Cut Finite Element Methods for Coupled Bulk–Surface Problems

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Abstract We develop a cut finite element method for a second order elliptic coupled bulk-surface model problem. We prove a priori estimates for the energy and L^2 norms of the error. Using stabilization terms we show that the resulting algebraic system of equations has a similar condition number as a standard fitted finite element method. Finally, we present a numerical example illustrating the accuracy and the robustness of our approach.

Mathematics Subject Classification (2000) 65N30, 65N12, 65N15

1 Introduction

Problems involving phenomena that take place both on surfaces (or interfaces) and in bulk domains occur in a variety of applications in fluid dynamics and biological applications. An example is given by the modeling of soluble surfactants. Surfactants are important because of their ability to reduce the surface tension. Examples of applications where the effects of surfactants are important in the modeling include detergents, oil recovery, and the treatment of lung diseases. A soluble surfactant is dissolved in the bulk fluid but also exists in adsorbed form on the interface. A computational challenge is then to properly account for the exchange between these two surfactant forms. The coupling between the dissolved form in the bulk and the adsorbed form on the interface involves computations of the gradient of the bulk surfactant concentration on a moving interface that may undergo topological changes, see e.g.[1]. In this context computational methods that allow the interface to be arbitrarily located with respect to a fixed background mesh are of great interest.

We consider a basic model problem of this nature that involves two coupled elliptic problems one in the bulk and one on the boundary of the bulk domain. The coupling term is defined in such a way that the overall bilinear form in the corresponding weak statement is coercive. A finite element method was proposed and analyzed for a similar model problem in [8]. See also [7], and the references therein for background on finite element methods for partial differential equations on surfaces. In [8] a polyhedral approximation of the bulk domain was used and its piecewise

This research was supported in part by EPSRC, UK, Grant No. EP/J002313/1, the Swedish Foundation for Strategic Research Grant No. AM13-0029, the Swedish Research Council Grants Nos. 2011-4992, 2013-4708, and 2014-4804, and Swedish strategic research programme eSSENCE

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polynomial boundary faces served as approximation of the surface. In this contribution we develop a method that is unfitted, that is, the surface is allowed to cut through a fixed background mesh in an arbitrary way. Such a finite element method was proposed in [16] for the Laplace–Beltrami operator. An overview of a general framework for this type of computational methods using finite element methods on cut meshes, so-called CutFEM methods, was recently given in [3]. The CutFEM approach is convenient since the same finite element space defined on a background grid can be used for solving both the partial differential equation in the bulk region and on the surface. However, a drawback of this type of methods is that the stiffness matrix may become arbitrarily ill conditioned depending on the position of the surface in the background mesh. In the case of the Laplace–Beltrami operator this ill conditioning has been addressed in [15] and [5]. For results on the stability of the bulk equation on cut meshes see [4, 12, 14]. We finally mention the application to advection diffusion equations on surfaces [17] and the extension to higher order methods [13] for bulk problems.

We use continuous piecewise linear elements defined on the background mesh to solve both the problem in the bulk domain and the problem on the surface. To stabilize the method we add gradient jump penalty terms as in [4, 5] that ensure that the resulting algebraic system of equations has optimal condition number. We also consider the approximation of the domain and prove a priori error estimates in both the H^1 and L^2 norms, taking both the approximation of the domain and of the solution into account.

The remainder of the paper is outlined as follows: In Section 2 we introduce the model problem and state the weak form, in Section 3 we introduce a discrete approximation of the domain, in Section 4 we prove a priori estimates for the energy and L^2 norm of the error, in Section 5 we prove an estimate of the condition number, and finally in Section 6 we present a numerical example.

2 The Continuous Coupled Bulk-Surface Problem

2.1 Strong Form

Let Ω be a domain in $\mathbb{R}^3$ with C^2 boundary Γ and exterior unit normal n. We consider the following problem: find $u_B : \Omega \to \mathbb{R}$ and $u_S : \Gamma \to \mathbb{R}$ such that

$$-\nabla \cdot (k_B \nabla u_B) = f_B \qquad \text{in } \Omega \tag{2.1}$$

$$-n \cdot k_B \nabla u_B = b_B u_B - b_S u_S \qquad \text{on } \Gamma \tag{2.2}$$

$$-\nabla_\Gamma \cdot (k_S \nabla_\Gamma u_S) = f_S - n \cdot k_B \nabla u_B \qquad \text{on } \Gamma \tag{2.3}$$

Here ∇ is the $\mathbb{R}^3$ gradient and ∇_Γ is the tangent gradient associated with Γ defined by

$$\nabla_\Gamma = P_\Gamma \nabla \tag{2.4}$$

with $P_\Gamma = P_\Gamma(x)$ the projection of $\mathbb{R}^3$ onto the tangent plane of Γ at $x \in \Gamma$, defined by

$$P_\Gamma = I - n \otimes n \tag{2.5}$$

Further, b_B, b_S, k_B, and k_S are positive constants, and $f_B : \Omega \to \mathbb{R}$ and $f_S : \Gamma \to \mathbb{R}$ are given functions. As mentioned above, this problem serves as a basic model for the concentration of surfactants interacting with a bulk concentration; it also models other processes, e.g., proton transport via a membrane surface [18].

2.2 Weak Form

Multiplying (2.1) by $v_B \in H^1(\Omega)$, integrating by parts, and using the boundary condition (2.2), we obtain

$$(f_B, v_B)_\Omega = (k_B \nabla u_B, \nabla v_B)_\Omega - (n \cdot k_B \nabla u_B, v_B)_\Gamma \tag{2.6}$$

$$= (k_B \nabla u_B, \nabla v_B)_\Omega + (b_B u_B - b_S u_S, v_B)_\Gamma \tag{2.7}$$

and thus we have the weak statement

$$(k_B \nabla u_B, \nabla v_B)_\Omega + (b_B u_B - b_S u_S, v_B)_\Gamma = (f_B, v_B)_\Omega \quad \forall v_B \in H^1(\Omega) \tag{2.8}$$

Next multiplying (2.3) by $v_S \in H^1(\Gamma)$, integrating by parts, and again using (2.2) we obtain

$$(k_S \nabla_\Gamma u_S, \nabla_\Gamma v_S)_\Gamma = (f_S - n \cdot k_S \nabla u_B, v_S)_\Gamma \tag{2.9}$$

$$= (f_S + (b_B u_B - b_S u_S), v_S)_\Gamma \tag{2.10}$$

and thus

$$(k_S \nabla_\Gamma u_S, \nabla_\Gamma v_S)_\Gamma - (b_B u_B - b_S u_S, v_S)_\Gamma = (f_S, v_S)_\Gamma \quad \forall v_S \in H^1(\Gamma) \tag{2.11}$$

We note that the solution to this system of equations is uniquely determined up to a pair of constant functions (c_B, c_S) such that $b_B c_B - b_S c_S = 0$. To obtain a unique solution we here choose to enforce $\int_\Gamma u_S = 0$.

Introducing the function spaces

$$V_B = H^1(\Omega), \quad V_S = H^1(\Gamma)/\langle 1_\Gamma \rangle, \quad W = V_B \times V_S \tag{2.12}$$

and choosing the test functions $b_B v_B$ and $b_S v_S$ we get the variational problem: find $u = (u_B, u_S) \in W$ such that

$$a(u, v) = l(v) \quad \forall v \in W \tag{2.13}$$

Here

$$a(u, v) = a_B(u_B, v_B) + a_S(u_S, v_S) + a_{BS}(u, v) \tag{2.14}$$

with

$$\begin{cases} a_B(u_B, v_B) = b_B(k_B \nabla u_B, \nabla v_B)_\Omega \\ a_S(u_S, v_S) = b_S(k_S \nabla_\Gamma u_S, \nabla_\Gamma v_S)_\Gamma \\ a_{BS}(u, v) = (b_B u_B - b_S u_S, b_B v_B - b_S v_S)_\Gamma = (b \cdot u, b \cdot v)_\Gamma \end{cases} \tag{2.15}$$

where we also introduced the notation $b = (b_B, -b_S)$ and

$$l(v) = l_B(v_B) + l_S(v_S) = b_B(f_B, v_B)_\Omega + b_S(f_S, v_S)_\Gamma \tag{2.16}$$

Introducing the energy norm

$$|||u|||^2 = a(u, u) \tag{2.17}$$

we directly obtain coercivity and continuity of the bilinear form $a(\cdot, \cdot)$ and continuity of $l(\cdot)$. Using the Lax-Milgram lemma there is a unique solution in W. If Γ is C^3 we additionally have the elliptic regularity estimate

$$\|u_B\|_{H^2(\Omega)} + \|u_S\|_{H^2(\Gamma)} \lesssim \|f_B\|_{L^2(\Omega)} + \|f_S\|_{L^2(\Gamma)} \tag{2.18}$$

see [8] for details. Here and below $\lesssim$ denotes less or equal up to a constant, $\| \cdot \|_{H^s(\omega)}$ denotes the standard Sobolev norm in $H^s(\omega)$ on the set ω, and $\| \cdot \|_{L^p(\omega)}$ denotes the $L^p(\omega)$ norm.

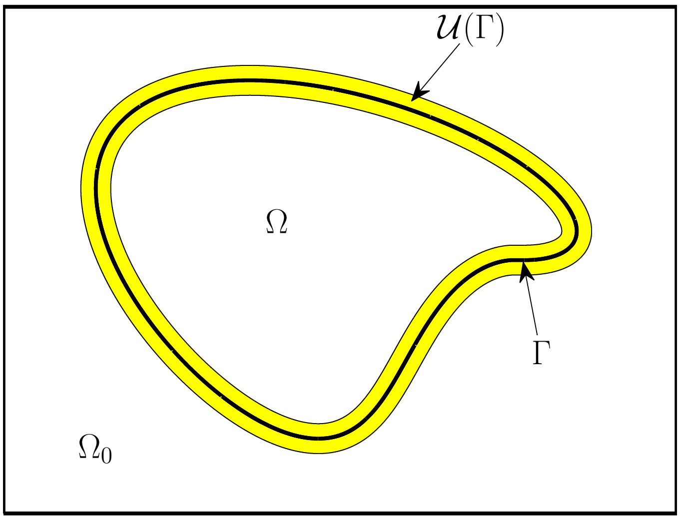


Fig. 1 Illustration of the domain Ω, Ω_0, $\mathcal{U}(\Gamma)$, and Γ. The domain $\mathcal{U}(\Gamma)$ is the yellow region where for each $x \in \mathcal{U}(\Gamma)$ there is a unique closest point on Γ.

3 The Finite Element Method

3.1 Approximation of the Domain

Let $\Gamma \in C^2$ and $p : \mathbb{R}^3 \ni x \mapsto \text{argmin}_{y\in\Gamma}|y-x| \in \Gamma$ denote the closest point mapping. Then there is an open neighborhood $\mathcal{U}(\Gamma)$ of Γ such that for each $x \in \mathcal{U}(\Gamma)$ there is a uniquely determined $p(x) \in \Gamma$. We let ρ be the signed distance function, $\rho(x) = |p(x)-x|$ in $\mathbb{R}^3\setminus\Omega$ and $\rho(x) = -|p(x)-x|$ in Ω. We define the extension of a function v defined on Γ to $\mathcal{U}(\Gamma)$ as follows

$$v^e = v \circ p \tag{3.1}$$

We refer to [11], in particular Appendix 14.6, for background on distance functions and closest point mappings.

