Rings of polynomials with Artinian coefficients

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Abstract

We study the extent to which the *weak Euclidean* and *stably free cancellation* properties hold for rings of Laurent polynomials $A[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ with coefficients in an Artinian ring A.

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Recall that a module S over a ring Λ is said to be *stably free* when $S \oplus \Lambda^a \cong \Lambda^b$ for some positive integers a, b. We say that Λ has *stably free cancellation* ($= SFC$) when any stably free Λ-module is free. Elementary duality considerations show this property is left-right symmetric. We show that Artinian rings have the SFC property. More generally, we study the extent to which the SFC property holds for the rings

$$L_n(A) \;=\; A[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$$

of Laurent polynomials in n variables $t_1 \dots, t_n$ with coefficients in an Artinian ring A. Here we do not assume that A is commutative but we do require that the variables t_i commute both amongst themselves and with the coefficients in A. When A is Artinian the Jacobson radical $\mathrm{rad}(A)$ is nilpotent ([9], p.81) and the quotient $A/\mathrm{rad}(A)$ is isomorphic to a product of matrix rings

(*) $$A/\mathrm{rad}(A) \;\cong\; M_{d_1}(D_1) \times \dots \times M_{d_m}(D_m)$$

where $D_1, \dots, D_m$ are division rings and $d_1, \dots, d_m$ are integers ≥ 1. The Artinian ring A is said to satisfy the *Eichler condition* (cf [11], pp. 174-175) when in the decomposition (*) above, D_i is commutative whenever $d_i = 1$. We strengthen this condition as follows; say that A is *strongly Eichler* when in (*) above each division algebra D_i is commutative; then we have:

Theorem I: If the Artinian ring A is strongly Eichler then $L_n(A)$ has property SFC for all $n \geq 1$.

There is a corresponding property which has strong stability implications for automorphisms of free modules. A ring Λ is *weakly Euclidean*(1) (cf [6] Chap.1) when for all $d \geq 2$, any $X \in GL_d(\Lambda)$ can be written as a product

$$X \;=\; E_1 \cdot \dots \cdot E_n \cdot \Delta_d(\lambda)$$

(1) The terminology arises from the classical theorem of H.J.S. Smith [10] which we may state as saying that an integral domain with a Euclidean algorithm is weakly Euclidean.

where each E_i is an elementary transvection and $\Delta_d(\lambda)$ is an elementary diagonal matrix with $\lambda \in \Lambda^*$. Here Λ^* denotes the group of invertible elements in the ring Λ. We say that the Artinian ring A is *very strongly Eichler* when in the decomposition (*) above each D_i is commutative and in addition each $d_i \geq 2$:

Theorem II: If the Artinian ring A is very strongly Eichler then $L_n(A)$ is weakly Euclidean for all $n \geq 1$.

Both Theorem I and Theorem II would seem to be best possible. In relation to Theorem I, a result of Ojanguren and Sridharan [8] shows that, for $n \geq 2$, $L_n(D)$ fails to have the SFC property whenever the division ring D is noncommutative. As regards Theorem II, when $n \geq 2$ the so-called 'Cohn matrix' (cf [2], p.26)

$$\begin{pmatrix} 1 + t_1 t_2 & -t_2^2 \\ t_1^2 & 1 - t_1 t_2 \end{pmatrix} \in GL_2(L_2(\mathbf{F}))$$

fails to decompose as a product of elementary matrices over any field $\mathbf{F}$. A direct proof of this result may be found on p.54 of Lam's book [7]. When $n = 1$ we nevertheless obtain the following useful result.

Theorem III : If the ring A is Artinian then $L_1(A)$ is weakly Euclidean; furthermore, if A also satisfies the Eichler condition then $L_1(A)$ has property SFC.

Finite rings are Artinian and strongly Eichler; thus we have:

Theorem IV : If the ring A is finite then

(i) $L_n(A)$ has property SFC for all $n \geq 1$; moreover

(ii) $L_1(A)$ is weakly Euclidean.

The results proved here all continue to hold if the rings $L_n(A)$ are replaced by the standard polynomial rings $P_m(A) = A[s_1, \dots, s_m]$ or even by rings of mixed type $A[s_1, \dots s_m, t_1, t_1^{-1}, \dots t_n, t_n^{-1}]$. However, as rings of the form $L_n(A)$ occur naturally as group rings $\mathbf{F}[\Phi \times C_\infty^n]$ when Φ is finite, the construction $L_n(A)$ seems more relevant to applications in non-simply connected homotopy theory (cf [6], Chap.11).

