Optimal Rates for the Random Fourier Feature Technique

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Outline

- Kernels.
- Random Fourier features (RFFs).
- Guarantees on RFF approximation: uniform, L^p.

Kernel, RKHS

- $k: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ kernel on $\mathcal{X}$, if
  - $\exists \varphi: \mathcal{X} \to H$(ilbert space) feature map,
  - $k(a,b) = \langle \varphi(a), \varphi(b) \rangle_H$ $(\forall a, b \in \mathcal{X})$.

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- Kernel examples: $\mathcal{X} = \mathbb{R}^d$ $(p > 0,\ \theta > 0)$
  - $k(a, b) = (\langle a, b \rangle + \theta)^p$: polynomial,
  - $k(a, b) = e^{-\|a-b\|_2^2/(2\theta^2)}$: Gaussian,
  - $k(a, b) = e^{-\theta\|a-b\|_2}$: Laplacian.

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  - $k(a, b) = e^{-\theta\|a-b\|_2}$: Laplacian.
- In the $H = H(k)$ RKHS ($\exists!$): $\varphi(u) = k(\cdot, u)$.

RKHS: evaluation point of view

- Let $H \subset \mathbb{R}^{\mathcal{X}}$ be a Hilbert space.
- Consider for fixed $x \in \mathcal{X}$ the $\delta_x : f \in H \mapsto f(x) \in \mathbb{R}$ map.
- The evaluation functional is linear:

$$\delta_x(\alpha f + \beta g) = \alpha \delta_x(f) + \beta \delta_x(g).$$

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- Def.: H is called *RKHS* if δ_x is continuous for $\forall x \in \mathcal{X}$.

RKHS: reproducing point of view

- Let $H \subset \mathbb{R}^{\mathcal{X}}$ be a Hilbert space.
- $k : \mathcal{X} \times \mathcal{X} \to$ is called a *reproducing kernel of H* if for $\forall x \in \mathcal{X}, f \in H$
  1. $k(\cdot, x) \in H$,
  2. $\langle f, k(\cdot, x) \rangle_H = f(x)$ (reproducing property).

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  Specifically, $\forall x, y \in \mathcal{X}$

$$k(x, y) = \langle k(\cdot, x), k(\cdot, y) \rangle_H .$$

RKHS: positive-definite point of view

- Let us given a $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ symmetric function.
- k is called *positive definite* if $\forall n \geq 1$, $\forall (a_1, \ldots, a_n) \in \mathbb{R}^n$, $(x_1, \ldots, x_n) \in \mathcal{X}^n$

$$\sum_{i,j=1}^{n} a_i a_j k(x_i, x_j) = \mathbf{a}^T \mathbf{G} \mathbf{a} \geq 0,$$

where $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^{n}$.

Kernel: example domains ($\mathcal{X}$)

- Euclidean space: $\mathcal{X} = \mathbb{R}^d$.
- Graphs, texts, time series, dynamical systems, distributions.

Kernel: application example – ridge regression

- Given: $\{(x_i, y_i)\}_{i=1}^{\ell}$, $H = H(k)$.
- Task: find $f \in H$ s.t. $f(x_i) \approx y_i$,

$$J(f) = \frac{1}{\ell}\sum_{i=1}^{\ell}[f(x_i) - y_i]^2 + \lambda \|f\|_H^2 \to \min_{f \in H} \quad (\lambda > 0).$$

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- Analytical solution, $\mathcal{O}(\ell^3)$ – expensive:

$$f(x) = [k(x_1, x), \ldots, k(x_\ell, x)](\mathbf{G} + \lambda\ell\mathbf{I})^{-1}[y_1; \ldots; y_\ell],$$
$$\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^{\ell}.$$

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- **Idea**: $\hat{\mathbf{G}}$, matrix-inversion lemma, fast primal solvers $\to$ RFF.

Focus

- $\mathcal{X} = \mathbb{R}^d$. k: continuous, shift-invariant $[k(\mathbf{x},\mathbf{y}) = \tilde{k}(\mathbf{x}-\mathbf{y})]$.
- By Bochner's theorem:

$$k(\mathbf{x},\mathbf{y}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})} \mathrm{d}\Lambda(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\Lambda(\boldsymbol{\omega}).$$

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- RFF trick [Rahimi and Recht, 2007] (MC): $\boldsymbol{\omega}_{1:m} := (\boldsymbol{\omega}_j)_{j=1}^m \overset{i.i.d.}{\sim} \Lambda$,

$$\hat{k}(\mathbf{x}, \mathbf{y}) = \frac{1}{m} \sum_{j=1}^m \cos\left(\boldsymbol{\omega}_j^T(\mathbf{x}-\mathbf{y})\right) = \int_{\mathbb{R}^d} \cos\left(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\Lambda_m(\boldsymbol{\omega}).$$

RFF – existing guarantee, basically

- Hoeffding inequality + union bound:

$$\left\|k - \hat{k}\right\|_{L^\infty(\mathcal{S})} = \mathcal{O}_p\left(\underbrace{|\mathcal{S}|}_{\text{linear}} \sqrt{\frac{\log m}{m}}\right).$$

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- Characteristic function point of view [Csörgő and Totik, 1983] (asymptotic!):
  1. $|\mathcal{S}_m| = e^{o(m)}$ is the optimal rate for a.s. convergence,
  2. For faster growing $|\mathcal{S}_m|$: even convergence in probability fails.

Today: one-page summary

1. Finite-sample L^∞-guarantee $\xrightarrow{\text{specifically}}$

$$\left\|k - \hat{k}\right\|_{L^\infty(\mathcal{S})} = \mathcal{O}_{a.s.}\left(\frac{\sqrt{\log|\mathcal{S}|}}{\sqrt{m}}\right)$$

$\Rightarrow \mathcal{S}$ can grow exponentially $[|\mathcal{S}_m| = e^{o(m)}]$ – optimal!

