (1999), Kocherlakota (2001) and Luttmer and Mariotti (2003) allow for the possibility of quasi-hyperbolic discounting: under the key assumption that agents have identical discount functions and identical CRRA period utility functions, whether discounting is quasi-hyperbolic is irrelevant and existence is not an issue. Closer to the work reported here, Luttmer and Mariotti (2006), Herings and Rohde (2006), Luttmer and Mariotti (2007) have shown that equilibria exist with general classes of dynamically inconsistent preferences[4].

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[1]Pollak (1968), Blackorby, Nissen, Primont, and Russell (1973), Peleg and Yaari (1973), Goldman (1980), Harris and Laibson (2001), Caplin and Leahy (2006) among others.

[2]Barro (1999), Kocherlakota (2001), Luttmer and Mariotti (2003), Luttmer and Mariotti (2006), Herings and Rohde (2006), Luttmer and Mariotti (2007), Herings and Rohde (2008).

[3]Caplin and Leahy (2001), Kocherlakota (2001), Luttmer and Mariotti (2003), among others, introduce dynamically inconsistent preferences in the representative agent economy.

[4]Observe that the preferences studied in Luttmer and Mariotti (2006) and Luttmer and Mariotti (2007) satisfy quasi-hyperbolic discounting (Laibson (1997)) and are, by construction, time separable. In Herings and Rohde (2006), preferences are independent of past consumption (like the preferences studied by us here) but, unlike us, they study existence in a market structure that is not sequentially complete.

In our example the dynamically inconsistent preferences that we study do not satisfy the assumption of time separability[5] and result in induced preferences over choices in first period markets that are non convex and satiated. At given prices, in our example, such induced preferences display satiation because the amount consumed by the second period "self"is a decreasing function of the amount saved by the first period "self". Further, at given prices, the induced preferences over first period choices fail to be convex as anticipated second and third period consumption are no longer concave functions of first period savings. We show that the market clearing allocation does not lie in the convex hull of demand even allowing for negative prices and hence the non-existence result.[6] Finally, with a naive representative agent, we show that perfect foresight is incompatible with market clearing and individual optimization.

The rest of the paper is structured as follows. In section 2 we introduce the three period representative agent economy, in section 3 we present the non existence example with a sophisticated representative agent, while in section 4 we study existence with a naive representative agent.

2. The Economy

We consider a simple representative agent economy over three periods, labeled by t, $t = 1, 2, 3$. There is a single asset (the tree) which delivers units of a consumption good (dividends or fruit) in every period. The consumption good is non storable, hence the asset provides the only way to transfer wealth across periods. Let c_t denote consumption in period t, $t = 1, 2, 3$. Let θ_{t+1} denote the amount of the asset held by the representative agent at the beginning of period $t + 1$. Then $\theta_{t+1}d_{t+1}$ denotes the amount of the consumption good available for consumption at $t + 1$.

[5]See Caplin and Leahy (2001). Among other specific properties, their model features both uncertainty and non-time separable preferences in an intrinsic way. As our analysis is limited to the deterministic case, we focus on the implications of non-time separable preferences that such a model may have. Time separability will also be violated in models of habit persistence although we do not explicitly focus on this case in this paper.

[6]Luttmer and Mariotti (2006), Herings and Rohde (2006), Luttmer and Mariotti (2007) deal only with potential non convexities, but not with satiation. Herings and Rohde (2006) prove existence in the case in which induced preferences are convex by assumption. Luttmer and Mariotti (2006) and Luttmer and Mariotti (2007) prove existence in a large economy by proving that there exist prices such that the market clearing allocation lies in the convex hull of demand.

We assume the representative agent is a price taker for both the consumption good and the asset. We normalize prices so that the price of the consumption good is fixed at 1 in each period, with p_t denoting the relative price of the asset in period t. The model is completely deterministic and the values of all fundamentals are known from the beginning by the agent. At the beginning of period 1, the agent is endowed with the entire asset ($\theta_1 = 1$) and the entire paid dividend d_1.

At each t, we assume that the agent has preferences ranking non negative commodity bundles. We assume that at each t, $t = 1, 2$, the preferences of the representative agent over consumption are represented by the utility function $u_t(c_t, ..., c_3)$. We assume that at each t, $t = 1, 2$ $u_t(c_t, ..., c_3)$[7] is smooth, strictly increasing and strictly quasi-concave.

We say preferences are dynamically inconsistent if given some non-negative c_1 the projection of preferences of the representative agent at $t = 1$ over $(c_2, c_3) \in \mathbb{R}^2_+$ are different from his preferences at $t = 2$ over $(c_2, c_3) \in \mathbb{R}^2_+$, or equivalently, for some non-negative c_1, $\frac{\frac{\partial u_1}{\partial c_3}(c_1,c_2,c_3)}{\frac{\partial u_1}{\partial c_2}(c_1,c_2,c_3)} \neq \frac{\frac{\partial u_2}{\partial c_3}(c_2,c_3)}{\frac{\partial u_2}{\partial c_2}(c_2,c_3)}$, $(c_2, c_3) \in \mathbb{R}^2_+$.[8]

In the remainder of the paper we assume that the preferences of the representative agent are dynamically inconsistent.

We consider the case where the representative agent is sophisticated, i.e. correctly anticipates that at $t = 2$ he will re-optimize, given his choices made at $t = 1$. At given prices p_t, $t = 1, 2$, the decision problem of the sophisticated representative agent is described by the following intra-personal game:

Players: each period t, $t = 1, 2$, the representative agent is considered as a distinct autonomous player.