Let Ω_0 be a domain in $\mathbb{R}^3$ that contains $\Omega \cup \mathcal{U}(\Gamma)$ and let $\mathcal{K}_{0,h}$ be a quasiuniform partition of Ω_0 into shape regular tetrahedra with mesh parameter h. See Fig. 1 for an illustration of the different domains. We consider a continuous piecewise linear approximation Γ_h of Γ such that $\Gamma_h \cap K$ is a subset of a hyperplane in $\mathbb{R}^3$ for each $K \in \mathcal{K}_{0,h}$.

We assume that $\Gamma_h \subset \mathcal{U}(\Gamma)$ and that the following approximation assumptions hold:

$$\|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^2 \tag{3.2}$$

and

$$\|n^e - n_h\|_{L^\infty(\Gamma_h)} \lesssim h \tag{3.3}$$

where n_h denotes the piecewise constant exterior unit normal to Γ_h. Finally, we define Ω_h as the domain enclosed by Γ_h. The assumptions (3.2) and (3.3) are consistent with the piecewise linear nature of the discrete surface. For instance, a common construction, is to let Γ_h be the zero levelset of a a piecewise linear approximation ρ_h of the exact distance function ρ. Then (3.2) and (3.3) follows directly from the approximation estimate

$$\|\rho - \rho_h\|_{L^\infty(\mathcal{U}(\Gamma))} + h\|\nabla(\rho - \rho_h)\|_{L^\infty(\mathcal{U}(\Gamma))} \lesssim h^2 \tag{3.4}$$

which holds for common interpolants, since $\nabla(\rho - \rho_h) = n - n_h$. We return to this construction in our numerical examples.

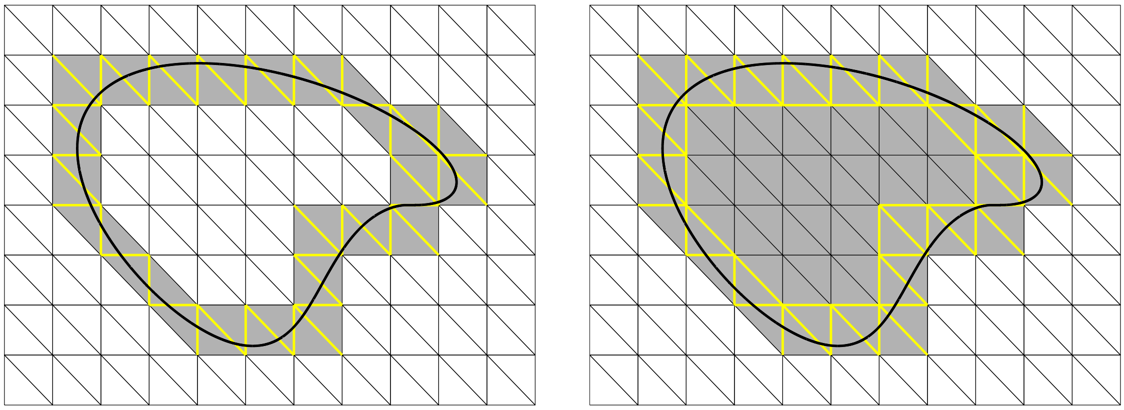


Fig. 2 Illustration of the surface and bulk meshes and the internal faces where stabilization is employed. Right: the shaded domain shows $\mathcal{N}^n_{S,h}$ and edges in $\mathcal{F}_{S,h}$ are marked yellow. Left: the shaded domain shows $\mathcal{N}^n_{B,h}$ and edges in $\mathcal{F}^n_{B,h}$ are marked yellow.

3.2 Finite Element Spaces

We define the following sets of elements

$$\mathcal{K}_{B,h} = \{K \in \mathcal{K}_{h,0} \, : \, K \cap \Omega_h \neq \emptyset\}, \qquad \mathcal{K}_{S,h} = \{K \in \mathcal{K}_{h,0} \, : \, K \cap \Gamma_h \neq \emptyset\} \tag{3.5}$$

and the corresponding sets

$$\mathcal{N}_{B,h} = \bigcup_{K \in \mathcal{K}_{B,h}} K, \qquad \mathcal{N}_{S,h} = \bigcup_{K \in \mathcal{K}_{S,h}} K \tag{3.6}$$

We let $V_{0,h}$ be the space of piecewise linear continuous functions defined on $\mathcal{K}_{0,h}$. Next let

$$V_{B,h} = V_{0,h}|_{\mathcal{N}_{B,h}}, \qquad V_{S,h} = V_{0,h}|_{\mathcal{N}_{S,h}} / \langle 1_{\Gamma_h} \rangle, \qquad W_h = V_{B,h} \times V_{S,h} \tag{3.7}$$

be the spaces of continuous piecewise linear polynomials defined on $\mathcal{N}_{B,h}$ and $\mathcal{N}_{S,h}$, respectively, where we also enforced $\int_{\Gamma_h} v_S = 0$ for $v \in V_{S,h}$.

3.3 The Finite Element Method

The finite element method takes the form: find $u_h = (u_{B,h}, u_{S,h}) \in W_h$ such that

$$A_h(u_h, v) = l_h(v) \quad \forall v \in W_h \tag{3.8}$$

Here the bilinear form is defined by

$$A_h(v, w) = a_h(v, w) + j_h(v, w) \tag{3.9}$$

with

$$a_h(v, w) = a_{B,h}(v_B, w_B) + a_{S,h}(v_S, w_S) + a_{BS,h}(v, w) \tag{3.10}$$

and

$$\begin{cases} a_{B,h}(u_B, v_B) = b_B(k_B \nabla u_B, \nabla v_B)_{\Omega_h} \\ a_{S,h}(u_S, v_S) = b_S(k_S \nabla_\Gamma u_S, \nabla_\Gamma v_S)_{\Gamma_h} \\ a_{BS,h}(u, v) = (b_B u_B - b_S u_S, b_B v_B - b_S v_S)_{\Gamma_h} = (b \cdot u, b \cdot v)_{\Gamma_h} \end{cases} \tag{3.11}$$

where $\nabla_{\Gamma_h} = P_h \nabla$ and $P_h = I - n_h \otimes n_h$. Next $j_h(v, w)$ is a stabilizing term of the form

$$j_h(v, w) = \tau_B h^3 j_B(v_B, w_B) + \tau_S j_S(v_S, w_S) \tag{3.12}$$

where τ_B, τ_S are positive parameters and, letting $[x]|_F$ denote the jump of x over the face F,

$$j_B(v_B, w_B) = \sum_{F \in \mathcal{F}_{B,h}} ([n_F \cdot \nabla v_B], [n_F \cdot \nabla w_B])_F \tag{3.13}$$

$$j_S(v_S, w_S) = \sum_{F \in \mathcal{F}_{S,h}} ([n_F \cdot \nabla v_S], [n_F \cdot \nabla w_S])_F \tag{3.14}$$

with $\mathcal{F}_{S,h}$ the set of internal faces (i.e. faces with two neighbors) in $\mathcal{K}_{S,h}$ and $\mathcal{F}_{B,h}$ denotes the set of faces that are internal in $\mathcal{K}_{B,h}$ and belong to an element in $\mathcal{K}_{S,h}$, see Fig. 2. Finally, the right hand side is defined by

$$l_h(v) = l_{B,h}(v_B) + l_{S,h}(v_S) = b_B(f_{B,h}, v_B)_{\Omega_h} + b_S(f_{S,h}, v_S)_{\Gamma_h} \tag{3.15}$$

with $f_{B,h}$ and $f_{S,h}$ discrete approximations of f_B and f_S that will be specified more precisely below.

The purpose of the stabilization terms is to ensure that the resulting algebraic system of equations is well conditioned.

4 A Priori Error Estimates

4.1 Outline of the proof

The main steps in the derivation of the a priori error estimates are as follows:

- We construct a bijective mapping $F_h : \Omega \to \Omega_0$ that maps the exact domain to the approximate domain, more precisely $F_h(\Omega) = \Omega_h$ and $F_h(\Gamma) = \Gamma_h$. The mapping F_h is used to lift the discrete solution onto the exact domain where the error is evaluated. The construction of F_h is based on a representation of the discrete boundary Γ_h as a normal function over the exact boundary Γ together with an extension to a small tubular neighborhood, with thickness δ, of the boundary. In the complement of the tubular neighborhood F_h is the identity mapping. In order to get control of the size of the derivative DF_h of F_h we find that the best choice of the thickness δ of the tubular neighborhood is $\delta \sim h$. As a consequence of the approximation assumption (3.2) this choice of δ is indeed also possible for small enough mesh size. If the boundary has large local curvature the meshsize must be smaller in order to resolve the boundary in the same way as for a standard finite element method using a triangulations of the domain.
- Next a Strang type lemma relates the error in the computed solution to an interpolation error and quadrature errors emanating from the approximation of the domain.
- Using the assumptions on the approximation properties of the discrete surface we derive bounds of the quadrature errors. The surface quadrature errors are $O(h^2)$ while the bulk quadrature error is $O(h)$ in the δ tubular neighborhood and zero elsewhere.
- To establish an optimal order energy norm error estimate only first order estimates of the quadrature errors are needed but for L^2 error estimates second order estimates are necessary. To achieve second order estimate of the bulk quadrature error we utilize the fact that $\delta \sim h$ together with control of derivatives of the dual problem and an additional assumption on f_B that provides control of the H^1 norm of f_B in the vicinity of the boundary.

4.2 Mapping the Exact Domain to the Approximate Domain

In this section we define the mapping F_h and prove estimates of its derivative DF_h and Jacobian JF_h. We refer to [11], in particular Appendix 14.6, for useful background on surfaces described by distance functions.

The Mapping $\boldsymbol{F_h}$*:* For $\delta > 0$ let $\mathcal{U}_\delta(\Gamma)$ be the open tubular δ neighborhood

$$\mathcal{U}_\delta(\Gamma) = \{x \in \mathbb{R}^3 : |\rho(x)| < \delta\} \tag{4.1}$$

For $0 < \delta \leq \delta_0$, where δ_0 is a constant, that only depend on the domain, chosen such that $\mathcal{U}_{\delta_0}(\Gamma) \subset \mathcal{U}(\Gamma)$, the mapping

$$\mathcal{U}_\delta(\Gamma) \ni x \mapsto (p(x), \rho(x)) \in \Gamma \times (-\delta, \delta) \tag{4.2}$$

is a bijection with inverse

$$\Gamma \times (-\delta, \delta) \ni (x, z) \mapsto x + zn(x) \in \mathcal{U}_\delta(\Gamma) \tag{4.3}$$

We next note that there is a function $\gamma_h : \Gamma \to \mathbb{R}$ such that the mapping

$$q_h : \ \Gamma \ni x \mapsto x + n(x)\gamma_h(x) \in \Gamma_h \tag{4.4}$$

is a bijection. Since for $x \in \Gamma_h$ there holds $p(x) = x - n^e(x)\rho(x)$ we may deduce that $q_h(x)$ is the inverse mapping to $p(x) : \Gamma_h \mapsto \Gamma$.

