Modeling of volatility-linked financial products

by

Lorenzo Torricelli

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
of
University College London
achieved under the supervision of
Professor William T. Shaw

Department of Mathematics
University College London

July 2015

I, Lorenzo Torricelli, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed:

Date:

Abstract

This thesis is the collation of four papers, adapted from their original versions as to form here four distinct chapters. In the first chapter we illustrate and solve the pricing problem of a target volatility option (TVO) using three different methodologies. In the second chapter we study the pricing PDE for a general contingent claim involving an asset and its realized volatility, and then solve it for a variety of actual models and payoffs. The third chapter introduces a class of time-changed stochastic processes based on which a martingale asset price evolution can be devised. Pricing equations for volatility-linked derivatives are also obtained in this framework. In the final chapter we analyze one specific model of this class; we conclude that it does show high flexibility in explaining the forward volatility skew dynamics and that it can capture certain interesting stylized facts.

This thesis is dedicated to Gabriella and Alberto

Acknowledgements

I would like to acknowledge everyone who contributed, at any level, in making this PhD thesis possible. In particular, my thanks go to my supervisor William T. Shaw, my co-authors Giuseppe Di Graziano and Martino Grasselli, Robb McDonald, Andrea Macrina, Valery Smyshlyaev, Helen Higgins, Lane P. Hughston, Louis-Faris Morganti and Yashoda Mahendran, Yue-Kuen Kwok, Stoyan Stoyanov, Cecilia Mancini, Rossella Agliardi, Mark and Angela Lim, Milan Kecman, Julien Hok, Alessandro Gnoatto, Federica Graziani, Julien Pantz, Elena Spinelli, Irene Guaraldo, Mariaenrica Giannuzzi, four anonymous referees, and most importantly my parents for their love and support during these years of study.

Contents

List of Figures

List of Tables

Preface

My PhD research stems from the dissertation I produced in 2009 as a task for completing the MSc on Mathematics and Finance at Imperial College London. Back then I was assigned to an intern cooperation at Deutsche Bank London with Dr Giuseppe Di Graziano; he introduced me to the topic he was researching at the time, the target volatility option, and very kindly shared some of his core results. Based on further developments, we were later able to produce the joint paper [30] on the subject.

As I obtained a full-time PhD position at King's College in 2010, I was fortunate enough to find in Professor William Shaw a supervisor who believed in the potential of the strain of research I was pursuing, and who constantly supported my efforts during these years. The following year I moved to UCL where I completed my research as a part-time student. The outcome of my studies is the present work.

This thesis is in the "collection of essays" format. It consists of four chapters written in the form of journal papers dealing with certain common topics. Although the work is spread over separate projects, I have put some effort in maximizing the cross-referencing. The content of this document is organized as follows. In the introduction we outline and provide motivation for the work done; also, we discuss some related issues and recent developments in the literature. Chapters 1 and 2 are edited versions of the published works [30] and [96]; the paper [30] has also appeared in the collection [56] based on a series of presentations made at the Fields Institute of Toronto. Chapter 3 is a slight improvement of the ArXiv eprint [97], a version of which is at the moment under peer review for a scientific journal. Chapter 4 is serving as a basis for a joint work with Professor Martino Grasselli. The four chapters are wrapped up by a one-page conclusion summarizing the contributions of our work.

To facilitate the comparability with the original works, the notation of each chapter has been kept the same as the corresponding paper. This may sometimes lead to the repeated definition of the same objects across the chapters, at times through different symbols: for example, the quadratic variation process is denoted by I_t in chapter 2 and by $\langle X \rangle_t$ in chapter 3. Appendices containing proofs are placed at the end of each individual chapter.

Introduction

The present thesis is the result of four years of research, one spent at King's College London as a full-time student, and the other three at UCL in a part-time non-resident PhD scheme. By the time I begun my work I did not have in mind a single subject to focus on. My starting point was my MSc dissertation on pricing a certain type of target/controlled volatility investment. As a first step, I assisted my former MSc supervisor Giuseppe Di Graziano in turning such a work into the paper [30]. However, it was unclear to me at that time whether such a topic could have been suitably developed into a full doctoral thesis.

I thus begun to look from a wider perspective at the elements defining the target volatility financial products: an equity -a martingale- and its realized (or historical) volatility -a quadratic variation process. The interplay of these two is not only relevant to the financial practice, as for instance it appears in the target volatility option (TVO) paper [30], but it is also a cornerstone of the modern martingale representation theory: examples are the Dambis and Schwarz's theorem [92] and Monroe's theorem [83]. These theorems reveal that, in some sense, continuous martingales are characterized by their quadratic variations. On the other hand, quadratic variations do appear in derivative payoffs, as it is the case of volatility derivatives, or in mixed payoffs, such as the TVO itself. Also, a strictly related concept is that of a *time change*, that is, a process describing the activity rate at which the trades occur, that generalizes the idea of accrued volatility.

The realized volatility process therefore plays two rather different roles for derivative pricing. As a historical measured variable, it can directly affect the price, for example by appearing as a payoff component. As an abstract stochastic process, it can be used to model the "market activity rate", that is, the speed and magnitude at which the markets reacts over a certain time span to the information flow. Broadly speaking, this PhD thesis consists of a series of separate works where these two concepts of "volatility" are brought to interact with each other, as well as with other classic asset modeling features, such as the presence of jumps in returns, correlation effects, or multifactor model specifications. In the final chapter all these elements are combined into a single novel asset pricing model constituting the culmination of this research.

Even though each chapter represents a self-consistent project, some common features are identifiable across the thesis as a whole. In first place, the approach to derivative pricing taken throughout is the classic Fourier/Laplace inversion technique. This is only natural since we analyze models whose characteristic functions are available in closed form. The distinctive feature here is assuming that a contribution to the derivative price is given by the quadratic variation of the returns, which requires an additional second variable to be included in the integral inversion process. Although efficiency issues are outside the scope of this work, it would be of course a remarkable achievement to solve the problem of finding an optimal integration contour in the multivariate setting presented here. Another similarity across the different chapters is the technical approach to the derivation of the characteristic functions. Those are computed by using either the classical differential/PDE/Feynman-Kac approach, or the change of measure technique used prominently by Carr and Wu in [15]. Finally, pricing equations are normally tested against a Monte Carlo simulation.

Chapter 1 is a slightly expanded version of the target volatility option paper [30] written jointly with Giuseppe Di Graziano. Back in 2008 markets started to show interest in target volatility options and asset allocation strategies, i.e. investments where the asset exposure is set, and possibly rebalanced, according to the volatility realized by the asset over a certain time frame, with the aim of keeping the volatility of the investment at a desired level. Two main types of target volatility investments can be distinguished: the target volatility option, a derivative, and target volatility portfolio strategies/asset allocations.

The study presented here mainly refers to the derivative product. The problem of asset allocations aiming at maintaining a fixed volatility has also been documented in recent times. Since target volatility portfolios can be associated with a constant volatility level, funds of this type can be of interest for the purpose of transparently meeting the risk levels prescribed by the regulatory agencies while limiting the portfolio downside (see [95]). Stoyanov, [95] proves some statistical properties of a target volatility fund in the Heston model; Morrison and Tadrowski [84] assess the guarantee costs of entering a target volatility investment; in [22], Coles provides additional results on pricing, hedging and managing the skew of options written on a target volatility portfolio. A number of other studies illustrating the performance of target volatility funds are also present [19, 82, 100]. Criticism over the risk reduction properties of target volatility portfolio strategies has been raised in [50, 101], pointing out that when the underlying asset is discontinuous and jumps at random times, a periodic rebalancing at fixed times cannot offset the volatility fluctuations. We maintain an interest in the topic, and believe that there are unanswered points in the extant literature; in section 1.7 we sketch a continuous-time portfolio model for the target volatility allocation that essentially follows [95].

Instead, a target volatility option is an option, normally written on an equity or an index, whose payoff is normalized at expiry with the volatility realized by that same index or equity and then multiplied by a contractually-agreed constant giving the subjective volatility view of the investors, the *target volatility*. The ratio between the target and the realized volatility hence determines the total notional exposure of the payoff. There are various financial motivations for buying a TVO. In first place the price of the option is generally lower than that of a vanilla call option written on the corresponding index or equity. Secondly, the price of the option, for short maturities and at-the-money contracts, is of Black-Scholes-type with implied volatility equal to the target volatility. Typically, an investor chooses to enter a TVO contract when plain vanilla quoted options are not affordable due to high implied volatilities. At the same time, if the "target" prediction is correct, she still retains chances of receiving a similar or even higher payoff. The main contribution of this first chapter is to provide practical pricing equations for this new type of derivative. To achieve this we relied on the simplifying assumption that the returns of the asset are independent from the stochastic volatility. Three methodologies are provided: one Taylor strike price expansion, one Laplace integral inversion, and a polynomial approximation. The assumption of independence further allows pricing the TVO by replication with claims on the underlying, which in turn can be decomposed in a portfolio of call and put options (by the Breeden and Litzenberger formula, [10]), an argument already endorsed by Carr and Lee [14] for the valuation of pure volatility derivatives. We also reintroduce a third approach, present in the Imperial College MSc dissertation but left out from the paper [30], consisting in a uniform approximation of the TVO payoff through polynomial claims on the underlying. Also, comparing to the published paper, the numerical part has been revised and expanded, and figures relating to the various methods have been added.

Having established several valuation methods mostly relying on an independence assumption, the relaxation of one such an assumption represented a clear further research objective. While it looked problematic to introduce correlation in the pricing by replication argument, the integral inversion method does not seem to suffer from the same drawback. Indeed, characteristic functions of analytical models naturally account for dependence between the various equations of a diffusion process.

In order to add correlation between price and volatility, in chapter 2 we restate the pricing problem of a TVO from a purely theoretical standpoint. We augment an Ito diffusion consisting of a correlated pair of equations for the price and the stochastic volatility with the ODE giving the dynamics of the realized volatility. By means of the the usual Feynman-Kac argument, we can associate to these equations a PDE with terminal condition given by the TVO payoff; the solution of such a PDE yields the derivative price. We soon realized that there is no specificity at all in assuming the terminal condition to be the TVO payoff, and that we can select as a contingent claim to be priced any sufficiently regular function of price and volatility. The discussion and the derivation follow quite closely the book by Lewis, [75]: by Fourier-transforming the PDE and using the unity as a terminal condition, we can in some significant cases solve the equation explicitly, thus obtaining the joint characteristic function of the asset and its accrued volatility. By then plugging this function in the inverse-Fourier representation for the derivative value, we produce the pricing equation we were aiming at.

However, in principle, the introduction of a second variable to be modeled calls for a revision of the discussion on the conditions that the diffusion must satisfy in order to guarantee the existence and uniqueness of the solution. It is well-known that when some growth constraint on the coefficients on the diffusion process are not met, problems arise in the correspondence between the stochastic and the analytical solution of the PDE. A financial interpretation of these issue is the case for the so-called "volatility explosions" and "volatility vanishings" (Lewis [75], Sin [94]). It has been long- known (Feller [39, 40]) that the martingale property of the stochastic solution is linked to the non-finiteness of a certain functional of the volatility process. Failing this, the Feynman-Kac argument identifying the stochastic and analytical solution for the PDE breaks down, meaning that the stochastic solution is a supermartingale only. In this case the difference in value with the true price represents the premium to be paid as to account for the possibility of a volatility explosion or vanishing. Although practitioners are normally well aware of the conditions under which the calibration procedure fails to meet volatility explosion/vanishing tests, we remark that a less recognized issue in the PDE solution/discounted expected value correspondence is that also regularity conditions on the payoff must be imposed (Friedman, [45]). This means that in principle not every payoff can be priced in a given stochastic volatility model; in the case of power claims in the Heston model, an important work is that of Andersen and Piterbarg [3]. We also take into consideration these conditions throughout our discussion.

Since the general set up taken allows to value a variety of joint payoffs, in the pricing problem we consider contingent claims different from the TVO. We thus end up introducing some derivatives written on the joint performance on an asset and its accrued volatility that we hope could be of interest for financial uses. On the other hand, the general valuation formula of chapter 2 does not account for the specificity of the TVO. By comparing equation (2.15) with the corresponding pricing relation (1.43) in chapter 1, one sees that in the latter a single complex integration variable is present, as opposed to the full complex bivariate inversion of equation (2.15). This is because in (1.43) we exploit the integral representation of the inverse square root, which is a specific feature of the Fourier transform of the TVO payoff. This significantly shortens both the computational burden and the truncation error entailed by (2.15).

So far the content of the paper [96]. In this thesis, we extend the treatment to jump diffusion models. The presence of jumps in the price has a major impact on the volatility realizations, since it adds to the continuous quadratic variation process random jumps whose magnitude is given by the sum of the squares of the log-price discontinuities. Therefore, by adding to the exponential Brownian equation a jump part, we also turn the SDE for the realized volatility into a jump diffusion. In full analogy with the purely diffusive case, we apply the Fourier transform to the associated PIDE and, by assuming the terminal condition to be 1, we then derive the characteristic function of the transition probability densities. The regularity conditions on the resulting jump diffusions must be then complemented by the usual mild requirements on the decay of the distribution of the jumps.

Finally, we added a part on hedging a joint asset and volatility derivative when the underlying follows a jump diffusion. The two arguments offered are both very standard: namely, the replication argument by "completing the market" with a traded instrument further to the underlying (e.g. Wilmott [99]) and the classic mean-variance hedging strategy for jump diffusions. In both cases, the formulae obtained are in line with those known for derivatives depending on an underlying asset only. The reason seems to us to lie in the fact that the quadratic variation is a random process completely dependent to the stock price movements. Therefore, the equations for hedging do not account for extra terms as it would happen if we were minimizing L^2 distances on spaces spanned by more than one martingale, or tried to replicate the pathwise movements of multiple Brownian components.

Chapter 3 has its foundations in a pattern that can be observed in the formulae of chapter 2. The equations in chapter 2 for the joint characteristic function of the log-price and its quadratic variation can be seen to retain the same functional structure of the univariate case of the log-returns, differing from it only by a substitution involving the second Fourier variable. This has to do with a probabilistic interpretation of the characteristic function of the log-price. As pointed out by Carr and Wu [15] among the others, such a transform can be written as a characteristic function of the quadratic variation after a change of measure keeping track of correlations. In our case, this relationship provides a direct analytical connection between the asset returns and their quadratic variation. This ultimately means that we are able to express the joint characteristic function of the returns and the quadratic variation in terms of a transform of the quadratic variation only, calculated for some combination of the two Fourier parameters.

A useful mathematical representation contributing to a financial view of this relationship is that of a *time change*, that is, an increasing almost surely finite process dictating the random arrival times of trades. Realized volatility, understood as the pathwise integral of an instantaneous volatility process, is one instance of a time change. The powerful feature of time-changed modeling is that it easily specifies to many different popular models from the literature (stochastic volatility models, jump diffusions, Lévy processes etc.). Time-changed asset models are typically applied to Lévy processes; also, under a certain condition, the time-changing operation preserves the local martingale property of the underlying process. In this respect, our contribution to the time-changing theory is that we found out that it is possible to time-change the continuous and jump part of some underlying Lévy process with two *separate* and possibly dependent processes, in such a way as to obtain a local martingale. If we assume the time changes to be absolutely continuous, this means that we can build a price process with *two* associated rates of activity: one for the stochastic variance and one for the jump rate. That is, we can postulate a pair of processes that respectively model the rate of occurrence of "normal" and "critical" price changes. It stands to reason that those effects in real markets may very well be linked or correlated, a stylized fact that we are then able to capture. We christened these models *decoupled time-changed (DTC) Lévy processes*.

To a certain extent, DTC asset pricing models were investigated before by Huang and Wu in [63]. However, their work leaves open questions on the theoretical standing of this kind of processes. For example, the martingale properties of the asset price are not shown in their work. By appealing to the theory of Jacod [65], we find that DTC Lévy processes are local martingales if subject to a synchronization property between the underlying Lévy process and the time changes. Namely, the underlying Lévy process must be constant on the discontinuity sets of the time changes. Clearly, the same applies to ordinary time changes, a fact that was not fully recognized in the work of Carr and Wu [15]. It is our view that to gain a better understanding of time-changed Lévy processes one must directly inspect their martingale representation given by their local characteristics (Jacod and Shiryaev [67], chapter 2). The bottom line is that the characteristic triplet of a Lévy process is well-behaved with respect to the time-changing operation only if the synchronization properties mentioned above hold. This ground of analysis starkly contrasts with the previous literature, where proofs normally rely on Doob's optional sampling theorem. Arguments of this kind seem to us to be fundamentally flawed, since time changes are not almost surely bounded processes, nor are the exponential martingales arising from Lévy processes uniformly integrable in general.

From the practical viewpoint, DTC processes are of importance because they not only specify to standard time changed models, but also to other important asset pricing models, like the stochastic volatility model with jumps (Bates, [6, 7]) and the stochastic volatility model with jumps and stochastic jump rate (Fang, and Huang and Wu [37, 63]). Moreover, joint asset and volatility payoffs can be priced within our proposed DTC framework, a simple property whose search led to the whole body of theory just mentioned. The third chapter of this thesis is a rather faithful adaptation of the paper [97].

Still, one may very well question whether, theoretical subtleties apart, a DTC theory could be of benefit for further asset modeling research. A fair objection would be that this kind of processes may just represent a mathematical shorthand of already known models. Furthermore, a full empirical analysis has been already conducted in [63], and our work does not add from that angle. As an answer to this objection, we believe to have found (section 3.6) a new way to asset modeling which makes full use of the new features allowed by a DTC process; most importantly, the ability to model dependency between the stochastic volatility and a stochastic jump rate within an analytical pricing framework.

The idea is to move away from the traditional choice for the instantaneous volatility processes as given by a classic system of SDEs, typically given by CIR-type equations. As Grasselli and Tebaldi [55] have shown, in the class of exponentially-affine models it is highly unclear if a multivariate set-up based on exogenously-correlated Ito diffusions can generate analytical transforms for the model. Instead, we assume the activity rates to follow the diagonal entries of a multivariate matrix process called the *Wishart process*, carrying a more sophisticated intrinsic correlation structure. The mathematics of the Wishart process have been extensively studied by Bru [12], and have been already used in financial modeling by Grasselli and da Fonseca [27, 28] and Gouriéroux and Sufana [52, 53], to name a few. The analytical properties of the Wishart process carry on to the Laplace transform of a DTC price model based on it. At the same time, the interdependencies between the entries of the Wishart process generate in our context an endogenous stochastic correlation between the jump arrival rate and the instantaneous variance of the process.

In chapter 4 we generalize and make more precise the ideas sketched on chapter 3, section 3.6. In order to model the instantaneous activity rates, we employ this time two projections of a common underlying Wishart process of arbitrary dimension. The model retains a dependence between the activity rates; moreover, it is a multifactor model, a property that allows to explain the volatility surface by means of several stochastic factors. In such a way not only we obtain a correlation between the jump activity and the stochastic volatility, but we are also able to generate a stochastic correlation between the log-asset returns and the volatility, a distinctive property of the multifactor models. This is a desirable feature if one intends to model forward volatility smiles, which is in turn essential for correctly pricing forward-starting derivatives (da Fonseca *et al.* [28], da Fonseca and Grasselli [26]). Also, the ability to capture the effects of sophisticated volatility/asset dependencies makes of this model an ideal candidate for the valuation of heavily volatility-dependent derivatives like the TVO.

Ultimately, in this final chapter we combine in a concrete model all of the elements introduced in this thesis: mixed payoffs of volatility and equity, activity rates and time changes, inverse-Fourier pricing, transform analysis by changes of measures, jumps, correlations, and so on. We believe that direct inroads to this new model would have been much more difficult if we did not have the theoretical DTC framework at hand to refer to.

At the time of writing this introduction some of the ideas contained in this thesis seem to have caught on. In [98] Wang and Wang add to the TVO analysis of [30] by studying the hedging problem from the point of view of the mean-variance optimization. They derive an explicit Föllmer-Schweizer decomposition for the TVO price (Föllmer and Schweizer, [43]), when the underlying follows an exponential Lévy process, in both a continuous and discrete time set-up. Since the Föllmer-Schweizer decomposition operates on semimartingales, if we assume the underlying to be a semimartingale in the real statistical measure, this should allow for a hedging strategy in which the orthogonal component gives the cost of the hedge. In contrast, we notice that the mean-variance optimization argument provided in chapter 2 minimizes the hedging error with respect to the risk-neutral measure, i.e. we directly assume martingale dynamics for the underlying, yielding to a Kunita-Watanabe type decomposition for the hedge (see e.g. Cont and Tankov [23]). This is a weaker result, since the hedging error in this case cannot be readily interpreted in terms of actual portfolio profits and losses, for the latter require to be calculated in the market measure.

Another relevant contribution to the theory of the target volatility options has been put forth by Grasselli and Marabel Romo [54], who make a case for pricing the TVO using a multifactor model. The authors argue that a TVO is highly sensitive to the volatility skew and correlation modeling. To this end, it can be for example noted that a linear correlation between returns and volatility induces a monotonic relationship between the option price and the movements of the underlying. Indeed, assuming the leverage to be negative as logical, a price increase is associated with a volatility reduction and thus to an even higher call TVO price. However, if we assume a variable correlation between returns and volatility, as truly observed in the markets, this no longer needs to be true. Typically, variable (and in particular, stochastic) correlation effects can be obtained by means of a multifactor model specification. The authors thus carry out an empirical analysis by calibrating both the two factor model by Christoffersen *et al.* [20] and the classic Heston [60] model, and use the parameters to price a call TVO. As expected, they find relevant price discrepancies between the two model. As a side result they also give formulae for the Vega and Vomma sensitivity of the TVO; in particular the TVO Vomma (sensitivity of Vega with respect to the volatility) is negative. Implied volatilities from the option increase precisely when Vega decreases. Therefore TVOs can be used to Vega-hedge other exotic derivatives, like e.g. the reverse cliquet option, whose Vomma has opposite sign. This property makes of the TVO a possible hedging instrument for this portfolio sensitivity measure. Finally, they provide pricing formulae for a forward starting TVO, that is, a TVO whose strike price is set at a date later than the inception of the contract.

The Fourier inversion technique applied to joint asset and volatility products used in [30, 96] and reproduced in this thesis has been recently improved by da Fonseca *et al.* [25]. By conditioning to the total variance paths the authors combine the inverse Fourier pricing equation also found in [96] (chapter 2, equation (2.15)) with the integral representation of the inverse square root of the total variance used for instance in (1.43). This yields a mixed Laplace/Fourier transform of the transition densities to be used in the semi-closed formula for the TVO price. They extend this technique to other path-dependent volatility derivatives, like the double digital call option introduced in chapter 2, section 2.6, and the corridor swap. Finally, they provide analytical formulae for pricing these derivatives under a number of models, including the Heston [60], Bi-Heston [20], and Multi-Heston [28] volatility models.

A further work related to the pricing of exotic derivatives through joint transforms is that of Zheng and Zeng [103]. In the context of a 3/2 volatility models the authors derive a formula for the triple joint characteristic function of the log-price, the realized volatility and the instantaneous volatility. En route to this result, the authors recover the formula for the joint log-price and integrated variance already presented in [96] (chapter 2, section 3.5). They also extend the treatment to the computation of joint forward characteristic functions for the 3/2 model, and use these for pricing forward-starting target volatility options and other volatility-dependent products.

These and other pieces of research confirm that the body of financial and mathematical theory on cross volatility/equity payoffs is enjoying a significant expansion in recent times. The introduction of these products in financial markets calls for a more accurate modeling of the possible interplay between these two factors. In particular, in order to treat these new payoffs one must improve the ability to handle the interactions between the evolution of an asset and that of its volatility. This is the general research area we are trying to contribute to with the present thesis. In writing this piece of research, we have followed a path that starts off with the analysis of a new volatility-linked payoff and terminates with an asset pricing model providing a natural environment for its valuation.

Chapter 1

Target volatility options and asset allocations

Variance and volatility swaps were the first instances of volatility derivatives. They were introduced in the late nineties to allow investors to trade pure volatility risk (see [29] for a detailed account). Over the past few years, volatility products have become very liquid and widely traded instruments. Investors use volatility derivatives to hedge the volatility risk of their portfolios or to speculate on future realized volatility levels.

Variance and volatility swaps have been extensively studied in the literature. Derman *et al.* [29] show how to price and statically hedge variance swaps in a model independent fashion by investing in a portfolio of call and put options when the underlying exhibits continuous sample paths. In a seminal paper, Carr and Lee [14] provide several methods for pricing and hedging a large class of functions on the quadratic variation. Prices and hedges of quadratic variation claims are expressed in terms of weighted portfolios of European contracts on the terminal value of the underlying. Fritz and Gatheral [46] instead study some ill-posed problems connected with the replication strategy suggested by Carr and Lee [14] for certain payoffs and propose some regularization schemes.

A new type of volatility derivatives was introduced around 2008 under the name of target volatility option (TVO). TVOs allow investors to take a joint view on the realized volatility of a given underlying and its price. For example, a target volatility call pays at maturity the terminal value of the underlying S_T minus the strike K, floored at zero, rescaled by the ratio of a given target volatility (an arbitrary constant, say $\bar{\sigma}$) and the realized volatility RV_T of the underlying over the life of the option:

$$\phi(S_T, RV_T) = \frac{\bar{\sigma}}{RV_T} (S_T - K)^+ . \tag{1.1}$$

TVOs are popular with investors and hedgers because they are typically cheaper than vanilla options. As long as realized volatilities are lower than the target volatility, the payoff of the former is higher than the payoff of the corresponding vanilla option.

During the 2008 financial crisis for example, implied volatilities across asset classes experienced a steep increase, with a significant impact on option (long vega) costs. The generalized increase in implied volatilities was in part a consequence of higher expected future realized volatilities, but was also connected with dealers limits/reluctance to increase their short vega positions. TVOs were then introduced to allow investors to take a bullish/bearish view on the underlying asset in an option format at a relatively low cost.

A target volatility asset allocation (TVA) is instead a portfolio strategy aiming at maintaining a constant investment volatility over the time horizon. In order to achieve this, the equity exposure is periodically adjust at pre-specified dates, according to an inverse relationship with the volatility realized by the underlying during the previous holding period. If the volatility falls, the equity exposure is increased to take advantage of the bullish market situation. In contrast, as volatility increases and equity prices fall, the portfolio tends to lock the earnings in a bond position. On balance, the asset allocation should offset the equity volatility shocks and maintain the total investment volatility at an approximately constant level $\overline{\sigma}$. This portfolio strategy is important for two reasons. Firstly, for risk management purposes the amount $\overline{\sigma}$ of volatility targeted does provide the desired risk exposure of the investment. Secondly, in a valuation framework, the price of a derivative written on a target volatility index is approximately the Black-Scholes value of volatility $\overline{\sigma}$.

In this chapter we provide three methodologies for the pricing of TVOs. We shall assume that volatility is independent from the Brownian motion driving the returns underlying asset. This assumption is in general quite restrictive and its relaxation will be treated in the next chapter. The assumption of independence however allows us to reduce the TVO pricing problem to calculating the expectation of a portfolio of quadratic variation claims. Secondarily, we give an overview of the continuous-time TVAs, by essentially following the approach suggested by Stoyanov [95], and add some ideas for further research.

This first chapter is structured as follows. In the next section we state the main assumptions and introduce some notation. In section 1.1 we illustrate some properties of a volatility payoff equivalent to that of the TVO when independence is assumed with the stochastic volatility process. In section 1.2 we derive the first approximation method which is based on Taylor expansions. The price of a TVO at inception is approximated by a sum of integrals of certain functions of the underlying asset variance. The subsequent section extends the result of section 1.2 to a generic time t to take into account the effect of the cumulated variance. A second pricing methodology based on the log-strike Laplace transform of the option payoff is introduced in section 1.4. The results of the section can be applied to a large class of stochastic volatility models to obtain TVO prices by inverting numerically the Laplace transform of the claim value. In section 1.5, we provide a representation of the price of the TVO in terms of a weighted portfolio of vanilla call and put options. In particular, we show that the price of a TVO is equal to the expectation of some linear combination of functions of the terminal value S_T of the underlying. We then apply the Breeden and Litzenberg [10] formula to decompose the expectation above in a weighted portfolio of calls and puts and provide a formula for the weights. In section 1.6 we illustrate a further pricing methodology which makes use of a uniform polynomial approximation of the equivalent variance claim. In section 1.7 we briefly turn our attention to the problem of modeling TVAs in a continuous-time framework. Numerical analysis is provided in section 1.8, whose graphical output is shown at the end of the chapter. All of the methods presented exhibit high levels of accuracy. Proofs of the main results are provided in the appendix.

Our market is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions. Throughout the chapter we will assume that there exist a money market account process B_t paying zero interest rates. We shall also assume that there exists a pricing measure $\mathbb{Q}$ under which any non-dividend-paying asset S_t satisfies the stochastic differential equation:

$$dS_t = \sigma_t S_t dW_t, \tag{1.2}$$

where W_t is a $\mathbb{Q}$-Brownian motion and $\sigma_t > 0$ is a stochastic volatility process. All expectations are taken with respect to the measure $\mathbb{Q}$.

We will restrict our attention to processes σ_t satisfying a diffusion equation of the type

$$d\sigma_t = \mu_t(\sigma_t, t)dt + \nu_t(\sigma_t, t)dZ_t, \quad \sigma_0 > 0 \tag{1.3}$$

where the $\mathbb{Q}$-Brownian motion Z_t is independent of W_t.

Let $X_t = \log(S_t/S_0)$. The *quadratic variation* of the process X_t is given by

$$\langle X \rangle_t = \int_0^t \sigma_u^2 du. \tag{1.4}$$

Define an arbitrary constant $\bar{\sigma} > 0$, which we shall refer to as the *target volatility*. A *target volatility call option* with strike K is a contingent claim on S_t and $\langle X \rangle_t$ with time-t price given by

$$C_t^{TV}(S_t, K, \langle X \rangle_t) = \mathbb{E}_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (S_T - K)^+ \right]. \tag{1.5}$$

Similarly the time-t price of a put TVO can be obtained by calculating the following expectation:

$$P_t^{TV}(S_t, K, \langle X \rangle_t) = \mathbb{E}_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (K - S_T)^+ \right]. \tag{1.6}$$

TVOs allow option buyers to take a joint bet on the price of the underlying and its volatility. The target volatility typically represents the option buyer's expectation of the future average realized volatility of S_t during the tenor of the option. In particular, if volatility realizes at or below the target level, the payoff of the option will be greater or equal than the payoff of the corresponding vanilla option. If implied volatilities are relatively high compared to the buyer's expectations, TVOs provide a way to gain exposure to the underlying at a reduced premium.

1.1 Properties of the equivalent payoff

One first important remark about (1.5) is that it is a joint function of the stock price and variance of the form $p(S_T)q(\langle X \rangle_T)$ for measurable functions p and q. Indeed, having chosen σ_t and W_t independent, a standard conditioning argument ensures that we can equivalently write (1.5) as a variance claim.

Proposition 1.1.1. *Let σ_t be independent of W_t. Let $C^{BS}(S_0, K, x)$ be the Black-Scholes price of the vanilla European call option of initial underlying value S_0, strike K and total realized variance x on $[0, T]$, that is*

$$C^{BS}(S_0, K, x) = S_0 N(d^+(x)) - K N(d^-(x)) \tag{1.7}$$

where $N(\cdot)$ is the cumulative normal distribution and

$$d^{\pm}(x) = \frac{\log(S_0/K) \pm x/2}{\sqrt{x}}. \tag{1.8}$$

Then the function

$$h(x) = \overline{\sigma}\sqrt{T}\frac{C^{BS}(S_0, K, x)}{\sqrt{x}} \tag{1.9}$$

is such that:

$$\mathbb{E}[H(S_T, K, \langle X\rangle_T)] = \mathbb{E}[h(\langle X\rangle_T)]. \tag{1.10}$$

Under independence, the pricing problem has been therefore reduced to the pricing of a claim on the stock's quadratic variation only. Extensive treatment of this kind of claims is given in Carr and Lee, [14]. Nevertheless, for $h(x)$ as in (1.9), the results of [14] cannot be directly applied in order to get a useful pricing formula. Depending on the parameters S_0 and K, the function $h(x)$ may or may not be bounded on the half real line, thus not falling under the cases accounted there. More generally, we have the following result for the asymptotics of the equivalent payoff $h(x)$:

Lemma 1.1.2. *Let $h(x)$ be as in (1.9). Then*

$$\lim_{x\to 0^+} h(x) = \begin{cases} 0 & \text{if } S_0 < K \\ \overline{\sigma}\sqrt{T}S_0/\sqrt{2\pi} & \text{if } S_0 = K \\ O(x^{-1/2}) & \text{if } S_0 > K \end{cases} \tag{1.11}$$

and

$$\lim_{x\to +\infty} h(x) = 0. \tag{1.12}$$

Lemma 1.1.2 is intuitively clear: if the option begins out-of-the-money and the volatility is sufficiently small the payoff will not be triggered, regardless of how big $1/\sqrt{\langle X\rangle_T}$ can get. On the other hand if the options begins in-the-money, for small values of volatility the difference between terminal stock and strike is going to be positive while the inverse square root of the volatility diverges. Interestingly, for at-the-money (ATM) options these effects balance out to yield a version of the Bachelier formula. Therefore we see that for the pricing problem to be well-defined for every moneyness regime we need the following integrability condition:

$$\mathbb{E}[\langle X\rangle_T^{-1/2}] < \infty. \tag{B}$$

As an example, this condition holds true for sensible parameter choices when the volatility process is given by the CIR equation (see e.g. [32], theorem 4.1).

1.2 Taylor expansion approximation

We begin this section with a simple motivating example. Using the well-known Bachelier approximation formula and proposition 1.1.1 and the independence of the volatility and price processes, it is straightforward to see that the price of the at the money TVO is approximately equal to the price of a vanilla option with implied volatility $\bar{\sigma}$:

$$\begin{aligned} C_0^{TV}(S_0, S_0, 0) &= \mathbb{E}\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} \mathbb{E}[(S_T - S_0)^+ \mid \mathcal{F}^\sigma{}_T] \right] \\ &= \mathbb{E}\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} C^{BS}(S_0, S_0, \langle X \rangle_T) \right] \\ &\simeq S_0 \mathbb{E}\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} \sqrt{\frac{\langle X \rangle_T}{2\pi}} \right] \\ &= S_0 \bar{\sigma} \sqrt{\frac{T}{2\pi}} \\ &\simeq C^{BS}(S_0, S_0, \bar{\sigma}^2 T). \end{aligned} \tag{1.13}$$

As stated, the price at inception of an at-the-money TVO is thus approximately the Black-Scholes price of the equivalent ATM vanilla call. For out and in-the-money options we can develop the Black and Scholes formula as a function of K in its Taylor series around the ATM level S_0. We shall show that each term of the expansion can be written as an integral of some exponential function of the quadratic variation $\langle X \rangle_T$. Expectations of such quantities can be explicitly calculated for a large class of parametric models or can be derived using a non-parametric approach *à la* Bredeen and Litzenberger [10]. The following lemma allows us to express the Black and Scholes price as a weighted sum of functions of the cumulative variance:

Lemma 1.2.1. *The Black and Scholes (call) equation admits the following Taylor expansion as a function of the strike K around the ATM point S,*

$$C^{BS}(S, K, x) = S - (S+K)N\left(-\frac{\sqrt{x}}{2}\right) + e^{-x/8} \sum_{j=0}^{f(n)} x^{-(1/2+j)} W^{n,j}(K) + O((K-S)^{n+3}), \tag{1.14}$$

where

$$W^{n,j}(K) = \frac{1}{\sqrt{2\pi}} \sum_{k=2j}^{n} (-1)^k c^{f(k)-j,k} \frac{(K-S)^{k+2}}{S^{k+1}(k+2)!}, \tag{1.15}$$

and

$$f(k) = \begin{cases} \frac{k}{2}, & k \text{ even;} \\ \frac{k-1}{2}, & k \text{ odd.} \end{cases} \tag{1.16}$$

The coefficients $c^{j,n}$ can be derived explicitly by solving a simple recursive equation (see the appendix for details). In order to calculate the TVO price it is convenient to simplify the functions of the quadratic variation obtained as a consequence of the previous step using the results below:

Lemma 1.2.2. *For any $x, r > 0$ the following equalities hold:*

$$\frac{1}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(z+1/8)x}}{\sqrt{z+1/8}} dz, \tag{1.17}$$

and

$$x^{-r} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r}x} dz. \tag{1.18}$$

Substituting equation (1.14) into (1.9) and using (1.17) and (1.18), we have the following proposition:

Proposition 1.2.3. *The price of a call TVO can be approximated by a linear combination of integrals of some exponential function of the quadratic variation,*

$$C_0^{TV}(K) \simeq \bar{\sigma}\sqrt{T} \left[\frac{2S_0}{\sqrt{\pi}} I_0^{1/2,0} - \frac{S_0+K}{2\sqrt{\pi}} \Phi_0^{1,1/8} + \sum_{j=0}^{f(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8} \right], \tag{1.19}$$

where we have defined:

$$I_0^{r,a} = \int_0^\infty \mathbb{E}\left[e^{\lambda^{r,a}(z)\langle X \rangle_T} \right] dz, \tag{1.20}$$

$$\Phi_0^{r,a} = \int_0^\infty \frac{\mathbb{E}\left[e^{\lambda^{r,a}(z)\langle X \rangle_T} \right]}{\sqrt{z+a}} dz, \tag{1.21}$$

$$\lambda^{r,a}(z) = -(z^{1/r} + a), \tag{1.22}$$

and

$$\tilde{W}^{n,j} = \frac{W^{n,j}(K)}{(j+1)!}. \tag{1.23}$$

The use of Fubini's theorem to interchange the order of integration in the formula above is justified whenever the decay of the characteristic function of $\langle X \rangle_t$ is sufficiently fast, as it is in the case for example of exponentially-affine models. Integrals $I_0^{r,a}$ and $\Phi_0^{r,a}$ can be calculated explicitly for a variety of parametric models for which the Laplace transform of the quadratic variation is known in closed form. For example we could model the instantaneous variance process σ_t^2 as a CIR process. More generally we can make use of the abundant literature on affine processes (see Duffie *et al.* [42] for a detailed treatment) to derive closed form solutions for the price of the TVO.

In section 1.5 we shall use a model independent approach in the spirit of Carr and Lee [14], to express integrals (1.20) and (1.21) and thus the TVO price as a weighted portfolio of traded options.

1.3 Taylor expansion for $t > 0$

So far we have dealt with the pricing problem at time zero. As variance cumulates during the life of the option, the pricing problem changes and formulae become slightly more involved, although the solution remains similar in nature.

Let us consider the price of a TVO at time $t > 0$. We need to solve an expression of the form

$$\begin{aligned} C^{TV}(S_t, K, \langle X\rangle_t) = &\ \mathbb{E}_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - K)^+\right] \\ = &\ \mathbb{E}_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\epsilon + \langle X\rangle_{T-t}}} C^{BS}(S_t, K, \langle X\rangle_{T-t})\right], \end{aligned} \tag{1.24}$$

where we have set $\epsilon = \langle X\rangle_t$. Thanks to the Markovian structure of the stock price and volatility, the $t > 0$ pricing problem is very similar to the one encountered in the previous section. However the presence of the term ϵ_t causes a lack of symmetry between the powers of $\langle X\rangle_{T-t}$ in the numerator and the square root in the denominator when substituting the Black and Scholes formula with its Taylor series. When $t > 0$ proposition 1.1.1 easily extends to

$$\mathbb{E}_t[H(S_T, K, \langle X\rangle_T)] = \mathbb{E}_t[h^\epsilon(\langle X\rangle_T)]. \tag{1.25}$$

where $h^\epsilon(x) = h(x + \epsilon)$, $\epsilon > 0$ are the shifted payoffs. After expanding the time-t Black and Scholes formula appearing in h^ϵ around the strike K, we are left with calculating expressions of the form:

$$q_1(x) = \frac{N(-\sqrt{x}/2)}{\sqrt{\epsilon + x}}, \tag{1.26}$$

$$q_2(x) = \frac{x^{-(j+1/2)}}{\sqrt{\epsilon + x}}. \tag{1.27}$$

In principle, we could represent $q_1(x)$ and $q_2(x)$ as double integrals of exponential functions of x by considering the numerator and denominator separately. However, because of singularities in some of the integrals involved, this approach does not allow us to derive model independent prices. An alternative approach is to consider a Taylor expansion of $N(-\sqrt{x}/2)$ and $x^{-(j+1/2)}$ around the point $x + \epsilon$:

$$q_1(x) = \frac{N(-\frac{\sqrt{\epsilon+x}}{2})}{\sqrt{\epsilon + x}} + \frac{e^{-(\epsilon+x)/8}}{\sqrt{2\pi}} \sum_{i=0}^{m} \omega^{i,m}(\epsilon)(\epsilon + x)^{-(i+1)} + \mathrm{O}(\epsilon + x)^{m+2}), \tag{1.28}$$

where

$$\omega^{i,m}(\epsilon) = \sum_{k=j}^{m} (-1)^{k+1} \gamma^{i,k} \frac{\epsilon^{k+1}}{k+1!}, \tag{1.29}$$

and $\gamma^{j,k}$ satisfies the following recursion[1]:

$$\begin{aligned} &\gamma^{0,0} = -1/4 \\ &\gamma^{0,k} = \left(-\tfrac{1}{8}\right)\gamma^{0,k-1}, \qquad k = 1\ldots m \\ &\gamma^{k,k} = (1/2 - k)\gamma^{k-1,k-1}, \qquad k = 1\ldots m \\ &\gamma^{j,k} = \left(-\tfrac{1}{8}\right)\gamma^{j,k-1} + (1/2 - j)\gamma^{j-1,k-1}, \qquad j = 1\ldots m,\ k = j+1\ldots m. \end{aligned} \tag{1.30}$$

[1]The closed form solution of the recursion is $\gamma^{j,k} = -\frac{1}{4}\prod_{i=1}^{j}\frac{1-2i}{2}\left(-\frac{1}{8}\right)^{k-j}\binom{k}{j}$ for $j \geq 1$ and $k \geq j$.