Actions: $A_t = \{(c_t, \theta_{t+1}) \in \mathbb{R}^2_+ : c_t + p_t\theta_{t+1} \leq (p_t + d_t)\theta_t\}$ constitutes the set of actions available to player t.

Histories: the set of possible histories at $t = 2$ is $H_1 = A_1$, while the set of histories at $t = 1$, H_0 is a singleton.

Strategies: a strategy for the date t consumer is a Borel measurable function $\gamma_t : H_{t-1} \to \Delta(A_t)$.

[7]Clearly, as $u_1(.)$ depends on c_1, c_2 and c_3 but $u_2(.)$ depends on c_2 and c_3 but not c_1, the preferences studied here are consistent with anticipatory feelings but not with habit persistence.

[8]As preferences are monotonic over consumption in each period, the optimal period 3 choice is to always choose maximum feasible consumption. It follows that the asset price in period 3 is zero. In this 3 period economy our exclusive focus is on the time inconsistency between periods 1 and 2.

Definition 1. At prices p_1, p_2, a *Sophisticated Solution (SS)* γ is a Strotz (1956) solution i.e. for each player t, γ_t induces a level of consumption which maximizes its own utility given any feasible history of choices and the utility maximizing strategies of the future.

Remark. From definition 1, at given prices, it follows that a SS is a subgame perfect Nash equilibrium of the intra-personal game, although, in general, the converse does not hold. In general, the two solution concepts would not coincide if there are multiple payoff maximizing consumption choices in some subgame for the period-2 consumer. However in our economy, as the second period utility is strictly quasi-concave guaranteeing a unique solution in each subgame, the two solution concepts coincide.

The market clearing condition for this economy is trivial: the agent must hold the entire unit of the asset in each period ($\theta_1 = \theta_2 = \theta_3 = 1$) and consumption must be equal to the entire dividend paid in each period ($c_1 = d_1, c_2 = d_2, c_3 = d_3$).

Definition 2. A *competitive equilibrium with a sophisticated representative agent* is a combination of prices (p_1^*, p_2^*) and allocation $(\theta_1^*, c_1^*, \theta_2^*, c_2^*, \theta_3^*, c_3^*)$ such that:
(i) $(\theta_1^*, c_1^*, \theta_2^*, c_2^*, \theta_3^*, c_3^*)$ is the outcome of SS at prices (p_1^*, p_2^*);
(ii) $(c_1^* = d_1, \theta_2^* = 1, c_2^* = d_2, \theta_3^* = 1, c_3^* = d_3)$.

Note that by construction at a competitive equilibrium with a sophisticated representative agent both selves of the representative agent face the same prices, i.e. the sophisticated representative agent at $t = 1$ must correctly forecast the asset price at $t = 2$. The definition of a competitive equilibrium with a sophisticated agent corresponds to the notion of a sophisticated equilibrium in Herrings and Rhode (2006) and to the notion of a competitive equilibrium in Luttmer and Mariotti (2006).

A weaker definition of competitive equilibrium with a sophisticated representative agent would allow for the possibility that the market clearing allocation lies in the convex hull of demand. To this end, at prices p_1, p_2, given a strategy γ, we define the demand correspondence: $D(p_1, p_2) = \{(c_1, \theta_2, c_2, \theta_3, c_3) \in \Re^5$: each $(c_1, \theta_2, c_2, \theta_3, c_3)$ is an outcome of SS at prices $(p_1, p_2)\}$. Even though preferences are strictly quasi-concave, the demand correspondence can be multi-valued in our setting as the induced preferences of the sophisticated representative agent at $t = 1$ may fail to be convex. Let $Conv(D(p_1, p_2))$ denote the convex hull of the demand correspondence i.e. the intersection of all convex sets containing $D(p_1, p_2; \gamma)$. A weaker notion of a competitive equilibrium follows:

Definition 3. A weak *competitive equilibrium with a sophisticated representative agent* is a combination of prices (p_1^*, p_2^*) such that:
(i) each $(c_1, \theta_2, c_2, \theta_3, c_3) \in D(p_1^*, p_2^*)$;
(ii) $(c_1^* = d_1, \theta_2^* = 1, c_2^* = d_2, \theta_3^* = 1, c_3^* = d_3) \in Conv(D(p_1^*, p_2^*))$.

At a weak competitive equilibrium with a sophisticated representative agent, as the market clearing allocation lies in the convex hull of the demand, market clearing is only obtained in expectation (equivalently, market clearing obtains in a reinterpretation of our model where the representative agent is a collection of a continuum of identical individuals).

Proposition 1. *(Non existence). Not only does a competitive equilibrium with a sophisticated representative agent not always exist but even a weak competitive equilibrium with a sophisticated representative agent does not always exist.*

In the following section we prove the proposition with a robust example.

3. An Example of Non Existence

In this section we construct a robust example, where utility is increasing, smooth and strictly quasi-concave, but where a competitive equilibrium with a sophisticated representative agent does not exist. In this example at any fixed configuration of asset prices, by backward induction, the representative agent at $t = 1$ anticipates how the demand of his future self at $t = 2$ for θ_3 varies as a function of the amount of θ_2 he chooses to hold. The resulting induced preferences over θ_2 at $t = 1$ are non-convex and satiated. We, then, show that there is no market clearing asset price at $t = 1$ for such an induced preference.

The non-existence result is due to the fact that, in our example, consumption does not always increase monotonically in wealth. In order to have a well behaved utility function such that consumption may be an inferior good over certain ranges of wealth, we use the *addilog* preferences which have been introduced by Houthakker (1960).[9]

We begin by specifying the utility function at each t for the representative agent. At $t = 1$ the utility function of the representative agent is:

(1) $$U_1(c_1, c_2, c_3) = c_1 + b\ln(c_2) + c\ln(c_3),$$

[9] Concavity of the single period utility functions together with time separability imply that every period consumption is a normal good. In our example this is not always the case as the period 2 player's preferences are not time separable.

where $b \in (0,1), c \in (0,1)$ and $b > c$.