Using the assumptions on the approximation properties (3.2) and (3.3) we obtain the following estimates (see Appendix for details)

$$\|\gamma_h\|_{L^\infty(\Gamma)} \lesssim h^2, \qquad \|\nabla_\Gamma \gamma_h\|_{L^\infty(\Gamma)} \lesssim h \tag{4.5}$$

Assuming that h is sufficiently small so that $\Gamma_h \subset \mathcal{U}_{\delta/3}(\Gamma)$ we may define the mapping

$$F_h : \Omega_0 \ni x \mapsto x + \chi(\rho(x))n^e(x)\gamma_h^e(x) \in \Omega_0 \tag{4.6}$$

where $\chi : (-\delta, \delta) \to [0, 1]$ is a smooth cut off function that equals 1 on $(-\delta/3, \delta/3)$ and 0 on $(-\delta, \delta) \setminus (-2\delta/3, 2\delta/3)$ and the derivative $D\chi$ satisfies the estimate

$$\|D\chi\|_{L^\infty(-\delta,\delta)} \lesssim \delta^{-1} \tag{4.7}$$

We note that by construction $F_h : \Omega_0 \to \Omega_0$ is a bijection such that

$$F_h(\Omega) = \Omega_h, \qquad F_h(\Gamma) = \Gamma_h \tag{4.8}$$

and

$$F_h = I \quad \text{in } \Omega_0 \setminus \mathcal{U}_\delta(\Gamma) \tag{4.9}$$

The Derivative $\boldsymbol{DF_h}$*:* The derivative $DF_h(x) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ of F_h at $x \in \Omega_0$ is given by

$$\begin{aligned} DF_h(x) &= I + \Big(\chi(\rho(x))n^e(x)\Big) D(\gamma_h^e(x)) && (4.10)\\ &\quad + \Big(D(\chi(\rho(x))n^e(x))\Big)\gamma_h^e(x) \\ &= I + \Big(\chi(\rho(x))n^e(x)\Big)(D\gamma_h)^e(x)Dp(x) && (4.11)\\ &\quad + \Big((D\chi)(\rho(x))D\rho(x)n^e(x)\Big)\gamma_h^e(x) \\ &\quad + \Big(\chi(\rho(x))(Dn)^e(x)Dp(x)\Big)\gamma_h^e(x) \end{aligned}$$

where we used the product and chain rules. Here and below $Dv(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the derivative (or the tangent map) at $x \in U \subset \mathbb{R}^n$ where $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of linear mappings $\mathbb{R}^n \to \mathbb{R}^m$ and U is an open set, see [6], Chapter VIII, for background on differential calculus. Next we note that

$$D\rho = n^e, \qquad Dn = \mathcal{H}_\Gamma, \qquad Dp = P_\Gamma^e - \rho\mathcal{H}_\Gamma \tag{4.12}$$

where we used the identity $p(x) = x - \rho(x)D\rho(x) = x - \rho(x)n^e(x)$ to conclude that $Dp(x) = I - (D\rho(x)) \otimes n^e(x) - \rho(x)D^2\rho(x) = I - n^e(x) \otimes n^e(x) - \rho(x)D^2\rho(x)$ and also introduced the curvature tensor $\mathcal{H}_\Gamma(x) = D^2\rho(x) = \nabla \otimes \nabla\rho(x), x \in \Gamma$. Note that it holds $\|\mathcal{H}_\Gamma\|_{L^\infty(\mathcal{U}_\delta(\Gamma))} \lesssim 1$ for δ small enough. Using these identities in (4.11) and the fact that $(D\gamma_h)^e(x)Dp(x) = (P_\Gamma\nabla\gamma_h)\cdot(P_\Gamma^e - \rho\mathcal{H}_\Gamma)$ we obtain

$$
\begin{aligned}
DF_h(x) = I &+ \chi(\rho(x))n^e(x) \otimes (\nabla_\Gamma\gamma_h)^e(x)(P_\Gamma^e(x) - \rho(x)\mathcal{H}_\Gamma(x)) \\
&+ \gamma_h^e(x)(D\chi)(\rho(x))n^e(x) \otimes n^e(x) \\
&+ \chi(\rho(x))\gamma_h^e(x)\mathcal{H}_\Gamma^e(x)(P_\Gamma^e(x) - \rho(x)\mathcal{H}_\Gamma(x))
\end{aligned}
\tag{4.13}
$$

which is the expression for the derivative or (tangent map) in Cartesian coordinates. On the surface Γ we have the simplified expression

$$
DF_h(x) = I + n(x) \otimes \nabla_\Gamma\gamma_h(x) + \gamma_h(x)\mathcal{H}_\Gamma(x) \tag{4.14}
$$

since $\chi = 1$ and $D\chi = 0$ in a neighborhood of Γ and $\rho(x) = 0$ for $x \in \Gamma$. We note that $DF_h(x)$ maps the tangent space $T_x(\Gamma)$ into the piecewise defined tangent space $T_{F_h(x)}(\Gamma_h)$. In other words we have the identity

$$
DF_hP_\Gamma = (P_{\Gamma_h} \circ F_h)DF_hP_\Gamma \tag{4.15}
$$

and the mapping

$$
DF_{h,\Gamma}(x) : T_x(\Gamma) \ni y \mapsto (P_{\Gamma_h} \circ F_h)DF_hP_\Gamma y \in T_{F_h(x)}(\Gamma_h) \tag{4.16}
$$

is invertible.

Next we have the following estimate

$$
\begin{cases}
DF_h = I + O(h) + \delta^{-1}O(h^2) + O(h^2) & \in \mathcal{U}_\delta(\Gamma) \\
DF_h = I & \in \Omega_0 \setminus \mathcal{U}_\delta(\Gamma)
\end{cases}
\tag{4.17}
$$

since the last three terms in (4.13) are zero in $\Omega_0 \setminus \mathcal{U}_\delta(\Gamma)$ and in $\mathcal{U}_\delta(\Gamma)$ they can be directly estimated using (4.5) and (4.7) as follows

$$
\begin{aligned}
&|\chi(\rho(x))n^e(x) \otimes (\nabla_\Gamma\gamma_h)^e(x)(P_\Gamma^e(x) - \rho(x)\mathcal{H}_\Gamma(x))| \\
&\quad\lesssim |\chi(\rho(x))|\,|n^e(x)|\,|(\nabla_\Gamma\gamma_h)^e(x)|(|P_\Gamma^e(x)| + |\rho(x)|\,|\mathcal{H}_\Gamma(x)| \lesssim h
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&|\gamma_h^e(x)(D\chi)(\rho(x))n^e(x) \otimes n^e(x)| \\
&\quad\lesssim |\gamma_h^e(x)|\,|D\chi(\rho(x))|\,|n^e(x)|\,|n^e(x)| \lesssim \delta^{-1}h^2
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&|\chi(\rho(x))\gamma_h^e(x)\mathcal{H}_\Gamma^e(x)(P_\Gamma^e(x) - \rho(x)\mathcal{H}_\Gamma(x))| \\
&\quad\lesssim |\chi(\rho(x))|\,|\gamma_h^e(x)||\mathcal{H}_\Gamma^e(x)|\,(|P_\Gamma^e(x)| + |\rho(x)|\,|\mathcal{H}_\Gamma(x)|) \lesssim h^2
\end{aligned}
\tag{4.20}
$$

uniformly for all $x \in \mathcal{U}_\delta(\Gamma)$. The estimate (4.17) holds for any $0 < \delta \leq \delta_0$ and h such that

$$
\Gamma_h \subset \mathcal{U}_{\delta/3}(\Gamma) \tag{4.21}
$$

Recall that (4.21) is required in the definition (4.6) of the mapping F_h. Now, if there is a constant $C_1 > 0$ such that $C_1h < \delta \leq \delta_0$, then there is a constant $h_0 > 0$, independent of δ, such that (4.21) holds for $0 < h \leq h_0$, since we have the estimate $\|\gamma_h\|_{L^\infty(\Gamma)} \leq C_2h^2 \leq (C_2h_0)h < C_1h/3 < \delta/3$, where we may choose h_0 such that $C_2h_0 < C_1/3$. We therefore conclude that we may choose $\delta \sim h$, ($\delta \lesssim h$ and $h \lesssim \delta$) and that the following estimate holds in that case

$$
\begin{cases}
DF_h = I + O(h) & \text{in } \mathcal{U}_\delta(\Gamma) \\
DF_h = I & \text{in } \Omega_0 \setminus \mathcal{U}_\delta(\Gamma)
\end{cases}
\tag{4.22}
$$

From here on the parameter δ in the definition (4.6) of F_h will be chosen such that $\delta \sim h$.

Using the estimate (4.22) we conclude that, for small enough h, we have the bounds

$$\|DF_h\|_{L^\infty(\Omega_0,\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3))} \lesssim 1, \qquad \|DF_h^{-1}\|_{L^\infty(\Omega_0,\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3))} \lesssim 1 \tag{4.23}$$

and starting from (4.16) estimating the norm of the composition of maps in terms of the norms of the individual maps, and using the fact that the projections have norm one together with (4.23) we obtain

$$\|DF_{h,\Gamma}\|_{L^\infty(\Gamma,\mathcal{L}(T_x(\Gamma),T_{F_h(x)}(\Gamma_h))} \lesssim 1, \qquad \|DF_{h,\Gamma}^{-1}\|_{L^\infty(\Gamma,\mathcal{L}(T_{F_h(x)}(\Gamma_h),T_x(\Gamma))} \lesssim 1 \tag{4.24}$$

Below we simplify the notation as follows $\|DF_h\|_{L^\infty(\Omega_0)} = \|DF_h\|_{L^\infty(\Omega_0,\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3))}$ for the mappings DF_h and $DF_{h,\Gamma}$ and their inverses.

The Jacobian Determinants $\boldsymbol{JF_h}$ *and* $\boldsymbol{JF_{h,\Gamma}}$*:* We have the following relations between the measures on the exact and approximate surface and domain

$$d\Omega_h = JF_h d\Omega, \qquad d\Gamma_h = JF_{h,\Gamma} d\Gamma \tag{4.25}$$

where the Jacobian determinants are defined by

$$JF_h(x) = |\det(DF_h(x))| \tag{4.26}$$

$$JF_{h,\Gamma}(x) = |DF_{h,\Gamma}(x)\xi_1 \times DF_{h,\Gamma}(x)\xi_2| \tag{4.27}$$

and $\{\xi_1, \xi_2\}$ is an orthonormal basis in $T_x(\Gamma)$. We note that $JF_h = 1$ on $\Omega_0 \setminus \mathcal{U}_\delta(\Gamma)$ and recall that $DF_h = I + O(h)$ in $\mathcal{U}_\delta(\Gamma)$, see (4.22). Thus we have the following estimates in the bulk

$$\|JF_h\|_{L^\infty(\Omega_0)} \lesssim 1, \quad \|JF_h^{-1}\|_{L^\infty(\Omega_0)} \lesssim 1, \quad \begin{cases} \|1 - JF_h\|_{L^\infty(\mathcal{U}_\delta(\Gamma))} \lesssim h & \text{in } \mathcal{U}_\delta(\Gamma) \\ \|1 - JF_h\|_{L^\infty(\mathcal{U}_\delta(\Gamma))} = 0 & \text{in } \Omega_0 \setminus \mathcal{U}_\delta(\Gamma) \end{cases} \tag{4.28}$$

Here we used the fact that the determinant is given by $\det(DF_h) = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^3 DF_{h,i\sigma(i)}$ and S_3 is the group of pertmutations of $\{1,2,3\}$. In our case the diagonal elements are of the form $1 + O(h)$ and the off diagonal elements are $O(h)$. It is then clear that all contributions to the sum except the product of the diagonal elements are (at most) $O(h)$ and the product of the diagonal elements are $1 + O(h)$ and therefore $JF_h = |\det(DF_h)| = 1 + O(h)$.

