# Intermittency and Localisation Phenomena in the Parabolic Anderson and Bouchaud Trap Models

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A dissertation submitted to

**University College London**

in partial fulfilment of the
requirements for the degree of

**Doctor of Philosophy**

Supervised by

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March 2016

## Declaration

I, Stephen Muirhead, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been clearly indicated in the thesis.

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## Relationship to published work

Substantial parts of this thesis are based on the published works [28, 33, 58] and the preprint [59]. A significant portion of the results in Chapter 2 appear in [33], although here we present a new proof based on the methods developed in [59], and additionally derive several new extensions to these results. Chapter 3 is closely based on [28, 58], although here we have heavily restructured the results in these works to present them in a unified manner. Chapter 4 is a slightly simplified version of [59]; here we have chosen to limit the full generality of the results in [59] to maximise clarity of presentation.

I am the sole author of the published work [58]. The published works [28, 33] and the preprint [59] are jointly authored by myself and, respectively, D. Croydon, A. Fiodorov and R. Pymar. Credit for the parts of this thesis based on these works lies with myself and the respective coauthor in equal measure.

# Abstract

This thesis studies intermittency and localisation phenomena in the *parabolic Anderson model* (PAM) and the *Bouchaud trap model* (BTM), models for random walks in a random branching environment and a random trapping landscape respectively.

In the PAM, we study the phenomenon of *complete localisation*, which describes the eventual concentration of the (renormalised) mass function on a single site with overwhelming probability. Our main result is that complete localisation holds for potential distributions with (i) Weibull tail decay, and (ii) fractional-double-exponential tail decay. Since complete localisation is strongly conjectured to break down for potentials with double-exponential tail decay, in a sense our work completes the program of establishing complete localisation in the PAM begun in [46, 52, 64]. In the Weibull case, we further give a detailed geometric description of the complete localisation behaviour.

In the BTM, we study the regime of slowly varying traps, that is, when the survival function of the trap distribution has a slowly varying tail at infinity. Our main result is that the BTM on the integers exhibits extremely strong localisation behaviour that is qualitatively different to the known localisation behaviour in the regularly varying case. More precisely, we demonstrate that (i) the mass function of the BTM concentrates on two-sites with overwhelming probability, and (ii) the rescaled BTM converges to a highly-singular process we call the *extremal FIN process*.

Finally, we explore the interaction between the localisation phenomena due to random branching and trapping mechanisms by studying a hybrid model which combines these mechanisms. Our main result is that, under certain natural assumptions, the localisation effects due to random branching and trapping mechanisms tend to (i) mutually reinforce, and (ii) induce a local correlation in the random fields (c.f. the 'fit and stable' hypothesis of population dynamics).

## Acknowledgements

First and foremost, my sincerest gratitude to my supervisor Dr. Nadia Sidorova, for the thoughtfully proposed research topic, the boundless patience and generosity with your time, for judiciously balancing guidance and independence, and for the countless other ways in which you helped me realise this thesis.

I gratefully acknowledge the financial support of the Leverhulme Trust, as well as the financial and wider support of the UCL Department of Mathematics, without which this thesis would not have been possible.

To my coauthors Dr. David Croydon, Artiom Fiodorov and Dr. Richard Pymar, thank you for the opportunity to collaborate with such capable, diligent mathematicians. To Artiom and Richard in particular, who always gave me an outlet for my questions, doubts and often vague ideas, it was a pleasure to work so closely with you. Thanks also to the many others who contributed in some way to this thesis – among them Prof. Wolfgang König, Dr. Onur Gün, Dr. Renato Soares dos Santos, Franziska Flegel, and in particular the thesis examiners, Prof. Peter Mörters and Dr. Codina Cotar – for your helpful suggestions, insights, and more than a few corrections.

To the other graduate students in the department, thank you for providing a motivating and supportive environment in which to work, with just the right level of distraction. To Adam, Artiom and Olly in particular, here right from the beginning, thanks for the companionship, and no hard feelings for the pranks.

To my parents, my sister, and the rest of my family and friends, thank you for your unwavering support and encouragement. You have all made sacrifices, large and small, to enable me to get to this point, and for that I will always be grateful.

Last but not least, to Annika, who gives me a home to go back to, thank you for always putting a smile on my face.

# General notation

For ease of reference, we collect here general notation that will be used throughout the thesis. All other notation will be introduced as needed.

**Asymptotic notation**: For real-valued functions $f, g$ we use $f(x) \sim g(x)$ to denote that

$$\lim_{x\to\infty} f(x)/g(x) = 1\,,$$

and $f(x) = o(g(x))$, $f(x) \ll g(x)$ or $g(x) \gg f(x)$ to denote that

$$\lim_{x\to\infty} f(x)/g(x) = 0\,.$$

We use $f(x) = O(g(x))$ to denote that, as $x \to \infty$, there exists a $c > 0$ such that eventually

$$|f(x)| < c|g(x)|\,.$$

**Notation for collections of random variables**: When we describe a collection $X = \{X_i\}_{i\in I}$ of random variables as *bounded above in probability* we mean that $\{X_i\}_{i\in I}$ is tight. In other words, for each $\varepsilon > 0$ there exists a $c > 0$ such that

$$\inf_{i\in I} P\left(X_i < c\right) > 1 - \varepsilon\,.$$

Similarly, when we describe a collection $X = \{X_i\}_{i\in I}$ of strictly-positive random variables as *bounded below in probability* we mean that $\{X_i^{-1}\}_{i\in I}$ is tight. In other words, for each $\varepsilon > 0$ there exists a $c > 0$ such that

$$\inf_{i\in I} P\left(X_i > c\right) > 1 - \varepsilon\,.$$

For random variables $X$ and $Y$, we say that the random variable $X$ *stochastically dominates* the random variable $Y$, denoted by $X \succ Y$ or $Y \prec X$, if for each $x \in \mathbb{R}$

$$F_X(x) \le F_Y(x)\,,$$

where $F_X$ and $F_Y$ denote the cumulative distribution functions of $X$ and $Y$ respectively.

**Notation for subsets of $\mathbb{Z}^d$**: We equip $\mathbb{Z}^d$ with the standard $\ell_1$ distance, denoted by $|\cdot|$. For a site $z \in \mathbb{Z}^d$ and a radius $r \ge 0$, define the closed ball

$$B(z, r) := \{y \in \mathbb{Z}^d : |z - y| \le r\}\,.$$

For a set $S \subseteq \mathbb{Z}^d$, we denote the complement of‘$S$ by $S^c$ (similarly, for a probability event $\mathcal{E}$ we denote the complement of $\mathcal{E}$ by $\mathcal{E}^c$.) For $S \subseteq \mathbb{Z}^d$ define its $r$-extension, for $r \ge 0$,

$$B(S, r) := \bigcup_{z\in S} B(z, r)\,,$$

its separation distance

$$\mathrm{sep}\,(S) := \min_{\substack{x,y\in\mathcal{S}\\ x\neq y}} |x-y|\,,$$

and its outer boundary

$$\partial S = \{y \in S^c : \text{there exists } x \in S \text{ such that } |x-y| = 1\}\,.$$

For a set $S \subseteq \mathbb{Z}^d$, the function $\mathbb{1}_S$ denotes the indicator function of $S$, in other words, the function $\mathbb{1}_S : \mathbb{Z}^d \to \{0,1\}$ that takes 1 on $S$ and 0 elsewhere (similarly, where the function $\mathbb{1}_\mathcal{E}$ is indexed by a probability event $\mathcal{E}$, it denotes the indicator function of the event $\mathcal{E}$.)

**Notation for operators**: Wherever a vector $v$ is naturally interpreted as an operator, it is understood as the multiplication operator associated to $v$, i.e. the operator $\mathcal{V}$ defined by

$$\mathcal{V}f(x) := v(x)f(x)\,.$$

For an operator $\mathcal{H}$ on functions $f : \mathbb{Z}^d \to \mathbb{R}$, the restriction of $\mathcal{H}$ to a set $S \subseteq \mathbb{Z}^d$ is the operator on functions $f|_S : S \to \mathbb{R}$ induced by the operator $\mathbb{1}_S\mathcal{H}\mathbb{1}_S$.

**Notation for paths**: For $k \in \mathbb{N}$ and sites $y, z \in \mathbb{Z}^d$, let $\Gamma_k(y,z)$ be the set of nearest neighbour paths in $\mathbb{Z}^d$ of length $k$ running from $y$ to $z$, with each $p \in \Gamma_k(y,z)$ indexed as

$$y =: p_0 \to p_1 \to p_2 \to \ldots \to p_k := z\,.$$

Similarly, denote the path collections

$$\Gamma_k(y) := \bigcup_{z\in\mathbb{Z}^d} \Gamma_k(y,z)\ , \quad \Gamma(y,z) := \bigcup_{k\in\mathbb{N}} \Gamma_k(y,z)\,,$$

$$\Gamma(y) := \bigcup_{k\in\mathbb{N}} \Gamma_k(y)\ , \quad \Gamma := \bigcup_{y\in\mathbb{Z}^d} \Gamma(y)\,.$$

For a site $z \in \mathbb{Z}^d$, denote by $n(z)$ the number of shortest paths from the origin to $z$, i.e., $n(z) := |\Gamma_{|z|}(0,z)|$. For a path $p \in \Gamma_k(y,z)$ denote the path set $\{p\} := \{p_0, p_1, \ldots, p_k\}$ and path length $|p| := k$. For a nearest neighbour random walk $X$ let $p(X_t) \in \Gamma(X_0)$ denote the path associated with the trajectory of $\{X_s\}_{s\le t}$, and let $p_k(X) \in \Gamma_k(X_0)$ denote the path associated with the random walk $\{X_s\}_{s\ge 0}$ up to and including its $k^{\text{th}}$ jump.

**Other notation**:

We use $\mathbf{1}$ and $\mathbf{0}$ to denote, respectively, the vector of ones and the zero vector.

For a real-valued unbounded càdlàg function $f$, we use the phrase *right-continuous inverse* to describe the function

$$f^{-1}(x) := \inf\{t : f(t) > x\}\,.$$

For $x, y \in \mathbb{R}$, define $x \wedge y := \min\{x, y\}$. Further, denote by $[x]$ and $x^+$ the integer and positive parts of $x$ respectively, i.e.,

$$[x] := \max\{z \in \mathbb{Z} : z \le x\} \quad \text{and} \quad x^+ := \max\{x, 0\}\,.$$

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# Chapter 1

# Introduction

This thesis studies intermittency and localisation phenomena in two distinct but related models of random walks in random media – the *parabolic Anderson model* (PAM) and the *Bouchaud trap model* (BTM) – which are models for random walks in a random branching environment and a random trapping landscape respectively. The PAM and BTM both have their origins in the statistical physics literature, the PAM as a model for electron localisation inside a semiconductor, and the BTM as a model for the dynamics of spin-glasses on certain intermediate time-scales.

The PAM and BTM are of great interest in the theory of stochastic processes because they are important examples of *intermittent* processes. In other words, over long periods of time these models develop pronounced spatial and temporal inhomogeneities. This may be contrasted with the tendency of many commonly-studied stochastic processes, such as the simple random walk, to *homogenise* over long periods of time. The term *localisation* refers to an extreme form of intermittency in which the models tend to concentrate on small subsets of the domain with high probability.

The overall goal of this thesis is to seek a better understanding of intermittency and localisation phenomena in the PAM and BTM, to both determine the conditions under which localisation occurs and to describe its qualitative and quantitative features. Our focus will mainly be on the Cauchy problem for the models: given an initial configuration of particles, we study the time-evolution of the (renormalised) probability mass function of the model, that is, the relative likelihood of finding a particle at a given site. For both the PAM and the BTM this evolution is governed by a parabolic partial differential equation with environment measurable random coefficients, whose (environment measurable) solution depends on the particular realisation of the random media. In this context, localisation refers to the concentration, with high probability, of the (suitably renormalised) probability mass function on small subsets of the domain.

A secondary aim of this thesis is to explore the interaction between the localisation phenomena exhibited by the PAM and BTM. To this end we introduce a hybrid model combining the dynamics of the PAM and BTM, and examine its localisation properties. We refer to this model as the Bouchaud–Anderson model (BAM). The BAM has several links to the existing literature, including in the study of population dynamics in mathematical biology, and in the study of quantum mechanics in the case of position-dependent mass.

This thesis is structured as follows. In Chapter 1, we introduce the PAM and BTM, provide a review of known intermittency and localisation results for these models, and briefly present the most salient aspects of our main results. We also introduce the BAM – our hybrid model combining the dynamics of the PAM and BTM – and briefly outline our main results on this model. In Chapters 2–4, which represent the bulk of the thesis, we explore in depth our results on the PAM, BTM and BAM respectively. Finally, in Chapter 5 we comment on some future directions for research on intermittency and localisation in the PAM, BTM and BAM.

Although most of the results contained in Chapters 2–4 can be found in the published works [28, 33, 58] and the preprint [59], we would like to remark on some important differences between what is presented here and what appears in these works:

- Perhaps the main innovation in this thesis is a new probabilistic proof of the results in [33], which by-passes many of the technicalities present in that paper (in particular, its reliance on the results and methods of proof of [2, 3, 4, 5]). Instead, our new proof is essentially self-contained, and maintains the probabilistic interpretation of objects associated to the PAM wherever possible. This proof is largely based on the method developed in [59] (and inspired by [38]), but adapted (and simplified) to the PAM. As a new application, we show how this method can be used to extend the results in [33] to the more challenging case of potential distributions with fractional-double-exponential tail decay.
- A secondary difference is the streamlining of the results and presentation of [28, 58] (in Chapter 3) and [59] (in Chapter 4) in order to unify the methods of proof and give as simple a presentation of the results as possible. Indeed, throughout the thesis we aim to give as much intuition and heuristic insight as we can, even at the expense of the full generality of the results found in the published versions. Most notably, in Chapter 4 we give a stream-lined presentation of the results in [59] for a particular special case of trap distribution. This avoids some of the technical difficulties of the general case, while still capturing the relevant phenomena of interest.

## 1.1 The parabolic Anderson model

The PAM is the Cauchy equation on the lattice $\mathbb{Z}^d$

$$\begin{aligned} \frac{\partial u(t,z)}{\partial t} &= (\Delta + \xi)\, u(t,z)\,, & (t,z) &\in [0,\infty) \times \mathbb{Z}^d\,, \\ u(0,z) &= \mathbb{1}_{\{0\}}(z)\,, & z &\in \mathbb{Z}^d\,, \end{aligned} \tag{1.1}$$

where $\Delta$ denotes the *discrete Laplacian* defined by

$$(\Delta f)(z) := \sum_{|y-z|=1} \left(f(y) - f(z)\right)\,,$$

and $\xi = \{\xi(z)\}_{z \in \mathbb{Z}^d}$ is an independent identically-distributed (i.i.d.) random field known as the (random) *potential field*. Let $\mathbf{P}$ denote the probability measure associated to the potential field $\xi$, and note that $u(t,z)$ is a $\mathbf{P}$-measurable random process that depends on the particular realisation of $\xi$. For a large class of potential field distributions,[2] equation (1.1) has, $\mathbf{P}$-almost surely, a unique non-negative solution defined for all time $t$.

The PAM is named after the physicist P.W. Anderson who used the random Schrödinger operator $\mathcal{H} := \Delta + \xi$ to model the evolution of the wave-function of an electron inside a semiconductor[3] [1] via the time-inhomogeneous (random) Schrödinger equation

$$\begin{aligned} \frac{\partial \psi(t,z)}{\partial t} &= -i\bar{h}(\Delta + \xi)\,\psi(t,z)\,, && (t,z) \in [0,\infty) \times \mathbb{Z}^d\,, \\ \psi(0,z) &= \mathbb{1}_{\{0\}}(z)\,, && z \in \mathbb{Z}^d\,, \end{aligned} \tag{1.2}$$

where $\psi(t,z)$ is complex valued, $\bar{h}$ is the reduced Planck length, and $i$ denotes the complex number $\sqrt{-1}$. Anderson discovered the remarkable fact that, for many choices of the random potential field $\xi$, the wave-function $\psi$ tends to concentrate, over long periods of time, on just a few sites of the domain – a phenomenon now known as *Anderson localisation* – in stark contrast with the general tendency of wave-functions to diffuse over time. Although the Cauchy equation (1.2) bears superficial similarities with the PAM equation (1.1), the presence of the complex number $i$ ensures that the dynamics of the two models are actually rather different, and we will not discuss equation (1.2) further. For a general overview of the PAM, see [39].

Although the PAM can be studied in the case that $\xi$ takes negative values, we shall restrict our attention to the case of *non-negative* potential $\xi$. The reason is that we wish to make a connection between the PAM and branching random walks in a random branching landscape. To this end, consider a system of diffusive, branching particles on $\mathbb{Z}^d$ specified by:

- *Initialisation*: A single particle at the origin;
- *Branching*: Each particle branches (i.e. duplicates) independently at the jump times of a time-inhomogeneous Poisson process, with the rate of the Poisson process for a particle at a site $z$ given by $\xi(z)$;[1]
- *Diffusion*: Each particle evolves as an independent continuous-time simple random walk on $\mathbb{Z}^d$, that is, the waiting time for each particle at each site is independent and distributed exponentially with unit mean, with the subsequent site chosen uniformly from among the nearest neighbours.

It is not hard to see that the expected number of particles in the above system (i.e. averaging over the jump times, trajectories and branching times of all of the particles in the

[2] More specifically, for all distributions satisfying a certain integrability condition on the upper-tail; see [39]. If the condition is not satisfied, the solution $\mathbf{P}$-almost surely 'blows-up' in finite time.

[3] A semiconductor consists of a base metal that has been 'doped' with small impurities. Anderson's ansatz was that the impurities are distributed as a homogeneous Poisson point process, and so the potential can be modelled as an i.i.d. random field.

[1] Of course, if $\xi$ takes negative values, the above interpretation remains valid as long as we also insist that if $\xi(z) < 0$ then particles at $z$ are killed at a rate given by $|\xi(z)|$.

system) at site $z$ at time $t$ solves the PAM equation (1.1). In other words the PAM is the *thermodynamic* limit of this particle system. Given the clear links between this system of diffusive, branching particles and certain models of population dynamics – i.e. interpreting the potential as the 'fitness' of certain geographic sites or genetic configurations etc. – this provides a strong additional motivation for studying the PAM.[1]

Moreover, it turns out that the above system of branching particles gives a key probabilistic insight into the PAM. Indeed, under the averaging described above (i.e. over the jump times, trajectories and branching times) such a system is equivalent to a single particle undertaking a continuous-time simple random walk $X_s$, whose 'effective mass' grows exponentially at rate $\xi(X_s)$. This interpretation is formalised in the Feynman-Kac representation of the solution to (1.1):

$$u(t,z) := \mathbb{E}_0\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\}\mathbb{1}_{\{X_t=z\}}\right], \tag{1.3}$$

where $X_s$ denotes a continuous-time simple random walk on $\mathbb{Z}^d$, and $\mathbb{E}_z$ denotes the corresponding expectation over $X_s$ given that $X_0 = z$.

A final motivation for studying the PAM is that it is an important example of an intermittent process; in other words, the solution $u(t,z)$ develops pronounced spatial and temporal inhomogeneities. The reason, broadly speaking, is that over long periods of time the dominant contribution to the solution $u(t,z)$ will come from regions of the potential $\xi$ that contain particularly high values. Since these large values are spatially inhomogeneous (and also temporally inhomogeneous, from the point of view of the diffusive particles), the solution $u(t,z)$ can develop corresponding inhomogeneities. The qualitative and quantitative features of intermittency in the PAM are one of the main topics of this thesis, and will be explored in depth in Section 1.4 and Chapter 2.

*Remark.* Note that elsewhere in the literature (see, e.g., [5, 33]) the convention $(\Delta f)(z) := \sum_{|y-z|=1} f(y)$ is used to define the discrete Laplacian in the PAM. This is equivalent to shifting the random potential field by the constant $2d$, and makes no qualitative difference to the model.

## 1.2 The Bouchaud trap model

To define the Bouchaud trap model (BTM), first let $\sigma = \{\sigma(z)\}_{z\in\mathbb{Z}^d}$ be an strictly-positive i.i.d. random field known as the (random) *trapping landscape*, and let $\mathbf{P}$ denote the probability measure associated to the field $\sigma$. The BTM in the trapping landscape $\sigma$ is the continuous-time Markov chain on the lattice $\mathbb{Z}^d$ defined by the jump rates

$$w_{z\to y} := \begin{cases} \frac{1}{2d\sigma(z)}\,, & \text{if } |z-y| = 1\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$$

[1] The system of diffusive, branching particles described above may also be studied on its own without the averaging procedure, which arguably makes its relevance to population dynamics even stronger. This 'unaveraged' system exhibits similar, although subtly distinct, intermittency phenomena to the PAM, see [60] and the comments in Chapter 5.

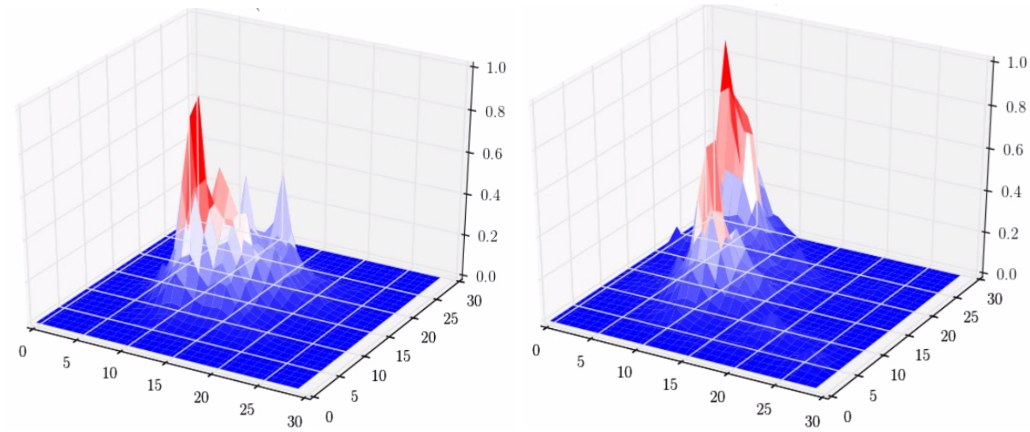


Figure 1: A simulation of the (suitably renormalised) probability mass function of the PAM with Weibull potential field at (i) a time close to $t = 0$, and (ii) a slightly later time. Note the concentration effects, which are markedly different to what would occur in the case of homogeneous potential. Credit: A. Fiodorov.

In other words, conditionally on $\sigma$, the BTM is the continuous-time symmetric random walk on $\mathbb{Z}^d$ with generator $\sigma^{-1}\Delta$, where here we use a slightly different convention for the discrete Laplacian $\Delta$, namely

$$(\Delta f)(z) := \frac{1}{2d} \sum_{|y-z|=1} (f(y) - f(z)) \, . \tag{1.4}$$

Note that this is equivalent to slowing down the evolution of the system by a factor $2d$, and makes no qualitative difference to the model.

The BTM is named after the physicist J.P. Bouchaud who used the model to study the long-term dynamics of certain spin-glass models (see [21]). Broadly speaking, spin-glass dynamics can be thought of as a random walk in an energy landscape which is globally flat, but characterised by sporadic deep wells. On short time-scales, the random walk cannot make it out of the deep wells; on long time-scales, the random walk reaches equilibrium among the set of deep wells. However, on intermediate time-scales, the dynamics of the random walk should roughly look like the BTM – that is, a random walk among random holding times. More recently, it has become clear that the BTM is an effective phenomenological model for a wide variety of more physically realistic trapping behaviour. For a general overview of the BTM and its relationship to spin-glasses see [12]. For a discussion of its application to a variety of physically realistic trap models see [10] or our comments in Chapter 5.

Under the initial condition $X_0 = 0$ and fixed trapping landscape $\sigma$, denote by $\mathbb{P}(X_t = z)$ the *quenched probability mass function* of the BTM. This is a $\mathbf{P}$-measurable random process that solves the Cauchy problem on the lattice $\mathbb{Z}^d$

$$\begin{aligned} \frac{\partial u(t,z)}{\partial t} &= \Delta \, \sigma^{-1} \, u(t,z) \, , && (t,z) \in [0,\infty) \times \mathbb{Z}^d \, , \\ u(0,z) &= \mathbb{1}_{\{0\}}(z) \, , && z \in \mathbb{Z}^d \, . \end{aligned} \tag{1.5}$$

We may also define an annealed probability mass function for the BTM, given by the semidirect product

$$P(X_t \in \cdot) := \int \mathbb{P}(X_t \in \cdot)\, d\mathbf{P}\,.$$

Note, however, that under the annealed law the BTM is not a Markov chain, since the trajectories of the BTM reveal information about the particular realisation of the trapping landscape $\sigma$.

Although the BTM can be defined in any dimension, and indeed on any graph, the BTM on the integers is of particular interest because it is an example of an *intermittent* process. Broadly speaking, over long periods of time the dynamics of the BTM on the integers are dominated by the deepest traps in the trapping landscape $\sigma$ that have been visited by the particle. Since these traps are spatially inhomogeneous (and also temporally inhomogeneous, from the point of view of the particle), the probability mass function $\mathbb{P}(X_t = z)$ can develop spatial and temporal inhomogeneities. The qualitative and quantitative features of intermittency in the BTM will be explored in depth in Section 1.4 and Chapter 3,

*Remark.* Common generalisations of the BTM include defining the model on arbitrary graphs (see [12]; indeed the BTM was originally defined on a complete graph [21]) and also to relax the requirement that the waiting-time of the random walk be exponentially distributed (see [9]; although in general this no longer defines a Markov process).

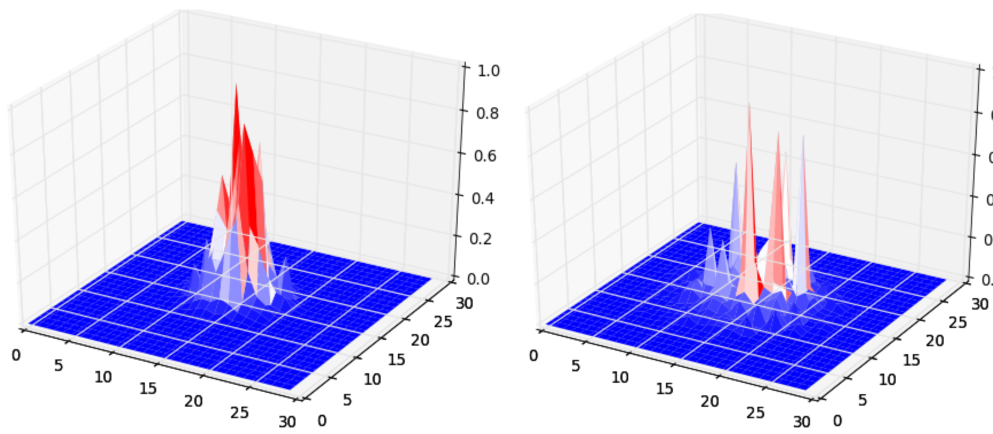


Figure 2: A simulation of the mass function of the BTM with Pareto trapping landscape at (i) a time close to $t = 0$, and (ii) a slightly later time. Note the concentration effects, which are both markedly different to what would occur in the case of homogeneous trapping landscape, and also distinct from the concentration effects in the PAM. Credit: A. Fiodorov.

## 1.3 The Bouchaud–Anderson model

As mentioned, a secondary aim of the thesis is to study how the localisation phenomena in the PAM and the BTM interact. To do this, we consider the Cauchy problem on the lattice $\mathbb{Z}^d$

$$\begin{aligned} \frac{\partial u(t,z)}{\partial t} &= (\Delta\sigma^{-1} + \xi)u(t,z)\,, && (t,z) \in [0,\infty) \times \mathbb{Z}^d\,, \\ u(0,z) &= \mathbb{1}_{\{0\}}(z)\,, && z \in \mathbb{Z}^d\,, \end{aligned} \tag{1.6}$$

derived by replacing the discrete Laplacian in equation (1.1) with the generator of the BTM in equation (1.5) (using the convention for the discrete Laplacian in (1.4)), where the **P**-measurable random fields $\xi$ and $\sigma$ are defined as above and are mutually independent.

To the best of our knowledge this thesis is the first work (along with [59] on which it is based) to consider such a hybrid model combining the dynamics of the PAM and BTM, and we shall refer to this model, for obvious reasons, as the *Bouchaud–Anderson model* (BAM). As we shall see, the interaction between the effect of the random potential field $\xi$ and the random trapping landscape $\sigma$ makes the behaviour of the BAM highly non-trivial.

By analogy with the PAM, the solution to equation (1.6) has a natural interpretation as the thermodynamic limit of a system of diffusive, branching particles on the lattice $\mathbb{Z}^d$ specified by:

- *Initialisation*: A single particle at the origin;
- *Branching*: Each particle branches (i.e. duplicates) independently at the jump times of a time-inhomogeneous Poisson process, with the rate of the Poisson process for a particle at a site $z$ given by $\xi(z)$;
- *Trapping*: Each particle evolves as an independent BTM, that is, the waiting time at each visit to a site $z$ is independent and distributed exponentially with mean $\sigma(z)$, with the subsequent site chosen uniformly from among the nearest neighbours.

As for the PAM, this interpretation of the BAM as the thermodynamic limit of a system of diffusive, branching particles results in the following Feynman-Kac representation of the solution to (1.6):

$$u(t,z) := \mathbb{E}_0 \left[ \exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}_{\{X_t = z\}} \right],$$

where $X$ is the BTM and, for $z \in \mathbb{Z}^d$, $\mathbb{E}_z$ denotes the expectation over $X$ given that $X_0 = z$.

Although to the best of our knowledge the BAM has not been considered before in the literature, there are clear connections between the BAM and other existing models. First recall that, as described above, the BAM can be interpreted as the thermodynamic limit of a particle system with random branching and trapping mechanisms. Similar systems with slightly different trapping mechanisms have been considered before in the mathematical literature, including systems in which the trapping mechanisms is given by asymmetric transition probabilities [8] and random conductances [71]. Nevertheless, these works have not considered the localisation properties of these models.

Such systems of trapped, branching particles also find an application in the study of population dynamics, and have received considerable attention in the mathematical biology literature (see, e.g., [23, 47, 54, 67]). In this context, the branching and trapping rates may be recast as the *fitness ('adaptedness')* and *stability ('adaptability')* respectively of individual states (e.g. geographic locations, genetic configurations etc.). Of primary interest in this literature is the tendency of populations to concentrate on states which are both fit and stable: the 'fit and stable' hypothesis. Although there has been numerical support for this hypothesis (see, eg., [23]), our results provide the first rigorous analysis of this phenomenon. Indeed, our results actually suggest a refinement of the hypothesis: that populations concentrate on states which are fit and stable, *but also* for which neighbouring sites are both fit and *unstable*.

Second, operators of the form $\Delta\sigma^{-1} + \xi$ have important applications in quantum mechanics, since their eigenvalues give the energy levels of a particle whose effective mass is position-dependent (see, e.g., [24, 32, 63]). To make the connection, consider the position-dependent mass Schrödinger equation for a particle with effective mass $\sigma$ in a potential field $\xi$. This equation has a Hamiltonian of general form (see [63])

$$\frac{1}{2}\left(\sigma^{-\alpha}\nabla\sigma^{-\beta}\nabla\sigma^{-\gamma} + \sigma^{-\gamma}\nabla\sigma^{-\beta}\nabla\sigma^{-\alpha}\right) + \xi\,, \qquad \alpha, \beta, \gamma \geq 0\,,\ \alpha + \beta + \gamma = 1\,.$$

Although there is no canonical choice for $\alpha, \beta, \gamma$, in the discrete setting a natural restriction is $\beta = 0$, which avoids symmetry breaking in the definition of $\nabla$. Specialising to the case $\alpha = \gamma = 1/2$ gives the operator

$$\sigma^{-\frac{1}{2}}\Delta\sigma^{-\frac{1}{2}} + \xi = \sigma^{-\frac{1}{2}}\left(\Delta\sigma^{-1} + \xi\right)\sigma^{\frac{1}{2}}\,. \tag{1.7}$$

We remark that (1.7) is the 'symmetrised' form of the operator $\Delta\sigma^{-1} + \xi$, and hence has equivalent spectral theory. In Section 2.2.1 we develop general theory for operators of the form $\Delta\sigma^{-1}+\xi$, including deriving path expansions and Feynman-Kac representations for the principal eigenvalue and eigenfunction respectively. This section is entirely self-contained, and is completely deterministic, and we expect that it will be of independent interest.

Third, there are connections between the BAM and the PAM in the case where the potential field distribution $\xi(0)$ is allowed to take on highly negative (or even infinitely negative) values, which may be interpreted as 'traps'. Previous work has noted the minimal influence of such 'traps' in $d \geq 2$ (see, e.g. [39, Section 2.4]), essentially due to percolation estimates, an observation that finds echoes in our results and methods. However, there are clear differences between this model and the BAM, primarily due to the fact that the traps in the BAM may coexist with sites of high potential; this coexistence underlies the phenomena of mutual reinforcement and correlation that we observe in the BAM (see Section 1.5 below). On the other hand, in dimension one the effect of highly negative potential values in the PAM is significant (see [16]). Indeed, since such sites cannot be avoided, their effect is to 'screen' off the growth that would otherwise occur from sites of high potential, and so the asymptotic growth of the solution depends heavily on the relationship between the upper and lower tails of $\xi(0)$. Again, this is reminiscent of the behaviour of the BAM in dimension one; indeed, in [59] we prove that our results on the BAM outlined in Section 1.5 below are only valid if the trap distribution $\sigma(0)$ decays sufficiently fast to ensure that 'screening' effects are negligible (see also our comments in Chapter 5).

## 1.4 Intermittency and localisation in the PAM and BTM

Broadly speaking, intermittency phenomena in the PAM and BTM are a consequence of the structure-forming effects of extremes in the respective random environments, and manifest if these structure-forming effects dominate over the smoothing effects of diffusion. Hence, as a general rule, intermittency occurs in random walks in random media if the extremes in the respective random media are both sufficiently pronounced and sufficiently regular.

Localisation is the most extreme manifestation of this phenomena, and refers to the tendency of the models to concentrate on small subsets of the domain. Note that this is a qualitative phenomena; as such there is no canonical way to define localisation. Indeed, one of the challenges of research on intermittency and localisation phenomena is precisely to develop well-adapted metrics to describe and quantify localisation.

We shall say that the PAM *localises* if, as $t \to \infty$, the solution of equation (1.1) is eventually concentrated on a small number of sites with overwhelming probability, i.e. if there exists a $\mathbf{P}$-measurable set-valued process $\Gamma_t$, called the *localisation set*, such that, as $t \to \infty$, $|\Gamma_t| = t^{o(1)}$ and

$$\frac{\sum_{z \in \Gamma_t} u(t,z)}{U(t)} \to 1 \qquad \text{in } \mathbf{P}\text{-probability}\,, \tag{1.8}$$

where $U(t) := \sum_{z \in \mathbb{Z}^d} u(t,z)$ is the total mass of the solution; see the section on 'General notation' at the start of the thesis for the definition of the asymptotic notation used here and throughout the thesis. The PAM is said to *localise almost surely* if the convergence in equation (1.8) holds almost surely as well as in probability.

Similarly, we shall say that the BTM *localises* if, as $t \to \infty$, the solution of equation (1.5) is eventually concentrated on a small number of sites with overwhelming probability, i.e. if there exists a $\mathbf{P}$-measurable set-valued process $\Gamma_t$, called the *localisation set*, such that, as $t \to \infty$, $|\Gamma_t| = t^{o(1)}$ and

$$\sum_{z \in \Gamma_t} \mathbb{P}(X_t = z) \to 1 \qquad \text{in } \mathbf{P}\text{-probability}\,, \tag{1.9}$$

or equivalently, in terms of the annealed law,

$$P(X_t \in \Gamma_t) \to 1\,.$$

The BTM is said to *localise almost surely* if the convergence in equation (1.9) holds $\mathbf{P}$-almost surely as well as in $\mathbf{P}$-probability.

Naturally, the primary measure of the strength of localisation in the PAM and BTM is the cardinality of the localisation set $\Gamma_t$. As such, the most extreme form of localisation is *complete localisation*, which occurs if the total mass is eventually concentrated at just one site, i.e. if $\Gamma_t$ can be chosen in equations (1.8) and (1.9) such that $|\Gamma_t| = 1$. We briefly outline some known results on localisation in the PAM and BTM.

### The parabolic Anderson model

The conditions under which the PAM localises in the sense of equation (1.8) has been the subject of intense and ongoing research over the last 25 years. Naturally, the strength of localisation in the PAM depends on (i) the asymptotic rate of decay, and (ii) the regularity of the upper-tail of the random variable $\xi(0)$. In this context, it is convenient to characterise

$\xi(0)$ by its *exponential tail decay rate* function

$$g_\xi(x) := -\log(\mathbf{P}(\xi(0) > x))$$

for then (i) and (ii) translate to the asymptotic growth and regularity of the non-decreasing function $g_\xi$. For simplicity, we shall assume here all necessary regularity conditions without further specification; determining the optimum regularity conditions under which localisation results holds is an interesting open question,[2] although not one we will focus on in this thesis.

The current understanding is that double-exponential tail decay ($g_\xi(x) \approx e^x$) forms the boundary of the complete localisation universality class. More precisely, it is conjectured that the PAM exhibits complete localisation as long as $\log g_\xi(x) \ll x$. This has been proven (in [52]) in the extremal[1] case of Pareto-like tail decay ($g_\xi(x) \sim \gamma \log x$, for $\gamma > d$), and more recently (in [64]) in the case of sub-Gaussian tail decay ($g_\xi(x) \sim x^\gamma$, for $\gamma < 2$). Nevertheless, this has left open the case of whether complete localisation occurs in the case of Weibull-like tail decay for arbitrary parameter, including the important case of Gaussian tails ($\gamma = 2$), as well as the case of fractional-double-exponential tail decay ($\log g_\xi(x) \sim x^\gamma$, for $\gamma < 1$).

On the other hand, if $\log g_\xi(x) \gg x$, then complete localisation is known not to hold (see [38]), although the PAM is still intermittent in a certain weaker sense, and may indeed localise in the sense of equation (1.8). What occurs in the interface regime of double-exponential tail decay ($\log g_\xi(x) \sim cx$, for $c > 0$) is not currently well-understood, although very recent work [17, 18] suggests that the PAM localises on a set $\Gamma_t$ consisting of a single connected island of bounded size.

The almost sure localisation behaviour (i.e. holding $\mathbf{P}$-almost surely) of the PAM is much less well-understood. Currently, only the extremal case of Pareto-like tail decay ($g_\xi(x) \sim \gamma \log x$, for $\gamma > d$) has been settled, in which it is known that the PAM almost surely localises on just two sites eventually. Note that this is the strongest possible almost sure localisation behaviour, since by simple continuity arguments there necessarily exist arbitrarily large times at which the solution is spread across at least two sites. It is an interesting open question as to whether two-site almost sure localisation holds in the entire class of sub-double-exponential tail decay; we explore this question further in Chapter 5.

### The Bouchaud trap model

The study of localisation in the Bouchaud trap model has also received considerable attention over the last 10 years. Again, the strength of localisation in the BTM depends on (i) the asymptotic rate of decay, and (ii) the regularity of the upper-tail of the random variable $\sigma(0)$.

A notable feature of the BTM is that localisation only occurs in dimension one. In higher dimensions, the traps either have negligible effect in the limit (if the tail of $\sigma(0)$ is integrable,

[2] Although this question is largely open in the case of the PAM, quite a lot is known in the related question of determining the behaviour of the top eigenvalues of the Anderson operator in a growing box, see e.g. [6].

[1] This case is extremal in the sense that if $g_\xi(x) \sim \gamma \log x$ for $\gamma > d$ (or even $\gamma \geq d$ if $d = 1$) then the solution to equation (1.1) $\mathbf{P}$-almost surely 'blows-up' in finite time, see [39].

essentially by virtue of the law of large numbers), or are visited in such a way that their overall effect is spatially-homogeneous (see [12] and [35] for a proof of this result in the case of Pareto-like tail decay, although the result is thought to hold more generally for arbitrary non-integrable tail decay; see Chapter 5 for further comments).

Previous studies of localisation in the BTM on the integers has focused on the case that the tail of $\sigma(0)$ is (i) integrable at infinity, or (ii) regularly varying with index $\alpha \in (0, 1)$ at infinity.[1] In the first case, the BTM is known to *homogenise* over large times; indeed the BTM, properly rescaled, converges to Brownian motion in the $t \to \infty$ limit. By contrast, it has been shown (in [34]) that in the second case the BTM is intermittent, in the sense that

$$\limsup_{t\to\infty} \sup_{z\in\mathbb{Z}} \mathbb{P}(X_t = z) > 0 \quad \mathbf{P}\text{-almost surely}\,.$$

In other words, for almost all trapping landscapes there exist arbitrarily large times at which the BTM has non-negligible probability mass located at a single site. On the other hand, this is a weaker property than the localisation in equation (1.9), and in fact it is known that in this case the quenched probability mass function is asymptotically supported by a certain 'dense' set (after suitable rescaling of the distance scale).

## 1.5 Outline of main results

The main results of this thesis concern localisation properties of the PAM and the BTM in various regimes: in the PAM, we prove that complete localisation holds in the case of potential distributions with Weibull tail decay for arbitrary parameter, as well as in the case of fractional-double-exponential (FDE) tail decay for parameters less than one; in the BTM, we establish strong localisation properties for the BTM on the integers in the regime of slowly varying traps. We also study the BAM in the regime in which both the potential field and trap distribution have Weibull tail, which turns out to be a natural regime in which to study the interaction between localisation effects due to the PAM and BTM.

In this section we give a brief overview of our results, highlighting the most important features. A full description of our results on the PAM, BTM and BAM follows in Chapters 2–4 respectively.

### 1.5.1 Complete localisation in the parabolic Anderson model

Recall that previous work on localisation in the PAM has established the complete localisation of the solution in the case of potential field with Pareto tail decay and, more generally, sub-Gaussian tail decay. Our main results focus on the PAM with Weibull potential (the *Weibull case*), that is, in the case that there exists a parameter $\gamma > 0$ such that

$$\mathbf{P}(\xi(0) > x) = e^{-x^\gamma}\,, \quad x > 0\,.$$

[1] Recall that a function $L$ is said to be *regularly varying* with index $\alpha > 0$ at infinity if $\lim_{u\to\infty} L(uv)/L(u) = v^\alpha$ for any $v > 0$.

Note that the Weibull case includes the important sub-cases of exponential tail decay ($\gamma = 1$) and Gaussian tail decay ($\gamma = 2$).

Although our results as stated hold only for precise, fully regular Weibull tail decay, we expect that they hold more generally for approximate Weibull tail decay as long as certain regularity assumptions are satisfied; determining the optimum regularity assumptions under which our results hold is an interesting open question, although not one we focus on in this thesis.

### Complete localisation and the 'radius of influence'

Our first main result is to confirm that the PAM with Weibull potential exhibits complete localisation, that is, its renormalised mass function is eventually localised at a single site with overwhelming probability. This extends the class of potential tail decay for which localisation in the PAM is known to hold, since the previous known results only covered sub-Gaussian tail decay, corresponding to $\gamma < 2$.

**Theorem 1.1** (Complete localisation for the PAM with Weibull potential)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *such that, as* $t \to \infty$*,*

$$\frac{u(t, Z_t)}{U(t)} \to 1 \qquad \text{in } \mathbf{P}\text{-probability}.$$

*The process* $Z_t$ *can be described explicitly; we defer this description to Chapter 2.*

One important by-product of our work on complete localisation in the Weibull case is to quantify a new measure of localisation strength, which we call the *radius of influence*. This measure allows us to distinguish various localisation strengths *within* the complete localisation universality class. Informally, the *radius of influence* measures the extent to which the localisation site itself is determined by purely local features of the random environment. More precisely, the radius of influence $\rho$ is the smallest integer for which the localisation site $Z_t$ can be determined by maximising a functional on $\mathbb{Z}^d$ that depends on the random environment $\xi$ only through its values in balls of radius $\rho$ around each site.

We remark that the concept of the *radius of influence* was not developed in previous studies of complete localisation in the PAM. This is quite natural, since in both the case of Pareto-like tail decay [52] and sub-Gaussian tail decay [64] it turns out that the localisation site can be determined by maximising a functional that depends on the potential field $\xi$ only through its value at individual lattice sites. In other words, interactions between neighbouring lattice sites have no influence on localisation, which in our language corresponds to a trivial radius of influence $\rho = 0$. In the case of Weibull potential with arbitrary parameter, the situation is more delicate, and we quantify this in the following result.

**Theorem 1.2** (Radius of influence for the PAM with Weibull potential field)**.** *The radius of influence of the PAM with Weibull potential field is the non-negative integer*

$$\rho := \left[\frac{\gamma - 1}{2}\right]^+ .$$

*To be more precise, for any* $p \geq 0$ *describe a* $\mathbf{P}$*-measurable functional* $\Psi_t : \mathbb{Z}^d \to \mathbb{R}$ *to be* *p-local if, for all $z \in \mathbb{Z}^d$, the value of $\Psi_t(z)$ depends on the potential field $\xi$ only through its values in the ball $B(z,p)$. Then, the following hold:*

1. *There exists a $\rho$-local $\mathbf{P}$-measurable functional $\Psi_t$ such that, as $t \to \infty$*

$$\frac{u\,(t, Z_t)}{U(t)} \to 1 \qquad \text{in } \mathbf{P}\text{-probability}\,, \tag{1.10}$$

   *where $Z_t := \operatorname{argmax}_{z\in\mathbb{Z}^d} \Psi_t(z)$, settling ties in the determination of* argmax *arbitrarily.*

2. *If $\rho \geq 1$, then for any $\eta \in [0,\rho)$ there does not exist a $\eta$-local functional $\Psi_t : \mathbb{Z}^d \to \mathbb{R}$ such that* (1.10) *holds.*

Note that the above result implies that $\rho = 0$ if and only if $\gamma < 3$, recovering the previously known results on completely localisation in the PAM with sub-Gaussian tail decay. Note also that $\rho \to \infty$ in the $\gamma \to \infty$ limit. We extract therefore an important and subtle feature of the PAM, that there exist potential fields for which the PAM eventually localises on a single site, but for which the exact location of this single site can depend on interactions between the potential field at an arbitrarily long range.

**Detailed description of complete localisation behaviour: Exponential decay, local profile, and ageing**

The remainder of our results give a more detailed description of the complete localisation behaviour of the PAM with Weibull potential. To state these results, we need to introduce several deterministic scales $r_t$, $a_t$ and $d_t$ which govern the asymptotic dynamics of the PAM. The first scale $r_t$ is the distance scale at which complete localisation occurs

$$r_t := \frac{t(d\log t)^{\frac{1}{\gamma}}}{\log\log t}\,.$$

Heuristics for determining the scale $r_t$ are given in Chapter 2. For now, let us remark that this scale is very close to linear; in other words, almost all of the mass of the solution to the PAM escapes to infinity at a speed that is ballistic up to a logarithmic correction. This is in contrast to the case of potentials with Pareto tail decay, in which the distance scale is a super-linear power of $t$.

The remaining scales $a_t$ and $d_t$ relate to the extreme value theory of the random variable $\xi(0)$, specifically the scale of (i) the largest value, and (ii) the gap between the top two values, of the potential $\xi$ in a ball $B(0,t)$ of radius $t$ around the origin. Standard extreme value theory suggests that these scales satisfy, for each $x \in \mathbb{R}$, as $t \to \infty$,

$$|B(0,t)|\;\mathbf{P}\left(\xi(0) > a_t + xd_t\right) \to e^{-x}\,.$$

In the Weibull case, a simple computation (via a Taylor expansion) gives the value of these scales as

$$a_t := (d\log t)^{\frac{1}{\gamma}} \quad \text{and} \quad d_t := \frac{1}{\gamma}(d\log t)^{\frac{1}{\gamma}-1}\,.$$

Note that, under this analysis, the scales that describe the asymptotic dynamics of the PAM would be expected to be $a_{r_t}$ and $d_{r_t}$, however since

$$a_{r_t} \sim a_t \quad \text{and} \quad d_{r_t} \sim d_t$$

it is sufficient to work with the scales $a_t$ and $d_t$.

Using the scales $r_t$, $a_t$ and $d_t$ we are able to give a much more detailed description of the complete localisation behaviour of the PAM. First, we prove that the solution to the PAM has exponential decay around the localisation site; the exponent of the decay is identified as $\log a_t$. Note, however, that the exponential decay of the solution holds only up to the distance scale $r_t$; it turns out that on the scale $r_t$ the solution has a more complicated profile. Rather than specify this profile precisely, we give a simple uniform bound which holds at, and beyond, the scale $r_t$; see Chapter 2 for an informal description of the full profile.

**Theorem 1.3** (Exponential decay of the renormalised solution)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *and a constant* $c > 1$ *such that, for any function* $\kappa_t \to 0$ *decaying sufficiently slowly, as* $t \to \infty$ *the following hold:*

*(a) For each* $z \in B(Z_t, r_t\kappa_t) \setminus \{Z_t\}$,

$$\mathbf{P}\left(c^{-|z-Z_t|} < \frac{u(t,z)}{U(t)} a_t^{|z-Z_t|} < c^{|z-Z_t|}\right) \to 1\,,$$

*and the convergence in probability holds also for the union of these events;*

*(b) Moreover,*

$$\mathbf{P}\left(e^{td_t\kappa_t} \sum_{z \notin B(Z_t, r_t\kappa_t)} \frac{u(t,z)}{U(t)} < c\right) \to 1\,.$$

The exponential decay result in Theorem 1.3 can be seen as a strengthening of the complete localisation result in Theorem 1.1 above; this type of result has not been proven before in previous work studying complete localisation in the PAM. As a corollary we deduce the rate of convergence of the solution of the PAM to a completely localised state.

**Corollary 1.4** (Rate of convergence of the PAM to complete localisation)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *and a constant* $c > 1$ *such that, as* $t \to \infty$,

$$\mathbf{P}\left(c^{-1} < \left(1 - \frac{1}{U(t)}\, u(t, Z_t)\right) (\log t)^{\frac{1}{\gamma}} < c\right) \to 1\,.$$

Next, we give a detailed description of the asymptotic properties of the localisation site $Z_t$, which establishes the distance scale of the localisation site as well as the local profile of the potential field around the localisation site. In order to state these results in full, we shall need to define the concept of *interface sites*, which are sites at a distance of *precisely* the radius of influence $\rho$ from the localisation site, and moreover at values of the parameter $\gamma$ for which the radius of influence is transitioning from one integer to the next. To this end, define the interface set

$$\mathcal{I} := \left\{ z \in \mathbb{Z}^d \setminus \{0\} : |z| = \frac{\gamma - 1}{2} \right\}\,,$$

remarking that $\mathcal{I}$ is non-empty if and only if $\gamma \in \{3, 5, \ldots\}$.

**Theorem 1.5** (Asymptotic description of the localisation site)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *satisfying* (1.10) *such that, as* $t \to \infty$*, the following hold:*

*(a) (Distance scale of the localisation site)*

$$\frac{Z_t}{r_t} \Rightarrow X \qquad \text{in law}\,,$$

*where* $X$ *is a random vector whose coordinates are independent and Laplace distributed random variables with absolute-moment one;*

*(b) (Local profile of the potential field)*
*Define the function* $q : \{0, \ldots, \rho\} \to [0,1]$ *by*

$$q(x) := 1 - \frac{2x}{\gamma - 1}\,,$$

*using the convention that* $0/0 := 0$*. For each* $z \in B(0,\rho) \setminus \mathcal{I}$ *there exists a* $c > 0$ *such that*

$$\frac{\xi(Z_t + z)}{a_t^{q(|z|)}} \to c \qquad \text{in } \mathbf{P}\text{-probability};$$

*on the other hand, for each* $z \in \mathcal{I}$ *there exists a* $c > 0$ *such that, uniformly on any compact set,*

$$f_{\xi(Z_t+z)}(x) \to \frac{e^{cx} f_{\xi(0)}(x)}{\mathbf{E}\left[e^{c\xi(0)}\right]}\,,$$

*where* $f_\xi(z)$ *denotes the density of the potential field* $\xi$ *at the site* $z$*. Note that each of the constants in the above can be described explicitly; we defer this description to Chapter 2.*

Our description of the local profile of the potential field establishes that the potential field inside the ball of radius $\rho$ around the localisation site grows, asymptotically, as a certain power of $a_t$ that decays with distance away from the site. At the interface sites (i.e. the sites for which the power is precisely 0) we provide a finer description of the potential field. Note that our description of the local profile is valid up to the radius of influence; this is completely natural, since the radius of influence describes precisely the range of potential field interaction, and hence the potential field beyond this range is necessarily independent of the localisation site.

Finally, we derive the ageing behaviour of the PAM in the Weibull case. Ageing refers to the tendency of a system to slow down its evolution over time, and is a common feature of models of statistical physics.[1] For the PAM in the Weibull case, ageing manifests in the spacing between the times at which the site of complete localisation is transitioning. As a consequence, we can also establish the ageing of the solution of the PAM.

**Theorem 1.6** (Ageing of the complete localisation site)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *satisfying* (1.10) *such that, as* $t \to \infty$,

$$\frac{T_t}{t} \Rightarrow \Theta \qquad \text{in law}$$

[1] See [57] for a detailed analysis of ageing in the PAM in the Pareto case.

*where*

$$T_t := \inf\{s > 0 : Z_{t+s} \neq Z_t\}\,,$$

*and* $\Theta$ *is a non-degenerate almost surely positive random variable.*

**Theorem 1.7** (Ageing of the renormalised solution)**.** *For any* $\varepsilon \in (0,1)$*, as* $t \to \infty$*,*

$$\frac{T_t^\varepsilon}{t} \Rightarrow \Theta \qquad \textit{in law}$$

*where*

$$T_t^\varepsilon := \inf\left\{ s > 0 : \left| \frac{u(t,\cdot)}{U(t)} - \frac{u(t+s,\cdot)}{U(t+s)} \right|_{\ell_\infty} > \varepsilon \right\}\,,$$

*and* $\Theta$ *is the same non-degenerate almost surely positive random variable as in Theorem 1.6.*

#### Extending our results to the fractional-double-exponential case

We also extend certain of our results to the case of potential distributions with fractional-double-exponential (FDE) tail decay (the *FDE case*), that is, the case when there exists a $\gamma \in (0,1)$ such that

$$\mathbf{P}(\xi(0) > x) = \exp\left\{-e^{x^\gamma}\right\}\,, \quad x > 0\,.$$

Our main result is that complete localisation also holds in the FDE case.

**Theorem 1.8** (Complete localisation for the PAM with FDE potential)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *such that, as* $t \to \infty$

$$\frac{u(t,Z_t)}{U(t)} \to 1\,. \tag{1.11}$$

*The process* $Z_t$ *can be described explicitly; we defer this description to Chapter 2.*

Recall that the FDE case sits just below the conjectured boundary of the complete localisation universality class, which is generally believed to be formed by potential distributions with exact double-exponential tail decay (corresponding to $\gamma = 1$). In this sense, Theorem 1.8 completes the program of establishing complete localisation in the PAM begun in [46, 52, 64]. As expected, our results and methods of proof break down in the limit as $\gamma \to 1$. Note that the FDE case has previously been studied in [7], in which the asymptotics of the top order statistics of the eigenvalues of the Anderson operator in a growing box were determined. Although this is strongly related to the localisation behaviour of the PAM, it does not necessarily imply the complete localisation of the model.

On the other hand, not all of our results from the Weibull case carry over to the FDE case. First, although we do not prove it, we strongly believe that the radius of influence in the FDE case is no longer a fixed integer (as in Theorem 1.2 above), but instead grows to infinity with $t$; in other words, the localisation site depends on interactions between the potential field $\xi$ at an unbounded range. Indeed, we conjecture that the radius of influence in the FDE case has order

$$\rho_t := \frac{\log\log t}{\log\log\log t}\,.$$

We provide some heuristics for this conjecture in Chapter 2.

Second, again although we do not prove it, we strongly conjecture that no exact exponential decay of the solution holds throughout the entire distance scale of the localisation site (as in the Weibull case; see Theorem 1.3 above). Instead, we expect that the solution has two regimes of exponential decay, one that holds on short scales (up to the radius of influence), and another that holds on longer scales (up to the distance scale).

Roughly speaking, the reason for this two-tiered exponential decay is because high sites of the potential cluster on short scales around the localisation site, indeed the potential around the localisation site is asymptotically flat on this scale (in the sense that $\xi(y) \sim \xi(Z_t)$); we make this heuristic more precise in Chapter 2. On the other hand, on longer scales the potential is independent of the localisation site, and so by the law of large numbers a stronger exponential decay should hold on average. Instead of proving such a detailed description of the exponential decay, we simply establish an upper bound for the exponential decay of the solution around the localisation site that holds on the entire distance scale; as in Theorem 1.3 we combine this with a weaker bound that holds across the entire domain. As a corollary, we deduce an upper bound for the rate of convergence of the solution of the PAM to a completely localised state.

To describe this upper bound on the exponential decay of the solution, we need to introduce the equivalent scales $r_t, a_t$ and $d_t$ in the FDE case, i.e.

$$r_t := \frac{t(\log\log t)^{\frac{1}{\gamma}-1}}{d\log t \log\log\log t}\,, \quad a_t := (\log\log t + \log d)^{\frac{1}{\gamma}} \quad \text{and} \quad d_t := \frac{(\log\log t)^{\frac{1}{\gamma}-1}}{\gamma\, d \log t}\,. \tag{1.12}$$

Heuristics for determining the scale $r_t$ are identical to in the Weibull case, and will be given in Chapter 2. As in the Weibull case, the scales $a_t$ and $d_t$ follow from standard extreme value theory of the random variable $\xi(0)$. Note that, in comparison to the Weibull case, the scales $r_t$, $a_t$ and $d_t$ replace logarithmic terms with double logarithmic terms, and double logarithmic terms with triple logarithmic terms. This is completely natural, since tails with FDE decay result from logarithmic transformations of tails with Weibull decay. Finally, we also need to introduce the scale of the upper bound on the rate of exponential decay

$$\hat{a}_t := (\log\log t)^{\frac{1}{\gamma}-1}\,. \tag{1.13}$$

**Theorem 1.9** (Distance scale of the localisation site)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *satisfying* (1.11) *such that, as* $t \to \infty$*,*

$$\frac{Z_t}{r_t} \Rightarrow X \qquad \textit{in law}\,,$$

*where* $X$ *is the same random vector as in part* (a) *of Theorem 1.5.*

**Theorem 1.10** (Upper bound on exponential decay of the renormalised solution)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *and a constant* $c > 0$ *such that, for any function* $\kappa_t \to 0$ *decaying sufficiently slowly, as* $t \to \infty$ *the following hold:*

*(a) For each* $z \in B(Z_t, \kappa_t r_t) \setminus \{Z_t\}$*,*

$$\mathbf{P}\left(\frac{u(t,z)}{U(t)}\hat{a}_t^{-|z-Z_t|} < c^{-|z-Z_t|}\right) \to 1\,,$$

*and the convergence in probability holds also for the union of these events;*

*(b) Moreover,*

$$\mathbf{P}\left(e^{td_t\kappa_t}\sum_{z\notin B(Z_t,\kappa_t r_t)}\frac{u(t,z)}{U(t)}<c\right)\to 1\,.$$

**Corollary 1.11** (Rate of convergence of the PAM to complete localisation)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *and a constant* $c>0$ *such that, as* $t\to\infty$*,*

$$\mathbf{P}\left(\left(1-\frac{1}{U(t)}\,u(t,Z_t)\right)(\log\log t)^{\frac{1}{\gamma}-1}<c\right)\to 1\,.$$

Although we do not prove it, we conjecture that the upper bound on the exponential rate of decay of the solution in part (a) of Theorem 1.10 is tight up to the choice of constant; we provide some heuristics for this conjecture in Chapter 2. As expected, the decay rate $\hat{a}_t$ degenerates as $\gamma\to 1$ (i.e. $\hat{a}_t\to\infty$ if and only if $\gamma<1$), which provides further evidence that double-exponential tail decay is the boundary of the complete localisation universality class.

### 1.5.2 Localisation in the Bouchaud trap model on the integers with slowly varying traps

Recall that previous work on localisation in the BTM has studied the case in which the trap distribution has regularly varying or integrable tails at infinity. In this thesis we consider the case of slowly varying traps, i.e. in which the càdlàg, non-decreasing and unbounded function

$$L(x):=\frac{1}{\mathbf{P}(\sigma(0)>x)}$$

satisfies the *slow variation* property

$$\lim_{u\to\infty}\frac{L(uv)}{L(u)}=1\,,\quad\text{for any } v>0\,. \tag{1.14}$$

Slowly varying trap models arise naturally in the study of certain random walks in random media, such as biased random walks on critical Galton-Watson trees [27], and spin-glass dynamics on subexponential time scales [11, 22]. They also have parallels with Sinai's random walk [65], as reflected in the logarithmic rate of escape to infinity and strong localisation properties of that model. With regards to the BTM with slowly varying traps in particular, recent work has studied the extremal ageing [43] of this model, which is qualitatively different from the equivalent phenomena in the case of integrable or regularly varying traps.

Note that all our results on the BTM relate exclusively to the BTM on the integers. This is since, as mentioned above, the BTM does not exhibit localisation in dimensions higher than one, even in the case of slowly varying traps (see also our discussion in Chapter 5). Henceforth in this section, and throughout Chapter 3, the BTM will always refer to the BTM on the integers.

### Two-site localisation

Recall that the localisation properties of the BTM have been previously studied in the case of trap distribution with regularly varying tails at infinity [34, 35], in which it was established that

$$\limsup_{t\to\infty} \sup_{z\in\mathbb{Z}} \mathbb{P}(X_t = z) > 0 \qquad \mathbf{P}\text{-almost surely}\,. \tag{1.15}$$

In [13] it was suggested that a stronger form of localisation than (1.15) should hold as the index of regularly variation tends to zero, namely that at large times the probability mass of the BTM should eventually be carried by just two sites with overwhelming probability (with respect to trapping landscape distribution $\mathbf{P}$). Although [13] gave heuristic justifications, to the best of our knowledge this has not yet been rigorously established in the literature.

Our first main result takes up this suggestion, confirming the prediction of [13] that the BTM exhibits two-site localisation with overwhelming probability.

**Theorem 1.12** (Two-site localisation in probability)**.** *There exists a $\mathbf{P}$-measurable set-valued process $\Gamma_t$ with $|\Gamma_t| = 2$ such that, as $t \to \infty$,*

$$\mathbb{P}(X_t \in \Gamma_t) \to 1 \quad \textit{in } \mathbf{P}\textit{-probability}\,,$$

*or equivalently, under the annealed law,*

$$P(X_t \in \Gamma_t) \to 1\,.$$

*The set-valued process $\Gamma_t$ can be described explicitly; see Theorem 1.13 below.*

Remark that Theorem 1.12 is a localisation result holding in probability. Almost sure localisation results for the BTM with slowly varying traps – i.e. localisation results that hold $\mathbf{P}$-almost surely, such as (1.15) in the regularly varying case – are more delicate, and turn out to depend on finer properties of the trap distribution. Such results will be the subject of upcoming work [29]; we make some remarks about this in Chapter 5.

We give some intuition for why two-site localisation in the BTM is a natural consequence of slowly varying traps. Recall the fundamental property of sequences of i.i.d. random variables with slowly varying tails: that the cumulative sum of such sequences are asymptotically dominated, with overwhelming probability, by the maximal term. Translated to the BTM, this means that the local time of the BTM is overwhelmingly likely to be dominated by the deepest trap the BTM has visited. Given that this trap could be located either on the positive or negative half-line, it is natural to expect the concentration of the mass function on two sites.

Let us make the above heuristics slightly more precise. In light of the above, it is natural to expect that, at time $t$, the BTM is likely to be located on the first traps on the positive and negative half-line whose depth exceeds a certain level $\ell_t$. This level should be determined in such a way that, by time $t$: (i) traps with depth exceeding $\ell_t$ are too deep to escape from; whereas (ii) traps with depth less than $\ell_t$ are very likely to have been escaped from (meaning that excursions away from the trap are of sufficient length to discover traps of a similar depth). These heuristics allow us to give a detailed description of the two-site localisation, both determining the two localisation sites explicitly, as well as the limiting proportion of probability mass located at each site. As a corollary, we deduce the single-time scaling limit, under the annealed law, of the BTM with slowly varying traps.

So first define, for each $t \geq 0$, the level

$$\ell_t := \min\{s \geq 0 : s\,L(s) \geq t\}\,, \tag{1.16}$$

remarking that this is well-defined since $L$ is càdlàg. Further, denote by $Z_t^{(1)}$ (respectively $Z_t^{(2)}$) the closest site to the origin on the positive (respectively negative) half-line where the trap value exceeds the level $\ell_t$, i.e.

$$Z_t^{(1)} := \min\{z \in \mathbb{Z}^+ : \sigma(z) > \ell_t\} \quad \text{and} \quad Z_t^{(2)} := \max\{z \in \mathbb{Z}^- : \sigma(z) > \ell_t\}\,,$$

and let $\Gamma_t := \{Z_t^{(1)}, Z_t^{(2)}\}$, remarking that $\Gamma_t$ is $\mathbf{P}$-measurable. Abbreviate $r_t := L(\ell_t)$, and note that $\ell_t, r_t \to \infty$ as $t \to \infty$.

**Theorem 1.13** (Detailed description of two-site localisation)**.** *For $i = 1, 2$, as $t \to \infty$,*

$$\mathbb{P}(X_t = Z_t^{(i)}) + \frac{|Z_t^{(i)}|}{\sum_{z \in \Gamma_t} |z|} \to 1 \quad \textit{in } \mathbf{P}\textit{-probability}\,.$$

**Theorem 1.14** (Distance scale of the localisation set)**.** *As $t \to \infty$,*

$$r_t^{-1}\left(Z_t^{(1)}, -Z_t^{(2)}\right) \Rightarrow (\mathcal{E}_1, \mathcal{E}_2) \quad \textit{in } \mathbf{P}\textit{-law}\,,$$

*where $\{\mathcal{E}_i\}_{i=1,2}$ are independent exponential random variables with unit mean.*

**Corollary 1.15** (Single-time scaling limit)**.** *Under the annealed law, as $t \to \infty$,*

$$r_t^{-1} X_t \Rightarrow y_1 \delta_{-x_1} + y_2 \delta_{x_2}\,,$$

*where $\{x_i\}_{i=1,2}$ are independent standard exponential random variables, $\delta_x$ is a Dirac measure at the point $x$, and each $i = 1, 2$ satisfies $y_i := 1 - x_i / \sum_{j=1,2} x_j$. Note that this implies that*

$$(y_1, y_2) \overset{d}{=} (\mathcal{U}, 1 - \mathcal{U})\,,$$

*where $\mathcal{U}$ is a uniform random variable on $[0, 1]$. Through a change of variable, the above is equivalent to*

$$\frac{1}{t} X_{tL^{-1}(t)} \Rightarrow y_1 \delta_{-x_1} + y_2 \delta_{x_2}\,, \tag{1.17}$$

*where $L^{-1}$ denotes the right-continuous inverse of $L$.*

Let us give some intuition for the level $\ell_t$ (and hence the localisation set $\Gamma_t$). As discussed above, recall that the slowly varying trap distribution implies that the dynamics of the BTM should be dominated by the effect of the deepest visited trap. By standard properties of i.i.d. sequences, the spacing between the successive record deepest traps on the positive (respectively negative) half-line grows linearly with the distance from the origin. Hence, for the BTM to venture from the record deepest trap $z$ to an even deeper trap, it must travel a distance approximately $|z|$, and so, by simple random walk local time estimates, will return to $z$ approximately $|z|$ times before doing so; such a displacement takes approximately a time $\sigma(z)|z|$. Finally, standard extreme value estimates give $L(\sigma(z))$ as the correct scale for the location $|z|$ of the first trap of depth $\sigma(z)$. Hence, this displacement takes approximately $\sigma(z)L(\sigma(z))$ time. As such, we expect the BTM to be located on the first site $z$ that it visits such that $\sigma(z)L(\sigma(z)) > t$, i.e. the first site in $\Gamma_t$ that it hits.