We wish to thank the referee whose careful observations have helped to clarify a number of statements.

§1 : The weak Euclidean property for $L_1(A)$:

Given a ring Λ and integer $d \geq 2$ there is a canonical Λ-basis $\{\epsilon^{(d)}(r,s)\}_{1 \leq r,s \leq d}$ for the ring of $d \times d$ matrices $M_d(\Lambda)$ given by

$$\epsilon^{(d)}(r,s)_{tu} = \delta_{rt}\delta_{su};$$

that is, $\epsilon^{(d)}(r,s)$ is the $d \times d$ matrix with '1' in the $(r,s)^{th}$ position and '0' elsewhere. By an *elementary matrix of Type I* in $M_d(\Lambda)$ we mean one of the form

$$E(r,s;\lambda) = I_d + \lambda\epsilon^{(d)}(r,s) \quad (r \neq s,\ \lambda \in \Lambda).$$

By an *elementary matrix of Type II* in $M_d(\Lambda)$ we mean one of the form

$$\Delta_d(\lambda) \;=\; \begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \qquad (\lambda \in \Lambda^*)$$

Formally we have $\Delta_d(\lambda) \;=\; I_d \;+\; (\lambda - 1)\epsilon^{(d)}(1,1)$ where $\lambda \in \Lambda^*$. We say that Λ is *weakly Euclidean* when for $d \geq 2$ each invertible matrix $X \in GL_d(\Lambda)$ can be written in the form

$$X \;=\; E \cdot \Delta_d(\lambda)$$

where E is a product of elementary matrices of type I over Λ and $\lambda \in \Lambda^*$. A ring homomorphism $\varphi : A \to B$ has the *lifting property for units* when the induced map $\phi_* : A^* \to B^*$ is surjective. We say φ has the *strong lifting property for units*(2) when in addition the following holds for $\alpha \in A$;

$$\alpha \in A^* \iff \varphi(\alpha) \in B^*.$$

It is straightforward to see that:

(1.1) Let $\varphi : A \to B$ be a surjective ring homomorphism; if $\mathrm{Ker}(\varphi)$ is nilpotent then φ has the strong lifting property for units.

Elsewhere ([6], Prop. 2.43, p.21) we have shown:

(1.2) Let $\varphi : A \to B$ be a surjective ring homomorphism where B is weakly Euclidean; if φ has the strong lifting property for units then A is also weakly Euclidean.

Thus we have:

(1.3) Let $\varphi : A \to B$ be a surjective ring homomorphism with nilpotent kernel; if B is weakly Euclidean then A is also weakly Euclidean.

Proposition 1.4 : Let $D_1, \dots, D_m$ be (possibly noncommutative) division rings; then $M_{d_1}(D_1[t, t^{-1}]) \times \dots \times M_{d_m}(D_m[t, t^{-1}])$ is weakly Euclidean for any positive integers $d_1, \dots, d_m$.

Proof : If D_i is a division ring then $D_i[t, t^{-1}]$ is a (possibly noncommutative) integral domain which admits a Euclidean algorithm (cf [3]). It is now straightforward to see that matrix rings $M_{d_i}(D_i[t, t^{-1}])$ are also weakly Euclidean. (cf.[6] p.22). The required conclusion now follows as the class of weakly Euclidean rings is closed under finite direct products. □

(2) The referee points out that the strong lifting property for φ may be re-stated as saying that φ has the lifting property and is a local morphism in the sense of Camps and Dicks [1].

Theorem 1.5: Let A be an Artinian ring; then $A[t,t^{-1}]$ is weakly Euclidean.