Similarly,

$$q_2(x) = \sum_{k=0}^{m} \zeta^{k,j}(\epsilon)(\epsilon + x)^{-(j+k+1)} + \mathrm{O}(m+1) \tag{1.31}$$

where we have defined $\zeta^{0,j}(\epsilon) = 1$ and

$$\zeta^{k,j}(\epsilon) = \frac{\epsilon^k}{k!} \prod_{i=0}^{k-1} (j + i + 1/2) \tag{1.32}$$

for $k \geq 1$.

These approximations can be substituted in the Taylor expansion of the time-t Black and Scholes price. By putting all terms $\langle X \rangle_T - \langle X \rangle_t$ as common factor, after some rearrangements we obtain the following proposition

Proposition 1.3.1.

$$C_t^{TV}(K) \simeq \bar{\sigma}\sqrt{T} \left[\frac{2S_t}{\sqrt{\pi}} I_t^{1/2,0,0} - \frac{S_t + K}{2\sqrt{\pi}} \Phi_t^{1,1/8} + \sum_{j=0}^{m+f(n)} \hat{W}_t^{n,m,j}(K, \langle X \rangle_t) I_t^{j+1,1/8,0} \right], \tag{1.33}$$

where

$$I_t^{r,a,b} = \int_0^\infty e^{-(z^{1/r}+b)\langle X \rangle_t} \mathbb{E}_t \left[e^{\lambda^{r,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] dz, \tag{1.34}$$

$$\Phi_t^{r,a} = \int_0^\infty \frac{e^{-(z+a)\langle X \rangle_t}}{\sqrt{z+a}} \mathbb{E}_t \left[e^{\lambda^{r,a}(\langle X \rangle_T - \langle X \rangle_t)} \right] dz, \tag{1.35}$$

$$\lambda^{r,a}(z) = -(z^{1/r} + a), \tag{1.36}$$

and the weights of the linear combination are given by:

$$\begin{aligned} \hat{W}_t^{n,m,j}(K,\epsilon) &= \frac{1}{(j+1)!} \Bigg\{ - \frac{S_t + K}{\sqrt{2\pi}} e^{-\epsilon/8} \omega^{j,m}(\epsilon) \\ &+ \sum_{k=0}^{\min(j,f(n))} W^{n,k}(K) \zeta^{j-k,k}(\epsilon) I_{j \leq m} \\ &+ \sum_{k=0}^{\min(m,f(n)-j+m)} W^{n,j-m+k}(K) \zeta^{m-k,j-m+k}(\epsilon) I_{j>m} \Bigg\}. \end{aligned} \tag{1.37}$$

Note that to simplify the notation, we have imposed that the summation in the Taylor expansion of $q_1(x)$ and $q_2(x)$ is up to m for both functions.

1.4 TVO pricing using Laplace transforms

An alternative approach to the use of Taylor series to derive the price of a TVO is based on Laplace transform techniques. In particular, we shall consider the Laplace transform of the payoff in the log-strike variable. As we will show later in this section, this approach leads to very simple semi-analytical solutions which are also efficient from a computational point of view. The main drawback of the methodology is that the model-independent approach cannot be applied.

Let us consider the pricing problem of a put TVO. It is convenient to express the payoff of the option is in terms of the log-strike $k = \log K$

$$P_t(S_t, e^k, \langle X \rangle_t) = \mathbb{E}_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}}(e^k - S_T)^+\right] = P(k). \tag{1.38}$$

By Laplace-transforming the option price in the log-strike k, we can eliminate the *max* function appearing in the payoff of the TVO. This will allow us to reduce the problem to the pricing of a quadratic variation claim rather than a joint claim on the terminal value of the stock and the quadratic variation.

For any complex α such that Re$(\alpha) > 1$, the Laplace transform of $P(k)$ is equal to:

$$\begin{aligned}\hat{P}_t(\alpha) &= \int_0^\infty e^{-\alpha k} P_t(k) dk \\ &= \bar{\sigma}\sqrt{T} S_t^{1-\alpha} \mathbb{E}_t\left[\frac{1}{\sqrt{\epsilon_t + \langle X \rangle_T - \langle X \rangle_t}} \frac{e^{(1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)}\right].\end{aligned} \tag{1.39}$$

Using formula (1.18) we can represent the denominator of (1.39) in integral form:

$$\frac{1}{\sqrt{\epsilon + x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2(\epsilon + x)} dz. \tag{1.40}$$

Under the assumption of independence of σ_t and W_t, after applying Fubini's theorem, we can write $\hat{P}(\alpha)$ in terms of S_t and the quadratic variation

$$\begin{aligned}\hat{P}_t(\alpha) &= 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty e^{-z^2 \langle X \rangle_t} \mathbb{E}_t\left[\frac{e^{-z^2(\langle X \rangle_T - \langle X \rangle_t) + (1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)}\right] dz \\ &= 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty \frac{e^{-z^2 \langle X \rangle_t} \mathbb{E}_t\left[e^{\lambda_{z,\alpha}(\langle X \rangle_T - \langle X \rangle_t)}\right]}{\alpha(\alpha-1)} dz,\end{aligned} \tag{1.41}$$

where we have defined the function:

$$\lambda_{z,\alpha} = -(z^2 + \alpha(1-\alpha)/2). \tag{1.42}$$

The Laplace transform (1.41) can be calculated explicitly for a variety stochastic volatility models (e.g. exponentially-affine models), and the price of the TVO can be then obtained by inverting (1.41) numerically. In particular, pricing the TVO amounts to calculating the following integral

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} \int_0^\infty \int_0^\infty e^{-z^2\langle X \rangle_t} \text{Re}\left(\frac{S_t^{1-a-iu} \mathbb{E}_t\left[e^{\lambda_{z,a+iu}(\langle X \rangle_T - \langle X \rangle_t)}\right]}{(a+iu)(a+iu-1)}\right) cos(uk) dz du. \tag{1.43}$$

Numerical integration can be achieved by using, for example, the Abate-Whitt method [1] which is based on an application of the trapezium rule combined with the Euler summation. The Laplace method is fast, easy to implement and produces accurate and stable results.

1.5 Robust pricing

Many authors in recent years (see for example Schoutens et al [93]) have highlighted some problems related to model dependence in the context of exotic option pricing and hedging. For example, local volatility models are known to lead to significantly different results from stochastic volatility models when pricing forward-starting and cliquet options. Even within the stochastic volatility class, different models lead to different prices for path-dependent options when calibrated to the same volatility surface.

In a pioneering paper, Breeden and Litzenberger [10] showed how to obtain the risk-neutral density of prices of the underlying asset from traded European option prices. Model-independent prices for European-style claims on the underlying with sufficiently smooth second derivatives may then be obtained by forming a portfolio of traded call and put options (see e.g. Carr and Madan [16]). Although these results are not immediately applicable to volatility derivatives, Carr and Lee [14] proved that under the assumption of independence, expectations of exponential functions of the quadratic variation are equal to expectations of some function of the terminal value of the underlying. The results of [14] allow us to calculate the price of a special class of highly path dependent claims as if they were European options. Indeed for any complex number λ we have:

$$\mathbb{E}_t[e^{\lambda(\langle X\rangle_T - \langle X\rangle_t)}] = \mathbb{E}_t\left[e^{(X_T - X_t)p(\lambda)}\right] = \mathbb{E}_t\left[\left(\frac{S_T}{S_t}\right)^{p(\lambda)}\right], \tag{1.44}$$

where

$$p(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}. \tag{1.45}$$

In sections 1.2 and 1.3, we showed how to approximate TVO prices by a linear combination of integrals involving expressions of the form of the central term of equation (1.44). We shall now show how to apply the Breeden-Litzenberger formula to represent the TVO price in terms of traded option prices. Let $f(S)$ be a twice-differentiable payoff. For some arbitrary constant η we have:

$$f(S) = f(\eta) + f^{'}(k)[S-\eta] + \int_{\eta}^{\infty} f^{''}(x)(S-x)^{+}dx + \int_{0}^{\eta} f^{''}(x)(x-S)^{+}dx. \tag{1.46}$$

By taking conditional expectations on both sides of (1.46), we obtain a representation of the price of the claim $f(S)$ in terms of vanilla call and put prices

$$\mathbb{E}_t[f(S_T)] = f(\eta)+f^{'}(\eta)[S_t-\eta]+\int_{\eta}^{\infty} f^{''}(x)C_t^M(S_t,x)dx+\int_{0}^{\eta} f^{''}(x)P_t^M(S_t,x)dx, \tag{1.47}$$

where $C_t^M(S_t,x)$ and $P_t^M(S_t,x)$ are respectively the prices of calls and puts with strike x.

1.5.1 Robust pricing via Taylor expansion

In order to apply formula (1.47) to price TVOs, we need to find a function $f(S_T)$ such that $\mathbb{E}_t[f(S_T)]$ equals the TVO price. As shown in formula (1.33), the price of a call TVO can be approximated by a linear combination of terms of the form $I_t^{r,a,b}$ and $\Phi_t^{1,a}$ for an integer r and real constants a and b. In order to derive model independent prices for TVOs it is thus sufficient to apply the Breeden-Litzenberger formula to $I_t^{r,a,b}$ and $\Phi_t^{1,a}$ and derive the portfolio of forwards, calls and puts yielding the model independent price.

Carr-Lee formula (1.44) can then be used to express $I_t^{r,a,b}$ and $\Phi_t^{1,a}$ as an expectation of some integral of the terminal value of the underlying S_T. In particular, it can be shown that:

$$\begin{aligned} I_t^{r,a,b} &= \int_0^\infty e^{-(z^{1/r}+b)\langle X\rangle_t}\mathbb{E}_t\left[e^{\lambda^{r,a}(\langle X\rangle_T-\langle X\rangle_t)}\right] \\ &= \mathbb{E}_t\left[\int_0^\infty e^{-(z^{1/r}+b)\langle X\rangle_t}\mathrm{Re}\left(\frac{S_T}{S_t}\right)^{p^{r,a}(z)} dz\right], \end{aligned} \tag{1.48}$$

where we have applied Fubini's theorem to interchange the order of integration and defined

$$p^{r,a} = 1/2 \pm \sqrt{1/4 - 2z^{1/r} - 2a}. \tag{1.49}$$

Similarly,

$$\begin{aligned} \Phi_t^{r,a} &= \int_0^\infty \frac{e^{-(z+a)\langle X\rangle_t}}{\sqrt{z+a}}\mathbb{E}_t\left[e^{\lambda^{r,a}(\langle X\rangle_T-\langle X\rangle_t)}\right] dz \\ &= \mathbb{E}_t\left[\int_0^\infty \frac{e^{-(z+a)\langle X\rangle_t}}{\sqrt{z+a}}\mathrm{Re}\left(\frac{S_T}{S_t}\right)^{p^{r,a}(z)} dz\right]. \end{aligned} \tag{1.50}$$

The last step is to define the following functions of the terminal value of the underlying S:

$$\tilde{I}_t^{r,a,b}(S) = \int_0^\infty e^{-(z^{1/r}+b)\langle X\rangle_t}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{r,a}(z)} dz \tag{1.51}$$

$$\tilde{\Phi}_t^{r,a,b}(S) = \int_0^\infty \frac{e^{-(z+a)\langle X\rangle_t}}{\sqrt{z+a}}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{r,a}(z)} dz. \tag{1.52}$$

For $t > 0$ the second derivative in S of functions $I_t^{r,a,b}(S)$ and $\Phi_t^{r,a,b}(S)$ is well-defined as $\langle X\rangle_t$ is strictly positive. The left-hand side of equalities (1.48) and (1.50) is equal to the conditional expectation of functions $\tilde{I}_t^{r,a,b}(S)$ and $\tilde{\Phi}_t^{r,a,b}(S)$ respectively. We can now apply formula (1.47) to the functions above with $\eta = S_t$ and substitute the result in (1.33) to obtain a representation of the TVO price in terms of traded call and put options.

Note that for $t = 0$ integrals in expressions (1.51) and (1.52) do not converge and we cannot use Fubini's theorem to interchange integrals in equations (1.48) and (1.50). It is therefore not always possible to calculate the TVO price using formula (1.47). However, if the TVO contract is redefined as

$$C_0^{TV}(S_0, K, 0) = \mathbb{E}\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{c + \langle X\rangle_T}}(S_T - K)^+\right] \tag{1.53}$$

for some small arbitrary constant c, a "robust price", in the sense previously defined, does exist.

1.5.2 Robust pricing via Laplace transforms

Can the robust pricing approach be applied to Laplace transform method introduced in section 1.4 ? Using the Carr-Lee formula (1.44) we can express the Laplace transform of the TVO as a conditional expectation of some function of the terminal value S_T,

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}}\mathbb{E}_t\left[\int_0^\infty e^{-z^2\langle X\rangle_t}\int_0^\infty \mathrm{Re}\left(\frac{S_t^{1-a-iu}(S_T/S_t)^{p^{\pm}_{z,\alpha}}}{(a+iu)(a+iu-1)}\right)cos(uk)dudz\right], \tag{1.54}$$

where we have set $\alpha = a + iu$ and defined:

$$p^{\pm}_{z,\alpha} = 1/2 \pm \sqrt{1/4 - 2z^2 - \alpha(1-\alpha)}. \tag{1.55}$$

In principle, we could define the function $f(S)$ as

$$f(S) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}}\int_0^\infty e^{-z^2\langle X\rangle_t}\int_0^\infty \mathrm{Re}\left(\frac{S_t^{1-a-iu}(S/S_t)^{p^{\pm}_{z,\alpha}}}{(a+iu)(a+iu-1)}\right)cos(uk)dudz. \tag{1.56}$$

However the second derivative of $f(S)$ does not exist, because the integral in the variable z obtained after differentiating (1.56) twice does not converge. It is therefore not possible to apply the Breeden-Litzenberger decomposition to derive a model independent price for the TVO using the Laplace transform method introduced in section 1.4.

1.6 Bernstein polynomial approximation

Applying the theory developed in [14], a third way to approach the problem of pricing TVOs is writing the equivalent claim (1.9) as a uniform limit of a polynomial sequence. The advantage of this kind of approach is that the approximating claims are much simpler and easy to manage mathematical expressions, and we do not need to compute hard integral transforms to find prices.

By the Weierstrass theorem continuous real-valued functions on a compact set are known to be uniformly approximated by some sequence of polynomials. To construct explicitly such an approximating sequence typically one makes use of the *Bernstein polynomials*. We have the following theorem:

Theorem 1.6.1 (Bernstein). *Let $f(x)$ be continuous function on $[0, 1]$. The Bernstein polynomials for f of order n are defined by*

$$B_n f(x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k} \tag{1.57}$$

and are a sequence such that $B_n f(x) \to f(x)$ uniformly on $[0, 1]$.

For all $c > 0$, by means of the transformation $y \mapsto -\log(x/c)$ we can extend the Weierstrass theorem to all the positive functions having finite limits. Recall that by lemma 1.1.2, $h(x)$ can be completed to a continuous function on $[0, +\infty)$ converging at $+\infty$ if and only if $S_0 \leq K$. In principle we are thus not able to uniformly approximate $h(x)$ with a polynomial. However, we may very well do so for the shifted polynomials $h^\epsilon(x)$, because the latter monotonically converge to h in ϵ. We have the following result:

Proposition 1.6.2. *Let* $h^\epsilon(x) = h(x+\epsilon)$, $\epsilon \geq 0$. *We have, for all* $c > 0$*:*

$$\mathbb{E}_t[h(\langle X\rangle_T)] = \lim_{\epsilon\to 0} \mathbb{E}_t[h^\epsilon(\langle X\rangle_T)], \tag{1.58}$$

$$\mathbb{E}_t[h^\epsilon(\langle X\rangle_T)] = \lim_{n\to+\infty} \sum_{k=0}^{n} C_k^n \mathbb{E}_t\left[P_k(S_T, S_0)\right] \tag{1.59}$$

with

$$P_k(x,y) = (x/y)^{p^\pm(-ck)}, \tag{1.60}$$

$$C_k^n = \sum_{j=1}^{k} (-1)^{k-j} h_*^\epsilon\left(j/n\right) \binom{n}{k}\binom{k}{j} \tag{1.61}$$

and

$$h_*^\epsilon(x) = h^\epsilon\left(-\log x/c\right). \tag{1.62}$$

Again, note that by lemma 1.1.2 if $S_0 \leq K$ the proposition above can be fully restated with $\epsilon = 0$ because in such case h will be continuous in 0. On the other hand, if $t = 0$ and $S_0 > K$, this proposition allows for an estimate of the option value by choosing ϵ sufficiently small and selecting in (1.59) the desired n. In either case, whenever $t > 0$ then ϵ can be taken to be the cumulated variance up to time t and as a consequence of equation (1.25) formula (1.59) directly gives an estimate for the value of the claim we need to approximate. Thus in some sense under this method the TVO valuation for $t > 0$ is a more natural problem.

We have noticed that acting on c varies the speed at which the algorithm converges. Also, for any given n choosing $c < 1/8n$ makes sure that the approximating value is a real number. Compared to those of previous sections, the calculations above are very easy to perform; all we need to do is computing the n values for the claims $e^{-k\langle X\rangle_T}$, the $n(n-1)/2$ binomial coefficients $\binom{n}{k}$, $\binom{k}{j}$ and the n values $h_*^\epsilon(k/n)$.

1.7 The target volatility asset allocation

A target volatility strategies is a dynamic portfolio allocation which aims at keeping the portfolio volatility constant at a desired target exposure $\overline{\sigma}$. Such an asset allocation retains several properties helpful for portfolio risk management and derivative valuation.

In case the risky asset follows a continuous diffusion model and if continuous-time trading is assumed, the main properties of the target volatility strategies have been given by Stoyanov in [95]. In this section we review an expand such an approach while also adding some considerations for possible improvements.

Our investment universe consists of a riskless bond B_t earning a constant fixed interest rate r, following the ODE

$$dB_t = rB_t dt, \quad B_0 = 1, \tag{1.63}$$

and of a risky equity S_t driven by the geometric Brownian motion

$$S_t = \mu_t S_t dt + \sigma_t S_t dW_t \tag{1.64}$$

satisfying the usual assumptions. The dynamics of (1.64) are assumed to be those of the historical market measure. By trading only in B_t and S_t we would like to devise a continuous time self-financing strategy

$$A_t = (\Theta_t^r, \Theta_t) \tag{1.65}$$

in such a way that the resulting portfolio process has a constant volatility $\overline{\sigma}$. The processes Θ_t^r, Θ_t are càglàd predictable processes defining respectively the quantities of bond and stock to be held at each time t.

Assume that C is our initial endowment. We initially set $\Theta_0 = C\overline{\sigma}/(\sigma_0 B_0)$ and $\Theta_0^r = (C - \Theta_0 B_0)/S_0$, so that $\pi_0 = C$. For $t > 0$, define the portfolio earning process:

$$\pi_t = \int_0^t \Theta_u^r dB_u + \int_0^t \Theta_u dS_u. \tag{1.66}$$

The cash values at time t of Θ_t^r and Θ_t are given respectively by $V_t^r = \Theta_t^r B_t$ and $V_t = \Theta_t S_t$. That the portfolio is self-financing means requiring that $V_t^r + V_t = \pi_t$ for all t; by setting $V_t^r = w_t^r \pi_t$ and $V_t = w_t \pi_t$, this condition is equivalent to $w_t^r + w_t = 1$. We therefore have:

$$\begin{aligned} d\pi_t =& \Theta_t dB_t + \Theta_t dS_t = \frac{V_t^r}{B_t} dB_t + \frac{V_t}{S_t} dS_t = w_t^r \pi_t \frac{dB_t}{B_t} + w_t \pi_t \frac{dS_t}{S_t} \\ =& r\pi_t dt + w_t(\mu_t - r)\pi_t dt + w_t \sigma_t \pi_t dW_t. \end{aligned} \tag{1.67}$$

We see that the portfolio process decomposes as a fixed riskless interest rate r paid on the whole portfolio, a stochastic risk premium proportional to $(\mu_t - r)$ and weighted by w_t, and a volatility term where w_t counterforces σ_t. We wish to set the percentage equity exposure w_t in such a way that the log-returns of π_t show constant volatility $\overline{\sigma}$. Clearly, this can be achieved by setting

$$w_t = \frac{\overline{\sigma}}{\sigma_t}. \tag{1.68}$$

It is important to observe that w_t can be greater than one, an occurrence which takes place if and only if $\sigma_t < \overline{\sigma}$. This means that there could be trading periods during which the equity exposure must be set to exceed 100% of the portfolio value: to achieve this the cash exposure must become negative. This is to say that in order to maintain the portfolio we need to borrow money each and every time that the instantaneous volatility exceeds $\overline{\sigma}$. The target volatility portfolio is therefore a leveraged portfolio.

Substituting the value (1.68) in (1.67) we have the SDE for π_t:

$$d\pi_t = \left(r + \frac{\overline{\sigma}}{\sigma_t}(\mu_t - r) \right) \pi_t dt + \overline{\sigma}\pi_t W_t, \quad \pi_0 = S_0. \tag{1.69}$$

The implications of equation (1.69) from the point of view of the derivative pricing are clear. If one wishes to write, say, a European derivative F on π_t, the theoretical price V of $F(\pi_t)$ will be exactly the Black-Scholes value $F^{BS}(\overline{\sigma})$ of F on a log-normal asset of volatility $\overline{\sigma}$. This can be shown by performing the usual measure change via the Girsanov exponential martingale which eliminates the stochastic risk premium from the expression above, and yields the portfolio distribution in an equivalent pricing measure. More precisely, the measure $\mathbb{Q}$ having Radon Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \frac{\mu_u - r}{\sigma_u} dW_u - \frac{1}{2}\int_0^t \left(\frac{\mu_u - r}{\sigma_u} \right)^2 du \right) \tag{1.70}$$

gives the risk-neutral distribution of the asset π_t as log-normal with parameters r and $\overline{\sigma}$.

The approach described guarantees the analytical tractability of the portfolio model, but it does however present several major shortcomings. In first place, using σ_t to devise a trading strategy is not fully convincing as the instantaneous volatility is not observed in the market and has to be statistically estimated, with the obvious related robustness issues. Secondly, continuous-time rebalancing directly linked to the spot market volatility may show wild fluctuations in periods of high volatility, leading to high transaction costs. Thirdly, no exposure limits are imposed for the equity component of the portfolio, which is clearly unrealistic. We are currently investigating alternative definitions for the equity portfolio exposure w_t overcoming these issues. One alternative definition for the trading strategy solving the first issue could be using a discrete time trading strategy involving the realized volatility. For example we could partition the investment horizon T in n holding periods of length $\delta_n = T/n$ during which the portfolio holdings are kept constant. At each rebalancing date the equity exposure is adjourned proportionally to the inverse of the volatility realized during the holding period. The process w_t^n would therefore be taken as

$$w_t^n := \overline{\sigma} \left(\frac{\delta_n}{\sigma_0^2 \delta_n \mathbb{1}_{\{0 \le t < t_1\}} + \sum_{i=1}^{n-1} I(t_{i-1}, t_i) \mathbb{1}_{\{t_i \le t < t_{i+1}\}}} \right)^{1/2} \tag{1.71}$$

where we let $I(t_{i-1}, t_i) = \langle X \rangle_{t_i} - \langle X \rangle_{t_i}$. One could then set up an SDE for the portfolio process π_t^n by solving (1.67) with w_t replaced by w_t^n. The main result should be that, as n gets larger, π_t^n tends to π_t in the mean-square sense. However, the process π_t^n thus constructed is a stochastic process whose discontinuities occur at fixed times, and increase with n. The implications of this fact are unclear and currently under analysis.

1.8 Numerical results

We have implemented and tested the pricing formulae presented in the previous sections using MATLAB. In particular, we have assumed Heston dynamics for the underlying asset process,

$$dS_t = \sqrt{v_t} S_t dW_t, \tag{1.72}$$

with CIR instantaneous variance process for the variance given by the SDE:

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dZ_t. \tag{1.73}$$

As usual, W_t and Z_t are independent Brownian motions. The following parameters for the variance process have been used:

$$S_0 = 100, \quad v_0 = 0.2, \quad \overline{\sigma} = 0.1, \quad \kappa = 0.5, \quad \theta = 0.2, \quad \eta = 0.3. \tag{1.74}$$

For a set of fixed market spot values we compared the TVO call values given by formula (1.43) with a Monte Carlo simulation of 100.000 samples across three maturities and five different moneynesses: $\pm 40\%, \pm 20\%, 0\%$. The results are shown in table 1.1 and achieve three-digit precision.

Table 1.2 compares the ATM TVO values calculated through equations (1.19), (1.43) and (1.59) for polynomials of various orders n, against a Monte Carlo simulation based on 10.000 sample paths. Again, the strike K ranges between -40% and $+40\%$ the spot price. Tables 1.3 and 1.4 contain similar computations where the free parameters are respectively taken to be the maturity T and the variance realized at the valuation time t. The graphical output of tables 1.2, 1.3 and 1.4 are shown in figures 1.1-1.6.

Overall a Taylor polynomial of order 3 performs better in terms of accuracy of the Bernstein polynomial of the maximum order considered, which is 30. However, even at such a high order, the computational time required for the Bernstein method is shorter than that of the Taylor method because it does not require numerical integration.

Appendix: proofs

Proof of proposition 1.1.1 . Let $\mathcal{F}_T^\sigma$ be the filtration generated by the process σ_t from time 0 up to time T. By the independence of W_t and σ_t we have that conditional on $\mathcal{F}^\sigma$ the process W_t is still a Brownian motion. Therefore, by the usual properties of the conditional expectation:

$$\begin{aligned}\mathbb{E}_t\left[\frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - K)^+\right] &= \mathbb{E}_t\left[\frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}\mathbb{E}_t\left[(S_T - K)^+\Big|\,\mathcal{F}_T^\sigma\right]\right] \\ &= \mathbb{E}_t\left[\frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}C^{BS}(S_0, K, \langle X\rangle_T)\right] = \mathbb{E}_t[h(\langle X\rangle_T)].\end{aligned} \tag{1.75}$$

□

Proof of lemma 1.1.2. Let $S_0 < K$. If $x \to 0^+$, $d^+(x)$ and $d^-(x)$ tend both to $-\infty$ and $N(d^\pm(x)) \to 0$. The asymptotic series for $N(z)$ as $z \to -\infty$ is

$$\frac{e^{-z^2/2}}{\sqrt{2\pi}}(z^{-1} + O(z^{-2})) \tag{1.76}$$

and so as x goes to 0 from the right we have

$$S_0 \frac{N(d^+(x))}{\sqrt{x}} = \frac{S_0}{\sqrt{2\pi}} e^{-\frac{d^+(x)^2}{2}} \left(\frac{1}{\log(S/K) + x/2} + O(\sqrt{x}) \right) \to 0. \tag{1.77}$$

The same holds for $KN(d^-(x))/\sqrt{x}$. If $S_0 = K$ then as $x \to 0^+$ the numerator of $h(x)$ tends to 0 because $N(d^\pm(x)) = N(\pm\sqrt{x}/2) \to 1$. The McLaurin series for $N(z)$ is

$$N(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}(z + O(z^2)) \tag{1.78}$$

therefore

$$h(x) = \frac{\overline{\sigma}\sqrt{T} S_0}{\sqrt{x}} \left(N(\sqrt{x}/2) - N(-\sqrt{x}/2) \right) \to \frac{\overline{\sigma}\sqrt{T} S_0}{\sqrt{2\pi}}. \tag{1.79}$$

Finally if $S_0 > K$ then $N(d^\pm(x)) \to 1$, so that the numerator remains bounded in 0 while $h(x)$ diverges as $x^{-1/2}$. Equation (1.12) is immediate since the Black-Scholes price is bounded in volatility. □

Proof of proposition 1.2.3. Consider first the Taylor expansion of the Black and Scholes formula $C^{BS}(S, K, x) = C(K)$ with respect to the strike K around the ATM point S

$$C(K) = C(S) + C^{(1)}(K - S_0) + \sum_{k=0}^{\infty} C^{(k+2)}(S) \frac{(K - S)^{k+2}}{(k+2)!}, \tag{1.80}$$

where $C^i(S)$ represents the i-th derivative with respect to the strike K evaluated at the ATM point S. Following Estrella [36], the Taylor series converges for $0 < K < 2S$, and it is possible to derive the generic expression for $C^{k+2}(S)$ for all $k \geq 0$:

$$C^{k+2}(S) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\hat{\sigma}^2}{8} \right) \frac{P_k(d^+)}{S^{k+1} \hat{\sigma}^{k+1}} (-1)^k, \tag{1.81}$$

where we have defined the time scaled volatility $\hat{\sigma} = \sigma\sqrt{t} = x$. It can be shown that $P_n(d^+)$ satisfies the following recursive equation: $P_0(d^+) = 1$ and:

$$P_k(d^+) = (d^+ + k\hat{\sigma}) P_{k-1}(d^+) - P'_{k-1}(d^+) \tag{1.82}$$

where d^+ is defined in (1.8). Noting that for $K = S$ it is $d^+ = \hat{\sigma}/2$, we can write the generic term P_n as a sum of powers of the volatility term, namely

$$P_k = \sum_{j=0}^{f(k)} c^{j,k} \hat{\sigma}^{\gamma(j,k)}, \tag{1.83}$$

where $c^{j,k}$ is the j-th term of the polynomial P_k, $f(k)$ is defined in (1.16), and

$$\gamma(j,k) = \begin{cases} 2j, & \text{k even;} \\ 2j+1, & \text{k odd.} \end{cases} \tag{1.84}$$

Polynomials P_k have degree k and consist of a sum of even (odd) powers of $\hat{\sigma}$ for k even (odd). The scaled polynomials

$$\bar{P}_k = \frac{P_k}{\hat{\sigma}^{k+1}} \tag{1.85}$$

consist only of odd powers of the volatility term $\hat{\sigma}$ and are equal to

$$\bar{P}_k = \sum_{j=0}^{f(k)} c^{f(k)-j,k} \hat{\sigma}^{-(1+2j)}. \tag{1.86}$$

Substituting (1.86) into (1.81) and using the result in the Taylor expansion of the Black and Scholes formula, we obtain

$$\begin{aligned} C(K) = C(S) &+ C^{(1)}(K-S) \\ &+ e^{-\hat{\sigma}^2/8} \sum_{k=0}^{n} (-1)^k \frac{(K-S)^{k+2}}{(k+2)!} \sum_{j=0}^{f(k)} c^{f(k)-j,n} \quad \hat{\sigma}^{-(1+2j)} + O((K-S)^{n+3}). \end{aligned} \tag{1.87}$$

By inverting the order of summation, taking $\hat{\sigma}$ as a common factor and using the definition of $C^0(S)$ and $C^1(S)$ we finally see that

$$\begin{aligned} C(K) = & S\left\{N\left(\frac{\hat{\sigma}}{2}\right) - N\left(-\frac{\hat{\sigma}}{2}\right)\right\} - N\left(-\frac{\hat{\sigma}}{2}\right)(K-S) \\ & + \frac{e^{-\hat{\sigma}^2/8}}{\sqrt{2\pi}} \sum_{j=0}^{f(n)} \hat{\sigma}^{-(1+2j)} \sum_{k=2j}^{n} (-1)^k c^{f(k)-j,k} \frac{(K-S)^{k+2}}{S^{k+1}(k+2)!} + O((K-S)^{n+3}) \end{aligned} \tag{1.88}$$

and formula (1.14) follows. □

Proof of lemma 1.2.2. For any $a > 0$, the Laplace transform of the function $g(z) = 1/\sqrt{\pi(z+a)}$ is equal to:

$$\begin{aligned} \hat{g}(x,a) &= \int_0^\infty \frac{e^{-xz}}{\sqrt{\pi(z+a)}} dz \\ &= \frac{e^{xa}}{\sqrt{x}} \text{erfc}(\sqrt{ax}) \\ &= \frac{2e^{ax}}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right). \end{aligned} \tag{1.89}$$

Setting $a = 1/8$ and rearranging we obtain formula (1.17); the proof of the second equality can be found in Schürger [90]. □

Proof of proposition 1.3.1. Let us denote the rescaled Black-Scholes price

$$\tilde{C}^{BS}(S,K,x,\epsilon) = \frac{C^{BS}(S,K,x)}{\sqrt{\epsilon + x}}. \tag{1.90}$$

Substituting (1.14) in the right-hand side of (1.90) and using (1.28) and (1.31), we have

$$\tilde{C}^{BS}(S,K,x,\epsilon) \simeq \frac{S}{\sqrt{\epsilon+x}} - \frac{(S+K)}{\sqrt{\epsilon+x}} N\left(-\frac{\sqrt{\epsilon+x}}{2}\right)$$
$$-(S+K)e^{-(x+\epsilon)/8}\sum_{j=0}^{m}\omega^{j,m}(\epsilon+x)^{-(1+j)} + e^{-x/8}\sum_{j=0}^{f(n)} W^{n,j}(K)\sum_{k=0}^{m}\zeta^{k,j}(\epsilon+x)^{-(1+k+j)}. \tag{1.91}$$

Expanding the sum in the equation above and putting all terms in $(\epsilon+x)^{-(1+j)}$ as a common factor, we obtain, after some algebra:

$$\begin{aligned}\tilde{C}^{BS}(S,K,x,\epsilon) \quad &\simeq \quad \frac{S}{\sqrt{\epsilon+x}} - \frac{(S+K)}{\sqrt{\epsilon+x}} N\left(-\frac{\sqrt{\epsilon+x}}{2}\right) \\ &- \quad (S+K)e^{-x/8}\sum_{j=0}^{m}(j+1)!\hat{W}(n,m,j)(K,\epsilon)(\epsilon+x)^{-(1+j)}. \end{aligned} \tag{1.92}$$

The time-t TVO price is equal to the t-conditional expectation

$$C_t^{TV}(K) = \mathbb{E}_t\left[\tilde{C}^{BS}(S_t, K, \langle X\rangle_T - \langle X\rangle_t, \langle X\rangle_t)\right]; \tag{1.93}$$

substituting (1.92) on the right-hand side of (1.93), making use of the integral representations (1.17) and (1.18) and applying Fubini's theorem yields the desired result.

□

Proof of the applicability of Fubini's theorem to formulae (1.48) and (1.50) . In order to justify the interchange of the order of integration in equation (1.48), it is sufficient to prove that

$$\int_0^{\infty} \mathbb{E}_t\left[\left| e^{-(z^{1/r}+b)\langle X\rangle_t}\mathrm{Re}\left(\frac{S_T}{S_t}\right)^{1/2\pm\sqrt{1/4-2z^{1/r}-4a}}\right|\right] dz < \infty. \tag{1.94}$$

Without loss of generality, set $b = 0$. For $z \to \infty$ the function

$$\begin{aligned} I(z) \quad &= \quad \mathbb{E}_t\left[\left| e^{-z^{1/r}\langle X\rangle_t}\mathrm{Re}\left(\frac{S_T}{S_t}\right)^{1/2\pm\sqrt{1/4-2z^{1/r}-4a}}\right|\right] \\ &\sim \quad e^{-z^{1/r}\langle X\rangle_t}\mathbb{E}_t\left[\left(\frac{S_T}{S_t}\right)^{1/2}\right] \to 0 \end{aligned} \tag{1.95}$$

decays exponentially, since $\mathbb{E}_t\left[(S_T/S_t)^{1/2}\right] < \infty$. The justification of equality (1.50) is similar.

□

Proof of proposition 1.6.1. [79], theorem 1.1.1. □

Proof of proposition 1.6.2. As $\epsilon \to 0$, the variables $h^{\epsilon}(\langle X\rangle_T)$ converge almost surely and monotonically to $h(\langle X\rangle_T)$ so (1.58) is clear. Being $h^{\epsilon}(x)$ continuous on $[0,+\infty)$ and $\lim_{x\to\infty} h^{\epsilon}(x) = 0$ we see that $h^{\epsilon}_*(x)$ is uniformly continuous on $[0,1]$ and $h^{\epsilon}_*(0) = 0$. But then by the Bernstein theorem

$$B_n h_*^\epsilon(y) \to h_*^\epsilon(y) \tag{1.96}$$

uniformly in $[0,1]$. Therefore if $y = e^{-cx}$ we have that

$$B_n h_*^\epsilon(e^{-cx}) \to h_*^\epsilon(e^{-cx}) = h^\epsilon(x) \tag{1.97}$$

uniformly in $[0,+\infty)$, so that $B_n h_*^\epsilon(e^{-c\langle X\rangle_T}) \to h_*^\epsilon(e^{-c\langle X\rangle_T})$ in mean. In particular:

$$\mathbb{E}_t[h^\epsilon(c\langle X\rangle_T)] = \lim_{n\to\infty} \mathbb{E}_t\left[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right]. \tag{1.98}$$

Finally, using the Newton binomial formula, shifting the j index, and changing the summation order we find that

$$\begin{aligned}
\mathbb{E}_t\left[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right] &= \sum_{j=1}^{n} h_*^\epsilon\left(j/n\right)\binom{n}{j}\mathbb{E}_t\left[e^{-cj\langle X\rangle_T}(1-e^{-c\langle X\rangle_T})^{n-j}\right] \\
&= \sum_{j=1}^{n} h_*^\epsilon\left(j/n\right)\binom{n}{j}\mathbb{E}_t\left[e^{-cj\langle X\rangle_T}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k e^{-ck\langle X\rangle_T}\right] \\
&= \sum_{k=1}^{n}\sum_{j=1}^{k} h_*^\epsilon\left(j/n\right)\binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}\mathbb{E}_t\left[e^{-ck\langle X\rangle_T}\right]
\end{aligned} \tag{1.99}$$

and then (1.59) follows from $\binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\binom{k}{j}$ and equation (1.44). □

Tables and figures

K	Laplace transform			Monte Carlo		
	T=1	T=2	T=3	T=1	T=2	T=3
60	8.34817	9.66239	10.60295	8.35267	9.66416	10.60701
80	5.21582	6.86439	8.04389	5.21659	6.86492	8.04662
100	3.03340	4.83608	6.13997	3.03152	4.83438	6.14030
120	1.69640	3.42085	4.73886	1.69431	3.41623	4.73499
140	0.93701	2.44696	3.70588	0.93408	2.44317	3.70266

Table 1.1: $S_{t_0} = 100$, $t = 0.25$, $\langle X \rangle_t = 0.1$. TVO prices for different strikes and maturities, using the Laplace method.

K	Taylor polynomial			Bernstein polynomial			Laplace transform	Monte Carlo
	n=1	n=2	n=3	n=10	n=20	n=30		
60	10.1534	11.1768	11.3814	21.3617	12.3200	11.6830	11.3919	11.4362
80	8.4475	8.7033	8.7289	13.5985	9.1500	8.8486	8.7301	8.7390
100	6.7416	6.7416	6.7416	6.7103	6.7348	6.7376	6.7416	6.7424
120	5.0357	5.2915	5.2659	5.0745	5.1940	5.2216	5.2672	5.2675
140	3.3298	4.3532	4.1485	3.9826	4.0867	4.1161	4.1699	4.1683

Table 1.2: $S_{t_0} = 100$, $T = 3$, $t = 0$, $\langle X \rangle_t = 0$. TVO prices for different strikes, using different pricing methods.

T	Taylor polynomial			Bernstein polynomial			Laplace transform	Monte Carlo
	n=1	n=2	n=3	n=10	n=20	n=30		
1	0.0776	0.7889	0.6822	0.6531	0.655	0.6557	0.6796	0.6765
2	1.7862	2.3835	2.2939	2.1247	2.2005	2.2306	2.2961	2.3028
3	3.1676	3.6829	3.6056	3.4280	3.5206	3.5510	3.6106	3.6023
4	4.3426	4.7952	4.7273	4.5311	4.6526	4.6818	4.7340	4.7254
5	5.3720	5.7746	5.7142	5.4329	5.6341	5.6718	5.7722	5.7166

Table 1.3: $S_{t_0} = 100$, $K = 130$, $t = 0.5$, $\langle X \rangle_t = 0.25$. TVO prices for different maturities, using different pricing methods.

$\langle X \rangle_{t_0}$	Taylor polynomial			Bernstein polynomial			Laplace transform	Monte Carlo
	n=1	n=2	n=3	n=10	n=20	n=30		
0.2	10.1790	10.7804	10.8706	10.9431	10.9183	10.9065	10.8849	10.8862
0.4	9.0426	9.5578	9.6351	9.5958	9.6298	9.6355	9.6467	9.6486
0.6	8.2371	8.6934	8.7618	8.6894	8.7370	8.7471	8.7667	8.7660
0.8	7.6247	8.0372	8.0991	8.0144	8.0643	8.0752	8.0963	8.0967
1	7.1374	7.5159	7.5727	7.4828	7.5314	7.5419	7.5623	7.5607

Table 1.4: $S_{t_0} = 100$, $K = 70$, $T = 5$, $t = 2$. TVO prices for different realized variance levels, using different pricing methods.

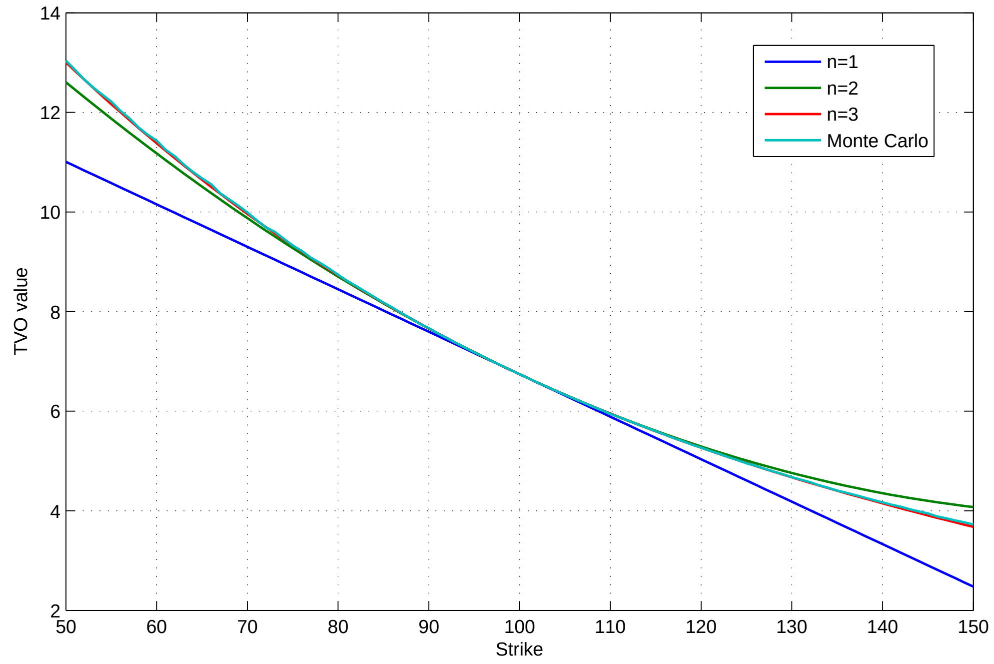

Figure 1.1: TVO value against the strike. Data from table 2, Taylor polynomials compared to the Monte Carlo simulation.