We assume that the utility function of the representative agent at $t = 2$ generates the following indirect *addilog* utility function:

$$V_2(p_2, \theta_2) = \alpha_2 \frac{(\theta_2(p_2 + d_2))^{\beta_2}}{\beta_2} + \alpha_3 \frac{(d_3\theta_2(p_2 + d_2)/p_2)^{\beta_3}}{\beta_3}, \tag{2}$$

where $\theta_2(p_2 + d_2)$ is the wealth of the representative agent at $t = 2$ and p_2/d_3 is the price of consumption at $t = 3$, $c_3 = \theta_3 d_3$.[10] This class of indirect utility functions was introduced by Houthakker (1960). Expression (2) draws on the work of Murthy (1982). Consistent with his assumptions we assume that the underlying preference and wealth parameters take the following values:

$$\beta_2 = -0.5, \beta_3 = 1, \alpha_2 = 0.2, \alpha_3 = 0.8, d_1 = d_2 = d_3 = 1. \tag{3}$$

de Boer, Bröcker, Jensen, and van Daal (2006) formally prove that when the β's are strictly greater than -1 and the α's add up to 1 the indirect utility function satisfies the following properties:

(i) homogeneous of degree zero in p_2 and θ_2,
(ii) non-increasing in p_2 and nondecreasing in θ_2,
(iii) strictly quasi-convex in p_2,
(iv) differentiable in p_2 and θ_2.

The fact that the indirect utility function is strictly quasi-convex in prices implies that the direct utility function, i.e. the dual of (2), is strictly quasi-concave by a well known result in duality theory [11].

Next we compute the asset demand functions at $t = 2$. Given that the utility function at $t = 2$ is strictly quasi-concave, we can apply Roy's Lemma and obtain:

$$c_2 = \frac{\alpha_2(\theta_2(p_2 + d_2))^{\beta_2+1}}{\alpha_2(\theta_2(p_2 + d_2))^{\beta_2} + \alpha_3(\theta_2(p_2 + d_2)/p_2)^{\beta_3}}. \tag{4}$$

It follows that as the period 2 budget constraint is satisfied with the equality, the demand for θ_3 at $t = 2$ as a function of θ_2, p_2 is

$$\theta_3(\theta_2, p_2) = \frac{\theta_2(p_2 + d_2) - c_2}{p_2}. \tag{5}$$

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[10]Since the optimal period 3 choice is to always choose maximum feasible consumption, without loss of generality, the asset price in period 3 is zero.
[11]See for example Mas-Colell, Whinston, and Green (1995), page 66.

Re-expressing c_1, c_2 and c_3 through the three inter-temporal budget constraints (satisfied in each case as an equality) and using $d_1 = d_2 = d_3 = 1$, we obtain the period 1 indirect utility function:

$$V_1(p_1, p_2, \theta_2) = p_1 - p_1\theta_2 + b\ln(p_2\theta_2 - p_2\theta_3(\theta_2, p_2)) + c\ln(\theta_3(\theta_2 p_2)). \tag{6}$$

Lemma 1. *The market clearing price at* $t = 2$ *such that* $\theta_2^* = \theta_3^* = 1$ *is* $p_2^* = 18.7$.

Proof. At the market clearing price vector it must be optimal for the representative agent to demand $\theta_2^* = \theta_3^* = 1$.

Using equation (4), we look for the p_2 such that the representative agent demands the market clearing quantities $c_2 = d_2, \theta_2 = 1$. Given the specified values $\beta_2 = -0.5, \beta_3 = 1, \alpha_2 = 0.2, \alpha_3 = 0.8, d_2 = 1$ we obtain the following equation:

$$(p_2^*)^2(p_2^* + 1)^{-3/2} = \alpha_3/\alpha_2. \tag{7}$$

Given that the utility function of the representative agent at $t = 2$ is strongly monotone, the market clearing price at $t = 2$ must be positive. There exists only one positive solution to (7), namely $p_2^* = 18.7$ and this is the market clearing price at $t = 2$. □

The preceding lemma computes the unique second period asset price that supports the market clearing allocation as the optimal choice of a period-2 consumer.

Lemma 2. *There exists a* K *strictly positive such that whenever* $b/c > K$ *then* $\frac{\partial V_1(p_1, p_2^*, \theta_2)}{\partial \theta_2} < 0, \theta_2 \geq 1$ *at each* $p_1 \geq 0$.

Proof. Plugging the values of the parameters and $p_2^* = 18.7$ into (4) we can re-express the demand for c_2 at $t = 2$, given $p_2^* = 18.7$, as a function of θ_2:

$$c_2(\theta_2, p_2^*) = \frac{.88\sqrt{\theta_2}}{.84\theta_2 + .04/\sqrt{\theta_2}}.$$

By computation note that $\frac{\partial c_2(\theta_2, p_2^*)}{\partial \theta_2} = -\frac{hy}{2}\frac{(\theta_2^{3/2} - \frac{2z}{y})}{(y\theta_2^{3/2} + z)^2}$, where $h = .88$,$y = .84$, $z = .04$. Notice that hy is strictly positive as it is the denominator of the fraction, however as $2z < y$, for $\theta_2 \geq 1$, $\theta_2^{3/2} - \frac{2z}{y} > 0$. Hence, c_2 is an inferior good at $t = 2$ over some range of income.