### A functional limit theorem for the BTM

Our second main result is to establish a functional limit extension of the single-time scaling limit in Corollary 1.15; this functional limit theorem reduces to the single time scaling-limit for any fixed time $t$. To describe this result, we need to introduce (i) the form of the scaling limit, and (ii) the topology in which the convergence takes place.

Let $\mathcal{P} = (x_i, v_i)_{i\in\mathbb{N}}$ be an inhomogeneous Poisson point process on $\mathbb{R}\times\mathbb{R}^+$ with intensity measure $v^{-2}dx\,dv$; this process will be interpreted as the scaling limit for the trapping landscape. Denote by $B = (B_t)_{t\geq 0}$ a standard Brownian motion (independent of $\mathcal{P}$). Let $m^B = (m^B_t)_{t\geq 0}$ be the $B$-explored extremal process for $\mathcal{P}$, defined by

$$m^B_t := \sup\left\{ v_i : \inf_{s\in[0,t]} B_s \leq x_i \leq \sup_{s\in[0,t]} B_s \right\}, \tag{1.18}$$

and let $I^B = (I^B_t)_{t\geq 0}$ be its right-continuous inverse, i.e. $I^B_t := \inf\{s : m^B_s > t\}$. We shall identify $B_{I^B_t}$ as the scaling limit of the BTM with slowly varying traps. Note that $B_{I^B}$ is a highly singular process. Indeed, conditionally on $\mathcal{P}$, its probability mass is concentrated, at each time $t > 0$, on the two sites

$$z^1_t := \min\{x_i \geq 0 : v_i > t\} \quad \text{and} \quad z^2_t := \max\{x_i \leq 0 : v_i > t\},$$

in proportion to their hitting probability with respect to $B$. In other words, conditionally on $\mathcal{P}$,

$$B_{I^B_t} = \begin{cases} z^1_t, & \text{with probability } |z^1_t|/(z^1_t - z^2_t), \\ z^2_t, & \text{with probability } |z^2_t|/(z^1_t - z^2_t). \end{cases}$$

This is consistent with the two-site localisation of the BTM in Theorem 1.12, although it does not imply it, since the scaling limit is not sensitive to the behaviour of the BTM on fine scales.

We claim that $B_{I^B}$ is the natural analogue of the FIN diffusion with parameter $\alpha \in (0,1)$ in the limiting case $\alpha = 0$ (the FIN diffusion is the scaling limit of the BTM with regularly varying traps; see [34, 35]). For this reason, we refer to the process $B_{I^B}$ as the *extremal FIN process.* In Chapter 3 we make precise the sense in which the FIN diffusion convergences to the extremal FIN process in the $\alpha \to 0$ limit.

Concerning topological issues, we need to introduce the Skorohod space of real-valued càdlàg functions on $\mathbb{R}^+$, $D(\mathbb{R}^+)$, equipped with the non-Skorohod $L_{1,\text{loc}}$ topology; a detailed definition of this topology, and its relationship to the stronger Skorohod $J_1$ and $M_1$ topologies, is provided in the appendix to Chapter 3. Our use of the non-Skorohod $L_{1,\text{loc}}$ topology for our functional limit theorem is motivated by the form of the limiting processes; we make this connection clear in Chapter 3. Indeed, our limit theorem would not hold in the stronger Skorohod $J_1$ and $M_1$ topologies.

**Theorem 1.16** (Functional limit theorem for the BTM)**.** *Under the annealed law, as $n \to \infty$,*

$$\left( \frac{1}{n} X_{nL^{-1}(nt)} \right)_{t \geq 0} \overset{L_1}{\Rightarrow} \left( B_{I_t^B} \right)_{t \geq 0} ,$$

*where $\overset{L_1}{\Rightarrow}$ denotes weak convergence in the $L_{1,\text{loc}}$ topology.*

Remark that setting $t = 1$ in Theorem 1.16 above results in the convergence in law

$$\frac{1}{t} X_{tL^{-1}(t)} \Rightarrow B_{I_1^B} ,$$

and by considering basic properties of the point process $\mathcal{P}$ and Brownian motion $B$ it may be seen that this is equivalent to the single-time scaling limit established in (1.17) above.

### Two-regimes of slowly varying traps

Finally, we observe that within the class of slowly varying tails we can distinguish two separate regimes that are relevant to the localisation behaviour of the BTM with slowly varying traps. Specifically, under a certain strengthening of the slow variation assumption we are able to give simplified versions of our main results. This is analogous to the simplified limit theorems for maximum and sums of sequences of i.i.d. random variables with slowly varying tails that are available under an analogous assumption, see [50]. Note that these two regimes are also relevant in distinguishing the almost sure localisation behaviour of the BTM (see the upcoming study [29], as well as the discussion in Chapter 5); indeed in terms of almost sure localisation, the two regimes actually give rise to qualitatively distinct localisation behaviour, rather than merely resulting in simplified scaling limits.

Let us introduce these two regimes by defining the following strengthening of the *slow variation* property (1.14):

**Assumption 1.** *It is the case that*

$$\lim_{u \to \infty} \frac{L(u/L(u))}{L(u)} = 1 .$$

Note that Assumption 1 is satisfied for $L(x) = (\log x)^\gamma$ for all $\gamma > 0$ (i.e. the class of *log-Pareto* distributions), but is only satisfied for $L(x) = \exp\{(\log x)^\gamma\}$ (i.e. the class of *log-Weibull* distributions) for the parameter range $0 < \gamma < 1/2$.

Under Assumption 1 we can simplify our main results on the BTM with slowly varying traps. First, we can simplify our description of two-site localisation by redefining the level $\ell_t$ to be identically $t$. Accordingly, define

$$\hat{Z}_t^{(1)} := \min\{z \in \mathbb{Z}^+ : \sigma(z) > t\} \quad \text{and} \quad \hat{Z}_t^{(2)} := \max\{z \in \mathbb{Z}^- : \sigma(z) > t\} .$$

We then have the following simplified version of our two-site localisation result in Theorem 1.12.

**Theorem 1.17** (Simplified description of two-site localisation)**.** *Suppose Assumption 1 holds. Then for $i = 1, 2$, as $t \to \infty$,*

$$\mathbb{P}(X_t = \hat{Z}_t^{(i)}) + \frac{|\hat{Z}_t^{(i)}|}{\sum_{i=1,2} |\hat{Z}_t^{(i)}|} \to 1 \quad \textit{in } \mathbf{P}\textit{-probability.}$$

Similarly, under the same assumption we also derive a version of our functional limit theorem for the BTM in Theorem 1.16 in which the non-linear scaling of time is simplified.

**Theorem 1.18** (Simplified functional limit theorem for the BTM)**.** *Suppose Assumption 1 holds. Then under the annealed law, as $n \to \infty$,*

$$\left( \frac{1}{n} X_{L^{-1}(nt)} \right)_{t \geq 0} \overset{L_1}{\Rightarrow} \left( B_{I_t^B} \right)_{t \geq 0} .$$

Let us briefly give some intuition behind these simplified results. In regards to two-site localisation, it can be seen that Assumption 1, combined with the definition of $\ell_t$, implies that $L(\ell_t) \sim L(t)$. Since $L(\cdot)$ gives the appropriate non-linear rescaling for exceedances by $\sigma$ of a certain level $\ell > 0$, this implies that, with overwhelming probability, $\hat{Z}_t^{(i)} = Z_t^{(i)}$ for $i = 1, 2$.

For the functional limit theorem, recall that the contribution to the time spent at deep traps on a distance scale $n$ is approximately $nL^{-1}(n)$, with $L^{-1}(n)$ the depth of deepest trap on this scale, and $n$ being the number of repeated visits to this site before escaping. Note, however, that under Assumption 1

$$L\left(nL^{-1}(n)\right) \sim n .$$

In other words, the contribution to the internal clock of the BTM from the repeat visits to the deepest trap is rendered negligible by the non-linear rescaling of time.

### 1.5.3 Localisation in the Bouchaud–Anderson model

In this thesis we restrict our study of the BAM to the case in which both the potential distribution $\xi(0)$ and the trap distribution $\sigma(0)$ have Weibull tail decay, i.e. there exist parameters $\gamma, \mu > 0$ such that

$$\mathbf{P}(\xi(0) > x) = e^{-x^\gamma} \quad \text{and} \quad \mathbf{P}(\sigma(0) > x) = e^{-x^\mu} , \qquad x > 0 . \tag{1.19}$$

As we shall see, this turns out to be a natural regime of the BAM to study, since the interaction between the potential field and trapping landscape exhibits certain phase transitions in $(\gamma, \mu)$. Note that equivalent results for more general classes of $\sigma(0)$ can be found in the preprint [59]. In particular, the results in [59] also include the boundary case $\mu = 0$; we shall make some remarks on this case below.

In addition to the above, for our results to hold we shall also need to assume that the trapping landscape $\sigma$ has *no quick sites*, that is, the trapping distribution $\sigma(0)$ is bounded away from zero. Hence, for simplicity, instead of (1.19) we shall assume that there exists a constant $\delta_\sigma > 0$ such that

$$\mathbf{P}(\sigma(0) > x) = e^{-x^\mu} \,, \qquad x > \delta_\sigma \,,$$

and moreover that

$$\mathbf{P}(\sigma(0) = \delta_\sigma) = 1 - e^{-\delta_\sigma^\mu} \,.$$

Again, the results in [59] allow for slightly more general assumptions on the lower tail.

### Complete localisation

Our first main result establishes the complete localisation of the BAM across the entire regime of Weibull potential and trapping landscape.

**Theorem 1.19** (Complete localisation in the BAM with Weibull potential and trapping landscape)**.** *There exists a* $\mathbf{P}$*-measurable process* $Z_t$ *such that, as* $t \to \infty$*,*

$$\frac{u(t, Z_t)}{U(t)} \to 1 \qquad \text{in } \mathbf{P}\text{-probability}. \tag{1.20}$$

*The process* $Z_t$ *can be described explicitly; we defer this description to Chapter 4.*

That the BAM completely localises for certain $(\gamma, \mu)$ is perhaps expected, since the PAM with Weibull potential also exhibits complete localisation. More surprising, however, is that complete localisation occurs regardless of the presence of very large traps, even in dimension one, since *a priori* it might be thought that large traps would draw probability mass away from the localisation site. Even more surprising is that, in dimensions higher than one, this result holds for arbitrarily heavy traps (i.e. it holds even in the $\mu \to 0$ limit; see [59]).

Note that, as in the PAM, we expect that the solution of the BAM decays exponentially around the site of complete localisation (at least the exponential decay should hold as an upper bound, although perhaps it is not matched exactly by a lower bound). For simplicity we do not attempt to prove this additional result here.

### Mutual reinforcement of localisation effects due to the PAM and the BTM

Since complete localisation holds in the entire regime, in order to probe the interaction between the potential field and trapping landscape we need a finer measure of localisation. Such a measure is provided by the *radii of influence* $\rho_\xi$ and $\rho_\sigma$ which, analogously to in the PAM, are the smallest integers for which the localisation site $Z_t$ can be determined by maximising a $\mathbf{P}$-measurable functional on $\mathbb{Z}^d$ that depends on $\xi$ and $\sigma$ only through their values in balls of radius $\rho_\xi$ and $\rho_\sigma$ respectively around each site. Our second main result is to determine the radii of influence $\rho_\xi$ and $\rho_\sigma$, and to prove that they are optimal.

**Theorem 1.20** (Radii of influence in the BAM with Weibull potential and trapping landscape)**.** *The radii of influence of the BAM with Weibull potential field and trapping landscape*

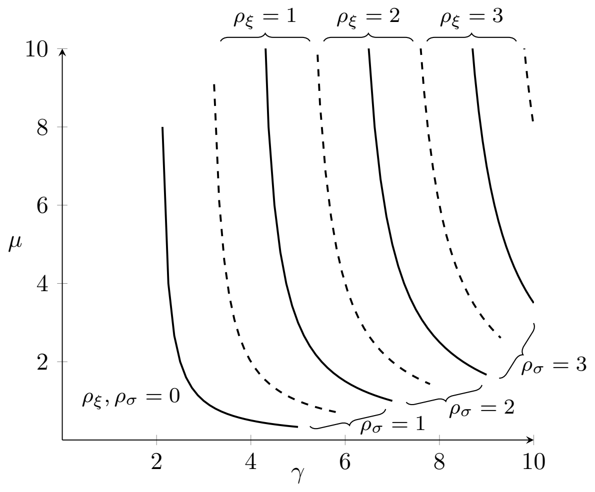


Figure 3: A partition of the parameter space of the BAM according to the radii of influence $\rho_\xi$ (dashed lines) and $\rho_\sigma$ (bold lines). The boundary curves are of the form $\mu = (2i-1)/(\gamma-2i)$ and $\mu = (2i)/(\gamma - 2i - 1)$, for $i \in \mathbb{N} \setminus \{0\}$.

*are the non-negative integers*

$$\rho_\xi := \left[ \frac{\gamma - 1}{2} \frac{\mu}{\mu + 1} \right]^+ \quad \textit{and} \quad \rho_\sigma := \left[ \frac{\gamma - 1}{2} \frac{\mu}{\mu + 1} + \frac{1}{2} \right]^+ \in \{\rho_\xi, \rho_\xi + 1\}.$$

*The precise meaning of this statement is the same as in Theorem 1.2. A partition of the parameter space of the BAM according to $\rho_\xi$ and $\rho_\sigma$ is depicted in Figure 3.*

Note that $\rho_\xi$ and $\rho_\sigma$ are decreasing functions of the strength of both the potential field and trapping landscape (i.e. increasings functions of $\gamma$ and $\mu$). Moreover, if we define $\rho_{PAM} := [(\gamma-1)/2]^+$ to be the radius of influence in the PAM for the same potential field $\xi$ (see Theorem 1.2), then we have that $\rho_\xi \le \rho_{\text{PAM}}$. In other words, the localisation effects due to the PAM and BTM are *mutually reinforcing*. The relationship between $\rho_\xi$, $\rho_\sigma$ and $\rho_{\text{PAM}}$ is depicted in Figure 4.

The conclusions of Theorem 1.20 also holds in the $\mu \to 0$ limit; see [59]. Surprisingly, it is not necessarily the case that $\rho_\sigma \to \rho_{\text{PAM}}$ in the $\mu \to \infty$ limit; indeed, if $\gamma \in [2i, 2i+1)$ for $i \in \mathbb{N}$, then in fact $\rho_\sigma \to \rho_{\text{PAM}} + 1$, meaning that influence of the trapping landscape $\sigma$ on the BAM is not continuous in the degenerate limit (i.e. as $\sigma(z) \to 1$ simultaneously for each $z$). On the other hand, $\rho_\xi \to \rho_{\text{PAM}}$ in the $\mu \to \infty$ limit, i.e. there is no discontinuity in the effect of the the trapping landscape $\sigma$ on the BAM on the level of the radius of influence of the potential field $\xi$.

### Reducibility of the BAM to the PAM

We next ask whether the BAM is 'reducible' to the PAM. There are actually two distinct notions of reducibility that are relevant. Strong reducibility describes the situation in which the trapping landscape $\sigma$ plays no role in determining the localisation site $Z_t$, and the macroscopic behaviour of the system is adequately approximated by the PAM with potential $\xi$. Weak reducibility describes the situation in which all necessary information to determine $Z_t$

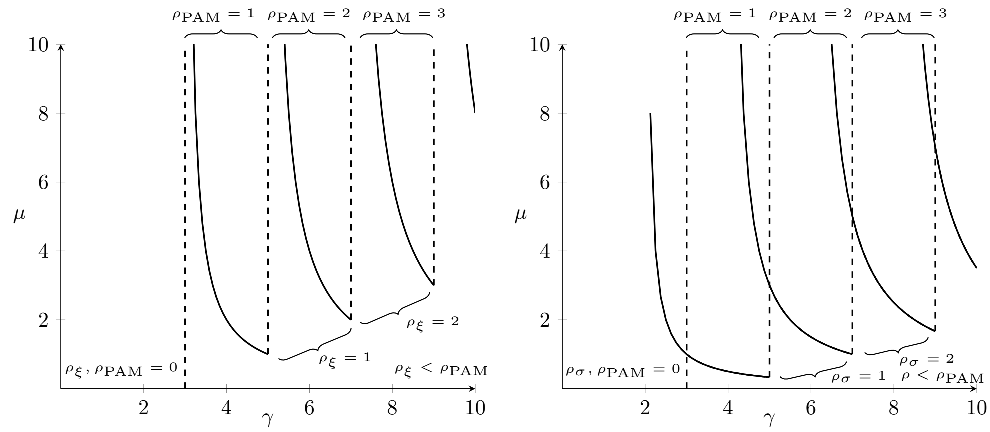


Figure 4: A partition of the parameter space of the BAM according to the relationship between $\rho_\xi$ (bold lines) and $\rho_{\text{PAM}}$ (dashed lines), and $\rho_\sigma$ (bold lines) and $\rho_{\text{PAM}}$ (dashed lines) respectively, where $\rho_{\text{PAM}}$ denotes the radius of influence in the equivalent PAM with identical potential field. The boundary curves are of the form $\mu = (2i-1)/(\gamma-2i)$ and $\mu = (2i)/(\gamma - 2i - 1)$ respectively, for $i \in \mathbb{N} \setminus \{0\}$.

is contained in the 'net growth rate' $\eta := \xi - \sigma^{-1}$, and moreover, the macroscopic behaviour of the BTM is adequately approximated by the PAM with potential replaced with $\eta$. The term 'net growth rate' comes from the interpretation of the BAM in terms of the system of diffusive, branching particles as introduced above. Our third main result is to determine the regimes in which the BAM is strongly and weakly reducibility to the PAM.

**Theorem 1.21** (Reducibility of the BAM to the PAM)**.** *The BAM is strongly reducible to the PAM if and only if $\gamma < 1$. The BAM is weakly reducible to the PAM if and only if $\gamma \geq 1$ and $\rho_\sigma = 0$. These regimes are depicted in Figure 5. The precise meaning of 'strongly reducible' and 'weakly reducible' are given in Theorem 4.3.*

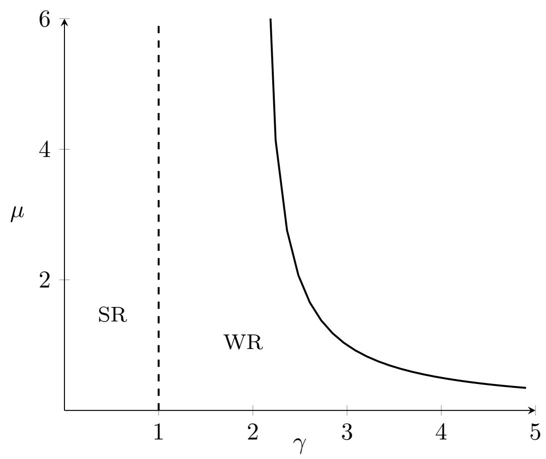


Figure 5: A partition of the parameter space of the BAM according to the whether the BAM is 'strongly reducible' to the PAM with the usual potential $\xi$ (left of the dashed line) or 'weakly reducible' to the PAM with the potential replaced with the 'net growth rate' $\eta$ (left of the bold curve). The boundary curve is $\mu = 1/(\gamma - 2)$, and is not included in the weakly reducible region.

### Local correlation between the potential field and trapping landscape: The 'fit and stable' hypothesis

Our final result is to establish the *local correlation* between the potential field and trapping landscape (where 'local' is from the perspective of the localisation site); this is the so-called 'fit and stable' hypothesis that has been predicted numerically in the mathematical biology literature (see, e.g., [23]), but never rigorously confirmed. Interestingly, the correlation that we observe is *positive* at the localisation site, but *negative* away from the localisation site, providing an unexpected extension to the 'fit and stable' hypothesis.

**Theorem 1.22** (Local correlation between the potential field and trapping landscape)**.** *Assume that $\gamma \geq 1$, so that the BAM is not strongly reducible to the PAM. Then there exists a $\mathbf{P}$-measurable process $Z_t$ satisfying* (1.20) *such that, as $t \to \infty$ eventually:*

1. *For all $z \in B(Z_t, \rho_\xi)$, the random variable $\xi(z)$ stochastically dominates $\xi(0)$;*
2. *The random variable $\sigma(Z_t)$ stochastically dominates $\sigma(0)$; and*
3. *For all $z \in B(Z_t, \rho_\sigma) \setminus \{Z_t\}$, the random variable $\sigma(z)$ is stochastically dominated by $\sigma(0)$.*

In Chapter 4 below we make the nature of this correlation explicit, as well as providing a full description of the localisation site, determining its distance scale, the local profile of the potential field and trapping landscape, and its ageing behaviour.

# Chapter 2

# Complete localisation in the parabolic Anderson model

## 2.1 Introduction

In Chapter 1 we presented our main results on the PAM with Weibull potential, that is, in the case where the potential field satisfies

$$\mathbf{P}(\xi(0) > x) = e^{-x^\gamma} \,, \quad x > 0 \,.$$

To summarise these results, we show that the PAM with Weibull potential exhibits the following localisation phenomena:

1. The renormalised solution of the PAM completely localises on a single site with overwhelming probability (Theorem 1.1);
2. The *radius of influence* is the non-negative integer (Theorem 1.2)
$$\rho := \left[ \frac{\gamma - 1}{2} \right]^+ \,;$$
3. The renormalised solution has exponential decay around the localisation site (Theorem 1.3), giving a lower bound on the rate of convergence to a completely localised state (Corollary 1.4).

We further provide a complete asymptotic description of the localisation site, presenting limit formulae for its distance scale and the local profile of the potential field, and also establish the ageing of the solution (Theorems 1.5 – 1.7).

The primary aim of this chapter is to give a full, essentially self-contained proof of these results. Although most of these results are contained in the published work [33], the method of proof we present here will be quite different, and is much closer in spirit to the probabilistic method developed in [59] (and inspired by [38]). In the final section of this chapter, we show

how this proof can be extended to establish complete localisation in the case of potential with fractional-double-exponential tail decay.

### 2.1.1 Defining the localisation site explicitly

To begin, let us define the localisation site explicitly. Recall the Feynman-Kac representation of the solution $u(t, z)$ in equation (1.3), and notice that we can decompose $u(t, z)$ into disjoint components by reference to the trajectories, or paths, of the underlying simple random walk in this representation. A natural question is then the following:

*Which collection of paths makes the dominant contribution to the solution?*

The essential feature of the complete localisation regime of the PAM, including the Weibull and FDE cases, is that the dominant contribution to the solution comes from a single collection of paths: those that travel 'quickly' to a certain $t$-dependent 'good' site $z$, stay within a ball of 'small' radius around this site, and end up back at the same site. (Precisely what constitutes 'quickly', 'good' and 'small' will be explored in detail below.) The contribution to the solution from such paths can be well approximated by two competing terms:

1. The *exponential growth rate* due to paths which stay near the site $z$ – this rate is essentially given by the *local principal eigenvalue* at the site $z$, which is the principal eigenvalue of the operator $\Delta + \xi$ restricted to a small ball around the site $z$; and

2. A *penalisation term* representing the probabilistic cost for the particle to diffuse to the site $z$ – this is primarily determined by a penalty factor

$$\frac{1}{\xi(z) - \xi(p_i)}$$

for each step $p_i$ along each of the shortest paths $p$ from the origin to the site $z$, and hence the probabilistic cost is

$$\sum_{p \in \Gamma_{|z|}(0,z)} \prod_{0 \le i < |z|} \frac{1}{\xi(z) - \xi(p_i)} \, ;$$

see the section on 'General notation' at the start of the thesis for the definition of the path notation used here and throughout the thesis.

By some simple *a priori* bounds, we may restrict our attention to sites $z$ such that $\xi(z) > \delta_1 a_t$ for any $\delta_1 \in (0, 1)$, recalling that the scale $a_t = (d \log t)^{\frac{1}{\gamma}}$ determines the largest value of the potential $\xi$ in a ball of radius $t$ around the origin. Moreover, we can show that the majority of the sites $y_i$ along any shortest path satisfy $\xi(y_i) < \delta_2 a_t$ for any $\delta_2 \in (0, \delta_1)$. Together these imply that the probabilistic penalty, on the exponential scale, is approximately

$$\log \left\{ a_t^{-|z|} \right\} \approx -|z| \log a_t \approx -\frac{|z|}{\gamma} \log \log t \, .$$

The site of complete localisation $Z_t$ is the site $z$ which possesses the best balance between these two competing terms.

Let us make this analysis precise. Recall the radius of influence $\rho$ and define the positive integer

$$j := \left[\frac{\gamma}{2}\right] \in \{\rho, \rho+1\},$$

which determines the radius that we will use to define the local principal eigenvalues. We shall see later that our choice of $j$ is largely arbitrary, and we could in fact replace it with any larger[1] integer without affecting the results. For each $z \in \mathbb{Z}^d$, define the operator

$$\mathcal{H}(z) := \Delta + \xi \mathbb{1}_{B(z,\rho)}$$

restricted to $B(z,j)$, where here $\xi$ and $\mathbb{1}_A$ are to be interpreted as the multiplication operators associated to the vectors $\xi$ and $\mathbb{1}_A$ respectively. In other words, $\mathcal{H}(z)$ is the operator on functions $f : B(0,j) \to \mathbb{R}$ induced by the operator

$$\mathbb{1}_{B(z,j)} \left(\Delta + \xi \mathbb{1}_{B(z,\rho)}\right) \mathbb{1}_{B(z,j)} = \mathbb{1}_{B(z,j)} \Delta \mathbb{1}_{B(z,j)} + \xi \mathbb{1}_{B(z,\rho)} .$$

Denote by $\lambda(z)$ the principal eigenvalue of this operator (which is real, since $\mathcal{H}(z)$ is symmetric). We refer to $\lambda(z)$ as the *local principal eigenvalue at* $z$. Note that the $\{\lambda(z)\}_{z\in\mathbb{Z}^d}$ are identically distributed, and have a dependency range bounded by $2j$, i.e. the random variables $\lambda(y)$ and $\lambda(z)$ are independent if and only if $|y-z| > 2j$. Remark also that in the special case $\gamma < 2$ (which implies $\rho = j = 0$), the principal local eigenvalue $\lambda(z)$ reduces to the 'net growth rate' $\eta(z) := \xi(z) - 2d$. For any sufficiently large $t$,[2] define a *penalisation functional* $\Psi_t : \mathbb{Z}^d \to \mathbb{R}$ by

$$\Psi_t(z) := \lambda(z) - \frac{|z|}{\gamma t} \log\log t .$$

Note that $\Psi_t$ represents precisely the trade-off between an exponential growth rate given by the local principal eigenvalue $\lambda(z)$ and a penalisation term that we described above.

As discussed above, it is natural to expect that localisation occurs on the maximiser of this functional. However, rather than maximising this functional over all of $\mathbb{Z}^d$, we first need to thin the space $\mathbb{Z}^d$ to allow us to focus on a set of suitable candidate sites. The reason for this thinning is two-fold. First, we wish to place some *a priori* bounds on the (i) the value of the potential at the candidate site, and (ii) the distance of the candidate sites to the origin. Second, we wish to avoid the fact that the local principal eigenvalues, as defined, are large at all points in a $j$-ball around sites of high potential; without thinning, this would lead to a cluster of sites for which this functional is large, all representing essentially the same region of the space.

To achieve the required thinning, introduce a large 'macrobox' $V_t := [-R_t, R_t]^d \cap \mathbb{Z}^d$, with $R_t := t(\log t)^{\frac{1}{\gamma}}$. The idea is that this macrobox is large enough that the solution to the PAM can easily be shown, by *a priori* estimates, to be overwhelmingly contained within this box; the precise value of $R_t$ is not so important. Second, fix a constant $0 < \theta < 1/2$ and define the macrobox level $L_t := ((1-\theta)\log|V_t|)^{\frac{1}{\gamma}}$. Let the subset $\Pi^{(L_t)} := \left\{z \in \mathbb{Z}^d : \xi(z) > L_t\right\} \cap V_t$

[1] Although we could not replace it with $\rho$, at least not in the case of general $\gamma > 0$; see the remarks in Section 2.3.3.

[2] By 'sufficiently large', we just mean any $t$ such that $\log\log t$ is well-defined.

consist of sites in $V_t$ at which $\xi$-exceedances of the level $L_t$ occur. We shall think of $\Pi^{(L_t)}$ as the set of suitable candidate sites of high potential.

We can now define the site of complete localisation to be the $\mathbf{P}$-measurable site

$$Z_t := \operatorname*{argmax}_{z \in \Pi^{(L_t)}} \Psi_t(z) \, .$$

The site $Z_t$ is well-defined eventually almost surely since, as we show in Lemma 2.18, the set $\Pi^{(L_t)}$ is non-empty (and of course finite) eventually almost surely. Moreover, for $t$ sufficiently large, $Z_t$ almost surely does not depend on the particular choice of $\theta$; indeed, as we show in part (b) of Theorem 1.5, the site $Z_t$ has potential on the order $a_t$, and so lies in $\Pi^{(L_t)}$ eventually for any choice of $\theta \in (0,1)$.[1]

With this explicit definition of $Z_t$, we present again a summary of our results in Theorems 1.1 and 1.3:

**Theorem 2.1** (Complete localisation and exponential decay). *There exists a constant $c > 1$ such that, for any function $\kappa_t \to 0$ decaying sufficiently slowly, as $t \to \infty$ the following hold:*

*(a) For each $z \in B(Z_t, \kappa_t r_t) \setminus \{Z_t\}$,*

$$\mathbf{P}\left( c^{-|z-Z_t|} < \frac{u(t,z)}{U(t)} a_t^{|z-Z_t|} < c^{|z-Z_t|} \right) \to 1 \, ,$$

*and the convergence in probability holds also for the union of these events;*

*(b) Moreover,*

$$\mathbf{P}\left( e^{t d_t \kappa_t} \sum_{z \notin B(Z_t, \kappa_t r_t)} \frac{u(t,z)}{U(t)} < c \right) \to 1 \, .$$

We make a comparison between the functional $\Psi_t$ and analogous functionals that have appeared in previous works on complete localisation in the PAM with Pareto potential [46, 52] and sub-Gaussian potential [64]. In all these works, the penalisation functionals replace the principal eigenvalue $\lambda(z)$ by the potential $\xi(z)$ – for instance, in [64] the penalisation functional

$$\Psi_t^*(z) := \xi(z) - \frac{|z|}{\gamma t} \log \log t$$

is used – which restricts the validity of the analysis to situations where there is an exact correspondence between the top order statistics of $\lambda(z)$ and $\xi$ in $V_t$. Clearly this holds[1] for $\gamma < 2$ by our definition of $\lambda(z)$, since then $\rho = 0$ and

$$\lambda(z) = \xi(z) + 2d \, .$$

On the other hand, the exact correspondence turns out to be false if $\gamma \geq 3$, and so an analysis based on these functionals fails for arbitrary $\gamma > 0$.

[1] Note that we restrict to $\theta < 1/2$ for other reasons, namely to ensure the almost sure separation in Lemma 2.19 below.

[1] It turns out that the correspondence is true for $\gamma < 3$, which means that the functional $\Psi_t^*$ is actually sufficient to determine $Z_t$ for any $\gamma < 3$. Nevertheless, to establish this in the regime $\gamma \in [2,3)$ requires the path expansion machinery that we develop in Section 2.2; this is why the proof in [64] broke down for $\gamma \geq 2$.

The above explicit description of the localisation site $Z_t$ allows us to give an informal derivation of the localisation distance scale $r_t$. Recall again that the scale $a_t$ determines the largest value of the potential $\xi(z)$ in a ball of radius $t$ around the origin, and hence also approximately determines the largest value of $\lambda(z)$ in this ball as well (see Lemma 2.3 below). With this fact, the maximiser of $\Psi_t$ should occur approximately at distance

$$\operatorname*{argmax}_r \left\{ a_r - \frac{r}{\gamma t} \log\log t \right\} = \operatorname*{argmax}_r \left\{ (d \log r)^{\frac{1}{\gamma}} - \frac{r}{\gamma t} \log\log t \right\} .$$

Differentiating and solving for $r$ gives precisely the scale $r_t$.

The above explicit description of the localisation site $Z_t$, along with our description of local profile of the potential field in Theorem 1.5, can also be used to deduce the radius of influence, i.e. to prove Theorem 1.2. First, the definition of $Z_t$ places an immediate upper bound on the radius of influence, since the penalisation functional $\Psi_t(z)$ depends on the potential field $\xi$ only through its values inside the set $\xi^{(\rho)}(z) := \{\xi(y)\}_{y \in B(z,\rho)}$. It remains to establish a lower bound. This is actually a simple consequence of our description of the local profile of the potential field, which states that the potential field $\xi$ is positively correlated with $\xi(0)$ within the entire ball $B(Z_t, \rho)$. Since we know that $Z_t$ is located within the set of candidate sites $\Pi^{(L_t)}$ with overwhelming probability, and since we show in Lemma 2.18 that $\Pi^{(L_t)}$ is $j$-separated eventually almost surely, this implies that any functional which can determine $Z_t$ cannot be $(\rho - 1)$-local, completing the proof.

Finally, we give a brief informal description of the shape of the solution $u(t, z)$ outside the ball $B(Z_t, \kappa_t r_t)$, explaining why it is natural that the exponential decay in Theorem 2.1 breaks down at the scale $r_t$. The key point is that near-maximisers of the functional $\Psi_t$ also lie at this scale. Since the solution is well-approximated by a superposition of mass due to the exponential growth of particles visiting all the near-maximiser, we expect the solution to have small 'peaks' at these sites, which are larger than the background exponential decay around $Z_t$.

### 2.1.2 Outline of proof of complete localisation

Before embarking on the proof of our main results, let us give a brief outline of our approach. As discussed above, the solution $u(t, z)$ can be decomposed into disjoint components by reference to the trajectories of the underlying simple random walk in the Feynman-Kac representation in (1.3). Using such a path decomposition, our strategy to prove complete localisation can be summarised as: (i) establish that a single component carries the entire non-negligible part of the solution; and (ii) show that the non-negligible component is localised at $Z_t$. To prove the exponential decay of the solution, instead of (ii) we actually establish the stronger result (iii) that the non-negligible component has exponential decay away from the localisation site $Z_t$.

To assist in the proof, we introduce auxiliary scaling functions $f_t, h_t, e_t, b_t \to 0$ and $g_t \to \infty$ as $t \to \infty$ that are convenient placeholders for negligibly decaying (respectively growing) functions. For technical reasons, we shall require some of these functions to be eventually monotone, and also satisfy certain relationships, as follows. First choose a $b_t \to 0$

decaying sufficiently slowly such that

$$1/\log\log t \ll b_t \ll 1\,.$$

Then choose $f_t, h_t, e_t$ and $g_t$ which are (i) eventually monotone, and (ii) satisfy

$$g_t/\log\log t \ll b_t \ll f_t h_t \ll g_t h_t \ll e_t\,. \tag{2.1}$$

It is easy to check that such a choice is always possible.

### Path decomposition

Here we explain how to construct the path decomposition; again, see the section on 'General notation' at the start of the thesis for the definition of the path notation. For a path $p \in \Gamma(0)$ such that $\{p\} \subseteq V_t$, let $z^{(p)}$ denote the largest local principal eigenvalue on the path, i.e.

$$z^{(p)} := \operatorname*{argmax}_{z\in\{p\}} \lambda(z)\,,$$

which is well-defined almost surely. Abbreviate the large ball

$$B_t := B\left(0, |Z_t|(1+h_t)\right) \cap V_t\,.$$

We partition the path set $\Gamma(0)$ into the following five disjoint components:

$$E_t^i := \begin{cases} \left\{p \in \Gamma(0) : \{p\} \subseteq B_t,\, z^{(p)} = Z_t\right\}\,, & i=1\,,\\ \left\{p \in \Gamma(0) : \{p\} \subseteq V_t,\, z^{(p)} \in \Pi^{(L_t)} \setminus Z_t\right\}\,, & i=2\,,\\ \left\{p \in \Gamma(0) : \{p\} \subseteq V_t,\, \{p\} \not\subseteq B_t,\, z^{(p)} = Z_t\right\}\,, & i=3\,,\\ \left\{p \in \Gamma(0) : \{p\} \subseteq V_t,\, z^{(p)} \notin \Pi^{(L_t)}\right\}\,, & i=4\,,\\ \left\{p \in \Gamma(0) : \{p\} \not\subseteq V_t\right\}\,, & i=5\,, \end{cases}$$

and associate each component $E_t^i$ with a portion of the total mass $U(t)$ of the solution. As such, for $z \in \mathbb{Z}^d$ and $p \in \Gamma$, define

$$u^p(t,z) := \mathbb{E}_{p_0}\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\} \mathbb{1}_{\{X_t=z\}} \mathbb{1}_{\{p(X_t)=p\}}\right]\,, \quad U^p(t) := \sum_{z\in\mathbb{Z}^d} u^p(t,z)\,,$$

and for each $1 \leq i \leq 5$, let

$$u^i(t,z) := \sum_{p\in E_t^i} u^p(t,z) \quad \text{and} \quad U^i(t) = \sum_{z\in\mathbb{Z}^d} u^i(t,z)\,.$$

Our strategy is to establish that each of $U^2(t)$, $U^3(t)$, $U^4(t)$ and $U^5(t)$ are negligible with respect to the total mass $U(t)$ of the solution. More than this, we seek to ensure they play no role in the exponential decay of the solution around the localisation site. For this, we show that

$$\frac{U^i(t)}{U(t)} e^{td_t b_t} \to 0 \qquad \text{in } \mathbf{P}\text{-probability}, \qquad \text{for } i = 2,3,4,5\,. \tag{2.2}$$

To complete the proof of localisation, we also prove that $U^1(t)$ is localised at $Z_t$, and moreover exhibits precise exponential decay inside the entire ball $B_t$. Finally, we have that

$$\mathbf{P}\left(B(Z_t, b_t r_t) \subseteq B_t\right) \to 1\,.$$

Putting these three facts together we have Theorem 1.3 satisfied with $b_t$ in place of $\kappa_t$. Since the scale $b_t$ was chosen arbitrarily as long as it decays slower than $1/\log\log t$, this implies the result.

Note that this strategy requires a balance to be struck in how $B_t$ is defined; it must be large enough that $U^3(t)$ is negligible, but small enough to ensure that the localisation and exponential decay hold inside the ball. The scale $h_t$ has been fine-tuned in (2.1) precisely to ensure this balance is struck correctly.

### Negligible paths

The negligibility of $U^4(t)$ and $U^5(t)$ are simple to establish, since these represent paths which either: (i) do not hit any site of high potential; or (ii) exit the macrobox. The main difficulty is establishing the negligibility of $U^2(t)$ and $U^3(t)$. Our proof is based on formalising two heuristics.

**First heuristic**: For each $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ define $\lambda^{(n)}(z)$ to be the principal eigenvalue of the $n$-local operator

$$\mathcal{H}^{(n)}(z) := \Delta + \xi$$

restricted to $B(z,n)$, and recall the positive integer $j := [\gamma/2]$. We expect that, for a path $p \in \Gamma(0) \setminus E_t^5$,

$$U^p(t) \approx \exp\left\{t\lambda^{(j)}(z^{(p)})\right\} a_t^{-|p|}\ , \tag{2.3}$$

which represents the balance between (i) the exponential growth of the solution at each site, and (ii) the probabilistic penalty for travelling each step along the path $p$.

The accuracy of this heuristic relies on some subtle observations about the PAM which we shall briefly discuss further. First is the claim that the $j$-local principal eigenvalues closely approximate the exponential growth rate of the solution at a site. This approximation, in turn, is based on the fact that there is a lack of resonance between the top eigenvalues of the operator $\Delta + \xi$ restricted to $V_t$.

Second is the claim that it is never beneficial for a path to visit other sites of high potential, other than $z^{(p)}$. This is proved by way of a 'cluster expansion' (see Lemma 2.15) which bounds the contribution to $U^p(t)$ between the time $p$ visits a site $z$ of high potential until it leaves the ball $B(z,j)$. Crucially, $j$ is chosen precisely to be the smallest integer for which this 'cluster expansion' bound is smaller than the probabilistic penalty associated with the path getting from outside the ball $B(z, j)$ to $z$ (see the proof of Proposition 2.36).

Finally is the claim that the probabilistic penalty for travelling along the path $p$ is approximately $1/a_t$ for each step of the path. This is a consequence of the well-separatedness of sites of high potential (see our description of the derivation of $Z_t$ above).

**Second heuristic**: We expect that, for $i = 1, 2, 3$,

$$U^i(t) \approx \max_{p \in E_t^i} U^p(t) \ , \tag{2.4}$$

which represents the idea that $U^i(t)$ should be dominated by the contribution from just a single path in the path set $E_t^i$. This is essentially due to the fact that the number of paths of length $k$ grows exponentially in $k$, whereas the probabilistic penalty associated with a path of length $k$ decays as $a_t^{-k}$, which dominates since $a_t \to \infty$.

Let us consider what these heuristics imply for $U^2(t)$ and $U^3(t)$. By analogy with $\Psi_t$ and $Z_t$, define

$$\Psi_t^{(j)}(z) = \lambda^{(j)}(z) - \frac{|z|}{\gamma t} \log \log t$$

and $Z_t^{(j)} := \operatorname{argmax}_{z \in \Pi^{(L_t)}} \Psi_t^{(j)}$. Clearly, by the two heuristics, the dominant contribution to $U(t)$ will come from a path $p \in \Gamma(0)$ that goes directly from the origin to $z^{(p)}$, and so we expect that

$$U(t) \approx \max_p \left\{ \exp\left\{ t\lambda^{(j)}(z^{(p)}) \right\} a_t^{-|z^{(p)}|} \right\} = \exp\left\{ t \max_z \Psi_t^{(j)}(z^{(p)}) \right\} = \exp\left\{ t\Psi_t^{(j)}(Z_t^{(j)}) \right\} \ .$$

Indeed, we formalise this approximation as a lower bound

$$\log U(t) > t\Psi_t^{(j)}(Z_t^{(j)}) + O(td_t b_t) \,. \tag{2.5}$$

Similarly for $U^2(t)$, the heuristics imply that the dominant contribution will come from the path $p \in E_t^2$ that goes directly from the origin to the site

$$Z_t^{(j,2)} = \operatorname*{argmax}_{z \in \Pi^{(L_t)} \setminus \{Z_t^{(j)}\}} \Psi_t^{(j)}(z) \,,$$

and so

$$U^2(t) \approx \exp\left\{ t\lambda^{(j)}(Z_t^{(j,2)}) \right\} a_t^{-|Z_t^{(j,2)}|} \approx \exp\left\{ t\Psi_t^{(j)}(Z_t^{(j,2)}) \right\} \ .$$

We formalise this approximation as an upper bound

$$\log U^2(t) < t\Psi_t^{(j)}(Z_t^{(j,2)}) + O(td_t b_t) \,,$$

which, together with equation (2.5), implies that

$$\log U^2(t) - \log U(t) < -t \left( \Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t^{(j,2)}) + O(d_t b_t) \right) \ .$$

Remark that the negligibility of $U^2(t)$ is then a consequence of the gap in the top order statistics of $\Psi_t^{(j)}$ being larger than $d_t b_t$ (which is the scale of both the error in these bounds, and the amount of negligibility we seek to establish in equation (2.2)).

At this point, our analysis indicates that $Z_t^{(j)}$ should be the localisation site. Indeed, it will turn out that $Z_t^{(j)} = Z_t$ with overwhelming probability (see Corollary 2.29). The radius of influence $\rho$ is precisely the smallest ball to which we can restrict the potential $\xi$ in the definition of $\mathcal{H}(z)$ in order to ensure that the top order statistic of $\lambda^{(j)}(\cdot)$ and $\lambda(\cdot)$ in the macrobox $V_t$ is identical.

Finally, the heuristics imply that the dominant contribution to $U^3(t)$ will come from a path $p$ that visits $Z_t$ but that also ventures outside $B_t$, and so

$$U^3(t) \approx \exp\left\{t\lambda^{(j)}(Z_t)\right\} a_t^{-|Z_t|(1+h_t)} .$$

We formalise this approximation as an upper bound

$$\log U^3(t) < t\lambda^{(j)}(Z_t) - \frac{1}{\gamma}|Z_t|(1+h_t)\log\log t + O(td_tb_t) \tag{2.6}$$

which, together with equation (2.5), implies that

$$\log U^3(t) - \log U(t) < -\frac{1}{\gamma}|Z_t|h_t \log\log t + O(td_tb_t)\,.$$

Remark that the negligibility of $U^3(t)$ is then satisfied if $|Z_t|h_t \log\log t > td_tb_t$.

In Section 2.3 we study extremal theory for $\lambda^{(j)}$ and $\Psi_t^{(j)}$, showing in particular that

$$\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t^{(j,2)}) > d_te_t \quad \text{and} \quad |Z_t^{(j)}|h_t \log\log t > td_te_t$$

both hold eventually with overwhelming probability. We also show that $Z_t^{(j)} = Z_t$ with overwhelming probability. Note that these are, as outlined above, precisely what we need to establish the negligibility of $U^2(t)$ and $U^3(t)$. In the process, we also establish the description of the localisation site $Z_t$ that is contained in Theorem 1.5, as well as the ageing of the localisation site in Theorem 1.6. In Section 2.4, we show how to formalise the heuristics in equations (2.3) and (2.4) into the bounds in equations (2.5) and (2.6), and so complete the proof of the negligibility of $U^2(t)$ and $U^3(t)$.

Note that the above analysis also suggests why the exact exponential decay of the solution in Theorem 2.1 breaks down at the distance scale $r_t$. At this scale, the solution has non-negligible contributions both from the exponential decay around the localisation site $Z_t$ and from the near-maximisers of the functional $\Psi_t$, the latter of which are rather delicate to control precisely. On the other hand, since these near-maximisers contribute approximately only

$$\exp\left\{-t\left(\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t^{(j,2)})\right)\right\}$$

proportion of the total mass, we can use the bound on $\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t^{(j,2)})$ to control their influence.

### Complete localisation, exponential decay and ageing

In Section 2.5 we establish that the solution component $U^1(t)$ has exponential decay around the localisation site $Z_t$, hence completing the proof of Theorem 2.1. The first step is to obtain an upper bound for the exponential decay of the solution around the localisation site $Z_t$, which is achieved by comparing the solution component $u^1(t,z)$ with the principal eigenfunction of the operator $\Delta + \xi$ restricted to the domain $B_t$. This is sufficient to give an upper bound on the exponential decay of the solution, and hence implies complete localisation, but does not yet give a lower bound on the exponential decay.

The second step is to complete the proof of the ageing of the solution in Theorem 1.7, showing in particular that this is a consequence of complete localisation and the corresponding ageing of the localisation site $Z_t$ in Theorem 1.6. We may then use the ageing of the solution to obtain a lower bound for the exponential decay of the solution, completing the proof of Theorem 2.1.

Throughout, we draw on the preliminary results established in Section 2.2, including general results on Schrödinger operators of the form $\Delta + \xi$, as well as properties of the high points of the random field $\xi$

## 2.2 Preliminary results: General theory for Schrödinger operators and the geometry of high points

In this chapter we present the preliminary results that will constitute the main input into our proofs. In the first part we develop the general theory of Schrödinger operators, that is, operators of the form $\Delta + \xi$. In the second part we study the potential field $\xi$, establishing asymptotics for the upper order statistics and studying the geometry of these high points.

### 2.2.1 General theory for Schrödinger operators

In this section we develop general theory for Schrödinger operators of the form $\Delta + \xi$ which is valid for arbitrary $\xi$. This section will be entirely self-contained and is completely deterministic, and may be of independent interest. While most of the results are known, they are often difficult to find in the literature and we collect the relevant theory for general reference.

Throughout this section, let $D \subseteq \mathbb{Z}^d$ be a (non-empty) bounded domain and let $\xi$ be an arbitrary function $\xi : \mathbb{Z}^d \to \mathbb{R}$, abbreviating $\eta := \xi - 2d$. Denote by $\mathcal{H}$ the operator $\Delta + \xi$ restricted to $D$, and let $\{\lambda_i\}_{i \le |D|}$ and $\{\varphi_i\}_{i \le |D|}$ be respectively the (finite) set of eigenvalues and eigenfunctions of $\mathcal{H}$, with eigenvalues in descending order and eigenfunctions $\ell_2$-normalised. Finally, let $X_s$ denote a simple continuous time random walk and define the stopping times

$$\tau_z := \inf\{t \ge 0 : X_t = z\} \quad \text{and} \quad \tau_{D^c} := \inf\{t \ge 0 : X_t \notin D\}\,.$$

We start by presenting representations and deriving simple bounds for $\lambda_1$ and $\varphi_1$.

**Lemma 2.2** (Principal eigenvalue monotonicity)**.** *For each $z \in D$ and $\delta > 0$, let $\bar{\lambda}_1$ be the* *principal eigenvalue of the operator*

$$\bar{\mathcal{H}} := \Delta + \xi + \delta \mathbb{1}_{\{z\}}$$

*restricted to* $D$*. Then* $\bar{\lambda}_1 > \lambda_1$*. Moreover, for each bounded domain* $\bar{D}$ *containing* $D$ *as a strict subset, let* $\bar{\lambda}_1$ *be the principal eigenvalue of the operator*

$$\bar{\mathcal{H}} := \Delta + \xi$$

*restricted to* $\bar{D}$*. Then* $\bar{\lambda}_1 > \lambda_1$*.*

*Proof.* These are general property of elliptic operators, which can be proved by considering the min-max (Reyleigh-Ritz) representation for $\bar{\lambda}_1$, and using $\varphi_1$ to obtain a lower bound for this representation. In particular, for the first statement we have

$$\begin{aligned} \bar{\lambda}_1 &= \sup_{\|v\|_{\ell_2(D)}=1} \langle \bar{\mathcal{H}} v, v\rangle_{\ell_2(D)} \geq \langle \bar{\mathcal{H}} \varphi_1, \varphi_1\rangle_{\ell_2(D)} \\ &= \langle \mathcal{H} \varphi_1, \varphi_1\rangle_{\ell_2(D)} + \langle \delta \mathbb{1}_{\{z\}} \varphi_1, \varphi_1\rangle_{\ell_2(D)} = \lambda_1 + \delta \varphi_1^2(z) > \lambda_1 \, , \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\ell_2(D)}$ denotes the (Euclidean) inner product on $\ell_2(D)$, and where the last inequality follows from the strict positivity of $\varphi_1$ (guaranteed by the Perron-Frobenius theorem). For the second statement, defining the natural extension of $\varphi_1$ to $\bar{D}$

$$\dot{\varphi}_1(y) := \begin{cases} \varphi_1(y) \, , & \text{if } y \in D \, , \\ 0 \, , & \text{if } y \in \bar{D} \setminus D \, , \end{cases}$$

we have instead that

$$\bar{\lambda}_1 = \sup_{\|v\|_{\ell_2(\bar{D})}=1} \langle \bar{\mathcal{H}} v, v\rangle_{\ell_2(\bar{D})} > \langle \bar{\mathcal{H}} \dot{\varphi}_1, \dot{\varphi}_1\rangle_{\ell_2(\bar{D})} = \langle \mathcal{H} \varphi_1, \varphi_1\rangle_{\ell_2(D)} = \lambda_1 \ ,$$

since equality in the min-max representation is only achieved by the principal eigenfunction of $\bar{\mathcal{H}}$ which, unlike $\dot{\varphi}_1$, is strictly positive on $\bar{D}$ (again by the Perron-Frobenius theorem). □

**Lemma 2.3** (Bounds on the principal eigenvalue)**.**

$$\max_{z \in D} \{\eta(z)\} \leq \lambda_1 \leq \max_{z \in D} \{\xi(z)\} \ .$$

*Proof.* The lower bound follows from the min-max representation for the principal eigenvalue

$$\lambda_1 = \sup_{\|v\|_{\ell_2(D)}=1} \langle \bar{\mathcal{H}} v, v\rangle_{\ell_2(D)} \geq \max_{z \in D} \langle \mathcal{H} \mathbb{1}_{\{z\}}, \mathbb{1}_{\{z\}}\rangle_{\ell_2(D)} = \max_{z \in D} \eta(z) \, .$$

The upper bound follows from the Gershgorin circle theorem. □

**Proposition 2.4** (Feynman-Kac representation for the principal eigenfunction)**.** *For each* $y, z \in D$ *the principal eigenfunction* $\varphi_1$ *satisfies the Feynman-Kac representation*

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_z} (\xi(X_s) - \lambda_1) \, ds \right\} \mathbb{1}_{\{\tau_{D^c} > \tau_z\}} \right] . \tag{2.7}$$

*Proof.* Consider $z$ fixed and define $v^z(y) := \varphi_1(y)/\varphi_1(z)$. Note that the function $v^z$ satisfies the Dirichlet problem

$$\begin{aligned} (\Delta + \xi - \lambda_1)\, v^z(y) &= 0\,, & y &\in D \setminus \{z\}\,, \\ v^z(y) &= \mathbb{1}_{\{z\}}(y)\,, & y &\notin D \setminus \{z\}\,. \end{aligned}$$

It is easy to check (i.e. by integrating over the first jump time) that the Feynman-Kac representation on the right-hand side of equation (2.7) also satisfies this Dirichlet problem; hence we are done if there is a unique solution. So assume another non-trivial solution $w$ exists. Then the difference $q := v^z - w$ satisfies the Dirichlet problem

$$\begin{aligned} (\Delta + \xi - \lambda_1)\, q(y) &= 0\,, & y &\in D \setminus \{z\}\,, \\ q(y) &= 0\,, & y &\notin D \setminus \{z\}\,. \end{aligned}$$

and so $q$ is not identically zero if and only if $\lambda_1$ is an eigenvalue of the operator $\Delta + \xi$ restricted to $D \setminus \{z\}$. By the domain monotonicity of the principal eigenvalue in Lemma 2.2, this is impossible. □

**Lemma 2.5** (Path-wise evaluation)**.** *For each $k \in \mathbb{N}$, $y, z \in D$, $p \in \Gamma_k(z, y)$ such that $p_i \neq y$ for $i < k$ and $\{p\} \subseteq D$, and $\zeta > \max_{0 \le i < k} \eta(p_i)$, we have*

$$\mathbb{E}_z \left[ \exp \left\{ \int_0^{\tau_y} (\xi(X_s) - \zeta)\, ds \right\} \mathbb{1}_{\{p_k(X) = p\}} \right] = \prod_{i=0}^{k-1} \frac{1}{\zeta - \eta(p_i)}\,.$$

*Proof.* This follows by integrating over the holding times at the sites $\{p_i\}_{0 \le i \le k-1}$, which are independent. The restriction on $\zeta$ ensures that the resulting integrals are finite. □

**Proposition 2.6** (Path expansion for the principal eigenvector)**.** *For each $y, z \in D$ the principal eigenfunction $\varphi_1$ satisfies the path expansion*

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \sum_{k \ge 1} \sum_{\substack{p \in \Gamma_k(y,z) \\ p_i \neq z,\, 0 \le i < k \\ \{p\} \subseteq D}} \prod_{0 \le i < k} \frac{1}{\lambda_1 - \eta(p_i)}\,.$$

*Proof.* The expectation on the right-hand side of equation (2.7) can be expanded path-wise using Lemma 2.5, which is valid by the lower bound in Lemma 2.3. □

**Proposition 2.7** (Path expansion for the principal eigenvalue)**.** *For each $z \in D$ the principal eigenvalue has the path expansion*

$$\lambda_1 = \eta(z) + \sum_{k \ge 2} \sum_{\substack{p \in \Gamma_k(z,z) \\ p_i \neq z,\, 0 < i < k \\ \{p\} \subseteq D}} \prod_{0 < i < k} \frac{1}{\lambda_1 - \eta(p_i)}\,.$$

*Proof.* Recalling that the eigenfunction relation evaluated at a site $z$ gives

$$\lambda_1 = \eta(z) + \sum_{|y - z| = 1} \frac{\varphi_1(y)}{\varphi_1(z)}\,,$$

the result follows from Proposition 2.6. □

We can use the path expansion for the principal eigenvalue to obtain a refined estimate for eigenvalue monotonicity under certain assumptions; while this estimate is not strictly needed for our main results, it appears to be new, and may be of independent interest. Fix a domain $E \subseteq D$, a site $x \in E$, and define the operator $\mathcal{H}^E := \Delta + \xi$ restricted to $E$, with $\lambda_1^E$ its principal eigenvalue.

**Proposition 2.8** (Refined principal eigenvalue monotonicity)**.** *Assume each $z \in E \setminus \{x\}$ satisfies*

$$\lambda_1^E - \xi(z) > 4d + \delta \,, \tag{2.8}$$

*for some $\delta > 0$. Then there exists a constant $c > 0$, depending only on $\delta > 0$, such that*

$$0 \le \lambda_1 - \lambda_1^E < c \sum_{y \in \partial E} \frac{\varphi_1(y)}{\varphi_1(x)} \,.$$

*Remark.* Note that this bound can be used to show that the localisation of the principal eigenfunction in $D$ at the site $x \in E$ implies that the principal eigenvalue of $D$ is well-approximated by the principal eigenvalue of the subdomain $E$.

*Proof.* Consider the respective path expansions for $\lambda_1$ and $\lambda_1^E$ in Proposition 2.7 and denote by $\tilde{\lambda}_t^E$ the perturbation to $\lambda_1^E$ that results from replacing $\lambda_t^E$ in the right-hand side of the path expansion for $\lambda_t^E$ with $\lambda_1$. By the eigenvalue monotonicity in Lemma 2.2 we have

$$\tilde{\lambda}_1^E \le \lambda_1^E \le \lambda_1 \,.$$

Moreover, from the path expansions it is clear that

$$\lambda_1 - \tilde{\lambda}_t^E = \sum_{k \ge 1} \sum_{\substack{p \in \Gamma_k(x,x) \\ p_i \ne x,\, 0<i<k \\ \{p\} \not\subseteq E, \{p\} \subseteq D}} \prod_{0 \le i < k} \frac{1}{\lambda_1 - \eta(p_i)} \,.$$

Splitting each path $p$ at the first index for which $p_i \notin E$, and collecting paths with identical subpaths from this index onward yields

$$\lambda_1 - \tilde{\lambda}_t^E = \sum_{y \in \partial E} \sum_{\ell \ge 1} \sum_{\substack{p \in \Gamma_\ell(y,x) \\ p_i \ne x,\, 0 \le i < \ell \\ \{p\} \subseteq D}} \left( \prod_{0 \le i < \ell} \frac{1}{\lambda_1 - \eta(p_i)} \left( \sum_{k \ge 1} \sum_{\substack{p \in \Gamma_k(z,y) \\ p_i \ne z,y,\, 0<i<k \\ \{p\} \subseteq E \cup \{y\}}} \prod_{0<i<k} \frac{1}{\lambda_1 - \eta(p_i)} \right) \right) .$$

Note that the condition (2.8) implies that $\lambda_1 - \eta(p_i) > 2d + \delta$ for each $p_i \in E$. Moreover, we have that $|\Gamma_k(z,y)| < (2d)^{k-1}$. Putting these together yields

$$\sum_{k \ge 1} \sum_{\substack{p \in \Gamma_k(z,y) \\ p_i \ne z,y,\, 0<i<k \\ \{p\} \subseteq E \cup \{y\}}} \prod_{0<i<k} \frac{1}{\lambda_1 - \eta(p_i)} \; < \; \sum_{k \ge 1} (2d)^{k-1} (2d+\delta)^{-(k-1)} < c \,,$$

from which we get the result. □

We now study the solution $u_z(t,y)$ to the Cauchy problem

$$\begin{aligned} \frac{\partial u_z(t,y)}{\partial t} &= \mathcal{H}\, u(t,y) \ , & (t,y) &\in [0,\infty) \times D \ , \\ u_z(0,y) &= \mathbb{1}_{\{z\}}(y) \ , & y &\in \mathbb{Z}^d \, . \end{aligned} \tag{2.9}$$

In particular, we give the spectral representation of $u_z(t,y)$ and deduce upper and lower bounds.

**Proposition 2.9** (Feynman-Kac representation of the solution)**.** *For each $y,z \in D$,*

$$u_z(t,y) = \mathbb{E}_z \left[ \exp\left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{X_t = y\}} \mathbb{1}_{\{\tau_{D^c} > t\}} \right] .$$

*Proof.* It can be directly verified (i.e. by integrating over the first jump time) that the Feynman-Kac representation satisfies (2.9). □

**Lemma 2.10** (Time-reversal)**.** *For each $y,z \in D$,*

$$u_z(t,y) = u_y(t,z) \, .$$

*Proof.* This follows from the fact that $\mathcal{H}$ is symmetric. □

**Proposition 2.11** (Spectral representation)**.** *For each $z \in D$ we have*

$$u_z(t,z) = \sum_i e^{\lambda_1 t} \varphi_1^2(z) \, .$$

*Proof.* This follows from the spectral theorem. □

**Corollary 2.12** (Bounds on the solution at the origin)**.** *For each $z \in D$ we have the bounds*

$$e^{\lambda_1 t} \varphi_1^2(z) \le u_z(t,z) \le e^{\lambda_1 t} \, .$$

**Proposition 2.13** (General bound on the solution)**.** *For each $y,z \in D$,*

$$u_z(t,y) \le e^{\lambda_1 t} \, \frac{\varphi_1(y)}{\varphi_1(z)} \, ,$$

*and therefore*

$$\sum_{y \in D} u_z(t,y) \le e^{\lambda_1 t} \sum_{y \in D} \frac{\varphi_1(y)}{\varphi_1(z)} \, .$$

*Proof.* First decompose the Feynman-Kac representation for $u_y(t,z)$ in Proposition 2.9 by conditioning on the stopping time $\tau_z$ and using the strong Markov property:

$$\begin{aligned} u_y(t,z) &= \mathbb{E}_{\tau_z} \left[ e^{\lambda_1 \tau_z} \, \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_z} (\xi(X_s) - \lambda_1) \ ds \right\} \mathbb{1}_{\{\tau_z < \tau_{D^c}\}} \middle| \, \tau_z \right] \right. \\ &\qquad \left. \times \, \mathbb{E}_z \left[ \exp\left\{ \int_0^{t-\tau_z} \xi(X'_s) \, ds \right\} \mathbb{1}_{\{X'_{t-\tau_z} = z, \tau'_{D^c} > t - \tau_z\}} \middle| \, \tau_z \right] \mathbb{1}_{\{\tau_z \le t\}} \right] \\ &= \mathbb{E}_{\tau_z} \left[ e^{\lambda_1 \tau_z} \, \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_z} (\xi(X_s) - \lambda_1) \ ds \right\} \mathbb{1}_{\{\tau_z < \tau_{D^c}\}} \middle| \, \tau_z \right] u_z(t - \tau_z, z) \mathbb{1}_{\{\tau_z \le t\}} \right] , \end{aligned}$$

where $\mathbb{E}_{\tau_z}$ denotes expectation taken over $\tau_z$, $X'_t$ is an independent copy of $X_t$, and $\tau'_{D^c} := \inf\{t \geq 0 : X'_t \notin D\}$. Using the upper bound in Corollary 2.12, deleting the condition that $\mathbb{1}_{\{\tau_z \leq t\}}$, and then combining with the Feynman-Kac representation for the principal eigenfunction in Proposition 2.4, we have that

$$u_y(t,z) \leq e^{\lambda_1 t}\mathbb{E}_y\left[\exp\left\{\int_0^{\tau_z} (\xi(X_s) - \lambda_1)\, ds\right\}\mathbb{1}_{\{\tau_z < \tau_{D^c}\}}\right] = e^{\lambda_1 t}\frac{\varphi_1(y)}{\varphi_1(z)}\,.$$

Applying the time-reversal in Lemma 2.10 and summing over $y \in D$ yields the result. □

**Proposition 2.14** (Bounds on the solution at different times)**.** *For each $y, z \in D$ and $s < t$, if*

$$\sum_{|x-y|=1} u_z(w,x) < u_z(w,y) \quad \textit{for all} \quad w \in (s,t)\,,$$

*then*

$$\frac{u_z(t,y)}{u_z(s,y)} \leq e^{(t-s)\xi(y)}\,.$$

*Proof.* This follows directly from the definition of the Cauchy problem for $u_z(t,y)$. □

Next we state a 'cluster expansion' that is useful for bounding expectations of the 'Feynman-Kac type'; this result can be found in [38, Lemma 4.2] and [40, Lemma 2.18], but we give the proof for completeness.

**Lemma 2.15** (Cluster expansion)**.** *For each $z \in D$ and for any $\zeta > \lambda_1$,*

$$\mathbb{E}_z\left[\exp\left\{\int_0^{\tau_{D^c}} (\xi(X_s) - \zeta)\, ds\right\}\right] < 1 + \frac{2d\,|D|}{\zeta - \lambda_1}$$

*Proof.* First abbreviate

$$u(y) := \mathbb{E}_y\left[\exp\left\{\int_0^{\tau_{D^c}} (\xi(X_s) - \zeta)\, ds\right\}\right]$$

and note that $u$ solves the boundary value problem

$$\begin{aligned}(\Delta + \xi - \zeta)u(y) &= 0\,, & y &\in D\,,\\ u(y) &= 1\,, & y &\notin D\,.\end{aligned} \tag{2.10}$$

With the substitution $w := u - \mathbf{1}$, where $\mathbf{1}$ denotes the vector of ones, (2.10) can be rewritten

$$\begin{aligned}(\Delta + \xi - \zeta)w(y) &= -\left((\Delta + \xi - \zeta)\mathbf{1}\right)(y)\,, & y &\in D\,,\\ w(y) &= 0\,, & y &\notin D\,.\end{aligned}$$

Since $\zeta > \lambda_1$, the solution exists and is given by

$$w(y) = \mathcal{R}_\zeta\left((\Delta + \xi - \zeta)\mathbf{1}\right)(y) = \mathcal{R}_\zeta(\xi - \zeta)(y)$$

where $\mathcal{R}_\zeta$ denotes the resolvent of $\mathcal{H}$ at $\zeta$. By Lemma 2.3 and since $\zeta > \lambda_1$ we have that

$$\xi(y) - \zeta < 2d$$

for all $y \in D$ and so by the positivity of the resolvent (guaranteed since $\mathcal{H}$ is elliptic and $\zeta > \lambda_1$) and the Cauchy-Schwartz inequality we obtain,

$$w(z) < 2d\,\mathcal{R}_\zeta(\mathbf{1})(z) \le 2d\,|D|^{\frac{1}{2}}\,\|\mathcal{R}_\zeta(\mathbf{1})\|_{\ell_2} \le 2d\,|D|\,\|\mathcal{R}_\zeta\|\,,$$

where $\|\cdot\|$ denotes the spectral norm. By considering the spectral representation of the resolvent, we have $\|\mathcal{R}_\zeta\| \le (\zeta - \lambda_1)^{-1}$, which gives the bound. $\square$

Finally, we give a general way to bound the contribution to the solution $u_z(t, y)$ from paths that hit a certain site $x \in D$ and then stay within a subdomain $E \subseteq D$ that contains $x$. In particular, we show that this contribution is bounded above by a quantity proportional to the principal eigenfunction of $\mathcal{H}$ restricted to $E$, which is crucial to proving the complete localisation of the solution. This result can be found in [38, Theorem 4.1]; we give the proof both for completeness and also with a view to extending it to case of 'Bouchaud–Anderson operators' in Chapter 4.