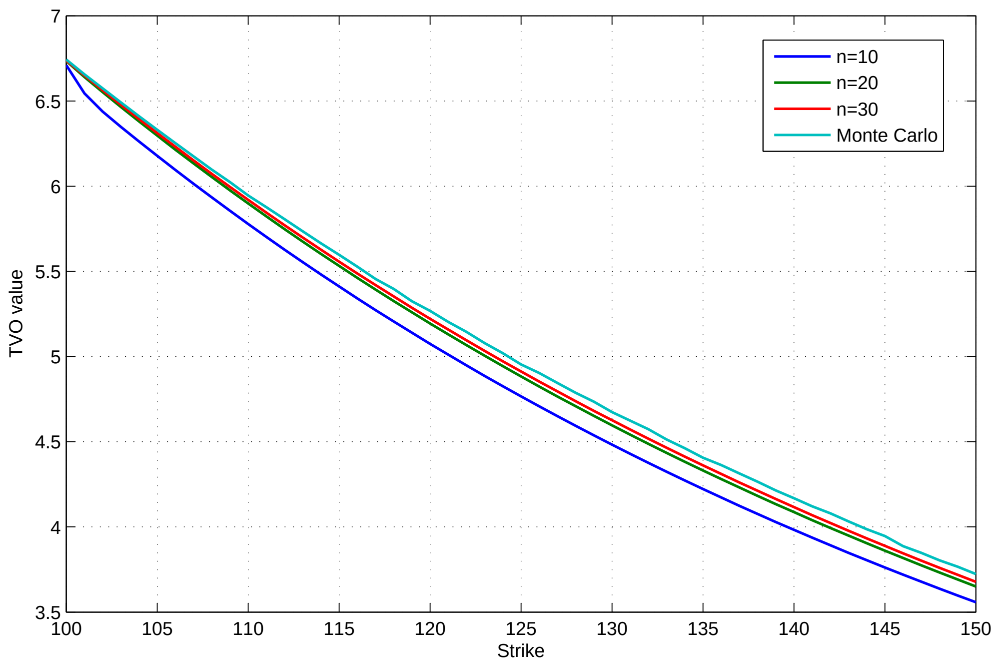

Figure 1.2: TVO values against the strike. Data from table 2, Bersntein polynomials compared to the Monte Carlo simulation, in-the-money strikes only (divergence occurs for OTM options; see section 1.6).

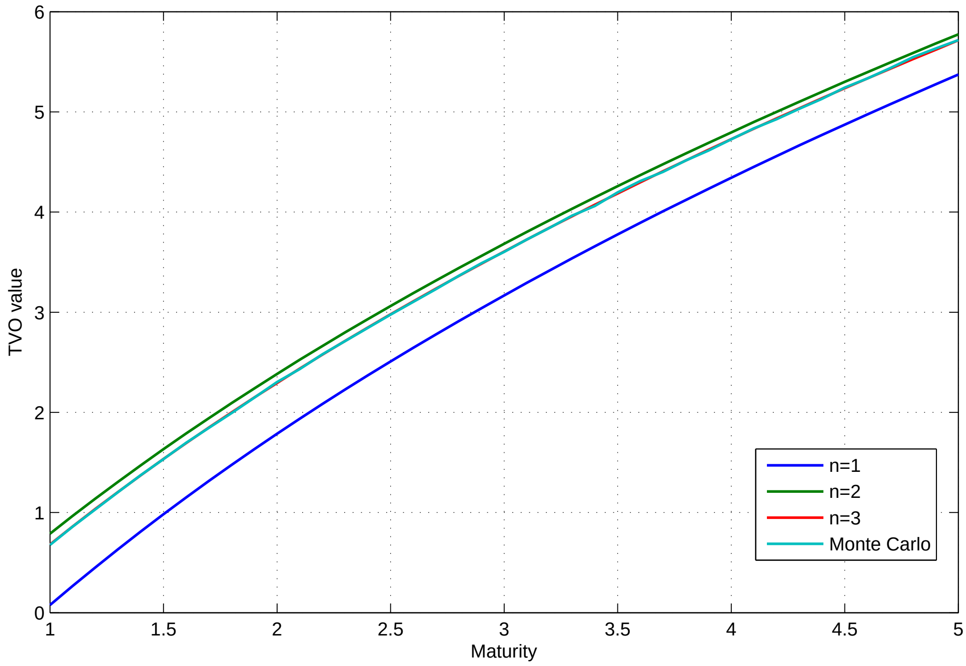

Figure 1.3: TVO values against maturity. Data from table 3, Taylor polynomials compared to the Monte Carlo simulation.

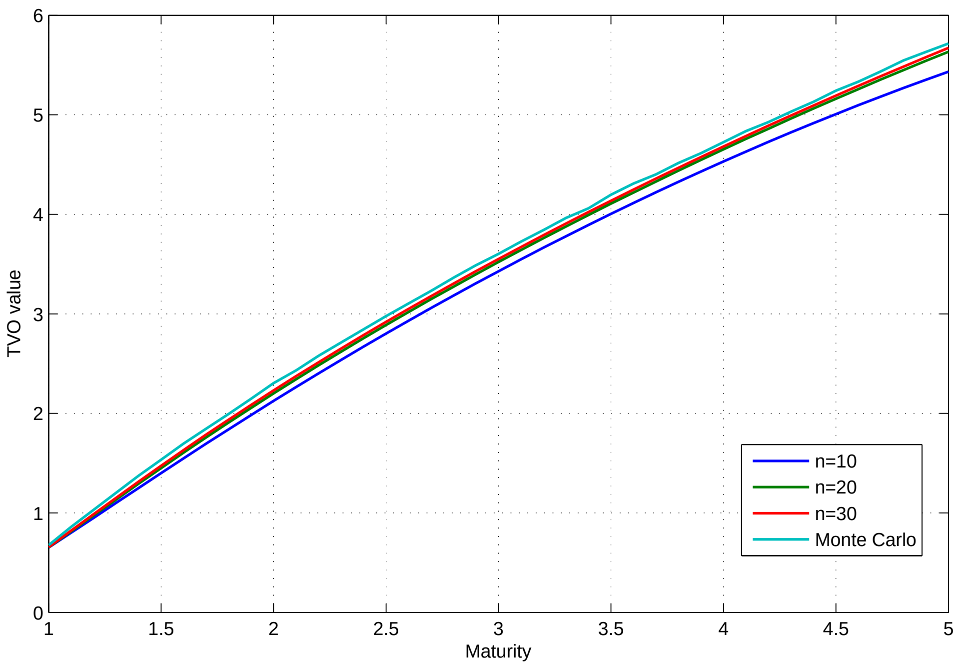

Figure 1.4: TVO values against maturity. Data from table 3, Bernstein polynomials compared to the Monte Carlo simulation.

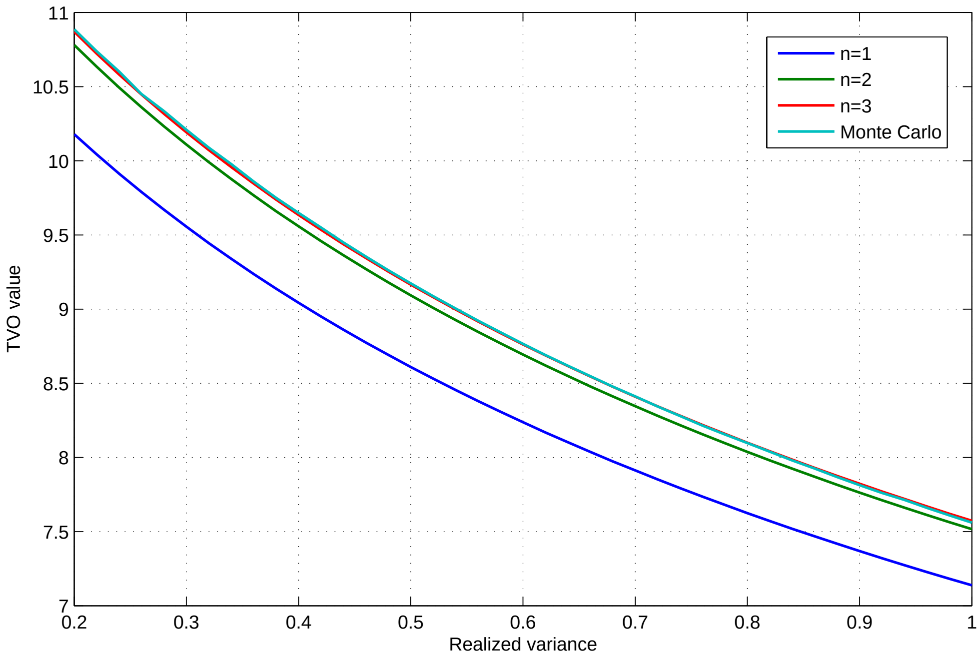

Figure 1.5: TVO values against realized variance. Data from table 4, Taylor polynomials compared to the Monte Carlo simulation.

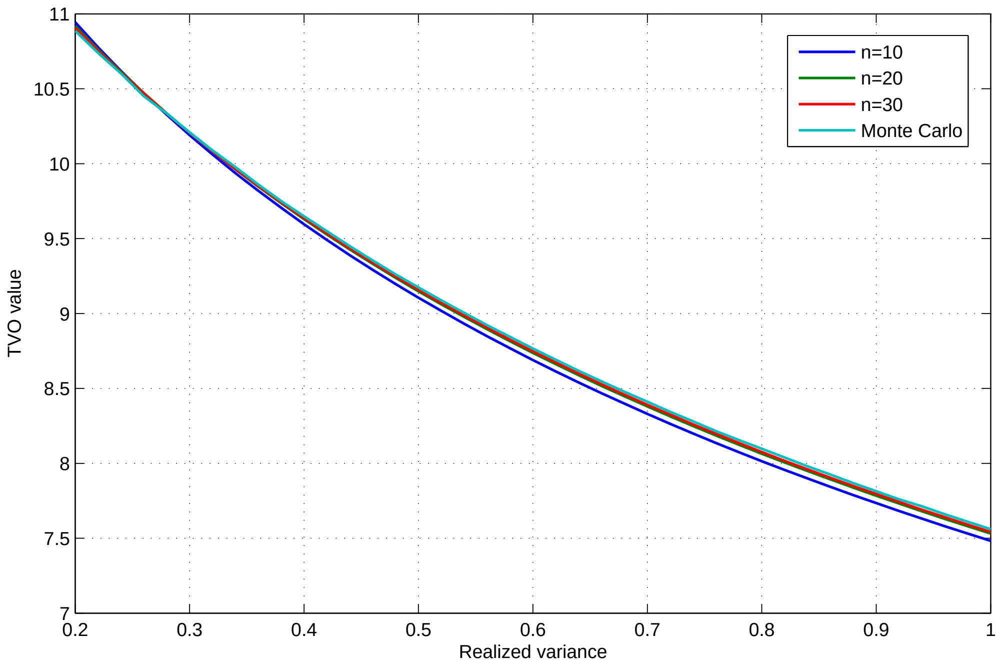

Figure 1.6: TVO values against realized variance. Data from table 4, Bernstein polynomials compared to the Monte Carlo simulation.

Chapter 2

Pricing joint claims on an asset and its realized variance in stochastic volatility models

In this chapter we further develop some of the ideas of chapter 1, and we do so in two different senses. In first place, we remove in the TVO pricing problem the assumption of independence between the asset and the stochastic volatility. Secondly, we find a general pricing technique valid not only for the TVO, but for any sufficiently regular joint asset and volatility payoff. In addition, we also consider a more general asset price model set-up.

In a general stochastic volatility framework, we find a partial differential equation giving the time-t price of a contract written jointly on an asset and its realized volatility, and solve it by Fourier transform methods. More precisely, denoting the quadratic variation of the log-returns of a market asset S_t by

$$I_t = \int_0^t v_u du, \tag{2.1}$$

where v_t is the instantaneous variance at time t, we are hereby interested in pricing European-style contingent claims maturing at time T of the form

$$F_T = F(S_T, I_T), \tag{2.2}$$

for some function F of two variables giving the underlying asset price and its quadratic variation. In a continuous stochastic volatility model the statistical realized variance and volatility of S_t are approximated respectively by I_T/T and $\sqrt{I_T/T}$, so that the class (2.2) is completely equivalent to that of the *joint asset and realized volatility (variance) claims*.

Far from being a mere theoretical exercise, introducing the historical variance in a payoff may very well lead to sensible real-life derivative products. As we shall see, coherently with the rationale standing behind the TVO, by means of a volatility correction it is possible to modify a European payoff f into a claim $\tilde{f}$, in such a way that taking a position in $\tilde{f}$ will be less costly, and still produce the same payoff as f if some predicted volatility event takes place. If an investor wishes to trade in f and has a strong belief about future volatility, he may choose to trade in $\tilde{f}$ instead, eventually being better off if his prediction was correct.

Although our main result is dependent upon the choice and calibration of the dynamics for the stochastic variance, it is universal in the sense that works with any sufficiently well-behaved stochastic volatility model. The possible alternative approach, a parameter-free replication pricing like the one advocated by Carr and Lee [14] and detailed in the previous chapter, although not prone to estimation errors, would evidently rely on the Breeden-Litzenberger formula replicating a claim on an asset through a portfolio of European call and put options. In practice, the issue with this approach is that the market may not offer a sufficient range or density of traded strikes, leading to truncation or discretization errors in the formula, especially for long maturities. In such instances a model-dependent choice, like the one suggested here, may be preferable.

Similar versions of our main equation are already present in literature. Lipton [77] gives the Fokker-Planck equation corresponding to the log-price version of (2.3). Fatone *et al.* [38] give a solution for such backward equation in the Heston model by means of a Fourier inversion, and then use the arising family of probability densities to obtain the price of pure volatility derivatives. Therefore, in their work a double integration is needed for a claim depending on a single state variable. Sepp [91] instead presents (the log-transformation of) equation (2.3) for jump diffusions and solves it with a method similar to ours, but then he excludes the price variable from the analysis and reverts to solutions for pure volatility derivatives in the Heston model. In contrast, we shall obtain pricing formulae for claims depending also on the final asset price at expiry for *any* well-behaved stochastic volatility model, while at the same time keeping the integration involved to a minimum.

In the spirit of the systematic study by Heath and Schweizer [58], care has been taken in emphasizing a series of sufficient conditions that make the pricing problem mathematically unambiguous. We do this by referring to the well-established theory of parabolic equations and SDEs (Friedman [44, 45], Feller [39], Kunita [73]).

The solution approach proposed is the natural two-dimensional extension of the pricing method found in Lewis [75]. Our *fundamental transform* will be taken with respect to the quadratic variation I_t, besides the log-price x. Strikingly, only minor modifications in the final formulae of Lewis are needed, which is indicative of how powerful such method is. It is noteworthy that the several-dimension Fourier transform idea can be in principle applied to the pricing of claims depending on other kinds of non-traded market factors (e.g. Asian options). As a matter of fact, this technique also applies to jump model. Hence, we do also state and solve the PIDE associated with pricing joint payoffs in a jump diffusion. As a further extra contribution, we briefly overview the hedging problem for the class of payoff we are studying, and give simple formulae for the hedging by replication and mean-variance hedging strategies.

In section 2.1 we define our model and derive the pricing equation, which is solved in section 2.2 together with a derivation for the Greeks. Section 2.3 shows the fundamental transforms for various models and deals with existence/uniqueness issues. In section 2.4 we treat and solve the PIDE for joint payoffs under a jump diffusion model while in section 2.5 we discuss the hedging problem. In section 2.6 some claims of the form $F(S_T, I_T)$ are introduced, which are then tested numerically in section 2.7. Technical details are provided in the appendix.

2.1 Setting up the equation

The single asset scenario that we are assuming consists of a three-factor Ito process $X_t = (S_t, v_t, I_t)$, $t \geq 0$, describing the evolution in time of a *risky asset* S_t, its *stochastic instantaneous variance* v_t and its *realized variance* I_t. A constant market risk-free rate r exists, and the asset S_t continuously pays to its owner a proportional constant dividend yield d. Valuations relying on such a stochastic variance model are unique modulo different choices of a market price of risk, which we hereafter assume to be fixed. This induces a risk-neutral pricing probability measure $\mathbb{P}$, which is the only one relevant in all that will follow. Under such a law the price S_t will therefore exhibit a log-return rate of $r - d$.

Let $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space satisfying the usual conditions, and let W_t^1, W_t^2 be two $\mathcal{F}_t$-adapted Brownian motions having correlation ρt. The underlying diffusion X_t is assumed to be of the form:

$$\begin{cases} dS_t = (r-d)S_t dt + \sqrt{v_t} S_t dW_t^1 \\ dv_t = \alpha(t, v_t)dt + \beta(t, v_t)dW_t^2 \\ dI_t = v_t dt. \end{cases} \tag{D}$$

We are interested in the behavior of this process in a finite time range $[0, T]$. We assume the coefficients $\alpha(t, x) : \mathbb{R}_+^2 \to \mathbb{R}$, $\beta(t, x) : \mathbb{R}_+^2 \to \mathbb{R}_+$ to be locally Lipschitz-continuous in x, uniformly in t; that is, for all compact sets $K \subset \mathbb{R}_+$, $\exists C_K > 0$ such that:

$$\sup_{0 \leq t \leq T} |\alpha(t, x) - \alpha(t, y)| + \sup_{0 \leq t \leq T} |\beta(t, x) - \beta(t, y)| < C_K |x - y|, \quad \forall x, y \in K. \tag{LL}$$

In general, (LL) is sufficient to ensure that a unique strong solution to (D) exists only up to a random exit time of $\mathbb{R}_+^3$. To obtain an everywhere well defined solution we must impose the (in the words of Feller [39]) *natural boundary conditions*:

$$\mathbb{P}_{x,t}\left(\sup_{t \leq s \leq T} v_s = +\infty\right) = \mathbb{P}_{x,t}\left(\inf_{t \leq s \leq T} v_s = 0\right) = 0, \quad \forall x \in \mathbb{R}_+, t \in [0, T). \tag{NB}$$

We also say that the 0 and $+\infty$ boundaries for the variance process must not be *attainable*. Most of the commonly used models for stochastic volatility satisfy (LL) but not necessarily (NB) for every possible choice of parameters. In practitioner's terms, failure to meet (NB) is interpreted as the possibility of "volatility explosions" or "volatility vanishings".

We wish now to find the PDE corresponding to the diffusion problem (D). Suppose one wishes to trade a derivative that pays off at the maturity date T a certain function of two variables: the underlying terminal asset value and the *quadratic variation* accumulated over $[0, T]$. The payoff is represented by the random variable $F(S_T, I_T)$, where $F(x, y)$ is an integrable function in the joint distribution of S_T and I_T. By the usual dynamic hedging argument we set up a portfolio that is long the contract, and short certain amounts of the underlying and another variance dependent contract. By choosing the hedge ratios so as to cancel the portfolio randomness, we argue that under the no-arbitrage condition, for the given market price of risk, the portfolio process must earn the risk-free rate r. The time-t value $V(S_t, I_t, v_t, t)$ of the contract can be thus seen to satisfy the following parabolic equation:

$$\frac{\partial V}{\partial t} + (r-d)S\frac{\partial V}{\partial S} + \alpha\frac{\partial V}{\partial v} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S \partial v} - rV = 0. \tag{2.3}$$

Pricing a cross asset-quadratic variation derivative $F(S_T, I_T)$ therefore amounts to solving the Cauchy free-boundary problem (2.3) in $\mathbb{R}^3_+ \times [0,T]$ with the following terminal condition:

$$V(S_T, I_T, v_T, T) = F(S_T, I_T). \tag{2.4}$$

This is the generalized version, in the price variable, of equation (13) of Sepp [91], and the dual of the Fokker-Planck equation appearing in Lipton [77] and Fatone *et al.* [38].

As it happens when dealing with parabolic equations arising from financial modeling, existence and uniqueness results for solutions may not be readily available from the standard theory of parabolic equations. Typically, this is for two reasons: coefficient constraints are not met, or terminal conditions (payoffs) are not continuous.

However, even if solvability remains an issue one has to live with[1], uniqueness of V essentially carries over from the uniqueness of a solution for (D), which is in turn enforced by assumptions (LL) and (NB). In addition, under these assumptions, we can invoke the Feynman-Kac theorem to link the discounted risk-neutral expectation of the payoff to the solution of the pricing equation. This is a standard requirement in the literature and it motivates our assumptions on (D).

Proposition 2.1.1. *Under assumptions (LL) and (NB) there exists at most one $C^{2,1}$ solution to problem (2.3)-(2.4); if such a solution does exist, for $x = (S_t, v_t, I_t)$ it is given by:*

$$V(x,t) = \mathbb{E}_{x,t}\left[e^{-r(T-t)}F(S_T, I_T)\right]. \tag{2.5}$$

Therefore under (LL) and (NB) pricing a claim of the form (2.2) is a well-posed problem, provided that (2.3) is solvable. We finally impose some growth constraints, this time directly on V:

$$V(S) < K_1(1 + S^{h_1}), \quad V(I) < K_2(1 + I^{h_2}) \tag{GC}$$

for some $K_1(I), K_2(S) > 0$, $h_1, h_2 \geq 0$.

The reason of this assumption is technical in nature and will allow us to perform the necessary reductions while solving the equation. The theory of parabolic equations (Friedman [44, 45]) provides sufficient conditions on the problem itself under which (GC) holds ([45], theorem 4.3, p. 147). However, it is difficult to give a comprehensive set of such assumptions in a financial setting, owing to the lack of the necessary regularity in many cases of interest: namely, superlinear growth of the coefficients of (D) or discontinuity of F. Alternatively, (GC) can be checked case by case, for example by using estimates along the lines of those derived by Bergman *et al.* [8], theorem 1. In any case, this condition is seen to hold for most of the cases accounted in section 2.3 (see appendix).

2.2 Solution to the PDE

We shall characterize the solution of the PDE by identifying a *fundamental transform*, which is a characteristic function for the model. To do this we will apply the Fourier transform to (2.3) with respect to both variables I and $\log S$. Once a fundamental transform has been found, we

[1] See, for example, Andersen and Piterbarg [3] on the non-existence of moments in the Heston model.

can invert it on a suitable domain in $\mathbb{C}^2$ and then conclude from proposition 2.1.1 the existence of a unique price for F.

Let $V(t,S,v,I)$ be the solution of (2.3) and consider the substitutions:

$$\begin{cases} \tau = T - t \\ x = \log S + (r-d)(T-t) \\ W(x,y,v,\tau) = \begin{cases} e^{r\tau} V(e^{-(r-d)\tau + x}, y, v, T-\tau) & \text{if } y > 0 \\ 0 & \text{if } y \leq 0. \end{cases} \end{cases} \tag{2.6}$$

Equation (2.3) can then be seen to be equivalent to the problem

$$\frac{v}{2}\left(\frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} + 2\frac{\partial W}{\partial y}\right) + \rho\beta\sqrt{v}\frac{\partial W}{\partial x \partial v} + \alpha\frac{\partial W}{\partial v} + \frac{\beta^2}{2}\frac{\partial^2 W}{\partial v^2} = \frac{\partial W}{\partial \tau}, \tag{2.7}$$

with initial condition:

$$W(x_0, y_0, v_0, 0) = \begin{cases} F(e^{x_0}, y_0) & \text{if } y_0 > 0 \\ 0 & \text{if } y_0 \leq 0. \end{cases} \tag{2.8}$$

For $(\eta, \omega) \in \mathbb{C}^2$, let the two-dimensional Fourier transform of $W(x,y,v,\tau)$ be:

$$\hat{W}(\omega, \eta, v, \tau) = \int_{\mathbb{R}^2} e^{ix\omega + iy\eta} W(x,y,v,\tau)\,dxdy. \tag{2.9}$$

We denote derivatives by subscripts. Consider the transform $\widehat{W_{\cdot}}$ of the partial derivatives of W. By substituting (2.7) in the integral above and integrating by parts we find that:

$$\begin{array}{lll} \widehat{W_\tau} = \hat{W}_\tau, & \widehat{W_x} = -i\omega\hat{W}, & \widehat{W_{xx}} = -\omega^2\hat{W} \\ \widehat{W_y} = -i\eta\hat{W}, & \widehat{W_v} = \hat{W}_v, & \widehat{W_{vv}} = \hat{W}_{vv} \\ \widehat{W_{xv}} = -i\omega\hat{W}_v & & \end{array} \tag{2.10}$$

provided that:

$$e^{i\omega x} W(x)|_{-\infty}^{+\infty} = e^{i\omega x} W_x(x)|_{-\infty}^{+\infty} = e^{i\eta y} W(y)|_{+\infty} = 0 \tag{2.11}$$

holds true for some ω, η. These relations are clear if we know V to satisfy (GC), which then yields (2.10) in a two-strip:

$$\Omega_1 = \{a_1 < Im(\omega) < a_2,\, Im(\eta) > 0\} \subset \mathbb{C}^2. \tag{2.12}$$

Fourier-transforming both sides of (2.7) and substituting the above relations, we have the fundamental PDE for $\hat{W}$:

$$\frac{\beta^2}{2}\frac{\partial^2 \hat{W}}{\partial v^2} + \frac{\partial \hat{W}}{\partial v}(\alpha - i\omega\sqrt{v}\rho\beta) - \frac{v}{2}(\omega^2 - i\omega + 2i\eta)\hat{W} = \frac{\partial \hat{W}}{\partial \tau}. \tag{2.13}$$

A *fundamental transform* $\hat{H}(\omega, \eta, v, \tau)$ for (2.7) is a solution to (2.13) such that $\hat{H}(\omega, \eta, v, 0) = 1$. Assume that such a solution exist. Up to a sign shift in the arguments, $\hat{H}$ is the characteristic function of the transition probability density associated with the process

$(\log S_t, I_t)$, and it is thus a holomorphic function[2] on a certain multi-strip $\Omega_2 \subset \mathbb{C}^2$.

Denote the Fourier transform of the payoff in the log-price and quadratic variation by $\hat{F}(\omega, \eta) := \hat{W}(\omega, \eta, v, 0)$, itself a holomorphic function on a third multi-strip $\Omega_3 \subset \mathbb{C}^2$. Since $\hat{F}(\omega, \eta)$ *does not depend on the variables* v *and* τ we see that the product $\hat{H}(\omega, \eta, v, \tau)\hat{F}(\omega, \eta)$ is *also a solution to (2.13) having initial condition* $\hat{F}(\omega, \eta)$. Therefore by taking the Fourier inverse of $\hat{H}(\omega, \eta, v, \tau)\hat{F}(\omega, \eta)$ on a multi-line

$$\Sigma = \{(\omega, \eta), \omega = s + ik_1, \eta = t + ik_2, s, t \in \mathbb{R}\} \subset \Omega = \bigcap_{i=1}^{3} \Omega_i, \quad k_1, k_2 \in \mathbb{R}, \tag{2.14}$$

and finally unwinding the variable change we are led to the solution of (2.3):

$$V(S, I, v, t) = \frac{e^{-r(T-t)}}{4\pi^2} \times \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} S^{-i\omega} e^{-i\omega(r-d)(T-t)} e^{-i\eta I} \hat{H}(\omega, \eta, v, T-t)\hat{F}(\omega, \eta) d\omega d\eta. \tag{2.15}$$

Finally, proposition 2.1.1 establishes that (2.15) is the unique price of $F(S_T, I_T)$. Of course, such an argument is meaningful provided that *a common domain of holomorphy* $\Omega \subset \mathbb{C}^2$ *of* $\hat{H}$ *and* $\hat{F}$ *actually exists*.

We summarize the above discussion in the following proposition:

Proposition 2.2.1. *Assume that the solution* X_t *of (D) is such that the dynamics for* v_t *satisfy (LL), (NB), and that (GC) holds. Further assume that* $\Omega \neq \emptyset$ *and let* $k_1, k_2 \in \mathbb{R}$ *be such that:*

$$\Sigma = \{(\omega, \eta), \omega = s + ik_1, \eta = t + ik_2, s, t \in \mathbb{R}\} \subset \Omega. \tag{2.16}$$

If a fundamental transform $\hat{H}(\omega, \eta, v, \tau)$ *can be found, the price of a claim* F *written on* S_t *and* I_t *is given by equation (2.15).*

This formula is completely general: in principle, under the given assumptions, it allows pricing under any stochastic volatility model. Another attractive feature of equation (2.15) is that it allows us to separate, by means of $\hat{H}$ and $\hat{F}$, the pricing information coming from the *model* from that coming from the *payoff*. This was one of the contributions of Lewis's work on stochastic volatility models and is equally valid here. Changing the stochastic volatility or the function to be valued only requires changing the corresponding transform to be used in (2.15), and not the whole re-computation of the solution.

2.2.1 Greeks

The representation found also allows for a straightforward computation of the Greeks. Calling $J(\omega, \eta, v, \tau)$ the integrand in (2.15) and differentiating V under integral sign we find that the

[2]See Lukacs [80], theorem 7.1.1. Since we are not confined to real arguments, we do not need to consider analytic continuations around 0, and may instead develop around any point in whose neighborhood $\hat{H}$ is holomorphic. This means that Ω_2 will exist somewhere, even if in general it may not contain the real axis.

Delta for the contract F is:

$$\Delta = \frac{\partial V}{\partial S} = -\frac{e^{-r(T-t)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \frac{i\omega}{S} J(\omega, \eta, v, \tau) d\omega d\eta. \tag{2.17}$$

Likewise, the Gamma is seen to be given by:

$$\Gamma = \frac{\partial^2 V}{\partial^2 S} = \frac{e^{-r(T-t)}}{4\pi^2} \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} \frac{i\omega - \omega^2}{S^2} J(\omega, \eta, v, \tau) d\omega d\eta. \tag{2.18}$$

The derivative Δ is one the two coefficients to be used in the hedge ratios yielding equation (2.3). The sensitivity to the initial instantaneous variance $\partial V/\partial v$ can be sometimes expressed in a similar fashion as (2.17) and (2.18), for example in exponentially-affine models. Clearly, the ability to fully hedge will depend also on the possibility to identify a fundamental set of securities for the market.

2.3 Model-specific fundamental transforms

We analyze here in more detail the fundamental transforms of the Heston model, the 3/2 model and GARCH models. The analytical tractability that characterizes these models in a standard stochastic volatility scenario carries over when realized volatility comes into the picture. Remarkably, the solution of (2.13) depends on the coefficient of the linear term as a parameter. As the variable η appears only in such coefficient, the derivations are identical to those already present in literature, to which our equations reduce when $\eta = 0$.

Below are the transforms, together with their domains of holomorphy, given as functions of two complex variables. Being a Fourier integral, $\hat{H}$ is everywhere holomorphic in its domain of definition. Complex square roots are always understood to be the positive determination. In accordance with our initial assumption, the parameters for the models already incorporate the market price of risk adjustment. For a sketch proof of the derivation of (2.20) consult the appendix; a complete treatment can be found in Lewis [75].

2.3.1 Heston model

In the model by Heston [60], a fundamental transform can easily be obtained. The dynamics of the instantaneous variance are:

$$dv_t = \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}dW_t, \tag{2.19}$$

with $\kappa, \theta, \epsilon > 0$. Volatility explosions never occur; taking $2\kappa\theta \geq \epsilon^2$ ensures that the 0 boundary is not attainable. Thus under these conditions assumption (NB) is met.

The fundamental transform of the Heston model is:

$$\hat{H}(\omega,\eta,v,\tau) = \exp[C(\omega,\eta,\tau) + vD(\omega,\eta,\tau)]$$

$$C(\omega,\eta,\tau) = \frac{\kappa\theta}{\epsilon^2}\left(\tau(b(\omega) - d(\omega,\eta)) - 2\log\left(\frac{e^{-d(\omega,\eta)\tau} - c(\omega,\eta)}{1 - c(\omega,\eta)}\right)\right)$$

$$D(\omega,\eta,\tau) = \frac{b(\omega) + d(\omega,\eta)}{\epsilon^2}\left(\frac{1 - e^{d(\omega,\eta)\tau}}{1 - c(\omega,\eta)e^{d(\omega,\eta)\tau}}\right) \tag{2.20}$$

$$c(\omega,\eta) = \frac{b(\omega) + d(\omega,\eta)}{b(\omega) - d(\omega,\eta)}$$
$$b(\omega) = \kappa + i\epsilon\omega\rho$$
$$d(\omega,\eta) = \sqrt{b(\omega)^2 + \epsilon^2(\omega^2 - i\omega + 2i\eta)}.$$

The expression for C uses the argument by Guo and Hung [57] and Lord and Kahl [78] to avoid discontinuity issues in the complex logarithm. The only singularities occur when $1 - c(\omega,\eta)e^{d(\omega,\eta)\tau} = 0$ causing divergence in both C and D; hence the domain of holomorphy of $\hat{H}$ is $\mathbb{C}^2 \setminus \mathcal{S}_{\kappa,\epsilon,\rho,\tau}$ where

$$\mathcal{S}_{\kappa,\epsilon,\rho,\tau} = \{(\omega,\eta) \in \mathbb{C}^2 |\ e^{-d(\omega,\eta)\tau} = c(\omega,\eta)\}. \tag{2.21}$$

2.3.2 3/2 model

We consider the general form as introduced by Lewis [75]. The instantaneous variance is given by:

$$dv_t = \kappa(\theta v_t - v_t^2)dt + \epsilon {v_t}^{3/2} dW_t. \tag{2.22}$$

Whenever $2\kappa \geq -\epsilon^2$ we have that $+\infty$ is unattainable; the 0 boundary is natural for any choice of parameters. We have the following fundamental transform for the model:

$$\hat{H}(\omega,\eta,v,\tau) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)} X\left(\frac{2\kappa\theta}{\epsilon^2 v}, \kappa\theta\tau\right)^{\alpha} {}_1F_1\left[\alpha,\beta,-X\left(\frac{2\kappa\theta}{\epsilon^2 v},\kappa\theta\tau\right)\right]$$
$$X(x,t) = \frac{x}{e^t - 1}$$
$$\alpha(\omega,\eta) = c(\omega,\eta) - b(\omega) \tag{2.23}$$
$$\beta(\omega,\eta) = 1 + 2c(\omega,\eta)$$
$$b(\omega) = (\kappa + \epsilon^2/2 + i\omega\rho\epsilon)/\epsilon^2$$
$$c(\omega,\eta) = \sqrt{b(\omega)^2 + d(\omega,\eta)}$$
$$d(\omega,\eta) = 2(\omega^2 - i\omega + 2i\eta)/\epsilon^2.$$

${}_1F_1(\alpha,\beta,z)$ is a confluent hypergeometric series and Γ is the Euler's Gamma function. Since β cannot be a negative integer the singularities of Γ are avoided, so that the domain of the transform is the whole $\mathbb{C}^2$.

2.3.3 GARCH model

We only take into account a particular instance of this model, namely when the dynamics are those of a geometric Brownian motion with drift. We also assume $\rho = 0$; the case $\rho \neq 0$ can be obtained by a simple modification of the derivation in [75]. The equation for the instantaneous variance is

$$dv_t = \theta v_t dt + \epsilon v_t dW_t, \tag{2.24}$$

with $\epsilon > 0$. Clearly, condition (NB) is always met. The fundamental transform is:

$$\begin{aligned}
&\hat{H}(\omega, \eta, v, \tau) = \\
&\frac{2^{\beta+1}}{d(\omega,\eta)^\beta} \left[\mathbb{I}_{\{\beta<0\}} \sum_{j=0}^{[-\beta/2]} \frac{-\beta - 2j}{j!\Gamma(1-\beta-j)} K_{-\beta-2j}(d(\omega,\eta)) e^{(\beta j + j^2)\epsilon^2 \tau/2} \right. \\
&\left. + \frac{1}{4\pi^2} \int_0^\infty \left| \Gamma\left(\frac{\beta + iz}{2}\right) \right|^2 z \sinh(z\pi) K_{iz}(d(\omega,\eta)) e^{-(\beta^2+z^2)\epsilon^2\tau/8} dz \right] \\
&\beta = 2\theta/\epsilon^2 - 1 \\
&d(\omega,\eta) = 2\sqrt{2(\omega^2 - i\omega + 2i\eta)v}/\epsilon.
\end{aligned} \tag{2.25}$$

Here $\mathbb{I}$ is the indicator function and K_x the modified Bessel function of second kind. By use of the appropriate series representation, K_x can be extended to an entire function. So we see that $\hat{H}$ is a holomorphic on $\mathbb{C}^2 \setminus \{\omega^2 - i\omega + 2i\eta\}$ whenever $2\theta > \epsilon^2$, and it is everywhere analytical on $\mathbb{C}^2$ otherwise.

2.4 Pricing in stochastic volatility models with jumps

We want now to turn the diffusion (D) into a jump model and see if some useful pricing differential relation can be extracted from the resulting jump diffusion. Namely, by adding jumps to the asset evolution in (D), we consider the following model:

$$\begin{cases} dS_t = (r - d - \kappa)S_{t^-}dt + \sqrt{v_t}S_{t^-}dW_t^1 + (\exp(J) - 1)S_{t^-}dN_t \\ dv_t = \alpha(t, v_t)dt + \beta(t, v_t)dW_t^2 \\ dI_t = v_t dt + J^2 dN_t. \end{cases} \tag{D*}$$

As usual, W_t^1 and W_t^2 are two linearly correlated Brownian motions with correlation coefficient ρt. The process N_t is a Poisson arrival process of rate λ and J a random variable giving the size of the jumps in the log-price process. The coefficient κ is the compensator for the jump part; since the jumps are of finite activity we have $\kappa = \int_{\mathbb{R}}(e^x - 1)\nu(dx)$ where ν is the associated Lévy measure. The constraints on α and β are the same as in (D). Moreover, a necessary condition on $\nu(dx)$ for S_t to be a square-integrable martingale is that:

$$\int_{|x|>1} e^{2x}\nu(dx) < \infty \tag{2.26}$$

(see e.g. [23]) which is in particular satisfied for the typical choice of J being normally distributed. Equation (2.26), together with condition (NB), is sufficient to ensure that S_t is a martingale. Indeed, in the equivalent martingale measure and assuming no dividends are paid, the solution of S_t is given by:

$$S_t = S_0 \exp\left(\int_0^t \sqrt{v_u} dW_u^1 - \int_0^t v_u d_u/2 + rt\right) e^{-\kappa t} \prod_{t \leq N_t} e^J. \tag{2.27}$$

The pair W_t^1, W_t^2 is orthogonal to $N_t - \kappa$ and thus, under the given assumptions, so are the continuous and pure jump martingales in (2.27). That is, S_t is the product of two orthogonal martingales, and therefore is itself a martingale.

It is important to compare the equations for I_t in (D) and (D*). When the log-process X_t of an underlying asset is a discontinuous semimartingale, its quadratic variation is discontinuous as well, and equals:

$$I_t = \langle X^c \rangle_t + \sum_t \Delta X_t^2 \tag{2.28}$$

where $\langle \cdot \rangle_t$ is the quadratic variation operator and X_t^c is the continuous part of X_t in the semimartingale representation of X_t, and $\Delta X_t = X_t - X_{t_-}$. In our situation equation (2.28) takes the form (D*).

It is well-known that the pricing PDEs associated to ordinary stochastic volatility models can be easily modified into partial integro-differential equations for some related jump diffusion (see e.g Bates [6, 7]); for the augmented diffusion (D*) the same idea applies. The argument runs as follows. In the diffusion above, let $V(S_t, I_t, v_t, t)$ be the value at time t of a claim paying off $F(S_T, I_T)$ at time T. Since discontinuous claims are not perfectly hedgeable, we are not able to apply a portfolio replication argument; however, risk-neutral measures still exist and by the fundamental theorem the expectation of the discounted value of V under every such measure must be a martingale. As usual, one first applies Ito's lemma for discontinuous functions to $e^{-r(T-t)}V(S_{t_-}, I_{t_-}, v_t, t) = \tilde{V}$, yielding:

$$\begin{aligned} d\tilde{V} =& \frac{\partial \tilde{V}}{\partial t} dt + (r - d - \kappa) S_{t_-} \frac{\partial \tilde{V}}{\partial S} dt + \sqrt{v_t} S_{t_-} \frac{\partial \tilde{V}}{\partial S} dW_t^1 + \alpha(t, v_t) \frac{\partial \tilde{V}}{\partial v} dt + \beta(t, v_t) \frac{\partial \tilde{V}}{\partial v} dW_t^2 \\ & + v_t \frac{\partial \tilde{V}}{\partial I} dt + \frac{v S_{t_-}^2}{2} \frac{\partial^2 \tilde{V}}{\partial S^2} dt + \frac{\beta(t, v_t)^2}{2} \frac{\partial^2 \tilde{V}}{\partial v^2} dt + \rho \beta(t, v_t) \sqrt{v_t} S_{t_-} \frac{\partial \tilde{V}}{\partial S \partial v} dt + \\ & \left(\tilde{V}(S_{t_-} e^J, I_{t_-} + J^2, v_t, t) - \tilde{V}(S_{t_-}, I_{t_-}, v_t, t)\right) dN_t \end{aligned} \tag{2.29}$$

We see that $\tilde{V}$ is a martingale if and only if the expectation of the above differential is zero. The PIDE for the stochastic volatility model with realized volatility (D*) satisfied by V is therefore

$$\begin{aligned} & \frac{\partial V}{\partial t} + (r - d - k) S \frac{\partial V}{\partial S} + \alpha \frac{\partial V}{\partial v} + v \frac{\partial V}{\partial I} + \frac{v S^2}{2} \frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2} \frac{\partial^2 V}{\partial v^2} + \rho \beta \sqrt{v} S \frac{\partial V}{\partial S \partial v} + \\ & \int_{\mathbb{R}} (V(S_{t_-} e^z, I_{t_-} + z^2, v_t, t) - V(S_{t_-}, I_{t_-}, v_t, t)) \nu(dz) - rV = 0. \end{aligned} \tag{2.30}$$

which is an equivalent of equation (2.3) that also accounts for the jumps in the asset and in the realized volatility, as well as the the Lévy compensator of the jumps.