On the surface we note that

$$DF_{h,\Gamma}(x)\xi = \xi + n(\xi \cdot \nabla_\Gamma \gamma_h) + \gamma_h \mathcal{H}_\Gamma \cdot \xi \qquad \forall \xi \in T_x(\Gamma) \tag{4.29}$$

where the last term is $O(h^2)$. The Jacobian determinant $JF_{h,\Gamma}$ is the norm of the cross product

$$\begin{aligned} |DF_{h,\Gamma}(x)\xi_1 \times DF_{h,\Gamma}(x)\xi_2| &= |(\xi_1 + n(\xi_1 \cdot \nabla_\Gamma \gamma_h)) \times (\xi_2 + n(\xi_2 \cdot \nabla_\Gamma \gamma_h))| + O(h^2) \\ &= |n - \xi_1(\xi_1 \cdot \nabla_\Gamma \gamma_h) - \xi_2(\xi_2 \cdot \nabla_\Gamma \gamma_h)| + O(h^2) \\ &= \Big(1 + (\xi_1 \cdot \nabla_\Gamma \gamma_h)^2 + (\xi_2 \cdot \nabla_\Gamma \gamma_h)^2\Big)^{1/2} + O(h^2) \\ &= 1 + O(h^2) \end{aligned} \tag{4.30}$$

where we used the identities $\xi_1 \times \xi_2 = n$, $n \times \xi_2 = -\xi_1$, $\xi_1 \times n = -\xi_2$, $n \times n = 0$, the fact that $\{\xi_1, \xi_2, n\}$ is a positively oriented orthonormal basis in $\mathbb{R}^3$ to compute the norm, and finally the estimate $(1+\gamma)^{1/2} \leq 1 + \gamma/2$, $\forall \gamma > 0$ in the last step. We thus have the following estimates for the surface Jacobian

$$\|JF_{h,\Gamma}\|_{L^\infty(\Gamma)} \lesssim 1, \qquad \|JF_{h,\Gamma}^{-1}\|_{L^\infty(\Gamma)} \lesssim 1, \qquad \|1 - JF_{h,\Gamma}\|_{L^\infty(\Gamma)} \lesssim h^2 \tag{4.31}$$

4.3 Lifting to the Exact Domain

We define the lifting or pullback of v^L with respect to F_h of a function v defined on Ω_0 as follows

$$v^L := v \circ F_h \tag{4.32}$$

We note in particular that any function defined on Ω_h and Γ_h may be lifted to a function on Ω and Γ. Using the chain rule

$$Dv^L = D(v \circ F_h) = (Dv \circ F_h) DF_h = (Dv)^L DF_h \tag{4.33}$$

and thus we obtain the identities

$$\nabla v^L = DF_h^T (\nabla v \circ F_h) = DF_h^T (\nabla v)^L \tag{4.34}$$

$$\begin{aligned} \nabla_\Gamma v^L &= P_\Gamma \nabla v^L = P_\Gamma DF_h^T (\nabla v)^L \\ &= P_\Gamma DF_h^T P_{\Gamma_h}^L (\nabla v)^L = (P_\Gamma DF_h^T P_{\Gamma_h}^L)(\nabla_{\Gamma_h} v)^L = DF_{h,\Gamma}^T (\nabla_{\Gamma_h} v)^L \end{aligned} \tag{4.35}$$

where $DF_{h,\Gamma}$ was defined in (4.16). Summarizing, we have the relations

$$\nabla v^L = DF_h^T (\nabla v)^L, \qquad (\nabla v)^L = DF_h^{-T} \nabla v^L \tag{4.36}$$

and

$$\nabla_\Gamma v^L = DF_{h,\Gamma}^T (\nabla_{\Gamma_h} v)^L, \qquad (\nabla_{\Gamma_h} v)^L = DF_{h,\Gamma}^{-T} \nabla_\Gamma v^L \tag{4.37}$$

Using the bounds (4.23) and (4.24) we conclude that the following equivalences hold

$$\|\nabla v^L\|_{L^2(\Omega)} \lesssim \|(\nabla v)^L\|_{L^2(\Omega)} \lesssim \|\nabla v^L\|_{L^2(\Omega)} \tag{4.38}$$

and

$$\|\nabla_\Gamma v^L\|_{L^2(\Gamma)} \lesssim \|(\nabla_{\Gamma_h} v)^L\|_{L^2(\Gamma)} \lesssim \|\nabla_\Gamma v^L\|_{L^2(\Gamma)} \tag{4.39}$$

4.4 Interpolation

Let $E_B : H^2(\Omega) \rightarrow H^2(\Omega_0)$ be an extension operator such that

$$\|E_B v\|_{H^2(\Omega_0)} \lesssim \|v\|_{H^2(\Omega)} \tag{4.40}$$

see [10] for details, and $E_S : H^2(\Gamma) \rightarrow H^2(\mathcal{U}(\Gamma))$ be the extension operator such that $E_S v = v \circ p$. Then we have the estimate

$$\|E_S v\|_{H^2(\mathcal{U}_\delta(\Gamma))} \lesssim \epsilon^{1/2} \|v\|_{H^2(\Gamma)} \tag{4.41}$$

for any $\epsilon > 0$ such that $\mathcal{U}_\epsilon(\Gamma) \subset \mathcal{U}_{\delta_0}(\Gamma)$. We finally define the extension operator

$$E : H^2(\Omega) \times H^2(\Gamma) \ni (u_B, u_S) \mapsto (E_B u_B, E_S u_S) \in H^2(\Omega_0) \times H^2(\mathcal{U}(\Gamma)) \tag{4.42}$$

When suitable we simplify the notation and write $u = Eu$. We let $\pi_{SZ,h} : L^2(\Omega_0) \rightarrow V_{0,h}$ denote the standard Scott-Zhang interpolation operator and recall the interpolation error estimate

$$\|v - \pi_{SZ,h} v\|_{H^m(K)} \leq C h^{2-m} \|v\|_{H^2(\mathcal{N}(K))}, \quad m = 1, 2, \quad K \in \mathcal{K}_{0,h} \tag{4.43}$$

where $\mathcal{N}(K) \subset \Omega_h$ is the union of the neighboring elements of K. We then define the interpolant

$$\pi_h u = (\pi_{B,h} u_B, \pi_{S,h} u_S) \tag{4.44}$$

where

$$\pi_{B,h} u_B = (\pi_{SZ,h} E_B u_B)|_{\mathcal{N}_{B,h}} \in V_{B,h} \tag{4.45}$$

and

$$\pi_{S,h} u_S = (\pi_{SZ,h} E_S u_S)|_{\mathcal{N}_{S,h}} \in V_{S,h} \tag{4.46}$$

We use the notation

$$\pi_h^L u = (\pi_h u)^L = (\pi_h u) \circ F_h \tag{4.47}$$

for the pullback of $\pi_h u$ to Ω by F_h. With these definitions we have the following lemma:

Lemma 41 *The following estimate holds*

$$|||u - \pi_h^L u||| \lesssim h\|u\|_{H^2(\Omega)\times H^2(\Gamma)} \tag{4.48}$$

Proof Using a trace inequality we obtain

$$\begin{aligned} |||u - \pi_h^L u|||^2 &= b_B k_B \|\nabla(u_B - \pi_{B,h}^L u_B)\|^2_{L^2(\Omega)} + b_S k_S \|\nabla(u_S - \pi_{S,h}^L u_S)\|^2_{L^2(\Gamma)} \qquad (4.49)\\ &\quad + \|b_B(u_B - \pi_{B,h}^L u_B) - b_S(u_S - \pi_{S,h}^L u_S)\|^2_{L^2(\Gamma)} \\ &\lesssim \|u_B - \pi_{B,h}^L u_B\|^2_{H^1(\Omega)} + \|u_S - \pi_{S,h}^L u_S\|^2_{H^1(\Gamma)} \qquad (4.50)\\ &= I + II \qquad (4.51)\end{aligned}$$

*Term **I**.* The first term may be estimated as follows

$$\begin{aligned} I &= \|u_B - \pi_{B,h}^L u_B\|_{H^1(\Omega)} = \|u_B - (\pi_{SZ,h} E_B u_B|_{\Omega_h})^L\|_{H^1(\Omega)} \\ &\leq \|u_B - (E_B u_B|_{\Omega_h})^L\|_{H^1(\Omega)} + \|((I - \pi_{SZ,h})E_B u_B|_{\Omega_h})^L\|_{H^1(\Omega)} \\ &\lesssim h\|u_B\|_{H^2(\Omega)} \qquad (4.52)\end{aligned}$$

Here we used the Sobolev Taylor's formula, see [2], to estimate the first term: consider first a function $v \in H^2(\Omega_0)$; then we have

$$\|v - v \circ F_h\|_{L^2(\Omega_0)} \lesssim \|I - F_h\|_{L^\infty(\Omega_0)} \|\nabla v\|_{L^2(\Omega_0)} \lesssim h^2 \|v\|_{H^1(\Omega_0)} \tag{4.53}$$

and for the derivative

$$\begin{aligned} &\|\nabla(v - v\circ F_h)\|_{L^2(\Omega_0)} \\ &\quad = \|\nabla v - DF_h^T(\nabla v \circ F_h)\|_{L^2(\Omega_0)} \qquad (4.54)\\ &\quad \leq \|\nabla v - (\nabla v \circ F_h)\|_{L^2(\Omega_0)} + \|(DF_h^T - I)(\nabla v \circ F_h)\|_{L^2(\Omega_0)} \qquad (4.55)\\ &\quad \lesssim \|I - F_h\|_{L^\infty(\Omega_0)} \|\nabla v\|_{H^1(\Omega_0)} + \|(DF_h^T - I)\|_{L^\infty(\Omega_0)} \|\nabla v\|_{L^2(\Omega_0)} \qquad (4.56)\\ &\quad \lesssim h^2 \|v\|_{H^2(\Omega_0)} + h\|\nabla v\|_{L^2(\Omega_0)} \qquad (4.57)\\ &\quad \lesssim h\|v\|_{H^2(\Omega_0)} \qquad (4.58)\end{aligned}$$

Now we may apply these inequalities with $v = E_B u_B$ and finally use the stability (4.40) of the extension operator E_B.

The second term in (4.52) is estimated by mapping to the discrete domain using the interpolation estimate (4.43) and then using the stability estimate (4.40).

*Term **II**.* Changing domain of integration from Γ to Γ_h and then using an element–wise trace inequality we obtain

$$\begin{aligned} \|\nabla_\Gamma(u_S - \pi_{S,h}^L u_S)\|^2_{L^2(\Gamma)} &= \|DF_{h,\Gamma}^T \nabla_{\Gamma_h}(u_S^e - \pi_{S,h} u_S)|JF_{h,\Gamma}|^{-1/2}\|^2_{L^2(\Gamma_h)} \qquad (4.59)\\ &\lesssim \sum_{K\in\mathcal{K}_{S,h}} h^{-1}\|u_S^e - \pi_{S,h}u_S\|_{H^1(K)} + h\|u_S^e - \pi_{S,h}u_S\|^2_{H^2(K)} \qquad (4.60)\\ &\lesssim \sum_{K\in\mathcal{K}_{S,h}} h\|u_S^e\|^2_{H^2(\mathcal{N}(K))} \qquad (4.61)\\ &\lesssim h^2\|u_S\|^2_{H^2(\Gamma)} \qquad (4.62)\end{aligned}$$

Here we used the interpolation estimate (4.43) followed by the stability estimate (4.41) for the extension operator with $\epsilon \sim h$, which is possible since it follows from quasi uniformity that there is $\epsilon \sim h$ such that $\cup_{K\in\mathcal{K}_{S,h}} \mathcal{N}(K) \subset \mathcal{U}_\epsilon(\Gamma)$.