Fix a domain $E \subseteq D$, a site $x \in E$, and recall the definition of the operator $\mathcal{H}^E := \Delta + \xi$ restricted to $E$, with $\lambda_1^E$ and $\varphi_1^E$ the principal eigenvalue and eigenfunction respectively. Define the stopping time

$$\tau_{x,E^c} := \inf\{t \ge \tau_x : X_t \notin E\}\,.$$

Then the contribution to the solution $u_z(t, y)$ from paths that hit $x$ and then stay within $E$ can be written

$$u_z^{x,E}(t,y) := \mathbb{E}_z\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\}\mathbb{1}_{\{X_t = y, \tau_x \le t, \tau_{x,E^c} > t, \tau_{D^c} > t\}}\right].$$

**Proposition 2.16** (Link between solution and principal eigenfunction; see [38, Theorem 4.1])**.** *For each $x \in E$, $y \in E \setminus \{x\}$ and $z \in D$,*

$$\frac{u_z^{x,E}(t,y)}{\sum_{y\in D} u_z(t,y)} \le \frac{1}{(\varphi_1^E(x))^3}\,\varphi_1^E(y)\,.$$

*Proof.* The first step is to make use of time-reversal, suitably adapted to $u_z^{x,E}(t, y)$. In particular, defining

$$\overleftarrow{u_y^{x,E}}(t,z) := \mathbb{E}_y\left[\exp\left\{\int_0^t \xi(X_s)\,ds\right\}\mathbb{1}_{\{X_t = z, \tau_x \le t, \tau_x < \tau_{E^c}, \tau_{D^c} > t\}}\right]$$

we can write

$$\frac{u_z^{x,E}(t,y)}{\sum_{y\in D} u_z(t,y)} \le \frac{u_z^{x,E}(t,y)}{u_z(t,x)} = \frac{\overleftarrow{u_y^{x,E}}(t,z)}{u_x(t,z)}\,. \tag{2.11}$$

Next we decompose the Feynman-Kac formula for $\overleftarrow{u_y^{x,E}}(t,z)$ as in the proof of Proposition 2.13, by conditioning on the stopping time $\tau_x$ and using the strong Markov property. More precisely, we write

$$\overleftarrow{u_y^{x,E}}(t,z) = \mathbb{E}_{\tau_x}\left[e^{\tau_x\lambda_1^E}\,\mathbb{E}_y\left[\exp\left\{\int_0^{\tau_x}\left(\xi(X_s) - \lambda_1^E\right)\,ds\right\}\mathbb{1}_{\{\tau_x < \tau_{E^c}\}}\,\middle|\,\tau_x\right]\right. \tag{2.12}$$

$$\times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t-\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{t-\tau_x}=z, \tau'_{D^c}>t-\tau_x\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \le t\}} \Bigg] ,$$

where $\mathbb{E}_{\tau_x}$, $X'_t$ and $\tau'_{D^c}$ are defined as in the proof of Proposition 2.13. Next, note that an application of Corollary 2.12 gives the bound

$$1 \le u_x^{x,E}(w,x)\, \frac{1}{(\varphi_1^E(x))^2}\, e^{-w\lambda_1^E} , \tag{2.13}$$

and recall the representation

$$u_x^{x,E}(w,x) = \mathbb{E}_x \left[ \exp\left\{ \int_0^{w} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_w = x, \tau'_{E^c}>w\}} \right] .$$

Combining the bound in (2.13) with equation (2.12) (setting $w=\tau_x$), gives

$$\begin{aligned}
\overleftarrow{u}_y^{x,E}(t,z) &\le \frac{1}{(\varphi_1^E(x))^2}\, \mathbb{E}_{\tau_x}\Bigg[ \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_x} \left(\xi(X_s) - \lambda_1^E\right) ds \right\} \mathbb{1}_{\{\tau_{E^c}>\tau_x\}} \middle| \tau_x \right] \\
&\qquad \times \mathbb{E}_x \left[ \exp\left\{ \int_0^{\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{\tau_x}=x, \tau'_{E^c}>\tau_x\}} \middle| \tau_x \right] \\
&\qquad \times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t-\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{t-\tau_x}=z, \tau'_{D^c}>t-\tau_x\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \le t\}} \Bigg] \\
&\le \frac{1}{(\varphi_1^E(x))^2}\, \mathbb{E}_{\tau_x}\Bigg[ \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_x} \left(\xi(X_s) - \lambda_1^E\right) ds \right\} \mathbb{1}_{\{\tau_{E^c}>\tau_x\}} \middle| \tau_x \right] \\
&\qquad \times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_t=z, \tau'_{D^c}>t\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \le t\}} \Bigg] \\
&\le \frac{1}{(\varphi_1^E(x))^3}\, \varphi_1^E(y)\; u_x(t,z) ,
\end{aligned}$$

where the inequality in the second step results from deleting the condition that $X'_{\tau_x} = x$, and where the last inequality results from deleting the condition that $\tau_x \le t$, and where we have used the Feynman-Kac representation for $\varphi_1^E$ given by Proposition 2.4. Combining this with equation (2.11) gives the result. ☐

### 2.2.2 High points of the potential field

In this section we establish some preliminary properties of the potential field $\xi$, particularly asymptotics for the upper order statistics and the geometry of these statistics. Note that several of these results are stated as almost sure asymptotics, although for our purposes it would be sufficient to have these results holding only in probability.

**Lemma 2.17** (Almost sure asymptotics for $\xi$). *Let $\{\xi_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables with distribution $\xi(0)$, and denote by $\xi_{n,i}$ the $i^{\text{th}}$ highest value in the set $\{\xi_j\}_{j\le n}$. Further, for each $a \le 1$ define the level $L_{n,a} := ((1-a)\log n)^{\frac{1}{\gamma}}$ and the set $\Pi^{(L_{n,a})} := \{\{j \le n : \xi_j > L_{n,a}\}$ of exceedances of this level. Then for $a \in [0,1)$ and $a' \in (0,1]$, as $n \to \infty$,*

$$\xi_{n,[n^a]} \sim L_{t,a} \qquad \textit{and} \qquad |\Pi^{(L_{n,a'})}| \sim n^{a'}$$

*hold almost surely.*

*Proof.* These follow from well-known results on sequences of i.i.d. random variables; they can be proved by observing that the random variables $\{\xi_i^\gamma\}$ are exponentially distributed, and then applying standard results on i.i.d. sequences of exponential random variables, see for instance the proof of [46, Lemma 4.7]. $\square$

For each $a \leq 1$, define the macrobox level $L_{t,a} := ((1-a)\log|V_t|)^{\frac{1}{\gamma}}$ and let the subset $\Pi^{(L_{t,a})} := \left\{z \in \mathbb{Z}^d : \xi(z) > L_{t,a}\right\} \cap V_t$ consist of sites in $V_t$ at which $\xi$-exceedances of the level $L_{t,a}$ occur. Recall that $L_t := L_{t,\theta}$.

**Corollary 2.18** (Almost sure asymptotics for $\xi$)**.** *Denote by $\xi_{t,i}$ the $i^{\text{th}}$ highest value of $\xi$ in $V_t$. Then for $a \in [0,1)$ and $a' \in (0,1]$, as $t \to \infty$,*

$$\xi_{t,[|V_t|^a]} \sim L_{t,a} \qquad \text{and} \qquad |\Pi^{(L_{t,a'})}| \sim |V_t|^{a'}$$

*hold almost surely.*

**Lemma 2.19** (Almost sure separation of high points of $\xi$)**.** *For any $a > 0$ and $n \in \mathbb{N}$ let*

$$\Pi_n^{(L_t,a)} := \{z \in B(V_t, n) : \xi(z) > L_{t,a}\}$$

*be the set of $L_{t,a}$ exceedances of $\xi$ in the $n$-extended macrobox $B(V_t, n)$. Then, for any $a' < a$, as $t \to \infty$*

$$sep\left(\Pi_n^{(L_{t,a})} \cup \{0\}\right) > |V_t|^{\frac{1-2a'}{d}}$$

*eventually almost surely.*

*Proof.* This result follows from the almost sure asymptotics for $\xi$ in Corollary 2.18, for instance as in [2, Lemma 1] (see also [52, Lemma 2.2]). $\square$

*Remark.* Note that we need the almost sure separation of high points in the $n$-extended macrobox $B(V_t, n)$ rather than just in $V_t$ because each $\lambda^{(n)}(z)$, for $z \in V_t$, depends on the random potential $\xi$ in the ball $B(z, n) \subseteq B(V_t, n)$. The result in Lemma 2.19 implies that, eventually almost surely, each $z \in \Pi^{(L_{t,a})}$ has the property that $\xi(y) < L_{t,a}$ for all $y \in B(z,n) \setminus \{z\}$.

**Corollary 2.20** (Paths cannot always remain close to high points of $\xi$)**.** *There exists a $c \in (0,1)$ such that, for each $n \in \mathbb{N}$, all paths $p \in \Gamma(0,z)$ such that $\{p\} \subseteq V_t$ satisfy, as $t \to \infty$,*

$$\left|\left\{i : p_i \notin B(\Pi^{(L_t)}, n)\right\}\right| > |z| - \frac{|z|}{t^c},$$

*eventually almost surely.*

*Proof.* Abbreviate $N := \text{sep}(\Pi^{(L_t)} \cup \{0\})$ and

$$Q := \left|\left\{i : p_i \notin B(\Pi^{(L_t)}, n)\right\}\right| .$$

Suppose a path $p$ passes through $m$ distinct $B(x,n)$ with $x \in \Pi^{(L_t)}$. Then, since there is a minimum distance of $(N-2n)$ between each such ball, the path $p$ satisfies

$$Q \geq m(N-2n) .$$

On the other hand, it is clear that $Q \geq |z| - (2n+1)m$. Therefore

$$Q \geq \min_{m \in \mathbb{N}} \max \{m(N-2n), |z| - (2n+1)m\} \geq \frac{(N-2n)|z|}{N+1} = |z| - \frac{(2n-1)|z|}{N+1}$$

and the result follows from Lemma 2.19. □

## 2.3 Extremal theory for local principal eigenvalues

In this section, we use point process techniques to study the random variables $Z_t^{(j)}$ and $\Psi_t^{(j)}(Z_t^{(j)})$, and generalisations thereof; the techniques used are similar to those found in [3, 33, 64], although we strengthen the results available in those papers. In the process, we complete the proof of Theorems 1.5 and 1.6. Throughout this section, let $\varepsilon$ be such that $0 < \varepsilon < \theta$.

### 2.3.1 Upper-tail properties of the local principal eigenvalues

The first step is to give upper-tail asymptotics for the distribution of the local principal eigenvalues $\lambda^{(n)}(z)$ for $z \in \Pi^{(L_t)}$ and $n \in \mathbb{N}$. These will allow us to study the random variables $Z_t^{(j)}$ and $\Psi_t^{(j)}(Z_t^{(j)})$ via point process techniques. For technical reasons, we shall actually consider a *punctured* version of $\lambda^{(n)}(z)$ which will coincide with $\lambda^{(n)}(z)$ eventually almost surely for each $z \in \Pi^{(L_t)}$.

To this end, let $\{\tilde{\xi}_z\}_{z \in V_t}$ be a collection of independent potential fields $\tilde{\xi}_z : \mathbb{Z}^d \to \mathbb{R}$ defined so that, for each $z \in V_t$, we have $\tilde{\xi}_z(z) = \xi(z)$, and, for each $y \in V_t \setminus \{z\}$, instead $\tilde{\xi}_z(y)$ is i.i.d. with common distribution

$$\tilde{\xi}(0) = \begin{cases} \xi(0), & \text{if } \xi(0) < L_t\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

Then, for each $z \in V_t$ and $n \in \mathbb{N}$, let $\tilde{\lambda}_t^{(n)}(z)$ be the principal eigenvalue of the *punctured* operator $\tilde{\mathcal{H}}^{(n)}(z) := \Delta + \tilde{\xi}_z$ restricted to $B(z,n)$.

**Proposition 2.21** (Path expansion for $\tilde{\lambda}_t^{(n)}$)**.** *For each $n \in \mathbb{N}$ and $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly, as $t \to \infty$,*

$$\begin{aligned} \tilde{\lambda}_t^{(n)}(z) &= \eta(z) + \sum_{2 \leq k \leq 2j} \sum_{\substack{p \in \Gamma_k(z,z) \\ p_i \neq z,\, 0<i<k \\ \{p\} \subseteq B(z,n)}} \prod_{0<i<k} \frac{1}{\tilde{\lambda}_t^{(n)}(z) - \eta(p_i)} + o(d_t e_t)\,, \\ &= \eta(z) + O(a_t^{-1})\,. \end{aligned}$$

*Moreover, as $t \to \infty$,*

$$\tilde{\lambda}_t^{(n)}(z) = \lambda^{(n)}(z)$$

*eventually almost surely.*

*Proof.* Applying Proposition 2.7 we have that

$$\tilde{\lambda}_t^{(n)}(z) = \eta(z) + \sum_{k\geq 2} \sum_{\substack{p\in\Gamma_k(z,z)\\ p_i\neq z,\, 0<i<k \\ \{p\}\subseteq B(z,n)}} \prod_{0<i<k} \frac{1}{\tilde{\lambda}_t^{(n)}(z) - \eta(p_i)} .$$

Now recall that, by Lemmas 2.19 and 2.3, for each $p_i \in B(z,n) \setminus \{z\}$,

$$\tilde{\lambda}_t^{(n)}(z) - \eta(p_i) > L_{t,\varepsilon} - L_t + 2d \sim (\theta - \varepsilon)a_t ,$$

eventually almost surely. Moreover, as $t \to \infty$,

$$a_t^{-(2j+2)} = o(d_t e_t) ,$$

by the definition of $j$. This means that, up to the error $o(d_t e_t)$, we can truncate the sum at paths with $2j$ steps. It also means that the total contribution from the sum over paths $p \in \Gamma_k(z,z)$ is $O(a_t^{-1})$. Finally, the fact that $\tilde{\lambda}_t^{(n)}(z) = \lambda^{(n)}(z)$ eventually almost sure follows directly from Lemma 2.19. □

**Proposition 2.22** (Extremal theory for $\tilde{\lambda}_t^{(n)}$; see [3, Section 6], [33, Proposition 4.2])**.** *For each $n \in \mathbb{N}$, there exists a scaling function $A_t = a_t + O(1)$ such that, as $t \to \infty$ and for each fixed $x \in \mathbb{R}$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) \to e^{-x} .$$

*Moreover, there exists a $c > 0$ such that, as $t \to \infty$ and uniformly for $x > 1$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) < e^{-cx^{\min\{1,\gamma\}}} .$$

*Proof.* First remark that, by Lemmas 2.19 and 2.3, as $t \to \infty$,

$$\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t \quad \text{implies that} \quad \xi(0) > L_{t,\varepsilon} ,$$

eventually almost surely, which means that we can apply the path expansion in Proposition 2.21 to $\tilde{\lambda}_t^{(n)}(0)$. Let $A_t$ be an arbitrary scale such that $A_t = a_t + O(1)$, and define the function

$$Q(A_t;\xi) := 2d + \sum_{2\leq k\leq 2j} \sum_{\substack{p\in\Gamma_k(0,0)\\ p_i\neq 0,\, 0<i<k \\ \{p\}\subseteq B(z,j)}} \prod_{0<i<k} \frac{1}{A_t - \eta(p_i)} ,$$

if $\xi(y) < L_t$ for each $y \in B(0,j) \setminus \{0\}$ and $Q(A_t;\xi) := 0$ otherwise. Note that, as $t \to \infty$, and uniformly in $\xi$, $Q(A_t;\xi) = O(1)$ and moreover

$$Q(A_t + x d_t;\xi) = Q(A_t;\xi) + o(d_t) .$$

Then, since $\tilde{\lambda}_t^{(n)}(0)$ is strictly increasing in $\xi(0)$ we have that, as $t \to \infty$,

$$\begin{aligned}\mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) &\sim \mathbf{P}\Big(\xi(0) > A_t + x d_t + Q(A_t + x d_t;\xi)\Big) \\ &\sim \mathbf{P}\Big(\xi(0) > A_t + x d_t + Q(A_t;\xi)\Big) \qquad (2.14)\end{aligned}$$

$$\begin{aligned}&\sim t^{-d}e^{-x}\int_{\xi}\exp\left\{a_t^\gamma-\Big(A_t+Q(A_t;\xi)\Big)^\gamma\right\}d\mu_\xi\\&\sim t^{-d}e^{-x}\int_{\xi}\exp\left\{a_t^\gamma-(A_t+O(1))^\gamma\right\}d\mu_\xi\end{aligned}$$

where the first asymptotic accounts for the error in the path expansion Proposition 2.21, the second and third asymptotics result from Taylor expansions, and are uniform in $\xi$ (as is the fourth asymptotic), and where $\mu_\xi$ stands for the joint probability density of $\{\xi(y)\}_{y\in B(0,n)\setminus\{0\}}$. Note that, for $C>0$ sufficiently large, eventually

$$a_t^\gamma-(a_t+C+O(1)^\gamma<0<a_t^\gamma-(a_t-C+O(1))^\gamma\,.$$

Hence, by continuity of $Q$, there exists an $A_t=a_t+O(1)$ such that, as $t\to\infty$,

$$\int_{\xi,\sigma}\exp\left\{a_t^\gamma-\Big(A_t+Q(A_t;\xi)\Big)^\gamma\right\}d\mu_\xi\to 1$$

which gives the first result. For the second, instead of (2.14) we bound $Q(A_t+xd_t;\xi)$ above, uniformly in $x>0$, by $Q(A_t;\xi)$, which produces the bound

$$t^{-d}\int_{\xi}\exp\left\{a_t^\gamma-\Big(A_t+Q(A_t;\xi)\Big)^\gamma\Big(1+\frac{x}{\gamma}(\log t)^{-1}\Big)^\gamma\right\}d\mu_\xi\,.$$

In the case $\gamma\geq 1$, we bound this expression above uniformly in $x>0$ by

$$t^{-d}\int_{\xi}\exp\left\{a_t^\gamma-\Big(A_t+Q(A_t;\xi)\Big)^\gamma\Big(1+\frac{x}{\gamma}(\log t)^{-1}\Big)\right\}d\mu_\xi\sim e^{-\frac{x}{\gamma}(1+o(1))}\,,$$

using the definition of $A_t$ and the fact that $A_t+Q(A_t;\xi)\sim a_t$ in the last step. The case $\gamma<1$ is simpler, since then we have simply

$$\mathbf{P}\Big(\xi(0)>A_t+xd_t+2d\Big)=\mathbf{P}\Big(\xi(0)>a_t+xd_t+O(1)\Big)$$

and the bound follows from the Weibull tail of $\xi(0)$. □

We now define the set-up we shall need to examine the correlation of the potential field near sites of high $\tilde\lambda^{(n)}$ in Theorem 1.5. Since the profile differs at the interface sites $\mathcal{I}$, our set-up shall necessarily need to take these sites into account.

For each $n\in\mathbb{N}$ and $y\in B(0,n\wedge\rho)$ define the positive constants

$$c_\xi(y):=\begin{cases}n(y)^{\frac{2}{\gamma-1}}, & \text{if } y\notin\mathcal{I}\,,\\ 0, & \text{else}\,,\end{cases}\qquad\text{and}\qquad \bar c_\xi(y):=\gamma\,n(y)^2\,,$$

recalling that $n(y):=|\Gamma_{|y|}(0,y)|$ denotes the number of shortest paths from the origin to $y$. Define the rectangles

$$E_\xi:=\prod_{y\in(B(0,n\wedge\rho)\setminus\{0\})\setminus\mathcal{I}}(-f_t,f_t)\;\times\prod_{y\in(B(0,n)\setminus B(0,n\wedge\rho))\cup\mathcal{I}}(f_t,g_t)\,,$$

$$S_\xi := \prod_{y\in(B(0,n\wedge\rho)\setminus\{0\})\setminus\mathcal{I}} a_t^{q(|y|)}(c_\xi(y)-f_t, c_\xi(y)+f_t) \;\times \prod_{y\in(B(0,n)\setminus B(0,n\wedge\rho))\cup\mathcal{I}} (f_t, g_t)\,,$$

the event

$$\mathcal{S}_t := \left\{\{\xi(Z_t+y)\}_{y\in B(0,n)\setminus\{0\}} \in S_\xi\right\}\,,$$

and, for each $x\in\mathbb{R}$ and the scaling function $A_t$ from Proposition 2.22, further define the event

$$\mathcal{A}_t := \left\{\tilde\lambda_t^{(n)}(0) > A_t + x d_t\right\}\,.$$

**Proposition 2.23** (Profile of the local potential field)**.** *For each $n\in\mathbb{N}$, as $t\to\infty$,*

$$\mathbf{P}\left(\mathcal{S}_t \middle| \mathcal{A}_t\right) \to 1\,.$$

*Moreover, as $t\to\infty$,*

$$f_{\xi(y)|\mathcal{A}_t}(x) \to \frac{e^{\bar c_\xi(y)x} f_\xi(x)}{\mathbf{E}[e^{\bar c_\xi(y)\xi(0)}]}\,, \quad \textit{for each } y\in\mathcal{I}\,, \tag{2.15}$$

*uniformly over $x\in(0,L_t)$.*

*Proof.* Define a field $s : B(0,n)\setminus\{0\}\to\mathbb{R}$. For a scale $C_t\sim a_t$ define the function

$$Q_t(C_t;s) := 2d - \sum_{2\le k\le 2j}\; \sum_{\substack{p\in\Gamma_k(0,0)\\ p_i\neq 0,\,0<i<k\\ \{p\}\subseteq B(0,n)}}\; \prod_{0<i<k} \frac{1}{C_t - a_t^{q(|p_i|)}(c_\xi(p_i)+s(p_i)) + 2d}\,,$$

if, for each $y\in B(0,n)\setminus\{0\}$,

$$a_t^{q(|y|)}(c_\xi(y)+s(y)) \in (0,L_t)$$

is satisfied, and $Q_t(C_t;s) := 0$ otherwise. Define further the function

$$R_t(C_t;s) := a_t^\gamma - (C_t + Q_t(C_t;s))^\gamma + \sum_{y\in B(0,n)} \left(\log f_\xi\left(a_t^{q_\xi(|y|)}(c_\xi(y)+s(y))\right) + \log a_t^{q(|y|)}\right)\,.$$

To motivate these definitions, consider that, similarly to the above, we can write

$$\mathbf{P}\left(\tilde\lambda_t^{(n)}(0) > A_t + x d_t\right) \sim t^{-d} e^{-x} \int_{\mathbb{R}^{|B(0,n)|-1}} \exp\left\{R_t(A_t;s)\right\}\, ds\,. \tag{2.16}$$

It remains to show that the integral in (2.16) is asymptotically concentrated on the set $E_\xi$ and that equation (2.15) is satisfied. This fact can be checked by a somewhat lengthy computation which we only sketch here.

The function $R_t(s)$ can be decomposed into two parts, one whose dependence on $s_\xi(y)$ is of order, as $t\to\infty$,

$$n(y)^2\,\gamma\, a_t^{q(|y|)}\, a_t^{\gamma-1-2|y|}\,(c_\xi(y)+s(y))\,(1+o(1))\,,$$

uniformly in $s$, another whose dependence is

$$a_t^{q(|y|)\gamma}(c_\xi(y)+s(y))^\gamma \,.$$

Hence, since we defined $q(|y|)$ precisely so that

$$q(|y|)+\gamma-1-2|y| = q(|y|)\gamma \,,$$

if $y \in B(0,\rho_\xi)$, the function $R_t$ has the asymptotic form, as $t \to \infty$,

$$R_t(s) = f(t;s) + a_t^\kappa \left(g(s(y)) + o(1)\right)$$

where $f(t;s)$ is some function not depending on $s(y)$, $\kappa$ is some non-negative constant with $\kappa > 0$ if any only if $y \in B(0,\rho_\xi) \setminus \mathcal{I}$, the function $g(x)$ satisfies

$$g(x) := \gamma\, n(y)^2 (c_\xi(y)+x) - (c_\xi(y)+x)^\gamma \,,$$

and where the error term $o(1)$ is uniform in $s$. Then we have, uniformly in $s$, as $t \to \infty$,

$$\int_{\mathbb{R}} e^{R_t(s)}\, ds(y) \sim e^{f(t;s)} \int_{\mathbb{R}} \exp\left\{a_t^\kappa g(s(y))\right\} ds(y) \,.$$

If $y \in B(0,\rho) \setminus \mathcal{I}$, and since $g(x)$ achieves a unique maximum at 0 (by the construction of $c_\xi(y)$), by the Laplace method this integral is asymptotically concentrated on $s(y) \in (-f_t, f_t)$. On the other hand, if $y \in \mathcal{I}$, then the integrand is asymptotically

$$e^{\bar{c}_\xi(y)s(y)} f_\xi(s(y)) \,,$$

uniformly over $s(y)$, which establishes (2.15). Trivially, if $y \notin B(0,\rho)$, then the integral is concentrated on $s(y) \in (f_t, g_t)$, and we have the result. □

### 2.3.2 Constructing the point process

The existence of asymptotics for the (punctured) local principal eigenvalues allows us to establish scaling limits for the penalisation functional $\Psi_t^{(j)}$. We start by constructing a point set from the pair $(z, \Psi_t^{(j)}(z))$ which will converge to a Poisson point process in an appropriate limit.

For technical reasons, we shall actually need to consider a certain generalisation of the functional $\Psi_t^{(j)}$. More precisely, for each $c \in \mathbb{R}$, define the functional $\Psi_{t,c}^{(j)} : V_t \to \mathbb{R}$ by

$$\Psi_{t,c}^{(j)}(z) := \lambda^{(j)}(z) - \frac{|z|}{\gamma t}\log\log t + c\frac{|z|}{t} \,.$$

Further, for each $z \in \Pi^{(L_t)}$ define

$$Y_{t,c,z}^{(j)} := \frac{\Psi_{t,c}^{(j)}(z) - A_{r_t}}{d_{r_t}} \qquad \text{and} \qquad \mathcal{M}_{t,c}^{(j)} := \sum_{z \in \Pi^{(L_t)}} \mathbb{1}_{(zr_t^{-1}, Y_{t,c,z}^{(j)})} \,.$$

Finally, for each $\tau \in \mathbb{R}$ and $\alpha > -1$ let

$$\hat{H}_\tau^\alpha := \{(x, y) \in \dot{\mathbb{R}}^{d+1} : y \geq \alpha|x| + \tau\},$$

where $\dot{\mathbb{R}}^{d+1}$ is the one-point compactification of $\mathbb{R}^{d+1}$. The set $\hat{H}_\tau^\alpha$ is the the domain on which we shall prove the point process convergence of $\mathcal{M}_{t,c}^{(j)}$; even though $\hat{H}_\tau^\alpha$ is unbounded, it turns out that $\hat{H}_\tau^\alpha$ contains a finite number of points of the set $\mathcal{M}_{t,c}^{(j)}$ eventually almost surely (here the restriction $\alpha > -1$ is crucial).

For clarity of exposition, we record here some remarks on the nature of the term $c|z|/t$ in the modified penalisation functional $\Psi_{t,c}^{(j_t)}$. First note that, as we shall see in Proposition 2.27 below, this term does not, with overwhelming probability, affect the determination of the maximiser of the functional. Nevertheless, there are two separate places in our proof that necessitate the addition of this term, one of which we could eliminate with a finer analysis of the potential field, the other unavoidable.

First, recall that our analysis of the potential field distinguishes three groups of sites inside the macrobox $V_t$: 'high' sites with potential above the level $L_{t,\varepsilon}$, typical sites with potential below $L_t$, and those in between. The gap between the levels $L_{t,\varepsilon}$ and $L_t$ is a constant multiple of $a_t$, and since the probabilistic penalisation due to diffusion to a site $z$ is calculated via a product of such gaps in potential along each shortest path to $z$ (see the discussion in Section 2.1 above), this leads precisely to a term of the form $c|z|/t$. On the other hand, the above analysis of the potential field is rather rough; if we instead introduce a fourth group of sites with potential negligible with respect to $a_t$, and show that most sites along any shortest path to $z$ belong to this group, we could avoid this term. Indeed, this kind of fine analysis turns out to be necessary in our extension to the case of fractional-double-exponential potential (see Section 2.6 below).

Second, as mentioned, the probabilistic penalty also includes a sum over *all* shortest paths to $z$. Since the number of shortest paths $n(z)$ is exponential in $|z|$, and since the base of the exponent differs among the candidate sites, this also introduces a term of the form $c|z|$ into the penalisation. This term is unavoidable, and indeed if we were to undertake an analysis of almost sure localisation such a term would play a crucial role in the determination of the localisation sites (see the discussion in Chapter 5 and the similar analysis in [52]).

**Proposition 2.24** (Point process convergence)**.** *For each $\tau, c \in \mathbb{R}$ and $\alpha > -1$, as $t \to \infty$,*

$$\mathcal{M}_{t,c}^{(j)}|_{\hat{H}_\tau^\alpha} \Rightarrow \mathcal{M} \quad \textit{in law},$$

*where $\mathcal{M}$ is a Poisson point process on $\hat{H}_\tau^\alpha$ with intensity measure $\nu(dx, dy) = dx \otimes e^{-y-|x|} dy$.*

*Proof.* The idea of the proof is to replace the set $\{\lambda^{(j)}(z)\}_{z \in \Pi^{(L_t)}}$ with the set of i.i.d. punctured principal eigenvalues $\{\tilde{\lambda}_t^{(j)}\}_{z \in V_t}$ and then apply standard results in i.i.d. extreme value theory to show convergence to $\mathcal{M}$ in $\hat{H}_\tau^\alpha$.

To this end, define $\tilde{\Psi}_{t,c}^{(j)}(z)$ and $\tilde{Y}_{t,c,z}^{(j)}$ equivalently to $\Psi_{t,c}^{(j)}(z)$ and $Y_{t,c,z}^{(j)}$ after replacing $\lambda^{(j)}(z)$ everywhere with $\tilde{\lambda}_t^{(j)}(z)$, and further define

$$\tilde{\mathcal{M}}_{t,c}^{(j)} = \sum_{v \in V_t} \mathbb{1}_{(z r_t^{-1}, \tilde{Y}_{t,c,z}^{(j)})}\,.$$

Recall that $\{\tilde{\lambda}_t^{(j)}\}_{z \in V_t}$ are i.i.d. with tail asymptotics and uniform tail decay governed by Proposition 2.22. By applying an identical argument as in [64, Lemma 3.1] and [45, Lemma 4.3], we have that, as $t \to \infty$,

$$\tilde{\mathcal{M}}_{t,c}^{(j)}\big|_{\hat{H}_\tau^\alpha} \Rightarrow \mathcal{M} \quad \text{in law} .$$

Note that the uniform tail decay is necessary here since it guarantees that $\hat{H}_\tau^\alpha$ contains a finite number of points almost surely (see [45, Lemma 4.3]). We claim that if $z \in V_t$ is such that

$$(zr_t^{-1}, \tilde{Y}_{t,c,z}^{(j)}) \in \hat{H}_\tau^\alpha ,$$

then, eventually almost surely,

$$z \in \Pi^{(L_t)} .$$

This is since $(zr_t^{-1}, \tilde{Y}_{t,c,z}^{(j)}) \in \hat{H}_\tau^\alpha$ is equivalent to

$$\tilde{\lambda}_t^{(j)}(z) \geq A_{r_t} + \frac{\alpha|z|d_{r_t}}{r_t} + \frac{|z|}{\gamma t}\log\log t - \frac{c|z|}{t} + \tau d_{r_t}$$

which implies that, as $t \to \infty$,

$$\begin{aligned} \tilde{\lambda}_t^{(j)}(z) &> a_t(1+o(1)) + (\alpha+1+o(1))\frac{|z|}{\gamma t}\log\log t + O(d_t) \\ &> a_t(1+o(1)) + O(d_t) \end{aligned}$$

since $A_{r_t} \sim a_{r_t} \sim a_t$, $d_{r_t} \sim d_t$ and $\alpha > -1$. The claim then follows by the upper bound in Lemma 2.3. As a consequence, we have that, as $t \to \infty$,

$$\sum_{z \in \Pi^{(L_t)}} \mathbb{1}_{(zr_t^{-1}, \tilde{Y}_{t,c,z}^{(j)})}\big|_{\hat{H}_\tau^\alpha} \Rightarrow \mathcal{M} \quad \text{in law} . \tag{2.17}$$

To complete the proof, we construct a coupling of the field $\xi$ with the fields $\{\tilde{\xi}_z\}_{z \in \Pi^{(L_t)}}$ with the property that

$$\left\{\lambda^{(j)}(z)\right\}_{z \in \Pi^{(L_t)}} = \left\{\tilde{\lambda}_t^{(j)}(z)\right\}_{z \in \Pi^{(L_t)}} , \tag{2.18}$$

for $t$ sufficiently large. In particular, by Lemma 2.19 there exists a $t_0$ such that almost surely, for all $t > t_0$, we have $r(\Pi^{(L_t)}) > 2j$. For such $t$ we define the coupling as follows: for $z \in \Pi^{(L_t)}$ and $y \in B(z,j)$ set $\tilde{\xi}_z(y) = \xi(y)$; otherwise choose $\tilde{\xi}_z(y)$ independently. Since $t > t_0$, $\{\tilde{\xi}_z\}_{z \in V_t}$ is indeed a set of independent fields and also (2.18) holds. Combining with (2.17) completes the proof. $\square$

We now use the point process $\mathcal{M}$ to analyse the joint distribution of top two statistics of the functional $\Psi_{t,c}^{(j)}$. So let

$$Z_{t,c}^{(j)} := \underset{z \in \Pi^{(L_t)}}{\operatorname{argmax}}\, \Psi_{t,c}^{(j)}(z) \quad \text{and} \quad Z_{t,c}^{(j,2)} := \underset{\substack{z \in \Pi^{(L_t)} \\ z \neq Z_{t,c}^{(j)}}}{\operatorname{argmax}}\, \Psi_{t,c}^{(j)} .$$

These are well-defined eventually almost surely, since $\Pi^{(L_t)}$ is finite and non-zero by Lemma 2.18.

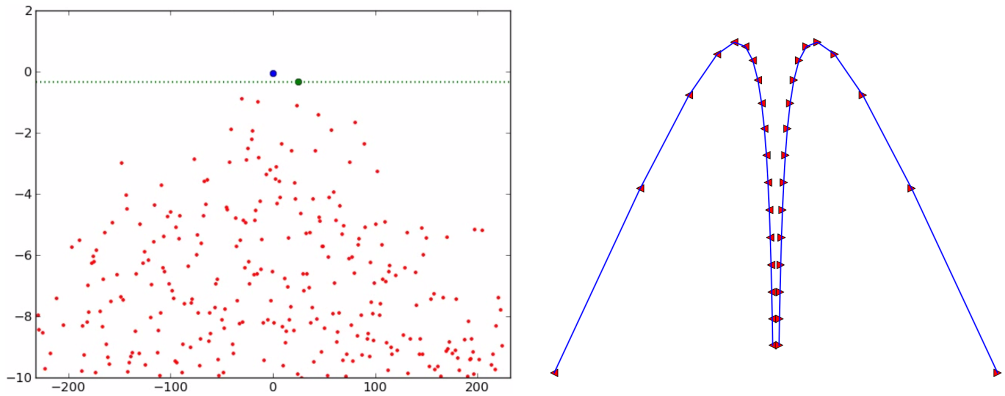


Figure 6: A simulation of the point set $\mathcal{M}_{t,c}^{(j)}$ in the case $d = 1$ with the top two points marked, and a depiction of the approximate trajectories $t \mapsto (z_i r_t^{-1}, Y_{t,c,z_i}^{(j)})$ of subsets of the process $(\mathcal{M}_{t,c}^{(j)})_{t \geq 0}$ corresponding to the pair $(z_i, \Psi_t^{(j)}(z_i))$, for fixed sites $z_1 < 0$ and $z_2 > 0$. Credit: A. Fiodorov.

**Corollary 2.25.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\left( \frac{Z_{t,c}^{(j)}}{r_t}, \frac{Z_{t,c}^{(j,2)}}{r_t}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j)}) - A_{r_t}}{d_{r_t}}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j,2)}) - A_{r_t}}{d_{r_t}} \right)$$

*converges in law to a random vector with density*

$$p(x_1, x_2, y_1, y_2) = \exp\{-(y_1 + y_2) - |x_1| - |x_2|) - 2^d e^{-y_2}\} \mathbb{1}_{\{y_1 > y_2\}}\,.$$

*Proof.* This follows from the point process density in Proposition 2.24 using the same computation as in [64, Proposition 3.2] (although note that the exact form of the density in [64] is slightly different, due to different normalisations for the scales $d_t$ and $r_t$). □

### 2.3.3 Properties of the localisation site

We now use the results from the previous subsection to analyse the localisation sites $Z_{t,c}^{(j)}$ and $Z_t$, and in the process complete the proof of Theorems 1.5 and 1.6. For each $c \in \mathbb{R}$, introduce the events

$$\mathcal{G}_{t,c} := \{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j)}) - \Psi_{t,c}^{(j)}(Z_{t,c}^{(j,2)}) > d_t e_t\}\,,$$

$$\mathcal{H}_t := \{r_t f_t < |Z_t^{(j)}| < r_t g_t\} \quad \text{and} \quad \mathcal{I}_t := \{a_t(1 - f_t) < \Psi_t^{(j)}(Z_t^{(j)}) < a_t(1 + f_t)\}\,,$$

and the event

$$\mathcal{E}_{t,c} := \mathcal{G}_{t,0} \cap \mathcal{G}_{t,c} \cap \mathcal{H}_t \cap \mathcal{I}_t \cap \mathcal{S}_t(Z_t^{(j)})$$

which acts to collect the relevant information that we shall later need.

**Proposition 2.26.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\mathbf{P}(\mathcal{E}_{t,c}) \to 1\,.$$

*Proof.* This follows from Propositions 2.22 and 2.23 and Corollary 2.25, since $A_{r_t} \sim a_t$ and $d_{r_t} \sim d_t$. □

In the next few propositions, we prove that the sites $Z_{t,c}^{(j)}$ and $Z_t^{(j)}$ are both equal to the localisation site $Z_t$ with overwhelming probability.

**Proposition 2.27.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$Z_{t,c}^{(j)} = Z_t^{(j)}$$

*holds eventually.*

*Proof.* Assume that $Z_{t,c}^{(j)} \neq Z_t^{(j)}$ and recall that $1/\log\log t < e_t/g_t$ eventually by (2.1). On the event $\mathcal{E}_{t,c}$, the statements

$$\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_{t,c}^{(j)}) > d_t e_t \quad \text{and} \quad \Psi_{t,c}^{(j)}(Z_{t,c}^{(j)}) - \Psi_{t,c}^{(j)}(Z_t^{(j)}) > d_t e_t$$

and, eventually,

$$|\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_{t,c}^{(j)}(Z_t^{(j)})| = |c|\frac{|Z_t^{(j)}|}{t} < \gamma \frac{d_t g_t}{\log\log t} < d_t e_t$$

all hold, giving a contradiction. □

**Lemma 2.28.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\lambda^{(j)}(Z_t^{(j)}) \geq \lambda(Z_t^{(j)}) \quad \textit{and} \quad \lambda^{(j)}(Z_t) \geq \lambda(Z_t)$$

*and*

$$\lambda^{(j)}(Z_t^{(j)}) - \lambda(Z_t^{(j)}) < d_t e_t$$

*all hold eventually.*

*Proof.* The first two statements follow from the domain monotonicity of the principal eigenvalue in Lemma 2.2. For the third statement, remark that the event $\mathcal{E}_{t,c}$ implies that $Z_t^{(j)} \in \Pi^{(L_{t,\varepsilon})}$, that $\xi(y) < L_t$ for all $y \in B(Z_t^{(j)}, \rho)$, and that $\xi(y) < g_t$ for all $y$ such that $j \geq |y - Z_t^{(j)}| > \rho$. Hence, by considering the path expansion in Proposition 2.21, we have that for some $c_1, c_2 > 0$,

$$\lambda^{(j)}(Z_t^{(j)}) - \lambda(Z_t^{(j)}) < \frac{c_1}{(L_{t,\epsilon} - L_t)^{2\rho}} \left( \frac{1}{L_{t,\varepsilon} - g_t} - \frac{1}{L_{t,\varepsilon}} \right) < \frac{c_2 g_t}{a_t^{2\rho+2}} < d_t e_t$$

eventually, with the last inequality holding since we defined $\rho$ precisely to be the smallest integer such that $-2\rho - 2 < 1 - \gamma$ holds □

*Remark.* Note that the above proof lies at the heart of why we need to define the localisation site $Z_t$ by reference to the principal eigenvalue $\lambda(z)$ of the operator $\Delta + \xi \mathbb{1}_{B(z,\rho)}$ restricted to $B(z, j)$, rather than simply using the principal eigenvalue $\lambda^{(\rho)}(z)$ of the operator $\Delta + \xi$ restricted to $B(z, \rho)$. In particular, if $\rho > 0$ (i.e. if $\gamma \geq 3$), then by considering the path expansion in Proposition 2.21 we can guarantee only that

$$\lambda^{(j)}(Z_t^{(j)}) - \lambda^{(\rho)}(Z_t^{(j)}) < \frac{c_1}{(L_{t,\epsilon} - L_t)^{2\rho+1}} = O\left(a_t^{-2\rho-1}\right),$$

and since the inequality $-2\rho - 1 < 1 - \gamma$ does not hold in general, this is insufficient to conclude the necessary bound in the general case $\gamma \geq 3$.

**Corollary 2.29** (Equivalence of $Z_t^{(j)}$ and $Z_t$)**.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$Z_t^{(j)} = Z_t$$

*eventually.*

*Proof.* Assume that $Z_t^{(j)} \neq Z_t$. On the event $\mathcal{E}_{t,c}$, Lemma 2.28 implies that

$$\begin{aligned} &\left(\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t(Z_t^{(j)})\right) - \left(\Psi_t^{(j)}(Z_t) - \Psi_t(Z_t)\right) \\ &\quad = \left(\lambda^{(j)}(Z_t^{(j)}) - \lambda(Z_t^{(j)})\right) - \left(\lambda^{(j)}(Z_t) - \lambda(Z_t)\right) < d_t e_t \end{aligned}$$

holds eventually. On the other hand, on the event $\mathcal{E}_{t,c}$, and by the definition of $Z_t$ and $Z_t^{(j)}$ as the argmax of $\Psi_t$ and $\Psi_t^{(j)}$ respectively,

$$\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t) > d_t e_t \quad \text{and} \quad \Psi_t(Z_t) - \Psi_t(Z_t^{(j)}) > 0$$

also hold, giving a contradiction. □

### 2.3.4 Completion of the proof of Theorems 1.5 and 1.6

Fix an arbitrary $c \in \mathbb{R}$. We prove Theorems 1.5 and 1.6 on the event $\mathcal{E}_{t,c}$, since by Proposition 2.26 this event holds with overwhelming probability eventually.

The distance scale of the localisation site (part (a) of Theorem 1.5) follows directly from the scaling limit in Corollary 2.25. The local profile of the potential field (part (b) of Theorem 1.5) is implied by the definition of the event $\mathcal{E}_{t,c}$, combining with Proposition 2.27 and Corollary 2.29. Finally, the ageing of the localisation site $Z_t$ (Theorem 1.6) is a simple consequence of the point process convergence in Proposition 2.24, and is proved in an identical manner to the corresponding result in [64].

## 2.4 Negligible paths: Upper bounds, lower bounds and cluster expansions

In this section we show that the contribution to the total mass $U(t)$ from the components $U^2(t)$, $U^3(t)$, $U^4(t)$ and $U^5(t)$ are all negligible. We proceed in two parts: first we prove a lower bound on the total mass $u^1(t, z)$ for each $z \in B_t$, which implies a lower bound on $U(t)$, and then we bound from above the contribution to the total mass from each $U^i(t)$. Throughout this section, let $\varepsilon$ be such that $0 < \varepsilon < \theta$.

We begin by proving a general result on eigenfunction decay around sites of high potential, which will be used in both the lower and upper bound. For each $z \in \Pi^{(L_{t,\varepsilon})}$, let $\varphi_1$ denote the principal eigenfunction of the operator $\mathcal{H}^{(j)}(z)$.

**Proposition 2.30.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly, as $t \to \infty$, almost surely*

$$\sum_{y \in B(z,j) \setminus \{z\}} \varphi_1(y) \to 0\,.$$

*Proof.* By Proposition 2.6, we have the path expansion

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \sum_{k \geq 1} \sum_{\substack{p \in \Gamma_k(y,z) \\ p_i \neq z,\, 0 \leq i < k \\ \{p\} \subseteq B(z,j)}} \prod_{0 \leq i < k} (2d)^{-1} \frac{1}{\lambda^{(j)}(z) - \eta(p_i)}\,, \qquad y \in B(z,j) \setminus \{z\}\,.$$

Since, by Lemmas 2.19 and 2.3, for each $y \in B(z,j) \setminus \{z\}$, almost surely

$$\lambda^{(j)}(z) - \eta(y_i) > L_{t,\varepsilon} - L_t\,,$$

the result follows. □

**Corollary 2.31** (Bound on total mass of the solution)**.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly and any $c > 1$, as $t \to \infty$, almost surely*

$$\mathbb{E}_z\left[e^{\int_0^t \xi(X_s)\,ds} \tau_{B(z,j)^c} > t\right] < c\, e^{t\lambda^{(j)}(z)}$$

*eventually.*

*Proof.* This follows by combining Propositions 2.13 and 2.30. □

### 2.4.1 Lower bounds on the solution

Recall that, by the discussion in Section 2.1.2, the contribution to the solution from paths can be approximated by considering both the benefit of being near a site of high potential and the probabilistic penalty from diffusing to that site. To formalise a lower bound for $u^1(t,z)$ we need a bound on both of these terms.

We begin by bounding from below the benefit to the solution from paths that start and end at a the site $Z_t$. Recall the definition of the path set $E_t^1$ and the solution component $u^1(t,z)$, and define by analogy the path set

$$E_{t,y}^1 := \{p \in \Gamma(y) \,:\, \{p\} \subseteq B_t, z^{(p)} = Z_t\}$$

and the solution component

$$u_y^1(t,z) := \sum_{p \in E_{t,z}^1} u^p(t,z)\,.$$

**Lemma 2.32.** *Abbreviate $z = Z_t$ for simplicity. On the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\log u_z^1(t,z) > t\lambda^{(j)}(z) + o(1)$$

*eventually almost surely.*

*Proof.* Recall the Feynman-Kac formula for $u^1_z(t,z)$ (see, e.g., Proposition 2.9), and note that since $B(z,j) \subseteq B_t$ the expectation is larger than the corresponding expectation taken only over paths that do not leave $B(z,j)$. Using Corollary 2.12, we then have that

$$u^1_z(t,z) \geq e^{\lambda^{(j)}(z)t} \varphi_1^2(z)\,,$$

where $\varphi_1$ denotes the principal eigenfunction of the operator $\mathcal{H}^{(j)}(z)$. Since the domain $B(z,j)$ is finite, the fact that the eigenfunction $\varphi_1$ is localised at $z$ (by Proposition 2.30) ensures that the square eigenfunction $\varphi_1^2$ is also localised at $z$, and the result follows. □

We are now ready to prove the lower bound on the solution; our proof follows closely the analogous arguments in [52, 64].

**Proposition 2.33.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\log u^1(t, Z_t) > t\lambda^{(j)}(Z_t) - \frac{|Z_t|}{\gamma} \log\log t + O(td_t b_t)$$

*almost surely.*

*Proof.* In the following proof set $z = Z_t$ and fix $r \in (0,1)$. By the Feynman-Kac formula (1.3), the non-negativity of $\xi$ and using the Markov property, we have, for each $y \in B_t$,

$$u^1(t,z) \geq u^1_z((1-r)t, z)\, \mathbb{P}(X_{rt} = z, \tau_{B_t^c} > rt)\,, \tag{2.19}$$

where $\mathbb{P}$ denotes the law of the simple continuous-time random walk $X = (X_s)_{s\geq 0}$ initialised at the origin. We bound the probability by using Poisson process jump rates and the fact that $X$ is symmetric. Applying Stirling's formula, we obtain

$$\log \mathbb{P}(X_{r_1 t} = z, \tau_{B_t^c} > rt) = -rt - |z| \log\left(\frac{2d\,|z|}{ert}\right) + O(\log t)\,. \tag{2.20}$$

Next note that on the event $\mathcal{E}_{t,c}$ we have that $Z_t \in \Pi^{(L_{t,\varepsilon})}$. Hence we can combine equations (2.19)–(2.20) and Lemma 2.32 to bound $\log u^1(t,y)$ below by

$$(1-r)t\lambda^{(j)}(z) - rt - |z| \log\left(\frac{2d\,|z|}{ert}\right) + O(\log t)\,.$$

We are now free to maximise the bound over $r$. Setting

$$r := \frac{|z|}{t\lambda^{(j)}(z)}\,,$$

it is clear that on event $\mathcal{E}_{t,c}$ we have $r \in (0,1)$. With this value of $r$ we obtain the bound

$$\log u^1(t,z) > t\lambda^{(j)}(z) - |z| \log \lambda^{(j)}(z) + O(|z|) + O(\log t)\,.$$

On event $\mathcal{E}_{t,c}$ we have that $\Psi_t^{(j)}(z) < a_t(1+f_t)$. Since also $|z| < r_t g_t$ on event $\mathcal{E}_{t,c}$ we find that

$$\lambda_t^{(j)}(z) < a_t(1 + f_t) + d_t g_t$$

and

$$\begin{aligned}\log u^1(t, z) &> t\lambda^{(j)}(z) - \frac{|z|}{\gamma} \log\log t + O\left(r_t g_t\right) \\ &> t\lambda^{(j)}(z) - \frac{|z|}{\gamma} \log\log t + O\left(t d_t b_t\right)\end{aligned}$$

with the last inequality following from our choice of scaling functions in (2.1). □

**Corollary 2.34.** *For each $c \in \mathbb{R}$ on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\log U(t) > t\lambda^{(j)}(Z_t) - \frac{|Z_t|}{\gamma} \log\log t + O(t d_t b_t)$$

*almost surely.*

### 2.4.2 Contribution from each $U^i(t)$ is negligible

In this section we prove that the contribution to $U(t)$ from the each of the components $U^i(t)$, for $i = 2, 3, 4, 5$, is negligible. The most difficult step is bounding the contribution from the components $U^2(t)$ and $U^3(t)$; these paths are permitted to visit sites of high potential that are not $Z_t$. Away from sites of high potential, there is a probabilistic penalty associated with each step of the path; this is easy to bound. However, close to these sites, the maximum contribution from the path may come from a complicated sequence of return cycles to the site, which are somewhat tricky to deal with.

This motivates the idea of grouping paths into equivalence classes depending only on their trajectory away from sites of high potential. So for each $t$, define a partition of paths into equivalence classes as follows. Suppose $p, \bar{p} \in \Gamma$ are two finite paths in $\mathbb{Z}^d$. Define inductively, $r^0 = 0$, and

$$s^\ell := \min\{i \geq r^{\ell-1} : p_i \in \Pi^{(L_t)}\} \quad \text{and} \quad r^\ell := \min\{i > s^\ell : p_i \in \partial B(p_{s^\ell}, j)\}$$

for each $\ell \in \mathbb{N}$, setting each to be $\infty$ if no such minimum $i$ exists, and define similarly $(\bar{s}^\ell, \bar{r}^\ell)_{\ell \geq 1}$ for path $\bar{p}$. Then we say that $p$ and $\bar{p}$ are in the same equivalence class if and only if, for all $\ell \geq 0$,

$$s^{\ell+1} - r^\ell = \bar{s}^{\ell+1} - \bar{r}^\ell \quad \text{and} \quad p_{r^\ell + i} = \bar{p}_{\bar{r}^\ell + i}\,, \quad \text{for each } i \in \{0, 1, \ldots, s^{\ell+1} - r^\ell\}\,.$$

Note that although $s^\ell$ and $r^\ell$ depend on $t$ (through the set $\Pi^{(L_t)}$) we suppress this dependence for clarity. If $p$ and $\bar{p}$ are in the same equivalence class at time $t$ we write $p \sim \bar{p}$. Denote by $P(p) := \{\bar{p} \in \Gamma : p \sim \bar{p}\}$. Informally, the equivalence class $P(p)$ consists of paths that have identical trajectory except for when they are in balls of radius $j$ around sites $z \in \Pi^{(L_t)}$ (or, more accurately, when they first hit a site $z \in \Pi^{(L_t)}$ until when they leave the ball $B(z, j)$).

It is natural to group these equivalence classes $P(p)$ according to (i) how many balls of radius $j$ around sites $z \in \Pi^{(L_t)}$ the path visits, and (ii) the total length of the path outside such balls. So for $m, n \in \mathbb{N}$, let $\mathcal{P}_{n,m}$ be the set of equivalence classes $P(p)$ of paths $p$ that satisfy

$$\max\{\ell : r^\ell < \infty\} = m \quad \text{and} \quad \sum_{\ell=0}^{m} (s^{\ell+1} \wedge |p|) - r^\ell = n\,.$$

Note that if a path $p$ satisfies these two properties for some $m$ and $n$ then any other path $\bar{p} \in P_p$ will also satisfy these properties for the same $m$ and $n$ and hence $\mathcal{P}_{n,m}$ is well-defined. The quantity $m$ counts the number of balls of radius $j$ around $z \in \Pi^{(L_t)}$ that the path *exits* (which is easier to work with than the number of balls the path enters); the quantity $n$ counts the total length of the path between leaving each of these balls and hitting the next site $z \in \Pi^{(L_t)}$.

Recalling the definitions of $p(X_t)$, define the event

$$\{p(X) \in P(p)\} := \bigcup_{s \geq 0} \{p(X_s) \in P(p)\}\,,$$

and remark that we have the relationship

$$\{p(X_t) \in P(p)\} \subseteq \{p(X) \in P(p)\}\,. \tag{2.21}$$

Denote by

$$U^{P(p)}(t) = \mathbb{E}_0 \left[ \exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}_{\{p(X_t) \in P(p)\}} \right]$$

the contribution to the total solution $U(t)$ from the path equivalence class $P(p)$.

The following lemma bounds the contribution of each $P(p) \in \mathcal{P}_{n,m}$ in terms of $m$ and $n$. The key fact motivating our set-up is that the contribution is decreasing in $n$.

**Lemma 2.35** (Bound on the contribution from each equivalence class)**.** *Let $m, n \in \mathbb{N}$ and $p \in \Gamma(0)$ such that $\{p\} \subseteq V_t$ and $P(p) \in \mathcal{P}_{n,m}$. Define $z^{(p)} := \operatorname{argmax}_{z \in \{p\}} \lambda^{(j)}(z)$ and let $\zeta > \max\{\lambda^{(j)}(z^{(p)}), L_{t,\varepsilon}\}$. Then there exist constants $c_1, c_2 > 0$ such that, for each $m, n, p$ and $\zeta$ uniformly, as $t \to \infty$,*

$$U^{P(p)}(t) < e^{\zeta t} \big(c_1(\zeta - L_t)\big)^{-n} \left(1 + c_2 \left(\zeta - \lambda^{(j)}(z^{(p)})\right)^{-1}\right)^m$$

*eventually almost surely.*

*Proof.* The strategy of the proof is to split $U^{P(p)}(t)$ into three components, corresponding to the contribution: (i) from when $X_s$ is outside $B(\Pi^{(L_t)}, j)$ until $X_s$ hits a site $z \in \Pi^{(L_t)}$; (ii) from when $X_s$ hits $z \in B(\Pi^{(L_t)}, j)$ until when $X_s$ leaves the ball $B(z, j)$; and (iii) if $X_s$ hits $z \in \Pi^{(L_t)}$ and does not subsequently leave $B(z, j)$, from this component separately. To bound the contribution from these components, we make use of Corollary 2.31, Lemma 2.15 and Lemma 2.5 respectively.

There are two cases to consider, depending on whether the event described in (iii) occurs, that is, if $s^{m+1} < \infty$. We begin with this case. To simplify notation in the following we abbreviate

$$I_a^b := \exp\left\{ \int_a^b (\xi(X_s) - \zeta)\, ds \right\}.$$

Recall the definition of $(s^\ell, r^\ell)_{\ell\in\mathbb{N}}$ and define the stopping times

$$R^0 := 0\ , \quad S^\ell := \inf\{s \ge R^{\ell-1} : X_s = p_{s^\ell}\} \quad \text{and} \quad R^\ell := \inf\{s \ge S^\ell : X_s = p_{r^\ell_t}\}$$

for each $\ell \in \{1,\dots,m\}$, and similarly define $S^{m+1}$ since $s^{m+1} < \infty$. We can then write

$$\begin{aligned} U^{P(p)}(t) &= \mathbb{E}_0\left[e^{\int_0^t \xi(X_s)\,ds}\mathbb{1}_{\{p(X_t)\in P(p)\}}\right] = e^{\zeta t}\,\mathbb{E}_0\left[I_0^t\,\mathbb{1}_{\{p(X_t)\in P(p)\}}\right] \\ &= e^{\zeta t}\,\mathbb{E}_0\left[\left(\prod_{\ell=0}^m I_{R^\ell}^{S^{\ell+1}}\right)\left(\prod_{\ell=1}^m I_{S^\ell}^{R^\ell}\right) I^t_{S^{m+1}}\mathbb{1}_{\{p(X_t)\in P(p)\}}\right]. \end{aligned}$$

Note that, conditionally on $S^{m+1}$, the quantity $I^t_{S^{m+1}}$ is independent of all other $I_a^b$ in this expectation. Thus we have

$$\begin{aligned} U^{P(p)}(t) = e^{\zeta t}\,\mathbb{E}\Bigg\{ &\mathbb{E}_0\left[\left(\prod_{\ell=0}^m I_{R^\ell}^{S^{\ell+1}}\right)\left(\prod_{\ell=1}^m I_{S^\ell}^{R^\ell}\right)\mathbb{1}_{\{p(X_t)\in P(p)\}}\,\Big|\, S^{m+1}\right] \\ &\times \mathbb{E}_0\left[I^t_{S^{m+1}}\mathbb{1}_{\{p(X_t)\in P(p)\}}\,\Big|\, S^{m+1}\right]\Bigg\} \end{aligned} \tag{2.22}$$

where the outside expectation is over the hitting time $S^{m+1}$. We use Corollary 2.31 to bound the expectation on the second line of (2.22); in the calculation that follows, abbreviate $s := s^{m+1}$ and $S := S^{m+1}$. We obtain, for some $C > 1$,

$$\mathbb{E}_0\left[I_S^t\,\mathbb{1}_{\{p(X_t)\in P_t(p)\}}\,\big|\, S\right] \le \mathbb{1}_{\{S\le t\}}\,\mathbb{E}_{p_s}\left[I_0^{t-S}\mathbb{1}_{\{\tau_{B(p_s,j)}>t-S\}}\Big| S\right] < Ce^{(t-S)(\lambda^{(j)}(p_s)-\zeta)} \le C$$

almost surely, since $\zeta > \lambda^{(j)}(p_s)$. Combining with (2.22) and using equation (2.21) we obtain

$$\begin{aligned} \mathbb{E}_0\Big[e^{\int_0^t \xi(X_s)\,ds}\mathbb{1}_{\{p(X_t)\in P(p)\}}\Big] &< C\,e^{\zeta t}\,\mathbb{E}_0\left[\left(\prod_{\ell=0}^m I_{R^\ell}^{S^{\ell+1}}\right)\left(\prod_{\ell=1}^m I_{S^\ell}^{R^\ell}\right)\mathbb{1}_{\{p(X)\in P(p)\}}\right] \\ &= Ce^{\zeta t}\,\mathbb{E}_0\left[\left(\prod_{\ell=0}^m I_{R^\ell}^{S^{\ell+1}}\right)\mathbb{1}_{\{p(X)\in P(p)\}}\right]\mathbb{E}_0\left[\left(\prod_{\ell=1}^m I_{S^\ell}^{R^\ell}\right)\mathbb{1}_{\{p(X)\in P(p)\}}\right]. \end{aligned} \tag{2.23}$$

Let $\xi^{(\ell)}_{\max} = \max_{r^\ell\le k<s^{\ell+1}}\xi(p_k)$, for $\ell = \{0,1,\dots,m\}$. By Lemma 2.5, which we can apply here since $\zeta > L_{t,\varepsilon} > L_t \ge \max_{0\le\ell\le m}\xi^{(\ell)}_{\max}$,

$$\begin{aligned} \mathbb{E}_0\left[\left(\prod_{\ell=0}^m I_{R^\ell}^{S^{\ell+1}}\right)\mathbb{1}_{\{p(X)\in P(p)\}}\right] &\le (2d)^{-n}\prod_{\ell=0}^m\prod_{k=r^\ell}^{s^{\ell+1}-1}\left(\zeta-\xi^{(\ell)}_{\max}+2d\right)^{-1} \\ &\le (2d)^{-n}\left(\zeta-L_t\right)^{-n}, \end{aligned} \tag{2.24}$$

almost surely, using the definition of $n$. Making the new abbreviation $s := s^\ell$, we have

$$\mathbb{E}_0\left[\left(\prod_{\ell=1}^m I_{S^\ell}^{R^\ell}\right)p(X)\in P(p)\right] = \prod_{\ell=1}^m \mathbb{E}_{p_s}\left[I_0^{\tau_{B(p_s,j)}}p(X)\in P(p)\right] \le \prod_{\ell=1}^m \mathbb{E}_{p_s}\left[I_0^{\tau_{B(p_s,j)}}\right].$$

Since $\zeta > \lambda^{(j)}(z^{(p)})$, we can apply the cluster expansion in Lemma 2.15 to deduce that

$$\prod_{\ell=1}^{m} \mathbb{E}_{p_s}\left[I_0^{\tau_{B(p_s,j)}}\right] < \left(1+\frac{2d\,|B(0,j)|}{\zeta-\lambda^{(j)}(z^{(p)})}\right)^m . \tag{2.25}$$

Using these two estimates, we obtain from equation (2.23) the desired bound.

We now deal with the case that $s^{m+1}=\infty$. Similarly to the above, we condition on $R^m$ to write $U^{P(p)}(t)$ as

$$e^{\zeta t}\,\mathbb{E}\left\{\mathbb{E}_0\left[\left(\prod_{\ell=0}^{m} I_{R^\ell}^{S^{\ell+1}}\right)\left(\prod_{\ell=1}^{m} I_{S^\ell}^{R^\ell}\right)\mathbb{1}_{\{p(X)\in P(p)\}}\Big|\,R^m\right]\mathbb{E}_0\left[I_{R^m}^{t}\,\mathbb{1}_{\{R^m\le t\}}\big|\,R^m\right]\right\}.$$

Set $l:=|p|-r^m>0$ and $\tau_{\text{end}}:=\inf\{s>0: X_s=X_t\}$. Observe that, since $\zeta>L_{t,\varepsilon}>L_t\ge \xi(X_t)$,

$$\mathbb{E}_0\left[I_{R^m}^{t}\,\mathbb{1}_{\{p(X_t)\in P(p)\}}\big|\,R^m\right]\le \mathbb{E}_0\left[I_{R^m}^{\tau_{\text{end}}}\,\mathbb{1}_{\{p(X_t)\in P(p)\}}\big|\,R^m\right]$$

and applying Lemma 2.5 (valid by Lemma 2.3) we get that

$$\mathbb{E}_0\left[I_{R^m}^{t}\,\mathbb{1}_{\{p(X_t)\in P(p)\}}\big|\,R^m\right]\le (2d)^{-l}(\zeta-L_t)^{-l}$$

almost surely. The rest of the proof proceeds similarly to the previous case. □

We can use Lemma 2.35 to bound the contribution from $U^2(t)$ and $U^3(t)$.