When the stochastic volatility model is specified as a square-root process, the equation above is a version of the PIDE derived by Bates in [6] (equation A4) by equilibrium arguments, only augmented with the additional terms for the realized volatility. In analogy with the Heston case, we can find the fundamental transform by setting the terminal condition to 1 and then taking the Fourier transform of (2.30) with respect to the log-price and realized variance. Let W be as in (2.6) with $r-)$ replaced by $r-d-\kappa$ in the second equation. Remembering the rules on the Fourier transformation of a shifted function, we recognize the analogous of (2.13) as

$$\frac{\beta^2}{2}\frac{\partial^2 \hat{W}}{\partial v^2}+\frac{\partial \hat{W}}{\partial v}(\alpha - i\omega\sqrt{v}\rho\beta)-\left(v\frac{\omega^2-i\omega+2i\eta}{2}+\int_{\mathbb{R}}(e^{i\omega z+i\eta z^2}-1)\nu(dz)\right)\hat{W}=\frac{\partial \hat{W}}{\partial \tau}. \tag{2.31}$$

We are thus looking for a solution $\hat{H}$ of the above equation subject to $\hat{H}(\omega,\eta,v,0)=1$. For example if v_t is given by a square root process, and the *joint* Laplace transform of J and J^2 is known, $\hat{H}$ is computable in closed form[3]. Again, we then reverse the transformation, apply equation (2.15) to a target claim $F(\omega,\eta)$, and we have the desired price of F under the jump-diffusion (D*). In chapter 3 we will recover the results of the foregoing analysis, as well as those from the previous sections, by using a probabilistic method.

2.5 Hedging a joint claim

In this section we outline and solve the problem of hedging a financial derivative $F(S_t, I_t)$ written on an asset S_t and its realized variance I_t in a typical incomplete market situation.

In many respects, the hedging problem under the class of models of the form above is akin to the hedging problem of a vanilla product in incomplete markets. The realized volatility is a state variable whose random evolution is completely determined by the evolution of the stochastic variance dynamics, and does not require independent modeling or sources of randomness, like the introduction of a new Brownian motion or jump components.

In what follows we describe the problem of hedging a joint payoff by using two well-known hedging strategies for derivative products in incomplete market scenarios, namely:

- hedging by replication using the underlying and a second traded instrument;
- the mean-variance hedging technique.

The first strategy is known to perfectly replicate a claim in a stochastic volatility setup, so we can apply it when the asset dynamics are given by (D). The second strategy is a popular method when exact replication is not allowed, as it is the case for a model with jumps, so we shall illustrate it in the context of the model (D*).

2.5.1 Hedging using a second traded instrument

Hedging in a continuous stochastic volatility model using a further liquidly traded derivative is the usual approach found in e.g. [99] and others. We show that this method extends at verbatim when the payoffs depend on the realized volatility.

[3]For explicit calculations see chapter 3, where the present set-up will be recovered through a different technique.

Assume that in the market there exists a sufficiently liquid instrument G_t, whose value depends on S_t, typically a plain vanilla instrument like a call option. We want to show that we can perfectly hedge F by using a quantity Γ_t of the options G_t, a number ϕ_t shares of the underlying S_t and an amount ψ_t of cash in the bank account B_t. To do so, we set-up the portfolio:

$$\Pi_t := \phi_t S_t + \Gamma_t G_t + \psi_t B_t \tag{2.32}$$

and set the hedge ratios as:

$$\Gamma_t = \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1} \tag{2.33}$$

$$\phi_t = \frac{\partial V}{\partial S} - \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}\frac{\partial G}{\partial S} \tag{2.34}$$

$$\begin{aligned}\psi_t = \frac{1}{rB_t}\left[\frac{\partial V}{\partial t} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S\partial v}\right.\\ \left. -\frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}\left(\frac{\partial G}{\partial t} + \frac{vS^2}{2}\frac{\partial^2 G}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 G}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial G}{\partial S\partial v}\right)\right].\end{aligned} \tag{2.35}$$

With these choices we have that Π_t is a self-financing portfolio, that is, a portfolio whose change in value is only due to the portfolio earning/losses, that perfectly replicates V_t, i.e. $V_t = \Pi_t$ for all t. Note that the hedge ratios of $(\phi_t, \psi_t, \Gamma_t)$ are exactly the same that should be used for a contingent claim $F(S_t)$ paying only on the asset terminal value (compare again with [99]).

Another possibility to try to attain a perfect hedge for $F(S_t, I_t)$ is using a pure volatility derivative P_t, whose value instantly depends only on t and v_t. We let P_t be the time-t value of such a volatility. This time we build the portfolio:

$$\Pi_t := \phi_t S_t + \Lambda_t P_t + \psi_t B_t \tag{2.36}$$

where $(\phi_t, \psi_t, \Lambda_t)$ are the hedging ratios to be considered: ϕ_t shares of S_t, a value of ψ_t in cash, and a position in Λ_t quantities of P_t. In this case we obtain that:

$$\Lambda_t = \frac{\partial V}{\partial v}\frac{\partial P}{\partial v}^{-1} \tag{2.37}$$

$$\phi_t = \frac{\partial V}{\partial S} \tag{2.38}$$

$$\begin{aligned}\psi_t = \frac{1}{rB_t}\left[\frac{\partial V}{\partial t} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S\partial v}\right.\\ \left. -\frac{\partial V}{\partial v}\frac{\partial P}{\partial v}^{-1}\left(\frac{\partial P}{\partial t} + \frac{\beta^2}{2}\frac{\partial^2 P}{\partial v^2} + v\frac{\partial P}{\partial I}\right)\right]\end{aligned} \tag{2.39}$$

Typically P_t will be a variance swap, as these are the most liquid volatility derivatives. It is noteworthy that a strategy in a stochastic volatility model where we "completed the market" using a volatility derivative instead of a derivative depending on the underlying price (such as a call option), yields a much simplified formula for the hedge ratio ϕ_t.

The hedging technique recalled in this section is a *perfect hedge*, which means that in a frictionless market where continuous-time trading is possible, the value of a claim on an asset following a stochastic volatility model can be exactly reproduced as a linear combination of the underlying, a money market account and an auxiliary liquid asset.

The formulae of this section are thus perfectly in line with those for hedging an asset-based derivative product in a stochastic volatility model, the reason being that the quadratic variance I_t is not a source of risk external to the log-returns and the stochastic variance.

2.5.2 Mean-variance hedging

In the more general case of the model (D*), the inclusion of jumps makes the market intrinsically incomplete. In the previous sections, we were able to devise a strategy to cancel the extra randomness due to the stochastic variance by adding another asset whose value is known. In contrast here the jump randomness can never be completely hedged away, and we have to rely on an approximate hedge using one of the many techniques proposed in the literature.

Let V_0 be the initial endowment for the hedge, i.e. the initial price of the security $F(S_t, I_t)$. What we shall consider is the *mean variance hedging strategy*, a strategy aimed at minimizing the hedging error that we briefly recall. Consider the self-financing portfolio (V_0, ϕ_t, ψ_t) with associated earning process

$$\pi_t(\phi) = V_0 + \int_0^t \psi_u dB_u + \int_0^t \phi_u dS_u. \tag{2.40}$$

This strategy is said to be *optimal* whenever ϕ_t is chosen as the minimizer at maturity of the $\mathcal{L}^2(\Omega)$ norm of the difference between the value V_t of the claim F and the portfolio π_t, for all $t \leq T$. That is, ϕ_t must be such that:

$$||V_t - \pi_t(\phi)||_{L^2(\Omega)} = \inf_{\theta \in \mathbb{A}} ||V_t - \pi_t(\theta)||_{L^2(\Omega)} \tag{2.41}$$

for all $t \leq T$. The set $\mathbb{A}$ is the set of all the *admissible trading strategies* involving the underlying S_t, that is, the set of all the predictable càglàd processes θ_t whose stochastic integral with respect to S_t is square-integrable. After discounting by r, the problem (2.40)-(2.41) can be restated as that of finding ϕ_t such that:

$$||\tilde{V}_t - \tilde{\pi}_t(\phi)||_{L^2(\Omega)} = \inf_{\theta \in \mathbb{A}} ||\tilde{V}_t - \tilde{\pi}_t(\theta)||_{L^2(\Omega)} \tag{2.42}$$

where

$$\tilde{\pi}_t(\phi) = V_0 + \int_0^t \phi_u d\tilde{S}_u. \tag{2.43}$$

From a theoretical perspective, the existence of a pair (V_0, ϕ_t), where ϕ_t is a process satisfying the above requirements, is proved by a direct application of the Kunita-Watanabe decomposition (see e.g. Cont and Tankov [23]). Unfortunately, the Kunita-Watanabe decomposition does not yield a constructive method to explicitly calculate an actual hedging strategy ϕ_t, and does not show that such strategy is indeed admissible , i.e. it is càglàd. However, by assuming $F(x, y)$ to be such that:

$$|F(x, y) - F(x_0, y_0)| \leq C||(x_0, y_0)||, \quad C > 0 \tag{2.44}$$

that is, the payoff F to be be Lipschitzian in $\mathbb{R}^2$, it is proved in the appendix that the minimal hedging strategy can be explicitly calculated by a standard differentiation argument, and it is attained by setting

$$\phi_t = \left(v_t \frac{\partial \tilde{V}}{\partial \tilde{S}} + \tilde{S}_t^{-1} \int_{\mathbb{R}} (\tilde{V}(\tilde{S}_{t_-} e^x, I_{t_-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t_-}, I_{t_-}, v_t, t))(e^x - 1)\nu(dx) \right) \times \left(v_t + \int_{\mathbb{R}} (e^x - 1)\nu(dx) \right)^{-1} . \quad (2.45)$$

In financial terms, ϕ_t is the position in the underlying asset minimizing the hedging error with the actual derivative value. If compared to the formula for the mean-variance hedging in a jump diffusion setup were the payoffs bear no dependence on I_t (see for example [23]), here we notice that there is an extra integration term arising from the presence of discontinuities in the value function caused by the jumps in the realized volatility. This is because in the Ito differential representation, the realized volatility in (D*) contributes with a stochastic term. This contrasts with the case of hedging under a pure stochastic volatility model treated in the previous section, were the contribution from the realized volatility was a drift term to be hedged with cash, and as such did not impact the holdings in the risky asset.

2.6 Some joint asset/volatility derivatives

We present here a list of European-style derivatives paying off a joint function of a terminal asset value and its realized variance or volatility. As explained in the introduction, these can all be considered as volatility-modified versions of well-known payoff structures, where the volatility factor reduces the initial price without affecting the payoff if the investor's volatility foresight happens to be correct. The target volatility option is one currently traded product of this kind.

2.6.1 Target volatility option

As we already discussed, a target volatility call option is the option to buy a certain fractional amount of shares if the underlying is worth more than the strike price at maturity. Such an amount is stochastic and depends upon both the target parameter $\overline{\sigma}$ set when writing the contract and the volatility realized by the asset in $[0, T]$. Under the independence hypothesis between the Brownian motion driving the underlying and the process for the instantaneous volatility, the value of an at the money call TVO is approximately the Black-Scholes price of a call option of constant volatility $\overline{\sigma}$ (see section 1.2). The payoff of a call TVO is:

$$F(S_T, I_T) = \overline{\sigma} \sqrt{\frac{T}{I_T}} (S_T - K)^+, \quad (2.46)$$

and its Fourier transform in the log-price and quadratic variation is given by:

$$\hat{F}(\omega, \eta) = \overline{\sigma}(1+i) \sqrt{\frac{\pi T}{2\eta}} \frac{K^{1+i\omega}}{(i\omega - \omega^2)}, \quad \text{for } Im(\omega) > 1 \,, Im(\eta) > 0 \ . \quad (2.47)$$

As it happens for vanilla options, a target volatility put will have the same payoff transform as a call, but in the domain we will instead have $\text{Im}(\omega) < 0$.

2.6.2 Double digital call

A derivative delivering at maturity a unit of cash if both the underlying asset and its realized variance at T are above two strike levels K_1 and K_2. In practice we are adding a further strike threshold to a digital call option, so it is intuitively clear that this derivative has to be priced less than it, and yet it must yield the same payoff if the terminal variance is higher than K_2. The claim is defined as:

$$F(S_T, I_T) = \mathbb{1}_{\{S_T \geq K_1,\, I_T/T \geq K_2\}}, \tag{2.48}$$

whereas the transform in log-price and quadratic variation is:

$$\hat{F}(\omega, \eta) = -\frac{K_1^{i\omega} e^{iTK_2\eta}}{\omega\eta}, \quad \text{for } Im(\omega) > 0\,, Im(\eta) > 0. \tag{2.49}$$

Clearly any other put/call combination in the two variables can be imagined.

2.6.3 Volatility capped call option

As in the previous example we can cheapen the price of a European call by adding the further constraint that the payoff is not triggered if the terminal realized volatility is not within an acceptable range. At maturity this product pays off:

$$F(S_T, I_T) = (S_T - K)^+ \mathbb{1}_{\{K_1 \leq \sqrt{I_T/T} \leq K_2\}}, \tag{2.50}$$

and has a log-price and quadratic variation transform given by:

$$\hat{F}(\omega, \eta) = \left(e^{i\eta K_1^2 T} - e^{i\eta K_2^2 T}\right) \frac{K^{1+i\omega}}{(\omega + i\omega^2)\eta}, \quad \text{for } Im(\omega) > 1\,, Im(\eta) > 0. \tag{2.51}$$

A more natural version of this product could be obtained by requiring that to get a positive payoff the volatility should never leave an interval $[K_1, K_2]$ at each given time $t < T$. The resulting derivative is a “volatility version” of a double barrier option; pricing it therefore amounts to solve problem (2.3)-(2.4) with an added boundary condition. This escapes the pricing framework presented; however, such a payoff could be of interest for future research in a context of volatility path-dependent claims.

2.6.4 Volatility-struck call option

This product gives the writer the option to buy an asset at maturity for a notional amount N, times the realized volatility of the underlying. The more the stock is subject to shocks, the less likely is the option to be triggered; hence investors could enter this contract if they are expecting low volatility levels. Just like for a TVO, predictions about the future realized volatility σ are reflected in setting the notional N. As a payoff we have:

$$F(S_T, I_T) = \left(S_T - N\sqrt{\frac{I_T}{T}}\right)^+, \tag{2.52}$$

with transform:

$$\hat{F}(\omega,\eta) = \left(\frac{N}{\sqrt{T}}\right)^{1+i\eta} \Gamma\left(\frac{3+i\omega}{2}\right) \frac{(-i\eta)^{-3/2-i\omega/2}}{i\omega - \omega^2},$$

$$\text{for } 1 < Im(\omega) < 3,\ Im(\eta) > 0. \qquad (2.53)$$

2.7 Numerical testing

Computations for the payoffs introduced in section 2.6 have been carried out in a Heston model[4] with parameters from chapter 1, in order to compare the values obtained there with the new methodology. The underlying process for the variance is given by

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dW_t, \qquad (2.54)$$

with:

$$\kappa = 0.5, \quad \theta = 0.2, \quad \eta = 0.3, \quad v_0 = 0.2. \qquad (2.55)$$

For different state variables and sets of parameters defining the claims, we compare a MATLAB Monte Carlo simulation based on an Euler scheme with exact sampling (Broadie and Kaya, [11]), against a MATHEMATICA implementation of (2.15). For the TVO, figures from the Laplace transform pricing method described in [30] are provided. A comparison of the prices of the products introduced with their vanilla counterparts is also given; it is striking how much cheaper the new claims are. Nevertheless, under favorable volatility scenarios, they produce the same payoffs as their standard versions.

K	Laplace transform	Monte Carlo simulation	PDE pricing	Vanilla call
60	11.3919	11.3897	11.3909	40.0061
80	8.7301	8.7281	8.7299	20.7211
100	6.7416	6.7415	6.7415	6.9013
120	5.2672	5.2618	5.2672	1.4252

Table 2.1: $T = 3$, $t = 0$, $\bar{\sigma} = 0.1$, $S_0 = 100$, $r = d = \rho = 0$. TVO valuation for different strikes. The Black-Scholes price of a call option of constant volatility $\bar{\sigma}$ is given for comparison (see chapter 1, section 1.4).

2.8 Conclusions

In this chapter we have explained possible reasons for the introduction of derivatives written jointly on an underlying asset and its accrued volatility. We have discussed the general problem of pricing European claims depending at maturity on such an asset and the total quadratic variation it exhibits. A pricing PDE has been derived, and a universal model-dependent solution has been found and characterized in terms of the model and the payoff.

Issues on the uniqueness and existence of such a solution have been addressed, and a de-

[4]For a numerical analysis of the jump diffusion in section 2.4 with the Bates [6] specifications, see chapter 3.

ρ	Monte Carlo	PDE pricing	Vanilla call
-0.8	10.3154	10.3975	41.5145
-0.4	9.9415	9.9505	41.3683
0	9.4398	9.4549	41.1688
0.4	8.9645	8.9059	40.8992
0.8	8.3136	8.3025	40.5433

Table 2.2: $T = 5$, $t = 2.5$, $r = 0.08$, $d = 0$, $\bar{\sigma} = 0.1$, $S_t = 100$, $K = 85$, $I_t = 0.46$. TVO prices for various correlations. Here we compare to a European call option in the Heston model having same parameters.

I_t	Monte Carlo	PDE pricing	Vanilla digital call
0.2	0.0951	0.0943	0.5358
0.3	0.2366	0.2426	0.5358
0.4	0.4393	0.4395	0.5358
0.5	0.5335	0.5330	0.5358

Table 2.3: $T = 2.5$, $t = 1$, $r = 0.1$, $d = 0.01$, $K_1 = 100$, $K_2 = 0.24$, $S_t = 120$, $\rho = 0.2$. Double digital call for different realized variance levels. Note how the prices converge to that of a digital call as K_2 becomes more likely to be hit.

K_2	Monte Carlo	PDE pricing	Vanilla call
0.35	7.7812	7.7743	37.2632
0.4	16.3226	16.3006	37.2632
0.45	25.1122	25.0732	37.2632
0.5	31.6069	31.5497	37.2632

Table 2.4: $T = 2, t = d = 0, r = 0.07$, $S_t = 110$, $K = 100$, $K_1 = 0.2$, $\rho = -0.3$. Volatility capped call option prices for different values of K_2. The reference call has same parameters as the volatility capped call. As the gap between K_1 and K_2 widens, the price approaches that of the vanilla call.

T	Monte Carlo	PDE pricing
2	4.8291	4.8810
3	8.9873	8.9383
4	11.8885	11.9086
5	14.1661	14.2002

Table 2.5: $I_t = 0.18$, $t = 1$, $r = 0.05$, $d = 0.02$, $S_t = 50$, $N = 150$, $\rho = -0.5$. Volatility-struck call option prices for different maturities.

tailed mathematical discussion of the domains of holomorphy of the involved functions has been carried out. Furthermore, we have provided an analytical representation for the Greeks, and have given formulae for specific models and payoffs. Pricing results have been extended to the case of an underlying being modeled through a jump diffusion, and hedging strategies for these derivatives have been discussed for both the continuous and discontinuous framework.

Numerical tests support our main result. In addition, figures confirm the intuition that it is possible to conceive volatility modifications of liquid instruments, less expensive than the original product, but paying off the same amount in market situations that an investor may want to exploit.

Appendix: proofs

Proof of proposition 2.1.1. The proof is an adaptation of Heath and Schweizer [58], theorem 1, the existence of a solution being assumed. See also Friedman [45], chapters 5-6.

Let D_n be a family of smooth bounded domains invading $\mathbb{R}^3_+$, and assume $V(x,t)$ is a $C^{2,1}(\mathbb{R}^3_+ \times [0,T))$ solution to (2.3)-(2.4). Let $\tau_n = \{\inf t | X_t \notin \overline{D}_n\}$. By Ito's formula it can be readily seen that if X_t is a solution to (D) then for all $x \in D_n$ and $t \leq T$,

$$V_n(x,t) = \mathbb{E}_{x,t}\left[e^{-r(T-t)}V(X_{T\wedge\tau_n}, T\wedge\tau_n)\right] \tag{2.56}$$

is a solution to the differential problem (2.3) with boundary condition $V_n(x,t) = V(x,t)$, $x \in \partial D_n$, and terminal condition $V(X_{T\wedge\tau_n}, T\wedge\tau_n)$; furthermore, it is the only solution there by the weak maximum principle. By (NB) and (LL) we have that the probabilistic family yielding (2.56) is strongly Markovian, so taking the conditional expectation at time τ_n (2.5) and then using the strong Markov property shows that $V(x,t)$ coincides with V_n on D_n for all n. Hence, V satisfies (2.3)-(2.4) on the whole space. Finally, again (LL) and (NB) imply (see for example Feller [39], Kunita [73]) that v_t, hence X_t, is (weakly) unique and finite almost surely, and this proves the claim. □

Proof of condition (GC) for the payoffs of section 2.6. Let $V(S,I) = \mathbb{E}_{x,t}\left[e^{-r(T-t)}F(S,I)\right]$ where F is a payoff from section 2.6. For a double digital call we have $F(S,I) \leq 1$ so that $V(S,I) \leq 1$ and (GC) is trivially verified. If F is a volatility capped call then for all I we have:

$$V(S,I) \leq \mathbb{E}_{x,t}\left[e^{-r(T-t)}(S_T - K)^+\right] \leq S, \tag{2.57}$$

where the last inequality follows at once from [8], theorem 1, and its generalization on page 1600. Also, by arguing that the *no-crossing* property (lemma, p. 1577) for S_t must also hold for the augmented diffusion (D), we see that theorem 1 can be extended to a payoff $F(S,I)$. In particular if F is given by (2.52) we see that:

$$\frac{\partial V(S,I)}{\partial S} \leq \sup_S \frac{\partial F(S,I)}{\partial S} = 1, \tag{2.58}$$

hence $V(S,I) \leq S, \forall S, I$. Similarly for a TVO we have:

$$\frac{\partial V(S,I)}{\partial S} \leq \overline{\sigma}\sqrt{T/I}, \tag{2.59}$$

which implies $V(S,I) \leq C_1 S$ for some $C_1 > 0$, because V is bounded around $I = 0$. □

Proof of equation (2.20) and solution to (2.31). We make the ansatz:

$$\hat{H}(\omega,\eta,v,\tau) = \exp[C(\omega,\eta,\tau) + vD(\omega,\eta,\tau)] \tag{2.60}$$

and substitute this in (2.13) with parameters from the Heston model. One obtains the decoupled ODEs for $C(\tau)$ and $D(\tau)$:

$$\begin{aligned} C' &= \kappa\theta D \\ D' &= \frac{\epsilon^2}{2}D^2 + D(\kappa + i\epsilon\rho\omega) - \frac{1}{2}(\omega^2 - i\omega + 2i\eta). \end{aligned} \tag{2.61}$$

The Riccati equation for D is solved by switching to an associated linear second order ODE for its logarithmic derivative. C is then found by direct integration.

In the case when $\hat{H}$ instead satisfies a PIDE of the form (2.31) with normally-distributed jumps of intensity λ, we modify the ansatz as:

$$\hat{H}(\omega,\eta,v,\tau) = \exp[C(\omega,\eta,\tau) + vD(\omega,\eta,\tau) + \lambda\tau(\phi_{J,J^2}(\omega,\eta) - 1)]. \tag{2.62}$$

By taking C and D like in (2.61), and letting ϕ_{J,J^2} be the joint characteristic function of J and J^2 we see that (2.62) satisfies (2.31). □

Condition (NB) for the accounted models. For the GARCH model the result is trivial. The most convincing way of checking (NB) for the other models described is using Feller's explosion test (Feller, [40]). In our case, the *scale function* for the variance dynamics in the Heston model is:

$$p(x) = \frac{2}{\epsilon^2}\int_1^x u^{-2\kappa\theta/\epsilon^2} e^{\frac{2\kappa}{\epsilon^2}(u-1)} \left(\int_1^u z^{2\kappa\theta/\epsilon^2-1} e^{-\frac{2\kappa}{\epsilon^2}(z-1)} dz\right) du \tag{2.63}$$

A necessary and sufficient condition for the process to attain 0 or $+\infty$ is that $p(0)$, $p(+\infty) < +\infty$. As $u \to +\infty$ the integrand is exponentially divergent, whereas convergence in 0 is happens if and only if $2\kappa\theta < \epsilon^2$. For the 3/2 model we have instead:

$$p(x) = \frac{2}{\epsilon^2}\int_1^x u^{2\kappa/\epsilon^2} e^{\frac{4\kappa\theta}{\epsilon^2}(1/u-1)} \left(\int_1^u z^{-2\kappa/\epsilon^2-3} e^{-\frac{4\kappa\theta}{\epsilon^2}(1/z-1)} dz\right) du. \tag{2.64}$$

so that this time it is $p(0) = +\infty$ for any choice of parameters, and $p(+\infty) < +\infty$ if and only if $2\kappa < -\epsilon^2$. □

Proof of the hedge by replication. Consider the hedging error

$$\epsilon_t = V_t - \Pi_t. \tag{2.65}$$

Assuming the self-financing property, and applying Ito's lemma, we see that Π_t evolves according to:

$$\begin{aligned} d\epsilon_t = &\left(\frac{\partial V}{\partial t} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S\partial v}\right)dt \\ &- \Gamma_t\left(\frac{\partial G}{\partial t} + \frac{vS^2}{2}\frac{\partial^2 G}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 G}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial G}{\partial S\partial v}\right)dt \\ &+ \left(\frac{\partial V}{\partial S} - \phi_t - \frac{\partial G}{\partial S}\Gamma_t\right)dS_t + \left(\frac{\partial V}{\partial v} - \frac{\partial G}{\partial v}\Gamma_t\right)dv_t - r\psi_t B_t dt. \end{aligned} \tag{2.66}$$

To perfectly hedge the position, we set Γ_t and ϕ_t as in (2.33)-(2.34), canceling the first two terms in the last line; whatever is left as a portfolio value must be borrowed from/invested in the cash position. That is, by further letting ψ_t as in equation (2.35) we have $\epsilon_t = 0$, implying $V_t = \Pi_t$. To verify that $(\phi_t, \psi_t, \Gamma_t)$ is indeed a self-financing portfolio, one just substitutes the ratios back in (2.32), obtaining

$$\begin{aligned} &\phi_t dS_t + \Gamma_t dG_t + \psi_t dB_t = \\ &\frac{1}{rB_t}\left[\left(\frac{\partial V}{\partial t} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S\partial v}\right)\right. \\ &\left.- \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}\left(\frac{\partial G}{\partial t} + \frac{vS^2}{2}\frac{\partial^2 G}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 G}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial G}{\partial S\partial v}\right)\right] rB_t dt \\ &+ \left(\frac{\partial V}{\partial S} - \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}\frac{\partial G}{\partial S}\right)dS_t + \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}dG_t = \\ &dV_t - \frac{\partial V}{\partial v}dv_t + \frac{\partial V}{\partial v}\frac{\partial G}{\partial v}^{-1}\frac{\partial G}{\partial v}dv_t = dV_t. \end{aligned} \tag{2.67}$$

In the case we are hedging with a volatility derivative P, notice that the Ito differential of P_t is given by

$$dP_t = \frac{\partial P}{\partial t}dt + \frac{\beta^2}{2}\frac{\partial^2 P}{\partial v^2}dt + v\frac{\partial P}{\partial I}dt + \frac{\partial P}{\partial v}dv_t. \tag{2.68}$$

Applying Ito's formula to the hedging error ϵ_t then yields:

$$\begin{aligned} d\epsilon_t = dV_t - \Pi_t = &\left(\frac{\partial V}{\partial t} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \rho\beta\sqrt{v}S\frac{\partial V}{\partial S\partial v}\right)dt \\ &- \Lambda_t\left(\frac{\partial P}{\partial t} + \frac{\beta^2}{2}\frac{\partial^2 P}{\partial v^2} + v\frac{\partial P}{\partial I}\right)dt \\ &+ \left(\frac{\partial V}{\partial S} - \phi_t\right)dS_t + \left(\frac{\partial V}{\partial v} - \Lambda_t\frac{\partial P}{\partial v}\right)dv_t - r\psi_t B_t dt. \end{aligned} \tag{2.69}$$

This time $\epsilon_t = 0$ is achieved by letting ϕ_t, Λ_t and ψ_t as in (2.37)-(2.39). The self-financing property is proved as in the previous case. □

Proof of the mean-variance hedging formula. Consider the hedging error $\epsilon(V_0, \phi)$ up to time T, given by

$$\epsilon(V_0, \phi) = V_T - V_0 - \int_0^T \phi_t d\tilde{S}_t. \tag{2.70}$$

In first place, we have that the optimal value for V_0 is $\mathbb{E}[F]$ because $\mathbb{E}[\epsilon(\phi_t)^2]$ can be rewritten as the sum of positive quantities

$$\mathbb{E}\left[\left(F - V_0 - \int_0^T \phi_t d\tilde{S}_t\right)^2\right] = \mathbb{E}[(\mathbb{E}[F] - V_0)^2] + Var\left[F - \int_0^T \phi_t d\tilde{S}_t\right] \tag{2.71}$$

Now applying Ito's lemma for discontinuous processes to $\tilde{V}$ and combining it to the (discounted version of) the pricing relation (2.30) we have:

$$\begin{aligned}\epsilon(V_0,\phi) &= \int_0^T \left(\frac{\partial \tilde{V}}{\partial \tilde{S}}(\tilde{S}_{t_-}, I_{t_-}, v_t, t) - \phi_t\right)\sqrt{v_t}\tilde{S}_{t_-} dW_t^1 \\ &+ \int_0^T \int_{\mathbb{R}} \left(\tilde{V}(\tilde{S}_{t_-}e^x, I_{t_-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t_-}, I_{t_-}, v_t, t) - \phi_t \tilde{S}_{t_-}(e^x - 1)\right)\gamma(dxdt) \\ &- \int_0^T \int_{\mathbb{R}} \left(\tilde{V}(\tilde{S}_{t_-}e^x, I_{t_-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t_-}, I_{t_-}, v_t, t) - \phi_t \tilde{S}_{t_-}(e^x - 1))\right)\nu(dx)dt,\end{aligned} \tag{2.72}$$

where we denoted with $\gamma(dxdt)$ the random jump measure associated with the jump part of the asset dynamics.

In principle, equation (2.72) is the sum of two *local* martingales. Square integrability of the integrands is required to prove that the summand are strict martingales. However, by assumption (2.44) F is Lipschitz, so that V is also Lipschitz; therefore $\partial\tilde{V}/\partial\tilde{S}$ is bounded. Combining this with the assumptions on ϕ_t, we deduce that the integrand in the first line of (2.72) is square integrable. To see that the compensated pure jump martingale in the second and third line are also square integrable, we directly use the Lipschitz property of V yielding:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} (\tilde{V}(\tilde{S}_{t_-}e^x, I_{t_-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t_-}, I_{t_-}, v_t, t))^2 \nu(dx)dt\right] \\ &\leq C_t \int_0^T \mathbb{E}[\tilde{S}_t^2]dt \int_{\mathbb{R}} (e^{2x} - 1)\nu(dx) + T\int_{\mathbb{R}} x^4 \nu(dx) < \infty\end{aligned} \tag{2.73}$$

where the finiteness of the integrals in $\nu(dx)$ follows by assumption (2.26) and the existence of the second moments of the price process is also assumed. By using Ito's isometries and the fact that W_t^1 and N_t are orthogonal, it is possible to compute the second moment of the hedging error:

$$\begin{aligned}\mathbb{E}\left[\epsilon(V_0,\phi_t)^2\right] &= \mathbb{E}\left[\int_0^T \left(\frac{\partial \tilde{V}}{\partial \tilde{S}}(\tilde{S}_{t_-}, I_{t_-}, v_t, t) - \phi_t\right)^2 v_t \tilde{S}_{t_-}^2 dt\right. \\ &+ \left.\int_0^T \int_{\mathbb{R}} (\tilde{V}(\tilde{S}_{t_-}e^x, I_{t_-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t_-}, I_{t_-}, v_t, t) - \phi_t \tilde{S}_{t_-}(e^x - 1))^2 \nu(dx)dt\right].\end{aligned} \tag{2.74}$$

We observe that the expectation above coincides with the variance of $\epsilon(V_0, \phi_t)$, being $\epsilon(V_0, \phi_t)$ a martingale starting at 0. Finally, by differentiating the quadratic expression (2.74) with respect to ϕ_t, passing the derivative inside the expectation, and equating the result to 0, we find that the global minimum for $\mathbb{E}[\epsilon(V_0, \phi_t)^2]$ must satisfy

$$\begin{aligned} 0 =& \left(\frac{\partial \tilde{V}}{\partial \tilde{S}} - \phi_t\right) v_t \tilde{S}_t^2 \\ &+ \int_{\mathbb{R}} (\tilde{V}(\tilde{S}_{t-} e^x, I_{t-} + x^2, v_t, t) - \tilde{V}(\tilde{S}_{t-}, I_{t-}, v_t, t) - \phi_t \tilde{S}_{t-}(e^x - 1))(e^x - 1)\tilde{S}_{t-} \nu(dx), \end{aligned} \tag{2.75}$$

from which (2.45) follows. *A fortiori*, we remark that ϕ_t is indeed càglàd. By reverting back to the spot price strategy π_t, we see that the equation above enforces $\psi_t = \phi_t S_t / B_t$.

Note that by minimizing a functional at the terminal time T, that is, by performing a *global* optimization, we were able to produce a *local* optimal strategy, i.e. a process minimizing the squared error at all times $t \leq T$. This is in general unfeasible when $\tilde{S}_t$ is not a strict martingale: more on this can be learned in [23] and [43]. □

Chapter 3

Valuation of asset and volatility derivatives using decoupled time-changed Lévy processes

The use of Lévy models in finance dates back to to the classic work of Merton [81], who proposed that the log-price dynamics of a stock return should follow an exponential Brownian diffusion punctuated by a Poisson arrival process of normally distributed jumps. In that work, two of the main shortcomings of the Black-Scholes model, the continuity of the sample paths and the normality of returns, were addressed for the first time. Over the years, Lévy processes have proved to be a flexible and yet mathematically tractable instrument for asset price modeling and sampling. One of the easiest ways of producing a Lévy process is to use the principle of *subordination* of a Brownian motion W_t. If T_t is an increasing Lévy process independent of W_t, then the subordinated process W_{T_t} will still be of Lévy type. Subordination is the simplest example of a *time change*, that is, the operation whereby one considers the time evolution of a stochastic process as occurring at a random time.

Return models depending on time-changed Brownian motions have been conjectured since Clark [21]; further theoretical support to the financial use of time-changed models is given by Monroe's theorem [83], asserting that any semimartingale can be viewed as a time change of a Brownian motion. Consequently, any semimartingale representing the log-price process of an asset can be considered as a re-scaled Wiener process. Empirical studies (Ane and Geman [4]) confirmed that normality of returns can be recovered in a new price density based on the quantity and arrival times of orders, which justifies the interpretation of T_t as "*business time*" or "*stochastic clock*"; the instantaneous variation of T_t is hence the "*activity rate*" at which the market reacts to the arrival of information. Further advances were made by Carr and Wu [15], who demonstrated that much more general time changes are potential candidates for asset price modeling, and effectively recovered many models from the standard literature by using a time-changed representation.

However, not all the possibilities in time change modeling are exhausted by the Carr and Wu framework. For example, the stochastic volatility model with jumps (SVJ) treated among the others by Bates [6], and the stochastic volatility model with jumps and stochastic jump rate (SVJSJ) studied by Fang [37], although retaining a time re-scaled structure, are not time-changed Lévy processes as they are understood in Carr and Wu [15]. Indeed, in these two classes of models the jump component does not follow the same time scaling as the continuous Brownian part: in the SVJ model the discontinuities have stationary increments, whereas in the SVJSJ model the jump rate is allowed to follow a stochastic process of its own. In other words, price models for which the "stochastic clock" runs at different paces for the "small" and "big" market movements have already been proposed and tested. The statistical analyses of Bates [6] and Fang [37] confirm that these models are capable of an excellent data fitting. As pointed out by Fang, there are various other reasons for conjecturing a stochastic jump rate. If activity rates are to be interpreted as the frequencies of arrival of new market information, it seems unlikely that such rates could be taken as constant, as this would imply a constant information flow. Moreover, a constant jump rate implies stationary jump risk premia, which also seems unreasonable. Another stylized fact potentially captured by a model with a stochastic jump rate is the slow convergence of returns to the normal distribution, which is not a feature of stationary jump models. Despite all these considerations, the idea of a stochastic jump rate has never really caught on.

On the other hand, if we want to exogenously model the market activity, the assumption of independence between the jump and the continuous instantaneous rates seems to be overly simplistic, as in reality the two corresponding information flows may very well influence each other. For example, a market crash or soaring certainly impacts the day-to-day volume of trading in the days following such an event. Conversely, a sustained high activity trend over a long period, typically associated to falling prices, may eventually lead to a sudden, panic-driven plunge in the shares' value. These and similar scenarios provide heuristic arguments for the assumption of a *correlated* pair of activity rates.

Motivated by these arguments, the natural question that arises is whether it is possible to manufacture consistent general time-changed martingale price processes in which the continuous and discontinuous parts of the underlying Lévy model follow two *different*, possibly correlated, stochastic time changes. We show in this chapter that the answer is affirmative. The family of stochastic processes we investigate is that obtained by time-modifying the continuous and jump parts of a given Lévy process X_t by two, in principle dependent, stochastic time scalings T_t and U_t satisfying a certain regularity condition (definition 3.2.1). We call such processes *decoupled time changes*. In a formula:

$$X_{T,U} := X^c_{T_t} + X^d_{U_t}, \tag{3.1}$$

where X^c_t and X^d_t represent respectively the Brownian and jump components of X_t.

The decoupled time-changed (DTC) approach allows to embed in a unifying mathematical framework many previously-known models or classes of models, so that the presented theory offers a natural generalization of some of the extant asset modeling research. In addition, the assumption of a pair of dependent activity rates can be captured by making use of decoupled time changes. In section 3.6 we illustrate an actual example of a model having this property by considering an explicit asset evolution based on a multivariate version of the square-root process known as the *Wishart process* (e.g. Bru [12]; Gourieroux [51]; da Fonseca *et al.* [28]), which we use to model the instantaneous activity rates.

Price processes based on DTC Lévy processes have been treated in the empirical analysis by Huang and Wu [63], who successfully calibrated several different Lévy and time-change specifications within the DTC framework. The authors focus on parameter estimation and goodness-of-fit rather than analyzing the jump/diffusive volatility dependence relations. Thus, their work does not provide any theoretical justification for the martingale property of the general asset price equation used. Furthermore, when coming down to the actual specifications, they effectively did not introduce a new analytically tractable model with true dependence between the activity rates. This is not unexpected, and it is an intrinsic issue that occurs when choosing the activity rates as belonging to the family of the affine jump diffusions (see section 3.5.4). Ultimately, as widely illustrated by Grasselli and Tebaldi [55], in such a framework it does not seem to be possible to explicitly derive the Laplace transform of the model when the instantaneous activity rate SDEs are correlated. In contrast, the model of section 3.6, which is *not* a jump diffusion in the sense of the classical Duffie *et al.* [31] framework, allows for a true dependence between the jump and the continuous activity rate, while at the same time retaining analytical tractability. Further to this, the extant literature on time changes in finance (e.g. [15, 63]), also seems to ignore a number of pitfalls that arise in time-changed modeling, for example:

- a time-changed (exponential) martingale is *not*, in general, a (an exponential) martingale (see the counterexample in the appendix);
- a time-changed local martingale is *not*, in general, a local martingale;
- a time-changed Markov process is *not*, in general, a Markov process (see footnote 3).

In setting up our framework, we prevent these issues by imposing sufficient conditions under which the good properties required for financial modeling are maintained when time changing, i.e. when considering a random time evolution of the underlying Lévy drivers. In this respect, the overall message of this chapter is that time-changed asset price processes are best understood by appealing to the theory of semimartingale representation (Jacod and Shiryaev, [67]).

In the spirit of the discussion carried out in the previous chapters, we use the DTC asset representation for the purpose of pricing financial derivatives depending jointly on the terminal price of the asset and its historical volatility. The pricing technique we use is the extension of equation (2.15) when the underlying is not given by a diffusion process but is rather based on a general exponential DTC evolution. Specific instances will be also analyzed; it should be apparent that in all the models we investigate there is no particular reason not to consider mixed price and volatility payoffs as the default input of pricing models e.g. for numerical implementation. This is because introducing the realized volatility in the characteristic function of the model does not cause the Fourier-inversion technique to break down. Clearly, pricing both vanilla and pure volatility derivatives is still possible within this framework, since the corresponding payoffs can be regarded as particular cases of our more general setting.

The remainder of the chapter is organized as follows. In section 3.1 we lay out the assumptions; in section 3.2 we derive martingale properties for a decoupled time-changed Lévy model. Section 3.3 shows the fundamental relation linking the characteristic function of the log-price and its quadratic variation and the joint Laplace transform of the time changes as computed in an appropriate measure. Section 3.4 is dedicated to the derivation of a pricing formula for products whose payoffs depend jointly on S_t and TV_t, including forward-starting ones. We devote section 3.5 to characterizing the DTC structure of a number of known models, and computing the joint characteristic function discussed in section 3.3 for each of them. In section 3.6 we introduce an exemplifying model of DTC type featuring correlation between the time changes/activity rates. Finally, in section 3.7 we implement our formulae to valuate different asset and volatility derivatives under various market conditions and asset price models, and briefly summarize our work. The more technical proofs have been placed in the appendix.