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1. Finite-sample L^∞-guarantee $\xrightarrow{\text{specifically}}$

$$\left\| k - \hat{k} \right\|_{L^\infty(\mathcal{S})} = \mathcal{O}_{a.s.}\left(\frac{\sqrt{\log |\mathcal{S}|}}{\sqrt{m}} \right)$$

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2. Finite sample L^p guarantees, $p \in [1, \infty)$.

. . . , where

- Uniform ($p = \infty$), L^p ($1 \le p < \infty$) norm:

$$\|k - \hat{k}\|_{L^\infty(\mathcal{S})} := \sup_{\mathbf{x},\mathbf{y}\in\mathcal{S}} \left| k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y}) \right|,$$

$$\|k - \hat{k}\|_{L^p(\mathcal{S})} := \left(\int_{\mathcal{S}} \int_{\mathcal{S}} |k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})|^p \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \right)^{\frac{1}{p}}.$$

Uniform bound: proof idea

1. Empirical process form $[\mathbb{P}g := \int g \mathrm{d}\mathbb{P}]$:

$$\sup_{\mathbf{x},\mathbf{y}\in\mathcal{S}} \left|k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})\right| = \sup_{g\in\mathcal{G}} |\Lambda g - \Lambda_m g| = \|\Lambda - \Lambda_m\|_{\mathcal{G}}.$$

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2. $f(\boldsymbol{\omega}_{1:m}) = \|\Lambda - \Lambda_m\|_{\mathcal{G}}$ concentrates (bounded difference):

$$\|\Lambda - \Lambda_m\|_{\mathcal{G}} \precsim \mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}} + \frac{1}{\sqrt{m}}.$$

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$$\|\Lambda - \Lambda_m\|_{\mathcal{G}} \lesssim \mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}} + \frac{1}{\sqrt{m}}.$$

3. $\mathcal{G}$ is 'nice' (uniformly bounded, separable Carathéodory) $\Rightarrow$

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}} \lesssim \mathbb{E}_{\boldsymbol{\omega}_{1:m}} \underbrace{\mathcal{R}\left(\mathcal{G}, \boldsymbol{\omega}_{1:m}\right)}_{\mathbb{E}_{\epsilon} \sup_{g\in\mathcal{G}} \left| \frac{1}{m} \sum_{j=1}^{m} \epsilon_j g(\omega_j) \right|}.$$

Uniform bound: proof idea – continued

4. Using Dudley's entropy bound:

$$\mathcal{R}\left(\mathcal{G}, \boldsymbol{\omega}_{1:m}\right) \lesssim \frac{1}{\sqrt{m}} \int_0^{|\mathcal{G}|_{L^2(\Lambda_m)}} \sqrt{\log \mathcal{N}(\mathcal{G}, L^2(\Lambda_m), u)} \mathrm{d}u.$$

Uniform bound: proof idea – continued

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5. $\mathcal{G}$ is smoothly parameterized by a compact set $\Rightarrow$

$$\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), u) \leq \left(\frac{4|\mathcal{S}|A}{u} + 1\right)^d, \; A(\boldsymbol{\omega}_{1:m}) = \sqrt{\frac{1}{m}\sum_{j=1}^m \|\boldsymbol{\omega}_j\|_2^2}.$$

Uniform bound: proof idea – continued

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6. Putting together [$|\mathcal{G}|_{L^2(\Lambda_m)} \leq 2$, Jensen inequality] we get the result.

Step-1: empirical process form

- Recall the notation:

$$\Lambda g = \int_{\mathbb{R}^d} g(\omega)\mathrm{d}\Lambda(\omega), \quad \Lambda_m g = \int_{\mathbb{R}^d} g(\omega)\mathrm{d}\Lambda_m(\omega) = \frac{1}{m}\sum_{j=1}^{m} g(\omega_j).$$

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- Reformulation of the objective:

$$\sup_{\mathbf{x},\mathbf{y} \in \mathcal{S}} \left| k(\mathbf{x}, \mathbf{y}) - \hat{k}(\mathbf{x}, \mathbf{y}) \right| = \sup_{g \in \mathcal{G}} |\Lambda g - \Lambda_m g| =: \|\Lambda - \Lambda_m\|_{\mathcal{G}} ,$$

where

$$\mathcal{G} = \{g_{\mathbf{z}} : \mathbf{z} \in \mathcal{S}_\Delta\},$$
$$\mathcal{S}_\Delta = \mathcal{S} - \mathcal{S} = \{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in \mathcal{S}\},$$
$$g_{\mathbf{z}} : \boldsymbol{\omega} \in \mathbb{R}^d \mapsto \cos\left(\boldsymbol{\omega}^T \mathbf{z}\right) \in \mathbb{R}.$$

Step-2: bounded diff. property of $f(\omega_{1:m}) = \|\Lambda - \Lambda_m\|_{\mathcal{G}}$

McDiarmid inequality: Let $\omega_1, \ldots, \omega_m \in D$ be independent r.v.-s, and $f : D^m \to \mathbb{R}$ satisfy the bounded diff. property ($\forall r$):

$$\sup_{u_1,\ldots,u_m,u_r' \in D} \left| f(u_1,\ldots,u_m) - f(u_1,\ldots,u_{r-1},u_r',u_{r+1},\ldots,u_m) \right| \leq c_r.$$

Then for $\forall \beta > 0$

$$\mathbb{P}\left(f(\omega_{1:m}) - \mathbb{E}\left[f(\omega_{1:m})\right] \geq \beta\right) \leq e^{-\frac{2\beta^2}{\sum_{r=1}^m c_r^2}}.$$

Step-2: bounded difference property of $\|\Lambda - \Lambda_m\|_{\mathcal{G}}$

Our choice: $f(\omega_1, \ldots, \omega_m) := \|\Lambda - \Lambda_m\|_{\mathcal{G}}$.

$$|f(\omega_1, \ldots, \omega_{r-1}, \omega_r, \omega_{r+1}, \ldots, \omega_m) - f(\omega_1, \ldots, \omega_{r-1}, \omega_r', \omega_{r+1}, \ldots, \omega_m)| =$$

$$= \left| \sup_{g \in \mathcal{G}} \left| \Lambda g - \frac{1}{m} \sum_{j=1} g(\omega_j) \right| - \sup_{g \in \mathcal{G}} \left| \Lambda g - \frac{1}{m} \sum_{j=1} g(\omega_j) + \frac{1}{m} \left[g(\omega_r) - g(\omega_r') \right] \right| \right|$$

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$$\overset{(*)}{\leq} \frac{1}{m} \sup_{g \in \mathcal{G}} \left| g(\omega_r) - g(\omega_r') \right|$$

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$$\leq \frac{1}{m} \left[\sup_{g \in \mathcal{G}} |g(\omega_r)| + \sup_{g \in \mathcal{G}} |g(\omega_r')| \right] \leq \frac{1+1}{m} = \frac{2}{m},$$

$(*)$: next slide.