Substituting the expression for $c_2(\theta_2, p_2^*)$ into (5) and (6) we obtain the period 1 indirect utility as a function of p_1 and θ_2 alone:

(8) $V_1(p_1, \theta_2) = p_1 + 1 - p_1\theta_2 + b\ln(\frac{.88\sqrt{\theta_2}}{.84\theta_2 + .04/\sqrt{\theta_2}}) + c\ln(1.05\theta_2 - \frac{.047\sqrt{\theta_2}}{.84\theta_2 + .04/\sqrt{\theta_2}})$.

Let

$$p_1 + 1 - p_1\theta_2 \equiv A,$$
$$b\ln(\frac{.88\sqrt{\theta_2}}{.84\theta_2 + .04/\sqrt{\theta_2}}) \equiv b\ln(\frac{h\theta_2}{y\theta_2^{3/2} + z}) \equiv B,$$

where $h \equiv .88$, $y \equiv .84$, $z \equiv .04$.

$$c\ln(1.05\theta_2 - \frac{.047\sqrt{\theta_2}}{.84\theta_2 + .04/\sqrt{\theta_2}}) \equiv c\ln(k\theta_2 - \frac{x\theta_2}{y\theta_2^{3/2} + z}) \equiv C,$$

where $k \equiv 1.05$, $x \equiv .047$, $y \equiv .84$, $z \equiv .04$.

By computation notice that as long as $p_1 \geq 0$, $\frac{\partial A}{\partial \theta_2} = -p_1 \leq 0$. Let

$$f(\theta_2) = \frac{\theta_2}{y\theta_2^{3/2} + z} > 0, \theta_2 \geq 1.$$

Hence, $B = b\ln(hf(\theta_2))$ and $C = c\ln(k\theta_2 - xf(\theta_2))$. Now

$$f'(\theta_2) = \frac{-y(\theta_2^{\frac{3}{2}} - \frac{2z}{y})}{2\left(y\theta_2^{3/2} + z\right)^2} < 0, \forall\, \theta_2 \geq 1$$

and

$$\frac{\partial B}{\partial \theta_2} = b\frac{f'(\theta_2)}{f(\theta_2)},$$
$$\frac{\partial C}{\partial \theta_2} = c\frac{(k - xf'(\theta_2))}{(k\theta_2 - xf(\theta_2))}.$$

Under the values of the parameters assumed so far, $(k - xf'(\theta_2)) > 0$ and $k\theta_2 - xf(\theta_2) > 0 \; \forall\, \theta_2 \geq 1$. Therefore, $\frac{\partial B}{\partial \theta_2} < 0$ and $\frac{\partial C}{\partial \theta_2} > 0$ for all $\theta_2 \geq 1$ i.e. second period consumption is an inferior good and third period consumption a normal good for the period-1 consumer. Further,

$$\frac{\partial(B + C)}{\partial \theta_2} < 0 \Leftrightarrow b\frac{f'(\theta_2)}{f(\theta_2)} < -c\frac{(k - xf'(\theta_2))}{(k\theta_2 - xf(\theta_2))}$$

or equivalently,

$$\frac{b}{c} > -\frac{f(\theta_2)}{f'(\theta_2)}\frac{(k - xf'(\theta_2))}{(k\theta_2 - xf(\theta_2))} > 0.$$

By substitution and simplification, it follows that

$$\frac{\partial(B+C)}{\partial\theta_2} < 0 \text{ iff } \frac{b}{c} > K(\theta_2) = \frac{k\left(y\theta_2^{3/2}+z\right)^2 + \left(\frac{xy}{2}\right)\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right)}{\left[k\left(y\theta_2^{3/2}+z\right) - x\right]\left(\frac{y}{2}\right)\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right)}.$$

As long as $\theta_2 \geq 1$, the denominator of $K(\theta_2)$ is bounded away from zero so that for any finite value of $\theta_2 \geq 1$, $K(\theta_2)$ is bounded. Let

$$\begin{aligned} K_1(\theta_2) &= k\left(y\theta_2^{3/2}+z\right)^2 + \left(\frac{xy}{2}\right)\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right), \\ K_2(\theta_2) &= \left[k\left(y\theta_2^{3/2}+z\right) - x\right]\left(\frac{y}{2}\right)\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right). \end{aligned}$$

By computation,

$$\begin{aligned} K_1'(\theta_2) &= \frac{3}{2}\theta_2^{1/2}\left[2k\left(y\theta_2^{3/2}+z\right) + \left(\frac{xy}{2}\right)\right], \\ K_2'(\theta_2) &= \left(\frac{y}{2}\right)\frac{3}{2}\theta_2^{1/2}\left[ky\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right) + \left(k\left(y\theta_2^{3/2}+z\right) - x\right)\right] \end{aligned}$$

As $K_1'(\theta_2) > 0$ and $K_2'(\theta_2) > 0$ for θ_2 large enough, both $\lim_{\theta_2\to+\infty} K_1(\theta_2) = \infty$ and $\lim_{\theta_2\to+\infty} K_2(\theta_2) = \infty$. By L'Hospital's rule, $\lim_{\theta_2\to+\infty} K(\theta_2) = \lim_{\theta_2\to+\infty} \frac{K_1'(\theta_2)}{K_2'(\theta_2)}$. Now,

$$\begin{aligned} & \lim_{\theta_2\to+\infty} \frac{K_1'(\theta_2)}{K_2'(\theta_2)} \\ = & \lim_{\theta_2\to+\infty} \frac{\left[2k\left(y\theta_2^{3/2}+z\right) + \left(\frac{xy}{2}\right)\right]}{\left(\frac{y}{2}\right)\left[ky\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right) + \left(k\left(y\theta_2^{3/2}+z\right) - x\right)\right]} \\ = & \lim_{\theta_2\to+\infty} \frac{\left[2 + \frac{\left(\frac{xy}{2}\right)}{k\left(y\theta_2^{3/2}+z\right)}\right]}{\left(\frac{y}{2}\right)\left[\frac{y\left(\theta_2^{\frac{3}{2}} - \frac{2z}{y}\right)}{\left(y\theta_2^{3/2}+z\right)} + \left(1 - \frac{x}{k\left(y\theta_2^{3/2}+z\right)}\right)\right]}. \end{aligned}$$