3.1 Assumptions and notation

As customary, our market is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions. Throughout the chapter we will assume that there exists a money market account process paying a riskless constant interest rate r.

Let S_t be a non dividend-paying market asset. $\tilde{S}_t$ will denote its time-zero discounted value $e^{-rt}S_t$. The *total realized variance* on $[0, t]$ of S_t is by definition the *quadratic variation* of the natural logarithm of S_t, that is:

$$TV_t := \langle \log S \rangle_t = \lim_{|\pi| \to 0} \sum_{t_i \in \pi} |\log S_{t_{i+1}} - \log S_{t_i}|^2. \tag{3.2}$$

The limit runs over the supremum norm of all the possible partitions π of $[t_0, t]$. The *total realized volatility* is $\sqrt{TV_t}$. The *period realized variance* and *volatility* (or realized variance/volatility *tout court*) are given respectively by $RV_t = TV_t/t$ and $\sqrt{RV_t}$. If $X_t = \log S_t$ is a semimartingale, by taking the limit in (3.2) it is easy to check that:

$$\langle X \rangle_t = X_t^2 - 2 \int_0^t X_{u^-} dX_u. \tag{3.3}$$

The algebra of the square matrices of order n with real-valued entries is indicated by $\mathcal{M}_n(\mathbb{R})$ and that of the symmetric real matrices by $Sim_n(\mathbb{R})$. Matrix product is denoted by juxtaposition; the scalar product between vectors is either indicated by multiplying on the left with the transposed vector $\cdot^T$ or by the usual dot notation. The symbol Tr stands for the trace operator.

If J is an absolutely continuous random variable, we denote by $f_J(x)$ its probability density function and by $\phi_J(z)$ its *characteristic function*

$$\phi_J(z) := \mathbb{E}[e^{iz^T J}]. \tag{3.4}$$

For a Fourier-integrable function $f : \mathbb{C}^n \to \mathbb{C}$ its Fourier transform will be denoted $\hat{f}$. For a complex-valued function or a complex plane subset, $\cdot^*$ indicates the complex conjugate function or set.

When we say that a process is a martingale we mean a martingale with respect to its natural filtration. The notation for the conditional expectation of a stochastic process X_t at time $t_0 < t$ with respect to $\mathcal{F}_{t_0}$ is $\mathbb{E}_{t_0}[\,\cdot\,]$. When the distribution of a process X_t depends on other state variables x_t (as in the case of a Markov process) the latter are implicitly understood to be given at time t_0 by x_{t_0}. If X_t is a process admitting conditional laws, the space of the integrable functions in the t_0-conditional distribution of X_t at time $t_0 < t$ is indicated by $L^1_{t_0}(X_t)$. The notation for the bilateral Laplace transform of the distribution of X_t conditional on $t_0 < t$ is:

$$\mathcal{L}_X(z) = \mathbb{E}_{t_0}[e^{-z^T X_t}] \tag{3.5}$$

where for brevity we drop the dependence on t and t_0 on the left hand side. The stochastic process of the left limits of X_t is indicated X_{t-}. The symbol ΔX_t stands for the difference $X_t - X_{t-}$ or $X_t - X_{t_0}$ for some prior time $t_0 < t$. Equalities are always understood to hold modulo almost sure equivalence.

If X_t is an n-dimensional Lévy process, the *characteristic exponent* of X_t is the complex-valued function $\psi_X : \mathbb{C}^n \to \mathbb{C}$ such that:

$$\mathbb{E}[e^{i\theta^T X_t}] = e^{t\psi_X(\theta)} \tag{3.6}$$

where θ lies in the subset of $\mathbb{C}^n$ and where the left-hand side is finite.

For a given choice of *truncation function* $\epsilon(x)$ (that is, a bounded function which is O$(|x|)$ around 0) the characteristic exponent has the unique *Lévy-Khintchine* representation:

$$\psi_X(\theta) = i\mu_\epsilon^T \theta - \frac{\theta^T \Sigma \theta}{2} + \int_{\mathbb{R}^n} (e^{i\theta^T x} - 1 - i\theta^T \epsilon(x))\nu(dx), \tag{3.7}$$

where $\mu_\epsilon \in \mathbb{R}^n$, Σ is a non-negative definite $n \times n$ matrix with real entries, and $\nu(dx)$ is a Radon measure on $\mathbb{R}^n$ having a density function that is integrable at $+\infty$ and O$(|x|^2)$ around 0. We shall make the standard choice $\epsilon(x) = x\mathbb{I}_{|x|\leq 1}$ and drop the dependence of μ on ϵ. The triplet (μ, Σ, ν) is then called the *characteristic triplet* or the *Lévy characteristics* of X_t.

A *stochastic time change* T_t is an $\mathcal{F}_t$-adapted càdlàg stochastic process, increasing and almost surely finite, such that T_t is an $\mathcal{F}_t$-adapted stopping time for each t. The *time change* of an n-dimensional Lévy process X_t according to T_t is the $\mathcal{F}_{T_t}$-adapted process $Y_t := X_{T_t}$.

3.2 Definition, martingale relations and asset price dynamics

In this first section we introduce the notion of DTC Lévy process and devise an exponential martingale structure naturally associated to it. This construct serves a twofold purpose. In first place it allows to formulate a DTC-based asset price evolution whose discounted value enjoys the martingale property. According to the general theory, this in turn enables to postulate the existence of a risk-neutral measure that correctly prices the market securities. Secondly, it defines a class of complex-valued martingales pivotal for the calculations in the next section.

Let $\mathcal{B}$ be the space of the n-dimensional $\mathcal{F}_t$-supported Brownian motions with drift starting at 0, and $\mathcal{J}$ be the space of the $\mathcal{F}_t$-supported *pure jump Lévy processes* starting at 0, that is, the class of the càdlàg $\mathcal{F}_t$-adapted processes with stationary and independent increments orthogonal[1] to all the elements of $\mathcal{B}$.

Every Lévy process X_t such that $X_0 = 0$ can be decomposed as the orthogonal sum

$$X_t = X_t^c + X_t^d, \tag{3.8}$$

[1]Two processes X_t and Y_t are said to be *orthogonal* if $\langle X, Y \rangle_t = 0$ for all t.

with $X_t^c \in \mathcal{B}$ and $X_t^d \in \mathcal{J}$. We shall refer to X_t^c and X_t^d respectively as the *continuous* and *discontinuous* parts of X_t.

Time changes are fairly general mathematical objects, so we have to introduce some additional requirements in order for our discussion to proceed. One property we shall assume throughout is the so-called *continuity with respect to the time change*.

Definition 3.2.1. Let T_t be a time change on a filtration $\mathcal{F}_t$. An $\mathcal{F}_t$-adapted process X_t is said to be T_t*-continuous*[2] if it is almost-surely constant on all the sets $[T_{t-}, T_t]$.

Obviously, a sufficient condition for T_t-continuity is the almost sure continuity of T_t. Hence, of particular relevance is the class of the *absolutely continuous* time changes, with respect to which every stochastic process is continuous. Given a pair of *instantaneous rate of activity* processes, that is, two exogenously-given càdlàg positive stochastic processes (v_t, u_t), valid time changes are given by the pathwise integrals:

$$T_t = \int_0^t v_{s-} ds, \tag{3.9}$$

$$U_t = \int_0^t u_{s-} ds. \tag{3.10}$$

The processes v_t and u_t describe the instantaneous impact of market trading and information arrival on the price, and formalize the concept of "business activity" over time.

A decoupled time change of a Lévy process is the sum of the (ordinary) time changes of its continuous and discontinuous part.

Definition 3.2.2. Let X_t be an n-dimensional Lévy process and T_t, U_t two time changes such that T_t is almost surely continuous and X_t^d is U_t-continuous. Then:

$$X_{T,U} = X_{T_t}^c + X_{U_t}^d \tag{3.11}$$

is the *decoupled time change* of X_t according to T_t and U_t.

By Jacod [65], corollaire 10.12, a first important property of $X_{T,U}$ is that it is an $\mathcal{F}_{T_t \wedge U_t}$ semimartingale. To avoid degenerate cases, in all that follows we always assume T_t and U_t to be such that $X_{T_t}^c$ and $X_{U_t}^d$ are Markov processes[3].

We now define the class of exponential martingales canonically associated with $X_{T,U}$ when the time changes are absolutely continuous. The following proposition represents the main theoretical tool of this chapter:

Proposition 3.2.3. *Let X_t^1 be an n-dimensional Brownian motion with drift and X_t^2 a pure jump Lévy process in $\mathbb{R}^n$. Let T_t^1 and T_t^2 be two absolutely continuous time changes and set*

[2] Jacod [65] uses T_t*-adapted*, and T_t*-synchronized* is sometimes found; however, T_t-continuous is also common in the literature, and in our view less ambiguous.

[3] In general, time changes of Markov processes are not Markovian; by using Dambis, Dubins and Schwarz's theorem (Karatzas and Shreve [71], theorem 4.6) one can manufacture a large class of counterexamples by starting from any continuous martingale that is not a Markov process.

$X_t = X_t^1 + X_t^2$ and $T_t = (T_t^1, T_t^2)$; define $X_{T_t} := X^1_{T_t^1} + X^2_{T_t^2}$ and denote by $\Theta \subseteq \mathbb{C}^n$ the domain of definition of $\mathbb{E}[\exp(i\theta^T X_t^2)]$. The process:

$$M_t(\theta, X_t, T_t) = \exp\left(i\theta^T X_{T_t} - T_t^1 \psi_{X^1}(\theta) - T_t^2 \psi_{X^2}(\theta)\right) \tag{3.12}$$

is a local martingale, and it is a martingale if and only if $\theta \in \Theta_0$, where:

$$\Theta_0 = \{\theta \in \Theta \text{ such that } \mathbb{E}[M_t(\theta, X_t, T_t)] = 1, \ \forall t \geq 0\}. \tag{3.13}$$

A first important remark is that the classic subordinated Lévy models of the form W_{T_t} *cannot* be recovered from this framework, because a Brownian motion does not enjoy the continuity property with respect to a Lévy subordinator.

Also, notice that when $T_t^1 = T_t^2$, the exponential M_t reduces to an ordinary time change of the type discussed by Carr an Wu [15]. Even in this simple case proposition 3.2.3 is not a consequence of the application of Doob's optional sampling theorem (unlike stated in [15], lemma 1) to the martingale $Z_t(\theta) = \exp(i\theta^T X_t - t\psi_X(\theta))$, because $Z_t(\theta)$ is not necessarily uniformly integrable. Indeed, time-transforming a process always preserves the semimartingale property, but the martingale property after a time change is only guaranteed to be maintained for uniformly integrable martingales. An actual example of an asset model of the form Z_{T_t} that is a strict supermartingale is given by Sin [94]. Other examples can be found in [72], subsection 3.8. This demonstrates that some choices of time changes are inherently unsuitable for time-changed asset price modeling. In the case of X_{T_t} being a one-dimensional Brownian integral, sufficient requirements for (3.13) to be satisfied are the well-known *Novikov* and *Kazamaki* conditions (Karatzas and Shreve [71], chapter 3). The set Θ_0 is sometimes called the *natural parameter set*.

Having obtained martingale relations for a stochastic exponential involving $X_{T,U}$, the risk-neutral dynamics for a DTC Lévy-driven asset are defined in the usual fashion. We have the following immediate corollary to proposition 3.2.3:

Corollary 3.2.4. *Let X_t be a scalar Lévy process of characteristic triplet (μ, σ^2, ν) and (T_t, U_t) a pair of absolutely continuous time changes. For a spot price value S_0 let, for $t > 0$:*

$$S_t = S_0 \exp(rt + i\theta_0 X_{T,U} - T_t\psi_X^c(\theta_0) - U_t\psi_X^d(\theta_0)) = S_0 e^{rt} M_t(\theta_0, X_t^c + X_t^d, (T_t, U_t)) \tag{3.14}$$

with $\theta_0 \in \Theta_0$ being such that (3.14) is a real number. The discounted process $\tilde{S}_t$ is a martingale, and therefore S_t is a price process consistent with the no-arbitrage condition.

The stochastic process in (3.14) is the fundamental asset model we shall use throughout the rest of the chapter.

3.3 Characteristic functions and the leverage-neutral measure

Characteristic functions of state variables are the essential component of the Fourier-inverse pricing methodology, because state price densities are analytically available only for a small number of models; in contrast, characteristic functions are computable in closed form in many instances (e.g. exponential Lévy models, Ito diffusions). This effectively means that in order to compute expectations (prices), the standard approach is not to integrate a payoff against a density function, but rather the payoff's Fourier transform against the characteristic functions of the price transition densities. Famous examples include the FFT paper [17] by Carr and Madan, Lewis's book [75] and subsequent paper [76].

The transform we are interested in is one associated with the price process (3.14). Compared to the usual inverse Fourier/Laplace framework the characteristic function we shall consider is not that of the discounted log-price alone, but one that incorporates also the quadratic variation of the log-process. Indeed, just as the characteristic function of the log-price allows for the derivation of pricing formulae for contingent claims $F(S_t)$, the joint characteristic function of $\log \tilde{S}_t$ and TV_t permits the valuation of payoffs of the form $F(S_t, TV_t)$. This fact has been envisaged before by Carr and Sun [18].

In the present section we compute this transform. There are normally two ways of computing characteristic functions/Laplace transforms of log-price densities. One is the analytical approach, which is popular for example in affine models, when the problem is ultimately reduced to solving a certain system of ODEs. The other is the probabilistic approach, in which the characteristic function of the log-price is linked with the Laplace transform of the integrated driving factors (where available) and then a change of measure is performed to keep track of correlations. As Carr and Wu [15] show this technique is intimately connected with time-changed asset modeling; in what follows we extend it to the case of the underlying being modeled through a full DTC Lévy process.

First of all we must verify that the quadratic variation operator respects the additivity and time-changed structure of $X_{T,U}$. We have the following "linearity/commutativity property", of independent interest:

Proposition 3.3.1. *A DTC Lévy process $X_{T,U}$ is such that $X^c_{T_t}$ and $X^d_{U_t}$ are orthogonal. Furthermore, its quadratic variation satisfies:*

$$\langle X_{T,U}\rangle_t = \langle X^c\rangle_{T_t} + \langle X^d\rangle_{U_t} = \Sigma T_t + \langle X^d\rangle_{U_t}. \tag{3.15}$$

That is, the quadratic variation of $X_{T,U}$ is the sum of the time changes of the quadratic variations of its continuous and discontinuous part.

Crucially, the processes $X^c_{T_t}$ and $X^d_{U_t}$ are orthogonal but not independent. Without the T_t and U_t-continuity assumption, this proposition would be false: a counterexample is provided in the appendix. Proposition 3.3.1 ensures that, in presence of time continuity of the Lévy continuous and jump parts with respect to the corresponding time changes, the quadratic variation of a DTC Lévy process is itself of DTC-type.

Now, for S_t as in (3.14) let us define:

$$\Phi_{t_0}(z,w) = \mathbb{E}_{t_0}[\exp(iz\log(\tilde{S}_t/S_{t_0}) + iw\,(TV_t - TV_{t_0}))]. \tag{3.16}$$

For each z, w for which the right hand side is finite, $\Phi_{t_0}(z,w)$ is the Fourier transform is the joint transition function from time t_0 to time t of $\log \tilde{S}_t$ and TV_t. The characteristic function

$\Phi_{t_0}(z,w)$ can be completely characterized in terms of the Lévy triplet of $X_t = X_t^c + X_t^d$ and the joint $\mathbb{Q}(z,w)$-distribution of T_t and U_t by virtue of the following proposition.

Proposition 3.3.2. *Let S_t be an asset evolution as in corollary 3.2.4, and define the family of $\mathbb{P}$ absolutely-continuous measures $\mathbb{Q}(z,w) << \mathbb{P}$ having Radon-Nikodym derivative:*

$$\frac{d\mathbb{Q}(z,w)}{d\mathbb{P}} = M_t((iz\theta_0, iw\theta_0), C_t + D_t, (T_t, U_t)) \tag{3.17}$$

where $C_t = (X_t^c, 0)$, $D_t = (X_t^d, i\theta_0 \langle X \rangle_t^d)$ and M_t is given by (3.12). For all (z,w) such that $(iz\theta_0, iw\theta_0) \in \Theta_0$, the characteristic function in (3.16) is given by:

$$\Phi_{t_0}(z,w) = \mathcal{L}^{\mathbb{Q}}_{\Delta T,\Delta U}(\zeta(z,w,\mu,\sigma,\theta_0), \xi(z,w,\nu,\theta_0)), \tag{3.18}$$

with the notation $\mathcal{L}^{\mathbb{Q}}_{\Delta T,\Delta U}(\cdot)$ indicating the bilateral Laplace transform of the conditional joint distribution of $T_t - T_{t_0}$ and $U_t - U_{t_0}$ taken under the measure $\mathbb{Q}(z,w)$, and

$$\zeta(z,w,\mu,\sigma,\theta_0) = \theta_0\mu(z - iz) - \theta_0^2\sigma^2(z^2 + iz - 2iw)/2, \tag{3.19}$$

$$\xi(z,w,\nu,\theta_0) = iz\psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0). \tag{3.20}$$

Notice that unlike the density processes used for standard numéraire changes, the new distributions implied by (3.17) also accounts for the quadratic variation as a factor. If we assume T_t and U_t to be pathwise integrals of the form (3.9) and (3.10), it is possible to interpret the Laplace transform (3.18) as being the analogue of a bivariate bond pricing formula, where the short rates are replaced by the instantaneous activity rates, and the pricing measure is not given once and for all, but varies as an effect of the correlation of (v_t, u_t) with the underlying Lévy process. The financial insight of (3.18) is that it is possible to formulate a valuation theory by just modeling the joint term structure of the activity rates v_t and u_t and their correlation with the stock.

Also of interest is the interpretation of the measure $\mathbb{Q}(z,w)$. Let us consider the special case of X_t being independent of T_t and U_t. In such a case it is straightforward to prove, by using the laws of the conditional expectation, that one obtains (3.18) with $\mathbb{Q}(z,w) = \mathbb{P}$. Therefore, whenever there is no dependence between the time changes and the underlying Lévy process, no change of measure is needed in order to extract the characteristic function $\Phi_{t_0}(z,w)$. In contrast, in presence of correlation between X_t and the time changes, the family $\mathbb{Q}(z,w)$ gives a measurement of the impact of leverage on the price densities. Furthermore, in some well-behaved cases this change of measure can be absorbed in the $\mathbb{P}$-dynamics of the asset through a suitable parameter alteration of the distributions of T_t and U_t. In accordance with Carr and Wu [15], we call $\mathbb{Q}(z,w)$ the *leverage-neutral measure* and $\Phi_{t_0}(z,w)$ the *leverage-neutral characteristic function*. Just as prices in a risky market can be equivalently computed in a risk-neutral environment according to a different price distribution, valuations in the presence of leverage can be performed in a different economy with no leverage by means of an appropriate distributional modification.

3.4 Pricing and price sensitivities

The characteristic function found in section 3.3 is needed to obtain analytical formulae for the valuation of European-type derivatives with a sufficiently regular payoff F. In the present section we find a semi-analytical formula based on an inversion integral that extends the standard Fourier-inversion machinery to our multivariate context.

Recall that since all the involved processes are Markovian, it makes sense to treat $\Phi_{t_0}(z,w)$ like a Gauss-Green integral kernel depending only on some given initial states at time t_0. The following proposition extends both theorem 1 of Lewis [75] and proposition 2.2.1.

Proposition 3.4.1. *Let $Y_t = \log S_t$, with S_t given in corollary 3.2.4. Let $F(x,y) \in L^1_{t_0}(Y_t, \langle Y\rangle_t)$ for all $t_0 < t$, be a positive payoff function having analytical Fourier transform $\hat{F}(z,w)$ in a multi-strip*

$$\Sigma_F = \{(z,w) \in \Theta,\ \alpha_1 < \ Im(z) < \alpha_2,\ \beta_1 < \ Im(w) < \beta_2,\ \alpha_1,\alpha_2,\beta_1,\beta_2 \in \overline{\mathbb{R}}\}. \tag{3.21}$$

Suppose further that $\Phi_{t_0}(z,w)$ is analytical in

$$\Sigma_\Phi = \{(z,w) \in \Theta,\ \gamma_1 < \ Im(z) < \gamma_2,\ \eta_1 < \ Im(w) < \eta_2,\ \gamma_1,\gamma_2,\eta_1,\eta_2 \in \overline{\mathbb{R}}\} \tag{3.22}$$

and that $\Phi_{t_0}(z,w) \in L^1(dz \times dw)$. If $\Sigma_F \cap \Sigma^*_\Phi \neq \emptyset$, then for every multi-line:

$$L_{k_1,k_2} = \{(x+ik_1, y+ik_2), (x,y) \in \mathbb{R}^2\} \subset \Sigma_F \cap \Sigma^*_\Phi \tag{3.23}$$

we have that the time-t_0 value of the contingent claim F maturing at time t is given by:

$$\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t,\langle Y\rangle_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2}\cdot \int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty} e^{-iw\langle Y\rangle_{t_0}} S_{t_0}^{-iz} e^{-r(t-t_0)iz}\Phi_{t_0}(-z,-w)\hat{F}(z,w)dzdw. \tag{3.24}$$

It is clear that modifying the asset dynamics specifications only acts on Φ_{t_0}, whereas changing the claim to be priced only influences $\hat{F}$. Also, by setting either variable to 0, we are able to extract from (3.24) the prices of both plain vanilla and pure volatility derivatives. For example, the pricing integrals by Lewis [75, 76] are special cases of the above equation when F does not depend on the realized volatility and Φ_{t_0} is either obtained from a diffusion or a Lévy process. Moreover, equation (2.15) is recovered when S_t is assumed to follow a stochastic volatility model.

In addition, this representation is useful if we are interested in the sensitivities of the claim value with respect to the underlying state variables. Let us consider for instance the Delta (sensitivity with respect to the change in the value of the underlying) and Gamma (sensitivity with respect to the rate of change in the value of the underlying) of valuations performed through formula (3.24). Call $I(r,t_0,z,w)$ the integrand on the right hand side of (3.24); by differentiating (if possible) under the integral sign and noting that Φ_{t_0} has no dependence on S_{t_0} we see that:

$$\Delta_t := \frac{\partial}{\partial S}\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t, \langle Y\rangle_t)] = -\frac{e^{-r(t-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}\frac{iz}{S_{t_0}}I(r,t_0,z,w)dzdw, \tag{3.25}$$

and

$$\Gamma_t := \frac{\partial^2}{\partial S^2}\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t, \langle Y\rangle_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}\frac{iz-z^2}{S_{t_0}^2}I(r,t_0,z,w)dzdw. \tag{3.26}$$

Mutatis mutandis we can repeat this argument if we want to determine the price sensitivity with respect to the quadratic variation $\langle Y\rangle_t$. Finally, as $\Phi_{t_0}(z,w)$ could also depend on other variables (e.g. an instantaneous rate of activity v_{t_0}) known at time t_0, by calling ν one such variable we have:

$$\mathcal{V}_t := \frac{\partial}{\partial \nu}\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t, \langle Y\rangle_t)] = \frac{e^{-r(t-t_0)}}{4\pi^2}\cdot \int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty} e^{-iw\langle Y\rangle_{t_0}}S_{t_0}^{-iz}e^{r(t-t_0)iz}\frac{\partial\Phi_{t_0}}{\partial\nu}(-z,-w)\hat{F}(z,w)dzdw. \tag{3.27}$$

This is especially well-suited to the case in which $\Phi_{t_0}(z,w)$ is exponentially-affine in ν, i.e.

$$\Phi_{t_0}(z,w) = \exp(A(z,w,t-t_0) + B(z,w,t-t_0)\nu_{t_0}), \tag{3.28}$$

for some functions A and B, when we have:

$$\frac{\partial\Phi_{t_0}}{\partial\nu}(-z,-w) = B(-z,-w,t-t_0)\Phi_{t_0}(-z,-w). \tag{3.29}$$

In section 3.5 we shall explicitly calculate Φ_{t_0} for a number of decoupled time-changed models.

3.4.1 Forward-starting and discretely-sampled payoffs

A forward-starting derivative is a contingent claim where the accrual of the underlying factors to be accounted for when receiving the payoff is set at a later date than initiation. If we assume, as usual, that the contingent claim F to be priced depends on the joint performance of S_t and TV_t, then for a maturity date T, times $t_0 < t^* < T$, and $Y_t = \log S_t$,

$$F_{t^*,T}(S_t, \langle X\rangle_t) = F(Y_T - Y_{t^*}, \langle Y\rangle_{T-t^*} - \langle Y\rangle_{t^*}) \tag{3.30}$$

is a forward-starting joint derivative activated at t^* written on S_t and TV_t. According to the general theory, the value function V_{t_0,t^*} of $F_{t^*,T}(Y_t, \langle Y\rangle_t)$ is:

$$V_{t_0,t^*} = \mathbb{E}_{t_0}\left[e^{-r(T-t_0)}F_{t^*,T}(Y_T, \langle Y\rangle_T)\right]. \tag{3.31}$$

In order to compute V_{t_0,t^*} first of all we define the *joint leverage-neutral forward-starting* *characteristic function* as:

$$\Phi_{t_0,t^*}(z,w) = \mathbb{E}_{t_0}[\exp(iz\log(\tilde{S}_T/\tilde{S}_{t^*}) + iw\,(TV_T - TV_{t^*}))]. \tag{3.32}$$

The interpretation of this characteristic function is the same as that in section 3.3. We can calculate this function as:

$$\begin{aligned}\Phi_{t_0,t^*}(z,w) =& \mathbb{E}_{t_0}[e^{iz\log(\tilde{S}_T/\tilde{S}_{t^*})+iw\,(TV_T-TV_{t^*})}]\\ =& \mathbb{E}_{t_0}[\mathbb{E}_{t^*}[e^{iz\log(\tilde{S}_T/\tilde{S}_{t^*})+iw\,(TV_T-TV_{t^*})}]]\\ =& \mathbb{E}_{t_0}[\Phi_{t^*}(z,w)].\end{aligned} \tag{3.33}$$

In the case where $\Phi_{t^*}(z,w)$ is exponentially-affine in some state variable ν_t we can further write:

$$\begin{aligned}\Phi_{t_0,t^*}(z,w) =& \mathbb{E}_{t_0}[\exp(A(z,w,t-t^*) + B(z,w,t-t^*)\nu_{t^*})]\\ =& \exp(A(z,w,t-t^*))\mathbb{E}_{t_0}[\exp(B(z,w,t-t^*)\nu_{t^*})].\end{aligned} \tag{3.34}$$

The characteristic function of ν_t in the last equality is available in closed form in some cases, and for popular models is itself exponentially-affine reflecting the activity rates.

By means of Φ_{t_0,t^*} we obtain the forward-starting equivalent of the pricing equation (3.24) as follows:

$$V_{t_0,t^*} = \frac{e^{-r(T-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty} e^{-izr(T-t_0)}\Phi_{t_0,t^*}(-z,-w)\hat{F}(z,w)dzdw \tag{3.35}$$

where the notation follows that of proposition 3.4.1. Unlike formula (3.24), the equation above bears no dependence on the spot price and the realized variance state variables.

Exotic derivatives that can be thought of as a portfolio of n forward-starting claims are rather popular among the investors. Given a partition $\Pi = \{t_0, t_1, \ldots, t_n = T\}$ of an investment horizon $[t_0, T]$, let us consider n forward claims set at t_{k-1} and expiring at t_k, that is:

$$F_\Pi(Y_t, \langle Y\rangle_t) = \sum_{k=1}^{n} a_k F_{t_{k-1},t_k}(Y_t, \langle Y\rangle_t). \tag{3.36}$$

We can compute prices for these payoffs by applying (3.35) and the linearity of the integral:

$$V_\Pi := \mathbb{E}_{t_0}[e^{-r(T-t_0)}F_\Pi(Y_T, \langle Y\rangle_T)] = \sum_{k=1}^{n} a_k V_{t_{k-1},t_k}. \tag{3.37}$$

To give some examples, a derivative of this kind is the so called *cliquet option*. A cliquet option consists of a sequence of forward-starting call options on the return of the asset between t_{k-1} and t_k, capped at $C > 0$ and floored at 0, i.e. in (3.30)

$$F(x,y) = N\max(0, \min(C, e^x - 1)) \tag{3.38}$$

where N is a notional amount. Also, a common alternative to the use of the total variance TV_t to model a volatility derivative is to discretely sample the variance in terms of the asset's squared log-returns. Indeed, this is the actual form of the volatility derivatives traded in the market. For example, from formula (3.36) one may recover a *discretely-sampled variance swap* by setting in (3.30)

$$F(x, y) = x^2/n. \tag{3.39}$$

Other discretely-sampled volatility derivatives can be priced in this way; for an exhaustive overview see e.g. Zheng and Kwok [102].

Finally, joint forward-starting payoffs can also be imagined. For example, a *forward-starting TVO* can be defined by letting in (3.30)

$$F(x, y) = \overline{\sigma}\sqrt{\frac{y}{T - t^*}}(e^x - K)^+. \tag{3.40}$$

The quantity $RV_{t^*,T} = \sqrt{(I_T - I_{t^*})/(T - t^*)}$ is the *forward-starting realized volatility*. Notice that such definition of a forward option entails a rescaling of the spot price at inception with the price at the activation date t^*.

3.5 Specific model analysis

We now determine the DTC Lévy structure (3.14) of various popular asset price processes, and find for each of them the corresponding leverage-neutral characteristic function $\Phi_{t_0}(z, w)$.

Such a derivation allows for the full implementation of equation (3.24) for the pricing of joint asset and volatility derivatives in all of the cases we shall deal with. What the discussion below should make apparent is that decoupled time changes offer a natural unifying framework for *a priori* different strains of financial asset models (e.g. continuous/jump diffusions, jump diffusions with stochastic volatility, Lévy processes). By classifying models through their DTC structure it is possible to recognize a "nesting" pattern linking different models, in which some can be considered particular cases of some others. This is of use for numerical purposes: as we shall see in section 3.7, one single implementation of equation (3.24) can produce values for several models, each one obtained by using a different instantiation of the code. Four categories of asset models are discussed: standard Lévy processes, stochastic volatility models, DTC jump diffusions and general exponentially-affine asset models.

3.5.1 Lévy processes

In case of the Lévy process the DTC structure coincides with the underlying Lévy process. To determine $\Phi_{t_0}(z, w)$ no change of measure is necessary, so this function represents the joint conditional characteristic function of the log-price and its quadratic variation as given in the risk-neutral measure. Below, are provided the calculations for some popular models.

Black-Scholes model

The classic SDE with constant parameters σ, r driven by a Brownian motion W_t:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3.41}$$

can be trivially recovered from (3.14) by setting the triplet for the underlying Lévy process $X_t = X_t^c$ to be $(0, \sigma, 0)$ and letting $T_t = t$, $U_t = 0$, so that $X_{T,U} = X_t$. From (3.18), we immediately have:

$$\Phi_{t_0}(z, w) = \exp(-(t - t_0)\sigma^2(z^2 + iz - 2iw)/2). \tag{3.42}$$

Jump diffusion models

In their classic works, Merton and Kou [81, 59] proposed to model the log-price dynamics as a finite-activity jump diffusion. The risk-neutral asset dynamics are given by:

$$dS_t = rS_{t^-}dt + \sigma_t S_{t^-}dW_t + S_{t^-}(\exp(J) - 1)dN_t - \kappa\lambda S_{t^-}dt \tag{3.43}$$

where W_t is a standard Brownian motion, N_t is a Poisson counter of intensity λ, and J is the jump size distribution. N_t and W_t are assumed to be independent, and the compensator κ equals $\phi_J(-i) - 1$. For the discounted price $\tilde{S}_t$ to be a true martingale, conditions on the asymptotic behavior of $f_J(x)$ must be imposed. In the Merton model J is normally distributed $J \sim \mathcal{N}(m, \delta^2)$, whereas Kou assumed for it an asymmetrically skewed double-exponential distribution, that is, the density function $f_J(x)$ as given by:

$$f_J(x) = \begin{cases} \alpha p e^{-\alpha x} \text{ if } x \geq 0 \\ \beta q e^{\beta x} \text{ if } x < 0 \end{cases} \tag{3.44}$$

for $\alpha > 1, \beta > 0$ and $p + q = 1$.

In these models no time change is involved, so $X_{T,U}$ coincides with the underlying Lévy process X_t having characteristic triplet $(0, \sigma^2, \lambda f_J(x)dx)$. To completely characterize $\Phi_{t_0}(z, w)$, observe that $(X_t^d, \langle X^d \rangle_t)$ is just a bivariate compound Poisson process of joint jump density $f_{J,J^2}(x, y)$ and intensity λ, whence:

$$\psi_D(z, w) = \lambda(\phi_{J,J^2}(z, w) - 1), \tag{3.45}$$

where $\phi_{J,J^2}(z, w)$ is the joint characteristic function of J and J^2. We conclude from (3.18) that Φ_{t_0} has the exponential structure:

$$\Phi_{t_0}(z, w) = \exp(-(t - t_0)(\sigma^2(z^2/2 + iz/2 - 2iw)/2 + \lambda(iz\kappa - \phi_{J,J^2}(z, w) + 1)). \tag{3.46}$$

Now for the Merton model we have

$$\phi_{J,J^2}(z, w) = \frac{\exp\left(\frac{imz - \delta^2 z^2/2 + im^2 w}{1 - 2i\delta^2 w}\right)}{\sqrt{1 - 2i\delta^2 w}}, \tag{3.47}$$

and the integral converges for $\text{Im}(w) > -1/2\delta^2$. For the Kou model we can write:

$$\phi_{J,J^2}(z, w) = \phi_{J_+,J_+^2}(z, w) + \phi_{J_-,J_-^2}(z, w); \tag{3.48}$$

the characteristic function of the positive and negative parts are:

$$\phi_{J_+,J_+^2}(z,w) = \alpha p\sqrt{\pi}e^{-\frac{(\alpha-iz)^2}{4iw}}\left(\frac{\text{Erfc}\left(\frac{\alpha-iz}{2\sqrt{-iw}}\right)}{2\sqrt{-iw}}\right), \tag{3.49}$$

$$\phi_{J_-,J_-^2}(z,w) = \beta q\sqrt{\pi}e^{-\frac{(\alpha-iz)^2}{4iw}}\left(\frac{\text{Erfc}\left(\frac{\beta-iz}{2\sqrt{-iw}}\right)}{2\sqrt{-iw}}\right), \tag{3.50}$$

which both converge for $\text{Im}(w) > 0$.

Tempered stable Lévy and CGMY

Another way of obtaining Lévy distributions for the asset price is to directly specify an infinite activity Lévy measure $\nu(dx)$. In such a case we have $X_{T,U} = X_t = X_t^d$, with X_t being a pure jump Lévy process of Lévy measure $\nu(dx)$. The two instances we analyze here are the tempered stable Lévy process (e.g. Cont and Tankov [23]), and the CGMY (Carr *et al.* [13]) models. Both of these are obtained as an exponential smoothing of stable distributions; the latter can be viewed as a generalization of the former allowing for an asymmetrical skew between the distribution of positive and negative jumps. The Lévy density for a CGMY process is:

$$\frac{d\nu(x)}{dx} = \frac{c_- e^{-\beta_-|x|}}{|x|^{1+\alpha_-}}\mathbb{I}_{\{x<0\}} + \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}}\mathbb{I}_{\{x\geq 0\}}. \tag{3.51}$$

which is well defined for all $c_+, c_-, \beta_+, \beta_- > 0$, $\alpha_+, \alpha_- < 2$. When $\alpha_+ = \alpha_-$ one has the tempered stable process. For simplicity in what follows we assume $\alpha_+, \alpha_- \neq 0, 1$; for such values the involved characteristic functions still exist, but lead to particular cases. Since

$$\Phi_{t_0}(z,w) = \exp((t-t_0)\xi(z,w,\nu(x)dx,-i)) \tag{3.52}$$

to fully characterize $\Phi_{t_0}(z,w)$ we only need to determine $\psi_X^d(\theta)$ and $\psi_D(z,w)$. Letting $\gamma_1 = \int_{-1}^{1} x d\nu(x)$, the exponent $\psi_X^d(\theta)$ is given by the standard theory (Cont and Tankov [23], proposition 4.2) as:

$$\begin{aligned}\psi_X^d(\theta) =& \gamma_1 + \Gamma(-\alpha_+)\beta_+^{\alpha_+}c_+\left(\left(1-\frac{i\theta}{\beta_+}\right)^{\alpha_+} - 1 + \frac{i\theta\alpha_+}{\beta_+}\right) + \\ & \Gamma(-\alpha_-)\beta_-^{\alpha_-}c_-\left(\left(1+\frac{i\theta}{\beta_-}\right)^{\alpha_-} - 1 - \frac{i\theta\alpha_-}{\beta_-}\right).\end{aligned} \tag{3.53}$$

Set $\gamma_2 = \int_{-1}^{1} x^2 d\nu(x)$; the positive part ψ_D^+ of ψ_D is then seen to be:

$$\psi_D^+(z,w) =$$
$$iz\gamma_1 + iw\gamma_2 + \int_0^{+\infty} (e^{izx+iwx^2} - 1 - (izx + iwx^2)) \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}} dx = iz\gamma_1 + iw\gamma_2 +$$
$$ic_+\beta_+^{\alpha_+} \left(-w\frac{\Gamma(2-\alpha_+)}{2i\beta_+^2} - z\frac{\Gamma(1-\alpha_+)}{2i\beta_+} + i\Gamma(-\alpha_+) \right) - c_+(\beta_+ - iz)^{\alpha_+} \left(\frac{i(\beta_+ - iz)^2}{w} \right)^{-\alpha_+/2}$$
$$\times \left(\sqrt{\frac{i(\beta_+ - iz)}{w}} \Gamma\left(\frac{1}{2} - \frac{\alpha_+}{2}\right) {}_1F_1\left[\frac{1-\alpha_+}{2}, \frac{3}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] - \right.$$
$$\left. \Gamma\left(-\frac{\alpha_+}{2}\right) {}_1F_1\left[-\frac{\alpha_+}{2}, \frac{1}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] \right). \tag{3.54}$$

Here Γ is the Euler Gamma function and ${}_1F_1$ the confluent hypergeometric function. The multi-strip of convergence of (3.54) is the set $\Sigma_\Phi = \{(z,w),\ \text{Im}(w) > 0,\ \text{Im}(z) > -\beta_+\}$. The determination ψ_D^- has a similar expression.

3.5.2 Stochastic volatility and the Heston model

In a stochastic volatility model the asset process is given, in a risk neutral-measure, by the SDE

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1 \tag{3.55}$$

where v_t is some continuous stochastic variance process. By the Dubins and Schwarz's theorem any continuous martingale M_t can be written as $M_t = W_{\langle M \rangle_t}$ for a certain Brownian motion W_t, which implies that the DTC structure of a stochastic volatility model corresponds to a standard Brownian motion W_t time-changed by T_t as in (3.9). In order to explicitly express the characteristic function $\Phi_{t_0}(z,w)$ we must make a specific choice for the dynamics in (3.55). For instance, we can make the popular choice of selecting a square-root (CIR) equation for the instantaneous variance:

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t} dW_t^2 \tag{3.56}$$

for positive constants α, θ, η and a Brownian motion W_t^2 linearly correlated with W_t^1 through a correlation coefficient ρ. For S_t to be well-defined, the parameters α, θ, and η need to satisfy the *Feller condition* $2\alpha\theta \geq \eta^2$. The system of SDEs (3.55)-(3.56) is the model by Heston [60]. As we change to the measure $\mathbb{Q}(z,w)$, the application of the complex-plane version of Girsanov's theorem and a simple algebraic manipulation reveals that the leverage-neutral dynamics v_t^z of v_t are of the same form as (3.56), but with parameters:

$$\alpha^z = \alpha - i\rho z\eta; \tag{3.57}$$

$$\theta^z = \alpha\theta/\alpha^z, \tag{3.58}$$

(see also [15]). Using equation (3.18), we determine Φ_{t_0} as follows[4]:

$$\Phi_{t_0}(z,w) = \mathcal{L}^{z}_{\Delta T}(z^2/2 + iz/2 - iw), \tag{3.59}$$

where $\mathcal{L}^{z}_{\Delta T}$ indicates the transform with respect to v^z_t which is well-known analytically (e.g. Dufresne [32]). The case $T_t = t$ reverts back to the Black-Scholes model, when (3.59) collapses to (3.42) with $\sigma^2 = v_0$.

Other choices for v_t are clearly possible, yielding different stochastic volatility models (the 3/2 model, GARCH, etc.). It is clear from the arguments above that for an analytical expression for Φ_{t_0} to exist it suffices that the Laplace transform of T_t is known in closed form[5] and that v_t belongs to a class of models that are stable under the Girsanov transformation.

3.5.3 DTC jump diffusions

When the underlying Lévy process is represented by a finite activity jump diffusion, operating a decoupled time change amounts to either introducing a stochastic volatility coefficient in the continuous Brownian part, or making the intensity of the compound Poisson process X^d_t stochastic, or both. Models carrying this structure have been prominently discussed by D.S. Bates [6, 7] and H. Fang [37].