We also need the face norm

$$|||v|||_{\mathcal{F}}^2 = h^3 j_B(v_{B,h}, v_{B,h}) + j_S(v_{S,h}, v_{S,h}) \tag{4.63}$$

$$= \sum_{F\in\mathcal{F}_{B,h}} h^3 \|[n_F\cdot\nabla v_B]\|^2_{L^2(F)} + \sum_{F\in\mathcal{F}_{S,h}} \|[n_F\cdot\nabla v_S\|^2_{L^2(F)} \tag{4.64}$$

for which we have the following interpolation error estimate.

Lemma 42 *The following estimate holds*

$$|||u-\pi_h u|||_{\mathcal{F}} \lesssim h\|u\|_{H^2(\Omega)\times H^2(\Gamma)} \tag{4.65}$$

Proof This estimate follows directly by using the element wise trace inequality $\|v\|^2_F \lesssim h^{-1}\|v\|^2_K + h\|\nabla v\|^2_K$ for all $v\in H^1(K)$, where F is a face of element K, followed by the interpolation estimate (4.43), and finally the stability estimates (4.40) and (4.41) for the extension operators.

4.5 Strang's Lemma

Lemma 43 *The following estimate holds*

$$\begin{aligned}\Big(|||u-u_h^L|||^2 + |||u-u_h|||^2_{\mathcal{F}}\Big)^{1/2} &\lesssim \Big(|||u-\pi_h^L u|||^2 + |||u-\pi_h u|||^2_{\mathcal{F}}\Big)^{1/2} \\ &\quad + \sup_{v\in W_h}\frac{a(u_h^L,v^L)-a_h(u_h,v)}{|||v^L|||} \\ &\quad + \sup_{v\in W_h}\frac{l(v^L)-l_h(v)}{|||v^L|||}\end{aligned} \tag{4.66}$$

Proof Adding and subtracting an interpolant $\pi_h^L u$, defined by (4.47), and using the triangle inequality we obtain

$$\begin{aligned}\Big(|||u-u_h^L|||^2 + |||u-u_h|||^2_{\mathcal{F}}\Big)^{1/2} &\leq \Big(|||u-\pi_h^L u|||^2 + |||u-\pi_h u|||^2_{\mathcal{F}}\Big)^{1/2} \\ &\quad + \Big(|||\pi_h^L u-u_h^L||| + |||\pi_h u-u_h|||^2_{\mathcal{F}}\Big)^{1/2}\end{aligned} \tag{4.67}$$

To estimate the second term we start from the coercivity

$$\Big(|||\pi_h^L u-u_h^L|||^2 + |||\pi_h u-u_h|||^2_{\mathcal{F}}\Big)^{1/2} \leq \sup_{v\in W_h\setminus\{0\}} \frac{a(\pi_h^L u-u_h^L, v^L) + j_h(\pi_h u-u_h,v)}{\Big(|||v^L|||^2 + |||v|||^2_{\mathcal{F}}\Big)^{1/2}} \tag{4.68}$$

Adding and subtracting the exact solution, and using Galerkin orthogonality the numerator may be written in the following form

$$a(\pi_h^L u-u_h^L, v^L) + j_h(\pi_h u-u_h,v)$$

$$= a(\pi_h^L u-u, v^L) + a(u-u_h^L, v^L) + j_h(\pi_h u-u_h,v) \tag{4.69}$$

$$= a(\pi_h^L u-u, v^L) + l(v^L) - a(u_h^L, v^L) + j_h(\pi_h u-u_h,v) \tag{4.70}$$

$$\begin{aligned}&= a(\pi_h^L u-u, v^L) + l(v^L) - l_h(v) \\ &\qquad + a_h(u_h,v) + j_h(u_h,v) - a(u_h^L,v^L) + j_h(\pi_h u-u_h,v)\end{aligned} \tag{4.71}$$

$$\begin{aligned}&= a(\pi_h^L u-u, v^L) + j_h(\pi_h u-u,v) \\ &\qquad + \Big(a_h(u_h,v) - a(u_h^L,v^L)\Big) + \Big(l(v^L)-l_h(v)\Big)\end{aligned} \tag{4.72}$$

Using (4.68) and estimating the first term using the Cauchy-Schwarz inequality the lemma follows directly.

4.6 Estimate of the Quadrature Errors

Lemma 44 *If h is small enough and δ in the definition (4.13) of F_h is chosen such that $\delta \sim h$, it holds*

$$\begin{aligned}|a(v^L, w^L) - a_h(v, w)| \lesssim{}& h^2 \|\nabla_\Gamma v_S^L\|_{L^2(\Gamma)} \|\nabla_\Gamma w_S^L\|_{L^2(\Gamma)} \\ &+ h^2 \|b \cdot v^L\|_{L^2(\Gamma)} \|b \cdot w^L\|_{L^2(\Gamma)} \\ &+ h \|\nabla v_B^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|\nabla w_B^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \quad \forall v, w \in W_h\end{aligned} \tag{4.73}$$

Proof Using the definition of the bilinear forms we have

$$\begin{aligned}a(v^L, w^L) - a_h(v, w) ={}& \underbrace{a_B(v_B^L, w_B^L) - a_{B,h}(v_B, w_B)}_{I} \\ &+ \underbrace{a_S(v_S^L, w_S^L) - a_{S,h}(v_S, w_S)}_{II} + \underbrace{a_{BS}(v, w) - a_{BS,h}(v, w)}_{III}\end{aligned} \tag{4.74}$$

$$= I + II + III \tag{4.75}$$

We now proceed with estimates of the three terms.

Term ***I***. Starting from the definition of the forms (2.15) and (3.11), changing domain of integration to Ω, and using (4.36), we obtain the following identity

$$\begin{aligned}&(b_B k_B)^{-1}(a_B(v_B^L, w_B^L) - a_{B,h}(v_B, w_B)) \\ &\quad = (DF_h^T (\nabla v_B)^L, DF_h^T (\nabla w_B)^L)_\Omega - (\nabla v_B, \nabla w_B)_{\Omega_h} \\ &\quad = (DF_h^T (\nabla v_B)^L, DF_h^T (\nabla w_B)^L)_\Omega - ((\nabla v_B)^L, (\nabla w_B)^L JF_h)_\Omega \\ &\quad = ((DF_h DF_h^T - JF_h I)(\nabla v_B)^L, (\nabla w_B)^L)_\Omega \\ &\quad = (\mathcal{A}_{h,\Omega} (\nabla v_B)^L, (\nabla w_B)^L)_\Omega\end{aligned} \tag{4.76}$$

In order to estimate $\mathcal{A}_{h,\Omega} = DF_h DF_h^T - JF_h I$ we note that $\mathcal{A}_{h,\Omega} = 0$ in $\Omega_0 \setminus \mathcal{U}_\delta(\Gamma)$ and in $\mathcal{U}_\delta(\Gamma)$ we have the identity

$$\mathcal{A}_{h,\Omega} = DF_h DF_h^T - JF_h I \tag{4.77}$$

$$= (DF_h - I)(DF_h - I)^T + (DF_h + DF_h^T) - I - JF_h I \tag{4.78}$$

$$= (DF_h - I)(DF_h - I)^T + (DF_h - I) + (DF_h - I)^T + (1 - JF_h) I \tag{4.79}$$

Estimating the right hand side we obtain

$$\begin{aligned}\|\mathcal{A}_{h,\Omega}\|_{L^\infty(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \lesssim{}& \|DF_h - I\|^2_{L^\infty(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \\ &+ \|DF_h - I\|_{L^\infty(\mathcal{U}_\delta(\Gamma)\cap\Omega)} + \|1 - JF_h\|_{L^\infty(\mathcal{U}_\delta(\Gamma)\cap\Omega)}\end{aligned} \tag{4.80}$$

$$\lesssim h^2 + h + h \lesssim h \tag{4.81}$$

where we used the estimates (4.22) and (4.28), and at last the fact that $h \in (0, h_0]$. Using the bound (4.81) for $\mathcal{A}_{h,\Omega}$ we obtain the estimate

$$|a_B(v^L, w^L) - a_{B,h}(v, w)| \lesssim h \|(\nabla v)^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|(\nabla w)^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.82}$$

$$\lesssim h \|\nabla v^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|\nabla w^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.83}$$

Here we used (4.23) to conclude that

$$\|(\nabla v)^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} = \|DF_h^{-T} (\nabla v^L)\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.84}$$

$$= \|DF_h^{-T}\|_{L^\infty(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|\nabla v^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.85}$$

$$\lesssim \|\nabla v^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.86}$$

Term ***II***. Proceeding in the same way and using (4.37) we obtain

$$
\begin{aligned}
&(b_S k_S)^{-1}\Big(a_S(v_S^L, w_S^L) - a_{S,h}(v_S, w_S)\Big)\\
&\quad = (\nabla_\Gamma v_S^L, \nabla_\Gamma w_S^L)_\Gamma - (\nabla_{\Gamma_h} v_S, \nabla_{\Gamma_h} w_S)_{\Gamma_h} &(4.87)\\
&\quad = (DF_{h,\Gamma}^T (\nabla_{\Gamma_h} v_S)^L, DF_{h,\Gamma}^T(\nabla_{\Gamma_h} w_S)^L)_\Gamma - ((\nabla_{\Gamma_h} v_S)^L, (\nabla_{\Gamma_h} w_S)^L JF_{h,\Gamma})_\Gamma &(4.88)\\
&\quad = ((DF_{h,\Gamma} DF_{h,\Gamma}^T - P_{\Gamma_h}^L JF_{h,\Gamma})(\nabla_{\Gamma_h} v_S)^L, (\nabla_{\Gamma_h} w_S)^L)_\Gamma &(4.89)\\
&\quad = (\mathcal{A}_{\Gamma,h}(\nabla_{\Gamma_h} v_S)^L, (\nabla_{\Gamma_h} w_S)^L)_\Gamma &(4.90)
\end{aligned}
$$

where we introduced

$$
\mathcal{A}_{\Gamma,h} = DF_{h,\Gamma} DF_{h,\Gamma}^T - P_{\Gamma_h}^L JF_{h,\Gamma} \tag{4.91}
$$

Using the definition (4.16) of $DF_{h,\Gamma}$ and the expression (4.14) for DF_h we have the identity

$$
\begin{aligned}
DF_{h,\Gamma} &= P_{\Gamma_h}^L DF_h P_\Gamma &(4.92)\\
&= P_{\Gamma_h}^L (I + n \otimes \nabla_\Gamma \gamma_h + \gamma_h \mathcal{H}_\Gamma) P_\Gamma &(4.93)\\
&= P_{\Gamma_h}^L P_\Gamma + (P_{\Gamma_h}^L n) \otimes \nabla_\Gamma \gamma_h + \gamma_h P_{\Gamma_h}^L \mathcal{H}_\Gamma P_\Gamma &(4.94)
\end{aligned}
$$

Here the second term can be estimated as follows

$$
\|(P_{\Gamma_h}^L n) \otimes \nabla_\Gamma \gamma_h\|_{L^\infty(\Gamma)} \lesssim \|P_{\Gamma_h}^L n\|_{L^\infty(\Gamma)} \|\nabla_\Gamma \gamma_h\|_{L^\infty(\Gamma)} \lesssim h^2 \tag{4.95}
$$

where we used the estimate

$$
\|P_{\Gamma_h}^L n\|_{L^\infty(\Gamma)} = \|P_{\Gamma_h}^L (n - n_h^L)\|_{L^\infty(\Gamma)} \lesssim \|n \circ p - n_h\|_{L^\infty(\Gamma_h)} \lesssim h \tag{4.96}
$$