Now, $\lim_{\theta_2 \to +\infty} \left[2 + \frac{\left(\frac{xy}{2}\right)}{k\left(y\theta_2^{3/2}+z\right)}\right] = 2 + \left(\lim_{\theta_2 \to +\infty} \frac{\left(\frac{xy}{2}\right)}{k\left(y\theta_2^{3/2}+z\right)}\right) = 2$ and

$$\begin{aligned}
&\lim_{\theta_2 \to +\infty} \left(\frac{y}{2}\right) \left[\frac{y(\theta_2^{\frac{3}{2}} - \frac{2z}{y})}{\left(y\theta_2^{3/2} + z\right)} + \left(1 - \frac{x}{k\left(y\theta_2^{3/2} + z\right)}\right)\right] \\
= &\left(\frac{y}{2}\right) \left[\left(\lim_{\theta_2 \to +\infty} \frac{y(\theta_2^{\frac{3}{2}} - \frac{2z}{y})}{\left(y\theta_2^{3/2} + z\right)}\right) + \left(\lim_{\theta_2 \to +\infty} \left(1 - \frac{x}{k\left(y\theta_2^{3/2} + z\right)}\right)\right)\right] \\
= &\left(\frac{y}{2}\right) \left[\left(\lim_{\theta_2 \to +\infty} \frac{y(\theta_2^{\frac{3}{2}} - \frac{2z}{y})}{\left(y\theta_2^{3/2} + z\right)}\right) + 1\right] \\
= &\ y
\end{aligned}$$

where the last equality follows as, by another application of L'Hospital's rule, $\lim_{\theta_2 \to +\infty} \frac{y(\theta_2^{\frac{3}{2}} - \frac{2z}{y})}{\left(y\theta_2^{3/2}+z\right)} = \lim_{\theta_2 \to +\infty} \frac{(3/2)y\theta_2^{\frac{1}{2}}}{(3/2)y\theta_2^{\frac{1}{2}}} = 1$. Therefore, $\lim_{\theta_2 \to +\infty} K(\theta_2) = \lim_{\theta_2 \to +\infty} \frac{K_1'(\theta_2)}{K_2'(\theta_2)} = \frac{2}{y} > 0$. Therefore, there exists a $K > 0$ such that $\underset{\theta_2 \geq 1}{sup} K(\theta_2) \leq K$ and $\frac{\partial(B+C)}{\partial\theta_2} < 0$ if $\frac{b}{c} > K$. It follows that there exists a K strictly positive, such that at any $p_1 \geq 0$, whenever $b/c > K$, $\frac{\partial V_1(p_1, p_2^*, \theta_2)}{\partial\theta_2} < 0, \forall\, \theta_2 \geq 1$. □

The preceding lemma establishes that, at any positive first period asset price, the (indirect) marginal utility of the period-1 consumer in θ_2, evaluated at p_2^*, is negative whenever $\theta_2 \geq 1$. Observe that we have to consider unbounded values of θ_2 in lemma 2 as we allow for the possibility that $p_2 = 0$.

In the next lemma we will allow for a negative asset price at $t = 1$. Observe that the reason for this is implicit in the calculations underlying lemma 2: it is that for each $p_1 \geq 0$, $V_1(p_1, \theta_2)$ attains a maximum at some value $\theta_2 < 1$. Note that in this case with $p_1 < 0$ the budget constraint at $t = 1$ is: $\theta_2 \geq 1 + d_1/p_1 - c_1/p_1$, which imposes a lower bound on θ_2.

Lemma 3. *There exists a K strictly positive such that whenever $b/c > K$, (a) there exists $p_1^* < 0$ such that $\frac{\partial V_1(p_1^*, p_2^*, \theta_2 = 1)}{\partial\theta_2} = 0$, however (b) $\underset{\theta_2 \to +\infty}{lim} \frac{\partial V_1(p_1^*, p_2^*, \theta_2)}{\partial\theta_2} = -p_1^* > 0$.*

Proof. By computation observe that $p_1^* = \frac{\partial(B+C)}{\partial\theta_2}|_{\theta_2=1} < 0$. Moreover $\frac{\partial V_1(p_1^*, p_2^*, \theta_2)}{\partial\theta_2} = -p_1^* + \frac{\partial(B+C)}{\partial\theta_2}$. By lemma 2 $\frac{\partial C}{\partial\theta_2} = c\frac{(k - xf'(\theta_2))}{(k\theta_2 - xf(\theta_2))} \geq 0$, $\theta_2 \geq 1$. It follows that $\frac{\partial B}{\partial\theta_2} \leq$

$\frac{\partial(B+C)}{\partial\theta_2} < 0$. Using the expressions derived in lemma 2,

$$\begin{aligned}\lim_{\theta_2\to+\infty}\frac{\partial B}{\partial\theta_2} &= \lim_{\theta_2\to+\infty} b\frac{f'(\theta_2)}{f(\theta_2)}\\ &= -b\left(\lim_{\theta_2\to+\infty}\frac{y(\theta_2^{\frac{3}{2}}-\frac{2z}{y})}{2\theta_2\left(y\theta_2^{3/2}+z\right)}\right)\end{aligned}$$