Step-2: (*) = reverse triangle inequality with sup

- Lemma: $\mathcal{G}$: set of functions, $a, b : \mathcal{G} \to \mathbb{R}$ maps; then

$$\left| \sup_{g \in \mathcal{G}} |a(g)| - \sup_{g \in \mathcal{G}} |a(g) + b(g)| \right|$$

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- Proof: combine

$$sup_{g\in\mathcal{G}} |a(g) + b(g)| \leq \sup_{g\in\mathcal{G}} \left(|a(g)| + |b(g)|\right) \leq \sup_{g\in\mathcal{G}} |a(g)| + \sup_{g\in\mathcal{G}} |b(g)|,$$

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- Proof: combine

$$\sup_{g\in\mathcal{G}} |a(g) + b(g)| \le \sup_{g\in\mathcal{G}} (|a(g)| + |b(g)|) \le \sup_{g\in\mathcal{G}} |a(g)| + \sup_{g\in\mathcal{G}} |b(g)|,$$

$$\begin{aligned} \sup_{g\in\mathcal{G}} |a(g)| &= \sup_{g\in\mathcal{G}} |a(g) + b(g) - b(g)| \\ &\le \sup_{g\in\mathcal{G}} |a(g) + b(g)| + \sup_{g\in\mathcal{G}} |b(g)|. \end{aligned}$$

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- Proof: combine

$$sup_{g\in\mathcal{G}} |a(g) + b(g)| \le \sup_{g\in\mathcal{G}} (|a(g)| + |b(g)|) \le \sup_{g\in\mathcal{G}} |a(g)| + \sup_{g\in\mathcal{G}} |b(g)|,$$

$$\sup_{g\in\mathcal{G}} |a(g)| = \sup_{g\in\mathcal{G}} |a(g) + b(g) - b(g)|$$

$$\le \sup_{g\in\mathcal{G}} |a(g) + b(g)| + \sup_{g\in\mathcal{G}} |b(g)|.$$

$$\Rightarrow \pm \left[\sup_{g\in\mathcal{G}} |a(g)| - \sup_{g\in\mathcal{G}} |a(g) + b(g)| \right] \le \sup_{g\in\mathcal{G}} |b(g)|.$$

- Our choice: $a(g) = \Lambda g - \frac{1}{m} \sum_{j=1} g(\omega_j)$, $b(g) = \frac{1}{m} [g(\omega_r) - g(\omega'_r)]$.

Step-2

Applying McDiarmid to f $(c_r = \frac{2}{m})$: for $\forall \tau > 0$ with probability $1 - e^{-\tau}$

$$\|\Lambda - \Lambda_m\|_{\mathcal{G}} \leq \underbrace{\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}}}_{\text{Step-3: bounding this term}} + \frac{\sqrt{2\tau}}{\sqrt{m}}.$$

Step-3: bounding $\mathbb{E}_{\omega_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}}$

$\mathcal{G} = \{g_{\mathbf{z}} : \mathbf{z} \in \mathcal{S}_\Delta\}$ is a separable Carathéodory family, i.e.

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Thus, by [Steinwart and Christmann, 2008, Prop. 7.10]

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|\Lambda - \Lambda_m\|_{\mathcal{G}} \le 2\mathbb{E}_{\boldsymbol{\omega}_{1:m}} [\underbrace{\mathcal{R}\left(\mathcal{G}, \boldsymbol{\omega}_{1:m}\right)}_{:=\mathbb{E}_\epsilon \sup_{g\in\mathcal{G}} \left| \frac{1}{m} \sum_{j=1}^m \epsilon_j g(\omega_j) \right|}]$$

using the uniformly boundedness of $\mathcal{G}$ $\left(\sup_{g\in\mathcal{G}} \|g\|_\infty \le 1 < \infty\right)$.

Step-4: bounding $\mathcal{R}$

Using Dudley's entropy integral [Bousquet, 2003, Eq. (4.4)]:

$$\mathcal{R}\left(\mathcal{G}, (\boldsymbol{\omega}_j)_{j=1}^m\right) \le \frac{8\sqrt{2}}{\sqrt{m}} \int_0^{|\mathcal{G}|_{L^2(\Lambda_m)}} \sqrt{\log \mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r)}\mathrm{d}r,$$

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- $|\mathcal{G}|_{L^2(\Lambda_m)} = \sup_{g_1,g_2\in\mathcal{G}} \|g_1 - g_2\|_{L^2(\Lambda_m)}$,
- $\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r)$: r-covering number.
  - r-net: $S \subseteq \mathcal{G}$, for $\forall g \in \mathcal{G}$ $\exists s \in S$ such that $\|g - s\|_{L^2(\Lambda_m)} \le r$.
  - $\mathcal{N}$: size of the smallest r-net of $\mathcal{G}$.