Proof: The radical $\mathrm{rad}(A)$ of the Artinian ring A is nilpotent (cf [9] p, 81). Consequently $\mathrm{rad}(A)[t,t^{-1}]$ is a nilpotent ideal in $A[t,t^{-1}]$. Moreover

$$A/\mathrm{rad}(A) \;\cong\; M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

for some division rings $D_1, \ldots, D_m$ so that

$$A[t,t^{-1}]/\mathrm{rad}(A)[t,t^{-1}] \;\cong\; M_{d_1}(D_1[t,t^{-1}]) \times \ldots \times M_{d_m}(D_m[t,t^{-1}]).$$

The desired conclusion now follows from (1.3) and (1.4). □

§2 : Suslin's Theorem and proof of Theorem II :

We shall use the following theorem of Suslin ([7], [12]):

Theorem 2.1: Let $\mathbf{F}$ be a field and let $k \geq 3$; then any $X \in GL_k(L_n(\mathbf{F}))$ can be written in the form

$$X \;=\; E_1 \cdots E_m \cdot \Delta_k(\lambda)$$

where $\lambda \in L_n(\mathbf{F})^*$ and each $E_i \in GL_k(L_n(\mathbf{F}))$ is an elementary matrix of type I.

We note that the unit group $L_n(\mathbf{F})^*$ consists simply of elements of the form $\alpha \cdot t_i^{e_i}$ where $\alpha \in \mathbf{F}^*$ and e_i is an integer ([6], Appendix C).

Fixing a ring Λ and an integer $q \geq 2$, we study elementary matrices over the rings $\Omega \;=\; M_d(M_q(\Lambda))$. Write

$$\mathcal{E}(i,j)_{kl} \;=\; \delta_{ik}\delta_{jl} I_q$$

where I_q is the identity matrix in $M_q(\Lambda)$; then $\{\mathcal{E}(i,j)\}_{1 \leq i,j \leq d}$ is a basis for $M_d(M_q(\Lambda))$ over $M_q(\Lambda)$. When $M_q(\Lambda)$ is considered as the base ring we write '$\bullet$' for matrix product over $M_q(\Lambda)$. Then elementary matrices of Type I in $GL_d(M_q(\Lambda))$ take the form

$$\overline{E}(i,j;Z) \;=\; \widetilde{I} \,+\, Z \bullet \mathcal{E}(i,j)$$

where $\widetilde{I}$ denotes the identity matrix in $M_d(M_q(\Lambda))$ and $Z \in M_q(\Lambda)$. Likewise elementary matrices of type II in $GL_d(M_q(\Lambda))$ take the form

$$\Delta_d(Z) \;=\; \begin{pmatrix} Z & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

where $Z \in GL_q(\Lambda) \;=\; M_q(\Lambda)^*$. In the special case where $Z \in GL_q(\Lambda)$ is itself an elementary matrix of Type II over Λ

$$Z \;=\; \Delta_q(\lambda) \;=\; \begin{pmatrix} \lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

with $\lambda \in \Lambda^*$ we write $\overline{\Delta}_{d,q}(\lambda) \ = \ \Delta_d(\Delta_q(\lambda)) \in GL_d(M_q(\Lambda))$.

When $d \geq 2$ there is a mapping, 'block decomposition', $\nu : M_{dq}(\Lambda) \to M_d(M_q(\Lambda))$ defined as follows: if $X = (x_{rs})_{1\leq r,s\leq dq} \in M_{dq}(\Lambda)$ and $1 \leq i,j \leq d$ then

$$\nu(X) \ = \ (X(i,j))_{1\leq i,j\leq d}$$

where $X(i,j) \in M_q(\Lambda)$ is given by $X(i,j)_{kl} \ = \ x_{q(i-1)+k,\, q(j-1)+l}$; moreover:

(2.2) For any ring Λ, $\nu : M_{dq}(\Lambda) \to M_d(M_q(\Lambda))$ is a ring isomorphism.

To record the relationship between the various elementary matrices under block decomposition we first observe that there are unique functions

$$\upsilon : \{1,\dots,dq\} \to \{1,\dots,d\} \quad ; \quad \rho : \{1,\dots,dq\} \to \{1,\dots,q\}$$

defined by the requirement $t + q \ = \ q\upsilon(t) \ + \ \rho(t)$ for $1 \leq t \leq dq$. It is straightforward to verify that:

(2.3) $\nu(\epsilon^{(dq)}(r,s)) \ = \ \epsilon^{(q)}(\rho(r),\rho(s)) \bullet \mathcal{E}(\upsilon(r),\upsilon(s))$.

The inverse relation is perhaps clearer, namely:

(2.4) $\nu^{-1}(\epsilon^{(q)}(a,b) \bullet \mathcal{E}(i,j)) \ = \ \epsilon^{(dq)}(q(i-1)+a, q(j-1)+b)$.