As the both the numerator and denominator of $\frac{y(\theta_2^{\frac{3}{2}}-\frac{2x}{y})}{2\theta_2\left(y\theta_2^{3/2}+z\right)}$ goes to $+\infty$ as $\theta_2\to+\infty$, using L'Hospital's rule

$$\begin{aligned}\lim_{\theta_2\to+\infty}\frac{y(\theta_2^{\frac{3}{2}}-\frac{2x}{y})}{2\theta_2\left(y\theta_2^{3/2}+z\right)} &= \frac{1}{2}\left(\lim_{\theta_2\to+\infty}\frac{y\left(\frac{3}{2}\right)\theta_2^{1/2}}{y\theta_2^{3/2}+z+\theta_2 y\left(\frac{3}{2}\right)\theta_2^{1/2}}\right)\\ &= \frac{1}{2}\left(\lim_{\theta_2\to+\infty}\frac{1}{\frac{y\theta_2^{3/2}+z}{y\left(\frac{3}{2}\right)\theta_2^{1/2}}+\theta_2}\right).\end{aligned}$$

Now, $\lim_{\theta_2\to+\infty}\left(\frac{y\theta_2^{3/2}+z}{y\left(\frac{3}{2}\right)\theta_2^{1/2}}+\theta_2\right) = \lim_{\theta_2\to+\infty}\left(\frac{y\theta_2^{3/2}+z}{y\left(\frac{3}{2}\right)\theta_2^{1/2}}\right)+\lim_{\theta_2\to+\infty}\theta_2$. Again, by using L'Hospital's rule

$$\lim_{\theta_2\to+\infty}\left(\frac{y\theta_2^{3/2}+z}{y\left(\frac{3}{2}\right)\theta_2^{1/2}}\right) = \lim_{\theta_2\to+\infty}\left(\frac{y\left(\frac{3}{2}\right)\theta_2^{1/2}}{y\left(\frac{3}{4}\right)\theta_2^{-(1/2)}}\right) = \lim_{\theta_2\to+\infty}(2\theta_2)$$

so that $\lim_{\theta_2\to+\infty}\left(\frac{y\theta_2^{3/2}+z}{y\left(\frac{3}{2}\right)\theta_2^{1/2}}+\theta_2\right) = \infty$. Therefore $\lim_{\theta_2\to+\infty}\frac{y(\theta_2^{\frac{3}{2}}-\frac{2x}{y})}{2\theta_2\left(y\theta_2^{3/2}+z\right)} = 0$ which, in turn, implies that $\lim_{\theta_2\to+\infty}\frac{\partial B}{\partial\theta_2} = 0^-$ and hence $\lim_{\theta_2\to+\infty}\frac{\partial(B+C)}{\partial\theta_2} = 0^-$. Therefore $\lim_{\theta_2\to+\infty}\frac{\partial V_1}{\partial\theta_2}(p_1^*, p_2^*, \theta_2) = -p_1^* > 0$. □

Lemma shows that at the unique period one asset price p_1^* (so that choosing $\theta_2 = 1$ satisfies the first-order condition characterizing an interior optima holds) also has the property that the marginal (indirect) utility of holding an extra unit of the period one asset is also strictly positive for large values of θ_2. In the following lemma, we show that that $\theta_2 = 1$ is never an optimal choice even allowing for a negative asset price at $t = 1$. In addition we also show that $\theta_2 = 1$ does not belong to the convex hull of demand even allowing for a negative asset price at $t = 1$. The latter statement implies that, even if we re-interpret the model so that the representative agent is a collection of a continuum of identical individuals, equilibrium existence is not restored.

Lemma 4. *Given lemmas 1, 2, 3, $\theta_2 = 1$ is not an element of the convex hull of demand even allowing for a negative asset price at $t = 1$ so that neither a competitive equilibrium, nor a weak competitive equilibrium, with a sophisticated representative agent exists.*

Proof. Lemma 1 implies that with a sophisticated representative agent there is a unique p_2^* candidate equilibrium price at period 2. For an equilibrium to exist, given p_2^*, there must be a p_1^* such that for the representative agent $\theta_2^* = 1$ is a SS.

There are two cases to consider.

1. $p_1 \geq 0$: fix a (p_1, p_2^*), $p_1 \geq 0$, by lemma 2 $\theta_2 = 1$ is never an optimal solution. Next, observe that a necessary condition for $\theta_2 = 1$ to be in the convex hull of individual demand is that $\frac{\partial V_1}{\partial \theta_2}(p_1, p_2^*, \theta_2') = 0$ for some $\theta_2' < 1$ and $\frac{\partial V_1}{\partial \theta_2}(p_1, p_2^*, \theta_2'') = 0$ for some $\theta_2'' > 1$, a possibility ruled out by lemma 2. It follows that $\theta_2 = 1$ is not in the convex hull of individual demand.

2. $p_1 < 0$: by lemma 3, in order to ensure that $\theta_2 = 1$ is chosen at $t = 1$ it necessarily follows that the only candidate equilibrium price is $p_1 = p_1^*$. Further by lemma 3 there exists $\underline{\theta}_2 > 1$ such that for all $\theta_2 > \underline{\theta}_2$, $\frac{\partial V_1}{\partial \theta_2}(p_1^*, p_2^*, \theta_2) > 0$. Therefore $\lim_{\theta_2 \to +\infty} V_1(p_1^*, p_2^*, \theta_2) = \lim_{\theta_2 \to +\infty} \int_{\underline{\theta}_2}^{\theta_2} \frac{\partial V_1}{\partial \theta_2}(p_1^*, p_2^*, \theta_2) + V_1(p_1^*, p_2^*, \underline{\theta}_2) = +\infty$ as $\lim_{\theta_2 \to +\infty} \frac{\partial V_1}{\partial \theta_2}(p_1^*, p_2^*, \theta_2) = -p_1^*$. It follows that at prices (p_1^*, p_2^*), $\theta_2 = 1$ cannot be an optimal choice for the representative agent.