Step-5: bound on $|\mathcal{G}|_{L^2(\Lambda_m)}$

$$|\mathcal{G}|_{L^2(\Lambda_m)} = \sup_{g_1, g_2 \in \mathcal{G}} \|g_1 - g_2\|_{L^2(\Lambda_m)} \leq \sup_{g_1, g_2 \in \mathcal{G}} \left(\|g_1\|_{L^2(\Lambda_m)} + \|g_2\|_{L^2(\Lambda_m)} \right)$$

$$\leq \sup_{g_1 \in \mathcal{G}} \|g_1\|_{L^2(\Lambda_m)} + \sup_{g_1 \in \mathcal{G}} \|g_2\|_{L^2(\Lambda_m)} \overset{(*)}{\leq} 2,$$

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$$\sup_{g \in \mathcal{G}} \|g\|_{L^2(\Lambda_m)} \overset{(*)}{=} \sup_{\mathbf{z} \in \mathcal{S}_\Delta} \sqrt{\frac{1}{m} \sum_{j=1}^{m} g_{\mathbf{z}}^2(\boldsymbol{\omega}_j)} = \sup_{\mathbf{z} \in \mathcal{S}_\Delta} \sqrt{\frac{1}{m} \sum_{j=1}^{m} \left[\cos\left(\boldsymbol{\omega}_j^T \mathbf{z}\right)\right]^2} \leq 1.$$

Step-5: bound on $\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r)$

Let $g_{\mathbf{z}_1}, g_{\mathbf{z}_2} \in \mathcal{G}$. We want to bound $\|g_{\mathbf{z}_1} - g_{\mathbf{z}_2}\|_{L^2(\Lambda_m)}$. One term:

$$\left|\cos\left(\boldsymbol{\omega}^T \mathbf{z}_1\right) - \cos\left(\boldsymbol{\omega}^T \mathbf{z}_2\right)\right| \overset{(*)}{\leq} \left\|\nabla_{\mathbf{z}} \cos\left(\boldsymbol{\omega}^T \mathbf{z}_c\right)\right\|_2 \|\mathbf{z}_1 - \mathbf{z}_2\|_2$$
$$= \left\|-\sin\left(\boldsymbol{\omega}^T \mathbf{z}_c\right)\boldsymbol{\omega}\right\|_2 \|\mathbf{z}_1 - \mathbf{z}_2\|_2$$
$$\leq \|\boldsymbol{\omega}\|_2 \|\mathbf{z}_1 - \mathbf{z}_2\|_2,$$

where $(*)$: mean-value theorem, $\mathbf{z}_c \in (\mathbf{z}_1, \mathbf{z}_2)$.

Step-5: bound on $\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r)$

- Smooth parameterization:

$$\left\|g_{\mathbf{z}_1} - g_{\mathbf{z}_2}\right\|_{L^2(\Lambda_m)} \leq \sqrt{\frac{1}{m}\sum_{j=1}^{m}\left(\left\|\boldsymbol{\omega}_j\right\|_2 \left\|\mathbf{z}_1 - \mathbf{z}_2\right\|_2\right)^2}$$

$$= \left\|\mathbf{z}_1 - \mathbf{z}_2\right\|_2 \underbrace{\sqrt{\frac{1}{m}\sum_{j=1}^{m}\left\|\boldsymbol{\omega}_j\right\|_2^2}}_{=:A=A(\boldsymbol{\omega}_{1:m})}.$$

- r-net on $(\mathcal{S}_\Delta, \|\cdot\|_2) \Rightarrow r' = rA$-net on $(\mathcal{G}, L^2(\Lambda_m))$.

Step-5: bound on $\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r)$

- Until now:

$$\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r) \le \mathcal{N}\left(\mathcal{S}_\Delta, \|\cdot\|_2, \frac{r}{A}\right).$$

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- ϵ-covering number of $\|\cdot\|_2$-ball in $\mathbb{R}^d$ with radius R: $\leq (\frac{4R}{\epsilon} + 1)^d$ [van de Geer, 2009, Lemma 2.5] $\Rightarrow$

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- We got [$\epsilon = \frac{r}{A}$, $R = |\mathcal{S}|$]:

$$\mathcal{N}(\mathcal{G}, L^2(\Lambda_m), r) \le \left(\frac{4|\mathcal{S}|A}{r} + 1\right)^d.$$

Step-6: Putting together

$$\mathcal{R}\left(\mathcal{G}, \boldsymbol{\omega}_{1:m}\right) \le \frac{8\sqrt{2d}}{\sqrt{m}} \int_0^2 \sqrt{\log\left(\frac{4|\mathcal{S}|A}{r} + 1\right)} \,\mathrm{d}r$$

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$$\stackrel{(b)}{\le} \frac{8\sqrt{2d}}{\sqrt{m}} \left[\int_0^2 \sqrt{\log\frac{2\left(2|\mathcal{S}|+1\right)}{r}} \,\mathrm{d}r + 2\sqrt{\log(A+1)}\right].$$

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Step 6: Putting together

$$\mathcal{R}\left(\mathcal{G}, \boldsymbol{\omega}_{1:m}\right) \stackrel{(a)}{\leq} \frac{16\sqrt{2d}}{\sqrt{m}}\left(\int_0^1 \sqrt{\log \frac{B+1}{r}}\,\mathrm{d}r + \sqrt{\log(A+1)}\right),$$

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(a): change of variables, $B := 2|\mathcal{S}|$, (b):
$\int_0^1 \sqrt{\log \frac{a}{\epsilon}} \, \mathrm{d}\epsilon \leq \sqrt{\log a} + \frac{1}{2\sqrt{\log a}}$ $(a > 1)$. $\Rightarrow$ by Jensen

Uniform guarantee

Let k be continuous, $\sigma^2 := \int_{\mathbb{R}^d} \|\omega\|^2 \, d\Lambda(\omega) < \infty$. Then for $\forall \tau > 0$ and compact set $\mathcal{S} \subset \mathbb{R}^d$

$$\Lambda^m \left(\|\hat{k} - k\|_{L^\infty(\mathcal{S})} \geq \frac{h(d, |\mathcal{S}|, \sigma) + \sqrt{2\tau}}{\sqrt{m}} \right) \leq e^{-\tau},$$

$$h(d, |\mathcal{S}|, \sigma) := 32\sqrt{2d \log(2|\mathcal{S}| + 1)} + 16\sqrt{\frac{2d}{\log(2|\mathcal{S}| + 1)}} + 32\sqrt{2d \log(\sigma + 1)}.$$

Consequence-1 (Borel-Cantelli lemma)

- A.s. convergence on compact sets: $\hat{k} \xrightarrow{m\to\infty} k$ at rate $\sqrt{\frac{\log|\mathcal{S}|}{m}}$.