**Proposition 2.36** (Upper bound on $U^2(t)$)**.** *There exists a constant $c\in\mathbb{R}$ such that, as $t\to\infty$,*

$$\log U^2(t) < t \max_{z\in\Pi^{(L_t)}\setminus\{Z_t\}} \Psi_{t,c}^{(j)}(z) + O(td_tb_t)$$

*almost surely.*

*Proof.* Recall the path set $E_t^2$, and for each $m,n\in\mathbb{N}$ define

$$\mathcal{P}_{n,m}^2 := \bigcup_{p\in E_t^2} P_t(p)\cap \mathcal{P}_{n,m}\,.$$

Note that $|\mathcal{P}_{n,m}^2|\le \kappa^{n+m}$, with $\kappa=\max\{2d,|\partial B(0,j)|\}$. We observe that

$$\begin{aligned} U^2(t) &= \sum_{n,m} U^{\mathcal{P}_{n,m}^2}(t) \le \sum_{n,m}\kappa^{n+m}\max_{P\in\mathcal{P}_{n,m}^2}\left\{U^P(t)\right\} = \sum_{n,m}\kappa^{-n-m}\max_{P\in\mathcal{P}_{n,m}^2}\left\{\kappa^{2(n+m)}U^P(t)\right\} \\ &\le \max_{n,m}\max_{P\in\mathcal{P}_{n,m}^2}\left\{\kappa^{2(n+m)}U^P(t)\right\}\sum_{n,m}\kappa^{-n-m}\,. \end{aligned} \tag{2.26}$$

For each $P\in\mathcal{P}_{n,m}^2$, denote by $z^{(P)}$ the site $y\in\Pi^{(L_t)}$ on a given path $p\in P$ which maximises $\lambda^{(j)}(y)$, remarking that this a class property of $P$ eventually almost surely by Lemma 2.19. Using Lemma 2.35, for each $P\in\mathcal{P}_{n,m}^2$ and for any $\zeta>\max\{\lambda^{(j)}(z^{(P)}),L_{t,\varepsilon}\}$, we have that there exist constants $c_1, c_2, c_3 > 0$ such that, eventually almost surely,

$$\kappa^{2(n+m)}\, U^P(t) < e^{\zeta t}\,(c_1(\zeta - L_t))^{-n}\left(c_2 + c_3(\zeta - \lambda^{(j)}(z^{(P)}))^{-1}\right)^m .$$

Set $\zeta = \max\{\lambda^{(j)}(z^{(P)}), L_{t,\varepsilon}\} + d_t b_t$. To lower bound $n$, observe that the number of steps between exiting a $j$-ball and hitting another site in $\Pi^{(L_t)}$ is at least $j+1$. We apply Corollary 2.20 to the balls $B(\Pi^{(L_t)}, j+1)$ to deduce that, eventually almost surely

$$n > m(j+1) + |z^{(P)}| - |z^{(P)}|^{c_4}\,, \tag{2.27}$$

for some $c_4 < 1$. Then, by monotonicity in $n$,

$$\begin{aligned}\kappa^{2(n+m)}\, U^P(t) < e^{t(\lambda^{(j)}(z^{(P)})+d_t b_t)}&(c_1(L_{t,\varepsilon} - L_t))^{-|z^{(P)}|+|z^{(P)}|^{c_4}} \\ &\times \left((c_1(L_{t,\varepsilon} - L_t))^{-j-1}(c_2 + c_3 d_t b_t)^{-1}\right)^m\end{aligned}$$

eventually almost surely. Note that $j$ was chosen precisely to be the smallest integer such that

$$(j+1)\log a_t + \log(d_t) \to \infty \tag{2.28}$$

which implies, since $b_t \gg 1/\log\log t$ by (2.1), that

$$(j+1)\log a_t + \log(c_2 + c_3 d_t b_t) \to \infty\,.$$

By Lemma 2.19, for $z \in \Pi^{(L_t)}$, as $t \to \infty$,

$$|z|^{c_4}\log\left(c_1(L_{t,\varepsilon} - L_t)\right) < \frac{t d_t b_t}{\log\log t}$$

eventually almost surely. Moreover,

$$\log\left(L_{t,\varepsilon} - L_t\right) > \log a_t + c_5$$

eventually for some $c_5 > 0$. So there exists a constant $c \in \mathbb{R}$ such that

$$2(n+m)\log\kappa + \log U^P(t) < c|z^{(P)}| + \lambda^{(j)}(z^{(P)})t - \frac{1}{\gamma}|z^{(P)}|\log\log t + t d_t b_t$$

eventually almost surely, which yields the result. □

**Proposition 2.37** (Upper bound on $U^3(t)$). *There exists a constant $c \in \mathbb{R}$ such that, as $t \to \infty$,*

$$\log U^3(t) < t\Psi^{(j)}_{t,c}(Z_t) - h_t\frac{1}{\gamma}|Z_t|\log\log t + O\left(t d_t b_t\right)$$

*almost surely.*

*Proof.* Recall the set of paths $E^3_t$ and define $\mathcal{P}^3_{n,m}$ by analogy with $\mathcal{P}^2_{n,m}$. The proof then follows as for Proposition 2.36 after strengthening the bound in (2.27) to give that for each

$p \in E_t^3$ and for some $c_1 < 1$, eventually almost surely

$$n > m(j+1) + (1+h_t)\frac{1}{\gamma}|Z_t| \log\log t - |Z_t|^{c_1} . \qquad \square$$

**Proposition 2.38** (Upper bound on $U^4(t)$)**.** *For all $t \geq 0$,*

$$U^4(t) \leq e^{tL_t} .$$

*Proof.* This follows trivially from the definition of $U^4(t)$. $\square$

**Proposition 2.39** (Negligibility of $U^5(t)$)**.** *As $t \to \infty$, almost surely,*

$$U^5(t) \to 0 .$$

*Proof.* This result was proved in [40, Section 2.5]; for completeness we repeat the proof here. For each $n \in \mathbb{N}$, let $e_n(X)$ denote the event that $\max_{s<t} |X_s|_{\ell^\infty} = n$. Let $U_n^5(t)$ denote the expectations in the definition of $U^5(t)$ restricted to the event $e_n(X)$. Then, if $\xi_1^{(n)}$ is the largest value of $\xi$ in the box $\{z \in \mathbb{Z}^d : |z|_{\ell^\infty} \leq n\}$, we have

$$U_n^5(t) \leq e^{t\xi_1^{(n)}} \mathbb{P}(e_n(X)) .$$

As $n \to \infty$, we can bound $\xi_1^{(n)}$ almost surely with Lemma 2.18:

$$\xi_1^{(n)} \sim (d \log n)^{\frac{1}{\gamma}} .$$

For $n \geq R_t$ and by Stirling's approximation, we can also bound the probability $\mathbb{P}(e_n(X))$ by

$$\log \mathbb{P}(e_n(X)) \leq \log \mathrm{Pn}_{2dt}(n) < -n \log n + n \log t + O(n)$$

where $\mathrm{Pn}_a(n)$ denotes the probability mass function for the Poisson distribution with mean $a$, evaluated at $n$. Combining these bounds, for $n \geq R_t$ and as $t \to \infty$ eventually

$$U_n^5(t) < \exp\{t(d \log n)^{\frac{1}{\gamma}}(1+\varepsilon) - n \log n + n \log t + Cn)\}$$

almost surely, for any $\varepsilon > 0$ and for some $C > 0$. Since $n \geq R_t = t(\log t)^{\frac{1}{\gamma}}$, for $t$ large enough this can be further bounded as

$$U_n^5(t) < \exp\{-(1-\varepsilon) n \log n\} .$$

This implies that, eventually

$$\sum_{n \geq R_t} U_n^5(t) < e^{-(1-\varepsilon) R_t \log R_t} \sum_{n \geq 0} e^{-(1-\varepsilon) n \log R_t} = o\left(e^{-R_t}\right)$$

holds almost surely, which implies the result. $\square$

**Corollary 2.40.** *There exists a constant $c \in \mathbb{R}$ such that, as $t \to \infty$,*

$$e^{td_t b_t} \frac{U^2(t) + U^3(t) + U^4(t) + U^5(t)}{U(t)} \mathbb{1}_{\mathcal{E}_{t,c}} \to 0$$

*almost surely.*

*Proof.* Let $c$ be the maximum of the constants appearing in Propositions 2.36 and 2.37. Combining Corollary 2.34 and Proposition 2.36, and recalling that $Z_{t,c}^{(j)} = Z_t$ eventually by Proposition 2.27 and Corollary 2.29, we have that, on the event $\mathcal{E}_{t,c}$, eventually almost surely

$$\log U^2(t) - \log U(t) < t\left(\Psi_{t,c}^{(j)}(Z_{t,c}^{(j,2)}) - \Psi_{t,c}^{(j)}(Z_{t,c}^{(j)})\right) + c|Z_t| + O(td_tb_t)\,.$$

Using the gap in the maximisers of $\Psi_{t,c}^{(j)}$ and since $|Z_t| < r_tg_t$, we have that, as $t \to \infty$,

$$\log U^2(t) - \log U(t) < -td_te_t + O(r_tg_t) + O(td_tb_t) \to -\infty$$

by the properties of the scaling functions in (2.1). Similarly, combining Corollary 2.34 and Propositions 2.33 and 2.37, we have that, on the events $\mathcal{E}_{t,c}$, eventually almost surely

$$\log U^3(t) - \log U(t) < -h_t\frac{1}{\gamma}|Z_t|\log\log t + c|Z_t| + O(td_tb_t)$$

and so, using that $|Z_t| > r_tf_t$ on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,

$$\log U^3(t) - \log U(t) < -r_tf_th_t\frac{1}{\gamma}\log\log t + O(td_tb_t) \to -\infty$$

by the properties in (2.1). Finally, combining Corollary 2.34 and Propositions 2.38 and 2.39, we get the result. □

## 2.5 Localisation, exponential decay and ageing

In this section we complete the proof of Theorems 1.3 and 1.7. We first obtain an upper bound for the decay of the solution $u(t,z)$ away from the localisation site $Z_t$, which implies the complete localisation of the solution. Next we establish the ageing of the solution in Theorem 1.7. Finally, we use the ageing of the solution to obtain a lower bound on the exponential decay, which completes the proof of Theorem 1.3. Throughout this section, fix the constant $c > 0$ from Corollary 2.40, and abbreviate $\mathcal{E}_t := \mathcal{E}_{t,c}$.

### 2.5.1 Upper bound for exponential decay and complete localisation

Our upper bound for the exponential decay of the solution relies on a comparison between $u^1(t,z)$ and the principal eigenfunction of the operator $\mathcal{H} := \Delta + \xi$ restricted to $B_t$. So let $\lambda_t$ and $v_t$ denote, respectively, the principal eigenvalue and eigenfunction of the operator $\mathcal{H}$, renormalising $v_t$ so that $v_t(Z_t) = 1$. Remark that, on the event $\mathcal{E}_t$, we have that $B_t \in V_t$. Hence we can apply Proposition 2.16 which implies that, for any $y \in B_t \setminus \{Z_t\}$,

$$\frac{u^1(t,y)}{u^1(t,Z_t)} \le \|v_t\|_{\ell_2}^2\, v_t(y)\,. \tag{2.29}$$

Hence, to establish the exponential decay of the solution it will be sufficient to prove the exponential decay of principal eigenfunction $v_t(z)$; we shall achieve this via the Feynman-Kac representation and a similar 'cluster expansion' analysis to in the upper bound.

**Lemma 2.41** (Gap in $j$-local principal eigenvalues in $B_t$)**.** *On the event $\mathcal{E}_t$, each $z \in B_t \setminus \{Z_t\}$ satisfies*

$$\lambda^{(j)}(Z_t) - \lambda^{(j)}(z) > d_t e_t + o(d_t e_t) .$$

*Proof.* On the event $\mathcal{E}_t$, we have that $\lambda^{(j)}(Z_t) > a_t(1 - f_t)$ and so the claim is true for $z \notin \Pi^{(L_t)}$ by Lemma 2.3. On the other hand, if $z \in \Pi^{(L_t)}$ then

$$d_t e_t < \Psi_t^{(j)}(Z_t) - \Psi_t^{(j)}(z) = \lambda^{(j)}(Z_t) - \lambda^{(j)}(z) + \frac{|z| - |Z_t|}{\gamma t} \log \log t .$$

To complete the proof, notice that, for each $z \in B_t$,

$$\frac{|z| - |Z_t|}{\gamma t} \log \log t < \frac{r_t g_t h_t}{\gamma t} \log \log t = d_t g_t h_t \ll d_t e_t$$

since $g_t h_t \ll e_t$ by (2.1). □

**Corollary 2.42.** *Eventually on the event $\mathcal{E}_t$, each $z \in B_t \setminus \{Z_t\}$ satisfies*

$$\lambda_t > \lambda^{(j)}(z) + d_t e_t + o(d_t e_t) .$$

*Proof.* First note that, on the event $\mathcal{E}_t$, the ball $B(Z_t, j) \subseteq B_t$. Hence, by the domain monotonicity in Lemma 2.2, we have $\lambda_t \geq \lambda^{(j)}(Z_t)$, and so the result follows from Lemma 2.41. □

**Proposition 2.43** (Feynman-Kac representation for the principal eigenfunction)**.** *Eventually on the event $\mathcal{E}_t$,*

$$v_t(z) = \mathbb{E}_z \left[ \exp \left\{ \int_0^{\tau_{Z_t}} (\xi(X_s) - \lambda_t) \, ds \right\} \mathbb{1}_{\{\tau_{B_t^c} > \tau_{Z_t}\}} \right] ,$$

*where*

$$\tau_{Z_t} := \inf\{t \geq 0 : X_t = Z_t\} \quad \textit{and} \quad \tau_{B_t^c} := \inf\{t \geq 0 : X_t \notin B_t\} .$$

*Proof.* This is an application of Proposition 2.4, valid precisely because of Corollary 2.42. □

Recall the partition of paths into equivalence classes in Section 2.4, the quantities $r^\ell$ and $s^\ell$ associated to each equivalence class, and, for $m, n \in \mathbb{N}$, the set of equivalence classes $\mathcal{P}_{n,m}$. Recall also the event $\{p(X) \in P(p)\}$.

Define the path set

$$\bar{E}_t^1 := \left\{ p \in E_t^1 : |p| = \min \{i : p_i = Z_t\} \right\} ,$$

and for each $m, n \in \mathcal{N}$ define

$$\bar{\mathcal{P}}_{n,m}^1 := \bigcup_{p \in \bar{E}_t^1} P_t(p) \cap \mathcal{P}_{n,m} .$$

Further, for each $P \in \bar{\mathcal{P}}^1_{n,m}$ and $y \in B_t$ define

$$v_t^P(y) := \mathbb{E}_y \left[ \exp \left\{ \int_0^{\tau_{Z_t}} (\xi(X_s) - \lambda_t) \, ds \right\} \mathbb{1}_{p(X) \in P} \right] . \tag{2.30}$$

For each $P \in \bar{\mathcal{P}}^1_{n,m}$ denote by $z^{(P)}$ the site $y \in \Pi^{(L_t)}$ on a given path $p \in P$, excluding the site $Z_t$, which maximises $\lambda^{(j)}(y)$, setting $z^{(P)} = \varnothing$ (and $\lambda^{(j)}(\varnothing) = 0$) if no such $y$ exists. Remark that, whenever $z^{(P)}$ is defined, it is a class property of $P$ eventually almost surely, by Lemma 2.19.

**Lemma 2.44** (Bound on the contribution from each equivalence class)**.** *Let $m, n \in \mathbb{N}$ and $P \in \bar{\mathcal{P}}^1_{n,m}$. Then there exist constants $c_1, c_2 > 0$ such that, for each $m, n, P$ and $y \in B_t \backslash \Pi^{(L_t)}$ uniformly, as $t \to \infty$,*

$$v_t^P(y) < (c_1(\lambda_t - L_t))^{-n} \left(1 + c_2(\lambda_t - \lambda^{(j)}(z^{(P)}))^{-1}\right)^{m-1}$$

*eventually almost surely.*

*Proof.* Starting with the Feynman-Kac representation for $v_t^P(y)$ in equation (2.30), the proof follows similarly as in Lemma 2.35 for $\zeta = \lambda_t$, which is a valid setting for $\zeta$ because of Corollary 2.42. One modification is necessary to adapt the proof, namely that, for any $p \in P$, the final site $Z_t$ gives no contribution to the expectation, and hence we have $m - 1$ instead of the $m$ in Lemma 2.35. □

**Proposition 2.45** (Exponential decay of principal eigenfunction)**.** *On the event $\mathcal{E}_t$ there exists a $C > 0$ such that, for each $y \in B_t \setminus \{Z_t\}$ uniformly, as $t \to \infty$,*

$$\log v_t(y) < -\frac{|y - Z_t|}{\gamma} \log \log t + C|y - Z_t|$$

*eventually almost surely.*

*Proof.* Abbreviate $z = Z_t$. Similarly to in the proof of Proposition 2.36, decomposing the Feynman-Kac representation in Proposition 2.43 we observe that there exists $\kappa > 1$ such that

$$v_t(y) = \sum_{n,m} \sum_{P \in \bar{\mathcal{P}}^1_{n,m}} v_t^P(y) \leq \max_{n,m} \max_{P \in \bar{\mathcal{P}}^1_{n,m}} \left\{ \kappa^{2(n+m)} v_t^P(y) \right\} \sum_{n,m} \kappa^{-n-m} .$$

Suppose $y \in B_t \setminus \Pi^{(L_t)}$. Then for each $P \in \bar{\mathcal{P}}^1_{n,m}$, by Lemma 2.44 there exist $c_1, c_2, c_3 > 0$ such that

$$\kappa^{2(n+m)} v_t^P(y) < (c_1(\lambda_t - L_t))^{-n} (c_2 + c_3(\lambda_t - \lambda^{(j)}(z^{(P)}))^{-1})^{m-1}$$

eventually almost surely. Note also that by Corollary 2.20 (similarly to (2.27)), eventually almost surely

$$n > (m-1)(j+1) + |y - z| + |y - z| \, t^{-c_4}$$

for some $0 < c_4 < 1$. Then, on the event $\mathcal{E}_t$ and for any $0 < \varepsilon < \theta$,

$$\kappa^{2(n+m)} v_t^P(y) \leq (c_1(L_{t,\varepsilon} - L_t))^{-|y-z|(1+t^{-c_4})} \left((c_1(L_{t,\varepsilon} - L_t))^{-j-1}(c_2 + c_3(d_t e_t)^{-1})\right)^{m-1}$$

eventually almost surely by monotonicity in $n$ and Corollary 2.42, and so, applying equation (2.28), we have that

$$2(n+m)\log\kappa + \log v_t^P(y) < -|y-z|\log\log t + C|y-z|$$

eventually almost surely, for some $C > 0$. Suppose then that $y \in \Pi^{(L_t)}$. Here we proceed similarly, but we now need the stronger bound $n > m(j+1) + |y-z| + |y-z|t^{-c_4}$ for some $0 < c_4 < 1$, valid eventually almost surely for $y \in \Pi^{(L_t)}$ by Lemma 2.19. Then,

$$\begin{aligned}\kappa^{2(n+m)} v_t^P(y) < &\left((c_1(L_{t,\varepsilon} - L_t))^{-j-1}(d_t e_t)^{-1}\right)(c_1(L_{t,\varepsilon} - L_t))^{-|y-z|(1+t^{-c_4})} \\ &\times \left((c_1(L_{t,\varepsilon} - L_t))^{-j-1}(c_2 + c_3(d_t e_t)^{-1})\right)^{m-1},\end{aligned}$$

and the rest of the proof follows as before. □

**Corollary 2.46.** *On the event $\mathcal{E}_t$, as $t \to \infty$,*

$$\sum_{z \in B_t} \|v_t\|_{\ell_2} \to 1\,.$$

We are now in a position to prove our upper bound for the exponential decay of the solution (i.e. the upper bound in parts (a) and (b) of Theorem 1.3), and hence to establish complete localisation in Theorem 1.1. We work on the event $\mathcal{E}_t$, since this holds eventually with overwhelming probability by Proposition 2.26. Combining Proposition 2.45 and Corollary 2.46 with equation (2.29), for $y \in B_t \setminus \{Z_t\}$ we have the bound

$$\frac{u^1(t,y)}{u^1(t,Z_t)} \leq \left(\frac{c_1}{a_t}\right)^{|y-Z_t|},$$

eventually almost surely, for some $c_1 > 0$. To finish the proof, pick a $\kappa_t$ sufficiently small such that $\kappa_t \leq b_t$ eventually. Combining the above with the negligibility results already established in Corollary 2.40, each $y \in B(Z_t, r_t\kappa_t) \setminus \{Z_t\}$ satisfies

$$\frac{u(t,y)}{u(t,Z_t)} \leq \left(\frac{c_1}{a_t}\right)^{|y-Z_t|} + e^{-td_t b_t} < \left(\frac{c_2}{a_t}\right)^{|y-Z_t|},$$

for some $c_2 > 0$, since

$$a_t^{-|y-Z_t|} \leq a_t^{-r_t\kappa_t} = e^{-td_t\kappa_t + o(td_t\kappa_t)} < 2e^{-td_t\kappa_t}$$

eventually. On the other hand,

$$\sum_{y \notin B(Z_t, r_t\kappa_t)} \frac{u(t,y)}{u(t,Z_t)} \leq \sum_{y \notin B(Z_t, r_t)} \left(\frac{c_1}{a_t}\right)^{|y-Z_t|} + e^{-td_t b_t} < Ce^{-td_t\kappa_t},$$

for the same reason, and the proof is complete. □

### 2.5.2 Completion of the proof of Theorem 1.7

We are now in a position to prove the ageing of the solution, and hence complete the proof of Theorem 1.7. We achieve this by a direct comparison between the ageing of the solution and the ageing of the localisation site $Z_t$, which we already established in Section 2.3.

Fix an $s > 0$. Note first that our proof of complete localisation in the above subsection demonstrates that, for sufficiently small $\varepsilon > 0$ and sufficiently large $t > 0$, we have the chain of implications

$$\bigcap_{w\in[1,1+s]} \{Z_t = Z_{tw}\} \cap \mathcal{E}_{tw} \implies \{T_t^\varepsilon > ts\} \implies \bigcap_{w\in[1,1+s]} \{Z_t = Z_{tw}\} \cup \mathcal{E}_t^c \cup \mathcal{E}_{t(1+s)}^c . \quad (2.31)$$

The next proposition bridges the gap between the two sides of this statement.

**Proposition 2.47.** *For each $s > 0$, as $t \to \infty$,*

$$\bigcap_{w\in[1,1+s]} \{Z_t = Z_{tw}\} \cap \mathcal{E}_{wt} = \bigcap_{w\in[1,1+s]} \{Z_t = Z_{tw}\} \cap \mathcal{E}_t \cap \mathcal{E}_{t(1+s)}$$

*eventually almost surely.*

*Proof.* Fix an $s > 0$. Recall that the event $\mathcal{E}_t$ consists of the events $\mathcal{G}_{t,c}$, $\mathcal{H}_t$ and $\mathcal{I}_t$. Since $\mathcal{H}_t$ and $\mathcal{I}_t$ involve properties of $Z_t$ and the (eventually) monotone scales $a_t$, $r_t$, $f_t$ and $g_t$, we have that, as $t \to \infty$ eventually

$$\bigcap_{w\in[1,1+s]} \{Z_t = Z_{tw}\} \cap \mathcal{H}_w \cap \mathcal{I}_w \Leftrightarrow \{Z_t = Z_{t(1+s)}\} \cap \mathcal{H}_t \cup \mathcal{H}_{t(1+s)} \cap \mathcal{I}_t \cap \mathcal{I}_{t(1+s)} .$$

Hence it remains to consider the events $\mathcal{G}_{t,c}$. Consider then two sites $z^1, z^2 \in \Pi^{(L_t)} \cup \Pi^{(L_{t(1+s)})}$, which by monotonicity of $L_t$ implies that

$$z^1, z^2 \in \bigcap_{w\in[1,1+s]} \Pi^{(L_{tw})} .$$

Denote by $\Delta_\lambda := \lambda(z^1)-\lambda(z^2)$, $\Delta_z := |z^1|-|z^2|$, and $\mathfrak{g}_w = \Psi_w(z^1)-\Psi_w(z^2)$ for $w \in [1, 1+s]$. Then, as $t \to \infty$, uniformly in $\Delta_\lambda$, $\Delta_z$ and $w \in [1, 1 + s]$,

$$\begin{aligned}
\mathfrak{g}_{tw} &= \frac{\Delta_\lambda}{d_{tw}} + \frac{\Delta_z}{r_{tw}} \\
&= \frac{\Delta_\lambda}{d_t(1+o(1))} + \frac{\Delta_z}{r_t(1+s)(1+o(1))} \\
&= \mathfrak{g}_t - \frac{w}{1+w}\frac{\Delta_z}{r_t}(1+o(1)) .
\end{aligned}$$

Hence $\mathfrak{g}_{tw}$ is monotone on $w \in [1, 1+s]$ eventually. Since this is true for all pairs $z^1$ and $z^2$, if the maximiser $Z_t = Z_{t(1+s)}$ exceeds all other sites at time $t$ by a gap $d_t g_t$, this gap must be preserved across the whole time interval $[t, t(1 + s)]$, which establishes the result. □

We can now complete the proof of ageing. Since the events $\mathcal{E}_t$ hold with overwhelming probability by Proposition 2.26, and applying the eventually event equivalence in Proposition

2.47, the chain of implications at (2.31) can be written as

$$\mathbf{P}(\{T_t > ts\})(1+o(1)) \geq \mathbf{P}(\{T_t^\varepsilon > ts\}) \geq \mathbf{P}(\{T_t > ts\})(1+o(1)),$$

and so

$$\mathbf{P}(\{T_t \leq ts\}) = \mathbf{P}(\{T_t^\varepsilon \leq ts\})(1+o(1)).$$

Hence the convergence result for $T_t$ implies the equivalent convergence result for $T_t^\varepsilon$.

### 2.5.3 Completion of the proof of Theorem 1.3

To finish the section, we now use the ageing result to complete the proof of the lower bound for the exponential decay of the solution, and hence complete the proof of Theorem 1.3. We work on the events $\mathcal{E}_t$ and

$$\mathcal{F}_t := \{T^\varepsilon_{t-t/\log t} > t/\log t\},$$

which hold with overwhelming probability as $t \to \infty$ by Proposition 2.26 and the ageing of the solution in Theorem 1.7 respectively. Abbreviate $z = Z_t$ and fix $y \in B_t$ and

$$r := \frac{|y-z|}{ta_t},$$

which satisfies $r < 1/\log t$ for all $y \in B_t$ on the event $\mathcal{E}_t$. First note that, by the strong Markov property

$$u(t,y) \geq u(t-rt,z)\, \mathbb{P}_z(X_{rt} = y)$$

where $\mathbb{P}_z$ denotes the probability law of the simple continuous-time random walk started at $z$. On the other hand, by Propositions 2.14 and 2.22 and the fact that we are working on the event

$$\mathcal{F}_t \implies \{T^\varepsilon_{t-rt} > rt\}$$

we have that

$$\frac{u(t-rt,z)}{u(t,z)} \geq e^{-rt\xi(z)}.$$

Combining these, and using the Poisson jump rates, we have

$$\frac{u(t,y)}{u(t,z)} \geq e^{-rt\xi(z)}\,(2d)^{-|y-z|}\,e^{-rt}\,\frac{(rt)^{|y-z|}}{|y-z|!}.$$

Using the above choice of $r$, and combining with the fact that $\xi(z) < 2a_t$ on the event $\mathcal{E}_t$, the bound simplifies to

$$\frac{u(t,y)}{u(t,z)} \geq \left(\frac{c_1}{a_t}\right)^{|y-z|} \frac{|y-z|^{|y-z|}}{|y-z|!} \geq \left(\frac{c_1}{a_t}\right)^{|y-z|},$$

for some $c_1 > 0$, as required.

## 2.6 Extending our results to the case of fractional-double-exponential potential

In this section we extend our results to cover the case of fractional-double-exponential potential, that is, the case when there exists a parameter $\gamma \in (0,1)$ such that

$$\mathbf{P}(\xi(0) > x) = \exp\left\{-e^{x^\gamma}\right\} \, , \quad x > 0 \, .$$

Let us begin by recalling our main results on the FDE case, namely establishing complete localisation (Theorem 1.8) and an upper bound on the exponential decay of the solution (Theorem 1.10). In the process we shall make the localisation site $Z_t$ explicit.

The localisation site $Z_t$ is defined similarly to in the Weibull case, although we have to make suitable modifications to take into account (i) the new scales $r_t, a_t$ and $d_t$, as introduced in (1.12), and (ii) the fact that it will be also be necessary to consider local principal eigenvalues inside balls of *growing* radius $\rho_t := \log\log t$ (note that, unlike in the Weibull case, we do not seek to identify $\rho_t$ as the optimum radius of influence).

Taking these modifications into account, the penalisation functional we consider is

$$\Psi_t(z) := \lambda(z) - \frac{|z|}{\gamma t} \log\log\log t \, ,$$

where $\lambda(z)$ is the principal eigenvalue of the operator $\mathcal{H}(z) := \Delta + \xi$ restricted to $B(z, \rho_t)$. Note that the double logarithmic term from the Weibull case is now triple logarithmic, which is natural since the scale $a_t$ is now double logarithmic rather than simply logarithmic. To thin the space, we keep the same macrobox $V_t$, but adjust the corresponding level to be

$$L_t := (\log\log |V_t|)^{\frac{1}{\gamma}} - \theta \, \hat{a}_t \, ,$$

for some $\theta > \frac{1}{\gamma} \log 2$ and $\hat{a}_t$ defined as at (1.13). As before, we let $\Pi^{(L_t)}$ be the set of $L_t$-exceedances in $V_t$, and define the site of complete localisation to be

$$Z_t := \underset{z \in \Pi^{(L_t)}}{\operatorname{argmax}} \, \Psi_t(z) \, .$$

The following is a summary of our main results in Theorems 1.8 and 1.10.

**Theorem 2.48** (Complete localisation and upper bound on the exponential decay of the solution)**.** *There exists a constant $c > 0$ such that, for any function $\kappa_t \to 0$ decaying sufficiently slowly, as $t \to \infty$ the following hold:*

*(a) For each $z \in B(Z_t, \kappa_t r_t) \setminus \{Z_t\}$,*

$$\mathbf{P}\left( \frac{u(t,z)}{U(t)} \hat{a}_t^{-|z - Z_t|} < c^{-|z-Z_t|} \right) \to 1 \, ,$$

*and the convergence in probability holds also for the union of these events;*

*(b) Moreover,*

$$\mathbf{P}\left(e^{td_t\kappa_t}\sum_{z\notin B(Z_t,\kappa_t r_t)}\frac{u(t,z)}{U(t)}<c\right)\to 1\,.$$

Let us give some intuition as to why $\hat{a}_t$ represents the upper bound on the rate of exponential decay, and moreover why the same scale appears in the definition of the macrobox level $L_t$. Indeed this goes to the heart of the difference between the Weibull case and the FDE, and will provide motivation for why our proof of complete localisation needs to be subtly adapted in the FDE case.

Recall that in the Weibull case the macrobox level $L_t$ was defined to be a constant multiple of the scale $a_t$, and recall also that the exceedances of this level $\Pi^{(L_t)}$ – the so-called 'high sites' – were well-separated eventually almost surely (see Section 2.2). There are two crucial points to note here. First, the well-separatedness of the high sites $\Pi^{(L_t)}$ was essential in our proof of complete localisation, since it allowed us to apply cluster expansions around these sites. On the other hand, the fact that the potential at sites not in $\Pi^{(L_t)}$ (or, more precisely, the gap between the potential at these sites and the maximum potential in $V_t$) was of order a constant multiple of $a_t$ was the origin of the exponential decay of the solution on the scale $a_t$ (see, e.g., the informal derivation of the penalisation functional $\Psi_t$ above).

In the FDE case, by contrast, in order to maintain the well-separatedness of the exceedances $\Pi^{(L_t)}$ we are forced to define the macrobox level $L_t$ to be much larger than a constant multiple of $a_t$, basically because the potential field $\xi$ is less inhomogeneous. Indeed, the scale $\hat{a}_t$ represents precisely the largest scale at which exceedances of the level

$$L_t := (\log\log|V_t|)^{\frac{1}{\gamma}} - \theta\,\hat{a}_t$$

are well-separated. On the other hand, the scale $\hat{a}_t$ is precisely the scale of the gap between the potential at sites not in $\Pi^{(L_t)}$ and the maximum potential in $V_t$. Since we expect a cluster of potentials of order $L_t$ around the localisation site, on short scales (i.e. up to the radius of influence) this suggests that the solution decays, at short scales, at an exponential rate given by the scale $\hat{a}_t$.

Note, however, that this analysis is insufficient on its own to establish complete localisation, since in order to prove complete localisation we need to control the exponential decay everywhere inside the macrobox $V_t$. Indeed, in order for the penalisation function $\Psi_t$ to accurately describe the solution profile, we need to show that the exponential decay at large scales is actually given by $a_t$. In order to achieve this we introduce a new 'semi-high' level inside the macrobox – precisely a constant multiple of $a_t$ – and control how many semi-high points can lie along any given path from the origin in the macrobox. This two-tiered system of high sites and semi-high sites represents the main innovation needed to prove complete localisation in the FDE case.

Finally, let us give some intuition about why we conjecture that the radius of influence is of order $\log\log t/\log\log\log t$. Recall that the radius of influence is precisely the scale $n_t$ at which the contribution to the local principal eigenvalues from sites at a distance larger than $n_t$ is approximately the same order as the gap $d_t$ in the top order statistics of the eigenvalues inside the macrobox. Given the discussion above, the contribution is approximately $\hat{a}_t^{-n_t}$, and so $n_t$ should solve the equation

$$\hat{a}_t^{-n_t} \approx d_t \,.$$

A brief computation confirms that this implies $n_t = O(\log\log t/\log\log\log t)$.

In the rest of the section we prove our main results in Theorems 1.9–2.1. Since a substantial portion of the proof is identical to in the Weibull case, we will keep our focus firmly on the aspects of the proof that differ. Wherever the proof of preliminary results is identical to in the Weibull case, we shall state these results without proof.

### 2.6.1 Preliminary results: High and semi-high sites

We start by recalling our results on the high sites, which as in the Weibull case are exceedances of the level $L_t$. Since a Weibull distributed random variable can be mapped to a FDE distributed random variable by the mapping

$$x \mapsto (\log x^\gamma)^{\frac{1}{\gamma}} \,,$$

we will be able to translate these results directly across from Section 2.2.2.

For each $a \in [0, 1)$ define

$$L_{t,a} = (\log\log|V_t|)^{\frac{1}{\gamma}} + \frac{1}{\gamma}\log(1-a)\hat{a}_t \,.$$

**Lemma 2.49** (Almost sure asymptotics for $\xi$)**.** *Denote by $\xi_{t,i}$ the $i^{\text{th}}$ highest value of $\xi$ in $V_t$. Then for each $a \in [0, 1)$ and $a' \in (0, 1]$, as $t \to \infty$,*

$$L_{t,0} - \xi_{t,[|V_t|^a]} \sim L_{t,0} - L_{t,a} \qquad \textit{and} \qquad |\Pi^{(L_{t,a'})}| \sim |V_t|^{a'}$$

*hold almost surely.*

**Lemma 2.50** (Almost sure separation of high points of $\xi$)**.** *For any $a > 0$, let*

$$\Pi_{\rho_t}^{(L_t,a)} := \{z \in B(V_t, \rho_t) : \xi(z) > L_{t,a}\}$$

*be the set of $L_{t,a}$ exceedances of $\xi$ in the $\rho_t$-extended macrobox $B(V_t, n)$. Then, for any $a' < a$, as $t \to \infty$*

$$sep\left(\Pi_{\rho_t}^{(L_{t,a})} \cup \{0\}\right) > |V_t|^{\frac{1-2a'}{d}}$$

*eventually almost surely.*

**Corollary 2.51** (Paths cannot always remain close to high points of $\xi$)**.** *There exists a $c \in (0, 1)$ such that all paths $p \in \Gamma(0, z)$ with $\{p\} \subseteq V_t$ satisfy, as $t \to \infty$,*

$$\left|\left\{p_i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\right\}\right| > |z| - \frac{|z|}{t^c},$$

*eventually almost surely.*

*Proof.* Note that the above result is slightly stronger than Corollary 2.20, since (i) we now consider paths that avoid certain balls of growing radius $B(\Pi^{(L_t)}, \rho_t)$, and (ii) the bound is for *distinct sites*, rather than total indices. However, despite these changes, the proof of Corollary 2.20 goes through exactly as before. For the first, this is since $\rho_t$ satisfies $\rho_t \ll t^c$ for any $c > 0$; for the second, this is immediate from the proof. □

We now introduce our semi-high sites, which as described above will be a constant multiple of the scale $a_t$. We aim to control how many semi-high sites can lie on a certain set of shortest, or near shortest, paths inside the macrobox.

For any $a \in (0,1)$, define the semi-high level $\hat{L}_{t,a} := (1-a)a_t$. For any constant $c > 1$ denote by $\mathcal{J}_{t,c,a}$ the event

$$\mathcal{J}_{t,c,a} := \bigcap_{z \in V_t} \bigcap_{\substack{p \in \Gamma(0,z),\, \{p\} \subseteq V_t, \\ |\{i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\}| < c|z|}} \left\{ \left| \left\{ p_i : p_i \notin B(\Pi^{(L_t)}, \rho_t),\, \xi(p_i) \le \hat{L}_{t,a} \right\} \right| > (1 - \rho_t^{-1})|z| \right\}.$$

In words, $\mathcal{J}_{t,c,a}$ is the event that all of the paths from the origin to sites $z \in V_t$ either (i) have a very large number of sites ($> c|z|$) lying outside balls around sites of high potential, or (ii) have a reasonably large number ($> (1-\rho_t^{-1})|z|$) of such sites that *also* have potential lying below the semi-high level. The decay term $\rho_t^{-1}$ here is not crucial, as long as it decays faster than $(\log a_t)^{-1}$. We use $\rho_t$ for convenience.

To assist in bounding the probability of this event we recall the following classical large deviations inequality.

**Lemma 2.52** (Chernoff-Hoeffding bound)**.** *For any $n \in \mathbb{N}$ and $p \in (0,1)$, let $S_{n,p}$ denote the sum of $n$ i.i.d. Bernoulli random variables with probability $p$. Then, for any $p < q < 1$,*

$$\mathbb{P}(S_{n,p} \ge qn) \le e^{-nH(q;p)}$$

*where $H(q;p) = q\log(q/p) + (1-q)\log((1-q)/(1-p))$ is the Kullback–Leibler divergence between Bernoulli distributed random variables with parameters $q$ and $p$ respectively.*

*Proof.* This is a well-known inequality, but we include a proof for completeness. By Markov's inequality, for any $\beta > 0$

$$\mathbb{P}(S_n \ge qn) = \mathbb{P}(\exp(\beta S_n) \ge e^{\beta qn}) \le \mathbb{E}[\exp(\beta S_n)]e^{-\beta qn} = \left(e^{-\beta q}(1 - p + pe^{\beta})\right)^n .$$

Setting

$$\beta = \log\left(\frac{q}{1-q}\right) - \log\left(\frac{p}{1-p}\right) > 0$$

yields the result. □

**Corollary 2.53.** *For any $n \in \mathbb{N}$ and $0 < q < p$, each $N \ge n$ satisfies*

$$\mathbb{P}(S_{N,p} > qn) \ge 1 - e^{-nH(q;p)}$$

*Proof.* Use the fact that $\mathbb{P}(S_{N,p} > qn) \ge \mathbb{P}(S_{n,p} > qn) = 1 - \mathbb{P}(S_{n,1-p} \ge (1-q)n)$. □

**Proposition 2.54** (Most sites on paths are below the semi-high level)**.** *For any $a \in (0,1)$ and $c > 1$, as $t \to \infty$,*

$$\mathbf{P}(\mathcal{J}_{t,c,a}) \to 1\,.$$

*Proof.* Fix a $c_1 \in (0,1)$ that satisfies Corollary 2.51, and consider a fixed $z \in V_t$ and path in $p \in \Gamma_n(0,z)$ with $n < c|z|$ and $\{p\} \subset V_t$. First note that since

$$\mathbf{P}(\xi(0) > \hat{L}_{t,a}) = \exp\{-(\log t)^{1-a}\}\,,$$

we have that, eventually as $t \to \infty$,

$$H\left(\rho_t^{-1}/(1-t^{-c_1}); \mathbf{P}(\xi(0) > \hat{L}_{t,a})\right) > \rho_t^{-1}(\log t)^{1-a} + O(1) > (\log t)^{c_2}\,,$$

for some constant $c_2 \in (0,1)$. Second, by Corollary 2.51, there are at least

$$|z|(1-t^{-c_1})$$

distinct sites on the path $p$ that satisfy $p_i \notin B(\Pi^{(L_t)}, \rho_t)$, eventually almost surely. Applying Corollary 2.53 with the (valid eventually) parameters

$$n = |z|(1+t^{-c_1})\,, \quad p = 1-\exp\{-(\log t)^{1-a}\} \quad \text{and} \quad q = 1-\rho_t^{-1}/(1-t^{-c_1})\,,$$

we deduce that

$$\begin{aligned}\mathbf{P}\Big(\Big|\Big\{p_i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\,,\ \xi(p_i) \le \hat{L}_{t,a}\Big\}\Big| &> (1-\rho_t^{-1})|z|\Big) \\ &> 1-\exp\left\{-|z|(1-t^{-c_1})(\log t)^{c_2}\right\} + o(1) \\ &> 1-\exp\left\{-|z|(\log t)^{c_3}\right\}\end{aligned}$$

eventually, for some $c_3 \in (0,1)$. Note that this is a bound on the probability of a *single* path $p \in \Gamma(0,z)$ having a large number of sites with potential below the semi-high level $\hat{L}_{t,a}$.

We next seek to limit the number of essentially distinct such paths we need to consider, in order that we may efficiently apply the union bound to complete the result. To do this, we introduce an equivalence class on paths; this is similar to our approach in Section 2.4 above. So let two paths $p, p' \in \Gamma(0,z)$ be considered equivalent if they have the same trajectories outside the balls $B(\Pi^{(L_t)}, \rho_t)$. Then the events

$$\Big\{\Big|\Big\{i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\Big\}\Big| < c|z|\Big\}$$

and

$$\Big\{\Big|\Big\{p_i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\,,\ \xi(p_i) \le \hat{L}_{t,a}\Big\}\Big| > (1-\rho_t^{-1})|z|\Big\}$$

are both class properties with respect to this equivalence class. Further, note that there are less than

$$\max\{2d, |\partial B(0,\rho_t)|\}^{c|z|} < e^{c|z|\rho_t}\,,$$

such equivalence classes with the property that

$$|\{i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\}| < c|z| .$$

Hence by applying the union bound over equivalence classes, and by the definition of $\rho_t$, we have that

$$\mathbf{P}(\mathcal{J}_{t,c,a} \text{ does not hold}) < \sum_{z \in V_t} \exp\{|z|c\rho_t - |z|(\log t)^{c_3}\} \to 0 ,$$

as $t \to \infty$, which implies the result. □

### 2.6.2 Extremal theory for local principal eigenvalues

The results and proofs in this section follow very closely the Weibull case but in simplified form – we do not seek to prove a local profile for the potential around $Z_t$, which was the source of many of the complications in the Weibull case. Note that, in place of the fixed integer $n$, we shall prove our results for the growing radius $n := \rho_t$. Here and throughout, we shall fix a $\varepsilon \in (0, 1)$ such that $-1/\gamma \log(1 - \varepsilon) = \theta + 1$. Note that our choice of $\varepsilon$ implies that

$$L_{t,\varepsilon} - L_t \sim \hat{a}_t .$$

This choice is essentially arbitrary, as long as $\varepsilon$ is sufficiently close to one. We also make use of the same set of auxiliary scaling functions $f_t, h_t, e_t, b_t \to 0$ and $g_t \to \infty$ in (2.1), although in the FDE case we shall need the slightly stronger assumptions that

$$1/\log\log\log t \ll b_t \ll 1$$

and

$$g_t/\log\log\log t \ll b_t \ll f_t h_t \ll g_t h_t \ll e_t . \tag{2.32}$$

Recall first $\tilde{\lambda}_t^{(n)}(z)$, the *punctured* version of $\lambda^{(n)}(z)$, and define completely analogously $\tilde{\lambda}(z)$ to be the punctured version of $\lambda(z) = \lambda^{(\rho_t)}(z)$.

**Proposition 2.55** (Path expansion for $\tilde{\lambda}_t$)**.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly, as $t \to \infty$,*

$$\begin{aligned}\tilde{\lambda}_t(z) &= \eta(z) + \sum_{2 \le k \le \rho_t} \sum_{\substack{p \in \Gamma_k(z,z) \\ p_i \neq z,\, 0<i<k \\ \{p\} \subseteq B(z,\rho_t)}} \prod_{0<i<k} \frac{1}{\tilde{\lambda}_t^{(n)}(z) - \eta(p_i)} + o(d_t e_t) ,\\ &= \eta(z) + O(\hat{a}_t^{-1}) .\end{aligned}$$

*Moreover, as $t \to \infty$,*

$$\tilde{\lambda}_t(z) = \lambda(z)$$

*eventually almost surely.*

*Proof.* The proof is the same as for Proposition 2.21, except that we now have, for each $p_i \in B(z, n) \setminus \{z\}$,

$$\tilde{\lambda}_t^{(n)}(z) - \eta(p_i) > L_{t,\varepsilon} - L_t + 2d \sim \hat{a}_t ,$$

eventually almost surely. Moreover, $\rho_t$ has been chosen to satisfy, as $t \to \infty$,

$$\hat{a}_t^{-\rho_t} = o(d_t e_t)\,,$$

by the definition of $\rho_t$. This means that, up to the error $o(d_t e_t)$, we can truncate the sum at paths with $\rho_t$ steps. It also means that the total contribution from the sum over paths $p \in \Gamma_k(z,z)$ is $O(\hat{a}_t^{-1})$. $\square$

**Proposition 2.56** (Extremal theory for $\tilde{\lambda}_t$; see [7])**.** *There exists a scaling function $A_t = a_t + O(1)$ such that, as $t \to \infty$ and for each fixed $x \in \mathbb{R}$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t(0) > A_t + x d_t\right) \to e^{-x}\,.$$

*Moreover, there exists a $c < 1$ such that, as $t \to \infty$ and uniformly for $x > 1$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t(0) > A_t + x d_t\right) < e^{-e^{cx^\gamma}}\,.$$

*Proof.* The proof of the first result is similar to Proposition 2.22. Defining $A_t$ to be an arbitrary scale $a_t + O(1)$, we still have that, by Lemmas 2.50 and 2.3, as $t \to \infty$,

$$\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t \quad \text{implies that} \quad \xi(0) > L_{t,\varepsilon}\,,$$

eventually almost surely. Define the function

$$Q(A_t;\xi) := 2d + \sum_{2 \le k \le \rho_t} \sum_{\substack{p \in \Gamma_k(0,0) \\ p_i \neq 0,\, 0<i<k \\ \{p\} \subseteq B(z,\rho_t)}} \prod_{0<i<k} \frac{1}{A_t - \eta(p_i)}\,,$$

if $\xi(y) < L_t$ for each $y \in B(0,j) \setminus \{0\}$ and $Q(A_t;\xi) := 0$ otherwise. Then since, as $t \to \infty$, uniformly on $\xi$,

$$Q(A_t + x d_t;\xi) = Q(A_t;\xi) + o(d_t)$$

and

$$\exp\left\{(A_t + Q(A_t;\xi))^{\frac{1}{\gamma}}\right\} \sim d \log t\,,$$

we can compute the asymptotics as

$$\begin{aligned}
\mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) &\sim \mathbf{P}\Big(\xi(0) > A_t + x d_t + Q(A_t + x d_t;\xi)\Big) \\
&\sim \mathbf{P}\Big(\xi(0) > A_t + x d_t + Q(A_t;\xi)\Big) \\
&\sim t^{-d} e^{-x} \int_\xi \exp\left\{e^{a_t^\gamma} - e^{(A_t + Q(A_t;\xi))^\gamma}\right\} d\mu_\xi \\
&\sim t^{-d} e^{-x} \int_\xi \exp\left\{e^{a_t^\gamma} - e^{(A_t + O(1))^\gamma}\right\} d\mu_\xi
\end{aligned}$$

where all the asymptotics are uniform in $\xi$. Since, for $C > 0$ sufficiently large, eventually

$$e^{a_t^\gamma} - e^{(a_t + C + O(1))^\gamma} < 0 < e^{a_t^\gamma} - e^{(a_t - C + O(1))^\gamma}\,,$$

and so, by the continuity of $Q$, there exists an $A_t = a_t + O(1)$ such that, as $t \to \infty$,

$$\int_{\xi,\sigma} \exp\left\{ e^{a_t^\gamma} - e^{(A_t + Q(A_t;\xi))^\gamma} \right\} d\mu_\xi \to 1$$

which gives the first result.

For the uniform bound, we again bound $Q(A_t + xd_t;\xi)$ above, uniformly in $x > 0$, by $Q(A_t;\xi,\sigma)$, which produces the bound

$$t^{-d} \int_\xi \exp\left\{ e^{a_t^\gamma} - \exp\left\{ \Big( A_t + Q(A_t;\xi) \Big)^\gamma \Big( 1 + \frac{x}{\gamma}(d \log t \log\log t)^{-1} \Big)^\gamma \right\} \right\} d\mu_\xi \, .$$

We bound this expression above uniformly in $x > 1$ using the bound, for some $c < 1$, that

$$(1+x)^\gamma \geq \begin{cases} 1 + cx\,, & \text{if } x \leq 1\,, \\ 1 + cx^\gamma\,, & \text{if } x > 1\,, \end{cases}$$

to get the upper bound

$$t^{-d} \int_\xi \exp\left\{ e^{a_t^\gamma} - e^{(A_t + Q(A_t;\xi))^\gamma} e^{cx^\gamma (d \log t)^{-1}} \right\} d\mu_\xi \, .$$

Combining with the bound $e^x > 1 + x$ and the definition of $A_t$ yields the result. $\square$

Using the same method from before, we can use the above results to determine the properties of the maximisers of the functionals $\Psi_{t,c}$, defined by

$$\Psi_{t,c}^{(j)}(z) := \lambda^{(j)}(z) - \frac{|z|}{\gamma t} \log\log\log t + c\frac{|z|}{t}\,,$$

as well as their top two statistics

$$Z_{t,c}^{(j)} := \operatorname*{argmax}_{z \in \Pi^{(L_t)}} \Psi_{t,c}^{(j)}(z) \quad \text{and} \quad Z_{t,c}^{(j,2)} := \operatorname*{argmax}_{\substack{z \in \Pi^{(L_t)} \\ z \neq Z_{t,c}^{(j)}}} \Psi_{t,c}^{(j)} \, .$$

In particular, using the scaling factor $A_t$ from Proposition 2.56, we have an identical point process convergence as in Proposition 2.24 (with the same limiting point process density), which implies the following scaling limit (and, as a consequence, Theorem 1.9).

**Corollary 2.57.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\left( \frac{Z_{t,c}^{(j)}}{r_t}, \frac{Z_{t,c}^{(j,2)}}{r_t}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j)}) - A_{r_t}}{d_{r_t}}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j,2)}) - A_{r_t}}{d_{r_t}} \right)$$

*converges in law to a random vector with density*

$$p(x_1,x_2,y_1,y_2) = \exp\{-(y_1+y_2) - |x_1| - |x_2|) - 2^d e^{-y_2}\} \mathbb{1}_{\{y_1 > y_2\}} \, .$$

We also introduce, for each $c \in \mathbb{R}$, the events

$$\mathcal{G}_{t,c} := \{\Psi_{t,c}(Z_{t,c}) - \Psi_{t,c}(Z^{(2)}_{t,c}) > d_t e_t\},$$

$$\mathcal{H}_t := \{r_t f_t < |Z_t| < r_t g_t\} \quad \text{and} \quad \mathcal{I}_t := \{a_t - g_t < \Psi_t(Z_t) < a_t + g_t\},$$

and the event

$$\mathcal{E}_{t,c} := \mathcal{G}_{t,0} \cap \mathcal{G}_{t,c} \cap \mathcal{H}_t \cap \mathcal{I}_t .$$

**Proposition 2.58.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\mathbf{P}(\mathcal{E}_{t,c}) \to 1 .$$

*Proof.* This follows as in Proposition 2.26; the tighter bounds for $\mathcal{I}_t$ are achievable (and turn out to be necessary) since now $A_{r_t} \sim a_t + O(1)$. □

**Corollary 2.59.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$Z_{t,c} = Z_t$$

*holds eventually.*

*Proof.* This holds just as in the proof of Proposition 2.27, using the fact that $1/\log\log\log t < e_t/g_t$ eventually by (2.32). □

### 2.6.3 Negligible paths: Upper bounds and lower bounds

Recall the division of the solution into path components $U^i(t)$. We prove the negligibility of $U^i(t)$, for $i = 2, 3, 4, 5$ in an identical manner as in the Weibull case, with a few minor differences as noted below. These differences relate mainly to the use of high sites and semi-high sites. Recalling our choice of $\varepsilon < 1$, define the constant $c_\alpha := (1-\gamma)^{-1} > 1$ and abbreviate the event $\mathcal{J}_t := \mathcal{J}_{t,c_\alpha,\varepsilon}$. Note that the constant $c_\alpha$ is chosen so that, as $t \to \infty$,

$$\hat{a}_t^{-c_\alpha |z|} \sim a_t .$$

We begin by restating the lower bound on the total solution.

**Proposition 2.60.** *For each $c \in \mathbb{R}$ on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\log U(t) > t\lambda(Z_t) - \frac{|Z_t|}{\gamma} \log\log\log t + O(t d_t b_t)$$

*almost surely.*

*Proof.* Identically as in the proof of Proposition 2.33, we have, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,

$$\log u^1(t, Z_t) > t\lambda(Z_t) - \frac{|Z_t|}{\gamma} \log \lambda(Z_t) + O(t d_t b_t) .$$

On event $\mathcal{E}_{t,c}$ we have that $\Psi_t(z) < a_t + g_t$. Since also $|z| < r_t g_t$ on event $\mathcal{E}_{t,c}$ we find that

$$\lambda_t(z) < a_t + g_t + d_t g_t$$

and so we have the result. $\square$

For the upper bound on $U^2(t)$ and $U^3(t)$, the proof proceeds as in the Weibull case. Namely, we start with a bound on the contribution from equivalence classes $P(p) \in \mathcal{P}_{n,m}$, and then we use this to prove the upper bound on the solution components. However, we need to make a small modification to this argument to take advantage of our introduction of 'semi-high' sites. To do this, for each equivalence class $P(p) \in \mathcal{P}_{n,m}$, denote the set of indices

$$\mathcal{N} := \left\{ i : p_i \notin B(\Pi^{(L_t)}, \rho_t) \right\}$$

which is a well-defined on each equivalence class $P(p) \in \mathcal{P}_{n,m}$.

**Lemma 2.61** (Bound on the contribution from each equivalence class)**.** *Let $m, n \in \mathbb{N}$ and $p \in \Gamma(0)$ such that $\{p\} \subseteq V_t$ and $P(p) \in \mathcal{P}_{n,m}$. Define $z^{(p)} := \operatorname{argmax}_{z \in \{p\}} \lambda(z)$ and let $\zeta > \max\{\lambda(z^{(p)}), L_{t,\varepsilon}\}$. Then, for each $m, n, p$ and $\zeta$ uniformly, there exist constants $c_1, c_2, c_3 > 0$ such that, as $t \to \infty$,*

$$U^{P(p)}(t) \ < e^{\zeta t} \prod_{i \in \mathcal{N}} \left(c_1(\zeta - \xi(p_i) + 2d)\right)^{-1} (c_2 \hat{a}_t)^{-(n-|\mathcal{N}|)} \left(1 + c_3 \left(\zeta - \lambda(z^{(p)})\right)^{-1}\right)^m$$

*eventually almost surely.*

*Proof.* Note first that $n \geq |\mathcal{N}| + m\rho_t$ eventually almost surely, since each visit to the centre of a ball in $B(\Pi^{(L_t)}, \rho_t)$ from outside requires at least $\rho_t$ steps, since moreover these balls are well-separated eventually almost surely by Lemma 2.50, and since $0 \notin B(\Pi^{(L_t)}, \rho_t)$ eventually almost surely. Then the rest of the proof follows as in Lemma 2.35, although we now, in equation 2.24, only apply the bound

$$L_{t,\varepsilon} - \xi(p_i) + 2d > L_{t,\varepsilon} - L_t + 2d \sim \hat{a}_t$$

to sites that lie inside $B(\Pi^{(L_t)}, \rho_t)$; for the remainder of the sites $p_i$, we keep track of the potential $\xi(p_i)$. $\square$

**Proposition 2.62** (Upper bound on $U^2(t)$)**.** *There exists a constant $c \in \mathbb{R}$ such that, on the event $\mathcal{J}_t$, as $t \to \infty$,*

$$\log U^2(t) < t \max_{z \in \Pi^{(L_t)} \setminus \{Z_t\}} \Psi^{(j)}_{t,c}(z) + O(t d_t b_t)$$

*almost surely.*

**Proposition 2.63** (Upper bound on $U^3(t)$)**.** *There exists a constant $c \in \mathbb{R}$ such that, on the event $\mathcal{J}_t$, as $t \to \infty$,*

$$\log U^3(t) < t \Psi^{(j)}_{t,c}(Z_t) - h_t \frac{1}{\gamma} |Z_t| \log\log\log t + O\left(t d_t b_t\right)$$

*almost surely.*

*Proof.* The proof of Propositions 2.62 and 2.63 is very similar to the proof of Propositions 2.36 and 2.37 in the Weibull case, with certain minor modifications relating to our introduction of semi-high sites. We outline below the necessary modification to the proof of Proposition 2.36; the modification to the proof of Proposition 2.37 are identical.

First, recall that in the proof of Proposition 2.62 we considered the pathset $\mathcal{P}^2_{n,m}$ whose cardinality we bounded above by $|\mathcal{P}^2_{n,m}| \leq \kappa^{n+m}$, with $\kappa = \max\{2d, |\partial B(0,j)|\}$. In the FDE case it will be necessary to instead bound this by $|\mathcal{P}^2_{n,m}| \leq \kappa_1^n \kappa_2^m$, with $\kappa_1 = 2d, \kappa_2 = |\partial B(0, \rho_t)|$. Then, as in (2.26), we have that

$$U^2(t) \leq \max_{n,m} \max_{P \in \mathcal{P}^2_{n,m}} \left\{ \kappa_1^{2n} \kappa_2^{2m} U^P(t) \right\} \sum_{n,m} \kappa_1^{-n} \kappa_2^{-m} .$$

Moreover, by Lemma 2.61, there exist $c_1, c_2, c_3 > 0$ such that

$$\kappa_1^{2n} \kappa_2^{2m} \, U^P(t) < e^{\zeta t} \prod_{i \in \mathcal{N}} \left(c_1(\zeta - \xi(p_i) + 2d)\right)^{-1} (c_2 \hat{a}_t)^{-(n - |\mathcal{N}|)} \left(\kappa_2 + c_3 \kappa_2 (\zeta - \lambda^{(j)}(z^{(P)}))^{-1}\right)^m .$$

Note that we have the 'naive' bound

$$L_{t,\varepsilon} - \xi(p_i) + 2d > L_{t,\varepsilon} - L_t + 2d \sim \hat{a}_t$$

and hence the above expression is monotonous decreasing in $n$. Moreover, as before, we can apply the cluster expansion to achieve monotonicity in $m$; the monotonicity holds precisely because we have

$$n - |\mathcal{N}| \geq m \rho_t$$

and since we defined $\rho_t$ to satisfy

$$\hat{a}_t^{-\rho_t} < d_t b_t |\partial B(0, \rho_t)|^{-1} =: d_t b_t / \kappa_2 .$$

After invoking the monotonicity in $m$, we are left with the bound

$$\kappa_1^{2n} \kappa_2^{2m} \, U^P(t) < e^{\zeta t} \prod_{\{i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\}} \left(c_1(\zeta - \xi(p_i) + 2d)\right)^{-1} .$$

Now recall that the event $\mathcal{J}_t$ guarantees that either

$$|\{i : p_i \notin B(\Pi^{(L_t)}, \rho_t)\}| > c_\alpha |z|$$

or else

$$|\{p_i : p_i \notin B(\Pi^{(L_t)}, \rho_t), \xi(p_i) < \hat{L}_{t,\varepsilon}\}| > (1 - \rho_t^{-1})|z| .$$

The former corresponds to the situation where the path is very long, and indeed long enough that the naive bound on $\xi(p_i)$ is enough to establish the correct penalty term. The latter corresponds to the situation where the path is short, but we know instead that a stronger bound holds for most $\xi(p_i)$ on the path.

If the first holds, then applying the naive bound that $\xi(p_i) < L_{t,\varepsilon}$ we have

$$\kappa_1^{2n}\kappa_2^{2m}\, U^P(t) < e^{\zeta t}\hat{a}_t^{-c_\alpha|z|}c_2^{|z|}$$

for some $c_2 > 0$. Since we defined $c_\alpha$ precisely so that $\hat{a}_t^{-c_\alpha|z|} \sim a_t$, this is equivalent to

$$\kappa_1^{2n}\kappa_2^{2m}\, U^P(t) < e^{\zeta t}a_t^{-|z|}c_3^{|z|}$$

for some $c_3 > 0$. If instead the second holds, then we may apply the stronger bound that

$$L_{t,\varepsilon} - \xi(p_i) + 2d > L_{t,\varepsilon} - \hat{L}_{t,\varepsilon} + 2d \sim \varepsilon a_t$$

to at least $(1-\rho_t^{-1})|z|$ of the indices. This leads to the bound

$$\kappa_1^{2n}\kappa_2^{2m}\, U^P(t) < e^{\zeta t}a_t^{-|z|(1-\rho_t^{-1})}c_2^{|z|}\,.$$

Since $\rho_t \ll \log a_t$, this is equivalent to

$$\kappa_1^{2n}\kappa_2^{2m}\, U^P(t) < e^{\zeta t}a_t^{-|z|}c_3^{|z|}\,.$$

This brings the proof back in line with the Weibull case, which we complete as before. □

**Corollary 2.64.** *There exists a constant $c \in \mathbb{R}$ such that, as $t \to \infty$,*

$$e^{td_tb_t}\frac{U^2(t)+U^3(t)+U^4(t)+U^5(t)}{U(t)}\mathbb{1}_{\mathcal{E}_{t,c}}\mathbb{1}_{\mathcal{J}_t} \to 0$$

*almost surely.*

*Proof.* The negligibility of $U^2$, $U^3$ and $U^5$ follow from the above as in the Weibull case. For $U^4$, the upper bound of $e^{tL_t}$ is still sufficient since the event $\mathcal{E}_{t,c}$ guarantees that $\log U(t) > t(a_t - g_t - d_tg_t) - td_tg_t$ and so

$$U^4(t)/U(t) > e^{t(L_t-a_t)}e^{tg_t+2td_tg_t} \ll e^{-td_tb_t}\,,$$

as required. Note that here we used the tighter bound on $\Psi_t(z)$ provided by the event $\mathcal{I}_t$, as well as the fact that $a_t - L_t = \theta\hat{a}_t + o(1) \gg d_tb_t$. □

### 2.6.4 Complete localisation and exponential decay

The completion of the proof of complete localisation (Theorem 1.8) and the upper bound on exponential decay (Theorem 1.10) follows almost identically as to the proof of the corresponding results in the Weibull case; the necessary modifications have already been addressed in our proof of the upper bounds in the previous subsection. On the other hand, to bound the exponential decay of the principal eigenfunction $v_t(y)$ we do not, as we did in the previous subsection, make use of the semi-high sites. This is because, due to the correlation in the potential field around $Z_t$, we are unable to control the number of semi-high sites on short scales near the localisation site $Z_t$. Instead, we establish an upper bound for the exponential decay given by the rate $\hat{a}_t$.

**Proposition 2.65** (Exponential decay of principal eigenfunction)**.** *Let $c \in \mathbb{R}$ be defined as is Corollary 2.64, and abbreviate $\mathcal{E}_t := \mathcal{E}_{t,c}$. Then there exists a $c_1 > 0$ such that, on the event $\mathcal{E}_t$ and for each $y \in B_t \setminus \{Z_t\}$ uniformly, as $t \to \infty$*

$$\log v_t(y) < -\left(\frac{1}{\gamma} - 1\right) |y - Z_t| \log\log\log t + c_1 |y - Z_t|$$

*eventually almost surely. As a corollary,*

$$\frac{u^1(t,y)}{u^1(t,Z_t)} < \left(\frac{c_1}{\hat{a}_t}\right)^{|y-Z_t|},$$

*eventually almost surely.*

Once we have established Proposition 2.65, the proof of Theorems 1.8 and 1.10 follow by combining with Corollary 2.64 and Propositions 2.58 and 2.54, just as in the Weibull case.

# Chapter 3

# The Bouchaud trap model with slowly varying traps

## 3.1 Introduction

In this chapter we study the BTM on the integers with slowly varying traps, that is, in the case in which the tail function of the trap distribution

$$L(x) := \frac{1}{\mathbf{P}(\sigma(0) > x)}$$

satisfies the slow variation property (1.14). In Chapter 1 we outlined our main results on the localisation properties this model, which can be summarised as follows:

1. The BTM eventually localises, under the annealed law, on the first trap on the positive and negative half-line respectively to exceed a certain deterministic level $\ell_t$ (defined at (1.16)), in other words, the BTM exhibits two-site localisation (Theorem 1.12);
2. The probability mass on each of the two localisation sites is, asymptotically, in inverse proportion to their distance from the origin (Theorem 1.13) – along with the scaling of the localisation set (Theorem 1.13) this implies a single-time scaling limit, under the annealed law, of the BTM (Corollary 1.15);
3. The BTM satisfies a certain functional limit theorem, under the annealed law, which generalises the above single-time scaling limit (Theorem 1.16); and
4. Under the strengthening of the slow-variation property in Assumption 1, we obtain simplified versions of our main results (Theorems 1.17 and 1.18).

In a sense each of our main results on the BTM can be viewed as manifestations of a single phenomena, namely that in the slowly varying case the dynamics of the BTM are dominated by the influence of the deepest trap the process has visited. This, in turn, is essentially due to the following classical limit theorem on the sum/max ratio of sequences of i.i.d. slowly varying random variables.

**Theorem 3.1** (Sum/max ratio of sequences of i.i.d. slowly varying random variables; see [30])**.** *Let $\sigma = \{\sigma_i\}_{i\in\mathbb{N}}$ denote a sequence of i.i.d. copies of $\sigma(0)$. Then, as $i \to \infty$,*

$$\frac{\sum_{i\leq n}\sigma_i}{\max_{i\leq n}\sigma_i} \to 1 \qquad \text{in probability}.$$

In words, this result states that sequences of i.i.d. slowly varying random variables have a partial sum which is asymptotically dominated by the maximal term with overwhelming probability. Our results will be based on certain functional limit analogues of this limit theorem (see Section 3.2.2), which are essentially due to Kasahara [50].

The rest of this chapter is organised as follows. In Section 3.2 we collect preliminary results which will act as the main input into our proofs. In Section 3.3 we complete the proof of two-site localisation (Theorem 1.12), as well as the ancillary results which lead to the single-time scaling limit (Theorems 1.13 and 1.14). We also prove the simplified versions of these results under Assumption 1 (Theorem 1.17). Finally in Section 3.4 we establish our functional limit theorem for the BTM (Theorem 1.16), as well as its simplified version under Assumption 1 (Theorem 1.18). Note that our functional limit theorems rely on certain convergence lemmas for the various topologies on the Skorohod space $D(\mathbb{R}^+)$ of real-valued càdlàg functions on $\mathbb{R}^+$; we collect these convergence lemmas in Appendix 3.5.

## 3.2 Preliminary results: Random walks and sequences of slowly varying random variables

In this section we collect preliminary results which will be crucial to our proofs. Most of these results are well-known, however certain extensions appear to be new (for example, the joint convergence in (3.2) below). We first present properties of random walks and finite-state Markov chains, and then establish properties of sequences of i.i.d. slowly varying random variables.

### 3.2.1 Random walks and Markov chains

We start by presenting some basic properties of simple discrete-time random walks (SRW). Let $D = (D_n)_{n\geq 0}$ denote a SRW on $\mathbb{Z}$ based at the origin. For a level $l > 0$ and a site $z \in \mathbb{Z}$ define the two-sided hitting time

$$a_l := \min\{n : |D_n| \geq l\}\,,$$

and the accumulated (discrete) local time up to this hitting time

$$\mathcal{L}_z^l := |\{n < a_l : D_n = z\}|\,.$$

Finally, let

$$d_n := \max_{i\leq\lfloor n\rfloor} D_i - \min_{i\leq\lfloor n\rfloor} D_i$$

denote the diffusion distance of $D$ after $n$ steps.

**Proposition 3.2** (Bounds for the local time of a SRW at the stopping time)**.** *As $l \to \infty$, both*

$$\frac{\max_z \mathcal{L}^l_z}{l} \quad \textit{and} \quad \frac{\mathcal{L}^l_0}{l}$$

*are bounded below and above in probability.*

*Proof.* These are simple consequences of invariance principles for random walk local times (see, e.g., [62, Chapter 10]). Indeed, these invariance principles actually imply the stronger result (see [20, Theorem 7.6]) that, as $l \to \infty$,

$$\left(l^{-1}\mathcal{L}^l_{\lfloor zl \rfloor}\right)_{z\in[-1,1]} \overset{J_1}{\Rightarrow} (\nu^1_z)_{z\in[-1,1]} \, ,$$

where $\nu^1_z$ denotes the local time of Brownian motion at the point $z$ accumulated up to the first hitting time of $\pm 1$, and $\overset{J_1}{\Rightarrow}$ denotes weak convergence in the Skorokhod space $D([-1,1])$ of real-valued càdlàg functions on $[-1,1]$ equipped with the $J_1$ topology; see [70] or Appendix 3.5 for a description. □

**Proposition 3.3** (Hitting probability for the SRW)**.** *For any $x \in \mathbb{Z}^+$ and $y \in \mathbb{Z}^-$,*

$$\mathbb{P}(b_x < b_y) = \frac{y}{x+y} \, ,$$

*where $b_z := \min\{n > 0 : D_n = z\}$.*

*Proof.* This well-known fact follows from the optional stopping theorem. □

**Proposition 3.4** (Bounds for the diffusion distance of a SRW)**.** *As $n \to \infty$,*

$$\frac{d_n}{\sqrt{n}}$$

*is bounded below and above in probability.*

*Proof.* This is a simple consequence of Donsker's invariance principle. □

Next, we shall establish some analogous results for continuous-time simple random walks (CTSRW). Consider a CTSRW on the integers, initialised at the origin, and let $\nu(n,x)$ be its (continuous) local time after $n$ steps, that is,

$$\nu(n,x) := \sum_{\{0 \le i \le \lfloor n \rfloor : D_i = x\}} \psi_i \, ,$$

where $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. unit-mean exponential distributions. Further, let $\nu_{\max}(n)$ and $\nu_{\min}(n)$ be respectively the maximum and minimum (continuous) local times among the sites visited after $n$ steps

$$\nu_{\max}(n) := \max_{x \in \{D_i : \, i \le n\}} \nu(n,x) := \max_{x\in\mathbb{Z}} \nu(n,x) \, ;$$

$$\nu_{\min}(n) := \min_{x \in \{D_i : \, i \le n\}} \nu(n,x) \, .$$

**Proposition 3.5** (Bounds for the local time of a CTSRW)**.** *As $n \to \infty$,*

$$\frac{\nu(n,0)}{\sqrt{n}} \quad \text{and} \quad \frac{\nu_{max}(n)}{\sqrt{n}}$$

*are both bounded below and above in probability.*

*Proof.* If $(B_t)_{t\geq 0}$ is a Brownian motion, and $(L_t(x))_{t\geq 0, x\in\mathbb{D}}$ is its local time process, then it is standard that $L_{\sigma_{\pm 1}}(0)$ has an exponential distribution with mean one, where $\sigma_{\pm 1}$ is the first hitting time of $\pm 1$. It follows that $(\nu(n,x))_{n\geq 0, x\in\mathbb{Z}}$ has the same distribution as $(L_{\sigma(n)}(x))_{n\geq 0, x\in\mathbb{Z}}$, where $\sigma(0) = 0$ and, for $n \geq 1$, $\sigma(n) := \inf\{t > \sigma(n-1) : B_t \in \mathbb{Z}\backslash\{B_{\sigma(n-1)}\}\}$. Since $n^{-1}\sigma(n) \to 1$ almost surely, it follows that

$$\frac{\nu(n,0)}{\sqrt{n}} \Rightarrow L_1(0) \quad \text{and} \quad \frac{\nu_{\max}(n)}{\sqrt{n}} \Rightarrow \sup_{x\in\mathbb{D}} L_1(x)\,,$$

in distribution, where to deduce this it is also helpful to recall the scaling property of Brownian local times, i.e. $(L_t(x))_{t\geq 0, x\in\mathbb{D}} \overset{d}{=} (\lambda^{-1/2}L_{\lambda t}(\lambda^{1/2}x))_{t\geq 0, x\in\mathbb{R}}$. ☐

**Proposition 3.6** (Bound for the minimum local time of a CTSRW)**.** *For any $T > \delta > 0$, as $n \to \infty$,*

$$n \min_{i\in[n^2\delta, n^2T]} \nu_{min}(i)$$

*is bounded below in probability.*

*Proof.* Combine the identity

$$n \min_{i\leq n} \psi_i \overset{d}{=} \psi_0\,, \quad n \in \mathbb{N}$$

with the bounds on the diffusion distance of Proposition 3.4. ☐

To complete this section, we state a useful aspect of the convergence of a finite-state Markov chain to equilibrium.