For the third term we have the estimate

$$
\|\gamma_h P_{\Gamma_h}^L \mathcal{H}_\Gamma P_\Gamma\|_{L^\infty(\Gamma)} \lesssim \|\gamma_h\|_{L^\infty(\Gamma)} \|P_{\Gamma_h}^L\|_{L^\infty(\Gamma)} \|\mathcal{H}_\Gamma\|_{L^\infty(\Gamma)} \|P_\Gamma\|_{L^\infty(\Gamma)} \lesssim h^2 \tag{4.97}
$$

Thus we conclude that

$$
DF_{h,\Gamma} = P_{\Gamma_h}^L P_\Gamma + O(h^2) \tag{4.98}
$$

Inserting this identity into the expression (4.91) for $\mathcal{A}_{\Gamma,h}$ and using the identity

$$
P_{\Gamma_h}^L JF_{h,\Gamma} = P_{\Gamma_h}^L + P_{\Gamma_h}^L (JF_{h,\Gamma} - 1) = P_{\Gamma_h}^L + O(h^2) \tag{4.99}
$$

where we used (4.31), we obtain

$$
\mathcal{A}_{\Gamma,h} = P_{\Gamma_h}^L P_\Gamma P_{\Gamma_h}^L - P_{\Gamma_h}^L + O(h^2) \tag{4.100}
$$

Now the following identity holds

$$
P_{\Gamma_h}^L P_\Gamma P_{\Gamma_h}^L - P_{\Gamma_h}^L = -P_{\Gamma_h}^L (P_\Gamma - P_{\Gamma_h}^L)(P_\Gamma - P_{\Gamma_h}^L) P_{\Gamma_h}^L \tag{4.101}
$$

which leads to the estimate

$$
\|P_{\Gamma_h}^L P_\Gamma P_{\Gamma_h}^L - P_{\Gamma_h}^L\|_{L^\infty(\Gamma)} \leq \|P_{\Gamma_h}^L\|_{L^\infty(\Gamma)}^2 \|P_\Gamma - P_{\Gamma_h}^L\|_{L^\infty(\Gamma)}^2 \lesssim h^2 \tag{4.102}
$$

where we used the bound

$$
\begin{aligned}
\|P_\Gamma - P_{\Gamma_h}^L\|_{L^\infty(\Gamma)} &= \|n \otimes n - n_h^L \otimes n_h^L\|_{L^\infty(\Gamma)} &(4.103)\\
&\lesssim \|(n - n_h^L) \otimes n\|_{L^\infty(\Gamma)} + \|n_h^L \otimes (n - n_h^L)\|_{L^\infty(\Gamma)} &(4.104)\\
&\lesssim \|n^e - n_h\|_{L^\infty(\Gamma_h)} &(4.105)\\
&\lesssim h &(4.106)
\end{aligned}
$$

Thus we finally arrive at

$$
\|\mathcal{A}_{\Gamma,h}\|_{L^\infty(\Gamma)} \lesssim h^2 \tag{4.107}
$$

and therefore we have the estimate

$$
\begin{aligned}
|a_S(v^L, w^L) - a_{S,h}(v, w)| &\lesssim h^2 \|(\nabla_{\Gamma_h} v)^L\|_{L^2(\Gamma)} \|(\nabla_{\Gamma_h} w)^L\|_{L^2(\Gamma)} &(4.108)\\
&\lesssim h^2 \|\nabla_\Gamma v^L\|_{L^2(\Gamma)} \|\nabla_\Gamma w^L\|_{L^2(\Gamma)} &(4.109)
\end{aligned}
$$

where at last we used (4.24).

Term ***III***. We have

$$a_{BS}(v^L, w^L) - a_{BS,h}(v,w) = (b \cdot v^L, b \cdot w^L)_\Gamma - (b \cdot v, b \cdot w)_{\Gamma_h} \tag{4.110}$$
$$= ((1 - JF_{\Gamma,h}) b \cdot v^L, b \cdot w^L)_\Gamma \tag{4.111}$$

and thus we obtain the estimate

$$|a_{BS}(v^L, w^L) - a_{BS,h}(v,w)| \lesssim h^2 \|b \cdot v^L\|_{L^2(\Gamma)} \|b \cdot w^L\|_{L^2(\Gamma)} \tag{4.112}$$

Conclusion. Combining (4.75) with the estimates (4.83), (4.109), and (4.112) of Terms $I - III$ the proof follows.

Lemma 45 *If h is small enough and δ in the definition (4.13) of F_h is chosen such that $\delta \sim h$, and the right hand side $f_h = (f_{B,h}, f_{S,h})$ satisfies the estimate*

$$\|f_B - f_{B,h}^L\|_{L^2(\Omega)} + \|f_S - f_{S,h}^L\|_{L^2(\Gamma)} \lesssim h^2 \tag{4.113}$$

then it holds

$$|l(v^L) - l_h(v)| \lesssim h^2 \|v^L\|_{L^2(\Omega)\times L^2(\Gamma)} + h\|f_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|v^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \quad \forall v \in W_h \tag{4.114}$$

Proof We have

$$l(v^L) - l_h(v) = b_B(f_B, v_B^L)_\Omega - b_B(f_{B,h}, v_B)_{\Omega_h} + b_S(f_S, v_S^L)_\Gamma - b_S(f_{S,h}, v_S)_{\Gamma_h} \tag{4.115}$$
$$= b_B(f_B - f_{B,h}^L JF_h, v_B^L)_\Omega + b_S(f_S - f_{S,h}^L JF_{h,\Gamma}, v_S^L)_\Gamma \tag{4.116}$$

which immediately leads to the estimate

$$|l(v^L) - l_h(v)| \lesssim h^2 \|v^L\|_{L^2(\Omega)\times L^2(\Gamma)} + h\|f_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|v^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.117}$$

4.7 Error Estimates

Theorem 41 *The following error estimate holds*

$$\left(|||u - u_h^L|||^2 + |||u - u_h|||_{\mathcal{F}}^2 \right)^{1/2} \lesssim h\|u\|_{H^2(\Omega)\times H^2(\Gamma)} \tag{4.118}$$

for small enough mesh parameter h.

Proof Using the Strang Lemma, Lemma 43, in combination with the quadrature error estimates in Lemma 44 and 45, we obtain

$$\left(|||u - u_h^L|||^2 + |||u - u_h|||_{\mathcal{F}}^2 \right)^{1/2} \lesssim \left(|||u - \pi_h^L u|||^2 + |||u - \pi_h u|||_{\mathcal{F}}^2 \right)^{1/2} \tag{4.119}$$
$$+ \sup_{v\in W_h} \frac{a(u_h^L, v^L) - a_h(u_h, v)}{|||v^L|||} + \sup_{v \in W_h} \frac{l(v^L) - l_h(v)}{|||v^L|||}$$
$$\lesssim \left(|||u - \pi_h^L u|||^2 + |||u - \pi_h u|||_{\mathcal{F}}^2 \right)^{1/2} + h|||u_h^L||| + h^2 \tag{4.120}$$
$$\lesssim h \tag{4.121}$$

Here we used the interpolation error estimates in Lemma 41 and Lemma 42, and the stability estimate

$$|||u_h^L||| \lesssim \|f\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.122}$$

in the last inequality.

Theorem 42 *If f_B satisfies the additional assumption*

$$\|f_B\|_{H^1(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)} \lesssim 1 \tag{4.123}$$

then the following error estimate holds

$$\|u - u_h^L\|_{L^2(\Omega)\times L^2(\Gamma)} \lesssim h^2 \|u\|_{H^2(\Omega)\times H^2(\Gamma)} \tag{4.124}$$

for small enough mesh parameter h.

Proof Let ϕ be the solution to the dual problem: find $\phi \in W$ such that

$$a(v,\phi) = (v,\psi)_{L^2(\Omega)\times L^2(\Gamma)} \quad \forall v \in W \tag{4.125}$$

where $\psi = (\psi_B, \psi_S) \in L^2(\Omega)\times L^2(\Gamma)$. Then we have the regularity estimate

$$\|\phi\|_{H^2(\Omega)\times H^2(\Gamma)} \lesssim \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.126}$$

Setting $v = u - u_h^L$, and adding and subtracting suitable terms we obtain

$$(u_B - u_{B,h}^L, \psi_B)_\Omega + (u_S - u_{S,h}^L, \psi_S)_\Gamma = a(u - u_h^L, \phi) \tag{4.127}$$

$$= a(u - u_h^L, \phi - \pi_h^L \phi) + a(u - u_h^L, \pi_h^L \phi) \tag{4.128}$$

$$= \underbrace{a(u - u_h^L, \phi - \pi_h^L \phi)}_{I} + \underbrace{\Big(l(\pi_h^L \phi) - l_h(\pi_h \phi)\Big)}_{II}$$

$$+ \underbrace{\Big(a_h(u_h, \pi_h \phi) - a(u_h^L, \pi_h^L \phi)\Big)}_{III} + \underbrace{j_h(u_h, \pi_h \phi)}_{IV}$$

$$= I + II + III + IV \tag{4.129}$$

Term ***I****.* Using Cauchy-Schwarz, the energy norm estimate (4.118), the interpolation estimate (4.48) we obtain

$$|I| \leq |||u - u_h^L|||\, |||\phi - \pi_h^L \phi||| \lesssim h^2 \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.130}$$

Term ***II****.* Using Lemma 45 we immediately get

$$|II| \lesssim h^2 \|\pi_h^L \phi\|_{L^2(\Omega)\times L^2(\Gamma)} + h \|f_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|\pi_{B,h}^L \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.131}$$

To show that the second term is actually of second order we shall use the Poincaré inequality

$$\|v\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \lesssim (\delta/\delta_0)^{1/2} \|v\|_{H^1(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)}, \quad 0 < \delta \leq \delta_0 \tag{4.132}$$

see [8] for a proof of this inequality. We proceed in the following way

$$\|\pi_{B,h}^L \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \leq \|\pi_{B,h}^L \phi_B - \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} + \|\phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma))} \tag{4.133}$$

$$\lesssim (h^2 + \delta^{1/2}) \|\phi_B\|_{H^2(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)} \tag{4.134}$$

$$\lesssim (h^2 + h^{1/2}) \|\phi_B\|_{H^2(\Omega)} \tag{4.135}$$

where we added and subtracted ϕ_B, used the interpolation error estimate (4.48) for the first term and the Poincaré inequality (4.132) for the second term, and finally we used the fact that $\delta \sim h$, see Section 4.2. Next using the assumption(4.123) on f_B and again using the Poincaré inequality (4.132) together with $\delta \sim h$ we get

$$\|f_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \lesssim \delta^{1/2} \|f_B\|_{H^1(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \lesssim h^{1/2} \|f_B\|_{H^1(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)} \tag{4.136}$$

Collecting the bounds (4.131), (4.135), and (4.136), we obtain

$$|II| \lesssim h^2 \|\phi\|_{L^2(\Omega)\times L^2(\Gamma)} + h^{3/2}(h^2 + h^{1/2}) \|f_B\|_{H^1(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)} \|\phi_B\|_{H^2(\Omega)} \tag{4.137}$$

$$\lesssim h^2 \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.138}$$

where we used L^2 stability of π_h^L and the stability (4.126) for the dual problem.