Stochastic volatility with jumps

The stochastic volatility model with jumps (SVJ) provides us with a first instance of a decoupled time change not otherwise obtainable as an ordinary time change. The SVJ model is in fact a Lévy decoupled time change with a time-changed continuous part and a time-homogeneous jump part. The dynamics for the asset price are given by the exponential jump diffusion:

$$dS_t = rS_{t^-}dt + \sqrt{v_t}S_{t^-}dW^1_t + S_{t^-}(\exp(J) - 1)dN_t - \kappa\lambda S_{t^-}dt; \tag{3.60}$$

for some Brownian motion W^1_t, stochastic variance process v_t, Poisson process N_t and jump size J having compensator κ. The underlying DTC structure of the SVJ model is given by $X_{T,U} = X^c_{T_t} + X^d_t$ with the characteristic triplet for X_t being $(0, 1, \lambda f_J(x)dx)$ and T_t taking the form (3.9). By assuming as a jump distribution a normal random variable, and as a variance process the square-root equation:

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW^2_t \tag{3.61}$$

we have the model by Bates. For the discounted asset value to be a martingale, the parameters of the driving stochastic volatility and jump process must be subject to the requirements of both subsection 3.5.2 and subsection 3.5.1. It is straightforward to see that $\Phi_{t_0}(z,w)$ decomposes into:

$$\Phi_{t_0}(z,w) = \Phi^c_{t_0}(z,w)\Phi^d_{t_0}(z,w), \tag{3.62}$$

[4]In chapter 2 we have found Φ_{t_0} for the Heston model by augmenting the SDE system (3.55)-(3.56) with the equation $dI_t = v_t dt$, and solved the associated Fourier-transformed parabolic equation via the usual Feynman-Kac argument. As it has to be, the two approaches coincide.

[5]See e.g. Lewis [75], chapter 2, for the Laplace transform of the cited models.

where $\Phi^c_{t_0}(z,w)$ and $\Phi^d_{t_0}(z,w)$ are given respectively by (3.59) and (3.46)-(3.47). Therefore:

$$\Phi_{t_0}(z,w) = \mathcal{L}^z_{\Delta T}(z^2/2 + iz/2 - iw)\exp(-(t-t_0)\lambda(iz\kappa - \phi_{J,J^2}(z,w)+1)). \tag{3.63}$$

So far, we have encountered either exponential Lévy models, or exponentially affine functions arising as solutions of a PDE problem. Here we have a mixture of the two: a time-homogeneous jump factor, modeled as a compound Poisson process, and a continuous diffusion factor, whose characteristic function solves a diffusion problem. The degenerate case $T_t = t$, yields a Merton jump diffusion with diffusion coefficient $\sqrt{v_0}$.

Stochastic volatility with jumps and a stochastic jump rate

Another way of obtaining a DTC model is obtained by introducing a stochastic jump frequency into the jump diffusion of the log-price. A jump process with stochastic volatility and stochastic jump rate (SVJSJ) has been suggested and empirically studied by Fang [37]. For a time change U_t, we assume N_t to be a pure jump process of finite activity such that conditionally on U_t, N_t is distributed like a Poisson random variable of parameter U_t, and is independent of every other involved process. Let λ_t be another continuous stochastic process; with the remaining notation as in subsection 3.5.3, we define the asset price dynamics as follows:

$$dS_t = rS_{t^-}dt + \sqrt{v_t}S_{t^-}dW^1_t + S_{t^-}(\exp(J)-1)dN_t - \kappa\lambda_t S_{t^-}dt; \tag{3.64}$$

This model has a clear DTC Lévy structure $X_{T,U}$ given by T_t, U_t as in (3.9) and (3.10) with $u_t = \lambda_t$, and characteristic triplet $(0,1,f_J(x)dx)$. The model by Fang is obtained by setting:

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW^2_t; \tag{3.65}$$

$$d\lambda_t = \alpha_\lambda(\theta_\lambda - \lambda_t)dt + \eta_\lambda\sqrt{\lambda_t}dW^3_t. \tag{3.66}$$

As usual we impose $\langle W^1, W^2\rangle_t = \rho dt$; in contrast, the Brownian motion W^3_t is assumed to be independent of all the other random variables. If both of the diffusion parameter sets obey Feller's condition and the density of J decays sufficiently fast, $\tilde{S}_t$ is a martingale. Like in the Bates model, the jumps J are normally distributed. The function Φ_{t_0} is then given by:

$$\Phi_{t_0}(z,w) = \mathcal{L}^z_{\Delta T}(z^2/2 + iz/2 - iw)\mathcal{L}_{\Delta U}(iz\kappa - \phi_{J,J^2}(z,w)+1). \tag{3.67}$$

Again we recognize that we can decompose $\Phi_{t_0}(z,w) = \Phi^c_{t_0}(z,w)\Phi^d_{t_0}(z,w)$, where $\Phi^c_{t_0}$ is the leverage-neutral characteristic function of a Heston process of variance v_t, and $\Phi^d_{t_0}$ that of a compound Poisson process time-changed with U_t, whose argument was computed in subsection 3.5.1. The Laplace transforms of the integrated-square root processes arising from v^z_t and λ_t are known, and the leverage-neutral version v^z_t of v_t has been given in subsection 3.5.2. Observe that there is no leverage effect in the jump part because of the assumptions on W^3_t. Finally, notice that the case $U_t = t$ reduces to the Bates model with a constant jump arrival rate equal to λ_0.

3.5.4 The Huang and Wu approach

In [63] Huang and Wu use a full DTC approach by selecting various jump processes X_t^d and using as a pair of time changes T_t and U_t the pathwise integrals of a bivariate square root process whose components have correlation ρ with the Brownian motion X_t^c. The equation for the log-returns provided is directly in the DTC form

$$\log S_t = \log S_0 + rt + \sigma X^c_{T_t} + X^d_{U_t} - \sigma^2 T_t - \psi^d(-i)U_t \tag{3.68}$$

The activity rates generating T_t and U_t are given by a bivariate diffusion of the CIR form

$$dv_t^i = \alpha^i(\theta^i - v_t^i)dt + \eta^i\sqrt{v_t^i}dW_t^i; \tag{3.69}$$

for $i = 1, 2$ and $\langle W^i, X^c\rangle_t = \rho$. Applying the usual machinery, after changing to the leverage-neutral measure and resorting to Girsanov's theorem we obtain the leverage-neutral characteristic function in the standard form

$$\Phi_{t_0}(z, w) = \mathcal{L}^z_{\Delta T,\Delta U}(\zeta(z, w, 0, 1, -i), \xi(z, w, \nu(dx), -i)). \tag{3.70}$$

Here $\mathcal{L}_{\Delta T,\Delta U}(\cdot)$ stands for the Laplace transform of the bivariate square root process having drift parameters whose coordinates are given by the analogous of equations (3.57) and (3.58). However, such a Laplace transform has no analytical solution, unless we assume the two components of (3.69) to be independent. The reason why this is the case can be understood by appealing to the theory of Grasselli and Tebaldi [55]. What the authors prove is that the system of Riccati ODEs canonically associated to the system (3.69) can only be solved in terms of the parameters of the system under the condition that the corresponding SDEs are independent ([55], theorem 17). This makes it unfeasible to instantiate (3.68)-(3.69) in such a way as to entail dependent continuous and jump activities and at the same time obtaining a closed formula for Φ_{t_0}.

3.5.5 General exponentially-affine activity rate models

A general theory of affine models for the discounted asset dynamics has been laid out by Duffie *et al.* [31], and Filipović [41]. We briefly illustrate how this ties in with decouple time-changed processes. Suppose we have a Markov process given by the stochastic differential equation:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t + dN_t \tag{3.71}$$

where W_t is an n-dimensional Brownian motion, N_t is an n-dimensional pure jump process of intensity $\lambda(Y_t)$ and joint jump size distribution $F(x_1, \dots, x_n)$ on $\mathbb{R}^n$. We fix a discount functional $R(x) = r_0 + r_1 \cdot x$, $(r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n$ and assume for the coefficients the following linear structure:

$$\begin{aligned} \mu(x) &= m_0 + m_1 x, \quad (m_0, m_1) \in \mathbb{R}^n \times \mathcal{M}_n(\mathbb{R}) \\ \sigma \cdot \sigma^T(x) &= \Sigma_0 + \Sigma_1 x, \quad (\Sigma_0, \Sigma_1) \in Sim_n(\mathbb{R}) \times Sim_n(\mathbb{R})^n \\ \lambda(x) &= l_0 + l_1 x, \quad (l_0, l_1) \in \mathbb{R} \times \mathbb{R}^n. \end{aligned} \tag{3.72}$$

For some one-dimensional DTC process $X_{T,U}$, let M_t be the change of measure martingale in (3.17) and assume Y_t to be two-dimensional, so that the marginals of Y_t represent the instantaneous activity rates v_t and u_t.

The leverage-neutral characteristic function Φ_{t_0} can be recovered as follows. By taking the Ito differential of $\log M_t$ one sees that M_t is itself a linear jump diffusion; we can thus define the three-dimensional augmented process $\tilde{Y}_t = (Y_t, \log M_t)$ having some associated extended parameters $\tilde{m}_0, \tilde{m}_1, \tilde{\Sigma}_0, \tilde{\Sigma}_1, \tilde{l}_0, \tilde{l}_1, \tilde{F}$ in (3.72). Furthermore, we can rewrite M_t as:

$$M_t = \exp(b \cdot \tilde{Y}_t) \tag{3.73}$$

where $b = (0,0,1)^T$. Now, according to the results of Duffie *et al.* [31], appendix C, under the measure $\mathbb{Q} = \mathbb{Q}(z,w)$ having Radon-Nikodym derivative M_t, we have:

$$\Psi_{t_0}^{\mathbb{Q}}(u) := \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\exp\left(-\int_{t_0}^{t} R(\tilde{Y}_s)ds\right) e^{u\tilde{Y}_t}\right] = e^{-\alpha(t_0)-\beta(t_0)\tilde{Y}_{t_0}} \tag{3.74}$$

for all u for which (3.74) is defined, and $\alpha(\tau)$, $\beta(\tau)$ following the Riccati system of ODEs[6]:

$$\beta'(\tau) = r_1 + (\tilde{m}_1^T + \tilde{\Sigma}_1 b)\beta(\tau) - \frac{1}{2}\beta(\tau)^T\tilde{\Sigma}_1\beta(\tau) - \tilde{l}_1(\mathcal{L}_{\tilde{F}}(\beta(\tau)+b) - \mathcal{L}_{\tilde{F}}(b)) \tag{3.75}$$

$$\alpha'(\tau) = r_0 + (\tilde{m}_0 + \tilde{\Sigma}_0 b)\beta(\tau) - \frac{1}{2}\beta(\tau)^T\tilde{\Sigma}_0\beta(\tau) - \tilde{l}_0(\mathcal{L}_{\tilde{F}}(\beta(\tau)+b) - \mathcal{L}_{\tilde{F}}(b)) \tag{3.76}$$

for $\tau \leq t$, with boundary conditions $\beta(t) = u$ and $\alpha(t) = 0$. By choosing

$$r_0 = 0, \quad r_1 = (\zeta(z,w,\mu,\sigma,\theta_0), \xi(z,w,\nu(dx),\theta_0), 0), \tag{3.77}$$

one notices that:

$$\Psi_{t_0}^{\mathbb{Q}}(0) = \Phi_{t_0}(z,w). \tag{3.78}$$

The solvability of equations (3.75)-(3.76) is discussed and characterized in Grasselli and Tebaldi [55]. What we have just shown is that the class of the exponentially-affine processes and that of the DTC Lévy processes intersect in the class of the DTC processes whose instantaneous activity rates are given by affine jump diffusions of the form (3.71)- (3.72).

We remark that $\tilde{Y}_t$ implicitly defines a price process S_t through the instantaneous activity rates and the change of measure martingale M_t accounting for the dependence structure between the time changes and the underlying Lévy process. The augmented diffusion $\tilde{Y}_t$ is an *exponentially-affine decoupled time change*; all the models reviewed so far fall under this category[7]. Another example of a model that can be represented in this form is the "double jump model" of Duffie *et al.* [31], given by a jump diffusion with stationary jump intensity, whose stochastic volatility is itself a jump diffusion process having the same driving Poisson process as the stock.

[6] $(\beta(\tau)^T\Sigma_1\beta(\tau))^k := \beta(\tau)^T\Sigma_1^k\beta(\tau),\ k = 1,\ldots,n.$

[7] A DTC model falling outside this intersection is the linear quadratic-affine model by Santa Clara and Yan, [88]. However, for the same reasons as those mentioned for the Huang and Wu model, such model does not possess an analytically tractable transform to be used for pricing purposes.

3.6 A novel DTC jump diffusion for derivative pricing

We now illustrate a theoretical model in the DTC framework admitting a closed formula for Φ_{t_0}. The price evolution we consider has several attractive features. In first place, it is a DTC jump diffusion and therefore allows for the presence of a stochastic jump rate and a stochastic volatility; also, the dynamics we assume carry the usual linear correlation between the stochastic volatility and the Brownian motion driving the stock, as well as a dependence structure between the instantaneous rates of activity. Thus, the hypothesis of a market jump and continuous activity correlated with each other finds room in this model. As we have seen, the asset modeling research reviewed in sections 3.5.3 and 3.5.4 cannot capture this effect.

The price process we analyze links to a modern and currently very active strain of research, which makes use of the so-called *Wishart process* for financial modeling. The Wishart process is a matrix-valued affine process, studied foremostly by M.F. Bru [12], that can be thought of as a multivariate extension of the CIR process. It has been used to model the driving factors of term structures and price processes by, among the others, da Fonseca *et al.* [28, 27], and Gouriéroux and Sufana [52, 53], among the others.

For two matrices Q and M in $\mathcal{M}_n(\mathbb{R})$, with Q invertible and M negative definite (to capture mean-reversion), some constant $c \geq n+1$ and an $n \times n$ matrix Brownian motion B_t, a Wishart process Σ_t is defined as being the only strong solution of the following multi-dimensional SDE:

$$d\Sigma_t = \sqrt{\Sigma_t} dB_t Q + Q^T dB_t^T \sqrt{\Sigma_t} + (M\Sigma_t + \Sigma_t M^T + cQ^T Q)dt. \tag{3.79}$$

Under these conditions the Wishart process is a symmetric matrix-valued process whose diagonal elements take only positive values.

We can use Σ_t to build a one-dimensional DTC jump diffusion model as follows. We choose $n = 2$ and let W_t be a two-dimensional Brownian motion such that $\langle W^1, B^{1,1}\rangle_t = \langle W^2, B^{2,1}\rangle_t = \rho t$ for some correlation parameter ρ and W_t is independent of all the other entries of B_t. Let N_t be a finite activity jump process like in subsection 3.5.3, that we further assume to be independent of both W_t and B_t. As usual, the jump distribution J is set to be independent of every other variable. Denoting by σ_t the positive-definite matrix square root of Σ_t, we can define the risk-neutral dynamics of the log-price process $Y_t = \log(S_t/S_0)$ as:

$$dY_t = (r - \Sigma_t^{1,1}/2 - \Sigma_t^{2,2}\kappa)dt + \sigma_t^{1,1} dW_t^1 + \sigma_t^{1,2} dW_t^2 + J dN_t, \quad Y_0 = 0 \tag{3.80}$$

where κ equals $\phi_J(-i) - 1$. The process S_t can be seen to be a local martingale of the form (3.14) by assuming the time changes in proposition 3.2.3 to be like those in equations (3.9) and (3.10) and letting $\theta_0 = (-i, -i)$ and

$$dX_t^c = \frac{\sigma_t^{1,1}}{\sqrt{\Sigma_t^{1,1}}} dW_t^1 + \frac{\sigma_t^{1,2}}{\sqrt{\Sigma_t^{1,1}}} dW_t^2, \quad X_t^d = \sum_{i=0}^{M_t} J, \quad v_t = \Sigma_t^{1,1}, \quad u_t = \Sigma_t^{2,2}. \tag{3.81}$$

In the above, M_t is a Poisson process of intensity 1.Let now w_t^j be the scalar Brownian motion driving $\Sigma_t^{j,j}$: it can be proved that

$$d\langle w^1, w^2\rangle_t = \frac{\Sigma_t^{1,2}(Q^{1,1}Q^{1,2} + Q^{2,1}Q^{2,2})}{\sqrt{\Sigma_t^{1,1}((Q^{1,1})^2 + (Q^{2,1})^2)}\sqrt{\Sigma_t^{2,2}((Q^{1,2})^2 + (Q^{2,2})^2)}}dt. \tag{3.82}$$

Observe that this correlation is *stochastic*. The correlation between Y_t and its instantaneous variance $\Sigma_t^{1,1}$ is instead determined by the interplay between ρ and Q; we have:

$$d\langle w^1, X^c\rangle_t = \frac{\rho Q^{1,1}}{\sqrt{(Q^{1,1})^2 + (Q^{2,1})^2}}dt. \tag{3.83}$$

By applying the Girsanov's transformation, we see that the $\mathbb{Q}(z, w)$-dynamics of (3.79) are given by the complex-valued Wishart process:

$$d\Sigma_t^z = \sqrt{\Sigma_t^z}dB_tQ + Q^T dB_t^T\sqrt{\Sigma_t^z} + (M^z\Sigma_t^z + \Sigma_t^z(M^z)^T + cQQ^T)dt \tag{3.84}$$

where

$$M^z = M + izQ^TR, \quad R = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.85}$$

whence:

$$\Phi_{t_0}(z, w) = \mathcal{L}^z_{\Delta T,\Delta U}(z^2/2 + iz/2 - iw, iz\kappa - \phi_{J,J^2}(z, w) + 1). \tag{3.86}$$

Notably, the Laplace transform $\mathcal{L}_{\Delta T,\Delta U}(\cdot)$ for v_t and u_t as in (3.81) can be derived in closed form (see the appendix), since it is a particular case of some well-studied transforms of the Wishart process.

It is therefore possible to price, and find price sensitivities of, joint price/volatility contingent claims on an asset whose log-price process follows Y_t. The model just presented is a particular DTC jump diffusion featuring not only the usual leverage effect between the underlying jump diffusion and the continuous/jump market activity, given by (3.83), but also a correlation structure between the rates of activities themselves, as shown by equation (3.82). In the spirit of the previous section, we remark that when all the Wishart matrices are diagonal an application of the Lévy theorem shows that this model can be embedded in an SVJSJ representation for some appropriate parameter choice.

This asset pricing model provides an example of how non-trivial DTC modeling (i.e. achieved by using *dependent* time changes) may work in practice. As a general approach, one could start from a multivariate stochastic process whose integrated marginals have a known joint Laplace transform, and use these as time changes for the continuous and discontinuous parts of some given Lévy process. The underlying Lévy triplet will only appear as an argument of such a transform, and the characteristic function of the process is then completely determined up to a measure change. If the underlying time changes are drawn from an analytically tractable joint distribution, the analyticity of the model will be preserved. This and similar models are currently the subject of further research; some advances are illustrated in the next chapter.

3.7 Numerical testing and final remarks

3.7.1 Implementation of the pricing formula

For validation purposes, we numerically implemented equation (3.24) in MATHEMATICA® for various models and payoffs, and compared the analytical prices so obtained to a MATLAB® simulation following an Euler scheme. The results confirm the consistence of the pricing formula with the risk-neutral valuation theory.

We analyzed three different contingent claims: one on S_t, one on TV_t, and one joint derivative on S_t and TV_t. Namely, we accounted for three different kinds of options: a vanilla call option, a call option on the realized volatility, and a call TVO.

For a plain call option of maturity t and strike K, the function F and its Fourier transform $\hat{F}$ to be used in (3.24) are:

$$F(z) = (e^z - K)^+, \quad \hat{F}(z) = \frac{K^{1+iz}}{(iz - z^2)}; \tag{3.87}$$

the function $\hat{F}$ exists and is analytic for $\text{Im}(z) > 1$.

A possible volatility investment is to write a call option using as an underlying the total realized volatility $\sqrt{TV_t}$ of an asset, or to buy a call option directly on a volatility index such as the VIX. Hence, we would like to price the contingent claim paying $(\sqrt{TV_t} - Q)^+$ at time t for some strike realized volatility level Q. In our equation we would then need to take:

$$F(w) = (\sqrt{w} - Q)^+, \quad \hat{F}(w) = \frac{\sqrt{\pi}\ \text{Erfc}(Q\sqrt{-iw})}{2(-iw)^{3/2}}; \tag{3.88}$$

the Fourier transform here is well-defined and holomorphic in $\text{Im}(w) > 0$.

The target volatility option mentioned in the introduction is a natural candidate for testing mixed-claim structures, being an instance of a currently traded joint asset/volatility derivative. The payoff function F and the Fourier transform for a call TVO of strike H, maturing at t with target volatility $\overline{\sigma}$ are:

$$F(z,w) = \overline{\sigma}\sqrt{\frac{t}{w}}(e^z - H)^+, \quad \hat{F}(z,w) = \overline{\sigma}(1+i)\sqrt{\frac{\pi t}{2w}}\frac{H^{1+iz}}{(iz - z^2)}. \tag{3.89}$$

Observe that, unlike the previous contracts, the payoff F of a TVO shows explicit dependence on the expiry t. The domain of holomorphy of $\hat{F}$ is the strip $\Sigma_F = \{(z,w) \in \mathbb{C}^2,\ \text{Im}(z) > 1,\ \text{Im}(w) > 0\}$.

We numerically tested these derivatives using five different stochastic models for the underlying asset processes: namely, the Black-Scholes, Heston, Merton, Bates and Fang models. All the prices have been produced with a single implementation of (3.24) with Φ_{t_0} given by (3.67). All we had to do is changing/voiding the relevant parameters, and replacing the module for $\hat{F}$ whenever switching payoff. The parameter estimates have been taken from Fang's [37] fitting of the S&P 500 index, and are illustrated in table 1. Tables 2 to 6 summarize the result obtained for five different sets of observable market conditions $(r, t_0, S_{t_0}, TV_{t_0})$ and contract parameters $t, K, Q, H, \overline{\sigma}$. For each given t_0, the maturity $t > t_0$ is the same for all the three options considered; a TVO is always compared to a vanilla call having same strike, and the target volatility is set to be constant across all the data sets.

A number of 100.000 paths of step size $(t - t_0)/1000$ have been simulated. The figures show a good overall match between the analytical value (AV) and the Monte Carlo value (MC); the relative error |AV−MC|/MC is shown in parentheses. For the call option on the volatility in some cases we almost attain four-digit precision. On the other hand, for some models and data sets the integrands for the TVO valuation remain highly oscillatory around the maximum integration range; when this occurs, a certain loss of accuracy is observed.

Parameters	Black-Scholes	Heston	Merton	Bates	Fang
σ_{t_0}	0.14	0.15	0.12	0.15	0.14
α		4.57		8.93	6.5
θ		0.0306		0.0167	0.0104
η		0.48		0.22	0.2
ρ		-0.82		-0.58	-0.48
λ_0			1.42	0.39	0.41
δ			0.0894	0.1049	0.2168
κ			-0.075	-0.11	-0.21
α_λ					5.06
θ_λ					0.13
η_λ					1.069

Table 3.1: parameters from the S&P estimations of Fang [37], section 4.

Model	Vanilla Call		Volatility Call		TVO Call	
	AV	MC	AV	MC	AV	MC
Black-Scholes	24.7627	24.7775(0.05%)	0.0847	0.0848(0.12%)	17.5441	17.6982(0.87%)
Heston	25.3893	25.3710(0.07%)	0.1088	0.1084(0.37%)	17.2248	17.6044(2.16%)
Merton	25.3243	25.2290(0.38%)	0.1192	0.1194(0.17%)	17.7529	17.7922(0.22%)
Bates	25.1166	25.0889(0.11%)	0.1002	0.1005(0.30%)	18.5980	18.7480(0.80%)
Fang	25.5686	25.6508(0.32%)	0.0907	0.0892(1.68%)	24.0494	24.0764(0.11%)

Table 3.2: Prices, parameter set 1: $S_{t_0} = 100$, $K = H = 80$, $Q = 0.05$, $t_0 = 0$, $t = 1$, $r = 0.06$, $\overline{\sigma} = 0.1$, $TV_{t_0} = 0$.

Model	Vanilla Call		Volatility Call		TVO Call	
	AV	MC	AV	MC	AV	MC
Black-Scholes	8.4801	8.4784(0.02%)	0.1672	0.1695(1.36%)	5.7622	5.6957(1.17%)
Heston	10.3063	10.3023(0.04%)	0.2167	0.2172(0.23%)	6.3815	6.7080(4.87%)
Merton	11.5845	11.5713(0.11%)	0.2357	0.2356(0.04%)	7.4564	7.4239(0.44%)
Bates	9.8607	9.8371(0.24%)	0.2002	0.2001(0.05%)	6.8180	6.9085(1.31%)
Fang	8.8630	8.8737(0.12%)	0.1827	0.1828(0.05%)	7.4173	7.5046(1.16%)

Table 3.3: Prices, parameter set 2: $S_{t_0} = 100$, $K = H = 120$, $Q = 0.1$, $t_0 = 0.5$, $t = 4$, $r = 0.039$, $\overline{\sigma} = 0.1$, $TV_{t_0} = 0.018$.

Model	Vanilla Call		Volatility Call		TVO Call	
	AV	MC	AV	MC	AV	MC
Black-Scholes	3.7627	3.7346(1.02%)	0.2300	0.2305(0.22%)	0.9771	0.9437(3.54%)
Heston	4.1390	4.1304(0.21%)	0.2318	0.2320(0.09%)	1.0480	1.0451(0.28%)
Merton	4.4169	4.4435(0.60%)	0.2348	0.2343(0.21%)	1.1254	1.1235(0.17%)
Bates	4.1842	4.1687(0.37%)	0.2327	0.2328(0.04%)	1.0593	1.0544(0.46%)
Fang	4.3219	4.3420(0.46%)	0.2362	0.2362(0.00%)	1.0919	1.0987(0.62%)

Table 3.4: Prices, parameter set 3: $S_{t_0} = 100$, $K = H = 100$, $Q = 0.25$, $t_0 = 1.25$, $t = 1.5$, $r = 0.072$, $\overline{\sigma} = 0.1$, $TV_{t_0} = 0.23$.

Model	Vanilla Call		Volatility Call		TVO Call	
	AV	MC	AV	MC	AV	MC
Black-Scholes	42.6506	42.6452(0.01%)	0.2670	0.2665(0.19%)	19.7252	19.9181(0.96%)
Heston	42.9595	43.0010(0.10%)	0.2859	0.2858(0.03%)	19.8454	19.6512(0.99%)
Merton	42.8984	42.8580(0.09%)	0.2955	0.2954(0.03%)	19.4192	19.3975(0.11%)
Bates	42.7768	42.7928(0.04%)	0.2804	0.2802(0.07%)	19.8042	19.8318(0.14%)
Fang	43.0039	43.0252(0.05%)	0.2793	0.2791(0.07%)	20.5992	20.5998(0.01%)

Table 3.5: Prices, parameter set 4: $S_{t_0} = 100$, $K = H = 60$, $Q = 0.2$, $t_0 = 3$, $t = 5$, $r = 0.0225$, $\overline{\sigma} = 0.1$, $TV_{t_0} = 0.19$.

3.7.2 Conclusions

In this chapter we suggested a theoretical pricing framework that can easily be made to represent popular settings, but whose full model and payoff generalities were not possible by using the previous theory. We achieved this by introducing the concept of decoupled time change and by considering payoffs on an asset and its accrued volatility as the default claims to be priced.

DTC processes provide a common time-changed representation for many models from the current literature, and helps to explain possible dependence relationships between the continuous and the jump market activities. We obtained martingale relations for stochastic exponentials of DTC Lévy processes, based on which we defined an asset price's dynamics. We then linked the joint characteristic function of the log-price dynamics and the quadratic variation to the joint Laplace transform of the time changes. As a by-product, we extended the measure change technique of Carr and Wu [15] to the class of DTC Lévy processes. In the DTC setup, we rigorously posed and solved the valuation problem of a derivative paying off on an asset S_t and its real-

Model	Vanilla Call		Volatility Call		TVO Call	
	AV	MC	AV	MC	AV	MC
Black-Scholes	2.3393	2.3080(1.36%)	0.1590	0.1588(0.13%)	1.9535	1.8622(4.90%)
Heston	2.5098	2.5071(0.11%)	0.1852	0.1862(0.54%)	2.2190	2.1317(4.10%)
Merton	3.7078	3.6843(0.64%)	0.1983	0.1981(0.10%)	3.0330	3.0165(0.55%)
Bates	2.7416	2.7380(0.13%)	0.1767	0.1769(0.11%)	2.3727	2.3798(0.30%)
Fang	1.9814	1.9410(2.08%)	0.1664	0.1668(0.24%)	1.9453	1.9167(1.49%)

Table 3.6: Prices, parameter set 5: $S_{t_0} = 100$, $K = H = 130$, $Q = 0.015$, $t_0 = 1$, $t = 2.5$, $r = 0.087$, $\overline{\sigma} = 0.1$, $TV_{t_0} = 0.009$.

ized volatility, by means of an inverse-Fourier integral relation that extends previously known formulae.

Several stochastic models and contingent claims have been analyzed. In all the accounted cases we outlined the underlying DTC structure and found the leverage-neutral characteristic function. Furthermore, we have introduced a novel DTC Lévy theoretical model which illustrates how equity modeling could benefit from the idea of decoupled time changes.

For numerical comparison and validation purposes, we focused on specific instances from the three payoff classes allowed by our equation: plain vanilla claims, volatility claims, and joint asset/volatility claims. The results confirm the validity of our method. From a computational standpoint, a single software implementation can output prices for several different combinations of models and payoffs.

Appendix: proofs

We begin by recalling some basic definitions from the semimartingale representation theory; in particular, we refer to Jacod and Shiryaev [67], chapters 2 and 3, and Jacod [65], chapitre X.

We define the *Doléans-Dade exponential* of an n-dimensional semimartingale X_t starting at 0 as:

$$\mathcal{E}(X_t) = e^{X_t - \langle X^c \rangle_t / 2} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \tag{3.90}$$

where X_t^c denotes the continuous part of X_t and the infinite product converges uniformly. This is known to be the solution of the SDE $dY_t = Y_{t-} dX_t$, $Y_0 = 1$.

Let $\epsilon(x)$ be a truncation function and $(\alpha_t, \beta_t, \rho(dt \times dx))$ be a triplet of predictable processes that are well-behaved in the sense of Jacod and Shiryaev [67], chapter 2, equations (2.12)-(2.14). For $\theta \in \mathbb{C}^n$, associate with $(\alpha_t, \beta_t, \rho(dt \times dx))$ the following complex-valued functional:

$$\Psi_t(\theta) = i\theta^T \alpha_t - \theta^T \beta_t \theta / 2 + \int_0^t \int_{\mathbb{R}^n} (e^{i\theta^T x} - 1 - i\theta^T x \epsilon(x)) \rho(ds \times dx). \tag{3.91}$$

This functional is well-defined on:

$$\mathcal{D} = \left\{ \theta \in \mathbb{C}^n \text{ such that } \int_0^t \int_{\mathbb{R}^n} e^{i\theta^T x} \epsilon(x) \rho(ds \times dx) < +\infty \text{ almost surely} \right\} \tag{3.92}$$

and because of the assumptions made it is also predictable and of finite variation.

Let X_t be an n-dimensional semimartingale. The *local characteristics* of X_t are the unique predictable processes $(\alpha_t, \beta_t, \nu(dt \times dx))$ as above, such that $\mathcal{E}(\Psi_t(\theta)) \neq 0$ and $\exp(i\theta^T X_t)/\mathcal{E}(\Psi_t(\theta))$ is a local martingale for all $\theta \in \mathcal{D}$. The process $\Psi_t^X(\theta)$ in (3.91) arising from the local characteristics of X_t is called the *cumulant process* of X_t, and it is independent of the choice of $\epsilon(x)$. It is clear that the local characteristics of a Lévy process X_t of Lévy triplet (μ, Σ, ν) are $(\mu t, \Sigma t, \nu dt)$.

If $\mathcal{B}$ is a Borel space, the time change of a random measure $\rho(dt \times dx)$ on the product measure space $\Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n)$ according to some time change T_t, is the random measure:

$$\rho(dT_t \times dx)(\omega, [0,t) \times B) = \rho(dt \times dx)(\omega, [0, T_t(\omega)) \times B) \tag{3.93}$$

for $\omega \in \Omega$, $t \geq 0$ and all sets $B \in \mathcal{B}(\mathbb{R}^n)$. A random measure $\rho(dt \times dx)$ is T_t-*adapted* if for all t, ω and B holds $\rho(dt \times dx)((T_{t-}, T_t], \omega, B) = 0$. This is equivalent to say that for each measurable random function W, the integral of W with respect to ρ is T_t-continuous (see [65], chapitre X); conversely, if X_t is a pure jump process that is T_t-continuous, then its associated jump measure $\rho(dt \times dx)$ is T_t-adapted (Kallsen and Shiryaev [70], proof of lemma 2.7).

A semimartingale X_t is said to be *quasi-left-continuous* if its local characteristic ν is such that $\nu(dt \times dx)(\omega, \{t\} \times B) = 0$ for all $t \geq 0$, Borel sets B in $\mathbb{R}^n$, and $\omega \in \Omega$. Essentially, quasi-left-continuity means that the discontinuities of the process cannot occur at fixed times.

The following theorem clarifies the importance of continuity/adaptedness under time changing, i.e. that stochastic integration and integration with respect to a random measure "commute" with the time changing operation.

Theorem A. *Let T_t be a time change with respect to some filtration $\mathcal{F}_t$.*

(i) Let X_t be a T_t-continuous semimartingale. For all $\mathcal{F}_t$-predictable integrands H_t, we have that H_{T_t} is $\mathcal{F}_{T_t}$-predictable, and:

$$\int_0^{T_t} H_s dX_s = \int_0^t H_{T_{s-}} dX_{T_s}; \tag{3.94}$$

(ii) Let $\rho(dt \times dx)$ be a T_t-adapted random measure on $\Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^n)$. For all measurable random functions $W(t, \omega, x)$ and $\omega \in \Omega$ it is:

$$\int_0^{T_t} \int_{\mathbb{R}^n} W(s, \omega, x) \rho(ds \times dx)(\omega) = \int_0^t \int_{\mathbb{R}^n} W(T_{s-}(\omega), \omega, x) \rho(dT_s \times dx)(\omega). \tag{3.95}$$

Proof. See Jacod [65], théorème 10.19, (a), for part *(i)*, and théorème 10.27, (a), for part *(ii)*. □

In particular, from part *(ii)* of theorem A follows that if X_t is a pure jump process with associated jump measure $\rho(dt \times dx)$ adapted to some time change T_t, then the time-changed process X_{T_t} has associated jump measure $\rho(dT_t \times dx)$.

It is essentially a consequence of theorem A that under the assumption of continuity with respect to T_t, the local characteristics of a time-changed semimartingale are well-behaved, in the sense of the next theorem.

Theorem B. *Let X_t be a semimartingale having local characteristics $(\alpha_t, \beta_t, \rho(dx \times dt))$ and cumulant process $\Psi_t^X(\theta)$ with domain $\mathcal{D}$, and let T_t be a time change such that X_t is T_t-continuous. Then the time-changed semimartingale $Y_t = X_{T_t}$ has local characteristics $(\alpha_{T_t}, \beta_{T_t}, \rho(dT_t \times dx))$ and the cumulant process $\Psi_t^Y(\theta)$ equals $\Psi_{T_t}^X(\theta)$, for all $\theta \in \mathcal{D}$.*

Proof. Kallsen and Shiryaev [70], lemma 2.7. □

We shall also need a result on linear transformation of a semimartingale and the corresponding change in the local characteristics.

Theorem C. Let X_t be an n-dimensional semimartingale having local characteristics $(\alpha_t, \beta_t, \rho(dx \times dt))$ and let M be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $Y_t = MX_t$ is an m-dimensional semimartingale whose local characteristics $(\alpha_t^Y, \beta_t^Y, \rho^Y(dx \times dt))$ are given by:

$$\begin{aligned} \alpha_t^Y &= M\alpha_t + \int_0^t \int_{\mathbb{R}^n} Mx(\mathbb{I}_{|Mx|<1} - \mathbb{I}_{|x|<1})\rho(dx \times ds) \\ \beta_t^Y &= M\beta_t M^T \\ \rho^Y(B \times [0,t]) &= \rho(\{x \in \mathbb{R}^n | Mx \in B\} \times [0,t]), \qquad \forall B \in \mathcal{B}(\mathbb{R}^m). \end{aligned} \tag{3.96}$$

Proof. Kallsen and Shiryaev, [70], lemma 2.5. □

Proof of proposition 3.2.3. Let $(\mu, \Sigma, 0)$ and $(0, 0, \nu)$ be the Lévy triplets of X_t^1 and X_t^2. Because of the T_t^1 and T_t^2-continuity assumption, we can apply theorem B and we immediately see that the local characteristics of $X^1_{T_t^1}$ and $X^2_{T_t^2}$ are respectively $(T_t^1\mu, T_t^1\Sigma, 0)$ and $(0, 0, dT_t^2\nu)$. By the application of theorem C with M taken as the juxtaposition of two $n \times n$ identity blocks, and $X_t = (X^1_{T_t^1}\ X^2_{T_t^2})^T$, we have that X_{T_t} has local characteristics[8] $(T_t^1\mu, T_t^1\Sigma, dT_t^2\nu)$

Let $\Psi_t(\theta)$ be the cumulant process of X_{T_t}; by definition the exponential $\mathcal{E}(\Psi_t(\theta))$ is well-defined if and only if $\theta \in \Theta$. But now the fact that T_t^1 and T_t^2 are continuous implies that X_{T_t} is quasi-left-continuous ([67], chapter 2, proposition 2.9), that in turn is sufficient for $\Psi_t(\theta)$ to be continuous ([67], chapter 3, theorem 7.4). Therefore, since Ψ_t is of finite variation, we have that $\mathcal{E}(\Psi_t(\theta)) = \exp(\Psi_t(\theta))$; in particular, this means that $\mathcal{E}(\Psi_t(\theta))$ never vanishes. By definition of the local characteristics, we then have that $M_t(\theta, X_t, T_t)$ is a local martingale for all $\theta \in \Theta$, and thus it is a martingale if and only if $\theta \in \Theta_0$. □

Proof of proposition 3.3.1. An immediate consequence of theorem B is that, under the present assumptions, the class of continuous and pure jump martingales are closed under time changing, so that orthogonality follows. Therefore:

$$\langle X_{T,U} \rangle_t = \langle X_T^c \rangle_t + \langle X_U^d \rangle_t. \tag{3.97}$$

The equation $\langle X_T^c \rangle_t = \Sigma T_t = \langle X^c \rangle_{T_t}$ can be established by the application of Dubins and Schwarz theorem. Regarding the discontinuous part, we notice that if $\rho(dt \times dx)$ is the jump measure associated to X_t^d we have that ρ is U_t-adapted because X_t^d is U_t-continuous. Hence, the application of theorem A, part *(ii)*, yields:

$$\langle X^d \rangle_{U_t} = \sum_{t<U_t} (\Delta X_t)^2 = \int_0^{U_t} x^2 \rho(ds \times dx) = \int_0^t x^2 \rho(dU_s \times dx) = \langle X_U^d \rangle_t. \tag{3.98}$$

[8]The process X_{T_t} is a particular instance of an *Ito semimartingale*: see Jacod and Protter [66].