**Proposition 3.7** (Monotonic convergence of a Markov chain to equilibrium)**.** *Let $M = (M_t)_{t\geq 0}$ be an irreducible, finite-state, time-homogeneous, continuous-time Markov chain, initialised at a state 0. Suppose further that the transition rates $w$ of $M$ satisfy the detailed balance condition, i.e. there exists a non-negative vector $\pi$ such that*

$$\pi(x)w_{x\to y} = \pi(y)w_{y\to x}$$

*for each pair of states $x$ and $y$. Then $\pi$ is the unique equilibrium distribution for $M$ and satisfies, as $t \to \infty$,*

$$\mathbb{P}(M_t = 0) \downarrow \pi(0) \quad \textit{monotonically}.$$

*Proof.* This is a well-known result from continuous-time Markov chain theory. It can be proved by considering the spectral representation of $\mathbb{P}(M_t = 0)$ in terms of the eigenvalues $\lambda_i$ and eigenfunctions $\varphi_i$ of the generator of $M_t$, i.e.

$$\mathbb{P}(M_t = 0) = \sum_i e^{\lambda_i t}\varphi_i^2(0)\,,$$

recalling that the detailed balance condition ensures that each $\lambda_i$ and $\varphi_i$ is real. Since $\mathbb{P}(M_t = 0)$ is bounded as $t \to \infty$, each $\lambda_i$ must satisfy $\lambda_i \le 0$, resulting in the monotonic convergence of $\mathbb{P}(M_t = 0)$ to its equilibrium density. □

### 3.2.2 Sequences of slowly varying random variables

Here we state general properties of sequences of i.i.d. random variables with slowly varying tail. Let $Y := \{Y_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. copies of $\sigma(0)$, and let $M := (M_n)_{n\ge0}$ and $\Sigma := (\Sigma_n)_{n\ge0}$ be, respectively, the extremal and sum processes for the sequence $Y$, i.e.

$$M_n := \max\{Y_i : i \le \lfloor n\rfloor\} \quad \text{and} \quad \Sigma_n := \sum_{i\le\lfloor n\rfloor} Y_i \,.$$

The key to our analysis of the trapping landscape is the fact that the extremal and sum processes for $Y$ have scaling limits that coincide; this can be considered as a functional limit extension of the classical result on the sum/max ratio of sequences of slowly varying random variables (see Theorem 3.1).

**Proposition 3.8** (Functional limit theorems for the extremal and sum process; see [50, 53])**.** *As $n \to \infty$,*

$$\left(\frac{1}{n}L(\Sigma_{nt})\right)_{t\ge0} \overset{J_1}{\Rightarrow} (m_t)_{t\ge0} \quad \textit{and} \quad \left(\frac{1}{n}L(M_{nt})\right)_{t\ge0} \overset{J_1}{\Rightarrow} (m_t)_{t\ge0}\ , \tag{3.1}$$

*where $m := (m_t)_{t\ge0}$ denotes the extremal process*

$$m_t := \max\{v_i : 0 \le x_i \le t\}$$

*for the set $\mathcal{T} := (x_i, v_i)_{i\in\mathbb{N}}$, an inhomogeneous Poisson point process on $\mathbb{R}^+ \times \mathbb{R}^+$ with intensity measure $x^{-2}dx\,dv$, and $\overset{J_1}{\Rightarrow}$ denotes weak convergence in the Skorokhod space $D(\mathbb{R}^+)$ equipped with the $J_1$ topology; see [70] or Appendix 3.5 for a description. Further, the convergence in equation* (3.1) *occurs jointly, in the sense that*

$$\left(\frac{1}{n}L(\Sigma_{nt}) - \frac{1}{n}L(M_{nt})\right)_{t\ge0} \overset{J_1}{\Rightarrow} (0)_{t\ge0}\ . \tag{3.2}$$

The first statement in equation (3.1) is the main result of [50]; the second statement may be derived by applying [53, Theorems 2.1, 3.2] to the sequences $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ of i.i.d. unit-mean exponential random variables, and then transforming using the inverse transform theorem,

$$Y_n \overset{d}{=} L^{-1}(\exp\{\psi_n\})\,.$$

Denoting by $M_n^\psi$ the extremal processes for $\psi$, i.e. $M_n^\psi := \max\{\psi_i : i \le \lfloor n\rfloor\}$, this yields

$$M_{nt}^\psi - \log n \overset{J_1}{\Rightarrow} \log m_t \tag{3.3}$$

(where we note that the limit process $(m(t))_{t\ge0}$ in [53] is $(\log m_t)_{t\ge0}$ in our notation). For the final step, we need to be slightly careful since $L(Y_i)$ is not identically distributed as $e^{\psi_i}$

in general. However, we do have that

$$\frac{1}{n}L\left(M_{nt}\right) \stackrel{d}{=} \frac{L\left(L^{-1}\left(e^{M^{\psi}_{nt}}\right)\right)}{e^{M^{\psi}_{nt}}} \times \frac{e^{M^{\psi}_{nt}}}{n}$$

as processes. By (3.3), the second product converges in distribution to $(m_t)_{t\geq 0}$ in the $J_1$ topology. As for the effect of multiplying by the first term, this can then be controlled using the facts that $e^{M^{\psi}_{n}} \to \infty$ almost surely, $L(L^{-1}(x)) \sim x$ as $x \to \infty$ (since $L$ is slowly varying), and also, for any $\varepsilon > 0$,

$$\lim_{t\to 0}\lim_{n\to\infty} \mathbf{P}\left(n^{-1}L\left(M_{nt}\right) \geq \varepsilon\right) \leq \lim_{t\to 0}\lim_{n\to\infty} \mathbf{P}\left(n^{-1}L\left(\Sigma_{nt}\right) \geq \varepsilon\right) = 0$$

by the first part of the proposition.

To establish the joint convergence in equation (3.2) we shall need the following two additional lemmas.

**Lemma 3.9** (Monotonicity implies joint convergence)**.** *Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be sequences of random variable defined on a common probability space, and suppose that, as $n \to \infty$,*

$$X_n \Rightarrow Z \quad \textit{and} \quad Y_n \Rightarrow Z \quad \textit{in law}$$

*for some limiting random variable $Z$. Assume further that $X_n \geq Y_n$ for each $n$. Then, as $n \to \infty$,*

$$X_n - Y_n \Rightarrow 0 \quad \textit{in law}.$$

*Proof.* For each $n \in \mathbb{N}$, $y \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\begin{aligned}
\mathbf{P}(Y_n > y) &= \mathbf{P}(Y_n > y,\, X_n - Y_n \geq \varepsilon) + \mathbf{P}(Y_n > y,\, |X_n - Y_n| < \varepsilon) \\
&\leq \mathbf{P}(X_n > y+\varepsilon,\, X_n - Y_n \geq \varepsilon) + \mathbf{P}(X_n > y,\, |X_n - Y_n| < \varepsilon) \\
&= \mathbf{P}(X_n > y+\varepsilon) + \mathbf{P}(X_n \in (y, y+\varepsilon],\, |X_n - Y_n| < \varepsilon)\,,
\end{aligned}$$

and so

$$\mathbf{P}(X_n \in (y, y+\varepsilon],\, |X_n - Y_n| < \varepsilon) \geq \mathbf{P}(Y_n > y) - \mathbf{P}(X_n > y+\varepsilon) \stackrel{n\to\infty}{\to} \mathbf{P}(Z \in (y, y+\varepsilon])\,.$$

To complete the proof note that, for arbitrary $C > 0$, we can cover $(-C, C]$ with a finite number of disjoint regions $(y_i, y_i + \varepsilon]$. Summing over these, we have that, for each $C > 0$,

$$\liminf_{n\to\infty} \mathbf{P}(|X_n - Y_n| < \varepsilon) \geq \liminf_{n\to\infty} \mathbf{P}(X_n \in (-C, C],\, |X_n - Y_n| < \varepsilon) \geq \mathbf{P}(Z \in (-C, C])\,.$$

Taking $C \to \infty$ establishes the result. □

**Lemma 3.10.** *For $\varepsilon > \varepsilon' > 0$ and non-decreasing functions $x_t, y_t \to \infty$, there exists a $t' > 0$ such that*

$$\{t > t' : L(x_t) > (1+\varepsilon)L(y_t)\} \subseteq \{t > t' : L(x_t - y_t) > (1+\varepsilon')L(y_t)\,.$$

*Proof.* By the slow variation property (1.14), as $t \to \infty$ eventually

$$(1+\varepsilon)L(y_t) > L(2y_t)\,.$$

Hence if $L(x_t) > (1+\varepsilon)L(y_t)$, then $x_t > 2y_t$ eventually since $L$ is non-decreasing, and so $x_t - y_t > x_t/2$. This means that

$$L(x_t - y_t) \geq L(x_t/2) > (1-\varepsilon'')L(x_t)$$

eventually for any $\varepsilon'' > 0$, again by the slow variation property (1.14). The claim then follows by choosing $\varepsilon''$ such that $(1-\varepsilon'')(1+\varepsilon) > (1+\varepsilon')$. □

We can now establish the joint convergence in equation (3.2).

*Proof.* By applying Lemma 3.9 component-wise, the convergence in (3.1) implies that the finite-dimensional distributions of

$$\left(\frac{1}{n}L(\Sigma_{nt}) - \frac{1}{n}L(M_{nt})\right)_{t\geq 0}$$

converge in law to the zero random vector; it remains to establish tightness in the topology of uniform convergence on compact sets. Using the criteria of [48, Proposition VI.3.26], we need only check that, for arbitrary $0 < \delta < T$ and $\varepsilon > 0$,

$$\lim_{C\to\infty}\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[\delta,T]}\left|\frac{1}{n}L(\Sigma_{nt}) - \frac{1}{n}L(M_{nt})\right| < C\right) = 0$$

and

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[\delta,T]}\sup_{u,v\in[t,t+\delta]}\left|\left(\frac{1}{n}L(\Sigma_{nu}) - \frac{1}{n}L(\Sigma_{nv})\right) - \left(\frac{1}{n}L(M_{nu}) - \frac{1}{n}L(M_{nv})\right)\right| > \varepsilon\right) = 0$$

both hold. The first criterion is trivially satisfied by the convergence in (3.1). For the second, since both $(n^{-1}L(\Sigma_{nt}))_{t\geq 0}$ and $(n^{-1}L(M_{nt}))_{t\geq 0}$ converge in the $J_1$ topology to the pure-jump process $m_t$, it is sufficient to show that the (finite) set of non-negligible jumps of

$$\left(n^{-1}L(\Sigma_{nt})\right)_{t\in[\delta,T]} \quad \text{and} \quad \left(n^{-1}L(M_{nt})\right)_{t\in[\delta,T]}$$

are eventually matched exactly, i.e. that

$$\lim_{n\to\infty} \mathbf{P}\left(\sup_{t\in[\delta,T]}\left|\left(\frac{1}{n}L(\Sigma_{nt}) - \frac{1}{n}L(\Sigma_{nt-})\right) - \left(\frac{1}{n}L(M_{nt}) - \frac{1}{n}L(M_{nt-})\right)\right| > \varepsilon\right) = 0\,.$$

Observe that, by the respective definitions of $M$ and $S$,

$$M_{nt} \geq \Sigma_{nt} - \Sigma_{nt-} \quad \text{and} \quad \Sigma_{nt-} \geq M_{nt-}\,.$$

Together with Lemma 3.10 and the fact that $L$ is non-decreasing, this implies that, for any

$\varepsilon > \varepsilon' > 0$, as $n \to \infty$ eventually we have the set inclusion

$$\left\{t \in [\delta, T] : \frac{1}{n}L(\Sigma_{nt}) > (1+\varepsilon)\frac{1}{n}L(\Sigma_{nt^-})\right\} \subseteq \left\{t \in [\delta, T] : \frac{1}{n}L(M_{nt}) > (1+\varepsilon')\frac{1}{n}L(M_{nt^-})\right\}.$$

Since the jumps are bounded in probability, the non-negligible jumps of $(n^{-1}L(\Sigma_{nt}))_{t\in[\delta,T]}$ are eventually matched exactly by non-negligible jumps of $(n^{-1}L(M_{nt}))_{t\in[\delta,T]}$. To complete the proof, note that if $t' > 0$ denotes the time of the first non-negligible jump in $(n^{-1}L(M_{nt}))_{t\in[\delta,T]}$ that is unmatched by a jump in $(n^{-1}L(\Sigma_{nt}))_{t\in[\delta,T]}$, then as $n \to \infty$ we would eventually have $M_{nt'} > \Sigma_{nt'}$, which is a contradiction. $\square$

Finally, we study some properties of the jump-set $\mathcal{J}$ of the maximum process $M$, i.e.

$$\mathcal{J} := \{n : M_n \neq M_{n^-}\} \subseteq \mathbb{N}\,.$$

We are interested in particular in the spacing of this jump-set. Note that this result does not depend on the fact that the tails are slowly varying.

**Proposition 3.11** (Jump-set spacing)**.** *For each $C > 0$, as $n \to \infty$,*

$$|\mathcal{J} \cap (n/C, nC]\,| \quad \textit{and} \quad \frac{\mathrm{sep}(\mathcal{J} \cap (n/C, nC])}{n}$$

*are respectively bounded above and bounded below in probability, recalling that* $\mathrm{sep}(\mathcal{S})$ *denotes the separation of the set* $\mathcal{S}$

$$\mathrm{sep}(\mathcal{S}) := \min_{\substack{i,j\in\mathcal{S}\\ i\neq j}} |i-j|\,.$$

*Proof.* Let $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. unit-mean exponential random variables, with $M_n^\psi$ the extremal processes for $\psi$, i.e. $M_n^\psi := \max\{\psi_i : i \leq \lfloor n \rfloor\}$, and $\mathcal{J}^\psi$ its associated jump-set, i.e. $\mathcal{J}^\psi := \{n : M_n^\psi \neq M_{n^-}^\psi\}$. Denote by $(k_i^n)_{i\geq 1}$ the ordered list of indices $i > n/C$ such that $\psi_i \geq M_{n/C}^\psi$ and abbreviate $K_n := |\{i : k_i^n \leq nC\}|$. Clearly we have that

$$|\mathcal{J}^\psi \cap (n/C, nC]\,| \leq K_n$$

and

$$\mathrm{sep}(\mathcal{J}^\psi \cap (n/C, nC]) \geq \mathrm{sep}(k_i^n : i = 1, 2, \ldots, K_n)\,.$$

Moreover, by the inverse transform theorem

$$Y_n \stackrel{d}{=} L^{-1}(\exp\{\psi_n\})\,,$$

and so there is a coupling of the sequences $Y$ and $\psi$ such that, for all $n$,

$$M_n \neq M_{n^-} \implies M_n^\psi \neq M_{n^-}^\psi\,.$$

Therefore, for this coupling, $\mathcal{J} \subseteq \mathcal{J}^\psi$, and so it is sufficient to prove that

$$K_n \quad \text{and} \quad \frac{\mathrm{sep}(k_i : i = 1, 2, \ldots, K_n)}{n}$$

are respectively bounded above and bounded below in probability.

For the first, note that, conditionally on $M^{\psi}_{n/C}$, the random variable $K_n$ is distributed as

$$\mathrm{Bi}\left(\lfloor nC\rfloor, \exp\{-M^{\psi}_{n/C}\}\right) , \tag{3.4}$$

where $\mathrm{Bi}(n,p)$ denotes a binomial random variable with mean $np$ and variance $np(1-p)$. It is a classical result of extreme-value theory (see, for example, [61]) that, as $n \to \infty$,

$$M^{\psi}_{n} - \log n \Rightarrow \mathcal{G} \quad \text{in law} ,$$

where $\mathcal{G}$ denotes the Gumbel distribution, and so

$$n \exp\{-M^{\psi}_{n/C}\} \tag{3.5}$$

is bounded above in probability. Together with equation (3.4) and Markov's inequality, this implies that $K_n$ is also bounded above in probability.

For the second, note that, conditionally on $M^{\psi}_{n/C}$, for any $i \geq 1$, the distance $k^n_{i+1} - k^n_i$ is distributed as $\mathrm{Geo}(\exp\{-M^{\psi}_{n/C}\})$, where $\mathrm{Geo}(p)$ denotes a geometric random variable (with support $1, 2, \dots$). Again, by equation (3.5), this implies that $n^{-1}(k^n_{i+1} - k^n_i)$ is bounded below in probability. By applying a union bound (conditionally on $K_n$, which we already know is bounded above in probability), we get the result. $\square$

## 3.3 Two-site localisation in the BTM with slowly varying traps

In this section we prove the two-site localisation of the BTM with slowly varying traps (Theorem 1.12). In particular, we prove that the BTM eventually localises on the set $\Gamma_t$ consisting of the first trap on the positive and negative half-line respectively with depth exceeding the level $\ell_t$. This allows us to establish the ancillary properties of the localisation set in Theorems 1.13 and 1.14. Finally, we prove the simplified description of the localisation result (Theorem 1.17) available under Assumption 1.

The rest of the section proceeds as follows: we first prove our results under the assumption that certain favourable inhomogeneity properties of the trapping landscape hold; we then analyse the trapping landscape, showing that these favourable inhomogeneity properties hold with overwhelming probability.

### 3.3.1 Two-site localisation under certain favourable inhomogeneity assumptions

In order to define the favourable inhomogeneity properties, we shall need an auxiliary function $h_t$ that tends to infinity (i.e. such that $h_t \to \infty$ as $t \to \infty$). We shall think of $h_t$ as being arbitrarily slowly growing, and indeed we shall require $h_t$ to satisfy $h_t^2 = o(r_t)$ as $t \to \infty$.

Further, define the quantities

$$S_t := \sum_{Z_t^{(2)} < z < Z_t^{(1)}} \sigma(z)\,, \qquad d_t := \max_{z \in \Gamma_t} |z| \qquad \text{and} \qquad m_t := \min_{z \in \Gamma_t} \sigma(z)\,,$$

and the $h$-dependent quantity

$$\bar{S}_t := \sum_{i=1,2} \sum_{1 \le |z - Z_t^{(i)}| < r_t/h_t} \sigma(z)\,.$$

We may now define the favourable inhomogeneity properties as the ($\mathbf{P}$-measurable, $h$-dependent) events

$$\mathcal{A}_t^h := \left\{ S_t d_t < \frac{t}{h_t} \right\}, \quad \mathcal{B}_t^h := \left\{ m_t > \frac{t h_t^2}{r_t} \right\} \quad \text{and} \quad \mathcal{C}_t^h := \left\{ \bar{S}_t < \frac{\ell_t}{h_t} \right\}.$$

In Section 3.3.2, we show that we can choose $h_t$ growing sufficiently slowly that, as $t \to \infty$,

$$\mathbf{P}\left( \mathcal{A}_t^h, \mathcal{B}_t^h, \mathcal{C}_t^h \right) \to 1\,. \tag{3.6}$$

For now we work under the assumption that (3.6) holds for a certain choice of $h_t$, showing how the main Theorems 1.12–1.14 follow from this assumption. The proof proceeds in the following stepsThis argument follows a similar structure to that used to show localisation of one-dimensional random walk in random environments in [72, Theorem 2.5.3].:

1. For a large fixed $t$, we show that the BTM is overwhelmingly likely to have hit the set $\Gamma_t$ before time $t$;

2. Assuming that the event in (1) occurs, let $\bar{y} \in \Gamma_t$ denote the first site in $\Gamma_t$ hit by the BTM. We then show that the BTM is very unlikely to have exited a certain narrow region $I_t^{\bar{y}}$ around the site $\bar{y}$ by time $t$;

3. Assuming that the events in (1) and (2) both occur, we use the equilibrium distribution of the BTM on an interval with periodic boundary conditions to show that the BTM is overwhelmingly likely to be located *at* the site $\bar{y}$ at time $t$, establishing Theorem 1.12.

4. Remark that (1)–(3) above imply that the BTM is overwhelmingly likely to be located at the site in $\Gamma_t$ that it first hits. To finish the proof of Theorems 1.13 and 1.14, we use simple properties of random walks and some basic extreme value theory.

**Step 1: Hitting the localisation set.** Fix a scaling function $h_t$ such that (3.6) holds. For each trapping landscape $\sigma$ and time $t > 0$, consider the BTM $(X_s)_{s \ge 0}$ in the trapping landscape $\sigma$ and define the hitting time

$$\tau_t^1 := \inf\{s \ge 0 : X_s \in \Gamma_t\}\,.$$

**Proposition 3.12.** *Assume $\mathcal{A}_t^h$ holds. As $t \to \infty$,*

$$\mathbb{P}(\tau_t^1 \le t) \to 1\,.$$

*Proof.* Let $Q_z$ denote the discrete local time at $z$ of the geometric path induced by $\{X_s : s < \tau_t^1\}$, and define

$$\bar{\Gamma}_t := \{z \in \mathbb{Z} : Z_t^{(2)} < z < Z_t^{(1)}\} .$$

Considering $\tau_t^1$ as the sum of waiting times along the geometric path induced by $\{X_s : s < \tau_t^1\}$, we have that

$$\tau_t^1 \overset{d}{=} \sum_{z \in \bar{\Gamma}_t} \mathrm{Gam}\left(Q_z, \sigma(z)\right) \prec \sum_{z \in \bar{\Gamma}_t} \mathrm{Gam}\left(\max_z Q_z, \sigma(z)\right) ,$$

where each $\mathrm{Gam}(n, \mu)$ is an independent gamma random variable with mean $n\mu$ and variance $n\mu^2$. Remark that, by the definition of $d_t$,

$$\frac{\max_z Q_z}{d_t} \prec \frac{\max_z \mathcal{L}_z^{d_t}}{d_t} , \tag{3.7}$$

and recall that, by Proposition 3.2, the right hand side of (3.7) is bounded above in probability. Since $h_t \to \infty$, this implies that, as $t \to \infty$,

$$\mathbb{P}\left(\tau_t^1 < \sum_{z \in \bar{\Gamma}_t} \mathrm{Gam}(d_t h_t/2, \sigma(z))\right) \to 1 . \tag{3.8}$$

Note that the factor of a half in the above equation is included purely for convenience in what follows. By Chebyshev's inequality,

$$\mathbb{P}\left(\mathrm{Gam}(n, \mu) \geq 2n\mu\right) \leq n^{-1} ,$$

and so, using the fact that

$$\mathbb{P}\left(\sum_i Y_i \geq \sum_i y_i\right) \leq \sum_i \mathbb{P}\left(Y_i \geq y_i\right)$$

for an arbitrary collection of random variables $\{Y_i\}_{i \in \mathbb{N}}$ and real numbers $\{y_i\}_{i \in \mathbb{N}}$, and also the fact that $|\bar{\Gamma}_t| < 2d_t$ by definition, we have

$$\begin{aligned} \mathbb{P}\Bigg(\sum_{z \in \bar{\Gamma}_t} \mathrm{Gam}(d_t h_t/2, \sigma(z)) \geq S_t d_t h_t\Bigg) &\leq \sum_{z \in \bar{\Gamma}_t} \mathbb{P}\Big(\mathrm{Gam}(d_t h_t/2, \sigma(z)) \geq \sigma(z) d_t h_t\Big) \\ &\leq \frac{2|\bar{\Gamma}_t|}{d_t h_t} < \frac{4}{h_t} \to 0 \quad \text{as } t \to \infty . \end{aligned} \tag{3.9}$$

Since $S_t d_t h_t < t$ on $\mathcal{A}_t^h$, combining equations (3.8) and (3.9) yields the result. □

**Step 2: Confining to a narrow region.** Define the random site $\bar{y} := X_{\tau_t^1} \in \Gamma_t$, a narrow region around $\bar{y}$

$$I_t^{\bar{y}} := \{z \in \mathbb{Z} : |z - \bar{y}| < r_t/h_t\} ,$$

and a second, strictly-later hitting time

$$\tau_t^2 := \inf\{s > \tau_t^1 : X_s \notin I_t^{\bar{y}}\} .$$

**Proposition 3.13.** *Assume $\mathcal{B}_t^h$ holds. As $t \to \infty$,*

$$\mathbb{P}(\tau_t^2 > t) \to 1\,.$$

*Proof.* Consider that

$$\mathbb{P}(\tau_t^2 > t) \geq \mathbb{P}(\tau_t^2 - \tau_t^1 > t)\,,$$

so it is sufficient to prove that the latter probability converges to one. Let $Q_0$ denote the discrete local time at $\bar{y}$ of the geometric path induced by $\{X_s : \tau_1^t \leq s < \tau_2^t\}$. Following the same reasoning as in the proof of Proposition 3.12, we have that

$$\tau_t^2 - \tau_t^1 \stackrel{d}{=} \sum_{z \in I_t^{\bar{y}}} \mathrm{Gam}(Q_z, \sigma(z)) \succ \mathrm{Gam}(Q_0, \sigma(\bar{y})) \succ \mathrm{Gam}(Q_0, m_t)\,.$$

By Proposition 3.2, as $t \to \infty$,

$$\frac{Q_0}{r_t/h_t} \stackrel{d}{=} \frac{\mathcal{L}_0^{r_t/h_t}}{r_t/h_t}$$

is bounded away from zero in probability, and so eventually

$$\mathbb{P}\left(\tau_t^2 - \tau_t^1 > \mathrm{Gam}\left(\frac{2r_t}{h_t^2}, m_t\right)\right) \to 1\,. \tag{3.10}$$

Note that the factor of two here is again included for convenience in what follows. By Chebyshev's inequality,

$$\mathbb{P}\left(\mathrm{Gam}(n, \mu) \leq n\mu/2\right) \leq 4n^{-1}\,,$$

and so, using the fact that $h_t^2 = o(r_t)$ as $t \to \infty$,

$$\mathbb{P}\left(\mathrm{Gam}\left(\frac{2r_t}{h_t^2}, m_t\right) < \frac{r_t m_t}{h_t^2}\right) < 2h_t^2/r_t \to 0 \quad \text{as } t \to \infty\,. \tag{3.11}$$

Since $r_t m_t/h_t^2 > t$ on $\mathcal{B}_t^h$, the result follows from combining (3.10) and (3.11). $\square$

**Step 3: Two-site localisation.** Introduce a new random process $(\hat{X}_s^t)_{s\geq 0}$ on the same probability space as $(X_s)_{s\geq 0}$ which is: (i) coupled to $(X_s)_{s\geq 0}$ until time $\tau_t^1$; and (ii) thereafter evolves as the BTM on $I_t^{\bar{y}}$ with periodic boundary conditions. In other words, $(\hat{X}^t_{\tau_t^1+s})_{s\geq 0}$ is a continuous-time Markov chain on $I_t^{\bar{y}}$, based at $\bar{y} := X_{\tau_t^1}$ by definition, with transition rates

$$w_{z\to y} := \begin{cases} \frac{1}{2\sigma(z)}, & \text{if } z \stackrel{*}{\sim} y\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

where $z \stackrel{*}{\sim} y$ denotes that $z$ and $y$ are either neighbours in $I_t^{\bar{y}}$ *or* that $z$ and $y$ are the two end points of $I_t^{\bar{y}}$.

**Proposition 3.14.** *Assume $\mathcal{C}_t^h$ holds. As $t \to \infty$,*

$$\mathbb{P}(\hat{X}_t^t = \bar{y}\,|\,\tau_t^1 \leq t) \to 1\,.$$

*Proof.* Remark first that the BTM defined on any locally-finite graph satisfies the detailed balance condition. Hence we can apply Proposition 3.7 to the irreducible, finite-state Markov chain $\{\hat{X}^t_{\tau^1_t+s}\}_{s\geq 0}$. We conclude that, as $s \to \infty$,

$$\mathbb{P}(\hat{X}^t_{\tau^1_t+s} = \bar{y}) \downarrow \pi(\bar{y}) \quad \text{monotonically}\,, \tag{3.12}$$

where $\pi$ is the equilibrium distribution of the BTM on $I^{\bar{y}}_t$ with periodic boundary conditions. We claim that this equilibrium distribution is proportional to the trapping landscape $\sigma$. To see why note that, by the definition of the BTM, $\pi$ satisfies

$$(\Delta\boldsymbol{\sigma}^{-1})\pi = \Delta(\boldsymbol{\sigma}^{-1}\pi) = \mathbf{0}\,,$$

where $\Delta$ is the discrete Laplacian on $I^{\bar{y}}_t$ with periodic boundary conditions, $\boldsymbol{\sigma}$ denotes point-wise multiplication by $\sigma$, and $\mathbf{0}$ denotes the zero vector. Since the equilibrium distribution of $\Delta$ is uniform, the vector $\boldsymbol{\sigma}^{-1}\pi$ is also uniform, and the claim follows.

As $\pi$ is proportional to $\sigma$, this implies that

$$\pi(z) = \frac{\sigma(z)}{\sigma(\bar{y})}\pi(\bar{y}) \leq \frac{\sigma(z)}{\sigma(\bar{y})}$$

for all $z \in I^{\bar{y}}_t$. Since, on the event $\mathcal{C}^h_t$,

$$\sum_{z\in I^{\bar{y}}_t\setminus\{\bar{y}\}} \sigma(z) \leq \bar{S}_t < \frac{\ell_t}{h_t} < \frac{\sigma(\bar{y})}{h_t} = o(\sigma(\bar{y})) \quad \text{as } t \to \infty\,,$$

we have that, as $t \to \infty$,

$$\sum_{z\in I^{\bar{y}}_t\setminus\{\bar{y}\}} \pi(z) \to 0\,. \tag{3.13}$$

Combining equations (3.12) and (3.13) gives the result. $\square$

*Proof of Theorem 1.12, assuming* (3.6) *holds.* We work on the event that each of $\mathcal{A}^h_t$, $\mathcal{B}^h_t$ and $\mathcal{C}^h_t$ holds, which is sufficient by (3.6). Note that, by the definition of $\{\hat{X}^t_s\}_{s\geq 0}$,

$$\mathbb{P}(\hat{X}^t_t \,|\, \tau^1_t \leq t < \tau^2_t) = \mathbb{P}(X_t \,|\, \tau^1_t \leq t < \tau^2_t)$$

Combining this with Propositions 3.12–3.14, as $t \to \infty$,

$$\mathbb{P}(X_t = \bar{y}) \to 1\,, \tag{3.14}$$

and we have the result. $\square$

**Step 4: Completion of the proof of Theorems 1.13 and 1.14.**

*Proof of Theorem 1.13, assuming* (3.6) *holds.* Again we work on the event that each of $\mathcal{A}^h_t$, $\mathcal{B}^h_t$ and $\mathcal{C}^h_t$ holds, which is sufficient by (3.6). Considering the BTM as a time-changed simple discrete-time random walk, it follows from Proposition 3.3 that

$$\mathbb{P}(\bar{y} = Z^{(1)}_t) = \frac{|Z^{(2)}_t|}{\sum_{z\in\Gamma_t}|z|}.$$

Combining with equation (3.14) completes the proof. □

*Proof of Theorem 1.14.* Each $\sigma(z)$ exceeds the level $l_t$ with probability

$$\mathbf{P}(\sigma(0) > l_t) = 1/L(l_t) = 1/r_t \,.$$

Hence for each $x, y > 0$, as $t \to \infty$,

$$\mathbf{P}(Z_t^{(1)} > xr_t, -Z_t^{(2)} > yr_t) = (1 - 1/r_t)^{\lfloor xr_t \rfloor + \lfloor yr_t \rfloor} \sim e^{-x-y} \,,$$

which proves the result. □

#### Simplified description of two-site localisation under the stronger assumption

We finish this section by proving Theorem 1.17, assuming (3.6) holds. Note that it is sufficient to prove that, under Assumption 1, as $t \to \infty$,

$$\mathbf{P}(\sigma(z) \in (\ell_t, t]) \to 0 \,,$$

since then $\hat{Z}_t^{(i)} = Z_t^{(i)}$ with overwhelming probability, and so Theorem 1.17 follows directly from Theorem 1.13. Note that this is equivalent to showing that, as $t \to \infty$, $L(t) \sim L(\ell_t)$.

To prove this, first note that Assumption 1 implies the slightly weaker assumption[2] that, as $u \to \infty$,

$$\frac{L(uL(u))}{L(u)} \to 1 \,.$$

Applying this to $\ell_t$ we deduce that $L(\ell_t L(\ell_t)) \sim L(\ell_t)$. On the other hand, by the definition of $\ell_t$ at (1.16), $L(\ell_t L(\ell_t)) \geq L(t) \geq L(\ell_t)$. Hence, combining the above, $L(t) \sim L(\ell_t)$, as required.

### 3.3.2 The trapping landscape is sufficiently inhomogeneous

In this section we prove that the trapping landscape $\sigma$ is sufficiently inhomogeneous, in the sense that the favourable events $\mathcal{A}_t^h, \mathcal{B}_t^h$ and $\mathcal{C}_t^h$ all hold eventually with overwhelming probability for a suitable choice of the slowly growing scaling function $h_t$. In other words, we prove that (3.6) holds. This analysis relies crucially on the fundamental properties of sequences of i.i.d. random variables that were established in Section 3.2.

#### Specifying the scaling function

Let us first specify an appropriate choice for $h_t$. The main condition we require is that $h_t \to \infty$ slowly enough so that, as $t \to \infty$,

$$L(\ell_t/h_t^3) > L(\ell_t)(1 - 1/h_t) \quad \text{and} \quad L(\ell_t h_t^3) < L(\ell_t)(1 + 1/h_t) \tag{3.15}$$

eventually, remarking that such a choice is possible by the slow variation property (1.14). For completeness, we construct an explicit scaling function $h_t$ satisfying (3.15). Define an

[2]The stronger form of the assumption is necessary for Theorem 1.18.

arbitrary, positive, increasing sequence $c := (c_i)_{i \in \mathbb{N}} \uparrow \infty$, and denote, for each $u > 0$,

$$f_t(u) := L(\ell_t u)/L(\ell_t)\,.$$

By the slow variation property (1.14), for each $u$ we know that $f_t(u) \to 1$ as $t \to \infty$. This means that, for each $i \in \mathbb{N}$, there exists a $t_i > 0$ such that

$$f_t(c_i^{-3}) > 1 - 1/c_i \quad \text{and} \quad f_t(c_i^3) < 1 + 1/c_i \quad \text{for all } t \geq t_i\,.$$

So we can simply define $h_t$, with increments only on the set $\{t_i\}_{i \in \mathbb{N}}$, satisfying $h_{t_i} := c_i$; it is easy to check that $h_t$ satisfies equation (3.15). Recall also that we imposed the condition that $h_t^2 = o(r_t)$ in Section 3.3. To construct a scaling function $h_t$ that satisfies these two conditions simultaneously, simply take the minimum of scaling functions that satisfy each separately.

**Properties of sequences of slowly varying random variables**

We now extract consequences of the results that were established in Section 3.2.2 above for sequences of i.i.d. slowly varying random variables. In particular, recall the definition of $Y, M$ and $\Sigma$ from that section, and for a level $l > 0$, define

$$n_l := \min\{n \in \mathbb{N} : M_n > l\} \quad \text{and} \quad s_l := \Sigma_{n_l^-} = \sum_{i < n_l} Y_i$$

to be respectively the index of the first exceedance of the level $l$ and the sum of all previous terms in the sequence. Further, for any $h > 0$, define

$$\bar{s}_l^h := \sum_{n : 1 \leq |n - n_l| < L(l)/h} Y_n\,.$$

Our aim is to analyse the four random variables $n_{\ell_t}$, $s_{\ell_t}$, $Y_{n_{\ell_t}}$ and $\bar{s}_{\ell_t}^{h_t}$. To assist in this analysis, we first need a preliminary asymptotic for $\ell_t$.

**Lemma 3.15** (Preliminary asymptotic for $\ell_t$)**.** *As $t \to \infty$,*

$$\ell_t \sim t/r_t\,.$$

*Proof.* From the definition of $\ell_t$ at (1.16), it holds that

$$\ell_t L(\ell_t^-) \leq t \leq \ell_t L(\ell_t)\,. \tag{3.16}$$

On the other hand, by the slow variation assumption (1.14), as $u \to \infty$,

$$L(u^-) \sim L(u)$$

which, combining with equation (3.16), gives the result. ∎

**Proposition 3.16** (Asymptotic law of the index of first exceedance)**.** *As $l \to \infty$,*

$$\frac{n_l}{L(l)} \Rightarrow \mathcal{E} \quad \textit{in law},$$

*where $\mathcal{E}$ is an exponential random variable with unit mean.*

*Proof.* Each $Y_i$ exceeds the level $l$ with probability

$$\mathbf{P}(\sigma(0) > l) = 1/L(l)\,.$$

Hence, for each $x > 0$, as $l \to \infty$,

$$\mathbf{P}(n_l > xL(l)) = (1 - 1/L(l))^{\lfloor xL(l) \rfloor} \to e^{-x}\,.$$

□

**Proposition 3.17** (Upper bound on sum prior to first exceedance)**.** *As $t \to \infty$,*

$$\mathbf{P}\left(s_{\ell_t} < \frac{t}{2r_t h_t^2}\right) \to 1\,.$$

*Proof.* By the joint scaling limits for $M$ and $\Sigma$ in Proposition 3.8, as $l \to \infty$,

$$L(s_l)/L(l)$$

converges in law to a certain $(0,1)$-valued random variable. Hence, as $t \to \infty$,

$$\mathbf{P}\left(L(s_{\ell_t}) < L(\ell_t)(1 - 1/h_t)\right) \to 1\,.$$

Combining with the first statement in equation (3.15) and the fact that $L$ is non-decreasing yields, as $t \to \infty$,

$$\mathbf{P}\left(s_{\ell_t} < \ell_t h_t^{-3}\right) \to 1.$$

Finally, applying Lemma 3.15 gives the result. □

**Proposition 3.18** (Lower bound on first exceedance)**.** *As $t \to \infty$,*

$$\mathbf{P}\left(Y_{n_{\ell_t}} > \frac{t h_t^2}{r_t}\right) \to 1\,.$$

*Proof.* By the scaling limit for $M$ in Proposition 3.8, as $l \to \infty$,

$$L(Y_{n_l})/L(l)$$

converges in law to a certain $(1,\infty)$-valued random variable. Hence, as $t \to \infty$,

$$\mathbf{P}\left(L(Y_{n_{\ell_t}}) > L(\ell_t)(1 + 1/h_t)\right) \to 1\,.$$

Combining with the second statement in equation (3.15), the fact that $L$ is non-decreasing, and Lemma 3.15 yields the result. □

**Proposition 3.19** (Bound on partial sum)**.** *As $t \to \infty$,*

$$\mathbf{P}\left( \bar{s}_{\ell_t}^{h_t} < \frac{\ell_t}{h_t^3} \right) \to 1 \,.$$

*Proof.* We first claim that $\bar{s}_{\ell_t}^{h_t}$ is stochastically dominated by $\Sigma_{2r_t/h_t}$. This is since

$$Y_i \overset{d}{=} \begin{cases} Y_1 \big| \{Y_1 \le \ell_t\} \ \prec \ Y_1, & \text{if } i < n_{\ell_t} \,, \\ Y_1, & \text{if } i > n_{\ell_t} , \end{cases}$$

where $Y_1 \big| \{Y_1 \le \ell_t\}$ denotes the random variable $Y_1$ conditioned on the event that $\{Y_1 \le \ell_t\}$, and moreover, for any $x > 0$ and $n \in \mathbb{N}$,

$$|\{i : 1 \le |i - n| < x\}| \le 2x.$$

By the scaling limit for $\Sigma$ in Proposition 3.8, as $l \to \infty$,

$$\frac{L(\Sigma_{2r_t/h_t})}{2r_t/h_t}$$

converges in law to a certain strictly-positive random variable. This implies that, as $t \to \infty$,

$$\mathbf{P}\left( L(\bar{s}_{\ell_t}^{h_t}) < r_t(1 - 1/h_t) \right) = \mathbf{P}\left( L(\bar{s}_{\ell_t}^{h_t}) < L(\ell_t)(1 - 1/h_t) \right) \to 1 \,.$$

Combining with the first statement in equation (3.15) and the fact that $L$ is non-decreasing yields the result. □

### The trapping landscape is sufficiently inhomogeneous

We are now in a position to prove that the events $\mathcal{A}_t^h, \mathcal{B}_t^h$ and $\mathcal{C}_t^h$ all hold eventually with overwhelming probability.

**Proposition 3.20.** *As $t \to \infty$,*

$$\mathbf{P}(\mathcal{A}_t^h) \to 1.$$

*Proof.* Applying Proposition 3.16 to the sequences $\{\sigma(z)\}_{z \in \mathbb{Z}^+}$ and $\{\sigma(z)\}_{z \in \mathbb{Z}^- \setminus \{0\}}$ we have that, as $t \to \infty$,

$$\mathbf{P}\left(d_t < r_t h_t\right) \to 1 \,. \tag{3.17}$$

Similarly, applying Proposition 3.17 to the same sequences, as $t \to \infty$,

$$\mathbf{P}\left(S_t < t/(r_t h_t^2)\right) \to 1 \,. \tag{3.18}$$

Combining equations (3.17) and (3.18) yields the result. □

**Proposition 3.21.** *As $t \to \infty$,*

$$\mathbf{P}(\mathcal{B}_t^h) \to 1 \,.$$

*Proof.* Similarly to the above, apply Proposition 3.18 to the sequences $\{\sigma(z)\}_{z\in\mathbb{Z}^+}$ and $\{\sigma(z)\}_{z\in\mathbb{Z}^-\setminus\{0\}}$. □

**Proposition 3.22.** *As $t \to \infty$,*

$$\mathbf{P}(\mathcal{C}_t^h) \to 1 .$$

*Proof.* By Proposition 3.16, as $t \to \infty$, neither of the sets

$$\{z : |z - Z_t^{(i)}| < r_t/h_t\}, \quad i = 1, 2$$

contains the origin with overwhelming probability. On this event, each of the sums

$$\sum_{0<|z-Z_t^{(i)}|<r_t/h_t} \sigma(z), \quad i = 1, 2$$

is distributed as an independent copy of the random variable $\bar{s}_{\ell_t}^{h_t}$ defined in Section 3.2.2. Applying Proposition 3.19 yields the result. □

## 3.4 Functional limit theorem for the BTM with slowly varying traps

In this section we prove the functional limit theorem for the BTM with slowly varying traps (Theorem 1.16). We also prove the simplified version of this limit theorem under Assumption 1 (Theorem 1.18). Finally, we make precise the sense in which the scaling limit of the BTM can be considered as the 'extremal FIN process', that is, the natural analogue of the FIN diffusion with parameter $\alpha \in (0, 1)$ in the limiting case $\alpha = 0$. Note that throughout this section we will work under the annealed law, denoted by $P$.

Let us begin by recalling our candidate for the scaling limit of the BTM, which we described in Chapter 1 as a time-changed (or *subordinated*) standard Brownian motion. Motivating this description, and key to the proof of our functional limit theorem, is the observation that the BTM can also be expressed as a time-changed simple random walk, where the time-change depends on the realisation of the underlying random walk. To see this, let $D = \{D_i\}_{i\in\mathbb{N}}$ be a SRW on the integers and let $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ be a collection of i.i.d. unit-mean exponential random variables, with $D$, $\psi$ and $\sigma$ independent. Define an $D$-dependent *clock process* $A = (A_n)_{n\geq 0}$ by setting

$$A_n := \sum_{i\leq\lfloor n\rfloor} \psi_i \, \sigma(D_i) ,$$

and let $I^D = (I_t^D)_{t\geq 0}$ be its right-continuous inverse, defined by

$$I_t^D := \inf\{n : A_n^D > t\} .$$

It is not hard to see that, under the annealed law, the BTM has an identical distribution to $D_{I^D}$. In other words, the BTM may equivalently be defined via a subordination of a simple random walk $D$ by the $D$-dependent clock process $A$.

Recall that our construction of scaling limit of the BTM followed an analogous pattern, proceeding by first constructing a clock process $m^B$ that depends on the exploration of a Brownian motion $B$ (see (1.18)), and then defining the scaling limit as $B_{I^B}$, where $I^B$ is the right-continuous inverse of $m^B$. Indeed, in order to prove the functional limit theorem, the main step is to analyse the clock process $A$, establishing its convergence to the limit process $m^B$. The functional limit theorem for the BTM then follows by an application of the continuous mapping theorem. In particular, it will be sufficient to prove the following theorems.

**Theorem 3.23** (Functional limit theorem for the clock process)**.** *Under the annealed law, as $n \to \infty$,*

$$\left(\frac{1}{n} L\left(\frac{1}{n} A_{n^2 t}\right)\right)_{t \geq 0} \overset{M_1}{\Rightarrow} \left(m_t^B\right)_{t \geq 0}$$

*where $\overset{M_1}{\Rightarrow}$ denotes weak convergence in the $M_1$ topology.*

**Theorem 3.24** (Simplified functional limit theorem for the clock process)**.** *Suppose Assumption 1 holds. Under the annealed law, as $n \to \infty$,*

$$\left(\frac{1}{n} L(A_{n^2 t})\right)_{t \geq 0} \overset{J_1}{\Rightarrow} \left(m_t^B\right)_{t \geq 0}$$

*where $\overset{J_1}{\Rightarrow}$ denotes weak convergence in the $J_1$ topology.*

Before proceeding, let us explain why the $M_1$ and $L_{1,\mathrm{loc}}$ topologies are the appropriate topologies for the convergence of the clock process in Theorem 3.23 and the convergence of the BTM in Theorems 1.16 and 1.18 respectively, but that the $J_1$ topology suffices for the convergence in Theorem 3.24. Recall that the $M_1$ topology extends the usual $J_1$ topology by allowing jumps in the limit process to be matched by multiple jumps of lesser magnitude in the limiting processes, as long as they are essentially monotone and occur in negligible time in the limit (see Appendix 3.5). With regards to Theorem 3.23, the need for the $M_1$ topology arises because the total amount of time that the BTM spends at the deepest-visited trap is a result of multiple visits to the trap, all of which can contribute in a non-negligible way to the jump in the limit clock process. Convergence in the stronger $J_1$ topology would only hold if only the first visit to the trap made a non-negligible contribution in the limit; this is precisely what is guaranteed by Assumption 1, but is not true in general.

Recall also that the non-Skorohod $L_{1,\mathrm{loc}}$ topology extends both the $J_1$ and $M_1$ topologies by allowing excursions in the limiting processes that are not present in the limit process, as long as they are of negligible magnitude in the $L_1$ sense (which, in particular, is the case if they are of bounded size and occur in negligible time in the limit). In regards to Theorems 1.16 and 1.18, the need for the $L_{1,\mathrm{loc}}$ topology arises because, during the time that the BTM is based at the deepest-visited trap, the BTM makes repeated excursions away from this site. Although these occur in negligible time in the limit, they are of a magnitude comparable to the distance scale, and so prevent convergence in the stronger Skorohod topologies. See [36, 56] for other examples of trap model convergence results that make use of the $L_{1,\mathrm{loc}}$ topology (or close variants). We remark that the convergence in the $L_{1,\mathrm{loc}}$ topology is too weak to imply the convergence of some commonly used functionals of the sample paths of $X$, including $\inf_{t \in [0,T]} X_t$ and $\sup_{t \in [0,T]} X_t$.

Finally, we believe that the convergence in the $L_{1,\mathrm{loc}}$ topology in Theorems 1.16 and 1.18 can actually be mildly strengthened to convergence in a topology that allows for zero-time excursions of bounded size in the limiting processes, but only if they occur at jump-times of the limit process (c.f. the space $E$ in [70, Section 15.4]). That we expect convergence to hold in this stronger topology is essentially due to the highly non-linear rescaling of time in the limit; since this is an artefact of the rescaling rather than an intrinsic property of the processes, we choose not to pursue the additional technical complications necessary to prove such a result here.

### 3.4.1 Scaling limits for the clock process

In this section we prove the convergence of the clock process $A$ to the limit process $m^B$, both in its general form in Theorem 3.23 and in its simplified form under Assumption 5.4 in Theorem 3.24. Our strategy is to 'squeeze' the clock process $A$ between the $D$-explored extremal and sum processes defined respectively by

$$M_n^D := \max\left\{\sigma(z) : \min_{i\leq\lfloor n\rfloor} D_n \leq z \leq \max_{i\leq\lfloor n\rfloor} D_n\right\} \quad \text{and} \quad \Sigma_n^D := \sum_{z=\min_{i\leq\lfloor n\rfloor} D_n}^{\max_{i\leq\lfloor n\rfloor} D_n} \sigma(z)\,.$$

We then apply a general squeeze convergence result for the Skorohod $M_1$ topology that we state and prove in Appendix 3.5 to complete the proof.

Throughout this section, fix constants $T > \delta > 0$. For technical reasons, we will additionally define an auxiliary function $h_n \to \infty$ growing sufficiently slowly that

$$\lim_{n\to\infty} \frac{L(L^{-1}(n/h_n)/h_n)}{L(L^{-1}(n/h_n))} = \lim_{n\to\infty} \frac{L(L^{-1}(n/h_n)\,h_n)}{L(L^{-1}(n/h_n))} = 1\,, \tag{3.19}$$

remarking that such an $h_n$ is guaranteed to exist by the slow-variation property (1.14) and since $\lim_{n\to\infty} L^{-1}(n) = \infty$. For completeness, we give an explicit construction of an $h_n$ satisfying the left-hand side of equation (3.19); the construction of an $h_n$ that simultaneously satisfies the right-hand side of equation (3.19) is analogous. Define an arbitrary increasing sequence $c = (c_i)_{i\in\mathbb{N}} \to \infty$, and denote, for each $u > 0$ and each $n > 0$,

$$f^n(u) := \frac{L(L^{-1}(nu)u)}{L(L^{-1}(nu))}\,.$$

By the slow-variation property and since $\lim_{n\to\infty} L^{-1}(n) = \infty$, we have that $f^n(u) \to 1$ for each $u$. This means that, for each $i \in \mathbb{N}$, there exists an $n_i \in \mathbb{N}$ such that

$$|1 - f^n(1/c_i)| < 1/c_i \quad \text{for all } n \geq n_i\,.$$

So define $h_n$, with increments only on the set $\{n_i\}_{i\in\mathbb{N}}$, satisfying $h_{n_i} := c_i$.

Similarly, under Assumption 1, we will additionally require that $h_n$ satisfies

$$\lim_{n\to\infty} \frac{L(L^{-1}(n/h_n)/(nh_n))}{L(L^{-1}(n/h_n))} = \lim_{n\to\infty} \frac{L(L^{-1}(n/h_n)\,n\,h_n)}{L(L^{-1}(n/h_n))} = 1\,, \tag{3.20}$$

which is again guaranteed to exist under Assumption 1 by analogous reasoning.

### Extremal processes associated to the BTM

The first step is to convert our results on general extremal processes stated in Section 3.2 into equivalent results for the analogous processes associated with the simple random walk $D$, in particular the processes $M^D$ and $\Sigma^D$.

We begin by focussing on the jump-set $\mathcal{J}^D$ associated to $M^D$

$$\mathcal{J}^D := \{n : M_n^D \neq M_{n-}^D\} \subseteq \mathbb{N} .$$

As in Section 3.2, our results on $\mathcal{J}^D$ do not depend on the fact that the tails of $\sigma(0)$ are slowly varying. Abbreviating $N_n^D := |\mathcal{J}^D \cap (\delta n^2, \lceil Tn^2 \rceil]|$, let $(j_i^n)_{1 \le i \le N_n^D}$ be the elements of $\mathcal{J}^D \cap (\delta n^2, \lceil Tn^2 \rceil]$ arranged in increasing order. Further, set

$$j_{N_n^D+1}^n := \min\{i > \lceil n^2 T \rceil : i \in \mathcal{J}^D\} ,$$

and write $J^n := \{j_i^n : \ i = 1, \dots, N_n^D + 1\}$.

**Proposition 3.25** (Jump-set spacing for $M^D$)**.** *As $n \to \infty$,*

$$N_n^D \qquad \textit{and} \qquad \frac{\operatorname{sep}(J^n)}{n^2}$$

*are respectively bounded above and bounded below in probability under the annealed law.*

*Proof.* Let $Y = \{Y_i\}_{i \in \mathbb{N}}$ be the sequence given by rearranging the elements of the trapping landscape $\sigma$ into the order that the relevant sites are visited by $D$, and let $\mathcal{J}$ be defined as in Section 3.2 for the sequence $Y$. Further, denote by $(k_i)_{i \ge 1}$ the ordered list of elements in $\mathcal{J} \cap (n/C, \infty)$ and abbreviate $K_n := |\mathcal{J} \cap (n/C, nC]|$. Let $d_n$ be defined as in Section 3.2.1 for the simple random walk $D$. Note that, by Proposition 3.4, for any $\varepsilon > 0$ there exists a $C > 0$ such that

$$\mathbb{P}\left(d_{\delta n^2} > n/C \quad \text{and} \quad d_{\lceil Tn^2 \rceil} < nC\right) > 1 - \varepsilon ,$$

which implies that

$$\mathbb{P}\left(N_n^D \le |\mathcal{J} \cap (n/C, nC]|\right) > 1 - \varepsilon .$$

Moreover, under $\mathbf{P}$, the sequence $Y$ is i.i.d. with common distribution $\sigma(0)$, and is independent of $D$. Hence we can apply Proposition 3.11 to bound

$$|\mathcal{J} \cap (n/C, nC]|$$

above under the annealed law, which proves the first result. Similarly, from Proposition 3.4 and the definition of $\operatorname{sep}(\cdot)$, it is possible to deduce that, for any $\varepsilon > 0$, there exists a $C > 0$ such that

$$\mathbb{P}\left(\operatorname{sep}(J^n) \ge \operatorname{sep}(\{k_i : i = 1, 2, \dots, K_n + 1\})^2/C\right) > 1 - \varepsilon .$$

Using the fact that $k_{K_n} \le nC$ and that $k_{K_n+1}$ is either in $(nC, n(C+1)]$ or in $(n(C+1), \infty)$, we have the trivial bound

$$\mathrm{sep}(\{k_i^n : i = 1, \dots, K_n + 1\}) \geq \min\left\{\mathrm{sep}(\mathcal{J} \cap (n/(C+1), n(C+1)]), n\right\},$$

and so Proposition 3.11 applied to $\mathcal{J} \cap (n/(C+1), N(C+1)]$ gives the result. □

**Proposition 3.26** (Local time at deepest-visited traps)**.** *For each $1 \leq i \leq N_n^D$, let $\nu^i(k,0)$ be defined similarly to $\nu(k,0)$ in Section 3.2.1 for the simple random walk $(D_{k+j_i^n} - D_{j_i^n})_{k\in\mathbb{N}}$. Then, as $n \to \infty$,*

$$P\left(\nu^i(j_{i+1}^n - j_i^n - 1, 0) > n/h_n \quad \textit{for all } 1 \leq i \leq N_n^D\right) \to 1 \tag{3.21}$$

*and*

$$P\left(\nu^i(\delta n^2 - 1, 0) > n/h_n \quad \textit{for all } 1 \leq i \leq N_n^D\right) \to 1\,. \tag{3.22}$$

*Proof.* By the time-homogeneity of a SRW and the fact that $j_i^n$ is a stopping time for each $i$,

$$\nu^i(n,0) \stackrel{d}{=} \nu(n,0),$$

and so it follows from Proposition 3.5 that, as $n \to \infty$,

$$\mathbb{P}\left(\nu^i(n^2/h_n, 0) > n/h_n\right) = \mathbb{P}\left(\nu(n^2/h_n, 0) > n/h_n\right) \to 1\,.$$

Since, by Proposition 3.25, $N_n^D$ is bounded above in $P$-probability, it follows that

$$P\left(\nu^i(n^2/h_n, 0) > n/h_n \quad \text{for each } 1 \leq i \leq N_n^D\right) \to 1\,.$$

This is sufficient to establish equation (3.22) since $\nu^i(\cdot, 0)$ is non-decreasing. For equation (3.21), simply apply the second part of Proposition 3.25, since $j_{i+1}^n - j_i^n \geq \mathrm{sep}(J^n)$ for each $1 \leq i \leq N_n^D$. □

We now turn to establishing functional limit theorems for the processes $M^D$ and $\Sigma^D$. In what follows, we make use of the product space $D(\mathbb{R}^+) \times D(\mathbb{R}^+)$. For a sequence of probability measures on $D(\mathbb{R}^+) \times D(\mathbb{R}^+)$, we denote by

$$\stackrel{J_1/J_1}{\Rightarrow} \quad \text{and} \quad \stackrel{M_1/J_1}{\Rightarrow}$$

weak convergence of the first component in the $J_1$ and $M_1$ topologies respectively, and the simultaneous weak convergence of the second component in the $J_1$ topology.

**Proposition 3.27** (Functional limit theorem for the $D$-explored extremal and sum processes)**.** *Under $P$, as $n \to \infty$,*

$$\left(\frac{1}{n}L(M^D_{n^2 t}), \frac{1}{n}D_{n^2 t}\right)_{t\geq 0} \stackrel{J_1/J_1}{\Rightarrow} \left(m_t^B, B_t\right)_{t\geq 0},$$

*and*

$$\left(\frac{1}{n}L(\Sigma^D_{n^2 t}), \frac{1}{n}D_{n^2 t}\right)_{t\geq 0} \stackrel{J_1/J_1}{\Rightarrow} \left(m_t^B, B_t\right)_{t\geq 0}$$

*in distribution.*

*Proof.* Let $Y = \{Y_i\}_{i\in\mathbb{N}}$ be the sequence defined in the proof of Proposition 3.25. Define $d_n$ as in Section 3.2.1 for the simple random walk $D$, and also define the equivalent diffusion distance for the standard Brownian motion

$$d_t^B := \sup_{s\le t} B_s - \inf_{s\le t} B_s\,. \tag{3.23}$$

Combining Proposition 3.8 and Donsker's invariance principle, we have under $\mathbb{P}$ that

$$\left(\frac{1}{n}L(M_{nt}), \frac{1}{n}D_{n^2t}\right)_{t\ge0} \overset{J_1/J_1}{\Rightarrow} (m_t, B_t)_{t\ge0} \tag{3.24}$$

in distribution, where $M_n$ and $m_t$ are defined as in Section 3.2.2. Note that, using the continuous mapping theorem (and the fact that $B$ is continuous almost surely), we also have that $(n^{-1}d_{n^2t})_{t\ge0}$ converges in distribution to $(d_t^B)_{t\ge0}$ (in the $J_1$ topology). Together with the composition result of Lemma 3.40, it follows that, under $\mathbb{P}$,

$$\left(\frac{1}{n}L\left(M_{d_{n^2t}}\right)\right)_{t\ge0} \overset{J_1}{\Rightarrow} \left(m_{d_t^B}\right)_{t\ge0} \tag{3.25}$$

in distribution (simultaneously with the convergence at (3.24)). Now, it is straightforward to check from the construction of the relevant processes that

$$\left(\frac{1}{n}L(M^D_{n^2t}), \frac{1}{n}D_{n^2t}\right)_{t\ge0} \overset{d}{=} \left(\frac{1}{n}L\left(M_{d_{n^2t}}\right), \frac{1}{n}D_{n^2t}\right)_{t\ge0}. \tag{3.26}$$

Moreover, we have that

$$\left(m_{d_t^B}, B_t\right)_{t\ge0} \overset{d}{=} (m_t^B, B_t)_{t\ge0}\,. \tag{3.27}$$

Indeed, by conditioning on $B$ and applying the spatial homogeneity of the underlying point process, checking that the finite dimensional distributions of the two above processes agree is easy, and (3.27) follows readily from this. Putting (3.25), (3.26) and (3.27) together completes the proof of the first claim of the proposition. The proof of the second claim is similar. □

**Corollary 3.28** (Lower bound for the $D$-explored extremal and sum processes)**.** *As $n\to\infty$,*

$$P\left(M^D_{n^2\delta} \ge L^{-1}(n/h_n)\right) \to 1 \quad \textit{and} \quad P\left(\Sigma^D_{n^2\delta} \ge L^{-1}(n/h_n)\right) \to 1\,.$$

*Proof.* By the existence of the scaling limit, as $n\to\infty$,

$$P\left(\frac{1}{n}L(M^D_{n^2\delta}) > 1/h_n\right) \to 1 \quad \text{and} \quad P\left(\frac{1}{n}L(\Sigma^D_{n^2\delta}) > 1/h_n\right) \to 1$$

both hold. The result then follows by the definition of $L^{-1}$. □

### Squeezing the clock process

The next step is to show that, under suitable rescaling, the clock process $A$ is squeezed (with high probability) between the extremal and sum processes $M^D$ and $\Sigma^D$; the squeezing is done in both time and space.

**Proposition 3.29.** *As $n \to \infty$,*

$$P\left(\frac{1}{n} A_{n^2 t} < \Sigma^D_{n^2 t}\, h_n \quad \text{for all } t \in [\delta, T]\right) \to 1\,. \tag{3.28}$$

*Moreover, for each $t \in [\delta, T]$ and $n \geq 0$ there exists a $P$-measurable random time $s^n_t \in [t, t+\delta]$ such that, as $n \to \infty$,*

$$P\left(\frac{1}{n} A_{n^2 s^n_t} > M^D_{n^2 s^n_t}/h_n \quad \text{for all } t \in [\delta, T]\right) \to 1\,. \tag{3.29}$$

*Proof.* Consider first the limit at (3.28). Let $\nu_{\max}(n)$ be defined as in Section 3.2.1 for the simple random walk $D$. Then, by the definition of $A_n$ and $\Sigma^D_n$,

$$A_n \leq \nu_{\max}(n)\, \Sigma^D_n\,,$$

for all $n \geq 0$, and so

$$\frac{1}{n} A_{n^2 t} \leq \frac{1}{n} \nu_{\max}(n^2 T)\, \Sigma^D_{n^2 t} \quad \text{for all } t \in [\delta, T]\,,$$

since $\nu_{\max}(\cdot)$ is non-decreasing. Equation (3.28) then follows by applying Proposition 3.5.

We now work towards equation (3.29), starting with an explicit construction of $s^n_t$ on the event

$$\mathcal{A}_{n,\delta_1} := \{\mathcal{J}^D \cap (\delta_1 n^2, \delta n^2] \neq \emptyset\}\,,$$

for each $n \geq 0$ and $\delta_1 \in (0, \delta]$. To this end, let $(j^{n,\delta_1}_i)^{N^D}_{i=1}$ be the elements of the set $\mathcal{J}^D \cap (\delta_1 n^2, \lceil T n^2 \rceil]$ arranged in increasing order. Note that, for simplicity, in what follows we will suppress the dependence of $j^{n,\delta_1}_i$ on $n$ and $\delta_1$. For any $t \in [\delta, T]$ let $i_t$ be the index of the last jump $j_i$ strictly less than $n^2 t + 1$, that is,

$$i_t := \max\{1 \leq i \leq N^D : j_i < n^2 t + 1\}.$$

Then, define $s^n_t$ by

$$s^n_t := \min\left\{\frac{1}{n^2}\,(j_{i_t+1} - 1)\,,\, t + \delta\right\}\,.$$

We note that by the monotonicity of the events $\mathcal{A}_{n,\delta_1}$, the above construction well-defines $s^n_t$ on the whole of $\mathcal{A}_n := \cup_{\delta_1 \leq \delta} \mathcal{A}_{n,\delta_1}$. Furthermore, by arbitrarily extending the definition of $s^n_t$ by setting $s^n_t = t$ for $t \in [\delta, T]$ on the event $\mathcal{A}^c_n$, we ensure that $s^n_t$ is $P$-measurable. We clearly also have that $s^n_t \in [t, t+\delta]$. Finally, this construction also guarantees that, on $\mathcal{A}_{n,\delta_1}$, for each $t \in [\delta, T]$,

$$i_{s_t} = i_t \tag{3.30}$$

and moreover that

$$n^2 s_t^n - j_{i_t} \geq \min\left\{j_{i_t+1} - j_{i_t} - 1\ ,\ \delta n^2 - 1\right\}\ . \tag{3.31}$$

Recalling the definition of $\nu^i(n,0)$ from Proposition 3.26 (substituting $\delta_1$ for $\delta$), we have by the definition of $A_n$ and $M_n^D$ that, on $\mathcal{A}_{n,\delta_1}$,

$$A_{n^2t} \geq \nu^{i_t}(n^2t - j_{i_t}, 0)\, M^D_{n^2t}$$

for each $t \in [\delta, T]$. Combining this with equations (3.30) and (3.31) gives, on $\mathcal{A}_{n,\delta_1}$,

$$\frac{1}{n} A_{n^2 s_t^n} \geq \frac{1}{n}\nu^{i_t}\left(\min\left\{j_{i_t+1} - j_{i_t} - 1, \delta n^2 - 1\right\}, 0\right)\, M^D_{n^2 s_t^n}\ ,$$

and so Proposition 3.26 yields that, for any $\delta_1 \leq \delta$

$$\liminf_{n\to\infty} P\left(\frac{1}{n} A_{n^2 s_t^n} > M^D_{n^2 s_t^n}/h_n \quad \text{for all } t \in [\delta, T]\right) \geq 1 - \limsup_{n\to\infty} P\left(\mathcal{A}^c_{n,\delta_1}\right)\ .$$

Finally, we have that

$$P\left(\mathcal{A}_{n,\delta_1}\right) = P\left(M^D_{\delta n^2} > M^D_{\delta_1 n^2}\right) \geq P\left(n^{-1} L(M^D_{\delta n^2}) > n^{-1} L(M^D_{\delta_1 n^2})\right)\ .$$

By Proposition 3.27, the liminf as $n \to \infty$ of the right-hand side above is bounded below by $P(m^B_\delta > m^B_{\delta_1}) = 1 - \delta_1/\delta$, or to put this another way

$$\limsup_{n\to\infty} P\left(\mathcal{A}^c_{n,\delta_1}\right) \leq \frac{\delta_1}{\delta}\ ,$$

which can be made arbitrarily small by adjusting the choice of $\delta_1$. □

**Proposition 3.30.** *As $n \to \infty$,*

$$P\left(\frac{1}{n} L\left(\frac{1}{n} A_{n^2t}\right) < \frac{1}{n} L\left(\Sigma^D_{n^2t}\right) + \delta \quad \textit{for all } t \in [\delta, T]\right) \to 1\,. \tag{3.32}$$

*Moreover, for each $t \in [\delta, T]$ and $n \geq 0$ there exists a $P$-measurable random time $s_t^n \in [t, t+\delta]$ such that, as $n \to \infty$,*

$$P\left(\frac{1}{n} L\left(\frac{1}{n} A_{n^2 s_t^n}\right) > \frac{1}{n} L\left(M^D_{n^2 s_t^n}\right) - \delta \quad \textit{for all } t \in [\delta, T]\right) \to 1\,. \tag{3.33}$$

*Proof.* Consider first equation (3.32). Starting from equation (3.28), applying $L$ to both sides of the inequality and then dividing by $n$ we get that, as $n \to \infty$,

$$P\left(\frac{1}{n} L\left(\frac{1}{n} A_{n^2t}\right) \leq \frac{1}{n} L\left(\Sigma^D_{n^2t}\, h_n\right) \quad \text{for all } t \in [\delta, T]\right) \to 1\,.$$

Note that by Corollary 3.28, and since $\Sigma^D_n$ is non-decreasing, as $n \to \infty$,

$$P\left(\Sigma^D_{n^2t} > L^{-1}(n/h_n) \quad \text{for all } t \in [\delta, T]\right) \to 1\,.$$

By equation (3.19), this means that for arbitrary $\eta > 0$, as $n \to \infty$,

$$P\left(\frac{1}{n}L\left(\Sigma^D_{n^2t}\,h_n\right) < \frac{1}{n}L\left(\Sigma^D_{n^2t}\right)(1+\eta) \quad \text{for all } t \in [\delta, T]\right) \to 1\,.$$

Since we have from Proposition 3.27 that

$$P\left(\frac{\eta}{n}L\left(\Sigma^D_{n^2T}\right) \geq \delta\right) \to P\left(\eta m^B_T \geq \delta\right)\,,$$

and the right-hand side converges to 0 as $\eta \to 0$, this is enough to yield the result.