From (2.3) we note that:

(2.5) $\nu(E(r,s;\lambda)) \ = \ \overline{E}(\upsilon(r),\upsilon(s);\ \lambda\,\epsilon(\rho(r),\rho(s))) \quad (\lambda \in \Lambda)$.

Likewise we have :

(2.6) $\nu(\Delta_{dq}(\lambda)) \ = \ \overline{\Delta}_{d,q}(\lambda) \quad (\lambda \in \Lambda^*)$.

We first consider the rings $L_n(\mathbf{F}) = \mathbf{F}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ where $\mathbf{F}$ is a field.

Theorem 2.7: Let $d,q \geq 1$ be integers such that $dq \geq 3$. If $X \in \mathrm{GL}_d(M_q(L_n(\mathbf{F}))$ then X can be expresssed as a product

$$X \ = \ \overline{E}_1 \bullet \dots \bullet \overline{E}_m \bullet \overline{\Delta}_{d,q}(\delta)$$

where $\overline{E}_1,\dots,\overline{E}_m \in GL_d(M_q(L_n(\mathbf{F})))$ are elementary of Type I and $\delta \in L_n(\mathbf{F})^*$.

Proof : Put $\Lambda = L_n(\mathbf{F})$. If $X \in \mathrm{GL}_d(M_q(\Lambda))$ put $\widehat{X} \ = \ \nu^{-1}(X) \in GL_{dq}(\Lambda)$. By Suslin's Theorem, $\widehat{X}$ can be expressed as a product

$$\widehat{X} \ = \ E_1 \cdots E_m \cdot \Delta(\lambda)$$

where $\lambda \in L_n(\mathbf{F})^*$ and each $E_i \in GL_{dq}(L_n(\mathbf{F}))$ is an elementary matrix of type I. Thus

$$\nu(\widehat{X}) \ = \ \nu(E_1) \bullet \dots \bullet \nu(E_m) \bullet \nu(\Delta(\lambda))$$

so that, writing $\overline{E_i} = \nu(E_i)$ we have $X = \overline{E}_1 \bullet \ldots \bullet \overline{E}_m \bullet \overline{\Delta}_{d,q}(\delta)$. □

Corollary 2.8 : If $\mathbf{F}$ is a field then $M_q(L_n(\mathbf{F}))$ is weakly Euclidean for each $q \geq 2$.

The weak Euclidean property is preserved under finite direct products. Moreover the construction L_n commutes with both direct products and with the functor $\Lambda \mapsto M_q(\Lambda)$; hence we have:

Corollary 2.9 : $L_n[M_{q_1}(\mathbf{F}_1) \times \ldots\ldots \times M_{q_m}(\mathbf{F}_m)]$ is weakly Euclidean whenever $\mathbf{F}_1, \ldots, \mathbf{F}_m$ are fields and $q_1, \ldots, q_m$ are integers ≥ 2.

Theorem 2.10: If the Artinian ring A is very strongly Eichler then $L_n(A)$ is weakly Euclidean for $n \geq 2$.

Proof : Write $A/\mathrm{rad}(A) \cong M_{q_1}(\mathbf{F}_1) \times \ldots \times M_{q_m}(\mathbf{F}_m)$ for some fields $\mathbf{F}_1, \ldots, \mathbf{F}_m$ and integers $q_1, \ldots, q_m \geq 2$. Then $L_n(\mathrm{rad}(A))$ is a nilpotent ideal in $L_n(A)$ and

$$L_n(A)/L_n(\mathrm{rad}(A)) \cong L_n[M_{q_1}(\mathbf{F}_1) \times \ldots\ldots \times M_{q_m}(\mathbf{F}_m)].$$

The desired conclusion now follows from (1.3) and (2.9). □

Theorem II is now the conjunction of (1.5) and (2.10).

§3: Proof of Theorems I, III and IV:

The following is a straightforward deduction from Nakayama's Lemma (cf [6] pp. 170-171).

Proposition 3.1 Let $\varphi : \Lambda \to \Omega$ be a surjective ring homomorphism such that $\mathrm{Ker}(\varphi)$ is nilpotent; if Ω satisfies SFC then so also does Λ.