□

Counterexample to propositions 3.2.3 and 3.3.1. Let X_t^c be a standard Brownian motion, and let T_t be an inverse Gaussian subordinator with parameters $\alpha > 0$ and 1, independent of X_t^c. The process $X_{T_t}^c$ is called a *normal inverse Gaussian process* of parameters $(\alpha, 0, 0, 1)$ and it is a pure jump process (Barndorff-Nielsen, [5]). Therefore by letting $X_t^d = X_{T_t}^c$ and $U_t = t$ we have $X_{T_t}^c = X_{U_t}^d$ so that orthogonality does not hold and it can be readily checked that (3.12) is not a martingale; moreover $\langle X_{T,U} \rangle_t = 2\langle X^d \rangle_t$ while the left hand side of (3.15) equals $\langle T \rangle_t + \langle X^d \rangle_t$. □

Proof of proposition 3.3.2. Since T_t and U_t are of finite variation, the total realized variance of an asset as in (3.14) satisfies $TV_t = -\theta_0^2 \langle X_{T,U} \rangle_t$, so that by proposition 3.3.1 we have:

$$TV_t = -\theta_0^2(\sigma^2 T_t + \langle X^d \rangle_{U_t}). \tag{3.99}$$

An application of proposition 3.2.3 guarantees that $C_t + D_t$ is a martingale for all $z, w \in \mathbb{C}$ such that $(iz\theta_0, iw\theta_0) \in \Theta_0$. By using relation (3.99) and operating the change of measure entailed by (3.17) we have:

$$\begin{aligned}
&\Phi_{t_0}(z, w) = \mathbb{E}_{t_0}[\exp(iz \log(\tilde{S}_t / S_{t_0}) + iw\,(TV_t - TV_{t_0})]\\
=&\mathbb{E}_{t_0}[\exp(iz(i\theta_0(\Delta X_{T_t}^c + \Delta X_{U_t}^d) - \Delta T_t \psi_X^c(\theta_0) - \Delta U_t \psi_X^d(\theta_0)) - iw\theta_0^2(\sigma^2 \Delta T_t + \Delta\langle X^d \rangle_{U_t}))]\\
=&\mathbb{E}_{t_0}[\exp(i(iz\theta_0, iw\theta_0) \cdot (\Delta C_{T_t} + \Delta D_{U_t}) - \Delta T_t(iz\psi_X^c(\theta_0) + iw\theta_0^2\sigma^2) - \Delta U_t iz\psi_X^d(\theta_0))]\\
=&\mathbb{E}_{t_0}^{\mathbb{Q}}[\exp(-\Delta T_t(\theta_0\mu(z - iz) - \theta_0^2\sigma^2(z^2 + iz - 2iw)/2) - \Delta U_t(iz\psi_X^d(\theta_0) - \psi_D(iz\theta_0, iw\theta_0)))].
\end{aligned}$$

To fully characterize Φ_{t_0} all that is left is expressing ψ_D in terms of ν. Since

$$\psi_D(z, w) = \log \mathbb{E}\left[\exp\left(\sum_{s<t} iz\Delta X_s^d + iw(\Delta X_s^d)^2\right)\right], \tag{3.100}$$

we have that:

$$\psi_D(z, w) = \int_{\mathbb{R}} (e^{izx + iwx^2} - 1 - i(zx + wx^2)\mathbb{1}_{|x|\leq 1})\nu(dx) \tag{3.101}$$

which completes the proof. □

Proof of proposition 3.4.1. We follow the proof of Lewis [76], theorem 3.2, lemma 3.3 and theorem 3.4. By writing the expectation as an inverse-Fourier integral (which can be done by the assumptions on F and because Φ_{t_0} is a characteristic function) and passing the expectation under the integration sign we have:

$$\mathbb{E}_{t_0}[e^{-r(t-t_0)}F(Y_t,\langle Y\rangle_t)] = \mathbb{E}_{t_0}\left[\frac{e^{-r(t-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty} S_t^{-iz}e^{-iw\langle Y\rangle_t}\hat{F}(z,w)dzdw\right]$$
$$= \frac{e^{-r(t-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty} e^{-iw\langle Y\rangle_{t_0}}S_{t_0}^{-iz}e^{-r(t-t_0)iz}\Phi_{t_0}(-z,-w)\hat{F}(z,w)dzdw. \tag{3.102}$$

All that remains to be proven is that Fubini's theorem application is justified. Let $N_t = \log(M_t(\theta_0, X_t, (T_t, U_t)))$ be the discounted, normalized log-price; define the probability transition densities $p_t(x,y) = \mathbb{P}(N_t < x, \langle N\rangle_t < y|\ t_0, N_{t_0}, \langle N\rangle_{t_0})\mathbb{I}_{\{x\in\mathbb{R}, y\geq\langle N\rangle_{t_0}\}}$, and let $\hat{p}_t(z,w)$ be their characteristic functions. For all $(z,w)\in L_{k_1,k_2}$ we have:

$$\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}\left|e^{-iw\langle Y\rangle_{t_0}}S_{t_0}^{-iz}e^{-r(t-t_0)iz}\Phi_{t_0}(-z,-w)\right|\hat{F}(z,w)dzdw$$
$$= \int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}\hat{p}_t(-z,-w)\hat{F}(z,w)dzdw$$
$$= \int_{\mathbb{R}^2}\hat{p}_t(-z+ik_1,-w+ik_2)\hat{F}(z+ik_1,w+ik_2)dzdw. \tag{3.103}$$

For $x\in\mathbb{R}$, $y\geq 0$, set $f(x,y) = e^{-k_1x-k_2y}F(x,y)$ $g(x,y) = e^{k_1x+k_2y}p_t(x,y)$. We see that the integrand in the right-hand side of (3.103) equals $\hat{g}^*(z,w)\hat{f}(z,w)$. But now f is $L^1(dx\times dy)$ because F is Fourier-integrable in Σ_F (for $(z,w)\in\Sigma_F$ take $\mathrm{Re}(z) = \mathrm{Re}(w) = 0$); similarly, $\hat{g}^*$ is $L^1(dz\times dw)$ because of the L^1 assumption on Φ_{t_0}. Therefore, the application of Parseval's formula yields:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{p}_t(-z+ik_1,-w+ik_2)\hat{F}(z+ik_1,w+ik_2)dzdw$$
$$= 4\pi^2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p_t(x,y)F(x,y)dxdy = 4\pi^2\mathbb{E}_{t_0}[F(N_t,\langle N\rangle_t)] < +\infty, \tag{3.104}$$

since $F\in L^1_{t_0}(N_t,\langle N\rangle_t)$. This proof straightforwardly adapts to forward-starting payoffs, because

$$V_{t_0,t^*} = \frac{e^{-r(T-t_0)}}{4\pi^2}\mathbb{E}_{t_0}\left[\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}e^{-izr(T-t_0)}e^{-iz(\tilde{Y}_T-\tilde{Y}_{t^*})-iw(\langle Y\rangle_t-\langle Y\rangle_{t^*})}\hat{F}(z,w)dzdw\right]$$
$$= \frac{e^{-r(T-t_0)}}{4\pi^2}\int_{ik_1-\infty}^{ik_1+\infty}\int_{ik_2-\infty}^{ik_2+\infty}e^{-izr(T-t_0)}\Phi_{t_0,t^*}(-z,-w)\hat{F}dzdw. \tag{3.105}$$

Since the integrability conditions of the transition probability functions are not altered when changing to their forward-starting counterparts, the application of Fubini's theorem can be justified as above.

□

Proof of the equations of section 3.6. We can endow Y_t with a correlation structure as follows.

Let Z_t be a two-dimensional matrix Brownian motion independent of W_t. The matrix process:

$$B_t = \begin{pmatrix} \rho W_t^1 + \sqrt{1-\rho^2} Z_t^{1,1} & Z_t^{1,2} \\ \rho W_t^2 + \sqrt{1-\rho^2} Z_t^{2,1} & Z_t^{2,2} \end{pmatrix} \tag{3.106}$$

is also a matrix Brownian motion enjoying the property that $\langle W^j, B^{j,1} \rangle_t = \rho t$ and W_t is independent of $B_t^{j,2}$ for $j = 1, 2$. Since $\Sigma_t^{i,i} = (\sigma_t^{i,i})^2 + (\sigma_t^{1,2})^2$, we have that X_t^c is indeed a Brownian motion and the activity rates are connected through the element $\sigma_t^{1,2}$.

To verify equations (3.82) and (3.83), observe that for some bounded variation processes S_t^j, $j = 1, 2$ we have that

$$d\Sigma_t^{j,j} = S_t^j dt + 2\sigma_t^{1,j}(Q^{1,j} dB_t^{1,1} + Q^{2,j} dB_t^{1,2}) + 2\sigma_t^{j,2}(Q^{1,j} dB_t^{2,1} + Q^{2,j} dB_t^{2,2}), \tag{3.107}$$

from which:

$$\begin{aligned} dw_t^j :=& \frac{d\Sigma_t^{j,j} - S_t^j dt}{2\sqrt{\Sigma_t^{j,j}((Q^{1,j})^2 + (Q^{2,j})^2)}} \\ =& \frac{\sigma_t^{1,j}(Q^{1,j} dB_t^{1,1} + Q^{2,j} dB_t^{1,2}) + \sigma_t^{j,2}(Q^{1,j} dB_t^{2,1} + Q^{2,j} dB_t^{2,2})}{\sqrt{\Sigma_t^{j,j}((Q^{1,j})^2 + (Q^{2,j})^2)}}. \end{aligned} \tag{3.108}$$

By taking the quadratic variation of the right-hand side we see that w_t^j are two Brownian motions such that $d\Sigma_t^{j,j} = S_t^j dt + 2\sqrt{\Sigma_t^{j,j}((Q^{1,j})^2 + (Q^{2,j})^2)} dw_t^j$. Equations (3.82) and (3.83) then follow from a direct computation.

Since X_t^d is orthogonal to all the entries of B_t, the change in the dynamics of Σ_t under $\mathbb{Q}(z, w)$ is only due to the correlation between X_t^c and B_t. Hence, for $(z, w) \in \Theta$, the Radon-Nikodym derivative M_t to be considered in (3.17) reduces to

$$M_t = \mathcal{E}\left(iz \int_0^t \sqrt{\Sigma_s^{1,1}} dX_s^c \right). \tag{3.109}$$

Furthermore, for $j = 1, 2$ we have:

$$d\left\langle \int_0^\cdot \sqrt{\Sigma_s^{1,1}} dX_s^c, B^{j,1} \right\rangle_t = \rho \sigma_t^{1,j} dt \tag{3.110}$$

$$d\left\langle \int_0^\cdot \sqrt{\Sigma_s^{1,1}} dX_s^c, B^{j,2} \right\rangle_t = 0 \tag{3.111}$$

so that application of Girsanov's theorem tells us that

$$d\tilde{B}_t = dB_t - iz\rho \begin{pmatrix} \sigma_t^{1,1} dt & 0 \\ \sigma_t^{1,2} dt & 0 \end{pmatrix} \tag{3.112}$$

is a $\mathbb{Q}(z, w)$-matrix Brownian motion. Solving the above for B_t and substituting in (3.79) yields (3.84). Equation (3.86) then follows from (3.18).

Finally, we give the formula for $\mathcal{L}_{T_t,U_t}(\cdot)$. For $\tau > 0$ and $n > 1$ consider the transform:

$$\phi_\Sigma(z) = \mathbb{E}\left[\exp\left(-\int_0^\tau \sum_{j=1}^n z_j \Sigma_s^{j,j} ds\right)\right] \tag{3.113}$$

for every vector of complex numbers $z = (z_1, \ldots, z_n)$ such that the above expectation is finite. The function $\phi_\Sigma(z)$ is exponentially-affine of the form

$$\phi_\Sigma(z) = \exp(-a(\tau) - Tr(A(\tau)\Sigma_0)), \tag{3.114}$$

since it is a particular case of the transforms studied in e.g. Grasselli and Tebaldi [55] or Gouriéroux [51]. The ODEs for $A(\tau), a(\tau)$ are given by:

$$A'(\tau) = A(\tau)M + M^T A(\tau) - 2A(\tau)Q^T QA(\tau) + D, \quad A(0) = 0 \tag{3.115}$$

$$a'(\tau) = Tr(cQ^T QA(\tau)), \quad a(0) = 0. \tag{3.116}$$

Here D is the diagonal matrix having the values $z_1, \ldots, z_n$ on the diagonal. The solution of (3.115)-(3.116) is obtainable through a linearization procedure that entails doubling the dimension of the problem, which yields:

$$A(\tau) = (A^{2,2}(\tau))^{-1} A^{2,1}(\tau) \tag{3.117}$$

$$a(\tau) = \frac{c}{2} Tr(\log\left(A^{2,2}(\tau)\right) + M^T \tau) \tag{3.118}$$

$$\begin{pmatrix} A^{1,1}(\tau) & A^{1,2}(\tau) \\ A^{2,1}(\tau) & A^{2,2}(\tau) \end{pmatrix} = \exp\left(\tau \begin{pmatrix} M & 2Q^T Q \\ D & -M^T \end{pmatrix}\right) \tag{3.119}$$

see for example [55], section 3.4.2, or [51], proposition 7. The formula for $\mathcal{L}_{\Delta T, \Delta U}$ follows from (3.117)-(3.119) when we choose $n = 2$, $(z_1, z_2) = (z, w)$ in (3.113), and set $\tau = t - t_0$, $\Sigma_0 = \Sigma_{t_0}$ in (3.114). □

Chapter 4

A multifactor DTC jump model with dependence between the stochastic volatility and the jump rate

In this last chapter we attempt at introducing an asset pricing model that combines several of the mathematical and financial elements treated in this thesis: joint volatility/asset derivatives, time changes, jumps, martingale theory, stochastic volatility and stochastic jump arrival rate, and so on. The model is inspired by the one sketched in section 3.6, to which it reduces for a certain parameter choice, and makes use of a Wishart process to specify the stochastic activity rates of a certain underlying jump diffusion. The rates of activity, the instantaneous variance and the stochastic jump rate, shall be given by two different positive linear combinations of the entries of some given Wishart process. This allows to model dependence between three factors: the two activity rates and the Brownian component of the log-returns. At the same time, the Fourier/Laplace transforms from the model are known in closed form, implying that an analytical pricing theory based on integral inversions of such transforms is at hand. Consistently with the rest of the thesis, the approach of considering contracts written on an asset and its realized volatility has been maintained.

The added value of the modeling approach adopted in this chapter improving on that of section 3.6, is that by considering projections from a state space of an arbitrary dimension we can describe each of the two activity rates by means of several stochastic factors. This is what is known as a *multifactor specification* of the state variables. Recent empirical analyses (e.g. [9, 20, 26]) suggest that a multifactor model specification allows to generate a term structure in the volatility surface with varying levels and slopes, as opposed to single factor models that can only reproduce a certain slope for any given volatility level. The ability of the model to generate realistic forward smiles is essential to correctly price certain forward-starting derivative products. On the other hand, calibrating pure diffusive asset models typically leads to parameter estimations inconsistent with the observed volatility time series (Bates, [7]). Adding jump components to the log-returns is a popular way to correct this, as this generally relaxes the diffusion parameters. A number of models dealing with one or both of these issues have been suggested (see e.g. [6, 7, 20, 81]). Retaining both a multifactor structure and a jump component, our model is able of capturing both of these stylized facts. Also, the hypothesis of a mutual influence between the diffusive volatility and the jump arrival rate (see the introductory part of chapter 3) can be accommodated by our model. To our knowledge, a focus on this particular relationship has not been suggested in the previous literature, and its economic implications have never been explored before. What we argue here is that this property has possible positive implications for the management of the the volatility surface. Indeed, it is now well understood that the standard setup where jumps are independent from the driving Brownian motions does reduce the overall control of the correlation coefficient on the surface. That is, in models with independent continuous and jump parts, steepness in the short term slope is achieved at the cost of a loss of sensitivity of the surface with respect to the diffusion parameters, in particular with regards to correlation (for a full discussion see [26]). This is an apparent drawback of jump models: the correlation in a diffusive model is a variable with a clear economic meaning, i.e. the quantity of leverage a firm relies on, whereas the jump parameters lack tangible financial significance. Still, jumps might be needed to model steep short term smiles in the volatility surface. In this respect, the model presented here potentially reconnects the long and short part of the surface, meaning that both of these should actively respond to a leverage variation. The time changes we operate are obtained by a transformation of a common Wishart component; therefore, changes in any of the Wishart parameters will simultaneously affect the jump and the diffusive activities. In particular, the stochastic volatility/returns correlation also establishes an implicit dependence between the jumps and the Brownian motion of the returns, since the jumps and the stochastic volatility are themselves correlated. Ultimately, this means that a leverage variation should impact the short as well as the long term sections of the volatility surface. In section 4.5 we provide some numerical evidence supporting these ideas.

From a valuation perspective, we envisage that the model analyzed is best suited for the pricing of joint volatility and asset derivatives, in particular forward-starting ones, for the same reasons as those expressed by Grasselli and Marabel Romo in [54]. A constant correlation coefficient between volatility and price, as generally assumed in pricing models, is too much of a rigid feature for the pricing of such products. As a first shortcoming, models with a constant correlation are unable to produce forward starting smiles, necessary for the valuation of forward-starting payoffs. Secondly, the volatility/asset relation when the correlation is constant may induce a valuation bias. For example, in the case of a TVO in a model with constant correlation, a price decrease has a magnified impact on the derivative value, because a drop in prices would then necessarily imply a volatility soaring, pushing further down the derivative value. A varying correlation as assumed here removes this undesirable monotonic relationship, and it is less binding on the value fluctuation of variance-linked derivatives like the TVO.

Models with non-stationary jumps and stochastic volatility have been analyzed before in the works of Fang [37] and Huang and Wu [63]. In the former the two SDEs driving the activity rates are assumed to be independent. In the second work, although the asset pricing model is first stated in a general way, thus in principle allowing for a correlated pair of activity rates, the assumption of independence is later on introduced. A similar model is that of Santa Clara and Yan [88]; however, in their work the dynamics of the volatility are chosen as a displaced Gaussian process, making it to fall outside the exponentially-affine class. Furthermore, the algebra of the model is extremely complex, and the ODEs for the affine parts does not seem to be analytically solvable.

The intrinsic problem with these approaches is that there is no obvious way of obtaining an exponentially-affine closed formula for the Riccati system of ODEs when the SDEs for the activity rates are linearly correlated. As we shall see, by means of the Wishart process -which is *not* a standard multidimensional Ito diffusion- it is possible to reintroduce closed formulae in models with dependent jumps and stochastic volatility.

Prominent multifactor models are those of Christoffersen *et. al* [20], who study a continuous version of the model by Bates [7] focusing on the stochastic correlation between returns and volatility, and the Multi-Heston model, by da Fonseca *et al.* [28], using an n-dimensional Wishart process to model the stochastic variance. Also, a less recognized fact is that the model by Bates [7], further to retaining a bi-factor volatility specification, is characterized by a jump factor with intensity given by a linear functional of the stochastic volatility. This means that even if the Bates [7] model is one with a stochastic jump rate, it cannot generate nontrivial correlation with the diffusive activity.

On a more general level, a systematic study of multifactor jump diffusions of affine type is that of Leippold and Trojani [74]. Effectively, the model we present falls under the class they investigate, which essentially consists of a multivariate/multifactor version of the classic jump diffusion framework of Duffie *et al.* [31]. However, the authors do not focus on the issue of the correlation between stochastic variance and jump intensity, which is central to our discussion. Furthermore, they are unable to identify a full analytical solution for the model transforms when the jump arrival intensity is stochastic. By using the DTC machinery of chapter 3 we make transparent how these solutions can be derived.

The point of this short literature review is that the extant research presents models characterized by either analytical tractability or dependence between the activity rates in a multifactor specification, but not both. Our work bridges this gap by proposing an asset pricing model presenting both of these features.

The material is organized as follows. In the next section we give the dynamics of the asset in both its multifactor and DTC forms; in section 4.2 we compute the stochastic correlation and covariances between the various driving factors; in sections 4.3 and 4.4 we offer a transform analysis by using respectively the DTC and the Feynman/Kac approach. Section 4.5 consists of a series of numerical tests supporting the model, and section 4.6 is a brief recapitulation of the work done. As usual, proofs are placed in the appendix.

4.1 The Wishart process and the asset model

We begin by recalling the definition of a Wishart process (Bru, [12]). On a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, let B_t be a matrix Brownian motion of order n, i.e. an $n \times n$ matrix whose entries are independent Brownian motions under $\mathbb{P}$. Let Q belong to the set of the real invertible matrices and M be a matrix of order n with all negative eigenvalues. For $c > n + 1$, a Wishart process is the only strong solution of the matrix SDE:

$$d\Sigma_t = (cQ^T Q + M\Sigma_t + \Sigma_t M^T)dt + \sqrt{\Sigma_t} dB_t Q + Q^T (dB_t)^T \sqrt{\Sigma_t}. \tag{4.1}$$

The condition $c > n+1$ (the Gindkin condition) also ensures that no eigenvalue of Σ_t ever vanishes in finite time; further properties of the Wishart process are explored in e.g. [12, 49, 51] among the others.

Let now W_t be another matrix Brownian motion of order n: a correlation relationship between and W_t and B_t can be established in the following way. Let R be a matrix such that $\langle W^{k,i}, B^{l,j}\rangle_t = \delta_{kl} R_{i,j} t$ for all $i,j,k,l = 1,\ldots,n$. It is proved in the appendix that W_t can be written as

$$W_t = B_t R^T + Z_t \sqrt{\mathbb{1} - RR^T}, \tag{4.2}$$

where Z_t is any Brownian motion independent of B_t. Clearly, for (4.2) to be real-valued, constraints must be imposed on R. Further limitations are present if one wants to grant the analytical tractability (i.e. affinity) of the model: see [28] for a full characterization.

Starting from the Wishart process in (4.1) it is possible to naturally generate a pair of interdependent instantaneous activity processes as follow. Let $C = c_{i,j}$ and $D = d_{i,j}$ be two symmetric positive-semidefinite matrices. Then

$$v_t = Tr[\Sigma_t C] = Tr[C\Sigma_t] = \sum_{i,j=1}^{n} c_{i,j}\Sigma_t^{j,i} \tag{4.3}$$

and

$$u_t = Tr[\Sigma_t D] = Tr[D\Sigma_t] = \sum_{i,j=1}^{n} d_{i,j}\Sigma_t^{j,i} \tag{4.4}$$

are positive continuous processes. Therefore the pathwise integrals

$$T_t = \int_0^t v_s ds = \sum_{i,j=1}^{n} c_{i,j} \int_0^t \Sigma_s^{j,i} ds, \tag{4.5}$$

$$U_t = \int_0^t u_s ds = \sum_{i,j=1}^{n} d_{i,j} \int_0^t \Sigma_s^{j,i} ds \tag{4.6}$$

are valid time changes. In substance, by combining a linear operator and the trace operator, the activity rates are defined as two different projections of a common generating Wishart process on the space of the real-valued positive processes. In the following, we build a finite activity jump diffusion model of DTC-type based on such activity rates specifications.

Let X_t be a finite activity Lévy process. According to the theory of chapter 3, we can write $X_t = X_t^c + X_t^d$ where X_t^c is a driftless Brownian motion and X_t^d is a compound Poisson process of intensity 1. Consider the decoupled time change

$$X_{T,U} = X_{T_t}^c + X_{U_t}^d. \tag{4.7}$$

Now, by Dambis and Schwarz's theorem we have that $X_{T_t}^c = \sqrt{Tr[C\Sigma_t]} dX_t^c$; on the other hand, by Lévy's theorem one could take X_t^c as the Brownian motion:

$$dX_t^c = \frac{Tr[\sqrt{C\Sigma_t} dW_t]}{\sqrt{Tr[C\Sigma_t]}} \tag{4.8}$$

so that

$$X^c_{T_t} = Tr[\sqrt{C\Sigma_t} dW_t], \tag{4.9}$$

where equalities hold in law. Equation (4.9) provides a link between a multifactor volatility specification, achieved through the matrix Brownian motion W_t, and a time-changed specification that makes use of the scalar Brownian motion X^c_t. Note that the principal matrix square roots above exist, being all the matrices involved positive semi-definite.

Regarding the jump parts, the time-changed compound Poisson process $X^d_{U_t}$ can be written as:

$$X^d_{U_t} = \sum_{i=0}^{P_{U_t}} J_i. \tag{4.10}$$

The J_is are i.i.d random variables with density $f_J(x)$ and P_t is a Poisson process of parameter 1. This means that, conditionally on U_t, the arrival of a jump has a Poisson distribution of parameter U_t. Alternatively, $X^d_{U_t}$ can be characterized as the driftless pure jump process with discontinuous local characteristic $f_J(x)u_t dx dt$ (see the appendix of chapter 3 and references therein).

Assume now that the market trades a riskless security earning a constant interest rate r. A price process S_t can be now introduced by setting $Y_t = \log S_t$ to be the process whose risk-neutral dynamics are given by

$$dY_t = (r - Tr[C\Sigma_t]/2 - Tr[D\Sigma_t]\kappa)dt + \sqrt{Tr[C\Sigma_t]} dX^c_t + dX^d_{U_t}. \tag{4.11}$$

By (4.9) this is equivalent, in terms of the matrix W_t, to:

$$dY_t = (r - Tr[C\Sigma_t]/2 - Tr[D\Sigma_t]\kappa)dt + Tr[\sqrt{C\Sigma_t} dW_t] + dX^d_{U_t}. \tag{4.12}$$

The constant κ is the Lévy compensator for the jumps J, i.e. $\kappa = \phi_J(-i) - 1$, where ϕ_J is the characteristic function of J. By applying the results of the previous chapter, we have the following fundamental result:

Proposition 4.1.1. *The discounted price process $S_t = S_0 \exp(Y_t - rt)$ is a local martingale; thus, under the usual mild integrability conditions, S_t is an asset model consistent with the no-arbitrage pricing theory.*

We refer to equations (4.11) and (4.12) respectively as the *DTC representation* and the *multifactor representation* of Y_t.

4.2 Correlations in the model

The log-asset price process defined in (4.11) and (4.12) carries a very rich dependence structure between its components. It is possible, and of interest, to identify the correlations between several of the underlying drivers, namely:

- the correlation between the Brownian motion X^c_t and the *scalar* Brownian motions driving v_t;
- the correlation between the activity rates v_t and u_t;

- the covariances between the log-price process Y_t and each of the components of Σ_t.

The fundamental feature of these correlations is that they are *stochastic*. Ultimately, a stochastic correlation means that the model is able to satisfactorily fit a whole term structure of the volatility surface, rather than a single cross-section (see [20, 28]). The three points above are addressed by the following proposition:

Proposition 4.2.1. *Let w_t^c and w_t^d be the scalar Brownian motions driving respectively v_t and u_t. A stochastic process as in (4.11) and (4.12) is such that:*

$$d\langle X^c, w^c\rangle_t = \frac{Tr[C\Sigma_t\sqrt{C}RQ]}{\sqrt{Tr[C\Sigma_t]}\sqrt{Tr[C\Sigma_t CQ^TQ]}}dt; \tag{a}$$

$$d\langle w^c, w^d\rangle_t = \frac{Tr[C\Sigma_t DQ^TQ]}{\sqrt{Tr[C\Sigma_t CQ^TQ]}\sqrt{Tr[D\Sigma_t DQ^TQ]}}dt; \tag{b}$$

$$d\langle \Sigma^{i,j}, Y\rangle_t = 2(\Sigma_t\sqrt{C}RQ)^{i,j}dt. \tag{c}$$

The correlation in (a) retains the usual interpretation of the leverage effect between the asset returns and the stochastic volatility, and involves the matrix R. As it generally happens in multifactor models, this quantity is itself stochastic. The expression in (b) provides instead a measurement of the interaction between the drivers of the activity rates in terms of the Wishart matrix Q and of the projection matrices C and D. The covariation calculated in (c) will be of use in the analytical follow-up of the next sections.

4.3 Computing the characteristic function using the DTC approach

According to equation (4.11) the log-dynamics Y_t retain a DTC structure; therefore, all of the theory developed in chapter 3 applies to the present situation. In particular, the Fourier semi-closed pricing equation of theorem 3.4.1 for derivative pricing can be used in the model under scrutiny. In this section we give the explicit expression of the leverage-neutral characteristic function $\Phi_{t_0}(z,w)$ introduced in section 3.3 in the case of an asset whose discounted log-price follows $\tilde{Y}_t = Y_t - rt$, thus allowing for the valuation of full joint asset/volatility payoffs under this model.

Recall that the *leverage-neutral characteristic function* of $(Y_t, \langle Y\rangle_t)$ is given by

$$\Phi_{t_0}(z,w) = \mathbb{E}_{t_0}[e^{iz(\tilde{Y}_t - \tilde{Y}_{t_0}) + iw(\langle Y\rangle_t - \langle Y\rangle_{t_0})}] \tag{4.13}$$

for some z, w in the domain of definition $\mathcal{D} \subset \mathbb{C}^2$. According to the results of the previous chapter, $\Phi_{t_0}(z,w)$ can be written as a Laplace transform of T_t and U_t under an equivalent measure $\mathbb{Q}(z,w)$ that reflects the leverage. This produces an exponentially-affine representation for Φ_{t_0} that can be calculated according to the following proposition.

Proposition 4.3.1. *Let Y_t be the log-price process in the equivalent equations (4.11) and (4.12). The leverage-neutral characteristic function $\Phi_{t_0}(z,w)$ is given by*

$$\Phi_{t_0}(z,w) = \exp(-a(t-t_0) - Tr[A(t-t_0)\Sigma_{t_0}]), \tag{4.14}$$

where, for $\tau > 0$

$$A(\tau) = A^{2,2}(\tau)^{-1} A^{2,1}(\tau) \tag{4.15}$$

$$a(\tau) = \frac{c}{2} Tr(\log\left(A^{2,2}(\tau)\right) + (M^T + iz\sqrt{C}RQ)\tau) \tag{4.16}$$

with

$$\begin{pmatrix} A^{1,1}(\tau) & A^{1,2}(\tau) \\ A^{2,1}(\tau) & A^{2,2}(\tau) \end{pmatrix} = \exp\left(\tau \begin{pmatrix} M + izQ^T R^T \sqrt{C} & 2Q^T Q \\ L(z,w) & -(M^T + iz\sqrt{C}RQ) \end{pmatrix}\right), \tag{4.17}$$

$$L(z,w) = \alpha(z,w)C + \beta(z,w)D \tag{4.18}$$

$$\alpha(z,w) = iw - (z^2 + iz)/2 \tag{4.19}$$

$$\beta(z,w) = \phi_{J,J^2}(z,w) - iz\kappa - 1, \tag{4.20}$$

the quantity $\phi_{J,J^2}(z,w)$ *being the joint characteristic function of* J *and* J^2.

As already remarked, this explicit construction would have been effectively unfeasible if the activity rates followed a standard Ito diffusion. Indeed, in one such framework it would not have been possible to recover an analytical solution for the joint characteristic function when the equations are correlated, because no closed formula for the associated system of Riccati equations is available in that case. This is a fundamental result by Grasselli and Tebaldi, [55]. In other words, in a multivariate setting where the jump rate and the stochastic volatility follow separate SDEs, introducing a constant correlation constraint breaks down the analytical tractability of the model. In contrast, the Wishart process is a system of connected SDEs showing a more sophisticated interaction, whose characteristic function is nevertheless exponentially-affine and admits a closed formula for the affine coefficients.

In the next section we provide an alternative derivation of the joint conditional asset/realized volatility transform in the classic PDE framework, and link it back to the leverage-neutral characteristic function.

4.4 The infinitesimal generator and the characteristic function

The standard route to derivative pricing for stochastic model is finding the solution of the associated PDE, which can be in turn obtained through the derivation of the infinitesimal generator of the associated jump diffusion. In this section we compute the infinitesimal generator associated of the triplet (Y_t, Σ_t, I_t), which leads to the pricing PDE for a payoff of the form $F(Y_T, I_T)$, where I_t is the quadratic variation of Y_t.

In the following, for $i, j = 1 \ldots n$, let

$$\delta_{i,j} = \frac{\partial}{\partial x_{i,j}}. \tag{4.21}$$

We indicate δ the matrix whose entries are $\delta_{i,j}$. The operator $\partial/\partial y$ stands for the derivative with respect to the log-price coordinate and $\partial/\partial I$ is that taken with respect to the quadratic variation coordinate. We have the following result:

Proposition 4.4.1. *Let $f(x) : \mathbb{R}^{n^2+2} \to \mathbb{R}$ be a twice differentiable function with continuous second derivatives. The infinitesimal generator $\mathcal{A}$ of (Y_t, Σ_t, I_t) is given by*

$$\mathcal{A}f = \left(r - \frac{1}{2}Tr[C\Sigma_t] - Tr[D\Sigma_t]\kappa\right)\frac{\partial f}{\partial y} + Tr[C\Sigma_t]\frac{\partial f}{\partial I} + \frac{1}{2}Tr[C\Sigma_t]\frac{\partial^2 f}{\partial^2 y}$$
$$+ Tr[(M\Sigma_t + \Sigma_t M^T + cQ^TQ)\delta + 2\Sigma_t\delta Q^TQ\delta]f + 2Tr[\Sigma_t\sqrt{C}RQ\delta]\frac{\partial f}{\partial y}$$
$$+ Tr[D\Sigma_t]\int_{\mathbb{R}}(f(Y_{t_-} + x, \Sigma_t, I_{t_-} + x^2) - f(Y_{t_-}, \Sigma_t, I_{t_-}))f_J(x)dx. \tag{4.22}$$

Therefore the value function $f(Y_t, \Sigma_t, I_t, t)$ of a sufficiently regular contingent claim F maturing at time T is a solution of the free boundary problem on $[0, T] \times \mathbb{R}^{n^2+1}$

$$\mathcal{A}f = \frac{\partial f}{\partial t} \tag{4.23}$$

with terminal condition $f(Y_T, \Sigma_T, I_T, T) = F(Y_T, I_T)$.

As clarified in chapter 2, a solution of the Dirichlet problem (4.23) can be used to determine the characteristic function of the model. Ultimately, this provides an approach to the Fourier-inversion valuation theory alternative to to that of section 4.3. For example, a fast way to calculate the conditional characteristic function $\phi_{t_0}(z)$ of Y_t is to make the exponential ansatz

$$\phi_{t_0}(z) = \mathbb{E}_{t_0}[e^{izY_t + iwI_t}] = \exp(c(t) + b(t)Y_{t_0} + Tr[A(t)\Sigma_{t_0}] + d(t)I_{t_0}); \tag{4.24}$$

substituting $f = \phi_{t_0}(z, w)$ in equation (4.23) yields the differential relation

$$0 = -d'(t)I_{t_0} - Tr[A'(t)\Sigma_{t_0}] - b'(t)Y_{t_0} - c'(t) + (r - Tr[C\Sigma_{t_0}]/2 - Tr[D\Sigma_{t_0}]\kappa)b(t)$$
$$+ \frac{1}{2}Tr[C\Sigma_{t_0}]b^2(t) + Tr[(M\Sigma_{t_0} + \Sigma_{t_0}M^T + cQ^TQ)A(t) + 2\Sigma_{t_0}A(t)Q^TQA(t)]$$
$$+2Tr[\Sigma_{t_0}\sqrt{C}RQ]A(t)b(t) + d'(t)Tr[C\Sigma_{t_0}] + Tr[D\Sigma_{t_0}]\int_{\mathbb{R}}(e^{xb(t)+x^2d(t)} - 1)f_J(x)dx \tag{4.25}$$

with initial conditions $A(t_0) = 0$, $b(t_0) = iz$, $c(t_0) = 0$ and $d(t_0) = iw$. Matching the coefficients of Y_{t_0} and I_{t_0} yields $b(t) = iz$ and $d(t) = iw$. By equating those of Σ_{t_0}, we then have

$$A'(t) = A(t)M + (M^T + 2iz\sqrt{C}RQ)A(t) - 2A(t)Q^TQA(t) - \frac{(z^2 + iz - 2iw)}{2}C +$$
$$(\phi_{J,J^2}(z, w) - 1 - iz\kappa)D. \tag{4.26}$$

Finally, from the linear terms we obtain:

$$c'(t) = Tr[cQ^TQA(t)] + izr. \tag{4.27}$$

A straightforward check shows that equation (4.26) is equivalent to (4.55) in the appendix, and that (4.57) is the discounted version of (4.27) when $Y_{t_0} = I_{t_0} = 0$, as expected.

By comparing the formal arguments given in this and the previous section, as well as the corresponding proofs in the appendix, it is apparent that the DTC "probabilistic" approach requires less analytical work than the usual "PDE/Feynman Kac" approach.

4.5 Model specifications and testing

In this section we instantiate our equations and test them against known models. In first place, we conduct a leverage sensitivity analysis showing that the proposed model performs better than the standard existing ones in terms of response of the volatility surface to a leverage variation. Secondly, we consider a true multifactor specification and compare some associated volatility skews to those derived by other jump and multifactor asset price evolutions.

4.5.1 Leverage sensitivity

As stated at the beginning of this chapter, the philosophy behind the presented model with correlated stochastic volatility and jump activity (CSVJA) is to be able to reproduce variations in the skew of the volatility surface by employing a global parameter affecting the short as well as the long term parts of the volatility surface. This is typically not feasible through models with independent jumps and stochastic volatility, where the jumps dominate the short term part and the stochastic volatility the long term one. We denote by ρ the usual correlation parameter between the log-returns and the Brownian activity of the stochastic variance. In the following we compare the variation of the volatility skew with respect to ρ in some existing models with that of the one presented: we expect to find a greater variation in the latter.

Examples of models featuring a stochastic jump rate are the one by Fang [37] and that by Huang and Wu [63]. Both of these can be embedded in the following representation:

$$\begin{aligned} dS_t &= rS_{t^-}dt + \sqrt{v_t}S_{t^-}dW_t^1 + S_{t^-}(\exp(J)-1)dN_t - \kappa\lambda_t S_{t^-}dt; \\ dv_t &= \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2; \\ d\lambda_t &= \alpha_\lambda(\theta_\lambda - \lambda_t)dt + \eta_\lambda\sqrt{\lambda_t}dW_t^3. \end{aligned} \tag{4.28}$$

where κ and r and retain the usual meaning, J is normally distributed $\mathcal{N}(\log(1+\kappa) - \delta^2/2, \delta)$, N_t is a jump process like those from equations (4.12)-(4.11), and we have that $\langle W^1, W^2\rangle_t = \rho t$. Fang assumed W_t^3 to be independent of the two other Brownian motions, whereas Huang and Wu also take $\langle W^1, W^3\rangle_t = t\rho_J$, but leave arbitrary the jump distribution. The Bates model can also be recovered from the above by just voiding the third equation.

Next, we observe that our CSVJA model can be instantiated to both the Bates and Fang model by an appropriate matrix specification. We choose a pair of jump parameters κ, δ in the Normal distribution, and set the following matrix parametrization for the CSVJA model:

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2. \end{pmatrix} \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} Q_1 & Q_0 \\ Q_0 & Q_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.29}$$

The choice of the matrices C and D ensures that $\Sigma_t^{1,1}$ and $\Sigma_t^{2,2}$ play the roles of v_t and λ_t respectively. If $Q_0 = Q_2 = M_2 = 0$ one has that the CSVJA model is equivalent to a Bates model of parameters

$$\alpha = -2M_1, \quad \theta = -\frac{bQ_{11}^2}{2M_1}, \quad \rho = R_1, \quad \eta = 2Q_{11}, \quad v_0 = \Sigma_1, \quad \lambda = \Sigma_2. \tag{4.30}$$

Similarly, specifications of the Fang model can be obtained by the parametrization (4.29) after having set $Q_0 = 0$, when:

$$\alpha = -2M_1, \quad \theta = -\frac{bQ_{11}^2}{2M_1}, \quad \eta = 2Q_{11}, \quad v_0 = \Sigma_1,$$

$$\alpha_\lambda = -2M_2, \quad \theta_\lambda = -\frac{bQ_{22}^2}{2M_2}, \quad \eta_\lambda = 2Q_{22}, \quad \lambda_0 = \Sigma_2, \quad \rho = R_1.$$

Also, it is easy to see that for the choice (4.29), the price process exactly coincides with the model introduced in section 3.6. Its activity and stochastic variance marginals are of pure CIR type, while their joint distribution is determined by the correlation process (3.82), also obtainable by substituting the values of C and D above in equation (b). Furthermore, the usual correlation between the Brownian motions of the activity rates and that driving the log-returns is in place, as given by equation (a) reducing to (3.83) under the current choice for the projection matrices.

In order to assess the effects of introducing a stochastic correlation between the activity rates on the leverage sensitivity of the short-term skew, we perform the following test. We select the entries in (4.29) as:

$$\begin{gathered} M_1 = M_2 = -0.33, \quad Q_{11} = Q_{22} = 0.25, \quad Q_0 = 0.3, \quad \Sigma_1 = \Sigma_2 = 0.01, \quad b = 3, \\ \kappa = -0.2, \qquad \delta = 0.3. \end{gathered} \tag{4.31}$$

In figures 4.1 to 4.3 we provide the 3-month volatility smiles extracted from the analytical call option prices for the Bates, Fang and CSVJA models specifications above. The values of ρ analyzed are 0 and -0.6. By equation (a), in order to work back this value of ρ for the CSVJA model, we have to take $R_1 = -0.7$. As well-expected, for both the Bates and the Fang the leverage variation has little impact on the volatility skew, especially for deep in-the-money call options. However, for the CSVJA model, a wider gap is shown. Since the marginal distributions of the activity rates for the CSVJA model are the same of the Fang model, the only possible reason explaining the additional skewness of the CSVJA model is the presence of the stochastic correlation (b)-(3.82) introduced by means of the extra volatility term $Q_0 \neq 0$. The effect of this "second" correlation on the skew can be intuitively explained as follows. Because of the leverage effect, as prices go down the volatility spikes up, generating negative skewness in the returns distribution. But now the volatility is correlated with the jump activity through the mean-reverting process $\Sigma_t^{1,2}$ which has positive mean reversion level. Therefore, when the volatility increases, the jump intensity is likely to increase as well, and hence so does the probability of observing a (negative on average) jump. The latter contributes to the negative skewness of the asset returns and thus reinforces the negative skew of the volatility smile.

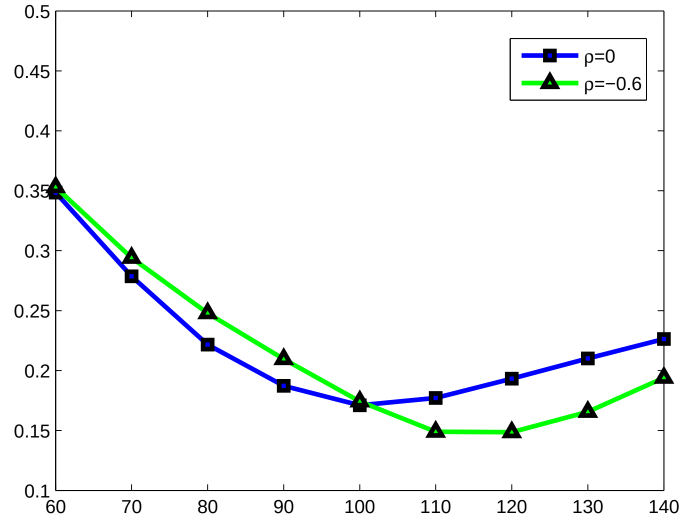

Figure 4.1: Comparison of the 3-month volatility skew in the Bates model for two different values of ρ. There is only a small difference in the skew of the two curves.

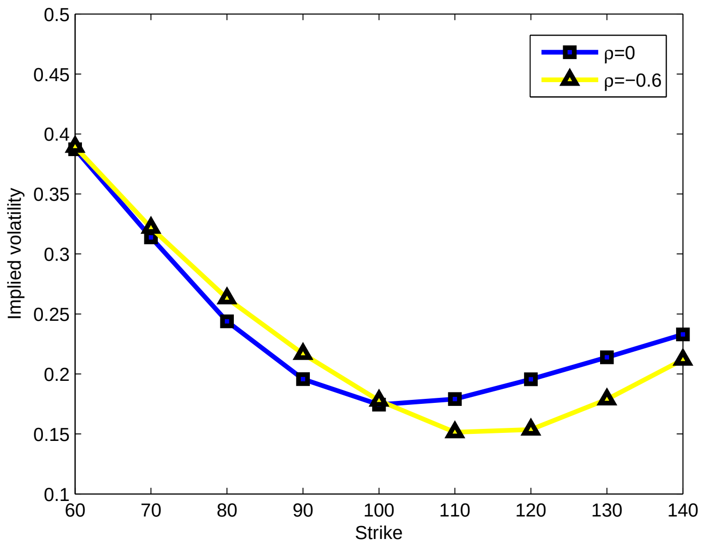

Figure 4.2: Comparison of the 3-month volatility skew in the Fang model for two different values of ρ. The situation is very similar to that of figure 4.1.