Consider then equation (3.33). Similarly, equation (3.29) gives that, as $n \to \infty$,

$$P\left(\frac{1}{n}L\left(\frac{1}{n}A_{n^2 s^n_t}\right) > \frac{1}{n}L\left(M^D_{n^2 s^n_t}/h_n\right) \quad \text{for all } t \in [\delta, T]\right) \to 1\,.$$

As before, Corollary 3.28, equation (3.19) and Proposition 3.27 then imply the result. □

Under Assumption 1, we establish the stronger uniform convergence (in space) of $A$ to $\Sigma^D$.

**Proposition 3.31.** *Under Assumption 1, as $n \to \infty$,*

$$\sup_{t\in[\delta,T]}\left|\frac{1}{n}L\left(A_{n^2t}\right) - \frac{1}{n}L\left(\Sigma^D_{n^2t}\right)\right| \to 0 \quad \text{in } P\text{-probability}\,.$$

*Proof.* Assume that $h_n \to \infty$ is growing sufficiently slowly that equation (3.20) holds and let $\nu_{\min}(n)$ be defined as in Section 3.2 for the simple random walk $S$. Then, by definition, $A_n \geq \nu_{\min}(n)\Sigma^D_n$, for all $n \geq 0$. Since, by Proposition 3.6, as $n \to \infty$,

$$P\left(\nu_{\min}(n^2t) > 1/(nh_n) \quad \text{for all } t \in [\delta, T]\right) \to 1\,,$$

together with equation (3.28), we have that, as $n \to \infty$,

$$P\left(\frac{1}{n}L\left(\Sigma^D_{n^2t}/(nh_n)\right) \leq \frac{1}{n}L\left(A_{n^2t}\right) \leq \frac{1}{n}L\left(\Sigma^D_{n^2t}\,(nh_n)\right) \quad \text{for all } t \in [\delta, T]\right) \to 1.$$

Finally, as in the proof of Proposition 3.30, Corollary 3.28 and equation (3.20) then jointly imply that for any $\eta > 0$, as $n \to \infty$,

$$P\left(\frac{1}{n}L\left(\Sigma^D_{n^2t}/(nh_n)\right) > \frac{1}{n}L\left(\Sigma^D_{n^2t}\right)(1-\eta) \quad \text{for all } t \in [\delta, T]\right) \to 1$$

and

$$P\left(\frac{1}{n}L\left(\Sigma^D_{n^2t}\,(nh_n)\right) < \frac{1}{n}L\left(\Sigma^D_{n^2t}\right)(1+\eta) \quad \text{for all } t \in [\delta, T]\right) \to 1\,.$$

By applying Proposition 3.27, it follows that for any $\eta, \varepsilon > 0$, as $n \to \infty$

$$\limsup_{n\to\infty} P\left(\sup_{t\in[\delta,T]}\left|\frac{1}{n}L\left(A_{n^2t}\right) - \frac{1}{n}L\left(\Sigma^D_{n^2t}\right)\right| \geq \varepsilon\right) \leq P\left(\eta m^B_T \geq \varepsilon\right)\,.$$

Letting $\eta \to 0$ completes the proof. □

### 3.4.2 Proofs of the main convergence results

We are now ready to prove the main result of this section, namely the convergence of the clock process in Theorems 3.23 and 3.24, and hence our main functional limit theorems (Theorems 1.16 and 1.18).

**Proposition 3.32** (Restated functional limit theorems for the clock process)**.** *Under $P$, as $n \to \infty$,*

$$\left(\frac{1}{n}L\left(\frac{1}{n}A_{n^2t}\right), \frac{1}{n}D_{n^2t}\right)_{t\geq 0} \stackrel{M_1/J_1}{\Rightarrow} \left(m_t^B, B_t\right)_{t\geq 0}$$

*in distribution. Moreover, if Assumption 1 holds, then under $P$, as $n \to \infty$,*

$$\left(\frac{1}{n}L(A_{n^2t}), \frac{1}{n}D_{n^2t}\right)_{t\geq 0} \stackrel{J_1/J_1}{\Rightarrow} \left(m_t^B, B_t\right)_{t\geq 0}$$

*in distribution.*

*Proof.* Recalling Proposition 3.27 and the bounds in Proposition 3.30, the first statement follows from the convergence result of Lemma 3.38. Similarly, recalling Proposition 3.27 and the bounds in Proposition 3.31, the second statement follows from the convergence result of Lemma 3.37. □

*Proof of Theorem 3.23 and Theorem 3.24.* The conclusions of Theorems 3.23 and 3.24 follow immediately from the previous result. □

To complete this section, we derive the convergence of the BTM to the limit process $B_{I^B}$, as stated in Theorems 1.16 and 1.18. The bulk of the work has already been done in establishing the convergence of the clock process above; only technicalities involving convergence results for the various topologies remain.

*Proof of Theorem 1.16 and Theorem 1.18.* Since the right-continuous inverse of the process $n^{-1}L(n^{-1}A_{n^{-2}t})$ is given by $n^{-2}I^D_{nL^{-1}(nt)}$, applying the inversion result of Lemma 3.39 to Proposition 3.32 yields that under $P$, as $n \to \infty$,

$$\left(n^{-2}I^D_{nL^{-1}(nt)}, n^{-1}D_{n^2t}\right)_{t\geq 0} \stackrel{M_1/J_1}{\Rightarrow} \left(I_t^B, B_t\right)_{t\geq 0} \tag{3.34}$$

in distribution. Similarly, noting that the right-continuous inverse of $n^{-1}L(A_{n^2t})$ is $n^{-2}I^D_{L^{-1}(nt)}$, we argue similarly to deduce that if Assumption 1 holds, then under $P$, as $n \to \infty$,

$$\left(n^{-2}I^D_{L^{-1}(nt)}, n^{-1}D_{n^2t}\right)_{t\geq 0} \stackrel{M_1/J_1}{\Rightarrow} \left(I_t^B, B_t\right)_{t\geq 0} \tag{3.35}$$

in distribution. Consequently, recalling that the law of $X$ under $P$ is identical to that of $D_{I^D}$, the second statement of Lemma 3.40 allows us to deduce the desired results by composing the two coordinates of (3.34) and (3.35). □

### 3.4.3 The extremal FIN process

In this section we prove that the scaling limit $B_{I^B}$ is the natural analogue of the FIN diffusion with parameter $\alpha \in (0,1)$ in the limiting case $\alpha = 0$. To make this precise, we first observe how the FIN diffusion can similarly be defined as the standard Brownian motion $B$, time-changed by a clock-process that is a function of $B$, the point-process $\mathcal{P}$, and the parameter $\alpha$. We then use this representation to show that the FIN diffusion with parameter $\alpha$ converges almost surely, under suitable rescaling, to the process $B_{I^B}$ as the parameter $\alpha \to 0$, where by 'almost surely' we mean with respect to the joint law of $\mathcal{P}$ and $B$.

So let us define the FIN diffusion (see [12]). Recall the definition of $\mathcal{P} = (x_i, v_i)_{i\in\mathbb{N}}$ and $B$. For $\alpha \in (0,1)$, let $\mathcal{P}^\alpha$ be the point process on $\mathbb{R} \times \mathbb{R}^+$ defined by the point set $(x_i, w_i) := (x_i, v_i^{1/\alpha})$, $i \in \mathbb{N}$. By simple change of variables, it is easy to see that $\mathcal{P}^\alpha$ is Poissonian with intensity measure $\alpha w^{-1-\alpha} dx\, dw$. Denote by $(L_t(x))_{t\geq 0, x\in\mathbb{R}}$ the local time process of $B$. Defining $m^{B,\alpha} = (m_t^{B,\alpha})_{t\geq 0}$ by

$$m_t^{B,\alpha} := \sum_i L_t(x_i) v_i^{1/\alpha} \tag{3.36}$$

with $I^{B,\alpha} = (I_t^{B,\alpha})_{t\geq 0}$ its right-continuous inverse, the FIN diffusion with parameter $\alpha$ is the process $B_{I^{B,\alpha}} = (B_{I_t^{B,\alpha}})_{t\geq 0}$. Note that in this definition we have built in a coupling, via the point process $\mathcal{P}$ and the Brownian motion $B$, of the FIN diffusion $B_{I^{B,\alpha}}$ and the extremal FIN process $B_{I^B}$. This allows us to state our convergence as an almost sure result.

**Theorem 3.33** (Convergence of the FIN diffusion to the extremal FIN process)**.** *As $\alpha \to 0$,*

$$\left(B_{I^{B,\alpha}_{t^{1/\alpha}}}\right)_{t\geq 0} \overset{L_1}{\to} \left(B_{I_t^B}\right)_{t\geq 0},$$

*where $\overset{L_1}{\to}$ denotes convergence in the $L_{1,\mathrm{loc}}$ topology, almost surely with respect to the joint law of $\mathcal{P}$ and $B$.*

*Remark.* That the convergence in Proposition 3.34 does not hold in the stronger $J_1$ topology can be easily seen from the fact that $m^{B,\alpha}$ is continuous for each $\alpha$ whereas the limit process $m^B$ is not continuous.

*Proof.* Recall the definitions of the processes $m^B$ and $m^{B,\alpha}$ from (1.18) and (3.36), which are the clock-processes for the extremal FIN process and the FIN diffusion with parameter $\alpha$, respectively. As in the proofs of Theorems 1.16 and 1.18, in order to establish Theorem 3.33 it is sufficient to prove the convergence of the clock-processes $m^{B,\alpha}$ to $m^B$, since then we may apply the inversion and composition results of Lemmas 3.39 and 3.40. Hence, the proof of Theorem 3.33 follows from Proposition 3.34 below. □

**Proposition 3.34** (Convergence of the clock-processes)**.** *As $\alpha \to 0$,*

$$\left(\left(m_t^{B,\alpha}\right)^\alpha\right)_{t\geq 0} \overset{M_1}{\to} \left(m_t^B\right)_{t\geq 0}$$

*where $\overset{M_1}{\to}$ denotes convergence in the $M_1$ topology, almost surely with respect to the joint law of $\mathcal{P}$ and $B$.*

Before proving Proposition 3.34, we shall establish a preliminary lemma which relates to the sum-to-max properties of certain rescaled point processes in the $\alpha \to 0$ limit.

**Lemma 3.35** (Sum-to-max). *Let $(c_i, v_i)_{i\in\mathbb{N}}$ be a set of points in $\mathbb{R}^+ \times (0,\infty)$ with the property that, for each $s \in (1,\infty)$,*

$$\sum_i c_i v_i^s < \infty .$$

*Then, as $s \to \infty$,*

$$\left(\sum_i c_i v_i^s\right)^{1/s} \to \sup_{i:c_i>0} v_i$$

*Proof.* Define the function $v : \mathbb{N} \to \mathbb{R}^+$ by the map $i \mapsto v_i$ and denote by $\mu$ the (possibly infinite) measure

$$\mu := \sum_i c_i \delta_i .$$

Then the claim is just the fact that the $L_s$ norm of $v$ with respect to the measure $\mu$ converges, if finite, to the $L_\infty$ norm of $v$ with respect to $\mu$ (see, for example, [55, Section 2.1]). □

*Proof of Proposition 3.34.* We start by proving convergence for a fixed $t$. By definition, we have that

$$m_t^{B,\alpha} := \sum_{i:L_t(x_i)>0} L_t(x_i) v_i^{1/\alpha} .$$

It is an elementary exercise to deduce from this, the fact that $\sup_{x\in\mathbb{R}} L_t(x) < \infty$ and $d_t^B < \infty$ almost surely (where $d_t^B$ was defined at (3.23)), and the definition of $\mathcal{P}$, that $m_t^{B,\alpha}$ is finite for any $\alpha \in (0,1)$, almost surely. We also have the identity

$$m_t^B = \sup_{i:L_t(x_i)>0} v_i$$

almost surely. Indeed, $L_t(x) > 0$ if and only if $x \in (\inf_{s\in[0,t]} B_s, \sup_{s\in[0,t]} B_s)$ almost surely (see, for example, [49, Corollary 22.18]). Moreover, we may assume that there are no points $(x_i, v_i)$ in $\mathcal{P}$ with $x_i \in \{\inf_{s\in[0,t]} B_s, \sup_{s\in[0,t]} B_s\}$ almost surely. Hence, applying Lemma 3.35 to the set of points $(L_t(x_i), v_i))_{i\in\mathbb{N}} \in \mathbb{R}^+ \times (0,\infty)$ yields that, for each fixed $t$, as $\alpha \to 0$,

$$\left(m_t^{B,\alpha}\right)^\alpha \to m_t^B$$

almost surely. By countability, we immediately deduce that this convergence holds for all rational times simultaneously. As the process $m^{B,\alpha}$ is non-decreasing for each $\alpha$ by definition, almost sure convergence in $M_1$ follows. □

*Remark.* A result corresponding to Theorem 3.33 can be also established for FIN diffusions in the $\alpha \to 1^-$ limit. In particular, we claim that as $\alpha \to 1^-$,

$$\left(B_{I^{B,\alpha}_{(1-\alpha)^{-1}t}}\right)_{t\geq 0} \xrightarrow{U} (B_t)_{t\geq 0} , \tag{3.37}$$

where $\xrightarrow{U}$ denotes uniform convergence over compact time intervals, almost surely with respect to the joint law of $\mathcal{P}$ and $B$. Since it is not directly related to the main results of this paper, we only sketch a proof. Defining $(x_i, v_i)_{i\in\mathbb{N}}$ as before and setting $\Sigma := \sum_{i:x_i\in[0,1],v_i\le 1} v_i^{1/\alpha}$, it is an elementary exercise to check (using Campbell's theorem for Poisson point processes, for example) that

$$\mathbf{E}(\Sigma) = \frac{\alpha}{1-\alpha}, \qquad \mathrm{Var}(\Sigma) = \frac{\alpha}{2-\alpha}.$$

Consequently,

$$\mathbf{P}\left(|(1-\alpha)\Sigma - \alpha| \ge \varepsilon\right) \le \frac{\alpha(1-\alpha)^2}{\varepsilon^2(2-\alpha)},$$

and a Borel-Cantelli argument yields

$$(1-\alpha)\Sigma \to 1, \tag{3.38}$$

along the subsequence $\alpha = 1 - n^{-1}$, almost surely. By the monotonicity of $\Sigma$ in $\alpha$, this is readily extended to almost sure convergence as $\alpha \to 1^-$. From this, we deduce that

$$(1-\alpha)\sum_{i:x_i\in[a,b]} v_i^{1/\alpha} \to (b-a), \qquad \text{as } \alpha\to 1^-,\ \forall a\le b, \tag{3.39}$$

almost surely (adding the finite number of terms with $v_i > 1$ clearly does not affect the limit at (3.38), and then a countability argument and monotonicity can be used to establish (3.39)). We note that the convergence at (3.39) implies almost sure vague convergence of the measures $(1-\alpha)\sum_i \delta_{x_i} v_i^{1/\alpha}$ to the Lebesgue measure on the real line. Thus, using the continuity of the Brownian local times, we obtain

$$(1-\alpha)m_t^{B,\alpha} = (1-\alpha)\sum_i L_t(x_i)v_i^{1/\alpha} \to \int L_t(x)dx = t$$

uniformly over compact intervals of $t$, almost surely. The claim at (3.37) then follows by taking inverses and composing with $B$, similarly to the proof of Theorem 3.33.

## 3.5 Appendix: Convergence of stochastic processes

In this appendex we collect information concerning the various topologies on the Skorohod space $D(\mathbb{R}^+)$ of real-valued càdlàg functions on $\mathbb{R}^+$. The first part consists of the basic definitions of these topologies; the second part contains the various convergence lemmas that we apply in Section 3.4.

### 3.5.1 Topologies on the space of real-valued càdlàg functions

The purpose of this section is to describe the various topologies in which our main results are proved, namely the Skorohod topologies $J_1$ and $M_1$, as introduced in [66], and the non-Skorohod topology $L_{1,\mathrm{loc}}$ on the Skorohod space $D(\mathbb{R}^+)$ of real-valued càdlàg functions on $\mathbb{R}^+$; see [14, 70] for a fuller account. Figure 3.5.1 gives a graphical illustration of the different kinds of discontinuities that the three topologies allow for.

We first define convergence in the respective topologies on the Skorohod space $D([0,T])$ of real-valued càdlàg functions on $[0,T]$, for fixed $T$.

$\mathbf{J}_1$: A sequence of functions $f_n \in D([0,T])$ converges to a function $f \in D([0,T])$ in the $J_1$ topology if there exists a sequence $\alpha_n : [0,T] \to [0,T]$ of continuous and one-to-one maps such that

$$\sup_{t\le T} |\alpha_n(t) - t| \to 0 \quad \text{and} \quad \sup_{t \le T} |f_n(\alpha_n(t)) - f(t)| \to 0.$$

Note that the $J_1$ topology extends the usual topology of uniform convergence over compact time intervals by allowing jumps in $f$ to be matched by jumps in $f_n$ that occur at slightly different times, as long as these differences are negligible in the limit.

$\mathbf{M}_1$: For a function $f \in D([0,T])$ define its graph $\mathcal{G}^f \subset \mathbb{R}^+ \times \mathbb{R}$ to be the ordered set consisting of the function $f$ and the line segments

$$\bigcup_{0\le t\le T} \{\lambda f(t^-) + (1-\lambda) f(t) : 0 \le \lambda \le 1\}$$

connecting each point of discontinuity of $f$. We remark that $\mathcal{G}^f$ can be continuously parameterised over $t \in [0,1]$ in the natural way; let such a parameterisation be $\mathcal{G}^f(t)$. A sequence of functions $f_n \in D([0,T])$ converges to a function $f \in D([0,T])$ in the $M_1$ topology if there exists a sequence $\alpha_n : [0,1] \to [0,1]$ of continuous and one-to-one maps such that

$$\sup_{t\le 1} \max_{i=1,2} |\pi_i \mathcal{G}^{f_n}(\alpha_n(t)) - \pi_i \mathcal{G}^f(t)| \to 0$$

where $\pi_1$ and $\pi_2$ are the projections of the graph onto the domain and codomain coordinate respectively. Note that the $M_1$ topology extends the $J_1$ topology by allowing jumps in $f$ to be matched by multiple jumps in $f_n$ of lesser magnitude as long as they are essentially monotone and occur in negligible time in the limit.

$\mathbf{L}_{1,\mathrm{loc}}$: A sequence of functions $f_n \in D([0,T])$ converges to a function $f \in D([0,T])$ in the $L_{1,\mathrm{loc}}$ topology if

$$\int_{t\le T} |f_n(t) - f(t)| dt \to 0.$$

Note that the $L_{1,\mathrm{loc}}$ topology extends both the $J_1$ and the $M_1$ topologies by allowing excursions in $f_n$ that are not present in $f$, as long as they are of negligible magnitude in the $L_1$ sense in the limit.

To extend the above definitions to the Skorohod space $D(\mathbb{R}^+)$, we say that a sequence of functions $f_n \in D(\mathbb{R}^+)$ converges to a function $f \in D(\mathbb{R}^+)$ in the $J_1$ (respectively $M_1$ and $L_{1,\mathrm{loc}}$) topology if and only if their restrictions to $[0,T]$ converge with respect to the $J_1$ (respectively $M_1$ and $L_{1,\mathrm{loc}}$) topology on $D([0,T])$ for every continuity point $T$ of $f$.

To summarise, we have that the $J_1$, $M_1$ and $L_{1,\mathrm{loc}}$ topologies are strictly ordered in the following sense, where we write $\stackrel{J_1}{\to}$, $\stackrel{M_1}{\to}$ and $\stackrel{L_1}{\to}$ for convergence in the relevant topologies.

**Proposition 3.36.** *For a sequence of functions $f_n \in D(\mathbb{R}^+)$ and a function $f \in D(\mathbb{R}^+)$,*

$$f_n \stackrel{J_1}{\to} f \implies f_n \stackrel{M_1}{\to} f \implies f_n \stackrel{L_1}{\to} f,$$

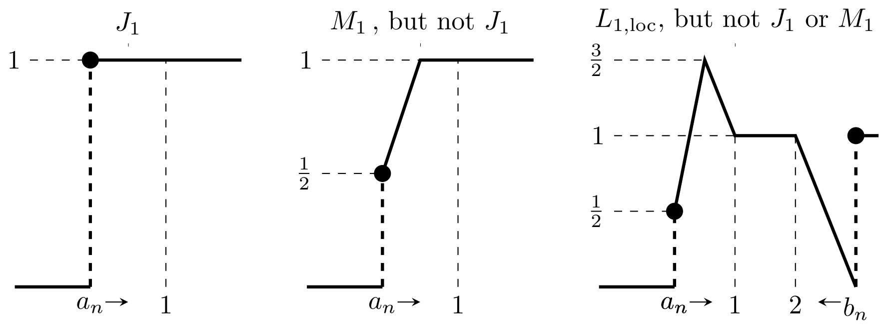


Figure 7: Examples of sequences of functions in $D(\mathbb{R}^+)$ that converge to the function $\mathbf{1}_{[1,\infty)}(\cdot)$ in the $J_1$, $M_1$ and $L_{1,\mathrm{loc}}$ topologies respectively, where $a_n := 1 - n^{-1}$ and $b_n := 2 + n^{-1}$.

*but none of the converse implications hold in general.*

### 3.5.2 Convergence lemmas

The purpose of this section is to present the convergence lemmas that we made use of in Section 3.4, including the squeeze convergence lemmas used to establish the convergence of the clock process (Theorems 3.23 and 3.24), and the inversion and composition lemmas used to complete the proof of our main functional limit theorems (Theorems 1.16 and 1.18).

Let $w_t^n, x_t^n$ and $y_t^n$ be sequences of stochastic processes in $D(\mathbb{R}^+)$. Throughout we will assume that each $w_t^n$, $x_t^n$ and $y_t^n$ is non-decreasing with

$$\lim_{n\to\infty} w_0^n = \lim_{n\to\infty} x_0^n = \lim_{n\to\infty} y_0^n = 0$$

in probability.

We will also suppose that there is a limiting stochastic process $x_t \in D(\mathbb{R}^+)$ such that, as $n \to \infty$,

$$(w_t^n)_{t\geq 0} \overset{J_1}{\Rightarrow} (x_t)_{t\geq 0} \quad \text{and} \quad (y_t^n)_{t\geq 0} \overset{J_1}{\Rightarrow} (x_t)_{t\geq 0},$$

where we recall that $\overset{J_1}{\Rightarrow}$, $\overset{M_1}{\Rightarrow}$ and $\overset{L_1}{\Rightarrow}$ denotes weak convergence in the $J_1$ and $M_1$ and $L_{1,\mathrm{loc}}$ topologies respectively.

We first give sufficient conditions under which the stochastic processes $x_t^n$ also converge weakly to the limit process $x_t$ in the $J_1$ and $M_1$ topologies respectively. For technical reasons we state our results in a way that allows for an auxiliary process to converge simultaneously. For a sequence of probability measures on $D(\mathbb{R}^+) \times D(\mathbb{R}^+)$, denote by

$$\overset{J_1/J_1}{\Rightarrow}, \quad \overset{M_1/J_1}{\Rightarrow} \quad \text{and} \quad \overset{L_1/J_1}{\Rightarrow}$$

weak convergence of the first component in the $J_1$, $M_1$ and $L_{1,\mathrm{loc}}$ topologies respectively, and the simultaneous weak convergence of the second component in the $J_1$ topology. Then let $z_t^n$ be an auxiliary sequence of stochastic processes in $D(\mathbb{R}^+)$ such that, as $n \to \infty$,

$$(w_t^n,\, z_t^n)_{t\geq 0} \overset{J_1/J_1}{\Rightarrow} (x_t,\, z_t)_{t\geq 0} \quad \text{and} \quad (y_t^n,\, z_t^n)_{t\geq 0} \overset{J_1/J_1}{\Rightarrow} (x_t,\, z_t)_{t\geq 0}, \tag{3.40}$$

for a limit process $z_t$ in $D(\mathbb{R}^+)$.

**Lemma 3.37** (Uniform convergence in space implies $J_1$ convergence)**.** *Assume that, for any $T > \delta > 0$, as $n \to \infty$,*

$$\sup_{t\in[\delta,T]} |x_t^n - w_t^n| \to 0 \qquad \textit{in probability.} \tag{3.41}$$

*Then, as $n \to \infty$,*

$$(x_t^n,\, z_t^n)_{t\geq 0} \overset{J_1/J_1}{\Rightarrow} (x_t,\, z_t)_{t\geq 0}.$$

*Proof.* The convergence of the finite dimensional distributions (where the time indices are points at which $x$ and $z$ are continuous almost surely) immediately follows from the convergence at (3.40) and (3.41); we need only show the tightness of the sequence $x_t^n$ in the $J_1$ topology. In particular, we need to show that, for each $T > 0$,

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{t\leq T} |x_t^n| \geq \lambda\right) = 0$$

and, for each $\varepsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathbf{P}\left(\inf_{\substack{0=t_0<t_1<\cdots<t_m=T:\\ \min_i(t_i-t_{i-1})>\delta}} \sup_{i=1,\ldots,m} \sup_{s,t\in[t_{i-1},t_i)} |x_s^n - x_t^n| \geq \varepsilon\right) = 0$$

(see [14, Theorem 16.8]). Now, the convergence of $w_t^n$ in the $J_1$ topology implies that the two conditions in [14, Theorem 16.8] are satisfied for $w_t^n$. It is an elementary exercise to check from this, the uniform convergence in equation (3.41), and the assumption that each $w_t^n$ and $x_t^n$ is non-decreasing with $w_0^n$ and $x_0^n$ converging to zero, that the conditions are also satisfied for $x_t^n$. □

**Lemma 3.38** (Squeeze convergence in space and time implies $M_1$ convergence)**.** *Assume that, for any $T > \delta > 0$, $t \geq 0$ and $n \in \mathbb{N}$ there exists a random, measurable $s_t^n \in [t, t+\delta]$ such that, as $n \to \infty$,*

$$\mathbb{P}\left(w_{s_t^n}^n - \delta < x_{s_t^n}^n < y_{s_t^n}^n + \delta \quad \textit{for all } t \in [\delta, T]\right) \to 1. \tag{3.42}$$

*Further, assume the limit process $(x_t, z_t)$ is almost surely continuous at each fixed time $t$. Then, as $n \to \infty$,*

$$(x_t^n,\, z_t^n)_{t\geq 0} \overset{M_1/J_1}{\Rightarrow} (x_t,\, z_t)_{t\geq 0}.$$

*Proof.* The convergence of the finite dimensional distributions immediately follows from (3.40), the squeeze convergence in equation (3.42), and the fact that $w_t^n$, $x_t^n$ and $y_t^n$ are all non-decreasing; we need only show the tightness of the sequence $x_t^n$ in the $M_1$ topology. Using the characterisation of tightness in [70, Theorem 12.12.3], we need to show in particular that, for each $T > 0$,

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbf{P}\left(\sup_{t\leq T} |x_t^n| \geq \lambda\right) = 0$$

and, for each $\varepsilon > 0$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbf{P}\left(\sup_{t\in[0,T]}\sup_{\substack{\max\{0,t-\delta\}\le t_1\\ <t_2<t_3\le\min\{t+\delta,T\}}}\left\{\left|\left|x^n_{t_2}-\left[x^n_{t_1},x^n_{t_3}\right]\right|\right|\right\}>\varepsilon\right)$$

where $\left|\left|x^n_{t_2}-\left[x^n_{t_1},x^n_{t_3}\right]\right|\right|$ denotes the Hausdorff distance in $\mathbb{R}^+\times\mathbb{R}$ between the point $(t_2, x^n_{t_2})$ and the line segment joining the points $(t_1, x^n_{t_1})$ and $(t_3, x^n_{t_3})$. Now, the convergence of $w^n_t$ and $x^n_t$ in the $J_1$ (and hence $M_1$) topology imply that the above two conditions are satisfied for $w^n_t$ and $y^n_t$. A combination of this, equation (3.42) and the fact that each $w^n_t$, $x^n_t$ and $y^n_t$ is non-decreasing with $w^n_0$, $x^n_0$ and $y^n_0$ converging to zero, then implies that the two conditions are also satisfied for $x^n_t$. □

Finally, we give basic inversion and composition lemmas for the topologies. We henceforth assume that $x^n_t$ and $z^n_t$ are deterministic functions in $D(\mathbb{R}^+)$ such that $x^n_t$ is non-decreasing with

$$\lim_{n\to\infty} x^n_0 = 0.$$

We further assume that, as $n\to\infty$,

$$(x^n_t,\, z^n_t)_{t\ge 0} \stackrel{J_1/J_1}{\to} (x_t,\, z_t)_{t\ge 0} \tag{3.43}$$

for some $x_t, z_t \in D(\mathbb{R}^+)$, where the notation means $\stackrel{J_1/J_1}{\to}$ that the first component converges in the $J_1$ topology and the second in the $J_1$ topology. (We will use similar notation when $J_1$ is replaced by $M_1$ or $L_{1,\text{loc}}$.) Furthermore, we assume that the limit process $z_t \in C(\mathbb{R}^+)$, the space of real-valued continuous functions on $\mathbb{R}^+$ (so that the convergence of the second coordinate can actually be considered as uniform convergence over compact time intervals). The reason that we insist on the presence of an auxiliary process is to take advantage of the composition lemma that we state below.

**Lemma 3.39** (Inversion lemma)**.** *Let $v^n_t$ and $v_t$ denote the right-continuous inverses of $x^n_t$ and $x_t$ respectively, and further assume that $v_t$ is based at the origin. Then, as $n\to\infty$,*

$$(v^n_t,\, z^n_t)_{t\ge 0} \stackrel{M_1/J_1}{\to} (v_t,\, z_t)_{t\ge 0}.$$

*The same conclusion holds if we weaken the assumption in equation* (3.43) *to*

$$(x^n_t,\, z^n_t)_{t\ge 0} \stackrel{M_1/J_1}{\to} (x_t,\, z_t)_{t\ge 0}.$$

*Proof.* This is a consequence of the continuity of right-continuous inverses in the $M_1$ topology (see [70, Corollary 13.6.5]; the continuity at zero follows from the assumptions that $x^n_0$ converges to zero and that $v_t$ is based at the origin). □

*Remark.* Note that, even under the assumption of equation (3.43), the conclusion of Lemma 3.39 does not hold in general in the stronger $J_1$ topology; see the discussion in [70, Example 13.6.1].

**Lemma 3.40** (Composition lemma)**.** *As $n \to \infty$,*

$$(z^n_{x^n_t})_{t\geq 0} \overset{J_1}{\to} (z_{x_t})_{t\geq 0}.$$

*If instead the assumption in equation* (3.43) *is weakened to*

$$(x^n_t,\, z^n_t)_{t\geq 0} \overset{M_1/J_1}{\to} (x_t,\, z_t)_{t\geq 0},$$

*then we may only conclude that, as $n \to \infty$,*

$$(z^n_{x^n_t})_{t\geq 0} \overset{L_1}{\to} (z_{x_t})_{t\geq 0}.$$

*Proof.* The first statement is standard, proven for example in [69, Theorem 3.1] (see also [70, Theorem 13.2.2]). As for the second statement, we start by noting that

$$\int_{t\leq T} \left| z^n_{x^n_t} - z_{x_t} \right| dt \leq \int_{t\leq T} \left| z^n_{x^n_t} - z_{x^n_t} \right| dt + \int_{t\leq T} \left| z_{x^n_t} - z_{x_t} \right| dt. \tag{3.44}$$

Now, since $x^n_t \to x_t$ in the $M_1$ topology, we must have that $\sup_n \sup_{t\leq T} x^n_t$ is bounded above by some $T_1 < \infty$. Hence the first term on the right-hand side of (3.44) satisfies

$$\int_{t\leq T} \left| z^n_{x^n_t} - z_{x^n_t} \right| dt \leq T \sup_{t\leq T_1} \left| z^n_t - z_t \right|,$$

which converges to zero as $n \to \infty$ by the uniform convergence of $z^n_t$ to $z_t$ over compact time intervals. We now deal with the second term on the right-hand side of (3.44). First note that we can also assume that $\sup_{t\leq T} x_t \leq T_1 < \infty$ (adjusting $T_1$ if necessary). Moreover, the continuity of $z$ yields that $\sup_{t\leq T_1} |z_t| \leq C < \infty$. Putting these bounds together, we find that, for every $\varepsilon > 0$,

$$\int_{t\leq T} \left| z_{x^n_t} - z_{x_t} \right| dt \leq T \sup_{\substack{s,t\leq T_1:\\ |s-t|<\varepsilon}} |z_s - z_t| + C \int_{t\leq T} \mathbf{1}_{\{|x^n_t - x_t| \geq \varepsilon\}} dt.$$

Since $x^n_t$ converges to $x_t$ in the $M_1$ topology, the same is true in the $L_{1,\mathrm{loc}}$ topology, which implies that the second term here converges to 0 as $n \to \infty$. Again appealing to the continuity of $z$, the first term can be made arbitrarily small by taking $\varepsilon$ small. This confirms that

$$\lim_{n\to\infty} \int_{t\leq T} \left| z^n_{x^n_t} - z_{x_t} \right| dt = 0,$$

as desired. □

*Remark.* The assumption that $z_t \in C(\mathbb{R}^+)$ is essential for the first conclusion of the previous result. Indeed, it no longer holds in general if we assume only that $z_t \in D(\mathbb{R}^+)$ with convergence in the $J_1$ topology; see the discussion in [70, Example 13.2.2].

*Remark.* The fact that the second convergence statement in Lemma 3.40 fails to hold in any of the Skorohod topologies lies at the heart of why we resort to the coarser non-Skorohod $L_{1,\mathrm{loc}}$ topology in Theorems 1.16 and 1.17; see the discussion in [70, Example 13.2.4].

# Chapter 4

# The Bouchaud–Anderson model

## 4.1 Introduction

In Chapter 1 we presented our main results on the BAM in the regime of Weibull potential and trapping landscape. In particular, we work under the assumption that there exist $\gamma, \mu, \delta_\sigma > 0$ such that

$$\mathbf{P}(\xi(0) > x) = e^{-x^\gamma}\,, \qquad x > 0\,,$$

and

$$\mathbf{P}(\sigma(0) > x) = e^{-x^\mu}\,,\ x > \delta_\sigma \qquad \text{and} \qquad \mathbf{P}(\sigma(0) = \delta_\sigma) = 1 - e^{-\delta_\sigma^\mu}\,,$$

where we interpret our restriction of the domain of $\sigma(0)$ to $[\delta_\sigma, \infty)$ as a 'no quick sites' assumption that is crucial to our proofs.

To summarise our main results, in the above regime we show that the BAM exhibits the following localisation phenomena:

- The renormalised solution of the BAM completely localises on a single site with overwhelming probability (Theorem 1.19;
- The *radii of influence* are the non-negative integers (Theorem 1.20)

$$\rho_\xi := \left[\frac{\gamma - 1}{2}\frac{\mu}{\mu+1}\right]^+ \quad \text{and} \quad \rho_\sigma := \left[\frac{\gamma-1}{2}\frac{\mu}{\mu+1} + \frac{1}{2}\right]^+ \in \{\rho_\xi, \rho_\xi + 1\},$$

  which are increasing functions of both $\gamma$ and $\mu$. Moreover, $\rho_\xi \leq \rho_{\text{PAM}}$ for identical potential field $\xi$. In other words the localisation effects due to the PAM and BTM are mutually reinforcing;
- The BAM is strongly reducible to the PAM if and only if $\gamma < 1$, and weakly reducible to the PAM if and only if $\rho_\sigma = 0$ and $\gamma \geq 1$ (Theorem 1.21); and
- The localisation induces a local correlation in the random fields (the 'fit and stable' hypothesis of population dynamics) (Theorem 1.22).

In this chapter we describe these results in full and give a self-contained proof. Note that this is a slightly specialised version of the results in a more general setting [59]; in particular, our restriction to the case of Weibull trapping landscape avoids some of the technical difficulties of the general case, while still capturing the relevant phenomena of interest.

### 4.1.1 Full description of main results

First we shall state our results in full, in particular by giving an explicit construction of the localisation site $Z_t$, which mirrors closely the construction in Chapter 2, and describing more precisely the nature of the correlation in Theorem 1.22.

Recall from Chapter 1 the radii of influence $\rho_\xi$ and $\rho_\sigma$, which satisfy $\rho_\sigma \geq \rho_\xi$. For each $z \in \mathbb{Z}^d$, define the operator

$$\mathcal{H}(z) := \Delta\sigma^{-1} + \xi \mathbb{1}_{B(z,\rho_\xi)}$$

restricted to $B(z, \rho_\sigma)$, denoting by $\lambda(z)$ its principal eigenvalue. In other words, $\mathcal{H}(z)$ is the operator on functions $f : B(0, \rho_\sigma) \to \mathbb{R}$ induced by the operator

$$\mathbb{1}_{B(z,\rho_\sigma)} \left(\Delta\sigma^{-1} + \xi \mathbb{1}_{B(z,\rho_\xi)}\right) \mathbb{1}_{B(z,\rho_\sigma)} = \mathbb{1}_{B(z,\rho_\sigma)} \Delta\sigma^{-1} \mathbb{1}_{B(z,\rho_\sigma)} + \xi \mathbb{1}_{B(z,\rho_\xi)} .$$

Note that each $\lambda(z)$ is real since the operator $\mathcal{H}(z)$ is similar to the symmetric operator

$$\sigma^{-\frac{1}{2}}\, \mathcal{H}(z)\, \sigma^{\frac{1}{2}} = \sigma^{-\frac{1}{2}} \Delta \sigma^{-\frac{1}{2}} + \xi \mathbb{1}_{B(z,\rho_\xi)} .$$

We refer to $\lambda(z)$ as the *local principal eigenvalue at* $z$, and remark that it is a certain functional of the sets $\xi^{(\rho_\xi)}(z) := \{\xi(y)\}_{y \in B(z,\rho_\xi)}$ and $\sigma^{(\rho_\sigma)}(z) := \{\sigma(y)\}_{y \in B(z,\rho_\sigma)}$. Note that the $\{\lambda(z)\}_{z \in V_t}$ are identically distributed, and have a dependency range bounded by $2\rho_\sigma$, i.e. the random variables $\lambda(y)$ and $\lambda(z)$ are independent if and only if $|y-z| > 2\rho_\sigma$. Remark also that in the special case $\rho_\sigma = 0$, $\lambda(z)$ reduces to the 'net growth rate' $\eta(z) = \xi(z) - \sigma^{-1}(z)$.

For any sufficiently large $t$, define a *penalisation functional* $\Psi_t : \mathbb{Z}^d \to \mathbb{R}$ by

$$\Psi_t(z) := \lambda(z) - \frac{|z|}{\gamma t} \log \log t .$$

Note that $\Psi_t$ has an identical form to the penalisation functional introduced in Chapter 2, representing the trade-off between the local principal eigenvalue $\lambda(z)$ and a certain probabilistic penalty; see the remarks in Chapter 2.

Finally, recall the 'macrobox' $V_t$, the constant $0 < \theta < 1/2$, the macrobox level $L_t$ and the subset $\Pi^{(L_t)}$ from Chapter 2. Further, define the random site

$$Z_t := \underset{z \in \Pi^{(L_t)}}{\operatorname{argmax}}\, \Psi_t(z) ,$$

which, as in Chapter 2, is well-defined eventually almost surely and does not depend on the particular choice of $\theta$. With this definition of the localisation site $Z_t$, we present again our main theorem (see Theorem 1.19).

**Theorem 4.1** (Complete localisation)**.** *As $t \to \infty$,*

$$\frac{u(t,Z_t)}{U(t)} \to 1 \qquad \textit{in } \mathbf{P}\textit{-probability}.$$

Before stating our remaining results we introduce some more notation. First we define exponents that explicitly describe the correlation of the fields $\xi$ and $\sigma$ around the localisation site $Z_t$. To this end, define the function $q_\xi : \mathbb{N} \to [0,1]$ and the non-negative constant $q_\sigma$ by

$$q_\xi(x) := \begin{cases} \left(1 - \frac{2x}{\gamma-1} - \frac{1}{\mu+1}\right)^+ & \text{if } \gamma > 1\,, \\ (1-x)^+ & \text{else,} \end{cases} \qquad \text{and} \qquad q_\sigma := \left(\frac{\gamma-1}{\mu+1}\right)^+ .$$

We shall also need to introduce the concept of 'interface sites' which, as in Chapter 1, are sites at a distance of precisely $\rho_\xi$ and $\rho_\sigma$ from the localisation site, and moreover at values of the parameters $(\gamma, \mu)$ for which $\rho_\xi$ and $\rho_\sigma$ are transitioning from one integer to the next. To this end define the sets

$$\mathcal{I}_\xi := \left\{ z \in \mathbb{Z}^d \setminus \{0\} : |z| = \frac{\gamma-1}{2}\frac{\mu}{\mu+1} \right\}, \; \mathcal{I}_\sigma := \left\{ z \in \mathbb{Z}^d \setminus \{0\} : |z| = \frac{\gamma-1}{2}\frac{\mu}{\mu+1} + \frac{1}{2} \right\}.$$

Note that $\mathcal{I}_\xi$ and $\mathcal{I}_\sigma$ are non-empty if and only if $(\gamma, \mu)$ lies on, respectively, the dashed and bold curves in Figure 3. Finally, define the random time $T_t$ and the scales $r_t$ and $a_t$ as in Chapter 1.

**Theorem 4.2** (Description of the localisation site)**.** *As $t \to \infty$ the following hold:*

*(a) (Localisation distance)*

$$\frac{Z_t}{r_t} \Rightarrow X \qquad \textit{in law},$$

*where $X$ is a random vector whose coordinates are independent and distributed as Laplace (two-sided exponential) random variables with absolute-moment one;*

*(b) (Local correlation of the potential field) For each $z \in B(0,\rho_\xi) \setminus \mathcal{I}_\xi$ there exists a $c > 0$ such that*

$$\frac{\xi(Z_t+z)}{a_t^{q_\xi(|z|)}} \to c \qquad \textit{in } \mathbf{P}\textit{-probability};$$

*on the other hand, for each $z \in \mathcal{I}_\xi$ there exists a $c > 0$ such that, uniformly over $x \in (0, L_t)$,*

$$f_{\xi(Z_t+z)}(x) \to \frac{e^{cx} f_\xi(x)}{\mathbf{E}[e^{c\xi(0)}]}\,,$$

*where $f_{\xi(z)}$ denotes the density of the potential field at site $z$;*

*(c) (Correlation of the trapping landscape at $Z_t$) If $\gamma > 1$, then there exists a $c > 0$ such that*

$$\frac{\sigma(Z_t)}{a_t^{q_\sigma}} \to c \qquad \textit{in } \mathbf{P}\textit{-probability};$$

*if $\gamma = 1$, then instead, uniformly in $x$,*

$$f_{\sigma(Z_t)}(x) \to \frac{e^{-1/x} f_\sigma(x)}{\mathbf{E}[e^{-1/\sigma(0)}]}\,,$$

*where $f_{\sigma(Z_t)}$ denotes the density of the trapping landscape at site $Z_t$;*

(d) *(Local correlation of the trapping landscape) For each $z \in B(0, \rho_\sigma) \setminus \{0\}$*

$$\sigma(Z_t + z) \to \delta_\sigma \qquad \text{in } \mathbf{P}\text{-probability};$$

*on the other hand, for each $z \in B(0, \rho_\sigma) \setminus \{0\} \setminus \mathcal{I}_\sigma$ there exists a $c > 0$ such that, uniformly over $x$,*

$$f_{\sigma(Z_t+z)}(x) \to \frac{e^{c/x} f_\sigma(x)}{\mathbb{E}[e^{c/\sigma(0)}]},$$

*where $f_{\sigma(z)}$ denotes the density of the trapping landscape at site $z$;*

(e) *(Ageing)*

$$\frac{T_t}{t} \Rightarrow \Theta \qquad \text{in law},$$

*where $\Theta$ is the same non-degenerate almost surely positive random variable as in Theorems 1.6 and 1.7.*

In Section 4.1.2 below we make some remarks as to why we must distinguish the 'interface sites', as well as the case $\gamma \leq 1$, in our results on the local correlation of the potential field and trapping landscape. Note that if we were also to consider the limiting case $\mu \to 0$, as in [59], we would need to further distinguish this case.

**Theorem 4.3** (Optimality results). *As $t \to \infty$ the following hold:*

(a) *(Optimality of the radii of influence) The radii of influence $\rho_\xi$ and $\rho_\sigma$ are optimal, in other words, there does not exist a functional $\psi_t$, depending either on $\xi$ only through its values in balls of radius $\rho_\xi - 1$ around each site $z$, or depending on $\sigma$ only through its values in balls of radius $\rho_\sigma - 1$ around each site $z$, such that*

$$\mathbf{P}\left(Z_t = \operatorname*{argmax}_{z \in \mathbb{Z}^d} \psi_t(z)\right) \to 1. \tag{4.1}$$

(b) *(Criterion for strong reducibility to the PAM) The localisation site is independent of the trapping landscape $\sigma$ if and only if $\gamma < 1$, in other words, if and only if $\gamma < 1$, there exists a random site $z_t \in \mathbb{Z}^d$, independent of $\sigma$, such that,*

$$\mathbf{P}(Z_t = z_t) \to 1. \tag{4.2}$$

(c) *(Criterion for weak reducibility to the PAM) The localisation site $Z_t$ depends on $\xi$ and $\sigma$ only through the value of $\eta$ if and only if $\rho_\sigma = 0$, in other words, if and only if $\rho_\sigma = 0$, there exists a random site $z_t \in \mathbb{Z}^d$, dependent on $\xi$ and $\sigma$ only through $\eta$, such that,*

$$\mathbf{P}(Z_t = z_t) \to 1. \tag{4.3}$$

### 4.1.2 Methods and techniques

Our approach to proving localisation in the BAM is loosely based on the proof presented in Chapter 2, although the complex interaction between the potential field and the trapping landscape means that these techniques cannot be trivially adapted. Instead, the presence of the trapping landscape requires the development of existing techniques on two main fronts.

First, establishing localisation in the BAM requires the development of the spectral theory of operators of the form $\Delta\sigma^{-1} + \xi$, including path expansions and Feynman-Kac representations for the principal eigenvalue and eigenfunction respectively. To the best of our knowledge, this theory has not appeared in the literature before, and may be of independent interest, including in the study of position-dependent mass Schrödinger operators. In the particular case of the BAM with Weibull tails, we also extend existing techniques to establish the max-class of local eigenvalues; this is necessary in order to extract extra information about the local correlation in the potential field and trapping landscape.

Second, in order to analyse the 'screening effect' of heavy traps, standard percolation estimates are insufficient: in dimension one, because of the geometry; in dimensions higher than one, because of complex dependencies between the potential field, the trapping landscape, and the localisation site $Z_t$. In dimension one we analyse heavy traps using coarse graining methods; in higher dimensions, we implement new ideas that allow us to apply percolation estimates in the presence of the dependencies.

We also provide some intuition for the shape of the local profile of the potential field and trapping landscape in parts (b)–(d) of Theorem 4.2. This shape is derived by considering the path expansion for $\lambda(0)$ (see Section 4.3) and determining the values of $\xi$ and $\sigma$ that appropriately balance: (i) the increase in $\lambda$ gained from favourable realisations of $\xi$ and $\sigma$; and (ii) the probabilistic penalty that results from such favourable realisations of $\xi$ and $\sigma$ if they are too unlikely. This balance is expressed through a convex function whose integral is asymptotically concentrated in the regions specified in Theorem 4.2. This computation is carried out in the proof of Proposition 4.24, identifying the constants in Theorem 4.2 explicitly.

We must distinguish the interface sites in $\mathcal{I}_\xi$ and $\mathcal{I}_\sigma$ in the correlation results in Theorem 4.3 since if $z \in B(0, \rho_\xi) \setminus \mathcal{I}_\xi$ then the value of $\xi(Z_t + z)$ is growing (with high probability) as $t \to \infty$. However, if $z \in \mathcal{I}_\xi$, the value of $\xi(Z_t + z)$ instead converges to a certain random variable with law distinct from the law of $\xi(0)$. Similarly, if $\gamma \geq 1$ and $z \in B(0, \rho_\sigma) \setminus \mathcal{I}_\sigma$, $\sigma(Z_t + z)$ converges to $\delta_\sigma$. However, for $z \in \mathcal{I}_\sigma$, then the value of $\sigma(Z_t + z)$ instead converges to a certain random variable with law distinct from the law of $\sigma(0)$. These properties are reflective of the fact that the correlation in the fields $\xi$ and $\sigma$ induced by the localisation site $Z_t$ decays away from the site.

The case $\gamma > 1$ must be further distinguished in our profile for $\sigma(Z_t)$ since if $\gamma > 1$ then the value of $\sigma(Z_t)$ is growing, and indeed growing with a deterministic leading order. However, if $\gamma = 1$, this is no longer true and instead $\sigma(Z_t)$ converges to a certain random variable with law distinct from the law of $\sigma(0)$.[1]

Note finally that if $(\gamma, \mu)$ is such that $\mathcal{I}_\sigma$ and $\mathcal{I}_\xi$ are both empty, then the probability in equations (4.1) actually converges to 0 for any such $\psi_t$; otherwise, the probability may converge to a constant $c \in (0, 1)$. Similarly, if $(\gamma, \mu)$ lies to the right of the dashed or bold line in Figure 5, the probabilities in (4.2) and (4.3) respectively converge to 0 for any such

[1] Of course, in the case $\gamma < 1$, with overwhelming probability $\sigma$ is independent of the localisation site $Z_t$ (cf. part (c) of Theorem 4.2) and so $\sigma(Z_t)$ has the same law as $\sigma(0)$.

$z_t$; if $(\gamma, \mu)$ lies on either line, the respective probability instead converges to a constant $c \in (0,1)$. We do not prove these additional results here.

### 4.1.3 Outline of proof and overview of the chapter

The proof of our main results follows a similar strategy as our proof in Chapter 2, namely to use a path decomposition to split the solution $U(t)$ into components and then prove that (i) only one of the components is non-negligible, and (ii) this component is localised at $Z_t$.

To this end, we may reuse a large quantity of the concepts and notation from Chapter 2, suitably adapted to the BAM. In particular, we define the ball $B_t$ as in Chapter 2, and make use of an identical path decomposition $\{E_t^i\}_{i=1}^5$ and associated component of the mass $U^i(t)$. We also make use of the same set of auxiliary scaling functions $f_t, h_t, e_t, b_t \to 0$ and $g_t \to \infty$ in (2.1), although in we shall need the slightly stronger assumptions that

$$1/\log\log\log t \ll b_t \ll 1$$

and

$$g_t/\log\log\log t \ll b_t \ll f_t h_t \ll g_t h_t \ll e_t \,. \tag{4.4}$$

The most significant modification we make to the notation in Chapter 2 is to redefine $j := [\gamma - 1]^+$, and to define the operator

$$\mathcal{H}^{(j)}(z) := \Delta\sigma^{-1} + \xi$$

restricted to $B(z, j)$, with $\lambda^{(j)}(z)$ its principal eigenvalue. We then define $\Psi_t^{(j)}(z)$ completely analogously to in Chapter 2.

Recall from Chapter 2 that our strategy to prove complete localisation is to formalise the heuristic that, for a path $p \in \Gamma(0) \setminus E_t^5$,

$$U^p(t) \approx \exp\left\{ t\lambda^{(j)}(z^{(p)}) \right\} a_t^{|p|} \,, \tag{4.5}$$

and further that, for $i = 1, 2, 3$,

$$U^i(t) \approx \max_{p \in E_t^i} U^p(t) \,. \tag{4.6}$$

Note that implicit in the first heuristic is the highly non-trivial fact that the trapping landscape $\sigma$ is not an obstacle to the diffusivity of the particle, in other words, that a sufficiently 'quick' path exists from 0 to the site $z^{(p)}$. If $d \geq 2$, this is essentially due to percolation estimates; if $d = 1$, then this relies crucially on an additional tail decay assumption on the distribution of $\sigma(0)$ which is satisfied in the case of Weibull tail decay. As outlined in [59], our proofs and methods break down when this assumption is not satisfied.

The rest of this chapter is organised as follows. In Section 4.3 we study extremal theory for $\lambda^{(j)}$ and $\Psi_t^{(j)}$, demonstrating, in particular, that

$$\Psi_t^{(j)}(Z_t^{(j)}) - \Psi_t^{(j)}(Z_t^{(j,2)}) > d_t e_t \quad \text{and} \quad |Z_t^{(j)}| h_t \log\log t > t d_t e_t$$

both hold eventually with overwhelming probability. We also show that $Z_t^{(j)} = Z_t$ with overwhelming probability. In the process, we establish the description of the localisation site $Z_t$ that is contained in Theorem 4.2, as well as the optimality results in Theorem 4.3. In Section 4.4, we show how to formalise the heuristics in equations (4.5) and (4.6) and so complete the proof of the negligibility of $U^2(t)$ and $U^3(t)$. In Section 4.5 we complete the proof of Theorem 4.1 by showing that $u^1(t,z)$ is localised at the site $Z_t$. The main idea is the same as in Chapter 2.

Throughout, we draw on the preliminary results established in Section 4.2. Section 4.2.1 contains a compilation of general results on operators of the form $\Delta\sigma^{-1} + \xi$. Section 4.2.2 contains a proof of the existence of 'quick' paths through the trapping landscape $\sigma$.

Note that our main goal in this chapter is to emphasise, as much as possible, which aspects of the proofs are similar to the equivalent proofs in the PAM case, and which aspects must be adapted to take into account the impact of the trapping landscape $\sigma$. As such, whenever the result follows identically as in the PAM case we will omit the proof.

## 4.2 Preliminary results: General theory for Bouchaud–Anderson operators and the existence of quick paths

In this section we collect preliminary results that will be essential in our proofs. The first set of results develops the general theory of operators of the 'Bouchaud–Anderson type', that is, operators of the form $\Delta\sigma^{-1} + \xi$. The second set of results deals with the existence of 'quick' paths through the trapping landscape $\sigma$.

### 4.2.1 General theory for Bouchaud–Anderson operators

In this section we develop general theory for operators of the form $\Delta\sigma^{-1} + \xi$ which is valid for arbitrary $\xi$ and positive $\sigma$; most of these results are generalisation of those appearing in Chapter 2, although some are specific to this setting. This section will be entirely self-contained and is completely deterministic, and may be of independent interest.

Throughout this section, let $D \subseteq \mathbb{Z}^d$ be a bounded domain and let $\xi$ and $\sigma$ be arbitrary functions $\xi : \mathbb{Z}^d \to \mathbb{R}$ and $\sigma : \mathbb{Z}^d \to \mathbb{R}^+$, with $\eta := \xi - \sigma^{-1}$. Denote by $\mathcal{H}$ the operator $\Delta\sigma^{-1} + \xi$ restricted to $D$, and let $\{\lambda_i\}_{i \le |D|}$ and $\{\varphi_i\}_{i \le |D|}$ be respectively the (finite) set of eigenvalues and eigenfunctions of $\mathcal{H}$, with eigenvalues in descending order and eigenfunctions $\ell_2$-normalised. Finally, recall that $X_s$ denotes the BTM and define the stopping times

$$\tau_z := \inf\{t \ge 0 : X_t = z\} \quad \text{and} \quad \tau_{D^c} := \inf\{t \ge 0 : X_t \notin D\}\,.$$

We start by presenting representations and deriving simple bounds for $\lambda_1$ and $\varphi_1$.

**Lemma 4.4** (Principal eigenvalue monotonicity). *For each $z \in D$ and $\delta > 0$, let $\bar{\lambda}_1$ be the principal eigenvalue of the operator*

$$\bar{\mathcal{H}} := \Delta\sigma^{-1} + \xi + \delta \mathbb{1}_{\{z\}}$$

*restricted to $D$. Then $\bar{\lambda}_1 > \lambda_1$. Moreover, for each bounded domain $\bar{D}$ containing $D$ as a strict subset, let $\bar{\lambda}_1$ be the principal eigenvalue of the operator*

$$\bar{\mathcal{H}} := \Delta\sigma^{-1} + \xi$$

*restricted to $\bar{D}$. Then $\bar{\lambda}_1 > \lambda_1$.*

**Lemma 4.5** (Bounds on the principal eigenvalue).

$$\max_{z\in D} \{\eta(z)\} \le \lambda_1 \le \max_{z\in D} \left\{ \eta(z) + \sum_{|y-z|=1} \frac{1}{2d} \sigma^{-1}(y) \right\}.$$

**Proposition 4.6** (Feynman-Kac representation for the principal eigenfunction). *For each $y, z \in D$ the principal eigenfunction $\varphi_1$ satisfies the Feynman-Kac representation*

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \frac{\sigma(y)}{\sigma(z)} \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_z} (\xi(X_s) - \lambda_1)\, ds \right\} \mathbb{1}_{\{\tau_{D^c} > \tau_z\}} \right]. \tag{4.7}$$

*Proof.* Consider $z$ fixed and define $v^z(y) := \varphi_1(y)/\varphi_1(z)$. Note that the function $v^z$ satisfies the Dirichlet problem

$$\begin{aligned} (\Delta\sigma^{-1} + \xi - \lambda_1)\, v^z(y) &= 0\,, & y &\in D \setminus \{z\}\,, \\ v^z(y) &= \mathbb{1}_{\{z\}}(y)\,, & y &\notin D \setminus \{z\}\,. \end{aligned}$$

It is easy to check (for instance, by integrating over the first holding time) that the Feynman-Kac representation on the right-hand side of equation (4.7) also satisfies this Dirichlet problem; hence we are done if there is a unique solution. So assume another non-trivial solution $w$ exists. Then the difference $q := v^z - w$ satisfies the Dirichlet problem

$$\begin{aligned} (\Delta\sigma^{-1} + \xi - \lambda_1)\, q(y) &= 0\,, & y &\in D \setminus \{z\}\,, \\ q(y) &= 0\,, & y &\notin D \setminus \{z\}\,. \end{aligned}$$

which is nonzero if and only if $\lambda_1$ is an eigenvalue of the operator $\Delta\sigma^{-1} + \xi$ restricted to $D \setminus \{z\}$. By the domain monotonicity of the principal eigenvalue in Lemma 4.4, this is impossible. □

**Lemma 4.7** (Path-wise evaluation). *For each $k \in \mathbb{N}$, $y, z \in D$, $p \in \Gamma_k(z, y)$ such that $p_i \neq y$ for $i < k$ and $\{p\} \subseteq D$, and $\zeta > \max_{0 \le i < k} \eta(p_i)$, we have*

$$\mathbb{E}_z \left[ \exp\left\{ \int_0^{\tau_y} (\xi(X_s) - \zeta)\, ds \right\} \mathbb{1}_{\{p_k(X) = p\}} \right] = \prod_{i=0}^{k-1} \frac{1}{2d} \frac{\sigma^{-1}(p_i)}{\zeta - \eta(p_i)}.$$

**Proposition 4.8** (Path expansion for the principal eigenvector). *For each $y, z \in D$ the* *principal eigenfunction* $\varphi_1$ *satisfies the path expansion*

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \frac{\sigma(y)}{\sigma(z)} \sum_{k\geq 1} \sum_{\substack{p\in\Gamma_k(y,z) \\ p_i\neq z,\, 0\leq i<k \\ \{p\}\subseteq D}} \prod_{0\leq i<k} \frac{1}{2d} \frac{\sigma^{-1}(p_i)}{\lambda_1 - \eta(p_i)} \, .$$

**Proposition 4.9** (Path expansion for the principal eigenvalue)**.** *For each* $z \in D$ *the principal eigenvalue has the path expansion*

$$\lambda_1 = \eta(z) + \sigma^{-1}(z) \sum_{k\geq 2} \sum_{\substack{p\in\Gamma_k(z,z) \\ p_i\neq z,\, 0<i<k \\ \{p\}\subseteq D}} \prod_{0<i<k} \frac{1}{2d} \frac{\sigma^{-1}(p_i)}{\lambda_1 - \eta(p_i)} \, .$$

*Proof.* Recalling that the eigenfunction relation evaluated at a site $z$ gives

$$\lambda_1 = \eta(z) + \sum_{|y-z|=1} \sigma^{-1}(y) \, \frac{\varphi_1(y)}{\varphi_1(z)} \, ,$$

the result follows from Proposition 4.8. □

We now study the solution $u_z(t,y)$ to the Cauchy problem

$$\begin{aligned} \frac{\partial u_z(t,y)}{\partial t} &= \mathcal{H}\, u(t,y) \ , && (t,y) \in [0,\infty) \times D \ ; \\ u_z(0,y) &= \mathbb{1}_{\{z\}}(y) \ , && y \in \mathbb{Z}^d \, . \end{aligned}$$

In particular, we give the spectral representation of $u_z(t,y)$ and deduce upper and lower bounds.

**Proposition 4.10** (Feynman-Kac representation of the solution)**.** *For each* $y, z \in D$*,*

$$u_z(t,y) = \mathbb{E}_z \left[ \exp\left\{ \int_0^t \xi(X_s) ds \right\} \mathbb{1}_{\{X_t = y\}} \mathbb{1}_{\{\tau_{D^c} > t\}} \right] \, .$$

**Lemma 4.11** (Time-reversal)**.** *For each* $y, z \in D$*,*

$$u_z(t,y)\, \sigma(z) = u_y(t,z)\, \sigma(y) \, .$$

*Proof.* Consider the symmetric operator

$$\tilde{\mathcal{H}} := \sigma^{-\frac{1}{2}} \mathcal{H} \sigma^{\frac{1}{2}} = \sigma^{-\frac{1}{2}} \Delta \sigma^{-\frac{1}{2}} + \xi$$

restricted to $D$, which can be viewed as the 'symmetrised' form of $\mathcal{H}$. Since,

$$e^{\tilde{\mathcal{H}}t} = e^{\sigma^{-\frac{1}{2}} \mathcal{H} \sigma^{\frac{1}{2}} t} = \sigma^{-\frac{1}{2}} e^{\mathcal{H}t} \sigma^{\frac{1}{2}} \, ,$$

we have, by the fact that $\tilde{\mathcal{H}}$ is symmetric,

$$u_z(t,y) = e^{\mathcal{H}t} \mathbb{1}_{\{z\}}(y) = \left( \frac{\sigma(y)}{\sigma(z)} \right)^{\frac{1}{2}} e^{\tilde{\mathcal{H}}t} \mathbb{1}_{\{z\}}(y) = \left( \frac{\sigma(y)}{\sigma(z)} \right)^{\frac{1}{2}} e^{\tilde{\mathcal{H}}t} \mathbb{1}_{\{y\}}(z)$$

$$= \frac{\sigma(y)}{\sigma(z)} e^{\mathcal{H}t} \mathbb{1}_{\{y\}}(z) = \frac{\sigma(y)}{\sigma(z)} u_y(t,z)\,. \qquad \square$$

**Proposition 4.12** (Spectral representation for the solution)**.** *For each $y, z \in D$, the solution $u_z(t,y)$ satisfies the spectral representation*

$$u_z(t,y) = \sigma^{-1}(z) \sum_i \frac{e^{\lambda_i t} \varphi_i(z) \varphi_i(y)}{\|\sigma^{-\frac{1}{2}} \varphi_i\|_{\ell_2}^2}\,.$$

*Proof.* Recall the symmetric operator $\tilde{\mathcal{H}}$ from the proof of Lemma 4.11. Note that each ($\ell_2$-normalised) eigenfunction $\tilde{\varphi}_i$ of $\tilde{\mathcal{H}}$ satisfies the relation

$$\tilde{\varphi}_i = \frac{\sigma^{-\frac{1}{2}} \varphi_i}{\|\sigma^{-\frac{1}{2}} \varphi_i\|_{\ell_2}}$$

with $\lambda_i$ the corresponding eigenvalue for $\tilde{\varphi}_i$. The proof then follows by applying the spectral theorem to $\tilde{\mathcal{H}}$. $\square$

**Corollary 4.13** (Bounds on the solution)**.** *For each $z \in D$ we have the bounds*

$$\frac{e^{\lambda_1 t} \sigma^{-1}(z) \varphi_1^2(z)}{\|\sigma^{-\frac{1}{2}} \varphi_1\|_{\ell_2}^2} \le u_z(t,z) \le e^{\lambda_1 t}\,.$$

**Proposition 4.14** (Bound on the total mass of the solution)**.** *For each $y, z \in D$,*

$$\sum_{y \in D} u_z(t,y) \le e^{\lambda_1 t} \sum_{y \in D} \frac{\varphi_1(y)}{\varphi_1(z)}\,.$$

*Proof.* As in the proof of Proposition 2.13 (this time applying the Feynman-Kac representation for the principal eigenfunction in Proposition 4.8), we have that

$$u_y(t,z) \le e^{\lambda_1 t} \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_z} (\xi(X_s) - \lambda_1)\, ds \right\} \mathbb{1}_{\{\tau_z < \tau_{D^c}\}} \right] = e^{\lambda_1 t} \frac{\varphi_1(y)}{\varphi_1(z)} \frac{\sigma(z)}{\sigma(y)}\,.$$

Applying the time-reversal Lemma 4.11, we have

$$u_z(t,y) = u_y(t,z) \frac{\sigma(y)}{\sigma(z)} \le e^{\lambda_1 t} \frac{\varphi_1(y)}{\varphi_1(z)}\,,$$

which, after summing over $y \in D$, yields the result. $\square$

Next we prove the analogue of the 'cluster expansion' in Lemma 2.15. To apply this to our analysis of the BAM, we shall actually need an additional form of the bound to accommodate the impact of the trapping landscape (see the proof of Lemma 4.42).