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- Specifically:
    - *asymptotic* optimality [Csörgő and Totik, 1983, Theorem 2] (if $k(\mathbf{z})$ vanishes at ∞).

Consequence-2: L^p guarantee $(1 \le p)$

Idea:

- Note that

$$\|\hat{k}-k\|_{L^p(\mathcal{S})} := \left(\int_{\mathcal{S}}\int_{\mathcal{S}} |\hat{k}(\mathbf{x},\mathbf{y})-k(\mathbf{x},\mathbf{y})|^p \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)^{\frac{1}{p}} \le \|\hat{k}-k\|_{L^\infty(\mathcal{S})} \mathrm{vol}^{2/p}(\mathcal{S}).$$

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- $\mathrm{vol}(B) = \frac{\pi^{d/2}|\mathcal{S}|^d}{2^d \Gamma\left(\frac{d}{2}+1\right)}$, $\Gamma(t) = \int_0^\infty u^{t-1} e^{-u} \, \mathrm{d}u$. $\Rightarrow$

L^p guarantee

Under the previous assumptions, and $1 \leq p < \infty$:

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Hence,

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Uniform guarantee: $|\mathcal{S}_m| = e^{m^{\delta<1}}$; now: $\frac{|\mathcal{S}_m|^{2d/p}}{\sqrt{m}} \to 0 \Rightarrow |\mathcal{S}_m| = o(m^{\frac{p}{4d}})$.

Direct L^p guarantee (proof after discussion)

Under the previous assumptions, and $1 < p < \infty$:

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C_p': universal constant; only p-dependent (not $|\mathcal{S}|$ or m-dep.).

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C_p': universal constant; only p-dependent (not $|\mathcal{S}|$ or m-dep.).
Note: if $2 \leq p$, then

1. $m^{1-\max\{\frac{1}{2},\frac{1}{p}\}} = \sqrt{m}$ [we got rid of $\sqrt{\log(\mathcal{S})}$],
2. $\|\hat{k}-k\|_{L^p(\mathcal{S}_m)} \stackrel{a.s.}{\to} 0$ if $|\mathcal{S}_m| = o\left(m^{\frac{p}{4d}}\right)$ as $m \to \infty$.

Direct L^p result: High-level idea

1. By the bounded difference property:

$$\|k - \hat{k}\|_{L^p(\mathcal{S})} \le \mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|k - \hat{k}\|_{L^p(\mathcal{S})} + \mathrm{vol}^{2/p}(\mathcal{S}) \sqrt{\frac{2\tau}{m}}.$$

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2. By $L^p \cong (L^{p'})^*$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$, the separability of $L^{p'}(\mathcal{S})$ and symmetrization [van der Vaart and Wellner, 1996, Lemma 2.3.1]:

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\|k-\hat{k}\|_{L^p(\mathcal{S})} \le \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \underbrace{\mathbb{E}_{\boldsymbol{\varepsilon}}\left\|\sum_{i=1}^m \varepsilon_i \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot\rangle)\right\|_{L^p(\mathcal{S})}}_{=:(*)}.$$

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3. Since $L^p(\mathcal{S})$ is of type $\min(2,p)$ $\exists C'_p$ such that

$$(*) \le C'_p\left(\sum_{i=1}^m \|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot\rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)}\right)^{\frac{1}{\min(2,p)}}.$$

Direct L^p result: Step-1

$f(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m) := \|k - \hat{k}\|_{L^p(\mathcal{S})}$ has bounded difference:

$$\hat{k}_i(\mathbf{x}, \mathbf{y}) = \frac{1}{m} \sum_{j \neq i} \cos(\boldsymbol{\omega}_j^T (\mathbf{x} - \mathbf{y})) + \frac{1}{m} \cos(\tilde{\boldsymbol{\omega}}_i^T (\mathbf{x} - \mathbf{y})),$$

$$\sup_{(\boldsymbol{\omega}_i)_{i=1}^m, \tilde{\boldsymbol{\omega}}_i} \left| \|k - \hat{k}\|_{L^p(\mathcal{S})} - \|k - \hat{k}_i\|_{L^p(\mathcal{S})} \right| \leq$$

$$\leq \sup_{(\boldsymbol{\omega}_i)_{i=1}^m, \tilde{\boldsymbol{\omega}}_i} \|\hat{k}_i - \hat{k}\|_{L^p(\mathcal{S})}$$

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$f(\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_m) := \|k-\hat{k}\|_{L^p(\mathcal{S})}$ has bounded difference:

$$\hat{k}_i(\mathbf{x},\mathbf{y}) = \frac{1}{m}\sum_{j\neq i}\cos(\boldsymbol{\omega}_j^T(\mathbf{x}-\mathbf{y})) + \frac{1}{m}\cos(\tilde{\boldsymbol{\omega}}_i^T(\mathbf{x}-\mathbf{y})),$$

$$\sup_{(\boldsymbol{\omega}_i)_{i=1}^m,\tilde{\boldsymbol{\omega}}_i}\left|\|k-\hat{k}\|_{L^p(\mathcal{S})} - \|k-\hat{k}_i\|_{L^p(\mathcal{S})}\right| \leq$$

$$\leq \sup_{(\boldsymbol{\omega}_i)_{i=1}^m,\tilde{\boldsymbol{\omega}}_i}\|\hat{k}_i-\hat{k}\|_{L^p(\mathcal{S})} \leq \frac{2}{m}\sup_{\boldsymbol{\omega}_i}\|\cos(\langle\boldsymbol{\omega}_i,\cdot-\cdot\rangle)\|_{L^p(\mathcal{S})}$$