It remains to check that $\theta_2 = 1$ is not in the convex hull of demand when $p_1 < 0$. By computation, observe that for any $\hat{\theta}_2 > 1$, a necessary condition for $\hat{\theta}_2$ to be an optimal choice is that $p_1 = p_1^*(\hat{\theta}_2) = \frac{\partial (B+C)}{\partial \theta_2}|_{\hat{\theta}_2} < 0$. Moreover using arguments analogous to lemma 3, it is verified that $\lim_{\theta_2 \to +\infty} \frac{\partial V_1(p_1^*(\hat{\theta}_2), p_2^*, \theta_2)}{\partial \theta_2} = -p_1^*(\hat{\theta}_2)$ and hence $\lim_{\theta_2 \to +\infty} V_1(p_1^*(\hat{\theta}_2), p_2^*, \theta_2) = +\infty$. Therefore, there is no $p_1 < 0$ for which there is some $\hat{\theta}_2 > 1$ such that $\hat{\theta}_2$ is an optimal choice. It follows that $\theta_2 = 1$ cannot be in the convex hull of individual demand. □

Note that the above non existence result is robust to small variations in parameter values by the continuity of the derivatives of the utility functions in these parameters.

Remarks. The feature that implies the nonexistence result in our example is the fact that period 2 consumption is an inferior good from the perspective of the period 2 decision maker, but it is a normal good from the perspective of the period 1 decision maker and this implies that the period 1 consumer demand for period 2 wealth, i.e. for θ_2, is satiated. The papers of Luttmer and Mariotti (2006), Herings and Rohde

(2006), Luttmer and Mariotti (2007) are able to prove existence because the issue of inferior goods and satiation and does not arise in their papers. With sequentially complete markets, concavity of the single period utility functions together with time separability imply that both c_2 and c_3 are normal goods for the period 2 consumer and this is enough to avoid satiation in period 1, because if both c_2 and c_3 increase in θ_2, then for the period 1 consumer utility monotonically increases in θ_2. In such a situation, the existence of an equilibrium price p_1^* such that the period 1 market clearing quantity $\theta_2 = 1$ belongs to the convex hull of the demand correspondence can be proved with a standard fixed point argument.[12] A necessary condition to re-establish the existence of the competitive equilibrium allowing for a large number of identical consumers is that there is at least one optimal quantity θ_2' which is smaller than the market clearing quantity $\theta_2 = 1$ and at least one optimal quantity θ_2'' which is greater. In our example, allowing for a large large number of identical consumers does not successfully reestablish the existence of a competitive equilibrium because, for any positive p_1, (8) decreases in θ_2 for all $\theta_2 \geq 1$ and this implies that $\theta_2 = 1$ cannot belong to the convex hull of the demand function. This happens because c_2 is an inferior commodity for all $\theta_2 \geq 1$ and, given the values of the discount factors b and c, the cost of a marginal decrease in c_2 is greater than the benefit of a marginal increase in c_3 for the period 1 decision maker.

Negative prices can generally reestablish the existence of a competitive equilibrium which fails to exist because of satiation, when free disposal is not allowed, as it happens in our example. In the case of negative p_1, the period 1 budget constraint is: $\theta_2 \geq 1 + d_1/p_1 - c_1/p_1$, i.e. with a negative p_1 there is no upper bound on the quantity of θ_2 which the consumer can demand and there is instead a lower bound. We have proved that any negative p_1 cannot re-establish the competitive equilibrium in our example showing that, given any negative p_1, the unique optimal choice for the period 1 consumer is to demand a quantity of θ_2 which goes to $+\infty$ implying non-vanishing excess demand.

4. Equilibrium with naive agents

In this section we study equilibria with a naive representative agent.

[12]The convex hull of the individual excess of demand of the period 1 consumer is convex (trivial) and has a closed graph (implied by the upper hemicontinuity and compact-valuedness of the demand function and the monotonicity of the preference for θ_2).

Fix p_t, $t = 1, 2$. When the representative agent is naive at $t = 1$, he does not anticipate that at $t = 2$ consumption and asset choices will be re-optimized. Therefore at $t = 1$ the representative agent solves

$$\begin{aligned} \max_{(c_1,c_2,c_3,\theta_2,\theta_3)} u_1(c_1, c_2, c_3) \\ \text{subject to:} \\ c_1 + p_1\theta_2 \le p_1 + d_1, \\ c_2 + p_2\theta_3 \le (p_2 + d_2)\theta_2, \\ c_3 = d_3\theta_3. \end{aligned} \tag{9}$$

Let $\hat{c}_t(p_1, p_2)$, $t = 1, 2, 3$ and $\hat{\theta}_t(p_1, p_2)$, $t = 2, 3$ denote the unique solution (if it exists) to the preceding maximization problem.

At $t = 2$ the representative agent solves

$$\begin{aligned} \max_{(c_2,c_3,\theta_3)} u_2(c_2, c_3) \\ \text{subject to:} \\ c_2 + p_2\theta_3 \le (p_2 + d_2)\hat{\theta}_2, \\ c_3 = d_3\theta_3. \end{aligned} \tag{10}$$

With a slight abuse of notation, the unique solution (if it exists) to the preceding maximization problem is denoted by $\tilde{c}_t(p_2, \hat{\theta}_2(p_1, p_2)) = \tilde{c}_t(p_1, p_2)$, $t = 2, 3$ and $\tilde{\theta}_3(p_2, \hat{\theta}_2(p_1, p_2)) = \tilde{\theta}_3(p_1, p_2)$.