**Lemma 4.15** (Cluster expansion)**.** *For each $z \in D$ and for any $\zeta > \lambda_1$,*

$$\mathbb{E}_z \left[ \exp\left\{ \int_0^{\tau_{D^c}} (\xi(X_s) - \zeta)\, ds \right\} \right] < 1 + \frac{\max_{z \in D} \{\sigma^{-1}(z)\}\, |D|}{\zeta - \lambda_1}$$

*and*

$$\mathbb{E}_z \left[ \exp\left\{ \int_0^{\tau_{D^c}} (\xi(X_s) - \zeta)\, ds \right\} \right] < \frac{\sigma^{-1}(z)}{\zeta - \lambda_1} \left( 1 + \frac{\max_{z \in D}\{\sigma^{-1}(z)\}\, |D|}{\zeta - \lambda_1} \right) .$$

*Proof.* We proceed by modifying the proof of Lemma 4.15. First abbreviate

$$u(y) := \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_{D^c}} (\xi(X_s) - \zeta)\, ds \right\} \right]$$

and remark that $u$ solves the boundary value problem

$$\begin{aligned} (\sigma^{-1}\Delta + \xi - \zeta)u(y) &= 0\,, & y &\in D\,; \\ u(y) &= 1\,, & y &\notin D\,. \end{aligned} \tag{4.8}$$

Note that, in contrast to in the proof of Proposition 4.6, in the above boundary value problem the relevant operator is the transpose of $\mathcal{H}$, since here we have not weighted the expectation by $\sigma$. We make the substitution $w := u - \mathbf{1}$, where $\mathbf{1}$ denotes the vector of ones, which turns (4.8) into

$$\begin{aligned} (\sigma^{-1}\Delta + \xi - \zeta)w(y) &= -\left((\sigma^{-1}\Delta + \xi - \zeta)\mathbf{1}\right)(y)\,, & y &\in D\,; \\ w(y) &= 0\,, & y &\notin D\,. \end{aligned}$$

Since $\zeta > \lambda_1$, the solution exists and is given by

$$w(y) = \mathcal{R}_\zeta((\sigma^{-1}\Delta + \xi - \zeta)\mathbf{1})(y) = \mathcal{R}_\zeta(\xi - \zeta)(y)$$

where $\mathcal{R}_\zeta$ denotes the resolvent of $\sigma^{-1}\Delta + \xi = \sigma^{-1}\mathcal{H}\sigma$, restricted to $D$, at $\zeta$. By Lemma 4.5 and since $\zeta > \lambda_1$ we have that $\xi(y) - \zeta < \sigma^{-1}(y)$ for all $y \in D$, and so by the positivity of the resolvent (guaranteed since $\sigma^{-1}\Delta + \xi$ is elliptic and $\zeta > \lambda_1$) we obtain

$$w(z) < \mathcal{R}_\zeta\left(\sigma^{-1}\right)(y) = \sigma^{-\frac{1}{2}}\tilde{\mathcal{R}}_\zeta\left(\sigma^{-\frac{1}{2}}\right)(y) \le \max_{z \in D}\{\sigma^{-1}(z)\}|D|\, \|\tilde{\mathcal{R}}_\zeta\|\,,$$

where $\tilde{\mathcal{R}}_\zeta$ denotes the resolvent of the symmetric operator $\sigma^{-\frac{1}{2}}\Delta\sigma^{-\frac{1}{2}} + \xi$, restricted to $D$, at $\zeta$, and $\|\cdot\|$ denotes the spectral norm. By considering the spectral representation of $\tilde{\mathcal{R}}_\zeta$, we have $\|\tilde{\mathcal{R}}_\zeta\| \le (\zeta - \lambda_1)^{-1}$, which gives the first bound. For the second bound, consider that (4.8) implies the identify

$$u(y) = \frac{\sigma^{-1}(y)}{\zeta - \xi(y) + \sigma^{-1}(y)} \sum_{|x-y|=1} \frac{1}{2d} u(x)\,. \tag{4.9}$$

Applying the first bound to each $u(x)$ in the sum in (4.9), the result follows by bounding $\xi(y) - \sigma^{-1}(y)$ in the denominator of (4.9) from above by $\lambda_1$, valid by the lower bound in Lemma 4.5. $\square$

Finally, we give a general way to bound the contribution to the solution $u_z(t, y)$ from paths that hit a certain site $x \in D$ and then stay within a subdomain $E \subseteq D$ that contains $x$; this is analogous to Proposition 2.16.

Fix a domain $E \subseteq D$, a site $x \in E$, and define the operator $\mathcal{H}^E := \Delta\sigma^{-1} + \xi$ restricted to $E$ with $\lambda_1^E$ and $\varphi_1^E$ respectively its principal eigenvalue and eigenfunction. Define the stopping time

$$\tau_{x,E^c} := \inf\{t \geq \tau_x : X_t \notin E\}\,.$$

Then the contribution to the solution $u_z(t, y)$ from paths that hit $x$ and then stay within $E$ can be written

$$u_z^{x,E}(t,y) := \mathbb{E}_z \left[ \exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}_{\{X_t = y, \tau_x \leq t, \tau_{x,E^c} > t, \tau_{D^c} > t\}} \right].$$

**Proposition 4.16** (Link between solution and principal eigenfunction)**.** *For each $x \in E$, $y \in E \setminus \{x\}$ and $z \in D$,*

$$\frac{u_z^{x,E}(t,y)}{\sum_{y \in D} u_z(t,y)} \leq \frac{\sigma(y) \|\sigma^{-\frac{1}{2}} \varphi_1^E\|_{\ell_2}^2}{(\varphi_1^E(x))^3}\, \varphi_1^E(y)\,.$$

*Proof.* We proceed by modifying the proof of Proposition 2.16. The first step is to make use of the time-reversal in Lemma 4.11, suitably adapted to $u_z^{x,E}(t,y)$. In particular, defining

$$\overleftarrow{u_y^{x,E}}(t,z) := \mathbb{E}_y \left[ \exp\left\{ \int_0^t \xi(X_s)\, ds \right\} \mathbb{1}_{\{X_t = z, \tau_x \leq t, \tau_x < \tau_{E^c}, \tau_{D^c} > t\}} \right]$$

we can write

$$\frac{u_z^{x,E}(t,y)}{\sum_{y \in D} u_z(t,y)} \leq \frac{u_z^{x,E}(t,y)}{u_z(t,x)} = \frac{\sigma(y)}{\sigma(x)} \frac{\overleftarrow{u_y^{x,E}}(t,z)}{u_x(t,z)}\,. \tag{4.10}$$

Next we decompose the Feynman-Kac formula for $\overleftarrow{u_y^{x,E}}(t,z)$ as in the proof of Proposition 4.14, by conditioning on the stopping time $\tau_x$ and using the strong Markov property. More precisely, we write

$$\begin{aligned}\overleftarrow{u_y^{x,E}}(t,z) = \mathbb{E}_{\tau_x} \Bigg[ e^{\tau_x \lambda_1^E}\, \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_x} \left(\xi(X_s) - \lambda_1^E\right)\, ds \right\} \mathbb{1}_{\{\tau_x < \tau_{E^c}\}} \middle| \tau_x \right] \\ \times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t-\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{t-\tau_x} = z, \tau'_{D^c} > t - \tau_x\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \leq t\}} \Bigg]\,,\end{aligned}$$

where $\mathbb{E}_{\tau_x}$, $X'_t$ and $\tau'_{D^c}$ are defined as in the proof of Proposition 4.14. Next, note that an application of Corollary 4.13 gives the bound

$$1 \leq u_x^{x,E}(w,x)\, \frac{\sigma(x) \|\sigma^{-\frac{1}{2}} \varphi_1^E\|_{\ell_2}^2}{(\varphi_1^E(x))^2}\, e^{-w\lambda_1^E}\,, \tag{4.11}$$

and recall the representation

$$u_x^{x,E}(w,x) = \mathbb{E}_x \left[ \exp\left\{ \int_0^w \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_w = x, \tau'_{E^c} > w\}} \right].$$

Combining the bound in (4.11) with equation (2.12) (setting $w = \tau_x$), gives

$$\overleftarrow{u_y^{x,E}}(t,z) \leq \frac{\sigma(x) \|\sigma^{-\frac{1}{2}} \varphi_1^E\|_{\ell_2}^2}{(\varphi_1^E(x))^2}\, \mathbb{E}_{\tau_x} \left[ \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_x} \left(\xi(X_s) - \lambda_1^E\right)\, ds \right\} \mathbb{1}_{\{\tau_{E^c} > \tau_x\}} \middle| \tau_x \right] \right.$$

$$
\begin{aligned}
&\times \mathbb{E}_x \left[ \exp\left\{ \int_0^{\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{\tau_x}=x, \tau'_{E^c}>\tau_x\}} \middle| \tau_x \right] \\
&\times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t-\tau_x} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_{t-\tau_x}=z, \tau'_{D^c}>t-\tau_x\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \le t\}} \Bigg] \\
\le{}& \frac{\sigma(x) \|\sigma^{-\frac{1}{2}} \varphi_1^E\|_{\ell_2}^2}{(\varphi_1^E(x))^2}\, \mathbb{E}_{\tau_x} \left[ \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_x} \left( \xi(X_s) - \lambda_1^E \right) ds \right\} \mathbb{1}_{\{\tau_{E^c}>\tau_x\}} \middle| \tau_x \right] \right. \\
&\times \mathbb{E}_x \left[ \exp\left\{ \int_0^{t} \xi(X'_s)\, ds \right\} \mathbb{1}_{\{X'_t=z, \tau'_{D^c}>t\}} \middle| \tau_x \right] \mathbb{1}_{\{\tau_x \le t\}} \Bigg] \\
\le{}& \frac{\sigma(x)^2 \|\sigma^{-\frac{1}{2}} \varphi_1^E\|_{\ell_2}^2}{\sigma(y)(\varphi_1^E(x))^3}\, \varphi_1^E(y)\; u_x(t,z)\,,
\end{aligned}
$$

where the inequality in the second step results from deleting the condition that $X'_{\tau_x} = x$, and where the last inequality results from deleting the condition that $\tau_x \le t$, and where we have used the Feynman-Kac representation for $\varphi_1^E$ given by Proposition 4.6. Combining this with equation (4.10) gives the result. □

### 4.2.2 Existence of quick paths

In this section we prove the existence of paths $p \in \Gamma(0,z)$ for certain $z \in V_t$ that have the property that (i) all $\sigma(p_i)$ are relatively small, and (ii) $p$ is not much longer than a direct path to $z$; what we mean by 'relatively small' and 'not much longer' will depend on the dimension. We shall informally refer to such paths as *quick paths*. The reason we are interested in quick paths is that they are intimately related to the probability that a particle undertaking the BAM reaches a certain site $z$ by time $t$, and so form an essential part of our proof that the trapping landscape does not prevent complete localisation in the BAM.

In dimension higher than one, we will additionally require that such paths do not travel too close to a certain well-separated set $S_t$. The reason for this additional requirement is that we will eventually seek to apply our results to the site $Z_t$, which depends in a complicated way on $\sigma(z)$ for $z \in B(\Pi^{(L_t)}, \rho_\sigma)$. We will wish to avoid this dependence, hence our insistence on the fact that the paths do not travel too close to $S_t$.

As a preliminary, we recall the asymptotics for the upper order statistics of $\sigma$ (since the trapping landscape is Weibull, this is identical to the asymptotics for $\xi$ in Chapter 2).

**Lemma 4.17** (Almost sure asymptotics for $\sigma$)**.** *Denote by $\sigma_n^1$ the largest value among $n$ i.i.d. copies of $\sigma(0)$. Then, as $n \to \infty$,*

$$\sigma_n^1 \sim (\log n)^{\frac{1}{\mu}}$$

*eventually almost surely.*

#### Quick paths in dimension one

In dimension one, there is only one shortest path from 0 to $z$ and this must pass through all intermediate sites. Hence, we seek to show that not too many traps on this path are too large. Clearly, the ability to do this depends on the tail decay of $\sigma(0)$, which is the origin of the extra tail decay condition for $d = 1$ that appears in [59].

To proceed, we must undertake a rather delicate analysis of the trapping landscape $\sigma$ in the region between 0 and $z$. We simplify this using coarse graining, essentially placing each site $y$ into a certain 'bin' depending on the value of $\sigma(y)$. We then seek to bound the number of sites in each bin, weighted by the depth of the traps corresponding to each bin. Note that the construction of the coarse graining scales is relatively simple in the case of Weibull traps; see [59] for the construction in a more general setting, as well as remarks as to how this construction breaks down if the tail of $\sigma(0)$ is too heavy.

Let $n_t$ be an arbitrary function such that $n_t \to \infty$ as $t \to \infty$. For a constant $c > 1$, define the scaling functions $\{\sigma_t^i\}_{i=0,1,2}$ by

$$\sigma_t^0 = 0\,, \ \sigma_t^1 := (\log\log n_t)^{c^{-1}} \quad \text{and} \quad \log \sigma_t^2 = (\sigma_t^1)^\mu\,,$$

and note that these satisfy, as $t \to \infty$,

$$\left(\sigma_t^1\right)^\mu \leq c^{-1} \log n_t \quad \text{and} \left(\sigma_t^2\right)^\mu \geq c \log n_t \tag{4.12}$$

eventually. For a path $p \in \Gamma_k$ define

$$N_i = \sum_{0 \leq i < k} \mathbb{1}_{\{\sigma(p_i) \in (\sigma_t^{i-1}, \sigma_t^i]\}} \quad \text{for each } 1 \leq i \leq 2\,.$$

The following proposition essentially bounds the number of sites in each coarse graining scale, weighted by the log of the scale. This will turn out to be the correct definition of a 'quick path' in Section 4.4.

**Proposition 4.18** (Existence of quick paths; $d = 1$)**.** *As $t \to \infty$, each path $p \in \Gamma_{|z|}(0, z)$ with $|z| < n_t$, satisfies*

$$\mathbf{P}\left( \sum_{i=1,2} N_i \log \sigma_t^i < n_t (\log\log\log n_t)^2 \right) \to 1$$

*and*

$$\max_{0 \leq i < |z|} \sigma(p_i) < \sigma_t^2\,,$$

*eventually almost surely.*

*Proof.* Abbreviating

$$\bar{F}_\sigma(x) := \mathbf{P}(\sigma(0) > x) = \exp\{-x^\mu\}\,,$$

we first prove that the event

$$\mathcal{N}_t := \bigcup_{i=1,2} \left\{ N_i \leq 2 n_t \, \exp\{F_\sigma(\sigma_t^{i-1})\} \right\}$$

satisfies $\mathbf{P}(\mathcal{N}_t) \to 1$ as $t \to \infty$. Note that each $N_i$ is stochastically dominated by a binomial random variable

$$\bar{N}_i \overset{d}{=} \text{Binom}(n_t, \bar{F}_\sigma(\sigma_t^{i-1}))\,,$$

with $\mathbf{E}\bar{N}_i = n_t \bar{F}_\sigma(\sigma_t^{i-1})$ and $\text{Var}\bar{N}_i \leq n_t \bar{F}_\sigma(\sigma_t^{i-1})$. By the union bound and Chebyshev's inequality,

$$\mathbf{P}\Big(\bigcup_i \{\bar{N}_i > 2\mathbf{E}\bar{N}_i\}\Big) \le \sum_i \mathbf{P}(\bar{N}_i > 2\mathbf{E}\bar{N}_i) \le \sum_i \frac{\mathrm{Var}\bar{N}_i}{(\mathbf{E}\bar{N}_i)^2} \le \sum_i \big(n_t \bar{F}_\sigma(\sigma_t^{i-1})\big)^{-1} . \quad (4.13)$$

Since the $\sigma_t^i$ are increasing in $i$ and recalling (4.12), for $i \in \{1,2\}$,

$$\bar{F}_\sigma(\sigma_t^{i-1}) \ge \bar{F}_\sigma(\sigma_t^2) \ge n_t^{-c^{-1}} .$$

Combining with (4.13), by the union bound, as $t \to \infty$, eventually

$$\mathbf{P}(\mathcal{N}_t) > 1 - 2\, n_t^{c^{-1}-1} \to 1 ,$$

since $c > 1$.

Now assume the event $\mathcal{N}_t$ holds and split the sum

$$\sum_{i=1,2} N_i \log \sigma_t^i = N_1 \log \sigma_t^1 + N_2 \log \sigma_t^2 .$$

For the first term, on the event $\mathcal{N}_t$ and recalling the definition of $\sigma_t^1$ we have

$$N_1 \log \sigma_t^1 \le 2n_t \bar{F}_\sigma(\sigma_t^0) \log \sigma_t^1 = 2n_t \log \sigma_t^1 = 2c^{-1} n_t \log\log\log n_t < n_t (\log\log\log n_t)^2/2$$

eventually. Similarly, for the second term, on the event $\mathcal{N}_t$ and recalling the definition of $\sigma_t^2$ we have

$$N_i \log \sigma_t^2 \le 2n_t \bar{F}_\sigma(\sigma_t^1) \log \sigma_t^2 \le 2n_t < n_t (\log\log\log n_t)^2/2$$

eventually. Finally, the fact that

$$\max_{0 \le i < |z|} \sigma(p_i) < \sigma_t^2$$

eventually almost surely follows from (4.12) and Lemma 4.17. ☐

### Quick paths in dimension higher than one

In dimensions higher than one we use percolation-type estimates to prove the existence of a path $p \in \Gamma(0,z)$ with $z \in S_t$ for some well-separated set $S_t$ that: (i) avoids all the deep traps; (ii) has $|p|$ not much more than $|z|$; and (iii) does not travel too close to sites in $S_t$. Since we use percolation-type arguments, it will turn out that we need no extra assumption on the tail decay of $\sigma(0)$. This allows us to greatly extend the validity of our result to apply to more general trapping landscapes, and in particular, for arbitrarily slowly-decaying trap distribution $\sigma(0)$; see [59].

So let us start with the relevant percolation-type estimates; for background on percolation theory see [19, 42]. Consider site percolation on $\mathbb{Z}^d$ with $\mathbb{P}(v \text{ open}) = q$ independently for every $v \in \mathbb{Z}^d$. We say that a subset of $\mathbb{Z}^d$ is $*$-connected if it is connected with respect to the adjacency relation

$$v \overset{*}{\sim} w \Leftrightarrow \max_{1 \le i \le d} |v_i - w_i| = 1 ,$$

where $v_i$ and $w_i$ denote the coordinate projections of $v$ and $w$ respectively. If $v \overset{*}{\sim} w$ we say that $w$ is a $*$-neighbour of $v$. A $*$-connected subset of $\mathbb{Z}^d$ is referred to as a $*$-cluster. The relevance of $*$-clusters is that they represent the blocking clusters for open paths in $\mathbb{Z}^d$. For $v \in \mathbb{Z}^d$ a closed site, denote by $\mathcal{C}(v)$ the largest $*$-cluster of closed sites containing $v$.

For two sites $u, v$ in $\mathbb{Z}^d$ denote by $d_\infty(u, v)$ their chemical distance (also known as the graph distance) with respect to site percolation, defined to be the length of the shortest open path from $u$ to $v$ (and defined to be infinite if no such path exists).

**Lemma 4.19** (Expected size and maximum of closed $*$-clusters)**.** *Let $q \in (1 - (3d)^{-1}, 1)$ and suppose $u_1, \dots, u_M$ are $M \in \mathbb{N}$ distinct closed sites in $\mathbb{Z}^d$. Then*

*(i)* $\mathbf{E}[|\mathcal{C}(u_1)|] \leq (1 - 3^d(1-q))^{-1}$*, and so in particular* $\mathbf{E}[|\mathcal{C}(u_1)|] \to 1$ *as* $q \to 1$*; and*

*(ii) For every* $x \in \mathbb{N}$*,*

$$\mathbf{P}(\max\{|\mathcal{C}(u_1)|, \dots, |\mathcal{C}(u_M)|\} < x) \geq 1 - M(3^d(1-q))^{[\log_{3^d} x]} .$$

*Proof.* Consider performing a breadth-first search on $\mathcal{C}(u_1)$ starting from the site $u_1$, by first discovering the closed $*$-neighbours $v_1, \dots, v_k$ of $u_1$, and then in turn discovering the closed $*$-neighbours of each of the $v_j$, $1 \leq j \leq k$, iterating this procedure to explore $\mathcal{C}(u_1)$. Suppose that the site $w$ has just been explored in this procedure. Then the number of closed $*$-neighbours of $w$ that have not already been discovered is stochastically dominated by a $\text{Binom}(3^d - 1, 1 - q)$ random variable. It follows that $|\mathcal{C}(u_1)| \prec Z$, where $Z$ is the total progeny of a branching process with offspring distribution $\text{Binom}(3^d, 1-q)$. Since $\mathbb{E}(Z) = (1 - 3^d(1-q))^{-1}$, this proves the first statement.

For the second statement, note that by the union bound we have

$$\mathbf{P}(\max\{|\mathcal{C}(u_1)|, \dots, |\mathcal{C}(u_M)|\} \geq x) \leq \sum_{i=1}^{M} \mathbf{P}(|\mathcal{C}(u_i)| \geq x) = M\,\mathbf{P}(|\mathcal{C}(u_1)| \geq x) .$$

Again by exploring $\mathcal{C}(u_1)$ we have

$$\mathbf{P}(|\mathcal{C}(u_1)| \geq x) \leq \mathbb{P}(Z \geq x) .$$

To finish the proof, note that by Markov's inequality we have

$$\mathbb{P}(Z \geq x) \leq \mathbb{P}(Z(\lfloor \log_{3^d} x \rfloor) > 0) \leq (3^d(1-q))^{[\log_{3^d} x]} ,$$

where $Z(n)$ denotes number of individuals in generation $n$ of the branching process. □

**Lemma 4.20** (Chemical distance)**.** *Fix two sites $u, v$ in $\mathbb{Z}^d$ and a function $c := c(q)$ with $c \to \infty$ as $q \to 1$. Then, as $q \to 1$,*

$$\mathbf{P}\left( \frac{d_\infty(u, v)}{|u - v|} < 1 + c(1-q) \right) \to 1 .$$

*Proof.* Denote by $\mathcal{C}_\infty$ the unique infinite open cluster, which exists almost surely for all $q$ sufficiently close to 1 (see [42]). Let $\hat{p} \in \Gamma_{|u-v|}(u, v)$ be any shortest path, denote by $K$ the subset of $\{\hat{p}\}$ consisting only of closed sites, and define

$$S := \Big| \bigcup_{x \in K} \mathcal{C}(x) \Big| \le \sum_{x \in K} |\mathcal{C}(x)| \,.$$

By part $(i)$ of Lemma 4.19 and the FKG inequality (see [42], Section 2.2), we have the bound

$$\mathbf{E}[S|\{u,v \in \mathcal{C}_\infty\}] \le \frac{\mathbf{E}[\,|K|\big|\{u,v\in\mathcal{C}_\infty\}]}{1-3^d(1-q)} \le \frac{|u-v|(1-q)}{1-3^d(1-q)} \,.$$

We now claim that, on the event $\{u,v \in \mathcal{C}_\infty\}$, it is possible to find a path $p \in \Gamma_k(u,v)$ for some $k \le |u-v| + (3^d-1)S$ such that every site in $\{p\}$ is open. To obtain the required path $p$ take the direct path $\hat{p}$ and divert it around $\mathcal{C}(u)$ for each closed $u \in \{\hat{p}\}$, so that every site in $\{p\}$ is either in $\{\hat{p}\}$ or in the outer boundary of some $\mathcal{C}(u)$, where by outer boundary we mean the set of sites $\{v \notin \mathcal{C}(u) : \exists u \in \mathcal{C}(u),\, u \overset{*}{\sim} v\}$. This procedure is possible since $u, v \in \mathcal{C}_\infty$. Then $\{p\}$ will consist of just open sites since the outer boundary of each $\mathcal{C}(u)$ is a path of open sites. The bound on $|p|$ follows from the fact that the size of the outer boundary of a $*$-cluster $A$ is at most $(3^d-1)|A|$.

We complete the proof of the Lemma with Markov's inequality:

$$\begin{aligned}\mathbf{P}\left(\frac{d_\infty(u,v)}{|u-v|} \ge 1 + c(1-q)\right) &\le \mathbf{P}\left(|S| > \frac{c(1-q)|u-v|}{(3^d-1)}\Bigg|\{u,v\in\mathcal{C}_\infty\}\right) + \mathbf{P}\left(\{u,v\in\mathcal{C}_\infty\}^c\right) \\ &\le \frac{3^d}{c(1-3^d(1-q))} + \mathbf{P}\left(\{u,v\in\mathcal{C}_\infty\}^c\right) \,.\end{aligned}$$

Since $\mathbf{P}(u,v \in \mathcal{C}_\infty) \to 1$ as as $q \to 1$, this completes the proof. □

We are now ready to show the existence of a quick path in dimensions higher than one. Let $S_t \subseteq \mathbb{Z}^d$ be such that

$$r(S_t) > t^\varepsilon \quad \text{and} \quad \min_{u \in S_t} |u| > t^\varepsilon$$

eventually for some $\varepsilon > 0$. Let $\sigma_t$ be an arbitrary function tending to infinity as $t \to \infty$. Define the set

$$\mathbb{Z}^d(\sigma_t, S_t) := \{z \in \mathbb{Z}^d : \sigma(z) \le \sigma_t, z \notin B(S_t, j)\} \,.$$

For a site $z \in \mathbb{Z}^d$, let $|z|_{\text{chem}}$ be the *chemical distance* of the ball $B(z,j)$ in this set, that is, the length of the shortest path from the origin to $\partial B(z,j)$ that lies exclusively in this subgraph (setting it as $\infty$ if such a path does not exist).

**Proposition 4.21** (Existence of quick paths; $d > 1$)**.** *Let $z_t \in S_t \cap V_t$ and let $c_t$ be a function such that $c_t \to \infty$ as $t \to \infty$ and $\bar{F}_\sigma(\sigma_t)c_t \ll 1$. Then, there exists a constant $c > 0$ such that, as $t \to \infty$,*

$$\mathbf{P}\left(\frac{|z_t|_{\text{chem}}}{|z_t|} \le 1 + \bar{F}_\sigma(\sigma_t)c_t + t^{-c}\right) \to 1 \,.$$

*Proof.* Let $q := 1 - \bar{F}_\sigma(\sigma_t)$. By Lemma 4.20, with probability tending to 1 as $t \to \infty$ there exists a path $\hat{p} \in \Gamma_{\ell_t}(0, z_t)$ for some

$$\ell_t \le |z_t|(1 + \bar{F}_\sigma(\sigma_t)c_t)$$

such that $\sigma(\hat{p}_i) \le \sigma_t$ for all $0 \le i < \ell_t$. Let $i = \min\{0 \le j < \ell_t : \hat{p}_j \in \partial B(z_t, j)\}$ and define $v_t := \hat{p}_i$ to be the first site in $\partial B(z_t, j)$ visited by path $\hat{p}$. We show how to modify $\hat{p}$ so that we obtain a new path $p \in \Gamma(0, v_t)$ for some $v_t \in \partial B(Z_t, j)$ with $\{p\} \subseteq \mathbb{Z}^d(\sigma_t, S_t)$.

The modification is done by diverting $\hat{p}$ around the balls of radius $j$ centred on sites in $S_t$. In doing so, we may encounter new closed sites $v$, and these too must be avoided if we wish to find a path $p$ with $\{p\} \subseteq \mathbb{Z}^d(\sigma_t, S_t)$. Formally, the set of these new closed sites is precisely

$$\{x \in \partial B\left(S_t \cap B(\{\hat{p}\}, j), j\right) : \sigma(x) > \sigma_t\}.$$

Denote by $M_t$ the size of this set and its elements as $w_1, \ldots, w_{M_t}$, and choose $0 < c_1 < \varepsilon$ where $\varepsilon$ is the constant appearing in the definition of $S_t$. Then by the separation of sites in $S_t$, we have

$$|S_t \cap B(\{\hat{p}\}, j)| \le \ell_t t^{-\varepsilon},$$

and so

$$M_t \le 3^d |B(0,j)| \ell_t t^{-\varepsilon} < |z_t| t^{-c_1} \tag{4.14}$$

for all $t$ sufficiently large. Choose now $0 < c_2 < c_1$, $\alpha < -1 - (1 - c_1)/c_2$, and $t$ sufficiently large so that

$$\bar{F}_\sigma(\sigma_t) < 3^{d\alpha}.$$

Applying part $(ii)$ of Lemma 4.19, we deduce that

$$\max\{|\mathcal{C}(w_1)|, \ldots, |\mathcal{C}(w_{M_t})|\} \le t^{c_2}$$

with probability tending to 1 as $t \to \infty$. We claim this implies that, by the separation of sites in $S_t$ and the fact that $c_2 < \varepsilon$, with overwhelming probability there exists a path $p \in \Gamma(0, v_t)$ which avoids all $j$-balls centred on sites in $S_t$ and all closed sites. Indeed to obtain this path we take path $\hat{p}$ and then divert around $j$-balls centred on sites in $S_t$ and then further divert around any new closed $*$-clusters we encounter. Since we know that no such cluster is too large, they cannot cut the origin off from $v_t$ in $\mathbb{Z}^d(\sigma_t, S_t)$, and furthermore we will not encounter any more sites in $S_t$ on the new path.

We can now bound $|p|$. The number of additional sites we must visit to obtain $p$ from $\hat{p}$ is at most $3^d M_t(|B(0,j)| + t^{c_2})$ with probability tending to 1 as $t \to \infty$; this comes from counting the diversions around each $j$-ball and the diversions around each closed cluster we then encounter. Using (4.14), we can choose a $0 < c < c_1 - c_2$ to yield the result. □

## 4.3 Extremal theory for local principal eigenvalues

In this section, we use point process techniques to study the random variables $Z_t^{(j)}$ and $\Psi_t^{(j)}(Z_t^{(j)})$, and generalisations thereof; the techniques used are similar to those in Chapter 2. In the process, we complete the proof of Theorems 4.2 and 4.3. Throughout this section, let $\varepsilon$ be such that $0 < \varepsilon < \theta$.

### 4.3.1 Upper-tail properties of the local principal eigenvalues

The first step is to give upper-tail asymptotics for the distribution of the local principal eigenvalues $\lambda^{(n)}(z)$ for $z \in \Pi^{(L_t)}$ and $n \in \mathbb{N}$. These will allow us to study the random variables $Z_t^{(j)}$ and $\Psi_t^{(j)}(Z_t^{(j)})$ via point process techniques. For technical reasons, we shall actually consider a *punctured* version of $\lambda^{(n)}(z)$ which will coincide with $\lambda^{(n)}(z)$ eventually almost surely for each $z \in \Pi^{(L_t)}$.

To this end, let $\{\tilde{\xi}_z\}_{z \in V_t}$ be a collection of independent potential fields $\tilde{\xi}_z : \mathbb{Z}^d \to \mathbb{R}$ defined so that, for each $z \in V_t$, we have $\tilde{\xi}_z(z) = \xi(z)$, and, for each $y \in V_t \setminus \{z\}$, instead $\tilde{\xi}_z(y)$ is i.i.d. with common distribution

$$\tilde{\xi}(0) = \begin{cases} \xi(0), & \text{if } \xi(0) < L_t\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

Then, for each $z \in V_t$ and $n \in \mathbb{N}$, let $\tilde{\lambda}_t^{(n)}(z)$ be the principal eigenvalue of the *punctured* operator $\tilde{\mathcal{H}}^{(n)}(z) := \Delta\sigma^{-1} + \tilde{\xi}_z$ restricted to $B(z,n)$.

**Proposition 4.22** (Path expansion for $\tilde{\lambda}_t^{(n)}$)**.** *For each $n \in \mathbb{N}$ and $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly, as $t \to \infty$,*

$$\begin{aligned} \tilde{\lambda}_t^{(n)}(z) &= \eta(z) + \sigma^{-1}(z) \sum_{2 \le k \le 2j} \sum_{\substack{p \in \Gamma_k(z,z) \\ p_i \neq z,\, 0<i<k \\ \{p\} \subseteq B(z,n)}} \prod_{0<i<k} (2d)^{-1} \frac{\sigma^{-1}(p_i)}{\tilde{\lambda}_t^{(n)}(z) - \eta(p_i)} + o(d_t e_t)\,, \\ &= \eta(z) + O(a_t^{-1})\,. \end{aligned}$$

*Moreover, as $t \to \infty$,*

$$\tilde{\lambda}_t^{(n)}(z) = \lambda^{(n)}(z)$$

*eventually almost surely.*

*Proof.* This follows just as in Proposition 2.21; the error asymptotics remain valid since $\sigma^{-1}(p_i)$ is bounded above by $\delta_\sigma^{-1}$. ☐

**Proposition 4.23** (Extremal theory for $\tilde{\lambda}_t^{(n)}$)**.** *For each $n \in \mathbb{N}$, there exists a scaling function $A_t = a_t + O(1)$ such that, as $t \to \infty$ and for each fixed $x \in \mathbb{R}$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) \to e^{-x}\,.$$

*Moreover, there exists a $c > 0$ such that, as $t \to \infty$ and uniformly for $x > 0$,*

$$t^d\, \mathbf{P}\left(\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t\right) < e^{-c x^{\min\{1,\gamma\}}}\,.$$

*Proof.* First remark that, by Lemmas 2.19 and 4.5, as $t \to \infty$,

$$\tilde{\lambda}_t^{(n)}(0) > A_t + x d_t \quad \text{implies that} \quad \xi(0) > L_{t,\varepsilon}\,,$$

eventually almost surely, which means that we can apply the path expansion in Proposition 4.22 to $\tilde{\lambda}_t^{(n)}(0)$. Let $A_t$ be an arbitrary scale such that $A_t = a_t + O(1)$, and define the function

$$Q(A_t;\xi,\sigma) := \sigma^{-1}(0) + \sigma^{-1}(0) \sum_{2\le k\le 2j} \sum_{\substack{p\in\Gamma_k(0,0)\\ p_i\neq 0,\,0<i<k\\ \{p\}\subseteq B(z,j)}} \prod_{0<i<k} (2d)^{-1} \frac{\sigma^{-1}(p_i)}{A_t-\eta(p_i)}\,,$$

if $\xi(y) < L_t$ for each $y \in B(0,j)\setminus\{0\}$ and $Q(A_t;\xi,\sigma) := 0$ otherwise. Note that, as $t\to\infty$, $Q(A_t;\xi,\sigma) = O(1)$ uniformly in $\xi$ and $\sigma$. Then, since $\tilde\lambda_t^{(n)}(0)$ is strictly increasing in $\xi(0)$ we have that, as $t\to\infty$,

$$\begin{aligned}
\mathbf{P}\left(\tilde\lambda_t^{(n)}(0) > A_t + xd_t\right) &\sim \mathbf{P}\Big(\xi(0) > A_t + xd_t + Q(A_t+xd_t;\xi,\sigma)\Big)\\
&\sim \mathbf{P}\Big(\xi(0) > A_t + xd_t + Q(A_t;\xi,\sigma)\Big) \qquad (4.15)\\
&\sim t^{-d}e^{-x}\int_{\xi,\sigma} \exp\left\{a_t^\gamma - \Big(A_t + Q(A_t;\xi,\sigma)\Big)^\gamma\right\} d\mu_\xi\, d\mu_\sigma\\
&\sim t^{-d}e^{-x}\int_{\xi,\sigma} \exp\left\{a_t^\gamma - (A_t+O(1))^\gamma\right\} d\mu_\xi\, d\mu_\sigma
\end{aligned}$$

where the first asymptotic accounts for the error in the path expansion Proposition 4.22, the second and third asymptotics result from Taylor expansions, and are uniform in $\xi$ and $\sigma$ (as is the fourth asymptotic), and where $\mu_\xi$ and $\mu_\sigma$ stand for the joint probability densities of $\{\xi(y)\}_{y\in B(0,n)\setminus\{0\}}$ and $\{\sigma(y)\}_{y\in B(0,n)}$ respectively. Note that, for $C>0$ sufficiently large, eventually

$$a_t^\gamma - (a_t + C + O(1)^\gamma < 0 < a_t^\gamma - (a_t - C + O(1))^\gamma\,.$$

Hence, by continuity of the path expansion in Proposition 4.22, there exists an $A_t = a_t + O(1)$ such that, as $t\to\infty$,

$$\int_{\xi,\sigma} \exp\left\{a_t^\gamma - \Big(A_t + Q(A_t;\xi,\sigma)\Big)^\gamma\right\} d\mu_\xi\, d\mu_\sigma \to 1$$

which gives the first result. For the second, instead of (4.15) we bound $Q(A_t+xd_t;\xi,\sigma)$ above, uniformly in $x>0$, by $Q(A_t;\xi,\sigma)$, which produces the bound

$$t^{-d}\int_{\xi,\sigma} \exp\left\{a_t^\gamma - \Big(A_t + Q(A_t;\xi,\sigma)\Big)^\gamma \left(1+\frac{x}{\gamma}(\log t)^{-1}\right)^\gamma\right\} d\mu_\xi\, d\mu_\sigma\,.$$

In the case $\gamma\ge 1$, we bound this expression above uniformly in $x>0$ by

$$t^{-d}\int_{\xi,\sigma} \exp\left\{a_t^\gamma - \Big(A_t + Q(A_t;\xi,\sigma)\Big)^\gamma \left(1+\frac{x}{\gamma}(\log t)^{-1}\right)\right\} d\mu_\xi\, d\mu_\sigma \sim e^{-\frac{x}{\gamma}(1+o(1))}\,,$$

using the definition of $A_t$ and the fact that $A_t + Q(A_t;\xi,\sigma)\sim a_t$ in the last step. The case $\gamma<1$ is simpler, since then we have simply

$$\mathbf{P}\Big(\xi(0) > A_t + xd_t + \sigma^{-1}(0)\Big) = \mathbf{P}\Big(\xi(0) > a_t + xd_t + O(1)\Big)$$

and the bound follows from the regularity of Weibull tail of $\xi(0)$. $\square$

We now define the set-up we shall need to examine the correlation of the potential field and trapping landscape near sites of high $\tilde{\lambda}^{(n)}$; since the nature of this correlation differs depending on $(\gamma, \mu)$, so does our set-up. Fix a constant $\nu \in (0,1)$. Recall first the definition of the 'interface sites' $\mathcal{I}_\xi$ and $\mathcal{I}_\sigma$ and also the definition of $n(y)$. For each $y \in \mathbb{Z}^d$ define the positive constants

$$c_\sigma := \begin{cases} \left(\frac{\gamma}{\mu}\right)^{\frac{1}{\mu+1}} & \text{if } q_\sigma > 0 \\ 0 & \text{else} \end{cases} , \quad c_\xi(y) := \begin{cases} \left(n(y)^2 (2d)^{-1} \delta_\sigma^{-1} c_\sigma^{-1}\right)^{\frac{1}{\gamma-1}} & \text{if } q_\xi(|y|) > 0 \\ 0 & \text{else} \end{cases} ,$$

$$\bar{c}_\sigma(y) := n(y)^2 (2d)^{-1} \gamma c_\sigma^{-1} \qquad \text{and} \qquad \bar{c}_\xi(y) := \bar{c}_\sigma(y)\, \delta_\sigma^{-1} .$$

Further, if $\gamma > 1$, for each $n \in \mathbb{N}$ define the rectangles

$$E_\xi := \prod_{y \in (B(0, n\wedge\rho_\xi) \setminus \{0\}) \setminus \mathcal{I}_\xi} (-f_t, f_t) \;\times \prod_{y \in (B(0,n) \setminus B(0, n\wedge\rho_\xi)) \cup \mathcal{I}_\xi} (f_t, g_t) ,$$

$$E_\sigma := (-f_t, f_t) \;\times \prod_{y \in (B(0,n) \setminus \{0\}) \setminus \mathcal{I}_\sigma} (0, f_t) \;\times \prod_{y \in (B(0,n) \setminus B(0, n\wedge\rho_\sigma)) \cup \mathcal{I}_\sigma} (0, g_t) ,$$

$$S_\xi := \prod_{y \in (B(0, n\wedge\rho_\xi) \setminus \{0\}) \setminus \mathcal{I}_\xi} a_t^{q_\xi(|y|)} (c_\xi(y) - f_t, c_\xi(y) + f_t) \;\times \prod_{y \in (B(0,n) \setminus B(0, n\wedge\rho_\xi)) \cup \mathcal{I}_\xi} (f_t, g_t) ,$$

and

$$S_\sigma := a_t^{q_\sigma} (c_\sigma - f_t, c_\sigma + f_t) \;\times \prod_{y \in (B(0,n) \setminus \{0\}) \setminus \mathcal{I}_\sigma} (\delta_\sigma, \delta_\sigma + f_t) \;\times \prod_{y \in (B(0,n) \setminus B(0, n\wedge\rho_\sigma)) \cup \mathcal{I}_\sigma} (0, g_t) .$$

If $\gamma \leq 1$, maintain the definitions of $E_\xi$ and $S_\xi$ but define instead

$$E_\sigma := (a_t^{-\nu}, \infty) \;\times \prod_{y \in B(0,n) \setminus \{0\}} (0, g_t) \quad \text{and} \quad S_\sigma := \prod_{y \in B(0,n)} (0, g_t) \,.t$$

For each $n \in \mathbb{N}$, define the event

$$\mathcal{S}_t := \left\{ \{\xi(y)\}_{y \in B(0,n) \setminus \{0\}} \in S_\xi \;,\; \{\sigma(y)\}_{y \in B(0,n)} \in S_\sigma \right\} ,$$

and, for each $x \in \mathbb{R}$ and the scaling function $A_t$ from Proposition 4.23, further define the event

$$\mathcal{A}_t := \left\{ \tilde{\lambda}_t^{(n)}(0) > A_t + x d_t \right\} .$$

**Proposition 4.24** (Correlation of potential field and trapping landscape)**.** *For each $n \in \mathbb{N}$, as $t \to \infty$,*

$$\mathbf{P}\left(\mathcal{S}_t \middle| \mathcal{A}_t\right) \to 1 .$$

*Moreover, as $t \to \infty$,*

$$f_{\xi(y)|\mathcal{A}_t}(x) \to \frac{e^{\bar{c}_\xi(y) x} f_\xi(x)}{\mathbf{E}[e^{\bar{c}_\xi(y) \xi(0)}]} , \qquad \textit{for each } y \in \mathcal{I}_\xi , \tag{4.16}$$

*uniformly over $x \in (0, L_t)$, and*

$$f_{\sigma(y)|\mathcal{A}_t}(x) \to \frac{e^{\bar{c}_\sigma(y)/x} f_\sigma(x)}{\mathbf{E}\big[e^{\bar{c}_\sigma(y)/\sigma(0)}\big]} \,, \quad \text{for each } y \in \mathcal{I}_\sigma \,, \tag{4.17}$$

*uniformly over $x$. Finally, if $\gamma = 1$, then for each $x \in \mathbb{R}^+$, as $t \to \infty$,*

$$f_{\sigma(0)|\mathcal{A}_t}(x) \to \frac{e^{-1/x} f_\sigma(x)}{\mathbf{E}\big[e^{-1/\sigma(0)}\big]} \,, \tag{4.18}$$

*uniformly over $x$.*

*Proof.* Define a field $s : B(0,n) \setminus \{0\} \cup B(0,n) \to \mathbb{R}$ with projections $s_\xi$ and $s_\sigma$ onto $B(0,n) \setminus \{0\}$ and $B(0,n)$ respectively. For a scale $C_t \sim a_t$ define the function

$$\begin{aligned} Q_t(C_t; s) &:= a_t^{-q_\sigma} (c_\sigma + s_\sigma(0))^{-1} - a_t^{-q_\sigma} (c_\sigma + s_\sigma(0))^{-1} \\ &\times \sum_{2 \le k \le 2j} \sum_{\substack{p \in \Gamma_k(0,0) \\ p_i \neq 0,\, 0<i<k \\ \{p\} \subseteq B(0,n)}} \prod_{0<i<k} \frac{(2d)^{-1} \, (\delta_\sigma + s_\sigma(p_i))^{-1}}{C_t - a_t^{q_\xi(|p_i|)} (c_\xi(p_i) + s_\xi(p_i)) + (\delta_\sigma + s_\sigma(p_i))^{-1}} \,, \end{aligned}$$

if, for each $y \in B(0,n) \setminus \{0\}$,

$$a_t^{q_\xi(|y|)} (c_\xi(y) + s_\xi(y)) \in (0, L_t) \,, \quad s_\sigma(y) > 0 \quad \text{and} \quad a_t^{q_\sigma} (c_\sigma + s_\sigma(0)) > 0$$

are satisfied, and $Q_t(C_t; s) := 0$ otherwise. Define further the function

$$\begin{aligned} R_t(C_t; s) := a_t^\gamma - (C_t + Q_t(C_t; s))^\gamma + \sum_{y \in B(0,n)} \Big( \log f_\xi \Big( a_t^{q_\xi(|y|)} (c_\xi(y) + s_\xi(y)) \Big) + \log a_t^{q_\xi(|y|)} \Big) \\ + \log f_\sigma \left( a_t^{q_\sigma} (c_\sigma + s_\sigma(0) \right) + \log a_t^{q_\sigma} + \sum_{y \in B(0,n) \setminus \{0\}} \log f_\sigma \left( \delta_\sigma + s_\sigma(y) \right) \,. \end{aligned}$$

To motivate these definitions, consider that, similarly to the above, we can write

$$\mathbf{P}\left( \tilde{\lambda}_t^{(n)}(0) > A_t + x d_t \right) \sim t^{-d} e^{-x} \int_{\mathbb{R}^{2|B(0,n)|-1}} \exp\left\{ R_t(A_t; s) \right\} \, ds \,. \tag{4.19}$$

It remains to show that the integral in (4.19) is asymptotically concentrated on the set $E_\xi \times E_\sigma$ and that equations (2.15)–(4.18) are satisfied. This fact can be checked by a somewhat lengthy computation which we only sketch here. We shall treat separately the cases (i) $\gamma > 1$, and (ii) $\gamma \le 1$; we begin with case (i), which is the most delicate.

We must analyse the variables $s_\sigma(0)$, $\{s_\sigma(y)\}_{y \in B(0,n) \setminus \{0\}}$, and $\{s_\xi(y)\}_{y \in B(0,n) \setminus \{0\}}$ separately; we start with $s_\sigma(0)$. In what follows abbreviate $R_t(A_t; s)$ by $R_t(s)$. Fix an arbitrary choice of the components of $s$ and consider how $R_t(s)$ varies with $s_\sigma(0)$. Notice that the function $R_t(s)$ can be decomposed into two parts, one of which decreases as $s_\sigma(0)$ increases (through $Q_t$) and another which increases as $s_\sigma(0)$ increases (through $f_\sigma$). The first part is analysed by Taylor expanding $(A_t + Q_t(A_t; s))^\gamma$, from which it can be seen that the dependence on $s_\sigma(0)$ is, as $t \to \infty$,

$$\gamma \, a_t^{-q_\sigma} \, a_t^{\gamma - 1} \, (c_\sigma + s_\sigma(0))^{-1} \, (1 + o(1))$$

where the error term $o(1)$ is uniform in $s$. The second part is given by $-\log f_\sigma(a_t^{q_\sigma}(c_\sigma + s_\sigma(0)))$ which is eventually

$$a_t^{q_\sigma \mu}(c_\sigma + s_\sigma(0))^\mu .$$

Hence, since we defined $q_\sigma$ precisely so that

$$-q_\sigma + \gamma - 1 = q_\sigma \mu ,$$

the function $R_t$ has the asymptotic form, as $t \to \infty$,

$$R_t(s) = f_1(t;s) + a_t^{\kappa_1} \left(g_1(s_\sigma(0) + o(1)\right)$$

where $f_1(t;s)$ is some function not depending on $s_\sigma(0)$, $\kappa_1$ is some positive constant, the function $g_1(x)$ satisfies

$$g_1(x) := -\gamma(c_\sigma + x)^{-1} - (c_\sigma + x)^\mu ,$$

and the error term $o(1)$ is uniform in $s$. Then we have, uniformly in $s$, as $t \to \infty$,

$$\int_{\mathbb{R}} e^{R_t(s)}\, ds_\sigma(0) \sim e^{f_1(t;s)} \int_{\mathbb{R}} \exp\left\{a_t^{\kappa_1} g_1(s_\sigma(0))\right\}\, ds_\sigma(0) . \tag{4.20}$$

Remark that $g_1(x)$ achieves a unique maximum at 0 (by the construction of $c_\sigma$). Therefore, by the Laplace method, the above integral is eventually asymptotically concentrated around 0 on the order $a_t^{\kappa_1}$, and hence the integral is concentrated on the domain $s_\sigma(0) \in (-f_t, f_t)$.

Consider now the variables $\{s_\sigma(y)\}_{y \in B(0,n) \setminus \{0\}}$. Fix an $s_\sigma(0) \in (-f_t, f_t)$ and an arbitrary choice of the remaining components of $s$. Again, similarly to the above, the function $R_t(s)$ can be decomposed into two parts, one whose dependence on $s_\sigma(y)$ is, as $t \to \infty$,

$$n(y)^2\, (2d)^{-1}\, \gamma\, c_\sigma^{-1}\, a_t^{\gamma - 2|y|}\, a_t^{-q_\sigma}\, (\delta_\sigma + s_\sigma(y))^{-1}\, (1 + o(1))$$

uniformly in $s$, and another whose dependence is

$$-\log f_\sigma(\delta_\sigma + s_\sigma(y)) .$$

Then we have, uniformly in $s$, as $t \to \infty$,

$$\int_{\mathbb{R}} e^{R_t(s)}\, ds_\sigma(y) \sim e^{f_2(t;s)} \int_{\mathbb{R}} \exp\left\{\gamma c_\sigma^{-1} a_t^{\kappa_2} (\delta_\sigma + s_\sigma(y))^{-1}\right\} f_\sigma(\delta_\sigma + s_\sigma(y))\, ds_\sigma(y) ,$$

where $f_2(t;s)$ is some function not depending on $s_\xi(y)$, $\kappa_2$ is some non-negative constant with $\kappa_2 > 0$ if and only if $y \in B(0, \rho_\sigma) \setminus \mathcal{I}_\sigma$, and where the error term $o(1)$ is uniform in $s$. Hence, if $y \in B(0, \rho_\sigma) \setminus \mathcal{I}_\sigma$, then along with the lower-tail assumption of $\sigma(0)$, it is clear that the above integral is asymptotically concentrated on $s_\sigma(y) \in (0, f_t)$. On the other hand, if $y \in \mathcal{I}_\sigma$, then the integrand is asymptotically

$$e^{\bar{c}_\sigma(y)/(s_\sigma(y) + \delta_\sigma)} f_\sigma(s_\sigma(y) + \delta_\sigma) ,$$

uniformly over $s_\sigma(y)$, which establishes (4.17). Trivially, if $y \notin B(0, \rho_\sigma)$, then the integral is concentrated on $s_\sigma(y) \in (f_t, g_t)$.

Finally, consider the variables $\{s_\xi(y)\}_{y \in B(0,n)\setminus\{0\}}$ and fix $s_\sigma(0) \in (-f_t, f_t)$, $s_\sigma(y) \in (0, f_t)$ for each $y \in B(0,\rho_\sigma) \setminus \mathcal{I}_\sigma$, and an arbitrary choice of the remaining components of $s$. The function $R_t(s)$ can be decomposed into two parts, one whose dependence on $s_\xi(y)$ is of order, as $t \to \infty$,

$$n(y)^2 \, (2d)^{-1} (\delta_\sigma + s_\sigma(y))^{-1} \gamma c_\sigma^{-1} \, a_t^{q_\xi(|y|)} \, a_t^{\gamma-1-2|y|} a_t^{-q_\sigma} \, (c_\xi(y) + s_\xi(y)) \, (1 + o(1)) \ ,$$

uniformly in $s$, another whose dependence is

$$a_t^{q_\xi(|y|)\gamma} (c_\xi(y) + s_\xi(y))^\gamma \, .$$

Hence, since we defined $q_\xi(|y|)$ precisely so that

$$q_\xi(|y|) + \gamma - 1 - 2|y| - q_\sigma = q_\xi(|y|)\gamma \, ,$$

if $y \in B(0, \rho_\xi)$, the function $R_t$ has the asymptotic form, as $t \to \infty$,

$$R_t(s) = f_3(t; s) + a_t^{\kappa_3} \, (g_3(s_\xi(y)) + o(1))$$

where $f_3(t; s)$ is some function not depending on $s_\xi(y)$, $\kappa_3$ is some non-negative constant with $\kappa_3 > 0$ if any only if $y \in B(0, \rho_\xi) \setminus \mathcal{I}_\xi$, the function $g_3(x)$ satisfies

$$g_3(x) := \gamma \, n(y)^2 \, (2d)^{-1} \, \delta_\sigma^{-1} \, c_\sigma^{-1} (c_\xi(y) + x) - (c_\xi(y) + x)^\gamma \, ,$$

and where the error term $o(1)$ is uniform in $s$. Then we have, uniformly in $s$, as $t \to \infty$,

$$\int_{\mathbb{R}} e^{R_t(s)} \, ds_\xi(y) \sim e^{f_3(t;s)} \int_{\mathbb{R}} \exp\left\{a_t^{\kappa_3} g_3(s_\xi(y))\right\} \, ds_\xi(y) \, .$$

If $y \in B(0, \rho_\xi) \setminus \mathcal{I}_\xi$, and since $g_3(x)$ achieves a unique maximum at 0 (by the construction of $c_\xi(y)$), again by the Laplace method this integral is also asymptotically concentrated on $s_\xi(y) \in (-f_t, f_t)$. On the other hand, if $y \in \mathcal{I}_\xi$, then the integrand is asymptotically

$$e^{\bar{c}_\xi(y) s_\xi(y)} f_\xi(s_\xi(y)) \, ,$$

uniformly over $s_\xi(y)$, which establishes (4.16). Trivially, if $y \notin B(0, \rho_\xi)$, then the integral is concentrated on $s_\xi(y) \in (f_t, g_t)$. Since we have now shown that each component of (4.19) is asymptotically concentrated on the respective component of the set $E_\xi \times E_\sigma$, integrating first over $s_\xi(y)$ and $s_\sigma(y)$ for $y \in B(0, n) \setminus \{0\}$, and then over $s_\sigma(0)$, we have the result.

Case (ii) is easier to handle. Now the integral in (4.20) becomes

$$e^{f_1(t;s)} \int_{\mathbb{R}} \exp\{-\gamma a_t^{\gamma-1} s_\sigma^{-1}(0) + o(1)\} \, f_\sigma(s_\sigma(0)) \, ds_\sigma(0) \, ,$$

with the error uniform in $s$. If $\gamma < 1$, then this integral is clearly concentrated on $s_\sigma(0) \in (0, g_t)$. If $\gamma = 1$, then the integrand of this integral is asymptotically

$$e^{s_\sigma^{-1}(0)} f_\sigma(s_\sigma(0)) \, ,$$

uniformly over $s_\sigma(0)$, which establishes (4.18). The remainder of the proof is identical. $\square$

### 4.3.2 Constructing the point process

Just as in Chapter 2, the existence of asymptotics for the (punctured) local principal eigenvalues allows us to establish scaling limits for the penalisation functional $\Psi_t^{(j)}$.

We start by constructing a point set from the pair $(z, \Psi_t^{(j)}(z))$ which will converge to a point process in the limit. Again, for technical reasons, we shall actually need to consider a certain generalisation of the functional $\Psi_t^{(j)}$. More precisely, for each $c \in \mathbb{R}$, define the functional $\Psi_{t,c}^{(j)} : V_t \to \mathbb{R}$ by

$$\Psi_{t,c}^{(j)}(z) := \lambda^{(j)}(z) - \frac{|z|}{\gamma t} \log\log t + c\frac{|z|}{t} .$$

Further, for each $z \in \Pi^{(L_t)}$ define

$$Y_{t,c,z}^{(j)} := \frac{\Psi_{t,c}^{(j)}(z) - A_{r_t}}{d_{r_t}} \qquad \text{and} \qquad \mathcal{M}_{t,c}^{(j)} := \sum_{z \in \Pi^{(L_t)}} \mathbb{1}_{(zr_t^{-1}, Y_{t,c,z}^{(j)})} .$$

Finally, for each $\tau \in \mathbb{R}$ and $\alpha > -1$ let

$$\hat{H}_\tau^\alpha := \{(x, y) \in \dot{\mathbb{R}}^{d+1} : y \geq \alpha|x| + \tau\}$$

where $\dot{\mathbb{R}}^{d+1}$ is the one-point compactification of $\mathbb{R}^{d+1}$.

**Proposition 4.25** (Point process convergence)**.** *For each $\tau, c \in \mathbb{R}$ and $\alpha > -1$, as $t \to \infty$,*

$$\mathcal{M}_{t,c}^{(j)}|_{\hat{H}_\tau^\alpha} \Rightarrow \mathcal{M} \quad \textit{in law},$$

*where $\mathcal{M}$ is a Poisson point process on $\hat{H}_\tau^\alpha$ with intensity measure $\nu(dx, dy) = dx \otimes e^{-y-|x|}dy$.*

We now use the point process $\mathcal{M}$ to analyse the joint distribution of top two statistics of the functional $\Psi_{t,c}^{(j)}$. So let

$$Z_{t,c}^{(j)} := \operatorname*{argmax}_{z \in \Pi^{(L_t)}} \Psi_{t,c}^{(j)}(z) \quad \text{and} \quad Z_{t,c}^{(j,2)} := \operatorname*{argmax}_{\substack{z \in \Pi^{(L_t)} \\ z \neq Z_{t,c}^{(j)}}} \Psi_{t,c}^{(j)} .$$

Note that eventually these are well-defined almost surely, since $\Pi^{(L_t)}$ is finite and non-zero by Lemma 2.18.

**Corollary 4.26.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\left( \frac{Z_{t,c}^{(j)}}{r_t}, \frac{Z_{t,c}^{(j,2)}}{r_t}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j)}) - A_{r_t}}{d_{r_t}}, \frac{\Psi_{t,c}^{(j)}(Z_{t,c}^{(j,2)}) - A_{r_t}}{d_{r_t}} \right)$$

*converges in law to a random vector with density*

$$p(x_1, x_2, y_1, y_2) = \exp\{-(y_1 + y_2) - |x_1| - |x_2|) - 2^d e^{-y_2}\} \mathbb{1}_{\{y_1 > y_2\}} .$$

### 4.3.3 Properties of the localisation site

In this subsection we use the results from the previous subsection to analyse the localisation sites $Z^{(j)}_{t,c}$ and $Z_t$, and in the process complete the proof of Theorems 4.2 and 4.3. For each $c \in \mathbb{R}$, introduce the events

$$\mathcal{G}_{t,c} := \{\Psi^{(j)}_{t,c}(Z^{(j)}_{t,c}) - \Psi^{(j)}_{t,c}(Z^{(j,2)}_{t,c}) > d_t e_t\}\,,$$

$$\mathcal{H}_t := \{r_t f_t < |Z^{(j)}_t| < r_t g_t\} \quad \text{and} \quad \mathcal{I}_t := \{a_t(1-f_t) < \Psi^{(j)}_t(Z^{(j)}_t) < a_t(1+f_t)\}\,,$$

and the event

$$\mathcal{E}_{t,c} := \mathcal{S}_t(Z^{(j)}_t) \cap \mathcal{G}_{t,0} \cap \mathcal{G}_{t,c} \cap \mathcal{H}_t \cap \mathcal{I}_t$$

which acts to collect the relevant information that we shall later need.

**Proposition 4.27.** *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\mathbf{P}(\mathcal{E}_{t,c}) \to 1\,.$$

In the next few propositions, we prove that the sites $Z^{(j)}_{t,c}$ and $Z^{(j)}_t$ are both equal to the localisation site $Z_t$ with overwhelming probability.

**Proposition 4.28.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$Z^{(j)}_{t,c} = Z^{(j)}_t$$

*holds eventually.*

**Lemma 4.29.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$\lambda^{(j)}(Z^{(j)}_t) \geq \lambda(Z^{(j)}_t) \quad \textit{and} \quad \lambda^{(j)}(Z_t) \geq \lambda(Z_t)$$

*and*

$$\lambda^{(j)}(Z^{(j)}_t) - \lambda(Z^{(j)}_t) < d_t e_t$$

*all hold eventually.*

*Proof.* The first two statements follow from the domain monotonicity of the principal eigenvalue in Lemma 4.4. For the third statement, remark that the event $\mathcal{E}_{t,c}$ implies that $Z^{(j)}_t \in \Pi^{(L_{t,\varepsilon})}$, that $\xi(y) < L_t$ for all $y \in B(Z^{(j)}_t, \rho_\sigma)$, that $\xi(y) < g_t$ for all $y$ such that $j \geq |y - Z^{(j)}_t| > \rho_\xi$, and that $\sigma(Z^{(j)}_t) > a_t^{q_\sigma} f_t$. Hence, by considering the path expansion in Proposition 4.22, and using the same argument as in the proof of Lemma 2.28, we have that for some $C > 0$,

$$\lambda^{(j)}(Z^{(j)}_t) - \lambda(Z^{(j)}_t) < \frac{C a_t^{-(\frac{\gamma-1}{\mu+1})^+} g_t}{f_t (L_{t,\epsilon} - L_t)^{2\rho_\sigma + 2}} < d_t e_t$$

eventually, with the last inequality holding since

$$-2\rho_\sigma - 2 - \left(\frac{\gamma-1}{\mu+1}\right)^+ < 1 - \gamma\,. \qquad \square$$

**Corollary 4.30** (Equivalence of $Z_t^{(j)}$ and $Z_t$)**.** *For each $c \in \mathbb{R}$, on the event $\mathcal{E}_{t,c}$, as $t \to \infty$,*

$$Z_t^{(j)} = Z_t$$

*eventually.*

Finally, we prove a criterion for the independence of $Z_t$ from the trapping landscape $\sigma$. Define $\psi_t(z) := \xi(z) - \frac{|z|}{\gamma t} \log\log t$ and let $z_t := \operatorname{argmax}_{z \in \Pi^{(L_t)}} \psi_t(z)$. Note that $z_t$ is independent of $\sigma$.

**Proposition 4.31** (Criterion for the independence of $Z_t$ from the trapping landscape $\sigma$)**.** *If $\gamma < 1$, then as $t \to \infty$,*

$$\mathbf{P}\left(Z_t = z_t\right) \to 1\,.$$

*Proof.* By Proposition 4.27 we may assume $\mathcal{E}_{t,c}$ holds. Observe that, on $\mathcal{E}_{t,c}$ and by Proposition 4.22, any $z \in \Pi^{(L_{t,\varepsilon})} \setminus \{Z_t^{(j)}\}$ satisfies

$$\psi_t(Z_t^{(j)}) > \Psi_t^{(j)}(Z_t^{(j)}) > \Psi_t^{(j)}(z) + d_t e_t > \psi_t(z) + O(1) + d_t e_t\ .$$

Moreover, by Lemma 4.5 and on $\mathcal{E}_{t,c}$, any $z \in \Pi^{(L_t)} \setminus \Pi^{(L_{t,\varepsilon})}$ also satisfies

$$\psi_t(Z_t^{(j)}) > \Psi_t^{(j)}(Z_t^{(j)}) > \psi_t(z) + O(1) + d_t e_t\ .$$

Since $d_t e_t \to \infty$ if $\gamma < 1$, this implies that $Z_t^{(j)} = \operatorname{argmax}_{z \in \Pi^{(L_t)}} \psi_t(z) =: z_t$. Corollary 4.30 completes the proof. □

### 4.3.4 Proof of Theorems 4.2 and 4.3

Theorem 4.2 is proved identically to in Chapter 2, so here we focus only on Theorem 4.3. Consider part (a) By definition, $Z_t$ depends only on the values of $\xi$ and $\sigma$ in balls of radius $\rho_\xi$ and $\rho_\sigma$ respectively around each site, and so the radii $\rho_\xi$ and $\rho_\sigma$ are certainly sufficient. To show necessity, consider that the results in parts (b)–(d) of Theorem 4.2 establish the correlation of the fields $\xi$ and $\sigma$ at a distance $\rho_\xi$ and $\rho_\sigma$ respectively around $Z_t$. Hence these radii are necessary as well.

Consider then part (b). The sufficient condition for the reduction to $\xi$ follows directly from Proposition 4.31. To show necessity, consider that the results in part (c) of Theorem 4.2 establish that, if $\gamma \geq 1$, the value of $\sigma(Z_t)$ is not an independent copy of $\sigma(0)$, and hence $Z_t$ must depend on $\sigma$.

It remains to prove part (c). If $\rho_\sigma = 0$ then $Z_t$ depends on $\xi$ and $\sigma$ only through $\eta$ by definition. On the other hand, suppose $\rho_\sigma \geq 1$ and, for the purposes of contradiction, that there exists a random site $z_t$, depending only on $\xi$ and $\sigma$ through $\eta$, such that, as $t \to \infty$,

$$\mathbf{P}(Z_t = z_t) \to 1\,.$$

Fix a site $y$ and a constant $c > \delta_\sigma$. We establish a contradiction by considering two bounds on the probability of the event

$$\{\sigma(y) < c,\, |Z_t - y| = 1\}\,.$$

We first consider the case that $\{z \in \mathcal{I}_\sigma : |z| = 1\}$ is empty. Then by part (d) of Theorem 4.2, conditionally on event $\{|Z_t - y| = 1\}$, we have that $\sigma(y) \to \delta_\sigma$ in probability as $t \to \infty$. This implies that there exists some $c_1 > 0$ such that

$$\mathbf{P}(\sigma(y) < c,\, |Z_t - y| = 1) > (\mathbf{P}(\sigma(y) < c) + c_1)\; \mathbf{P}(|Z_t - y| = 1) \tag{4.21}$$

eventually. In the case that $\{z \in \mathcal{I}_\sigma : |z| = 1\}$ is not empty, conditionally on event $\{|Z_t - y| = 1\}$ and again by part (d) of Theorem 4.2,

$$f_{\sigma(y)}(x) \to c_2 e^{\bar{c}_\sigma / x} f_\sigma(x)$$

for some $c_2 > 0$, and so (4.21) holds in this case as well.

We now work on the event $\{Z_t = z_t\}$ and show how to obtain a lower bound on the probability of the event $\{\sigma(y) < c,\, |z_t - y| = 1\}$. Let $\bar{\eta} = \{\eta(v) : v \neq y\}$. Remark first that, since $z_t \in \Pi^{(L_t)}$, by Proposition 4.22 we have that $\lambda_t(z_t)$ is increasing in $\eta(y)$ for $|y - z_t| = 1$. Hence there exists a function $\beta_t : \bar{\eta} \to \mathbb{R} \cup \{\infty\}$ such that, conditionally on $\bar{\eta}$,

$$\{|z_t - y| = 1\} \quad \text{and} \quad \{\eta(y) \geq \beta_t(\bar{\eta})\}$$

agree almost surely. To see this, set $\beta_t(\bar{\eta})$ to be the minimum $\eta(y)$ such that with such a value of $\eta(y)$, we have $|z_t - y| = 1$ (and setting it to be infinity if no such value exists). Then clearly, if $\eta(y) < \beta_t(\bar{\eta})$ we cannot have $|z_t - y| = 1$, and on the other hand we claim that if $\eta(y) \geq \beta_t(y)$ we have $|z_t - y| = 1$. This follows by the almost-sure separation of Lemma 2.19, which ensures that $\{y = z_t\}$ has probability 0. Then, eventually almost surely,

$$\begin{aligned}
\mathbf{P}(\sigma(y) < c,\, |z_t - y| = 1) &= \mathbf{E}_{\bar{\eta}}\left[\mathbf{E}[\mathbb{1}_{\{|z_t - y| = 1\}} \mathbb{1}_{\{\sigma(y) < c\}} \,|\, \bar{\eta}]\right] \\
&= \mathbf{E}_{\bar{\eta}}\left[\mathbf{E}[\mathbb{1}_{\{\eta(y) > \beta_t(\bar{\eta})\}} \mathbb{1}_{\{\sigma(y) < c\}} \,|\, \bar{\eta}]\right] \\
&\leq \mathbf{E}_{\bar{\eta}}\left[\mathbf{E}[\mathbb{1}_{\{\eta(y) > \beta_t(\bar{\eta})\}} \,|\, \bar{\eta}]\, \mathbf{E}[\mathbb{1}_{\{\sigma(y) < c\}} \,|\, \bar{\eta}]\right] \\
&= \mathbf{P}(\sigma(y) < c)\, \mathbf{P}(|z_t - y| = 1)\,,
\end{aligned}$$

where the second equality uses the fact that $z_t$ depends on $\sigma$ only through $\eta$, and the inequality holds since, conditionally on $\bar{\eta}$, the events $\{\eta(y) > \beta_t(\bar{\eta})\}$ and $\{\sigma(y) < c\}$ are negatively correlated. Since $z_t = Z_t$ with probability going to one, combining with (4.21) gives the required contradiction.