Direct L^p result: Step-1

$f(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_m) := \|k - \hat{k}\|_{L^p(\mathcal{S})}$ has bounded difference:

$$\hat{k}_i(\mathbf{x}, \mathbf{y}) = \frac{1}{m} \sum_{j \neq i} \cos(\boldsymbol{\omega}_j^T (\mathbf{x} - \mathbf{y})) + \frac{1}{m} \cos(\tilde{\boldsymbol{\omega}}_i^T (\mathbf{x} - \mathbf{y})),$$

$$\sup_{(\boldsymbol{\omega}_i)_{i=1}^m, \tilde{\boldsymbol{\omega}}_i} \left| \|k - \hat{k}\|_{L^p(\mathcal{S})} - \|k - \hat{k}_i\|_{L^p(\mathcal{S})} \right| \leq$$

$$\leq \sup_{(\boldsymbol{\omega}_i)_{i=1}^m, \tilde{\boldsymbol{\omega}}_i} \|\hat{k}_i - \hat{k}\|_{L^p(\mathcal{S})} \leq \frac{2}{m} \sup_{\boldsymbol{\omega}_i} \|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle)\|_{L^p(\mathcal{S})}$$

$$\leq \frac{2}{m} \mathrm{vol}^{2/p}(\mathcal{S}) =: c_r.$$

$\Rightarrow$ We can apply the McDiarmid inequality.

We write $\|\cdot\|_{L^p}$ as a countable sup

Let $1 < p' < \infty$.

- Let $(X, \mathcal{A}, \mu)$, $\mu(X) < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\left[L^{p'}(X, \mathcal{A}, \mu)\right]^* = \{F_f : f \in L^p(X, \mathcal{A}, \mu)\},$$

$$F_f(u) = \int_X uf \mathrm{d}\mu,$$

and $\|f\|_{L^p} = \|F_f\| = \sup_{\|g\|_{L^{p'}}=1} |F_f(g)| =: (*)$.

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and $\|f\|_{L^p} = \|F_f\| = \sup_{\|g\|_{L^{p'}}=1} |F_f(g)| =: (*)$.

- Moreover, since for $X = \mathcal{S}$, $L^{p'}(\mathcal{S})$ is separable [Cohn, 2013, Prop. 3.4.5] $\Rightarrow \exists\, \mathcal{G} \subseteq S_{L^{p'}(\mathcal{S})}(0, 1)$ countable [Carothers, 2004, Lemma 6.7]: $(*) = \sup_{g \in \mathcal{G}} |F_f(g)|$.

Direct L^p result: Step-2

By the previous rewriting

$$\|k-\hat{k}\|_{L^p(\mathcal{S})} = \left\|F_{k-\hat{k}}\right\| = \sup_{g\in\mathcal{G}}\left|\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y})\left[k(\mathbf{x},\mathbf{y})-\hat{k}(\mathbf{x},\mathbf{y})\right]\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right| =: (*)$$

Direct L^p result: Step-2

By the previous rewriting

$$\|k - \hat{k}\|_{L^p(\mathcal{S})} = \left\|F_{k-\hat{k}}\right\| = \sup_{g\in\mathcal{G}} \left| \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x}, \mathbf{y}) \left[k(\mathbf{x}, \mathbf{y}) - \hat{k}(\mathbf{x}, \mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right| =: (*)$$

$$\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x}, \mathbf{y}) \left[k(\mathbf{x}, \mathbf{y}) - \hat{k}(\mathbf{x}, \mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

Direct L^p result: Step-2

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$$\|k-\hat{k}\|_{L^p(\mathcal{S})} = \left\|F_{k-\hat{k}}\right\| = \sup_{g\in\mathcal{G}} \left| \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right| =: (*)$$

$$\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$
$$= \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[\int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y}))\mathrm{d}(\Lambda-\Lambda_m)(\boldsymbol{\omega})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

Direct L^p result: Step-2

By the previous rewriting

$$\|k-\hat{k}\|_{L^p(\mathcal{S})} = \left\|F_{k-\hat{k}}\right\| = \sup_{g\in\mathcal{G}}\left|\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y})\left[k(\mathbf{x},\mathbf{y})-\hat{k}(\mathbf{x},\mathbf{y})\right]\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right| =: (*)$$

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$$= \int_{\mathbb{R}^d}\underbrace{\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y})\cos(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y}))\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}}_{=:\tilde{g}_g(\boldsymbol{\omega}):\ \text{measurable}\ \Leftarrow\ \text{dominated convergence}}\ \mathrm{d}(\Lambda-\Lambda_m)(\boldsymbol{\omega}) \Rightarrow$$

Direct L^p result: Step-2

By the previous rewriting

$$\|k-\hat{k}\|_{L^p(\mathcal{S})} = \left\|F_{k-\hat{k}}\right\| = \sup_{g\in\mathcal{G}} \left| \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right| =: (*)$$

$$\begin{aligned}
&\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[k(\mathbf{x},\mathbf{y}) - \hat{k}(\mathbf{x},\mathbf{y})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[\int_{\mathbb{R}^d} \cos(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y}))\mathrm{d}(\Lambda-\Lambda_m)(\boldsymbol{\omega})\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \int_{\mathbb{R}^d} \underbrace{\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \cos(\boldsymbol{\omega}^T(\mathbf{x}-\mathbf{y}))\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}}_{=:\tilde{g}_g(\boldsymbol{\omega}):\ \text{measurable} \Leftarrow \text{dominated convergence}} \mathrm{d}(\Lambda-\Lambda_m)(\boldsymbol{\omega}) \Rightarrow
\end{aligned}$$

$$(*) = \sup_{\tilde{g}\in\tilde{\mathcal{G}}:=\{\tilde{g}_g:g\in\mathcal{G}\}} \left|(\Lambda-\Lambda_m)\tilde{g}\right|,$$