We say preferences are strongly dynamically inconsistent if for all non-negative c_1 the preferences of the representative agent at $t = 1$ over $(c_2, c_3) \in \mathbb{R}^2_+$ are different from his preferences at $t = 2$ over $(c_2, c_3) \in \mathbb{R}^2_+$, or equivalently, for all non-negative c_1, $\frac{\frac{\partial u_1}{\partial c_3}(c_1,c_2,c_3)}{\frac{\partial u_1}{\partial c_2}(c_1,c_2,c_3)} \neq \frac{\frac{\partial u_2}{\partial c_3}(c_2,c_3)}{\frac{\partial u_2}{\partial c_2}(c_2,c_3)}$, $(c_2, c_3) \in \mathbb{R}^2_+$.[13]

The assumption that in every period the utility function is strictly monotone in consumption implies that inter-temporal budget constraints are satisfied at equalities in either maximization problem. As before, in a competitive equilibrium, it must be optimal for both selves of the naive representative agent to hold the entire unit of

[13]An example of a utility function satisfying this stronger definition would be one where there is a systematic shift in marginal rates of substitution between c_2, c_3 when the representative agent enters period 2, for example, $u_1(c_1, c_2, c_3) = \log c_1 + \log c_2 + \log c_3$ and $u_2(c_2, c_3) = 2\log c_2 + \log c_3$.

the asset in each period ($\theta_1 = \theta_2 = \theta_3 = 1$) and consumption must be equal to the entire paid dividend in each period ($c_1 = d_1, c_2 = d_2, c_3 = d_3$).

At this point we define two different notions of competitive equilibrium with a naive representative agent.

Definition 4. A *perfect foresight competitive equilibrium* is a combination of prices (p'_1, p'_2) and allocations $(\theta'_1, c'_1, \theta'_2, c'_2, \theta'_3, c'_3)$ such that $c'_1 = \hat{c}_1(p'_1, p'_2)$, $\theta'_2 = \hat{\theta}_2(p'_1, p'_2)$, $c'_2 = \tilde{c}_2(p'_1, p'_2)$,$\theta'_3 = \tilde{\theta}_3(p'_1, p'_2)$, $c'_3 = \tilde{c}_3(p'_1, p'_2)$ and $\theta'_1 = \theta'_2 = \theta'_3 = 1$, $c'_1 = d_1, c'_2 = d_2, c'_3 = d_3$.

Definition 5. A *temporary competitive equilibrium* is a combination of prices (p'_1, p'_2, p''_2) and allocations $(\theta'_1, c'_1, \theta'_2, c'_2, \theta'_3, c'_3)$ such that $c'_1 = \hat{c}_1(p'_1, p'_2)$, $\theta'_2 = \hat{\theta}_2(p'_1, p'_2)$, $c'_2 = \tilde{c}_2(p'_1, p''_2)$,$\theta'_3 = \tilde{\theta}_3(p'_1, p''_2)$, $c'_3 = \tilde{c}_3(p'_1, p''_2)$ and $\theta'_1 = \theta'_2 = \theta'_3 = 1$, $c'_1 = d_1, c'_2 = d_2, c'_3 = d_3$.

The definition of a perfect foresight competitive equilibrium with a naive agent is new. The definition of a temporary competitive equilibrium corresponds to the notion of a naive equilibrium in Herings and Rohde (2006). The following proposition establishes that although a perfect foresight competitive equilibrium with a naive representative agent does not exist, a temporary competitive equilibrium does.

Proposition 2. *A perfect foresight competitive equilibrium with a naive representative agent does not exists, however a temporary competitive equilibrium does.*

Proof. At $t = 1$ as the utility function $u_t()$ of the representative agent is smooth and strictly concave, $\hat{\theta}_2 = \hat{\theta}_3 = 1$ if and only if asset prices satisfy the following equations:

$$p'_1 = (p'_2 + d_2)\frac{\frac{\partial u_1}{\partial c_1}(d_1, d_2, d_3)}{\frac{\partial u_1}{\partial c_2}(d_1, d_2, d_3)},$$

$$p'_2 = d_3\frac{\frac{\partial u_1}{\partial c_3}(d_1, d_2, d_3)}{\frac{\partial u_1}{\partial c_2}(d_1, d_2, d_3)}.$$

Next, observe that at $t = 2$, $\tilde{\theta}_3 = 1$ if and only if asset prices satisfy the following equations:

$$p''_2 = d_3\frac{\frac{\partial u_2}{\partial c_3}(d_2, d_3)}{\frac{\partial u_2}{\partial c_2}(d_2, d_3)}.$$

As preferences are strongly dynamically inconsistent $\frac{\frac{\partial u_1}{\partial c_3}(d_1,d_2,d_3)}{\frac{\partial u_1}{\partial c_2}(d_1,d_2,d_3)} \neq \frac{\frac{\partial u_2}{\partial c_3}(d_2,d_3)}{\frac{\partial u_2}{\partial c_2}(d_2,d_3)}$ and therefore $p'_2 \neq p''_2$. Therefore market clearing and individual optimization with a naive representative agent are mutually incompatible if the asset price in the spot market at $t = 2$ is the same as the forecast asset price at $t = 1$. Finally observe that if the representative agent forecasts asset prices p_1',p_2' while the prevailing asset prices at $t = 2$ is p_2'', individual optimization and market clearing are mutually compatible. □

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