## 4.4 Negligible paths

In this section we show that the contribution to the total mass $U(t)$ from the components $U^2(t)$, $U^3(t)$, $U^4(t)$ and $U^5(t)$ are all negligible. We proceed in two parts: first we prove a lower bound on the total mass $U(t)$, and then we bound from above the contribution to the total mass from each $U^i(t)$. Throughout this section, let $\varepsilon$ be such that $0 < \varepsilon < \theta$.

### 4.4.1 Preliminaries

We begin by proving a general result on eigenfunction decay around sites of high potential, which will be used in both the lower and upper bound. For each $z \in \Pi^{(L_{t,\varepsilon})}$, let $\varphi_1$ denote the principal eigenfunction of the operator $\mathcal{H}^{(j)}(z)$.

**Proposition 4.32.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly, as $t \to \infty$, almost surely*

$$\sum_{y \in B(z,j) \setminus \{z\}} \varphi_1(y) \to 0 \qquad \textit{and} \qquad \sum_{y \in B(z,j) \setminus \{z\}} \frac{\sigma(y)^{-\frac{1}{2}} \varphi_1(y)}{\|\sigma^{-\frac{1}{2}} \varphi_1\|_{\ell_2}} \to 0.$$

*Proof.* By Proposition 4.8, we have the path expansion

$$\frac{\varphi_1(y)}{\varphi_1(z)} = \frac{\sigma(y)}{\sigma(z)} \sum_{k \geq 1} \sum_{\substack{p \in \Gamma_k(y,z) \\ p_i \neq z,\, 0 \leq i < k \\ \{p\} \subseteq B(z,j)}} \prod_{0 \leq i < k} (2d)^{-1} \frac{\sigma^{-1}(p_i)}{\lambda^{(j)}(z) - \eta(p_i)} \,, \quad y \in B(z,j) \setminus \{z\} \,.$$

Since, by Lemmas 2.19 and 4.5, for each $y \in B(z,j) \setminus \{z\}$, almost surely

$$\lambda^{(j)}(z) - \eta(y_i) > L_{t,\varepsilon} - L_t - \delta_\sigma^{-1} \,,$$

and moreover since $\sigma^{-1}(y) \leq \delta_\sigma^{-1}$ for all $y \in B(z,j)$, the result follows. □

**Corollary 4.33** (Bound on total mass of the solution)**.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly and any $c > 1$, as $t \to \infty$, almost surely*

$$\mathbb{E}_z \left[ e^{\int_0^t \xi(X_s)\, ds} \mathbb{1}_{\{\tau_{B(z,j)^c} > t\}} \right] < c\, e^{t\lambda^{(j)}(z)}$$

*eventually.*

### 4.4.2 Lower bound on the total mass $U(t)$

Recall that by the discussion in Section 2.1.2, the total mass $U(t)$ can be approximated by considering both the benefit of being near a site of high potential and the probabilistic penalty from diffusing to that site. To formalise a lower bound for $U(t)$ we need a bound on both of these terms.

We begin by bounding from below the benefit to the solution from paths that start and end at a site of high potential.

**Lemma 4.34.** *For each $z \in \Pi^{(L_{t,\varepsilon})}$ uniformly,*

$$\log u_z(t,z) > t\lambda^{(j)}(z) + o(1)$$

*eventually almost surely.*

*Proof.* Recall the Feynman-Kac formula for the solution $u_z(t,z)$ (see, e.g., Proposition 4.10), and note that the expectation is larger than the corresponding expectation taken only over paths that do not leave $B(z,j)$. Using Corollary 4.13, we then have that

$$u_z(t,z) \geq \frac{e^{\lambda^{(j)}(z)t}\sigma^{-1}(z)\varphi_1^2(z)}{\|\sigma^{-\frac{1}{2}}\varphi_1\|_{\ell_2}^2} \,,$$

where $\varphi_1$ denotes the principal eigenfunction of the operator $\mathcal{H}^{(j)}(z)$. Since the domain $B(z,j)$ is finite, the fact that the eigenfunction $\sigma^{-\frac{1}{2}}\varphi_1$ is localised at $z$ (by Proposition 2.30) ensures that the square eigenfunction $\sigma^{-1}\varphi_1^2$ is also localised at $z$, and the result follows. □

The next step is to bound from above the probabilistic penalty incurred by diffusing to a certain site. Naturally this penalty takes into account both the distance of the site from the origin, as well as the size of the traps on paths from the origin to the site. In bounding this, we shall make use of the existence of quick paths that we established in a general setting in Section 4.2.2. So let us first define explicitly the event $\Theta_t^d$ that guarantees us sufficiently quick paths, and show that it holds with overwhelming probability.

In the case $d = 1$, for $n_t := r_t g_t$ and $c := \varepsilon^{-1}$, recall the definition of the coarse-graining scales $\{\sigma_t^i\}_{i=0}^2$ from Proposition 4.18. Let $p \in \Gamma_{|Z_t|}(0, Z_t)$ be the (unique) shortest path from 0 to $Z_t$ and define

$$N_i^p := \sum_{0 \leq l < |Z_t|} \mathbb{1}_{\{\sigma(p_l) \in (\sigma_t^{i-1}, \sigma_t^i]\}} \,, \quad i = 1, 2 \,.$$

In the case $d \geq 2$, abbreviate $s_t := \log\log\log t$, and for $z_t := Z_t$, $\sigma_t := \log\log\log t$ and $S_t := \Pi^{(L_t)}$ recall the definition of $|Z_t|_{\text{chem}}$ from Proposition 4.21. Define the event

$$\Theta_t^d := \begin{cases} \left\{\sum_{i=1}^2 N_i^p \log \sigma_t^i < t d_t b_t \,,\ \max_{0 \leq l < |Z_t|} \sigma(p_l) < \sigma_t^2 \right\} \,, & d = 1 \,, \\ \{|Z_t|_{\text{chem}} < |Z_t| + r_t b_t\} \,, & d \geq 2 \,. \end{cases}$$

**Proposition 4.35** (Existence of quick paths). *For each $c \in \mathbb{R}$, as $t \to \infty$,*

$$\mathbf{P}(\Theta_t^d, \mathcal{E}_{t,c}) \to 1 \,.$$

*Proof.* Recall that on event $\mathcal{E}_{t,c}$ we have that $|Z_t| < r_t g_t$. Suppose $d = 1$. Then the result follows immediately from Proposition 4.18 and the properties of the scaling functions in (4.4), since

$$\log\log r_t g_t \sim \log\log t \,.$$

Suppose then $d \geq 2$. Note that conditioning on $\xi$ determines $\Pi^{(L_t)}$ and also that, by Lemma 2.19, eventually almost surely $\Pi^{(L_t)}$ satisfies the properties required by the set $S_t$. Since $Z_t \in \Pi^{(L_t)}$, conditioning on the values of $\sigma$ in $B(\Pi^{(L_t)}, j)$ therefore determines $Z_t$. Given $Z_t$ and $\Pi^{(L_t)}$, the event $\Theta_t^d$ is fully determined by the values of $\sigma$ in $\mathbb{Z}^d \setminus B(\Pi^{(L_t)}, j)$. Hence we can apply Proposition 4.21 with $z_t = Z_t$, $\sigma_t = s_t$ and $S_t = \Pi^{(L_t)}$, to deduce that there exists a $c_1 < 1$ such that, for all functions $c_t \to \infty$,

$$|Z_t|_{\text{chem}} < |Z_t|(1 + \bar{F}_\sigma(s_t)c_t + t^{-c_1})$$

with probability tending to 1. By (4.4) and the Weibull tail of $\sigma(0)$ we can pick a $c_t$ such that

$$r_t g_t \bar{F}_\sigma(s_t) c_t \ll r_t g_t c_t / s_t \ll r_t b_t \,,$$

and so we have the result. □

We are now ready to prove the lower bound.

**Proposition 4.36.** *For each $c \in \mathbb{R}$, on the events $\mathcal{E}_{t,c}$ and $\Theta_t^d$, as $t \to \infty$,*

$$\log U(t) > t\lambda^{(j)}(Z_t) - \frac{|Z_t|}{\gamma} \log\log t + O(td_t b_t)$$

*almost surely.*

*Proof.* In the following proof set $z = Z_t$. We first consider the case of $d \geq 2$. As in the proof of Proposition 2.33, for each $r \in (0,1)$,

$$U(t) \geq u_z((1-r)t, z)\, \mathbb{P}(\tau < rt) \,,$$

where $\mathbb{P}$ denotes the law of the BTM initialised at the origin and $\tau$ denotes the first hitting time of the site $Z_t$. We seek to bound $\mathbb{P}(\tau < rt)$. Since we are on event $\Theta_t^d$, there exists a path

$$p \in \bigcup_{y \in \partial B(z,j)} \Gamma_{\ell_t}(0,y)$$

for some $\ell_t < |z| + r_t b_t$ such that $\sigma(x) < s_t$ for all $x \in \{p\}$. Moreover, since we are on event $\mathcal{E}_{t,c}$, each $\sigma(x) \in B(z,j) \setminus \{z\}$ is such that $\sigma(x) < a_t^\nu$ for some $\nu \in (0,1)$. Hence if we denote by $(\tilde{X}_t)_{t \geq 0}$ a continuous-time random walk, initialised at the origin, with generator $\Delta \tilde{\sigma}^{-1}$, where $\tilde{\sigma}(x) = s_t$ for all $x \in \{p\}$, $\tilde{\sigma}(x) = a_t^\nu$ for all $x \in B(z,j) \setminus \{z\}$, and $\tilde{\sigma}(x) = \sigma(x)$ otherwise, then by a simple coupling argument we have that

$$\mathbb{P}(\tau < rt) \geq \mathbb{P}(\tilde{\tau} < rt) \,,$$

where $\tilde{\tau}$ is the first hitting time of $z$ by $\tilde{X}$. Using a similar calculation to in the proof of Proposition 2.33, for any $r_1, r_2 > 0$ such that $r_1 + r_2 \leq r$,

$$\log \mathbb{P}(\tilde{\tau} < rt) > -r_1 t s_t^{-1} - r_2 t a_t^{-\nu} - \ell_t \log\left(\frac{2d\,\ell_t}{e r_1 t s_t^{-1}}\right) + j \log r_2 + O(\log t) \,. \tag{4.22}$$

Now note that on the event $\mathcal{E}_{t,c}$ we have that $Z_t \in \Pi^{(L_{t,\varepsilon})}$. Hence we can combine equations (2.19)–(4.22) and Lemma 2.32 to get that

$$\log U(t) > (1 - r_1 - r_2) t\lambda^{(j)}(z) - r_1 t s_t^{-1} - r_2 t a_t^{-\nu} - \ell_t \log\left(\frac{2d\,\ell_t}{e r_1 t s_t^{-1}}\right) + j \log r_2 + O(\log t) \,.$$

We now use the bound $\ell_t < |z| + r_t b_t$ and choose $\{r_i\}_{i=1,2}$ to optimise the bound, that is, set

$$r_1 := \frac{|z| + r_t b_t}{t(\lambda^{(j)}(z) + s_t^{-1})} \quad \text{and} \quad r_2 := \frac{j}{t(\lambda^{(j)}(z) + a_t^{-\nu})} \,.$$

It is clear that on event $\mathcal{E}_{t,c}$ we have $r_1, r_2 \in (0,1)$. With these values of $r_1$ and $r_2$ we obtain

$$\log U(t) > t\lambda^{(j)}(z) - (|z| + r_t b_t)\Big\{ \log\Big(\frac{\lambda^{(j)}(z) + s_t^{-1}}{s_t^{-1}}\Big) + O(1)\Big\} + O(\log t)\,.$$

On event $\mathcal{E}_{t,c}$ we have that $\lambda^{(j)}(z) < a_t(1+f_t)$. Since also $|z| < r_t g_t$ on event $\mathcal{E}_{t,c}$ we find that

$$\begin{aligned}\log U(t) &> t\lambda^{(j)}(z) - |z|\log(\lambda^{(j)}(z)) - r_t b_t \log(\lambda^{(j)}(z)) + O\left(r_t g_t \log(s_t)\right)\\ &> t\lambda^{(j)}(z) - \frac{|z|}{\gamma}\log\log t + O\left(t d_t b_t\right)\end{aligned}$$

with the last inequality following since $s_t := \log\log\log t$ and by the choice of the scaling functions in equation (4.4).

Next, we turn to the case $d = 1$. Denote by $(\bar{X}_t)_{t\geq 0}$ a simple continuous-time random walk on the integers with generator $\Delta\bar{\sigma}^{-1}$, where $\bar{\sigma}(x) = \sigma_t^i$ if $\sigma(x) \in (\sigma_t^{i-1}, \sigma_t^i]$. Again, by a simple coupling argument

$$\mathbb{P}(\tau < rt) \geq \mathbb{P}(\bar{\tau} < rt)\,,$$

where $\bar{\tau}$ is the first hitting time of $z$ by $\bar{X}$ and $r \in (0,1)$. Furthermore, we have

$$\mathbb{P}(\bar{\tau} < rt) > 2^{-|Z_t|}\prod_{i=1}^{2}\mathbb{P}(\mathrm{Poi}(r_i t(\sigma_t^i)^{-1}) = N_i^p\,,$$

for any $r_1, r_2 > 0$ satisfying $r_1 + r_2 \leq r$. By a similar calculation to the $d \geq 2$ case, we have

$$\log U(t) > t(1-r)\lambda^{(j)}(z) + \sum_{i=1}^{2}\left(-r_i t(\sigma_t^i)^{-1} - N_i^p\log(2N_i^p/(e r_i t(\sigma_t^i)^{-1})\right) + O(\log t)\,.$$

Choose $r$ and $\{r_i\}_{i=1,2}$ to maximise this equation, that is, set

$$r_i = \frac{N_i^p}{t(\lambda^{(j)}(z) + (\sigma_t^i)^{-1})} \quad \text{and} \quad r = \sum_{i=1,2} r_i$$

noting that $r \in (0,1)$ for the same reason as in the $d \geq 2$ case. Then,

$$\begin{aligned}\log U(t) &> t\lambda^{(j)}(z) + \sum_{i=1}^{2}\left(-N_i^p\left(\log\left(\lambda^{(j)}(z)\sigma_t^i\right)\right) + O(1)\right) + O(\log t)\\ &= t\lambda^{(j)}(z) - |z|\log\left(\lambda^{(j)}(z)\right) - \sum_{i=1}^{2}\left(N_i^p\log\sigma_t^i + O(|z|)\right) + O(\log t)\,.\end{aligned}$$

The result follows since we are on event $\Theta_t^d$. □

### 4.4.3 Contribution from each $U^i(t)$ is negligible

In this section we prove that the contribution to $U(t)$ from the each of the components $U^i(t)$, for $i = 2, 3, 4, 5$, is negligible. The most difficult step is bounding the contribution from the components $U^2(t)$ and $U^3(t)$. We use the same approach as in Chapter 2.

For each path recall the integers $(r^\ell)_{\ell\geq 0}$ and $(s^\ell)_{\ell\geq 1}$, and the equivalence class $P(p)$ consisting of paths that have identical trajectory except for when they are in balls of radius $j$ around sites $z \in \Pi^{(L_t)}$ (or, more accurately, when they first hit a site $z \in \Pi^{(L_t)}$ until when they leave the ball $B(z,j)$). Recall also, for $m,n \in \mathbb{N}$, the set $\mathcal{P}_{n,m}$ of equivalence classes $P(p)$ of paths $p$ that satisfy

$$\max\{\ell : r^\ell < \infty\} = m \quad \text{and} \quad \sum_{\ell=0}^{m-1}(s^{\ell+1} - r^\ell) + s^{m+1}\mathbb{1}_{\{s^{m+1}<\infty\}} + |p|\mathbb{1}_{\{s^{m+1}=\infty\}} - r^m = n\,,$$

and that the quantity $m$ counts the number of balls of radius $j$ around $z \in \Pi^{(L_t)}$ that the path *exits*, and the quantity $n$ counts the total length of the path between leaving each of these balls and hitting the next site $z \in \Pi^{(L_t)}$. Recall finally the event $\{p(X) \in P(p)\}$ and the contribution $U^{P(p)}(t)$ to the total solution $U(t)$ from the path equivalence class $P(p)$.

The following lemma bounds the contribution of each $P(p) \in \mathcal{P}_{n,m}$ in terms of $m$ and $n$. The key fact motivating our set-up is that the contribution is decreasing in $n$.

**Lemma 4.37** (Bound on the contribution from each equivalence class)**.** *Let $m,n \in \mathbb{N}$ and $p \in \Gamma(0)$ such that $\{p\} \subseteq V_t$ and $P(p) \in \mathcal{P}_{n,m}$. Define $z^{(p)} := \operatorname{argmax}_{z\in\{p\}} \lambda^{(j)}(z)$ and let $\zeta > \max\{\lambda^{(j)}(z^{(p)}), L_{t,\varepsilon}\}$. Then there exist constants $c_1, c_2 > 0$ such that, for each $m,n,p$ and $\zeta$ uniformly, as $t \to \infty$,*

$$U^{P(p)}(t) < e^{\zeta t}\big(c_1(\zeta - L_t)\big)^{-n}\left(1 + c_2\left(\zeta - \lambda^{(j)}(z^{(p)})\right)^{-1}\right)^m$$

*eventually almost surely.*

*Proof.* This follows as in Chapter 2, since we have that $\sigma^{-1}(p_i)$ is bounded above by $\sigma_\sigma^{-1}$. ☐

Just like in Chapter 2 we can use Lemma 4.37 to bound the contribution to the total mass $U(t)$ from $U^2(t)$ and $U^3(t)$. The negligibility of $U^2(t)$ and $U^3(t)$ are then a consequence of the lower bound on the total mass $U(t)$ in Lemma 4.34. Since the negligibility of $U^4(t)$ and $U^5(t)$ are proved identically (noting that we have a lower bound on $\sigma$, which is crucial in $U^5(t)$), as a corollary we have the following.

**Corollary 4.38.** *There exists a constant $c$ such that, as $t \to \infty$,*

$$\frac{U^2(t) + U^3(t) + U^4(t) + U^5(t)}{U(t)}\mathbb{1}_{\mathcal{E}_{t,c}}\mathbb{1}_{\Theta_t^d} \to 0$$

*almost surely.*

## 4.5 Complete localisation

In this section we complete the proof of Theorem 4.1; that is, we show that the non-negligible component of the total solution, $u^1(t,z)$, is eventually localised at $Z_t$. Throughout this section, fix the constant $c > 0$ from Corollary 4.38.

The idea of the proof is identical to in the PAM, that: (i) the solution $u^1(t,z)$ is closely approximated by the principal eigenfunction of the operator $\mathcal{H} := \Delta\sigma^{-1} + \xi$ restricted to $B_t$, and; (ii) the principal eigenfunction decays exponentially away from $Z_t$. So let $\lambda_t$ and $v_t$ denote, respectively, the principal eigenvalue and eigenfunction of the operator $\mathcal{H}$, renormalising $v_t$ so that $v_t(Z_t) = 1$. Remark that, on the event $\mathcal{E}_t$, we have that $B_t \in V_t$. Hence we can apply Proposition 4.16 which implies that, for any $y \in B_t \setminus \{Z_t\}$,

$$\begin{aligned} \frac{u^1(t,y)}{u^1(t,Z_t)} &\le \sigma(y) \|\sigma^{-\frac{1}{2}} v_t\|_{\ell_2}^2 \, v_t(y) \\ &\le \delta_\sigma^{-1} \sigma(y) \|v_t\|_{\ell_2}^2 \, v_t(y) \, . \end{aligned} \tag{4.23}$$

Hence it is sufficient to prove that $v_t$ decays exponentially away from $Z_t$.

**Lemma 4.39** (Gap in $j$-local principal eigenvalues in $B_t$)**.** *On the event $\mathcal{E}_{t,c}$, each $z \in B_t \setminus \{Z_t\}$ satisfies*

$$\lambda^{(j)}(Z_t) - \lambda^{(j)}(z) > d_t e_t + o(d_t e_t) \, .$$

**Corollary 4.40.** *Eventually on the event $\mathcal{E}_{t,c}$, each $z \in B_t \setminus \{Z_t\}$ satisfies*

$$\lambda_t > \lambda^{(j)}(z) + d_t e_t + o(d_t e_t) \, .$$

**Proposition 4.41** (Feynman-Kac representation for the principal eigenfunction)**.** *Eventually on the event $\mathcal{E}_{t,c}$,*

$$v_t(z) = \frac{\sigma(z)}{\sigma(Z_t)} \mathbb{E}_z \left[ \exp\left\{ \int_0^{\tau_{Z_t}} (\xi(X_s) - \lambda_t) \, ds \right\} \mathbb{1}_{\{\tau_{B_t^c} > \tau_{Z_t}\}} \right] ,$$

*where*

$$\tau_{Z_t} := \inf\{t \ge 0 : X_t = Z_t\} \quad \text{and} \quad \tau_{B_t^c} := \inf\{t \ge 0 : X_t \notin B_t\} \, .$$

*Proof.* This is an application of Proposition 4.6, valid precisely because of Corollary 4.40. □

Recall the partition of paths into equivalence classes in Section 2.4, the quantities $r^\ell$ and $s^\ell$ associated to each equivalence class, and, for $m, n \in \mathbb{N}$, the set of equivalence classes $\mathcal{P}_{n,m}$. Recall also the event $\{p(X) \in P(p)\}$.

As in Chapter 2, define the path set $\bar{E}_t^1$, the set of equivalence classes $\bar{\mathcal{P}}_{n,m}^1$, the function

$$v_t^P(y) := \frac{\sigma(y)}{\sigma(Z_t)} \mathbb{E}_y \left[ \exp\left\{ \int_0^{\tau_{Z_t}} (\xi(X_s) - \lambda_t) \, ds \right\} \mathbb{1}_{\{p(X) \in P\}} \right] . \tag{4.24}$$

and the site $z^{(P)}$.

**Lemma 4.42** (Bound on the contribution from each equivalence class)**.** *Let $m, n \in \mathbb{N}$ and $P \in \bar{\mathcal{P}}_{n,m}^1$. Then there exist constants $c_1, c_2 > 0$ such that, for each $m, n, P$ and $y \in B_t \backslash \Pi^{(L_t)}$ uniformly, as $t \to \infty$,*

$$v_t^P(y) \, \sigma(Z_t) < (c_1(\lambda_t - L_t))^{-n} \left(1 + c_2(\lambda_t - \lambda^{(j)}(z^{(P)}))^{-1}\right)^{m-1}$$

*and, for each $m, n, P$ and $y \in \Pi^{(L_t)}$ uniformly, as $t \to \infty$,*

$$v_t^P(y) \, \sigma(Z_t) < \left(\lambda_t - \lambda^{(j)}(z^{(P)})\right)^{-1} (c_1(\lambda_t - L_t))^{-n} \left(1 + c_2(\lambda_t - \lambda^{(j)}(z^{(P)}))^{-1}\right)^{m-1}$$

*both hold eventually almost surely.*

*Proof.* The proof is similar to Lemma 2.44, however a small modification is necessary to take into account the additional $\sigma(y)$ factor present in the Feynman-Kac representation for $v_t^P$ in (4.24), which *a priori* could be arbitrarily large. How we take this into account depends on whether the path $p$ starts at a site of high potential. If $y \notin \Pi^{(L_t)}$, we simply modify equation (2.24) by pulling out the factor $\sigma(y)$ and bounding the right-hand side by

$$(2d)^{-n}\sigma^{-1}(y)(\lambda_t - L_t)^{-1}\left(1 + \delta_\sigma(\lambda_t - L_t)\right)^{-n+1},$$

and the claimed result follows. If $y \in \Pi^{(L_t)}$, we instead modify equation (2.25) by using the *second* bound in Lemma 4.15 on the product factor for $\ell = 1$, which yields (abbreviating $s := s^\ell$)

$$\mathbb{E}_y[I_0^{\tau_{B(y,j)}}]\prod_{\ell=2}^{m-1}\mathbb{E}_{p_s}\left[I_0^{\tau_{B(p_s,j)}}\right] \leq \sigma^{-1}(y)(\lambda_t - \lambda^{(j)}(z))^{-1}\left(1 + \frac{\delta_\sigma^{-1}|B(0,j)|}{\lambda_t - \lambda^{(j)}(z^{(P)})}\right)^{m-1},$$

and again the claimed result follows. $\square$

**Proposition 4.43** (Exponential decay of principal eigenfunction)**.** *On the event $\mathcal{E}_{t,c}$ there exists a constant $C > 0$ such that, for each $y \in B_t$ uniformly, as $t \to \infty$,*

$$\log v_t(y) + \log \sigma(Z_t) < -C|y - Z_t| \log\log t$$

*eventually almost surely.*

We are now in a position to establish Theorem 4.1. We work on the events $\mathcal{E}_{t,c}$ and $\Theta_t^d$, since by Proposition 4.35 these hold eventually with overwhelming probability. First, remark that Proposition 4.43 implies that, as $t \to \infty$,

$$\sigma(Z_t)\sum_{z \in B_t \setminus \{Z_t\}} v_t(z)^2 \to 0$$

almost surely, and so in particular $\|v_t\|_{\ell_2}^2 \to 1$, since we know $\sigma(Z_t) \geq \delta_\sigma$. Combining with equation (4.23), and the negligibility results already established in Corollary 4.38 on the events $\mathcal{E}_{t,c}$ and $\Theta_t^d$, completes the proof. $\square$

# Chapter 5

# Future directions

In this chapter we briefly outline some future directions for research on intermittency and localisation phenomena in random walk models. In particular, we focus on questions relating to intermittency and localisation phenomena in: (i) randomly branching random walks (such as the PAM); (ii) random walks in random trapping landscapes (such as the BTM); and (iii) hybrid models combining both branching and trapping dynamics (such as the BAM).

## 5.1 Localisation phenomena in randomly branching random walks

In Chapter 2 of this thesis we addressed the localisation properties of the PAM in the regime of potentials with sub-double-exponential tail decay, namely in the Weibull case and in the FDE case. We showed that the PAM completely localises, with overwhelming probability, in each of these regimes. This agrees with the conjecture that double-exponential potentials form the boundary of the complete localisation universality class. Here we consider some extensions to these results, including extensions to other models of randomly branching random walks.

### 5.1.1 Beyond the complete localisation regime of the PAM

A long-standing open question is to determine the localisation properties of the PAM beyond the complete localisation universality class, including in the case of potentials with (i) double-exponential tail decay, and (ii) tail decay that is lighter than double-exponential.

In the case of potentials with double-exponential tail decay, it has long been believed that localisation occurs on a certain number of connected sets of sites of high potential (often called *relevant islands*) whose size remains bounded in probability as $t \to \infty$ (although not bounded almost surely). The intuition is analogous to the discussion in Chapter 2, namely that in the double-exponential case there are connected clusters of sites whose potential differs from the maximum potential in the macrobox only by a bounded amount. This naturally leads to eigenfunctions (and hence peaks of the solution) which are approximately flat on this cluster. On the other hand, it is not clear *a priori* how many such relevant islands are needed to carry the solution; a strong conjecture is that only a *single* island is necessary.

**Conjecture 1** (Localisation on one relevant island in the double-exponential case)**.** *Assume there exists a $\gamma > 0$ such that $\xi(0)$ satisfies*

$$\mathbf{P}(\xi(0) > x) = \exp\{-e^{\gamma x}\}\,, \quad x > 0\,.$$

*Then, for any function $h_t \to \infty$, there exists a set-valued process $\Gamma_t$ such that $\Gamma_t$ is connected, $|\Gamma_t| \ll h_t$, and*

$$\frac{\sum_{z \in \Gamma_t} u(t,z)}{U(t)} \to 1 \qquad \textit{in probability}\,. \tag{5.1}$$

*On the other hand, there does not exist constant $c > 0$ and a set-valued process $\Gamma_t$ with $|\Gamma_t| \leq c$ such that* (5.1) *holds.*

It appears that this conjecture, at least in substantial part, has now been established in very recent work [18] that builds on the methods[1] presented in this thesis. Nevertheless, certain questions about the precise nature of localisation in the double-exponential case remain open. In particular, a detailed description of the shape of the solution, such as we provide in Theorem 1.3, has not yet been achieved. Such a description, for instance, would provide an upper bound on the rate of convergence to the localised state, such as we give in Corollaries 1.4 and 1.11 for instance.

In the case of potentials with tail decay that is lighter than double-exponential, it appears much more difficult to give a clear geometric picture of intermittency, and indeed it is unknown whether localisation in the sense of (1.8) occurs at all. One aspect that is not in doubt is that any localisation set $\Gamma_t$ must be growing in size as $t \to \infty$. This is essentially for the same reason as why localisation islands are of bounded size in the double-exponential case. What is unclear, however, is whether localisation occurs on a single such relevant island, a growing number of such relevant islands, or in fact not at all.

**Question 1.** *What are the localisation properties of the PAM in the case of potentials with tail decay that is lighter than double-exponential? In particular, does the solution localise on a single relevant island of sites of high potential, or a growing number of relevant islands, or not at all? How does the rate of growth of the size of the relevant islands depend on the tail of $\xi$?*

### 5.1.2 Almost sure localisation in the PAM

Recall that the only regime in which the almost sure (i.e. holding $\mathbf{P}$-almost surely) localisation properties of the PAM have been determined is the 'extremal'[1] case of Pareto tail decay. In that case, it has been established that the PAM localises on two-sites almost surely, as per the following theorem.

[1] Personal communication.

[1] Recall that beyond a certain threshold of heavy tail decay, the solution $\mathbf{P}$-almost surely 'blows-up' in finite time; see [39].

**Theorem** (Two-site almost sure localisation in the PAM with Pareto potential; see [52])**.** *Assume that there exists a $\gamma > d$ such that $\xi(0)$ satisfies*

$$\mathbf{P}(\xi(0) > x) = x^{-\gamma}\,, \quad x > 0\,.$$

*Then there exists a set-valued process $\Gamma_t$ such that $|\Gamma_t| = 2$ and*

$$\frac{\sum_{z\in\Gamma_t} u(t,z)}{U(t)} \to 1 \qquad \textit{almost surely}\,. \tag{5.2}$$

*On the other hand, there does not exist a site-valued process $Z_t =: \Gamma_t$ such that* (5.2) *holds.*

Along with the complete localisation in probability of the solution, this almost sure localisation result provides a very complete geometric picture of the asymptotic behaviour of the PAM, as follows. At typical large times the solution of the PAM is localised at a certain 'good' site, with negligible mass on other sites. However, at certain rare times a 'better' site becomes accessible, at which point the solution transitions to this new site, but in such a way that no non-negligible amount of mass is on any other site during the transition. Keeping in mind the interpretation of the PAM in terms of a certain particle system, this has been evocatively described as a 'two-cities' theorem.

A significant open question is to establish the almost sure localisation properties of the PAM in other regimes. In particular, we would like to answer the following.

**Question 2.** *Does two-site almost sure localisation hold in the entire complete localisation universality class? Alternatively, for potentials with tail decay that is almost double-exponential, does almost sure localisation require a bounded, or even a growing, number of sites or relevant islands? What are the almost sure localisation properties of the PAM for potentials with double-exponential, or lighter than double-exponential, tail decay?*

Here we make some comments on the difficulties involved in answering such questions, even in the case of Weibull potential. To a large extent, the existing method of proof used to establish almost sure localisation in the PAM is very similar to that for establishing localisation in probability, namely to identify a suitability criteria to access candidate sites – expressed through the penalisation functional $\Psi_t$ – and then to establish a gap in the top order statistics of this criteria. One significant difference for almost sure localisation is that the gap in the top order statistics must be controlled *across all large times*, rather than just at a certain large, but fixed, time. This introduces a need to evaluate (i) how precisely the penalisation functional $\Psi_t$ expresses the suitability of a candidate site, and (ii) how large a gap we can establish between the top order statistics of the penalisation functional across all large times. Note that, in order to establish two-site almost sure localisation, we must establish such a gap between the first and third order statistics of the functional – of course, by continuity, no almost sure gap exists between the top two order statistics.

Let us examine these two points more closely in the Weibull case. On the first point, recall from the discussion in Chapter 2 that the probabilistic penalty for diffusion to a certain candidate site $z$ can be well-approximated by

$$\sum_{p \in \Gamma_{|z|}(0,z)} \prod_i \frac{1}{\xi(z) - \xi(p_i)}$$

which, since $\xi(z) > \delta_1 a_t$ for some $\delta_1 \in (0,1)$ for all suitable candidate sites, we are able to simplify to

$$\exp\{|z| \log a_t + c|z| + o(td_t)\}$$

for some $c > 0$. Note that the error term $o(td_t)$ is smaller than ($t$ times the) gap $d_t$ that we are able to establish in the top order statistics of the penalisation functional $\Psi_t$, which was the crucial fact underlying complete localisation.

On the second point, through a simple Borel-Cantelli type analysis on the trajectories of the points in the point process $\Psi_t$ (i.e. using a similar discretisation scheme as in [52]) it can be seen that the largest almost sure gap we can hope to establish between the first and third order statistics of the functional $\Psi_t$ is of order

$$\frac{d_t}{\log t} = (\log t)^{\frac{1}{\gamma}-2} .$$

Note that the extra logarithmic factor, relative to $d_t$, in this gap then poses a problem, since it is now smaller than the error term of $d_t$ in the penalisation functional above. In other words, the above approximation to the probabilistic penalty is no longer fine enough for our purposes.

To remedy this we might use the Taylor expansion

$$\frac{1}{\xi(z) - \xi(p_i)} = \frac{1}{\xi(z)} - \frac{\xi(p_i)}{\xi(z)^2} + \frac{\xi(p_i)^2}{\xi(z)^3} + \dots$$

which, after taking a logarithm, results in an expression for the probabilistic penalty of the form

$$|z| \log a_t - \log n(z) - \frac{1}{a_t}\frac{1}{n(z)} \sum_{p \in \Gamma_{|z|}(0,z)} \sum_i \xi(p_i) + \frac{1}{a_t^2}\frac{1}{n(z)} \sum_{p \in \Gamma_{|z|}(0,z)} \sum_{i,j} \xi(p_i)\xi(p_j) + \dots \tag{5.3}$$

The first term in (5.3) is the usual probabilistic penalty that we used in our proof of localisation in probability. The second term counts the number of shortest paths to $z$; this is fairly easy to control since it depends only on local information (indeed this term also appears in the Pareto case; see [52]). However, the remaining terms in (5.3) are much more complicated to control because they are *non-local*: they depend on the potential field across significant portions of the domain. This induces dependencies in the values of the penalisation functional $\Psi_t$ at the candidate sites, making it harder to establish gaps in the top order statistics. Note that these non-local terms did not need to be taken into account in the Pareto case.

We remark finally that in the case $\gamma < 1$, it might actually be possible to determine the almost sure localisation properties of the PAM without needing to analyse the extra non-local terms in (5.3); instead we can simply control their influence using some rough bounds. Indeed similar expressions have previously been studied in the literature under the name 'greedy lattice animals', and in [37] it was shown that a law of large numbers holds for

$$\max_{p \in \Gamma_{|z|}(0,z)} \sum_i \xi(p_i) \, .$$

This implies that

$$\frac{1}{n(z)} \sum_{p \in \Gamma_{|z|}(0,z)} \sum_i \xi(p_i) = O(|z|)$$

and hence, in the regime $\gamma < 1$,

$$\frac{1}{a_t} \frac{1}{n(z)} \sum_{p \in \Gamma_{|z|}(0,z)} \sum_i \xi(p_i) \ll \frac{t d_t}{\log t} \, .$$

This suggests that an analysis involving only the first two terms may be sufficient to determine almost sure localisation, just as it was in the Pareto case. In light of this, we make the following conjecture.

**Conjecture 2.** *Assume that $\xi(0)$ satisfies, for some $\gamma \in (0,1)$,*

$$\mathbf{P}(\xi(0) > x) = \exp\{-x^\gamma\} \, , \quad x > 0 \, .$$

*Let $\Pi^{(L_t)}$ be defined as in Chapter 2, define the penalisation functional*[1]

$$\Psi_t(z) := \xi(z) - \frac{|z|}{\gamma t} \log\log t - \log n(z) \, ,$$

*and denote by $Z_t^{(1)}$ and $Z_t^{(2)}$ the maximisers*

$$Z_t^{(1)} := \operatorname*{argmax}_{z \in \Pi^{(L_t)}} \Psi_t(z) \quad \text{and} \quad Z_t^{(2)} := \operatorname*{argmax}_{z \in \Pi^{(L_t)} \setminus \{Z_t^{(1)}\}} \Psi_t(z) \, .$$

*Then, as $t \to \infty$,*

$$\frac{u(t, Z_t^{(1)}) + u(t, Z_t^{(2)})}{U(t)} \to 1 \quad \text{almost surely} \, .$$

The case $\gamma \geq 1$ appears far more challenging, and it is not obvious to us whether two-site almost sure localisation will hold throughout this case. Regardless, it seems clear that any result in this direction will necessarily require a detailed analysis of the non-local terms identified above, as well as the development of new techniques that can handle the induced dependencies in the values of $\Psi_t$.

### 5.1.3 Localisation in the 'unaveraged' PAM

Recall that the PAM can be considered as the thermodynamic limit of the system of particles on $\mathbb{Z}^d$ specified by:

- *Initialisation*: A single particle at the origin;

[1] Note that it should be sufficient to use the potential $\xi(z)$, rather than the local principal eigenvalue $\lambda(z)$, since if $\gamma < 1$ then $t d_t / \log t$ is also smaller than $t$ times the gap between $\xi(z) - 2d$ and $\lambda(z)$, for all candidate sites $z \in \Pi^{(L_t)}$, which is $O(1/a_t)$.

- *Branching*: Each particle branches (i.e. duplicates) independently at the jump times of a time-inhomogeneous Poisson process, with the rate of the Poisson process for a particle at a site $z$ given by $\xi(z)$;
- *Diffusion*: Each particle evolves as an independent continuous-time simple random walk on $\mathbb{Z}^d$, that is, the waiting time for each particle at each site is independent and distributed exponentially with unit mean, with the subsequent site chosen uniformly from among the nearest neighbours.

An interesting alternative to studying the PAM is to consider this system of branching, diffusive particles *before* averaging over the jump times, trajectories and branching times of the particles in this system. To this end, denote by $N(t, z)$ the counting function for the number of particles at time $t$ and site $z$, which is a random variable measurable with respect to: (i) the random potential field; (ii) the randomness in the Poisson processes determining the branching; and (iii) the randomness driving the diffusion of the particles. The PAM results from averaging $N$ over the second and third sources of randomness, in other words, $u(t, z) := \mathbb{E}(N(t, z))$.

Note that studying the counting function $N$ provides much more detailed information about the underlying particle system than is available through a study of the PAM. In applications, especially to population dynamics (see the discussion in Chapter 1), this particle system is often of primary interest.

The study of localisation properties of the above particle system was initiated in [60] where the case of Pareto potential was considered. It was proven that the (renormalised) counting function $N$ localises, under the annealed measure,[2] on a certain small ball around a certain site $Z_t$ that is measurable with respect to the potential field. Surprisingly, the site $Z_t$ is not always the same site as the localisation site of the PAM (i.e. under the same realisation of the potential field). In other words, the probability that $Z_t$ is equal to the corresponding site of complete localisation of the PAM is strictly between 0 and 1.

This work left open the question as to whether the random variable $N$, in the Pareto case, completely localises under the annealed measure; in [60] it was strongly conjectured that it does. Beyond the Pareto case, essentially all questions on the localisation properties of this model remain open.

**Question 3.** *In the case of Pareto potential field, does the (renormalised) counting function $N$ of the above particle system completely localise under the annealed measure? What are the localisation properties of the above particle system in the Weibull case, or in the case of potential with even lighter tail decay? Are the localisation sites in these regimes comparable to the localisation sites of the PAM? What are the almost sure localisation properties of the above particle system?*

The above questions are likely to be challenging to answer, and in general the above model is considerably more difficult to analyse than the PAM. The main reason is that instead of controlling the diffusion of a single particle, now all particles in the system must be controlled precisely. Further, the spectral methods that we used in Chapter 2 to analyse

[2] Given the multiple sources of randomness in the definition of $N$, this is the equivalent of studying 'localisation in probability' in the PAM.

the growth rate of the solution are no longer available; in [60] a variety of new techniques were developed to compensate for this.

## 5.2 Localisation phenomena in trap models

In Chapter 3 of this thesis we studied the localisation properties of the BTM on the integers in the case of slowly varying trap distribution. Our main result was that this model exhibits two-site localisation with overwhelming probability. Here we explore some possible extensions to this result, including some applications to other, more physically realistic, trap models.

### 5.2.1 Almost sure localisation in the slowly varying BTM

Following on from our study of the BTM on the integers with slowly varying traps, a natural question is to consider the almost sure localisation properties of this model. By simple symmetry and continuity considerations, the strongest localisation behaviour that we might expect to observe is that there exists an almost sure localisation set $\Gamma_t$ such that $|\Gamma_t| = 3$, although it is not a priori clear whether any slowly varying tail should actually attain this bound.

**Question 4.** *What are the almost sure localisation properties of the BTM on the integers with slowly varying traps? For which, if any, slowly varying tail decay does the BTM exhibit the strongest possible form of almost sure localisation (i.e. on three-sites)?*

A partial answer to this question will be given in upcoming work [29], the main thrust of which we briefly outline here. In particular, in [29] we show that the strongest form of almost sure localisation is actually attained for certain examples of the BTM with slowly varying traps. More surprisingly perhaps, we show that for each sufficiently large integer $N$ there exists a slowly varying tail such that almost sure localisation occurs on exactly $N$ sites. As such, the BTM with slowly varying traps is an example of a model that exhibits almost sure localisation on a finite number of sites, with the exact number of localisation sites able to be tuned by adjusting the parameters of the model.

Our answer is only partial because, for simplicity, we choose to work in the one-sided case (i.e. on the positive integers, rather than on the integers); this avoids some of the technical difficulties present in the two-side case, yet still exhibits the relevant phenomena that interests us. Note that in the one-sided case, the analogue to Theorem 1.12 holds with $|\Gamma_t| = 1$, and so the strongest form of almost sure localisation that one might hope to observe is a localisation set $\Gamma_t$ such that $|\Gamma_t| = 2$.

We shall now explain the sense in which the almost sure localisation behaviour of the BTM depends on the precise decay of the slowly varying tail of $\sigma(0)$. This requires us to introduce the concept of *second-order slow variation.* A function $L$ is said to be *second-order slowly varying with rate* $g$ if there exist functions $g, k$ such that $g(u) \to 0$ as $u \to \infty$ and

$$\lim_{u\to\infty} \frac{\frac{L(uv)}{L(u)} - 1}{g(u)} = k(v)\,, \quad \text{for any } v > 0\,, \tag{5.4}$$

and where there exists a $v$ such that $k(v) \neq 0$ and $k(uv) \neq k(u)$ for all $u > 0$; of course, the rate $g$ is only defined up to a constant multiple, but this will be unimportant in what follows. Second-order slow-variation is a natural strengthening of the slow-variation property, giving more precise information about the fluctuations of $L$ at infinity; see [15, Chapter 3] for an overview of second-order slow-variation.

Now recall the slowly varying function $L$ from Chapter 1 and assume that (5.4) holds. Abbreviate the function

$$d(u) := g(L^{-1}(u)),$$

where $L^{-1}$ denotes the right-continuous inverse of $L$, noting that $d(u) \to 0$ as $u \to \infty$. Define the integer

$$N := \min\left\{ l \in \{2, 3, \ldots\} : \sum_{n \in \mathbb{N}} (d(e^n) \log n)^{l-1} < \infty \right\}, \tag{5.5}$$

setting $N = \infty$ if no such $l$ exists.

The main result of [29] identifies $N$ as the number of almost sure localisation sites of the BTM. Before stating this result, we first need to introduce an additional assumption that acts to exclude certain boundary cases, in which the number of localisation sites falls in some sense intermediate between two integers.

**Assumption 2** (Exclusion of boundary cases)**.** *It is the case that $N < \infty$, and*

$$\sum_{n \in \mathbb{N}} d(e^n)^{N-1} (\log n)^N < \infty \quad \textit{and} \quad \sum_{n \in \mathbb{N}} d(e^n)^{N-2} = \infty .$$

**Theorem** (Almost sure localisation in the BTM on the positive integers; see [29])**.** *Let $\mathbb{P}$ denote the quenched probability mass function of the BTM on the positive integers with reflective boundary conditions at the origin. Assume that $L$ is continuous, and that* (5.4) *holds for a rate function $g$ that is eventually monotone decreasing. Then there exists a $\mathbf{P}$-measurable set-valued process $\Gamma_t$ satisfying $|\Gamma_t| \leq N$ such that, as $t \to \infty$,*

$$\mathbb{P}(X_t \in \Gamma_t) \to 1 \quad \mathbf{P}\textit{-almost surely}. \tag{5.6}$$

*Moreover if Assumption 2 also holds, then there is no set-valued process $\Gamma_t$ satisfying $|\Gamma_t| < N$ such that* (5.6) *holds.*

We note that the assumptions on $L$ are satisfied for a wide range of slowly varying distributions $\sigma(0)$. The main examples we have in mind are distributions satisfying

$$L(u) := \exp\{(\log u)^\gamma\}, \quad \gamma \in (0, 1), \tag{5.7}$$

for which $g(u) = (\log u)^{\gamma - 1}$, $k(v) = \gamma \log v$, and

$$N = 2 + \left\lfloor \frac{\gamma}{1 - \gamma} \right\rfloor .$$

In this example, we observe that $N = 2$ if and only if $\gamma < 1/2$. Hence the almost sure localisation behaviour of the BTM distinguishes precisely the two regimes of slowly varying tail that we identified in our results in Chapter 3. Note further that $N \to \infty$ as $\gamma \to 1$, and moreover that any $N \in \{2, 3, \ldots\}$ is attainable by selecting an appropriate $\gamma \in (0, 1)$. Other classes of slowly varying distribution for which our results hold are those with logarithmic decay ($L(u) = (\log u)^\gamma, \gamma > 0$), or double logarithmic decay ($L(u) = (\log \log u)^\gamma, \gamma > 0$); in both cases $L$ satisfies each of the assumptions with $N = 2$.

To give some intuition as to why the number of localisation sites depends on the second-order slow-variation rate $g$ in the way determined by (5.5), consider that $g(u)$ gives a measure of how likely records, or near records, of the sequence $\{\sigma(0)\}_{i \in \mathbb{N}}$ are to cluster on the same scale $u$. In particular, $g(u)^k$ gives the approximate probability that such a cluster consists of at least $k$ sites. Next, consider that the height of the $n^{\text{th}}$ record of the sequence $\{\sigma(i)\}_{i \in \mathbb{N}}$ is approximately $L^{-1}(e^n)$. Hence, by a Borel-Cantelli argument, the summability of $d(e^n)^k$ determines whether a cluster of $k$ records, or near records, occurs eventually almost surely. From here, notice that a cluster of records, or near records, on the same scale naturally gives rise to a division of the probability mass function of the BTM across this cluster. Together with the site from which the BTM eventually escapes after leaving the cluster, we see that this indeed suggests almost sure localisation on $N$ sites. In regards to the extra logarithmic factors appearing in the definition of $N$ in (5.5) and in Assumption 2, it is possible that these are artefacts of our proof which could be removed (or at least relaxed).[3]

**Question 5.** *Are the extra logarithmic factors appearing in the definition of $N$ in* (5.5) *and in Assumption 2 necessary for our results to hold? More generally, what is the localisation behaviour in the 'boundary cases'? Does this behaviour depend on even finer properties than the second-order slowly varying functions $g$ and $k$?*

The above heuristics also allow us to conjecture the almost sure localisation properties of the BTM on the integers. The intuition is that the clustering argument described above is valid across the whole positive and negative half-lines, although an extra localisation site is needed to take into account the fact that the BTM can now escape, after leaving the cluster, in two directions. Nevertheless, formalising this heuristic presents additional technical challenges not present in the one-sided case.

**Conjecture 3.** *Let $\mathbb{P}$ denote the quenched probability mass function of the BTM on the integers and let $N^* := N + 1$, with $N$ defined as above. If* (5.4) *holds, then there exists a $\mathbf{P}$-measurable set-valued process $\Gamma_t$ satisfying $|\Gamma_t| \le N^*$ such that, as $t \to \infty$,*

$$\mathbb{P}(X_t \in \Gamma_t) \to 1 \quad \mathbf{P}\text{-almost surely}. \tag{5.8}$$

*Moreover if Assumption 2 also holds, then there is no set-valued process $\Gamma_t$ satisfying $|\Gamma_t| < N^*$ such that* (5.8) *holds.*

We remark finally that our results on the almost sure localisation of the BTM rely crucially on a certain almost sure analogue of classical results on the sum/max ratio of i.i.d. sequences of slowly varying random variables (in particular, the result stated as Theorem 3.1 above). To the best of our knowledge this result is new, and may be of independent interest.

[3]The same comment applies to our hypothesis that $L$ is continuous and $g$ is eventually monotone decreasing.

**Theorem** (Sum/max ratio for sequences of slowly varying random variables; see [29])**.** *Assume that $L$ is continuous, and that* (5.4) *holds for a rate function $g$ that is eventually monotone decreasing. Let $\{\sigma_i\}_{i \in \mathbb{N}}$ be an i.i.d. sequence of copies of $\sigma_0$, and denote by $m_i$ and $S_i$ the maximum and sum respectively of the partial sequence $\{\sigma_j\}_{j \le i}$. Then, almost surely,*

$$\liminf_{i \to \infty} \frac{S_i}{m_i} = 1 \quad \textit{and} \quad \limsup_{i \to \infty} \frac{S_i}{m_i} \le N - 1 \, .$$

*Moreover, if Assumption 2 also holds, then almost surely*

$$\limsup_{i \to \infty} \frac{S_i}{m_i} = N - 1 \, .$$

In the special case of $L$ satisfying (5.7), this implies that, almost surely.

$$\liminf_{i \to \infty} \frac{S_i}{m_i} = 1 \quad \text{and} \quad \limsup_{i \to \infty} \frac{S_i}{m_i} = 1 + \left\lfloor \frac{\gamma}{1-\gamma} \right\rfloor .$$

Hence, in this case, $\lim_{i \to \infty} S_i/m_i = 1$ almost surely if and only if $\gamma < 1/2$. For comparison, recall that the latter limit holds in probability for all slowly varying tails (see Theorem 3.1); an observation which (together with Fatou's lemma) already yields the $\liminf$ part of the previous result.

### 5.2.2 The higher-dimensional BTM in the case of slowly varying traps

As mentioned in Chapter 1, in the case of regularly varying traps the BTM in higher dimensions (in particular on $\mathbb{Z}^d$) does not exhibit localisation. Instead, the BTM rescales to a process known as *fractional kinetics* (FK), which can be defined as a Brownian motion that is time-changed by an independent stable subordinator (i.e. a non-decreasing stable Lévy process). We note a counterintuitive feature of this scaling limit: although the traps continue to influence the dynamics of the BTM in the limit, the overall effect of the traps is asymptotically *independent* of the trajectory of the BTM, and only affects the clock process.

A similar asymptotic independence of the trajectory and clock process is expected to hold in the case of slowly varying traps, although to the best of our knowledge this is yet to be explored in detail in the literature. Here we formalise this statement, and conjecture the equivalent scaling limit for the BTM in the slowly varying case.

First we introduce the conjectured scaling limit, which can be thought of as the $\alpha \to 0$ limit of the usual FK process with parameter $\alpha \in (0, 1)$ (i.e. the *extremal FK process*). This process has already appeared in the literature before, notably in [27] where it was shown to be the limit of a 'directed' version of the BTM on the integers with slowly varying traps. This process will also appear in upcoming work on more general randomly trapped random walks in the slowly varying case (c.f. [9] in the regularly varying case), as one of the main classes of scaling limits that may arise in general i.i.d. trap models. Let $\mathcal{P}$ denote the inhomogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}$ with intensity measure $v^{-2} \, dx \, dv$, denote by

$m = (m_t)_{t \geq 0}$ the extremal process for $\mathcal{P}$, defined by

$$m_t := \sup \{v_i : x_i \leq t\} ,$$

and let $I = (I_t)_{t \geq 0}$ be its right-continuous inverse. Let $B = (B_t)_{t \geq 0}$ be an independent Brownian motion on $\mathbb{Z}^d$. We shall identify the process $B_I = (B_{I_t})_{t \geq 0}$ as the scaling limit of the higher-dimensional BTM.

**Conjecture 5** (Convergence of the higher-dimensional BTM with slowly varying traps to the extremal FK process)**.** *Suppose $d \geq 2$ and assume that $\sigma(0)$ has a slowly varying tail at infinity. Denote by $X = (X_t)_{t \geq 0}$ the BTM on $\mathbb{Z}^d$ (defined analogously to the BTM on the integers). Then, under the annealed law, as $n \to \infty$,*

$$\left( \frac{1}{n} X_{L^{-1}(nt)} \right)_{t \geq 0} \overset{L_1}{\Rightarrow} (B_{I_t})_{t \geq 0} ,$$

*where $\overset{L_1}{\Rightarrow}$ denotes weak convergence in the $L_{1,loc}$ topology (defined analogously to in the one-dimensional case in Appendix 3.5).*

Intuition for the above result is the same as in the regularly varying case (see the discussion in [12] and [68]). More precisely, recall that higher-dimensional simple random walks are either transient ($d \geq 3$) or weakly null-recurrent ($d = 2$), in the sense that the return time distribution has a slowly varying tail.[4] This transience, or weak null-recurrence, has the consequence that repeat visits to deep traps make a negligible contribution to the scaling limit, since in all likelihood much deeper traps have already been visited prior to any significant number of repeat visits. This has the effect that (i) the clock-process of the BTM is asymptotically independent of its trajectory, and (ii) unlike in the one-dimensional case, the scaling of time does not require an extra factor of $n$ to account for repeat visits to each deep trap.

### 5.2.3 Connections to other trap models

One of the main motivations for studying the BTM is the strong connections to more physically realistic trap models. Indeed the BTM is an effective phenomenological model for a variety of trapping behaviour (see the general discussion in [10, 12]). In light of this, we explore here some potential applications of our results and methods to other trap models, in particular to models in the 'extremal' regime of slowly varying traps.

#### Biased random walks on critical structures

Biased random walks on critical structures give a natural example of extremal trapping phenomena. The main examples of such structures are the incipient infinite percolation cluster (IIC) and the critical Galton-Watson tree conditioned to survive (CGWT). The

[4] Note that the slowly varying tail of the return time distribution is crucial to the asymptotic independence of the trajectory and the clock process, and mere null-recurrence is insufficient; see [36, 68]. Indeed the simple random walk on the integers is null-recurrent, but not weakly so, since the tail of the return time distribution decays polynomially.

IIC is a central object of interest throughout probability and physics, but its complicated geometry makes its analysis challenging. The CGWT is an attractive alternative model, which replicates many of the features of the IIC but is more analytically tractable. Since the geometry of critical structures directly is hard to describe explicitly, a central tool with which to probe the geometry is through the behaviour of induced random walks.

The link between random walks on critical structures and the BTM comes from the unique geometry of critical structures, which consist of a small number of 'paths to infinity' (sometimes described as the 'backbone') with a certain number of finite branches (or 'dead-ends' in the IIC) attached. Projecting a random walk onto the 'backbone' produces a trap model, and it is known that biasing the walk (in any fixed direction for the IIC; towards the leaves for the CGWT) results in a trap model that is in the extremal regime. In light of the above, we propose some possible applications of our work on the BTM to the study of the localisation properties of the biased random walk on the CGWT and IIC.

Turning first to the CGWT, recent work has established the scaling limit of a biased random walk on the CGWT [27], confirming that this model lies in the extremal regime. Although not made explicit, this work also suggests the annealed localisation behaviour of the model, namely that a single branch of the CGWT asymptotically carries the entire probability mass of the walk. One possible application of our results and methods would be to complete the description of the localisation properties of this model by determining its almost sure localisation behaviour, that is, by establishing the number of branches which asymptotically carry the probability mass eventually almost surely. Based on [27], the work in this thesis, as well as the upcoming work [29] on almost sure localisation in the BTM discussed above, we expect that just two branches of the CGWT suffice to carry the probability mass eventually almost surely.

**Question 6.** *Does the biased random walk on the critical Galton-Watson tree exhibit almost sure localisation on two branches? That is, can we identify a (CGWT-measurable) set of two branches that asymptotically carry the probability mass of the biased random walk eventually almost surely?*

By contrast to the CGWT, the behaviour of the biased random walk on the IIC is currently poorly-understood, especially in low dimensions.[5] Aside from the complex dependency between the 'backbone' and the 'dead-ends' , an additional complication is that a second trapping effect arises out of loops in the 'backbone' of the IIC, for instance if they are orientated in the same direction as the bias. Previous work has made incremental progress by approximating the 'backbone' of the (high-dimensional) IIC by the range of a random walk [25, 26], and studying the resulting trapping effect due to loops; interestingly, this effect was also found to lie in the extremal regime. In other words, biased random walks on the IIC actually experience two distinct extremal trapping phenomena. In light of this, one possible direction of study is to approximate the IIC by the range of a random walk equipped with an i.i.d. trapping landscape in the extremal regime (so-called 'transparent' traps would be a natural choice, see [10, 27]). It would be interesting to determine the localisation behaviour of biased random walks in such a model, in particular to gain insight into how the trapping effects due to the 'backbone' and 'dead-ends' of the IIC might interact.

[5] Indeed, apart from in the case $d = 2$, the IIC has so far only been rigorously constructed in $d \geq 19$, although it has also been constructed in the case $d > 6$ for sufficiently 'spread out' percolation; see [44]

**Question 7.** *What is the localisation behaviour of the biased random walk on the incipient infinite critical percolation cluster? What is the localisation behaviour of an approximation of this model consisting of a biased random walk on the range of a random walk equipped with transparent, slowly varying traps?*

**Random walks in random environment**

The classical random walk in random environment (RWRE) is the random walk on $\mathbb{Z}$ with transition probabilities

$$q_{z \to z+1} := p_z \in [0, 1]$$

where $\{p_z\}_{z \in \mathbb{Z}}$ is an i.i.d. random environment. Trapping in the RWRE arises out of deep valleys in the potential function $P$, defined by $P(0) := 0$ and

$$P(z) - P(z-1) := \log(p_z/(1-p_z)) .$$

Two distinct regimes exist – transient and recurrent – depending on whether $P(z)$ diverges as $z \to \infty$. In the recurrent case, the RWRE is known to exhibit extremal localisation. In the transient case, the RWRE can lie in the stable or extremal regimes of localisation; as far as we are aware, only the stable regime has been studied.

One possible direction of future research would be to establish the almost sure localisation of the recurrent RWRE (also known as Sinai's random walk). A classical result is that the RWRE exhibits extremal localisation in the sense that, after $n$ steps, it is very likely to lie in a small window around the bottom of the deepest valley in the potential on the scale $(\log n)^2$ [41, 65]. This result establishes the annealed localisation of the RWRE. What has not yet been established, to the best of our knowledge, is the almost sure localisation of the RWRE, that is, the precise number of valleys, and the tightest 'windows', that carry the probability mass of the walk at all large times. Based on the results of this thesis, we expect that the quenched probability mass will be carried by just two valleys eventually almost surely. Note that similar questions have previously been studied in [31], in which the local time *of a single trajectory* of the RWRE was shown to concentrate on two sites almost surely.

**Question 8.** *What is the almost sure localisation behaviour of Sinai's random walk? Are two valleys sufficient to asymptotically carry the quenched probability mass of the walk eventually almost surely?*

Another possible application of our results and methods would be to analyse the extremal regime of localisation in the transient RWRE. It is well-known that the transient RWRE exhibits different scaling limit regimes depending on the tail decay of $-\log p_z$. Classical results cover the 'homogeneous' (or 'integrable') regime (e.g. central limit theorems, no localisation), and the 'stable' (or 'regularly varying') regime (e.g. convergence to stable processes, weak localisation) [51]. One possible application of our results would be to study the 'extremal' regime of the transient RWRE, that is, the regime in which the tails of $-\log p_z$ are slowly varying at infinity. To our knowledge this regime has not yet been studied in the literature, and we expect to observe interesting extremal limits and corresponding strong localisation phenomena.

**Question 9.** *What are the possible scaling limit of the transient RWRE in the 'extremal' regime? Does the RWRE in this regime exhibit strong localisation behaviour?*

## 5.3 Hybrid models combining branching and trapping mechanisms

In Chapter 4 of this thesis we introduced a hybrid model, the BAM, combining the random branching mechanism of the PAM and the random trapping mechanism of the BTM, and initiated the study of its localisation properties. Our main finding was that the localisation effects due to the random branching and random trapping landscapes are mutually reinforcing, and moreover induce a local correlation in the two random fields. Here we outline some of the questions provoked by this initial study, both on the topic of the BAM, as well as on the topic of other, alternative, hybrid models.

### 5.3.1 Removing our assumptions on the trapping landscape

Recall that our results on the BAM relied on two major assumptions on the trapping landscape (this is true of the results in this thesis, as well as in their more general form in [59]). First, we assumed that the trapping landscape contained no 'quick sites', that is, the trap distribution $\sigma(0)$ was bounded away from zero. Second was our assumption, necessary only in $d = 1$, that the tail of the trap distribution $\sigma(0)$ did not decay too slowly, and as such there are no 'very deep traps' inside the macrobox. Note that second assumption was not relevant in this thesis since we assumed a Weibull tail decay, but appears in the general version of our results [59]. Note also that in $d \geq 2$ this assumption is not necessary, since our percolation estimates are able to control the deep traps for arbitrarily slowly decaying tail of $\sigma(0)$. Here we explore the possible consequences of removing these assumptions.

#### Removing the 'no quick sites' assumption

A close reading of Chapter 4 reveals that substantial parts of our proof remains valid in the presence of quick sites, that is, sites $z$ for which $\sigma(z)$ is small. However, in at least two places the presence of quick sites would have a significant impact on our proof.

First, in determining the upper tail of the distribution of the local principal eigenvalue $\lambda(0)$, we repeatedly use the fact that we can control the influence of quick sites near the origin by the simple bound $\sigma^{-1}(0) \leq \delta_\sigma^{-1}$. On the other hand, such quick sites would tend to increase $\lambda(0)$, since they allow for very quick traversals of paths from origin, and hence they might actually strengthen the localisation behaviour of the model. What is unclear, however, is whether the upper tail of $\lambda(0)$ would remain sufficiently regular to ensure our results remain valid.

Second, in proving the upper bound on the solution components $U^i(t)$ for $i = 2, 3, 4$ and 5, the presence of quick sites could potentially reduce the probabilistic penalty to diffuse to certain parts of the domain, leaving us unable to match the upper bound with the lower bound on $U(t)$. Crucially, in proving the upper bounds we have to control the number of quick sites on *every* path in the domain, rather than just a single path. This makes the situation analogous to our extension of our results on the PAM the FDE case, in which we needed to control the number of semi-high sites lying on every path (as opposed to how we use percolation estimates to control deep traps in the BAM, which only requires us to establish the existence of a single 'good' path).

**Question 10.** *Do our results on the BAM with Weibull potential and trapping landscape remain valid if we remove the 'no quick sites' assumption? Are there circumstances in which the presence of 'quick sites' actually strengthens the localisation properties of the model?*

#### Very deep traps in the one-dimensional BAM

In one-dimension, the presence of very deep traps can introduce 'screening effects' which act to drastically increase the probabilistic penalty associated with accessing some portions of the domain. Past a certain threshold, this starts to affect the lower bound on the solution, and hence necessitates a modification in the form of the localisation functional $\Psi_t$ we use to determines the 'best' sites. In [59] we identified the threshold on the tail decay of the trap distribution as

$$\mathbf{P}(\sigma(0) > x) < 1/\log x\,. \tag{5.9}$$

Past this threshold, we might conjecture the localisation behaviour as follows. Depending on the exact tail decay of $\sigma(0)$, the domain $\mathbb{Z}$ can be thought to possess certain $\mathbf{P}$-measurable barriers, corresponding to very deep traps, which cannot be crossed by time $t$. These barriers will restrict the mass function to a certain $\mathbf{P}$-measurable accessible region $\mathcal{R}_t$ around the origin. We might conjecture that the BAM localises on the top order statistics of the (usual) penalisation functional $\Psi_t$ restricted to the accessible region $\mathcal{R}_t$.

**Question 11.** *What are the localisation properties of the one-dimensional BAM in the case of very heavy-tailed trap distribution (that is, trap distributions $\sigma(0)$ that do not satisfy* (5.9)*)? Does localisation occur on the top order statistic of the functional $\Psi_t$ restricted to a suitable $\mathbf{P}$-measurable accessible region $\mathcal{R}_t$?*

### 5.3.2 The BAM in the case of potentials with double-exponential-or-lighter tail decay

In this thesis we studied the BAM in the case in which both the potential field and trapping landscape have Weibull tail decay. In this regime we observe a mutual reinforcement of localisation effects. On the other hand, since complete localisation holds throughout this regime, the mutual reinforcement manifests only through the radii of influence, rather than the cardinality of the localisation set.

As mentioned above, in the PAM potential distributions with double-exponential tail decay form the conjectured boundary of the complete localisation universality class. Since it is natural to expect that the mutual reinforcement of localisation effects still holds in this regime, this raises the question whether the BAM with double-exponential potential exhibits a phase transition from localisation on an island of bounded size to complete localisation, as the inhomogeneities in the trapping landscape become sufficiently pronounced.

**Question 12.** *What are the localisation properties of the BAM with double-exponential potential (and arbitrary trapping landscape)? In this regime does the BAM exhibit a phase transition from localisation on an island of bounded size to complete localisation, as the tail decay in the trapping landscape becomes sufficiently heavy?*

**Question 13.** *What are the localisation properties of the BAM with lighter-than-double exponential potential? For potentials with arbitrarily light tail decay, does complete localisation hold provided that the tail decay of the trap distribution is sufficiently heavy?*

### 5.3.3 Other hybrid models combining branching and trapping mechanisms

Finally, as mentioned in Chapter 1, we note that there are many alternative hybrid models, other than the BAM, that combine random branching and trapping mechanisms. It would be interesting to see to what extent the qualitative features that we uncovered in the BAM – namely the mutual reinforcement of localisation effects due to the branching and trapping mechanisms, and the local correlation in the random fields (i.e. the 'fit and stable hypothesis) – are universal, in the sense that they are exhibited by many varieties of hybrid models, regardless of their microscopic dynamics.

Examples of such alternative hybrid models include: (i) the underlying particle system that gives rise to the BAM as the thermodynamic limit (see the discussion of the 'unaveraged' PAM above); and (ii) hybrid models where the trapping landscape is given by asymmetric transition probabilities [8], random conductances [71], or more general random holding times that are not necessarily exponentially distributed.

**Question 14.** *What are the localisation properties of other hybrid models that combine random branching and random trapping mechanisms? Do these models all exhibit the same qualitative localisation phenomena that we identified in the BAM? Are there other qualitative features of these models that are universal?*

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