Direct L^p result: Step-2

By symmetrization [(a)] we get

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|k - \hat{k}\|_{L^p(\mathcal{S})} \overset{(a)}{\leq} 2\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{\tilde{g} \in \tilde{\mathcal{G}}} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i \tilde{g}(\boldsymbol{\omega}_i) \right|$$

Direct L^p result: Step-2

By symmetrization [(a)], $\tilde{g}$ def. [(b)] we get

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \|k - \hat{k}\|_{L^p(\mathcal{S})} \overset{(a)}{\leq} 2\mathbb{E}_{\boldsymbol{\omega}_{1:m}} \mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{\tilde{g} \in \tilde{\mathcal{G}}} \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i \tilde{g}(\boldsymbol{\omega}_i) \right|$$

$$\overset{(b)}{=} \frac{2}{m} \mathbb{E}_{\boldsymbol{\omega}_{1:m}} \mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{m} \varepsilon_i \int_{\mathcal{S} \times \mathcal{S}} g(\mathbf{x}, \mathbf{y}) \cos\left(\boldsymbol{\omega}_i^T (\mathbf{x} - \mathbf{y})\right) \mathrm{d}\mathbf{x} \mathrm{d}\mathbf{y} \right|$$

Direct L^p result: Step-2

By symmetrization [(a)], $\tilde{g}$ def. [(b)] we get

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\|k-\hat{k}\|_{L^p(\mathcal{S})} \overset{(a)}{\leq} 2\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{\tilde{g}\in\tilde{\mathcal{G}}} \left| \frac{1}{m}\sum_{i=1}^m \varepsilon_i \tilde{g}(\boldsymbol{\omega}_i) \right|$$

$$\overset{(b)}{=} \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^m \varepsilon_i \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \cos\left(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right|$$

$$= \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{g\in\mathcal{G}} \left| \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[\sum_{i=1}^m \varepsilon_i \cos\left(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y})\right)\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right|$$

Direct L^p result: Step-2

By symmetrization [(a)], $\tilde{g}$ def. [(b)] and $L^p \cong (L^{p'})^*$ [(c)], we get

$$\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\|k-\hat{k}\|_{L^p(\mathcal{S})} \overset{(a)}{\leq} 2\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{\tilde{g}\in\tilde{\mathcal{G}}}\left|\frac{1}{m}\sum_{i=1}^m \varepsilon_i \tilde{g}(\boldsymbol{\omega}_i)\right|$$

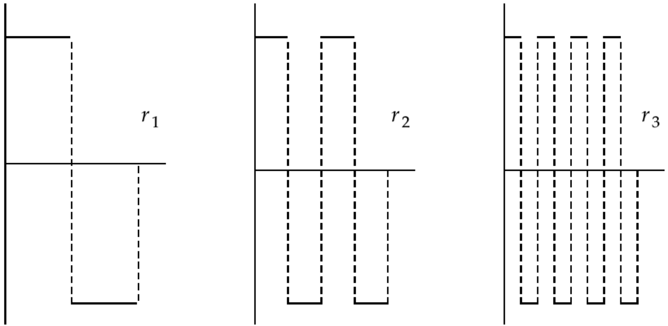
$$\overset{(b)}{=} \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{g\in\mathcal{G}}\left|\sum_{i=1}^m \varepsilon_i \int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \cos\left(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y})\right) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right|$$

$$= \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}} \sup_{g\in\mathcal{G}}\left|\int_{\mathcal{S}\times\mathcal{S}} g(\mathbf{x},\mathbf{y}) \left[\sum_{i=1}^m \varepsilon_i \cos\left(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y})\right)\right] \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right|$$

$$\overset{(c)}{=} \frac{2}{m}\mathbb{E}_{\boldsymbol{\omega}_{1:m}}\mathbb{E}_{\boldsymbol{\varepsilon}}\left\|\sum_{i=1}^m \varepsilon_i \cos(\langle\boldsymbol{\omega}_i, \cdot - \cdot\rangle)\right\|_{L^p(\mathcal{S})}.$$

Direct L^p result: Step-3

Rademacher functions: $r_j(s) = sgn\left(\sin\left(2^j \pi s\right)\right) \in L^2[0,1]$ $(j = 1, \ldots)$.



Direct L^p result: Step-3

Properties of Rademacher functions:

1. ONS in $L^2[0,1]$.

Direct L^p result: Step-3

Properties of Rademacher functions:

1. ONS in $L^2[0,1]$.
2. $[r_1(t);\dots;r_m(t)] = [\epsilon_1;\dots;\epsilon_m] \in \{-1,1\}^m$ Rademacher vector, where $t \sim U[0,1] \Rightarrow$

$$\mathbb{E}_{\boldsymbol{\varepsilon}} \left\| \sum_{j=1}^{m} \varepsilon_j f_j \right\| = \int_0^1 \left\| \sum_{j=1}^{m} r_j(s) f_j \right\| \mathrm{d}s.$$

Direct L^p result: Step-3

A $(Z, \|\cdot\|)$ Banach space is of type $q \in (1, 2]$ if $\exists C \in \mathbb{R}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(s) f_j \right\| \mathrm{d}s \le C \left(\sum_{j=1}^m \|f_j\|^q \right)^{\frac{1}{q}}, \forall m, \forall \{f_j\}_{j=1}^m \subseteq Z.$$

Direct L^p result: Step-3

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Notes:

1. q choice: $\forall$ ($\nexists$) B-space is of type 1 (> 2).