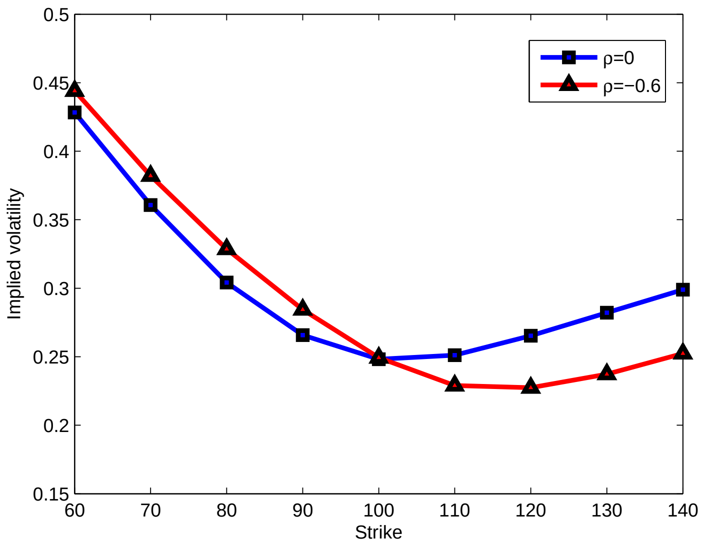

Figure 4.3: Comparison of the 3-month volatility skew in the CSVJA model for two different values of ρ. The negative skew increases much more than in the other two models.

4.5.2 Multifactor analysis

Even though the CSVJA model makes use of a multivariate process to model the driving factors, in its simplest form of the previous subsection it is not a true multifactor model. Effectively, we are using the individual one-dimensional factors $\Sigma_t^{1,1}$ and $\Sigma_t^{2,2}$ to model the diffusive and jump activities, and the off-diagonal term $\Sigma_t^{1,2}$ only impacts the correlation between them. In this section we instead analyze a true multifactor specification, by assuming that the dynamics of Y_t are driven by a 3-dimensional Wishart process. Two of the diagonal factors, say $\Sigma_t^{1,1}$ and $\Sigma_t^{2,2}$, are taken to model the multifactor structure: one will impact the short term part and the other one the long term part of the volatility surface. The factor $\Sigma_t^{3,3}$ will instead dictate the jump intensity of the process. We refer to this model as a *multifactor jump model with correlated stochastic volatility and jump activity* (MCSVJA).

We shall choose the matrices as follows:

$$M = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & \Sigma_3 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}. \tag{4.32}$$

The jump size distribution parameters are the same as in the previous section. We see that $\Sigma_t^{1,1} + \Sigma_t^{2,2}$ drives the stochastic variance and $b\Sigma_t^{3,3}$ the stochastic jump rate. Under the current choice of C, the possible different values of M_1 and M_2 imply that the process $Tr[C\Sigma_t]$ driving the stochastic volatility comprises of a short-term factor $\Sigma_t^{1,1}$ and a long-term one $\Sigma_t^{2,2}$, each responsible for generating the corresponding parts of the surface. Further, we can explicitly compute the correlations (a) and (b) as:

$$d\langle w^c, X^c\rangle_t = R_1 \frac{\Sigma_t^{1,1}Q_{11} + \Sigma_t^{1,2}(Q_{12}+Q_{21}) + \Sigma_t^{2,2}Q_{22}}{\sqrt{\Sigma_t^{1,1}+\Sigma_t^{2,2}}\sqrt{\Sigma_t^{1,1}Q_{11} + \Sigma_t^{1,2}(Q_{12}+Q_{21}) + \Sigma_t^{2,2}Q_{22}}}dt. \tag{4.33}$$

and

$$d\langle w^c, w^d\rangle_t = \frac{\Sigma_t^{1,3}q_{13} + \Sigma_t^{2,3}q_{23}}{\sqrt{\Sigma_t^{1,1}q_{11} + 2\Sigma_t^{1,2}q_{12} + \Sigma_t^{2,2}q_{22}}\sqrt{b\Sigma_t^{3,3}q_{33}}}dt. \tag{4.34}$$

where

$$\begin{aligned} q_{11} &= Q_{11}^2 + Q_{21}^2 + Q_{31}^2 \\ q_{12} &= Q_{11}Q_{12} + Q_{21}Q_{22} + Q_{31}Q_{32} \\ q_{22} &= Q_{12}^2 + Q_{22}^2 + Q_{32}^2 \\ q_{13} &= Q_{11}Q_{13} + Q_{21}Q_{23} + Q_{31}Q_{33} \\ q_{23} &= Q_{12}Q_{13} + Q_{22}Q_{23} + Q_{32}Q_{33} \\ q_{33} &= Q_{13}^2 + Q_{23}^2 + Q_{33}^2. \end{aligned} \tag{4.35}$$

From the above we deduce that the elements $Q_{i,j},\ i,j = 1,2$ affect the correlation between the Brownian components of the log-returns and that of the underlying Wishart factors, whereas Q_{13}, Q_{23}, Q_{31} and Q_{32} act on the relationship between the activity rates only.

What we want to do is to visually compare the volatility skews arising from a multifactor diffusive model and a single factor jump model with our MCSVJA model. It is well-known that jump models show a strong convexity in the short term smile and a flat term structure in the long term part of the surface; on the other hand, a diffusive multifactor jump model should have a less steep short term skew but the convexity of the smile persists for longer time-to-maturities. In principle, the MCSVJA should feature both of these properties. We expect the short term part of volatility surface from this model to behave like a jump model and the long term part like a multifactor one.

The pure diffusive model in the literature closest to ours is the multifactor Heston model (MH) of da Fonseca *et al.* [28], which is obtainable by simply setting $b = 0$ in (4.32); instead, a full MCSVJA parametrization is given when $b = 1$. A specification of the Bates mode can also embedded in (4.32) by letting again $b = 1$ and specifying as the only non-vanishing elements in the matrices M and Q to be respectively M_1 and Q_{11}. The full parametrization is taken as:

$$M = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -0.33 & 0 \\ 0 & 0 & -0.33 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{pmatrix},$$

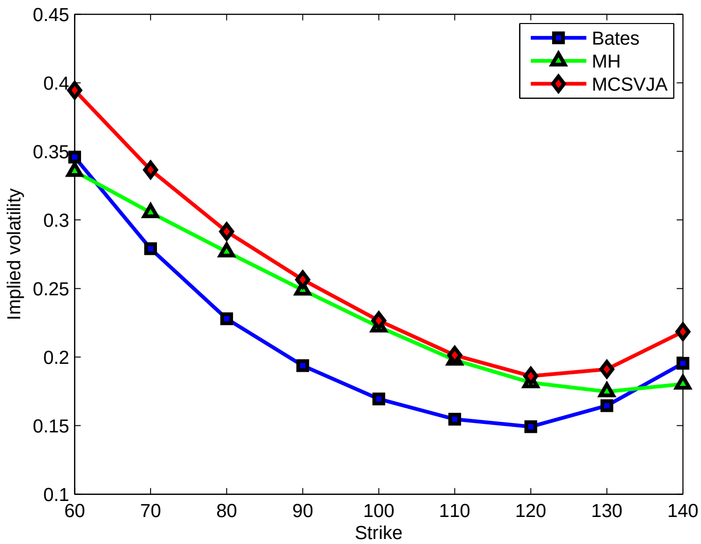

Figure 4.4: Comparison of the 3-months volatility skew from the Bates, MH and MCSVJA models instantiated by (4.36). The skew of the MCSVJA model is similar to that of the Bates model, a typical shape of a skew generated by a jump diffusion model.

$$R = \begin{pmatrix} -0.7 & 0 & 0 \\ 0 & -0.7 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{pmatrix}. \tag{4.36}$$

Hence, by assuming Q to be diagonal we are for the purposes of this test reverting to the case of independent rates. The mean reversion parameter M_1 affects the short term section of the volatility surface, and M_2 the long term one.

In figures 4.4 and 4.5 are shown respectively the 3 and 18-month volatility skews extracted from the analytical call option prices from the Bates, MH and MCSVJA parametrizations. We notice that the skew of the 3-month volatility smile of the MCSVJA model is sensibly more pronounced than that of the MH model, and resembles to that of the Bates model, up to a level shift. However, as time-to-maturity progresses, the shape of the long term smile departs from that of the Bates model and becomes instead similar to that of the MH model, which is more skewed for this maturity range (figure 4.5). The reason of this happening is that, as commonly reported, the jumps have little effect on the surface for longer maturities, where the volatility surface is instead more deeply influenced by the mean reversion of the long term diffusive component. Thus, in some sense, up to a shift of the surface level, the MCSVJA "interpolates" between the short term skew of the Bates model and the long term one of the MH model.

4.6 Conclusions and further work

To summarize the discussion so far, in this final chapter we have proposed and carried out an analysis of a potential financial asset pricing model featuring many desirable characteristics. In particular, the following properties of the model are widely documented to be able to capture certain stylized facts:

- stochastic volatility;

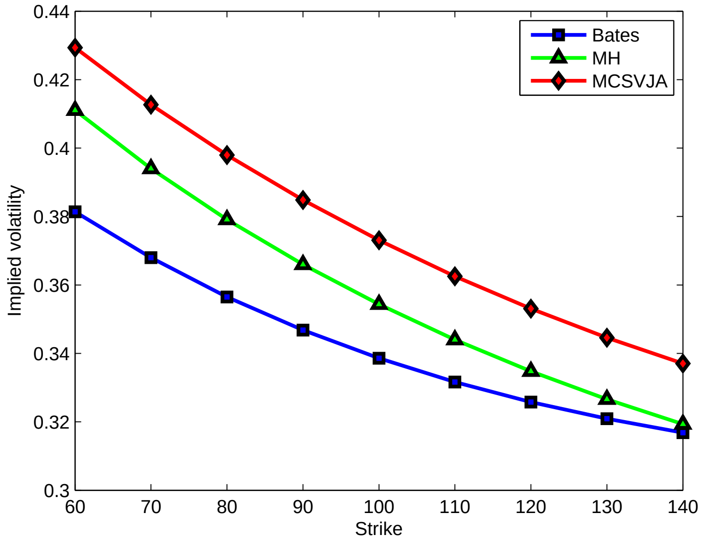

Figure 4.5: Comparison of the 18-months volatility skew from the Bates, MH and MCSVJA models instantiated by (4.36). The MCSVJA skew looks like that of the MH model, since both retain the same diffusive specification and the jump component in the MCSVJA does not have a strong impact on implied volatilities here. The volatility skew of the Bates model has been shifted 15% upwards.

- jumps;
- a multivariate specification of the stochastic volatility;
- stochastic correlation between the log-returns and the stochastic volatility (leverage effect).

In addition, the model retains a stochastic correlation dependency between the Brownian motion driving the stochastic volatility and that driving the stochastic jump rate: to our knowledge this is a novel feature, and it may be worth further research. Despite its apparent complexity, the model is not only analytically tractable, but also financially significant. Indeed, it combines the ability of different models to reproduce many features of the market data, such as the short term steepness of the smile for short maturities, the non-stationarity of the returns and the term structure of the volatility surface. Moreover, numerical tests confirm the intuition that we can shape the (M)CSVJA model in such a way as to generate skews that are more sensitive to a leverage change compared to those from jump models with independent activities. This is an effect of having introduced a correlated pair of activity rates indirectly connecting the jump arrival rate with the log-returns. Finally, when multifactoriality of the driving factors is modeled, the proposed asset evolution can capture both a steep short term volatility skew and a forward volatility smile, thus combining features of both jump and multifactor models.

Overall, the discussion so far supports the view that the model presented is a good candidate to overcome some of the rigidities in the volatility surface management presented by the existing models. However, much more testing has to be done in order to fully understand the effects of an interaction between the activity rates, including a proper empirical study based on real market data.

Appendix: proofs

Proof that W_t is a matrix Brownian motion. By using the characterization of a matrix Brownian motion (see e.g. Gouriéroux [51])

$$\mathbb{E}[(W_t\alpha)(W_t\beta)^T] = t\alpha^T\beta\mathbb{I}, \tag{4.37}$$

for all $\alpha, \beta \in \mathbb{R}^n$, we have:

$$\begin{aligned}\mathbb{E}\left[W_t\alpha, W_t\beta\right] =& \mathbb{E}[(B_tR^T\alpha)(B_tR^T\beta)^T] + \mathbb{E}[(Z_t\sqrt{\mathbb{I} - RR^T}\alpha)(Z_t\sqrt{\mathbb{I} - RR^T}\beta)^T] \\ =& t\alpha^T RR^T\beta\mathbb{I} + t\alpha^T(\mathbb{I} - RR^T)\beta\mathbb{I} = t\alpha^T\beta\mathbb{I}.\end{aligned} \tag{4.38}$$

and thus W_t is a matrix Brownian motion. Furthermore, for all $i, j, k, l = 1, \ldots, n$:

$$\langle W^{k,i}, B^{l,j}\rangle_t = \langle\sum_{h=1}^{n} B^{k,h}R^{i,h}, B^{l,j}\rangle_t = \delta_{kl}R_{i,j}t. \tag{4.39}$$

□

Proof of proposition 4.1.1. Apply proposition 3.2.3 to equation (4.11). □

In some of the proofs that follow, the lemma below will be useful:

Lemma 4.6.1. *For any $A, B \in M_n(\mathbb{R})$ and a matrix Brownian motion Z_t we have*

$$\langle Tr[AdZ], Tr[BdZ]\rangle_t = Tr[AB^T]dt. \tag{4.40}$$

Proof. We have

$$\langle Tr[AdZ], Tr[BdZ]\rangle_t = \left\langle \sum_{i,k} a_{i,k}dW^{k,i}, \sum_{i,h} b_{i,h}dW^{h,i} \right\rangle_t = \sum_{i,k} a_{i,k}b_{i,k}dt = Tr[AB^T]dt. \tag{4.41}$$

□

Proof of proposition 4.2.1. For (a) notice that

$$dTr[C\Sigma_t] = F_t dt + 2\sqrt{Tr[C\Sigma_t CQ^TQ]}dw_t^c \tag{4.42}$$

where, by the Lévy characterization theorem

$$dw_t^c = \frac{Tr[C\sqrt{\Sigma}_t dB_t Q]}{\sqrt{Tr[C\Sigma_t CQ^TQ]}}, \tag{4.43}$$

is the Brownian motion driving $Tr[C\Sigma_t]$. Setting $K = (\sqrt{Tr[C\Sigma_t CQ^TQ]}\sqrt{Tr[C\Sigma_t]})^{-1}$, we then have that the equality

$$d\langle X^c, w^c\rangle_t = K\langle Tr[C\sqrt{\Sigma}dBQ], Tr[\sqrt{C\Sigma}dBR^T]\rangle_t = KTr[C\Sigma_t\sqrt{C}RQ]dt \tag{4.44}$$

easily follows from lemma 4.6.1 and the elementary properties of the trace operator.

Regarding (b) we have that

$$w_t^d = \frac{Tr[D\sqrt{\Sigma}_t dB_t Q]}{\sqrt{Tr[D\Sigma_t DQ^T Q]}} \tag{4.45}$$

is the planar Brownian motion driving $Tr[D\Sigma_t]$. Therefore, after letting

$$H = \left(\sqrt{Tr[C\Sigma_t CQ^T Q]}\sqrt{Tr[D\Sigma_t DQ^T Q]}\right)^{-1} \tag{4.46}$$

lemma 4.6.1 produces

$$\begin{aligned} d\langle w^c, w^d\rangle_t =& H\left\langle Tr[D\sqrt{\Sigma}dBQ], Tr[C\sqrt{\Sigma}dBQ]\right\rangle_t \\ =& \frac{Tr[C\Sigma DQ^T Q]}{\sqrt{Tr[C\Sigma_t CQ^T Q]}\sqrt{Tr[D\Sigma_t DQ^T Q]}}dt. \end{aligned} \tag{4.47}$$

Finally, for (c), let $e^{i,j}$ be the matrix having 1 as the (i,j)-th element and 0 everywhere else. With the help of a final application of lemma 4.6.1, we calculate:

$$\begin{aligned} d\langle \Sigma_t^{i,j}, Y\rangle_t &= \langle dTr[e^{j,i}\Sigma], Tr[\sqrt{C\Sigma}dW]\rangle_t \\ &= \langle Tr[e^{j,i}d\Sigma], Tr[\sqrt{C\Sigma}dW]\rangle_t \\ &= 2\langle Tr[e^{j,i}\sqrt{\Sigma}dBQ], Tr[\sqrt{C\Sigma}dBR^T]\rangle_t \\ &= 2Tr[Qe^{j,i}\Sigma_t\sqrt{C}R]dt \\ &= 2(\Sigma_t\sqrt{C}RQ)^{i,j}dt. \end{aligned} \tag{4.48}$$

□

Proof of proposition 4.3.1. Proposition 3.3.2 implies that we can write:

$$\begin{aligned} \Phi_{t_0}(z,w) =& \mathbb{E}_{t_0}^{\mathbb{Q}(z,w)}\left[\exp\left(\alpha(z,w)(T_t - T_{t_0}) + \beta(z,w)(U_t - U_{t_0})\right)\right] \\ =& \mathbb{E}_{t_0}^{\mathbb{Q}(z,w)}\left[\exp\left(\alpha(z,w)\int_{t_0}^t Tr[C\Sigma_s]ds + \beta(z,w)\int_{t_0}^t Tr[D\Sigma_s]ds\right)\right] \\ =& \mathbb{E}_{t_0}^{\mathbb{Q}(z,w)}\left[\exp\left(Tr\left[\int_{t_0}^t (\alpha(z,w)C + \beta(z,w)D)\Sigma_s ds\right]\right)\right] \end{aligned} \tag{4.49}$$

We must now apply the Girsanov theorem to find the change of parameters in the distributions needed push back the change to the measure $\mathbb{Q}(z,w)$ to the original risk-neutral measure. Since the Wishart process is a continuous process, the measure change can be reduced to one with respect to a measure $\mathbb{Q}(z)$ depending only on z. This is because when changing to the leverage-neutral measure, the jump parameters in the measure change martingale induce no change in the activity rates distribution when applying Girsanov's theorem, being continuous processes by definition uncorrelated with every jump process. As a consequence, the relevant Radon-Nikodym derivative in proposition 3.3.2 is

$$\frac{d\mathbb{Q}(z)}{d\mathbb{P}} = \mathcal{E}\left(iz\int_0^t \sqrt{Tr[C\Sigma_s]}dX_s^c\right). \tag{4.50}$$

$\mathcal{E}$ being the usual stochastic exponential. Using lemma 4.6.1, for all $i,j = 1,\dots,n$ gives the correlations:

$$\begin{aligned}
&d\left\langle B^{i,j}, \int_0^{\cdot} \sqrt{Tr[C\Sigma_s]}dX_s^c, \right\rangle_t = d\left\langle B^{i,j}, \int_0^{\cdot} Tr[\sqrt{C\Sigma_s}dW_s]\right\rangle_t \\
=&\langle Tr[e^{j,i}dB], Tr[\sqrt{C\Sigma}dW]\rangle_t = \langle Tr[e^{j,i}dB], Tr[\sqrt{C\Sigma}dBR^T]\rangle_t \\
=&Tr[e^{j,i}\sqrt{\Sigma_t C}R]dt = (\sqrt{\Sigma_t C}R)^{i,j}dt.
\end{aligned} \tag{4.51}$$

Therefore, Girsanov's theorem applied to the Brownian motion matrix B_t guarantees that under the measure $\mathbb{Q}(z)$ the process

$$dB_t^z := dB_t - iz\sqrt{\Sigma_t C}Rdt \tag{4.52}$$

is a matrix Brownian motion. By solving for B_t in the equation for the Wishart process we obtain that the $\mathbb{Q}(z)$-dynamics of Σ_t^z are those of a Wishart process of matrices $Q^z = Q$ and

$$M^z = M + izQ^T R^T \sqrt{C}. \tag{4.53}$$

The transform in (4.13) therefore equals

$$\Phi_{t_0}(z,w) = \mathbb{E}_{t_0}\left[\exp\left(Tr\left(\int_{t_0}^t (\alpha(z,w)C + \beta(z,w)D)\Sigma_s^z ds\right)\right)\right] \tag{4.54}$$

which is analytically computable for any given $z, w \in \mathcal{D}$: see for example [28, 49, 51]. In particular, the function $\Phi_{t_0}(z,w)$ is exponentially-affine of the form (4.14) with $A(\tau)$ and $a(\tau)$ satisfying, for $\tau > 0$:

$$\begin{aligned}
&A'(\tau) = A(\tau)M^z + (M^z)^T A(\tau) - 2A(\tau)Q^T QA(\tau) + L(z,w) = \\
&A(\tau)(M + izQ^T R^T\sqrt{C}) + A(\tau)(M^T + iz\sqrt{C}RQ) - 2A(\tau)Q^T QA(\tau) + L(z,w), \quad (4.55)\\
&A(0) = 0, \quad (4.56)
\end{aligned}$$

$$a'(\tau) = Tr[cQ^T QA(\tau)], \quad a(0) = 0. \tag{4.57}$$

Applying e.g. [49], proposition 9, finally yields equations (4.15)-(4.17). □

Proof of proposition 4.4. The Ito differential of the realized volatility I_t of Y_t is given by

$$dI_t = Tr[C\Sigma_t]dt + \Delta Y_t^2. \tag{4.58}$$

Therefore, the linear term of the infinitesimal generator due to the quadratic variation equals $Tr[C\Sigma_t]$; for convenience we set $X_t = (Y_t, \Sigma_t, I_t)$ so that by the above $\Delta X_t = (\Delta Y_t, 0, \Delta Y_t^2)$.

The quadratic term corresponding to the process Σ_t is, according to the calculation in Bru

[12]:

$$Tr[(M\Sigma_t + \Sigma_t M^T + cQ^T Q)\delta + 2\Sigma_t \delta Q^T Q\delta]f(x). \tag{4.59}$$

The coefficient of the quadratic cross-term due to ∂y and δ follows from proposition 4.2.1 (c), and equals

$$2Tr[\Sigma_t\sqrt{C}RQ\delta]\frac{\partial}{\partial y}. \tag{4.60}$$

All of the remaining terms are trivial. By using the definition of infinitesimal generator we then have

$$\begin{aligned}\mathcal{A}f = \lim_{h\to 0}\frac{\mathbb{E}_t[f(X_{t+h})|X_t = x] - f}{h} = (r - \frac{1}{2}Tr[C\Sigma_t] - Tr[D\Sigma_t]\kappa)\frac{\partial f}{\partial y} + \\ + Tr[D\Sigma_t]\frac{\partial f}{\partial I} + \frac{1}{2}Tr[C\Sigma_t]\frac{\partial^2 f}{\partial^2 y} + Tr[(M\Sigma_t + \Sigma_t M^T + cQ^T Q)\delta + 2\Sigma_t\delta Q^T Q\delta]f \\ + 2Tr[\Sigma_t\sqrt{C}RQ\delta]\frac{\partial f}{\partial y} + \lim_{h\to 0}\frac{1}{h}\mathbb{E}_t\left[\sum_{t\le s}^{t+h} f(X_{s_-} + \Delta X_s) - f(X_{s_-})\right].\end{aligned} \tag{4.61}$$

Regarding the jump part, we can write the incremental process above as a random integral with respect to some jump measure $\rho(ds \times dz)$ in the following way:

$$\sum_{t\le s}^{t+h} f(X_{s_-} + \Delta X_s) - f(X_{s_-}) = \int_t^{t+h}\int_{\mathbb{R}} (f(Y_{s_-} + z, \Sigma_s, I_{s_-} + z^2) - f(X_{s-}))\rho(ds \times dz). \tag{4.62}$$

By definition the compensating measure of ρ is the unique predictable random measure μ such that

$$\int_0^t\int_{\mathbb{R}} (f(Y_{s_-} + z, \Sigma_s, I_{s_-} + z^2) - f(X_{s-}))(\rho - \mu)(ds \times dz) \tag{4.63}$$

is a local martingale. By Kallsen and Shiryaev [70], lemma 2.6 (and as explained in the appendix of chapter 3), in the present situation we have that $\mu(dt \times dz) = dT_t\nu(dz)$. Here $\nu(dz)$ is the Lévy measure associated with a compound Poisson process of intensity 1 and jump distribution J, and therefore $\mu(dt \times dz) = Tr[D\Sigma_t]f_J(z)dtdz$. Thus, by an application of Fubini's theorem we obtain:

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}\int_{\mathbb{R}}\mathbb{E}_t\left[\int_t^{t+h}(f(Y_{s_-} + z, \Sigma_s, I_{s_-} + z^2) - f(X_{s-})\rho(ds \times dz)\right] \\ &= \int_{\mathbb{R}}\lim_{h\to 0}\frac{1}{h}\mathbb{E}_t\left[\int_t^{t+h} Tr[D\Sigma_s](f(Y_{s_-} + z, \Sigma_s, I_{s_-} + z^2) - f(X_{s-}))ds\right] f_J(z)dz \\ &= Tr[D\Sigma_t]\int_{\mathbb{R}}(f(Y_{t_-} + z, \Sigma_t, I_{t_-} + z^2) - f(X_{t-}))f_J(z)dz.\end{aligned} \tag{4.64}$$

Under the usual exponential decay condition on f_J and by choosing f as growing slow enough, the last term is finite, so that the interchange between the integral and the expectation is justified.

□

Conclusion

This PhD thesis revolves around the interactions between price and volatility, from both a valuation and asset modeling viewpoints. The single topics we dealt with in form of separate projects/journal papers eventually combined to provide the background material for the stochastic model of chapter 4. These are, in our view, the main contributions of our studies.

We begun by reporting the first comprehensive mathematical study of the target volatility option, a financial derivative that serves as a model for certain type of investments aiming at controlling the investment risk by using the realized volatility as a correction factor.

As a second step, we rigorously tackled the problem of pricing an arbitrary joint asset and volatility contingent claim. The corresponding PDE turned out to be analytically solvable in terms of the characteristic function, so long as the underlying stochastic model for the asset price is. In doing so, we extended the well-known Fourier inversion pricing equations to joint asset and volatility derivatives.

In the attempt of extending this strain of research to Lévy models, we focused on time-changed processes. We contributed to this theory by showing that it is possible to separately apply time changes to the Brownian and jump part of a Lévy random evolution and still obtain processes suitable for equity modeling: we christened these *DTC Lévy processes*. This has been achieved by providing a line of analysis relying on the martingale representation theory, which we believe to be the correct theoretical framework corroborating the consistency of time-changed models with the risk-neutral pricing theory. Furthermore, we extended the existing transform analysis techniques for standard time-changed-based exponentials to DTC ones.

This modeling idea broke the ground for the introduction of a new class of models with semi-closed pricing formulae, those with coupling between the jump and diffusive activity. We took into account one instance of such models, found analytical pricing equations based on it, and performed some preliminary numerical testing suggesting that this model retains some desirable properties for the financial practice.

The topics we have dealt with in this work represent in our view a relatively unexplored and potentially large area of research. More financial and mathematical insight is necessary for the TVO, in particular with regards to hedging. It would also be interesting to deepen the understanding of certain mixed equity and volatility payoffs, for example those introduced in chapter 2, and see if there is any demand for such products in the financial markets. Moreover, the target volatility strategy/asset allocations, nowadays a rather popular investment solution, require more accurate statistical and mathematical analysis.

In terms of asset modeling the decoupled time-changed approach is a promising methodology for the introduction of new models. The model in chapter 4 certainly needs a great more deal of empirical analysis, part of which is currently under way. Furthermore, it is envisaged that decoupled time changing could also be helpful in designing a realistic multi-asset market framework. We hope to answer some of these points in future research.

Bibliography

[1] Abate, J., and Whitt, W. Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal on Computing 7* (1995), 36–43.

[2] Abramowitz, M., and Stegun, I. A. *Handbook of Mathematical Functions*. Dover, 1970.

[3] Andersen, L. B. G., and Piterbarg, V. V. Moment explosions in stochastic volatility models. *Finance and Stochastics 11*, 1 (2007), 29–50.

[4] Ane, T., and Geman, H. Order flow, transaction clock, and normality of asset returns. *The Journal of Finance 55* (2000), 2259–2284.

[5] Barndorff-Nielsen, O. E. Processes of normal inverse Gaussian type. *Finance and Stochastics 2* (1997), 41–68.

[6] Bates, D. S. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. *Review of Financial Studies 9* (1996), 69–107.

[7] Bates, D. S. Post-'87 crash fears in the S&P 500 futures option market. *Journal of Econometrics 14*, 1 (2000), 181–238.

[8] Bergman, Y. Z., Grundy, B. D., and Wiener, Z. General properties of option prices. *Journal of Finance 51*, 5 (1996), 1573–1610.

[9] Bergomi, L. Simle dynamics I & II. *Risk 94* (2005).

[10] Breeden, D., and Litzenberger, R. Prices of state contingent claims implicit in options prices. *Journal of Business 51* (1978), 621–651.

[11] Broadie, M., and Kaya, O. Exact simulation of stochastic volatility and other affine jump processes. *Operations Research 54*, 2 (2006), 217–231.

[12] Bru, M. F. Wishart processes. *Journal of Theoretical Probability 4* (1991), 725–743.

[13] Carr, P., Geman, H., Madan, D. B., and Yor, M. The fine structure of asset returns: An empirical investigation. *Journal of Business 75* (2002), 305–332.

[14] Carr, P., and Lee, R. Robust replication of volatility derivatives. *Mathematics in finance working paper series, New York University, Courant Institute of Mathematical Sciences 7* (2008).

[15] CARR, P., AND L.WU. Time-changed Lévy processes and option pricing. *Journal of Financial Economics 75* (2004), 113–141.

[16] CARR, P., AND MADAN, D. Towards a theory of volatility trading. In *Option Pricing, Interest Rates, and Risk Management*. University Press, 1998, pp. 417–427.

[17] CARR, P., MADAN, D., AND SMITH, R. H. Option valuation using the fast fourier transform. *Journal of Computational Finance 2* (1999), 61–73.

[18] CARR, P., AND SUN, J. A new approach for option pricing under stochastic volatility. *Journal of Theoretical Probability 10* (2007), 87–150.

[19] CHEW, L. Target volatility asset allocation strategy. *Society of Actuaries International News 53* (2011).

[20] CHRISTOFFERSEN, P., HESTON, S., AND JACOBS, K. The shape and term structure of the index option smirk: why multifactor stochastic volatility models work so well. *Management science 55* (2009), 1914–1932.

[21] CLARK, P. K. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica 41* (1973), 135–155.

[22] COLES, R. Vol target options, February 2014.

[23] CONT, R., AND TANKOV, P. *Financial Modelling with Jump Processes*. CRC Financial Mathematics Series. Chapman & Hall, 2004.

[24] COX, J. C., INGERSOLL, J. E., AND ROSS, J. S. A. A theory of the term structure of interest rates. *Econometrica 53*, 2 (1985), 385–407.

[25] DA FONSECA, J., GNOATTO, A., AND GRASSELLI, M. Analytic pricing of volatility-equity options within Wishart-based stochastic volatility models. *Operation Research Letters 43* (2015), 601–607.

[26] DA FONSECA, J., AND GRASSELLI, M. Riding on the smiles. *Quantitative Finance 11*, 11 (2011), 1609–1632.

[27] DA FONSECA, J., GRASSELLI, M., AND TEBALDI, C. Option pricing when correlation are stochastic: an analytical model. *Review of Derivatives Research 10*, 2 (2007), 151–180.

[28] DA FONSECA, J., GRASSELLI, M., AND TEBALDI, C. A multifactor volatility Heston model. *Quantitative Finance 8*, 6 (2008), 591–604.

[29] DERMAN, E., KAMAL, M., DEMETERFI, K., AND ZOU, J. More than you ever wanted to know about volatility swaps. *The Journal of Derivatives 6*, 4 (1999), 9–22.

[30] DI GRAZIANO, G., AND TORRICELLI, L. Target volatility option pricing. *International Journal of Theoretical and Applied Finance 15*, 1 (2012).

[31] DUFFIE, D., PAN, J., AND SINGLETON, K. Transform analysis and asset pricing for affine jump diffusions. *Econometrica 68* (2000), 1343–1376.

[32] DUFRESNE, D. *The integrated square-root process*. University of Montreal research paper number 90, 2001.

[33] DUPIRE, B. Arbitrage pricing with stochastic volatility. In *Proceedings of AFFI Conference* (1992).

[34] EBERLEIN, E., PAPAPANTOLEON, A., AND SHIRYAEV, A. N. Esscher transforms and the duality principle for multidimensional semimartingales. *The Annals of Applied Probability 19* (2009), 1944–1971.

[35] ELLIOTT, R., SIU, T., AND CHAN, L. Pricing volatility swaps under Heston's stochastic volatility model with regime switching. *Applied Mathematical Finance* (2007), 41–62.

[36] ESTRELLA, A. *Taylor, Black and Scholes: series approximations and risk management pitfalls*. Research Paper 9501, Federal Reserve Bank of New York,, 1995.

[37] FANG, H. Option pricing implications of a stochastic jump rate. *University of Virginia Working Paper* (2000).

[38] FATONE, L., MARIANI, F., RECCHIONI, M. C., AND ZIRILLI, F. Pricing realized variance options using integrated stochastic variance options in the Heston volatility model. *Discrete and Continuous Dynamical Systems Supplements* (2007), 354–363.

[39] FELLER, W. Two singular diffusion problems. *The Annals of Mathematics 54*, 1 (1951), 173–182.

[40] FELLER, W. The parabolic differential equations and the associated semi-groups of transformations. *The Annals of Mathematics*, 3 (1952), 468–519.

[41] FILIPOVIĆ, D. A general characterization of one factor affine term structure models. *Finance and Stochastics 5* (2001), 389–412.

[42] FILIPOVIĆ, D., DUFFIE, D., AND SCHACHERMAYER, W. Affine processes and applications in finance. *The Annals of Applied Probability 13* (2003), 984–1053.

[43] FÖLLMER, H., AND SCHWEIZER, M. Hedging of contingent claims under incomplete information. University of Bonn working paper, 1990.

[44] FRIEDMAN, A. *Partial Differential Equations of Parabolic Type*. Prentice Hall, 1964.

[45] FRIEDMAN, A. *Stochastic Differential Equations and Applications*. Dover, 1975.

[46] FRIZ, P., AND GATHERAL, J. Valuation of volatility derivatives as an inverse problem. *Quantitative Finance 5* (2005), 531–542.

[47] GALLANT, A. R., HSU, C. T., AND TAUCHEN, G. Using daily range data to calibrate volatility diffusions and extract the forward integrated variance. *The Review of Economics and Statistics 81* (2013), 617–631.

[48] GATHERAL, J. *The Volatility Surface: a Practitioner's Guide*. Wiley Finance. Wiley, 2006.

[49] GNOATTO, A., AND GRASSELLI, M. The explicit Laplace transform for the wishart process. *ArXiv eprint* (2013).

[50] GOLDSTICKER, R. P. Volatility targeting may miss the mark. BNY Mellon asset management, 2012.

[51] GOURIÉROUX, C. Continuous time Wishart process for stochastic risk. *Econometric Review 25* (2003), 177–217.

[52] GOURIÉROUX, C., AND SUFANA, R. Wishart quadratic term structure models. *CREF 03-10, HEC Montreal* (2003).

[53] GOURIÉROUX, C., AND SUFANA, R. Derivative pricing with Wishart multivariate stochastic volatility. *Journal of Business and Economic Statistics 3* (2010), 438–451.

[54] GRASSELLI, M., AND MARABEL ROMO, J. Stochastic skew and target volatility options. *Journal of Futures Market 36* (2015), 174–193.

[55] GRASSELLI, M., AND TEBALDI, C. Solvable affine term structure models. *Mathematical Finance 18* (2007), 135–153.

[56] GRASSELLI, M. R., AND HUGHSTON, L. P., Eds. *Finance at Fields* (2012), World Scientific.

[57] GUO, J. H., AND HUNG, M. W. A note on the discontinuity problem in Heston's stochastic volatility model. *Applied Mathematical Finance 14*, 4 (2007), 339–347.

[58] HEATH, D., AND SCHWEIZER, M. Martingales versus PDEs in finance: an equivalence result with examples. *Journal of Applied Probability 37* (2000), 947–957.

[59] HESTON, S. A jump-diffusion model for option pricing. *Management Science 48* (2002), 1086–1101.

[60] HESTON, S. L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies 6* (1993), 327–343.

[61] HONG, G. *Forward smile and derivative pricing*. UBS Working paper, 2004.

[62] HOWISON, S., RAFAILIDIS, A., AND RASMUSSEN, H. A note on the pricing and hedging of volatility derivatives. *Applied Mathematical Finance 11*, 4 (2004), 317–346.

[63] HUANG, J., AND WU, L. Specification analysis of option pricing models based on time-changed Lévy processes. *The Journal of Finance 59*, 3 (2004).

[64] HULL, J., AND WHITE, A. The pricing of options on assets with stochastic volatility. *Journal of Finance 42*, 2 (June 1987), 281–300.

[65] JACOD, J. *Calcul Stochastique et Problèmes de Martingales*. Lecture notes in Mathematics. Springer, 1979.

[66] JACOD, J., AND PROTTER, P. *Discretization of Processes*. No. 67 in Stochastic Modelling and Applied Probability. Springer, 2011.

[67] JACOD, J., AND SHIRYAEV, A. N. *Limit Theorems for Stochastic Processes*. No. 288 in Grundlehren der Mathematischen Wissenschaften. Springer, 1987.

[68] JANSON, S., AND TYSK, J. Feynman-Kac formulas for Black-Scholes-type operators. *Bullettin of the London Matematical Society 38*, 2 (2006), 269–282.

[69] JAVAHERI, A., WILMOTT, P., AND HAUG, E. Garch and volatility swaps. *Wilmott Magazine 1*, 17 (2002), 589–595.

[70] KALLSEN, J., AND SHIRYAEV, A. N. Time change representation of stochastic integrals. *Theory of Probability and its Applications 46* (2002), 522–528.

[71] KARATZAS, I., AND SHREVE, S. E. *Brownian Motion and Stochastic Calculus*. Springer, 2000.

[72] KÜCHLER, U., AND SØRENSEN, M. *Exponential Families of Stochastic Processes*. Springer Series in Statistics. Springer, 1997.

[73] KUNITA, H. Stochastic differential equations and stochastic flows of diffeomorphisms. In *Lecture Notes in Mathematics 1097* (1984), École d'été de Probabilités de Saint-Flour XII-1982, pp. 143–303.

[74] LEIPPOLD, M., AND TROJANI, F. Asset pricing with matrix jump diffusions. *Available at SSRN 1572576* (2008).

[75] LEWIS, A. *Option Valuation under Stochastic Volatility*. Finance Press, 2000.

[76] LEWIS, A. A simple option formula for general jump-diffusion and other exponential Lévy processes. *OptionCity.net Publications* (2001).

[77] LIPTON, A. *Mathematical Methods for Foreign Exchange: A Financial Engineer's Approach*. World Scientific Publishing Company, 2001.

[78] LORD, R., AND KAHL, C. Complex logarithms in Heston-like models. *Mathematical Finance 20*, 4 (2010), 671–694.

[79] LORENTZ, G. G. *The Bernstein Polynomials*. AMS, 1986.

[80] LUKACS, E. *Characteristic Functions*. Griffin, 1970.

[81] MERTON, S. C. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics 3* (1976), 125–144.

[82] MIKULSKIS, D., AND DAS, A. Risk-controlled volatility strategies. Tech. rep., Readington teach/in report, 2013.

[83] MONROE, I. Processes that can be embedded in brownian motions. *The Annals of Probability 6* (1978), 42–56.

[84] MORRISON, S., AND TADROWSKI, L. Guarantees and target volatility funds. Moody analytics, September 2013.

[85] MUHLE-KARBE, J., PFAFFEL, O., AND STELZER, R. Option pricing in multivariate stochastic volatility models of OU type. *ArXiv eprint* (November 2011).

[86] NEUBERGER, A. The log-contract: a new instrument to hedge volatility. *Journal of Portfolio Management 20*, 2 (1994), 74–80.

[87] ØKSENDAL, B. *Stochastic Differentail Equations: An Intorduction with Applications*. Universitext. Springer, 2007.

[88] SANTA CLARA, P., AND YAN, S. Crashes, volatility, and the equity premium: lessons from S & P 500 options. *The Review of Economics and Statistics 92*, 2 (2010), 435–451.

[89] SCHACHERMAYER, W., AND TEICHMANN, T. How close are the option pricing formulas of Bachelier and Black-Merton-Scholes? *Mathematical Finance 18*, 1 (January 2008), 155–170.

[90] SCHÜRGER, K. *Laplace transforms and suprema of stochastic processes*. University of Bonn, 2008.

[91] SEPP, A. Pricing options on realized variance in heston model with jumps in return and volatility. *Journal of Computational Finance 11*, 4 (2008), 33–70.

[92] SHREVE, S. E. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Finance. Springer, 2008.

[93] SIMONS, E., SCHOUTENS, W., AND TISTAERT, J. A perfect calibration! Now what? *Wilmott Magazine* (March 2004).

[94] SIN, C. Complications with stochastic volatility models. *Advances in Applied Probability 30* (1998), 256–268.

[95] STOYANOV, S. Structured equity investment strategies for long-term Asian investors. *EDHEC Risk Publications* (August 2011).

[96] TORRICELLI, L. Pricing joint claims on an asset and its realized variance in stochastic volatility models. *International Journal of Theoretical and Applied Finance 16*, 1 (2013).

[97] TORRICELLI, L. Valuation of asset and volatility derivatives using decoupled time-changed Lévy processes. *Review of derivatives research, Online First* (July 2015).

[98] WANG, X., AND WANG, Y. Variance-optimal hedging for target volatility options. *Journal of Industrial and Management Optimization 10* (2013), 207–218.

[99] WILMOTT, P. *Quantitative finance*. Wiley, 2006.

[100] WODALL, L. Insurers look to volatility controls to support long-term guarantees. *Risk.net* (June 2013).

[101] XUE, Y. Target volatility: an effective risk management tool for VA? Society of Actuaries, October 2012.

[102] ZHENG, W., AND KWOK, L. K. Closed form pricing formulas for discretely sampled generalized variance swaps. *Mathematical Finance 24*, 4 (2014), 855–881.

[103] ZHENG, W., AND ZENG, P. Closed-form partial transform of triple joint density for pricing exotic options and variance derivatives under the 3/2 model. *SSRN eprint* (2014).