*Term **III**.* Using Lemma 44 we obtain

$$
\begin{aligned}
|a(u_h^L, \pi_h^L \phi) - a_h(u_h^L, \pi_h^L \phi)| \lesssim\ & h^2 \|\nabla_\Gamma u_{S,h}^L\|_{L^2(\Gamma)} \|\nabla_\Gamma \pi_{S,h}^L \phi_S\|_{L^2(\Gamma)} \\
& + h^2 \|b \cdot u_h^L\|_{L^2(\Gamma)} \|b \cdot \pi_h^L \phi\|_{L^2(\Gamma)} \\
& + h \|\nabla u_{B,h}^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \|\nabla \pi_{B,h}^L \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)}
\end{aligned} \tag{4.139}
$$

To show that the third term is of second order we use the Poincaré inequality (4.132) in a similar way as for Term II. More precisely, we proceed in the following way

$$\|\nabla \pi_{B,h}^L \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \leq \|\nabla(\pi_{B,h}^L \phi_B - \phi_B)\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} + \|\nabla \phi_B\|_{L^2(\mathcal{U}_\delta(\Gamma))} \tag{4.140}$$

$$\lesssim (h + \delta^{1/2}) \|\phi_B\|_{H^2(\mathcal{U}_{\delta_0}(\Gamma)\cap\Omega)} \tag{4.141}$$

$$\lesssim (h + h^{1/2}) \|\phi_B\|_{H^2(\Omega)} \tag{4.142}$$

where again we used the fact that $\delta \sim h$, and the interpolation error estimate (4.48). The term $\|\nabla u_{B,h}^L\|_{L^2(\mathcal{U}_\delta(\Gamma))}$ can be estimated using the same technique but we employ the energy norm error estimate (4.118) instead

$$\|\nabla u_{B,h}^L\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \lesssim \|\nabla(u_{B,h}^L - u_B)\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} + \|\nabla u_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.143}$$

$$\lesssim |||u - u_h||| + \delta^{1/2} \|\nabla u_B\|_{L^2(\mathcal{U}_\delta(\Gamma)\cap\Omega)} \tag{4.144}$$

$$\lesssim (h + h^{1/2}) \|u\|_{H^2(\Omega)\times H^2(\Gamma)} \tag{4.145}$$

where we also used $\delta \sim h$. Combining (4.139) with estimates (4.142) and (4.145), and finally the stability estimate (4.126) for the dual problem we obtain

$$|III| \lesssim (h^2 + h(h + h^{1/2})^2) \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \lesssim h^2 \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.146}$$

*Term **IV**.* Using the fact that the jump term is consistent we obtain

$$|IV| = |j_h(u - u_h, \phi - \pi_h \phi)| \leq |||u - u_h|||_{\mathcal{F}} |||\phi - \pi_h \phi|||_{\mathcal{F}} \lesssim h^2 \|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} \tag{4.147}$$

where we used the energy estimate in Theorem 41 and the interpolation estimate in Lemma 41.

Conclusion. We conclude the proof by estimating the right hand side of (4.129) using the triangle inequality together with estimates (4.130), (4.138), (4.146), and (4.147) of Terms $I - IV$, and finally taking the supremum over all ψ such that $\|\psi\|_{L^2(\Omega)\times L^2(\Gamma)} = 1$.

5 Estimate of the Condition Number

Due to the different dimensions of the two coupled differential equations at the surface we shall see that it is natural to precondition the system in such a way that we seek $(v_{B,h}, v_{S,h})$ such that the solution $(u_{B,h}, u_{S,h})$ of (3.8) is given by

$$(u_{B,h}, u_{S,h}) = (v_{B,h}, h^{1/2} v_{S,h}) \tag{5.1}$$

The corresponding variational problem for $v_h = (v_{B,h}, v_{S,h})$ takes the form: find $v = (v_B, v_S) \in W_h$ such that

$$\tilde{A}_h(v, w) = \tilde{l}_h(w) \quad \forall w \in W_h \tag{5.2}$$

where the bilinear forms are defined by

$$\tilde{A}_h(v, w) = A_h((v_B, h^{1/2} v_S), (w_B, h^{1/2} w_S)), \quad \tilde{l}_h(w) = l_h((w_B, h^{1/2} w_S)) \tag{5.3}$$

We shall now estimate the condition number of the stiffness matrix $\tilde{A}$ associated with the bilinear form $\tilde{A}_h(\cdot,\cdot)$. Let $\{\varphi_{B,i}\}_{i=1}^{N_B}$ and $\{\varphi_{S,i}\}_{i=1}^{N_S}$ be the standard piecewise linear basis functions in $V_{B,h}$

and $V_{S,h} \oplus \langle 1_{\Gamma_h}\rangle$, respectively. Note that we have added the one dimensional space $\langle 1_{\Gamma_h}\rangle$ of constant functions on Γ_h. Define the following basis in the product space $V_{B,h} \times V_{S,h} \oplus \langle 1_{S,h}\rangle$:

$$\varphi_i = \begin{cases} (\varphi_{B,i}, 0) & 1 \leq i \leq N_B \\ (0, \varphi_{S,i-N_B}) & 1 + N_B \leq i \leq N = N_B + N_S \end{cases} \tag{5.4}$$

The expansion $v = \sum_{i=1}^N \widehat{v}_i \varphi_i$ defines an isomorphism

$$V_{B,h} \times V_{S,h}/\langle 1_{\Gamma_h}\rangle \oplus \langle 1_{S,h}\rangle \to \mathbb{R}^{N_B} \times \mathbb{R}^{N_S}/\langle 1_{\mathbb{R}^{N_S}}\rangle \oplus \langle 1_{\mathbb{R}^{N_S}}\rangle \tag{5.5}$$

$$(v_B, v_S \oplus \overline{v}_S 1_{S,h}) \mapsto (\widehat{v}_B, \widehat{v}_S \oplus \overline{v}_S 1_{\mathbb{R}^{N_S}}) \tag{5.6}$$

where v_S is the unique element in the equivalence classes of $V_{S,h}/\langle 1_{\Gamma_h}\rangle$ with $\int_{\Gamma_h} v_S = 0$ and $\overline{v}_S = |\Gamma_h|^{-1} \int_{\Gamma_h} v_S$ is the meanvalue of v_S. If we introduce the mesh dependent L^2 norm

$$\|v\|_h^2 = \|v_B\|^2_{L^2(\mathcal{N}_{B,h})} + \|v_S\|^2_{L^2(\mathcal{N}_{S,h})} \tag{5.7}$$

where the sets $\mathcal{N}_{B,h}$ and $\mathcal{N}_{S,h}$ are defined in (3.6), we have the following standard estimate

$$ch^{-d}\|v\|_h^2 \lesssim |\widehat{v}|_N^2 \lesssim Ch^{-d}\|v\|_h^2 \tag{5.8}$$

Let $\tilde{A}$ be the stiffness matrix with elements $a_{ij} = \tilde{A}_h(\varphi_i, \varphi_j) + J_h(\varphi_i, \varphi_j)$. The stiffness matrix is symmetric and has a one dimensional kernel consisting of a constant functions $v = (v_B, v_S)$, that satisfy $b \cdot v = b_B v_B - b_S v_S = 0$. We shall estimate the condition number of $\tilde{A}$ as an operator on the invariant space $V = \mathbb{R}^{N_B} \times \mathbb{R}^{N_S}/\langle 1_{\mathbb{R}^{N_S}}\rangle$ defined by

$$\kappa(\tilde{A}) = |\tilde{A}|_V |\tilde{A}^{-1}|_V \tag{5.9}$$

where $|x|_N^2 = \sum_{i=1}^N x_i^2$ for $x \in \mathbb{R}^N$ and $|\tilde{A}|_V = \sup_{x \in V\setminus\{0\}} \frac{|\tilde{A}x|_N}{|x|_N}$ for $\tilde{A} \in \mathbb{R}^{N\times N}$. Next we introduce the discrete energy norm

$$|||v|||_h^2 = A_h(v, v) = a_h(v, v) + j_h(v, v) \tag{5.10}$$

The proof of the estimate of the condition number follow the approach presented in [9] and rely on a Poincaré and an inverse inequality which we prove next.

Lemma 51 *(Poincaré inequality) Independently of the mesh/boundary intersection it holds that*

$$\|(v_B, v_S)\|_h \lesssim |||(v_B, h^{1/2} v_S)|||_h \quad \forall (v_B, v_S) \in W_h \tag{5.11}$$

Proof We have the following estimates

$$\|v_S\|^2_{L^2(\mathcal{N}_{S,h})} \lesssim h\|v_S\|^2_{L^2(\Gamma_h)} + hj_S(v_S, v_S) \tag{5.12}$$

$$\lesssim h\|\nabla_{\Gamma_h} v_S\|^2_{L^2(\Gamma_h)} + hj_S(v_S, v_S) \tag{5.13}$$

$$\lesssim \|\nabla_{\Gamma_h} h^{1/2} v_S\|^2_{L^2(\Gamma_h)} + j_S(h^{1/2} v_S, h^{1/2} v_S) \tag{5.14}$$

$$\lesssim |||(v_B, h^{1/2} v_S)|||_h^2 \tag{5.15}$$

where inequality (5.12) is provided by Lemma 4.4 in [5], in (5.13) we used the fact that $\int_{\Gamma_h} v_S = 0$ to apply a Poincaré inequality on Γ_h, see Lemma 4.1 in [5] for a proof, in (5.15) we wrote the estimate in terms of the scaled variable $h^{1/2} v_S$, and finally in (5.15) we estimated the left hand side by the full energy norm for $(v_B, h^{1/2} v_S)$.

Next using the control, see (5.23) below, provided by the jump term $j_B(\cdot,\cdot)$, and then adding and subtracting the L^2-projection $P_0 v_B$ of v_B onto constant functions on Ω_h followed by a Poincaré inequality we obtain

$$\begin{aligned}
\|v_B\|^2_{L^2(\mathcal{N}_{B,h})} &\lesssim \|v_B\|^2_{L^2(\Omega_h)} + h^3 j_B(v_B,v_B) &(5.16)\\
&\lesssim \|P_0 v_B\|^2_{L^2(\Omega_h)} + \|\nabla v_B\|^2_{L^2(\Omega_h)} + h^3 j_B(v_B,v_B) &(5.17)\\
&\lesssim \|P_0 v_B\|^2_{L^2(\Gamma_h)} + |||(v_B, h^{1/2} v_S)|||^2_h &(5.18)\\
&\lesssim \|(I-P_0)v_B\|^2_{L^2(\Gamma_h)} + \|v_B\|^2_{L^2(\Gamma_h)} + |||(v_B, h^{1/2} v_S)|||^2_h &(5.19)\\
&\lesssim \|\nabla v_B\|^2_{L^2(\Omega_h)} + b_B^{-2}\|b_B v_B - b_S h^{1/2} v_S\|^2_{L^2(\Gamma_h)} &(5.20)\\
&\qquad + b_B^{-2} b_S^2 \|h^{1/2} v_S\|^2_{L^2(\Gamma_h)} + |||(v_B, h^{1/2} v_S)|||^2_h \\
&\lesssim b_B^{-2}\|h^{1/2} v_S\|^2_{L^2(\Gamma_h)} + |||(v_B, h^{1/2} v_S)|||^2_h &(5.21)
\end{aligned}$$

Here and we added and subtracted v_B in order to control $P_0 v_B$ using the coupling term together with the estimate $\|h^{1/2} v_S\|^2_{\Gamma_h} \lesssim |||(v_B, h^{1/2} v_S)|||^2_h$, provided by (5.12)-(5.15), and the fact that the constant $b_B > 0$. Furthermore, in (5.16) is a consequence of the inverse inequality

$$\|v\|^2_{L^2(K_1)} \lesssim \|v\|^2_{L^2(K_2)} + h^3 \|[n_F \cdot \nabla v]\|^2_{L^2(F)} \quad \forall v \in V_{B,h} \tag{5.22}$$

that holds for each pair of elements K_1 and K_2 that share a face F. Iterating the inequality (5.22) we may control the elements at the boundary in terms of the elements in the interior of Ω_h as follows

$$\|v\|^2_{L^2(K_1)} \lesssim \|v\|^2_{L^2(K_N)} + \sum_{i=1}^{N-1} h^3 \|[n_F \cdot \nabla v]\|^2_{L^2(F_i)} \quad \forall v \in V_{B,h} \tag{5.23}$$

see [14] for further details. Note that for sufficiently small mesh size the length N of the shortest chain of elements that share an edge between an element that intersects the boundary and an interior element is uniformly bounded.