Direct L^p result: Step-3

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Direct L^p result: Step-3

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Notes:

1. q choice: $\forall$ ($\nexists$) B-space is of type 1 (> 2).
2. $\forall$ Hilbert space is of type 2.
3. $Z = L^p(X, \mathcal{A}, \mu)$ is of type $q = \min(2, p)$ [Lindenstrauss and Tzafriri, 1979, page 73] $\Rightarrow$

Direct L^p result: Step-3

$\exists C'_p$ such that

$$\mathbb{E}_{\varepsilon}\left\|\sum_{i=1}^{m}\varepsilon_i \cos(\langle\omega_i,\cdot-\cdot\rangle)\right\|_{L^p(\mathcal{S})} \le C'_p\left(\sum_{i=1}^{m}\|\cos(\langle\omega_i,\cdot-\cdot\rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)}\right)^{\frac{1}{\min(2,p)}} =: (*)$$

$$\sum_{i=1}^{m}\|\cos(\langle\omega_i,\cdot-\cdot\rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)} =$$

Direct L^p result: Step-3

$\exists C'_p$ such that

$$\mathbb{E}_{\varepsilon}\left\|\sum_{i=1}^{m}\varepsilon_i \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot\rangle)\right\|_{L^p(\mathcal{S})} \le C'_p \left(\sum_{i=1}^{m}\|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot\rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)}\right)^{\frac{1}{\min(2,p)}} =: (*)$$

$$\sum_{i=1}^{m}\|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot\rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)} = \sum_{i=1}^{m}\left(\int_{\mathcal{S}\times\mathcal{S}}\underbrace{\left|\cos(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y}))\right|^p}_{\le 1} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right)^{\frac{\min(2,p)}{p}}$$

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$\exists C'_p$ such that

$$\mathbb{E}_{\boldsymbol{\varepsilon}} \left\| \sum_{i=1}^m \varepsilon_i \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle) \right\|_{L^p(\mathcal{S})} \le C'_p \left(\sum_{i=1}^m \| \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle) \|_{L^p(\mathcal{S})}^{\min(2,p)} \right)^{\frac{1}{\min(2,p)}} =: (*)$$

$$\sum_{i=1}^m \| \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle) \|_{L^p(\mathcal{S})}^{\min(2,p)} = \sum_{i=1}^m \left(\int_{\mathcal{S}\times\mathcal{S}} \underbrace{\left| \cos(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y})) \right|^p}_{\le 1} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \right)^{\frac{\min(2,p)}{p}}$$

$$\le m \left[\mathrm{vol}^2(\mathcal{S}) \right]^{\frac{\min(2,p)}{p}} \Rightarrow$$

Direct L^p result: Step-3

$\exists C'_p$ such that

$$\mathbb{E}_{\varepsilon}\left\|\sum_{i=1}^m \varepsilon_i \cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle)\right\|_{L^p(\mathcal{S})} \le C'_p \left(\sum_{i=1}^m \|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)}\right)^{\frac{1}{\min(2,p)}} =: (*)$$

$$\sum_{i=1}^m \|\cos(\langle \boldsymbol{\omega}_i, \cdot - \cdot \rangle)\|_{L^p(\mathcal{S})}^{\min(2,p)} = \sum_{i=1}^m \left(\int_{\mathcal{S}\times\mathcal{S}} \underbrace{\left|\cos(\boldsymbol{\omega}_i^T(\mathbf{x}-\mathbf{y}))\right|^p}_{\le 1} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right)^{\frac{\min(2,p)}{p}}$$

$$\le m \left[\mathrm{vol}^2(\mathcal{S})\right]^{\frac{\min(2,p)}{p}} \Rightarrow$$

$$(*) \le C'_p m^{\frac{1}{\min(2,p)} = \max\{\frac{1}{2},\frac{1}{p}\}} \mathrm{vol}^{2/p}(\mathcal{S}).$$

Summary

Finite sample

- $L^\infty(\mathcal{S})$ guarantee $\xrightarrow{\text{spec.}} |\mathcal{S}_m| = e^{o(m)}$ – optimal!
- $L^p(\mathcal{S})$ results ($\Leftarrow$ uniform, type of L^p).

Summary

Finite sample

- $L^\infty(\mathcal{S})$ guarantee $\xrightarrow{\text{spec.}} |\mathcal{S}_m| = e^{o(m)}$ – optimal!
- $L^p(\mathcal{S})$ results ($\Leftarrow$ uniform, type of L^p).

Thank you for the attention!

Contents

- Borel-Cantelli lemma.
- McDiarmid inequality.
- $L^\infty(\mathcal{S})$ is *not* separable.

Borel-Cantelli lemma

- Assume: $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$.
- Then $\mathbb{P}(\infty$-ly many of them occur$) = 0$. Formally,

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0,$$

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

McDiarmid inequality [Shawe-Taylor and Cristianini, 2004]

Let $\omega_1, \ldots, \omega_m \in D$ be independent r.v.-s, and $f : D^m \to \mathbb{R}$ satisfy the bounded diff. property $(\forall r)$:

$$\sup_{u_1,\ldots,u_m,u'_r \in D} \left| f(u_1, \ldots, u_m) - f(u_1, \ldots, u_{r-1}, u'_r, u_{r+1}, \ldots, u_m) \right| \leq c_r.$$

Then for $\forall \beta > 0$

$$\mathbb{P}\left(f(\omega_1, \ldots, \omega_m) - \mathbb{E}\left[f(\omega_1, \ldots, \omega_m)\right] \geq \beta\right) \leq e^{-\frac{2\beta^2}{\sum_{r=1}^m c_r^2}}.$$

Note: specifically, if $c = c_r$ $(\forall r)$, $\tau = \frac{2\epsilon^2}{\sum_{r=1}^m c_r^2} = \frac{2\epsilon^2}{mc^2} \Leftrightarrow \epsilon = c\sqrt{\frac{\tau m}{2}}$ gives $\mathbb{P}\left(f(X_1, \ldots, X_m) < \mathbb{E}\left[f(X_1, \ldots, X_m)\right] + c\sqrt{\frac{\tau m}{2}}\right) \geq 1 - e^{-\tau}$.

$L^\infty(\mathcal{S})$ is *not* separable

- Assume that $0 \in \mathcal{S}$.
- Take $S := \{I_{B(0,r)}\}_{r>0} \subseteq L^\infty(\mathcal{S})$.
- $|S| >$ countable, and for $\forall s_1 \neq s_2 \in S$: $\|s_1 - s_2\|_{L^\infty(\mathcal{S})} = 1$.

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