Suppose that A is an Artinian ring such that

$$A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$$

where $D_1, \ldots, D_m$ are division rings. We shall apply (3.1) in the case $\Lambda = L_n(A)$, $\Omega = L_n(A)/L_n(\mathrm{rad}(A))$ and φ is the natural mapping. Then

$$\Omega \cong M_{d_1}(L_n(D_1)) \times \ldots \times M_{d_m}(L_n(D_m)).$$

We showed in [5] that Ω has property SFC provided each D_i is commutative; that is, provided A is strongly Eichler. Thus from (3.1) we obtain:

Proposition 3.2 : If the ring A is Artinian and strongly Eichler then $L_n(A)$ has property SFC.

As we observed in the Introduction, Ojanguran and Sridharan proved in [8] that $L_n(D)$ fails the SFC property whenever $n \geq 2$ and the division ring D is noncommutative. However, in the case $n = 1$ one may show that $L_1(\mathcal{D}) = \mathcal{D}[t, t^{-1}]$ has

SFC regardless of whether the division ring D is commutative or not. Indeed, in that case, $\mathcal{D}[t, t^{-1}]$ is projective free (cf [4] or [5] Prop 2.9). Now the SFC property is preserved under finite direct products and passage to matrix rings ([6] p. 171-173). Thus $M_{d_1}(L_1(D_1)) \times \ldots \times M_{d_m}(L_1(D_m))$ has property SFC. From (3.1) we get:

Proposition 3.3 : If the ring A is Artinian then $L_1(A)$ has property SFC.

The conjunction of (3.2) and (3.3) is Theorem I of the Introduction.

Any finite ring A is trivially Artinian so that $A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$ where $D_1, \ldots, D_m$ are finite division rings. However, a celebrated theorem of Wedderburn (cf [13] p.1) now shows that each D_i is commutative; that is :

(3.4) Any finite ring is Artinian and strongly Eichler.

Thus from (1.5), (3.2) and (3.4) we have:

Corollary 3.5 : Let A be a finite ring ; then

(i) $L_n(A)$ has property SFC for all $n \geq 1$;

(ii) $L_1(A)$ is weakly Euclidean.

We may regard the coefficient ring A as a degenerate case $A = L_0(A)$. Thus suppose that A is Artinian and write $A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \ldots \times M_{d_m}(D_m)$ where $D_1, \ldots, D_m$ are division rings. Then each $M_{d_1}(D_1)$ is weakly Euclidean and has property SFC. As both these properties are closed under finite direct products then $A/\mathrm{rad}(A)$ is weakly Euclidean and has property SFC. However, $\mathrm{rad}(A)$ is nilpotent so that, from (1.3) and (3.1), we conclude the following which should be well known but is difficult to locate explicitly in the literature.

(3.6) Any Artinian ring is weakly Euclidean and has property SFC.

REFERENCES

[1] : R. Camps and W. Dicks; On semilocal rings.
Israel J. Math. 81 (1993) 203-211.

[2] : P.M. Cohn; On the structure of the GL_2 of a ring.
Publications Mathématiques IHES 30 (1966) 5-53.

[3] : P.M. Cohn; Free rings and their relations. Academic Press 1985.

[4] : W. Dicks and E.D. Sontag; Sylvester domains.
Jour. Pure Appl. Alg. 13 (1978) 243-275.

[5] : F.E.A. Johnson; Stably free modules over rings of Laurent polynomials: Arch. Math. 97 (2011), 307- 317.

[6] : F.E.A. Johnson; Syzygies and homotopy theory.
Springer Verlag 2011.

[7] : T.Y. Lam; Serre's problem on projective modules.
Springer-Verlag 2006.

[8] : M. Ojanguran and R. Sridharan; Cancellation of Azumaya algebras. J. Algebra. 18 (1971) 501-505.

[9] : I. Reiner; Maximal orders. Academic Press 1975.

[10] : H.J.S. Smith; On systems of linear indeterminate equations and congruences. Phil. Trans. 151 (1861) 293-326.

[11] : R.G. Swan; Strong approximation and locally free modules; pp. 153-223 of 'Ring Theory and Algebra III', edited by B. McDonald. Marcel Dekker, New York, 1980.

[12] : A.A. Suslin; On the structure of the special linear group over polynomial rings. Math USSR Isvestiya 11 (1977) 221-238.

[13] : A. Weil; Basic number theory. Springer-Verlag 1973.

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