Combining the two estimates (5.15) and (5.21) the lemma follows directly.

Lemma 52 *(Inverse inequality) Independently of the mesh/boundary intersection it holds that*

$$|||(v_B, h^{1/2} v_S)|||^2_h \lesssim h^{-2} \|(v_B, v_S)\|^2_h \quad \forall (v_B, v_S) \in W_h \tag{5.24}$$

Proof Using standard estimates we obtain the following three estimates

$$b_B k_B \|\nabla v_B\|^2_{L^2(\Omega_h)} + \tau_B h^3 j_B(v_B, v_B) \lesssim h^{-2}\|v_B\|^2_{L^2(\mathcal{N}_{B,h})} \lesssim h^{-2}\|(v_B, v_S)\|^2_h \tag{5.25}$$

$$\|b_B v_B - b_S h^{1/2} v_S\|^2_{L^2(\Gamma_h)} \lesssim h^{-1} b_B \|v_B\|^2_{L^2(\mathcal{N}_{S,h})} + b_S \|v_S\|^2_{L^2(\mathcal{N}_{S,h})} \lesssim h^{-2}\|(v_B, v_S)\|^2_h \tag{5.26}$$

$$b_S k_S h \|\nabla_{\Gamma_h} v_S\|^2_{L^2(\Gamma_h)} + \tau_S h j_S(v_S, v_S)) \lesssim (b_S k_S + \tau_S)\|\nabla v_S\|^2_{L^2(\mathcal{N}_{S,h})} \lesssim h^{-2}\|v_S\|^2_{L^2(\mathcal{N}_{S,h})}$$

and thus the proof is complete.

Finally, we are ready to prove our final estimate of the condition number.

Theorem 51 *The following estimate of the condition number of the stiffness matrix holds independently of the mesh/boundary intersection*

$$\kappa(\tilde{A}) \lesssim h^{-2} \tag{5.27}$$

Proof We need to estimate $|\tilde{A}|_V$ and $|\tilde{A}^{-1}|_V$. Starting with $|\tilde{A}|_V$ we have

$$|\tilde{A}\widehat{v}|_V = \sup_{\widehat{w}\in V\setminus\{0\}} \frac{(\widehat{w},\tilde{A}\widehat{v})_N}{|\widehat{w}|_N} \tag{5.28}$$

$$= \sup_{w\in W_h\setminus\{0\}} \frac{A_h((v_B,h^{1/2}v_S),(w_B,h^{1/2}w_S))}{|||(w_B,h^{1/2}w_S)|||_h} \frac{|||(w_B,h^{1/2}w_S)|||_h}{\|(w_B,w_S)\|_h} \frac{\|(w_B,w_S)\|_h}{|\widehat{w}|_N} \tag{5.29}$$

$$\lesssim h^{(d-2)/2}|||(v_B,h^{1/2}v_S)|||_h \tag{5.30}$$

$$\lesssim h^{d-2}|\widehat{v}|_N \tag{5.31}$$

where at last we used the estimate

$$|||(v_B,h^{1/2}v_S)|||_h \lesssim h^{-1}\|(v_B,v_S)\|_h \lesssim h^{(d-2)/2}|\widehat{v}|_N \tag{5.32}$$

together with (5.24) and (5.8). Thus

$$|\tilde{A}|_V \lesssim h^{d-2} \tag{5.33}$$

Next we turn to the estimate of $|\tilde{A}^{-1}|_V$. Using (5.8) and (5.11), we get

$$\begin{aligned} |\widehat{v}|_N^2 &\lesssim h^{-d}|||(v_B,h^{1/2}v_S)|||_h^2 \lesssim h^{-d}A_h((v_B,h^{1/2}v_S),(v_B,h^{1/2}v_S)) \\ &\lesssim h^{-d}(\widehat{v},\tilde{A}\widehat{v})_N \lesssim h^{-d}|\widehat{v}|_N|\tilde{A}\widehat{v}|_N \end{aligned} \tag{5.34}$$

and thus we conclude that $|\widehat{v}|_N \leq Ch^{-d}|\tilde{A}\widehat{v}|_N$. Setting $\widehat{v} = \tilde{A}^{-1}\widehat{w}$ we obtain

$$|\tilde{A}^{-1}|_N \lesssim h^{-d} \tag{5.35}$$

Combining estimates (5.33) and (5.35) of $|\tilde{A}|_N$ and $|\tilde{A}^{-1}|_N$ the theorem follows.

6 Numerical results

We consider an example where the domain Ω is the unit sphere, $k_B = k_S = 1$, $b_B = b_S = 1$, and f_B and f_S are chosen such that the exact solution is as in [8] given by

$$\begin{aligned} u_B &= e^{(-x(x-1)-y(y-1))} \\ u_S &= (1 + x(1-2x) + y(1-2y))e^{(-x(x-1)-y(y-1))} \end{aligned} \tag{6.1}$$

We study the convergence rate of the numerical solution $u_h = (u_{B,h}, u_{S,h})$ and the condition number of the system matrix using the proposed finite element method. A direct solver is used to solve the linear systems. The stabilization parameters $\tau_B = \tau_S = 10^{-2}$. We use a structured mesh for Ω_0 and the mesh parameter $h = h_x = h_y = h_z$.

To represent the boundary Γ we use the standard level set method. We define a piecewise linear approximation to the distance function on $\mathcal{K}_{0,h}$ and Γ is approximated as the zero level set of this approximate distance function. Thus, Γ_h is represented by linear segments on $\mathcal{K}_{0,h}$. The normal vectors are computed from the linear segments.

The solution $u_{S,h}$ with $h = 0.13125$ and the triangulation of Γ_h are shown in Fig. 3. The convergence of u_h in both the L^2 norm and the H^1 norm are shown in Fig. 4. We have as expected first order convergence in the H^1 norm and second order convergence in the L^2 norm. The spectral condition number of the matrix $\tilde{A}$ associated with the bilinear form $\tilde{A}_h(\cdot,\cdot)$ (see equation (5.3)) is shown for different mesh sizes in Fig. 5.

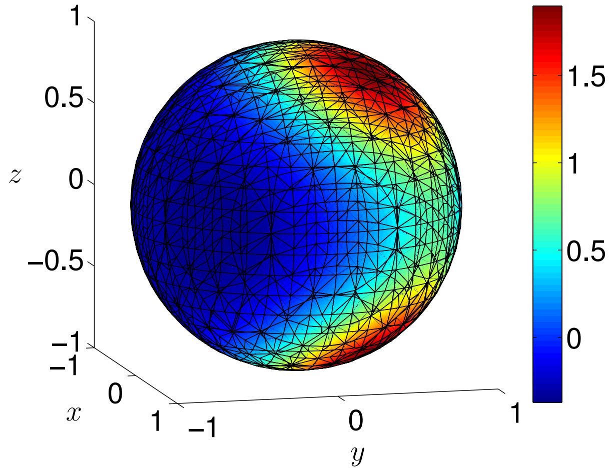
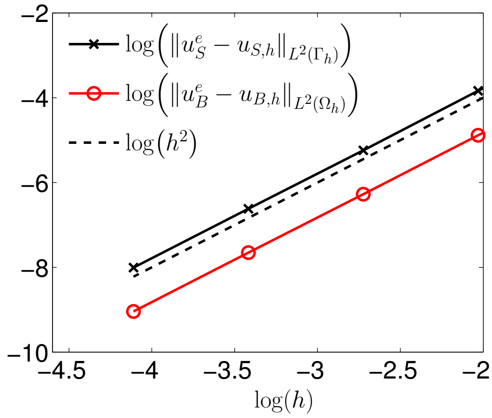


Fig. 3 The solution $u_{S,h}$ with $h = 0.13125$.



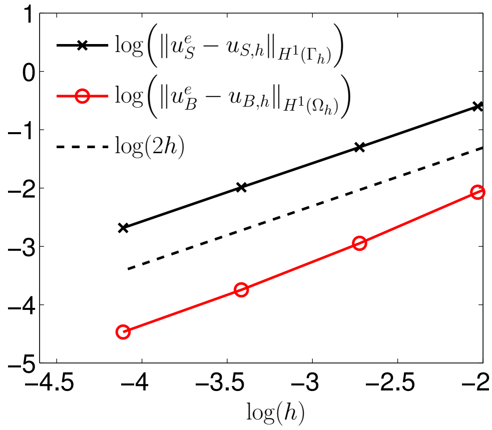


Fig. 4 Convergence of u_B and u_S. Upper panel: The error measured in the L^2 norm versus mesh size. Lower panel: The error measured in the H^1 norm versus mesh size.

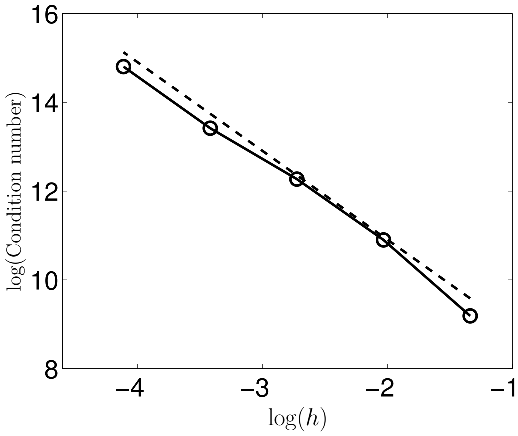


Fig. 5 The spectral condition number of the matrix $\tilde{A}$ versus mesh size. The dashed line is proportional to h^{-2}.

Appendix

Here we will give some details on the inequalities (4.5). First we recall that

$$q_h(x) = x + \gamma_h(x)n(x) \quad x \in \Gamma \tag{6.2}$$

Now using the definition of the closest point mapping

$$y = p(y) + \rho(y)n^e(y) \quad y \in \Gamma_h \tag{6.3}$$

Setting $x = p(y)$ in (6.2) we have

$$y = p(y) + \gamma_h(p(y))n^e(y) \quad y \in \Gamma_h \tag{6.4}$$

and therefore, by uniqueness, $\rho(y) = \gamma_h(p(y)), \forall y \in \Gamma_h$. Thus we have $\gamma_h = \rho^L$ and we immediately obtain the first inequality in (4.5) since

$$\|\gamma_h\|_{L^\infty(\Gamma)} = \|\rho^L\|_{L^\infty(\Gamma)} = \|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^2 \tag{6.5}$$

Next using (4.37) we have the identity

$$\nabla_\Gamma \gamma_h = \nabla_\Gamma \rho^L = DF^T_{h,\Gamma}(\nabla_{\Gamma_h}\rho)^L = DF^T_{h,\Gamma}(P_{\Gamma_h} n^e)^L \tag{6.6}$$

Estimating the right hand side using (4.24) and (4.96) we finally obtain

$$\|\nabla_\Gamma \gamma_h\|_\Gamma \lesssim \|\nabla_{\Gamma_h}\rho\|_{\Gamma_h} \lesssim \|P_{\Gamma_h} n^e\|_{\Gamma_h} \lesssim h \tag{6.7}$$

which is the second bound in (4.5).

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