Innovation Tournaments with Multiple Contributors

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Abstract

This paper studies innovation tournaments in which an organizer seeks solutions to an innovation-related problem from a number of agents. Agents exert effort to improve their solutions but face uncertainty about their solution performance. The organizer is interested in obtaining multiple solutions - agents whose solutions contribute to the organizer's utility are called contributors. Motivated by mixed policies observed in practice, where some tournaments are open and others restrict entry, we study when it is optimal for the organizer to conduct an open tournament or to restrict entry. Our analysis shows that whether an open tournament is optimal is tied to: (1) the variance of uncertainty as compared to the impact of effort; (2) the number of contributors, and (3) the skewness of the uncertainty distribution. Our results help explain mixed policies about restricting entry observed in practice as well as recent empirical and experimental findings.

Key words: Contest, Crowdsourcing, Incentive, Online Platforms, Technology, Uncertainty

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1. Introduction

As organizations increasingly look beyond their boundaries towards outsourcing research and development activities, innovation tournaments have emerged as one popular and cost-effective tool. In an innovation tournament, an organizer elicits solutions to a problem from a group of agents, but awards only the best solutions. One of the key decisions in the design of an innovation tournament is how many agents to let in (Boudreau et al. 2011). More agents in a tournament allow the organizer to tap into a more diverse set of solutions. However, more agents in a tournament also affect agents' incentives to exert effort towards improving their solutions by reducing their chances of winning an award. Thus, the organizer should carefully choose between an open tournament where any agent can freely participate or a restricted-entry tournament where only a subset of agents can participate. In this paper, we aim to understand why open tournaments are prevalent in practice and to also provide insights into when open tournaments are undesirable.

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We encounter many open innovation tournaments in practice. For instance, since 2012, Samsung has organized several open tournaments, called the Samsung Smart App Challenge, soliciting innovative applications for its online app store. At the crowdsourcing platform InnoCentive, organizers run open ideation and reduction-to-practice (RTP) challenges that seek innovative ideas and innovative solutions with working prototypes, respectively. Similar open tournaments are organized at crowdsourcing platforms Tongal and TopCoder in several categories such as concept projects and coding challenges. For instance, in the Arcelik Exploratory Testing Challenge, agents compete by identifying issues in Arcelik's website (Topcoder 2020). At the opposite end of the spectrum, there are also quite a few innovation tournaments with restricted entry. For instance, it is not uncommon in architectural design tournaments to restrict the number of participants (e.g., RAIC 2019). As a starting point of understanding these mixed policies observed in practice, we focus on two dimensions in which innovation tournaments differ: the uncertainty faced by agents that participate in a tournament and the estimated number of solutions utilized by the tournament organizer.

In innovation tournaments, agents face uncertainty about the quality of their solution due to the stochastic nature of the innovation process. This uncertainty is associated with the specific problem at hand and has two important properties: variance and skewness. First, the variance of agents' uncertainty can differ across tournaments. For instance, InnoCentive RTP challenges that seek innovative solutions (e.g., developing 3D-printable robots for bomb squads) may entail larger uncertainty than a Topcoder coding challenge such as the Arcelik Exploratory Testing Challenge. Second, beyond variance, skewness and tail properties of uncertainty distribution also vary across tournaments. It may be reasonable to expect most tournaments to feature symmetric (e.g., uniform as in Mihm and Schlapp 2019 or normal as in Hu and Wang 2019) or right-skewed (e.g., Gumbel as in Terwiesch and Xu 2008) distributions. Yet, in some tournaments, a left-skewed distribution for the solution uncertainty may be suitable. According to Dahan and Mendelson (2001), a left-skewed (Weibull-type) distribution is suitable when there are "predictably finite bounds on the upside profit potential of a new product ... Such might be the case for a product that serves a small market, upgrades an existing user base, conforms to a fixed-price contract, or is capacity-constrained" (page 110). For instance, in the Arcelik Exploratory Testing Challenge, the upside potential for developed solutions is limited by the severity of issues a user can encounter in Arcelik's website.[1]

The second dimension in which innovation tournaments differ is the estimated number of solutions that a tournament organizer will utilize. We refer to agents whose solutions contribute to the organizer's utility as *contributors*. Some tournaments, given the nature of the problem at hand,

[1] It is worth noting that not all coding challenges feature small variance or limited upside potential. For instance, in a bug-hunt challenge where very serious issues (e.g., security vulnerabilities) can be revealed, the upside potential can be high and the quality of solutions could be highly variable.

can have only a single contributor. For instance, this is the case for an architectural design contest where only a single design will be adopted. Other tournaments may feature multiple contributors. For instance, an organizer that runs an InnoCentive ideation challenge or a Tongal concept project may utilize or further develop multiple viable ideas or concepts instead of only the best one. The Samsung Smart App Challenge sought many useful applications to contribute to Samsung's objective of enriching its app marketplace. Our interview with Samsung revealed that the organizer of the Samsung Smart App Challenge 2013 estimated that 150 apps (among hundreds of submissions) could be uploaded to Samsung App marketplace. While the organizer in some tournaments may end up utilizing a different number of solutions than estimated, in some tournaments, the number of contributors has to be determined at the beginning of the tournament and cannot be changed. For instance, in Tongal concept projects, organizers often commit to receiving the intellectual property rights of a fixed number of concepts (see, e.g., Tongal 2020). Importantly, an organizer designs its tournament based on the expected number of contributors estimated before the tournament begins rather than the actual number of solutions used at the end. It is worth noting that the organizer does not necessarily pay all contributors. For instance, in ideation challenges, organizers usually have perpetual rights to use or further develop any submitted idea, but they award only the best idea(s). Similarly, in the Samsung Smart App Challenge 2013, although practitioners estimated 150 contributors, only the best few apps were given awards.

Our paper develops a model that is sufficiently general to capture the two key features of tournaments described above. In particular, we model agents' uncertainty with a general class of distributions that have log-concave or increasing density functions (e.g., normal, uniform, exponential, Weibull, and Gumbel distributions). This allows us to characterize the impact of variance and skewness in agents' uncertainty on the design of an optimal innovation tournament. In addition, we assume that the organizer's ex-ante utility depends explicitly on the best K submitted solutions, where K can be any number between one and the total number of participants. It turns out that the difference between a tournament with a single contributor and a tournament with many contributors plays an important role in a tournament's design.

Our analysis shows that whether an open tournament is optimal is closely tied to: (1) the variance of uncertainty as compared to the impact of effort (in short, uncertainty-effort ratio); (2) the number of contributors, and (3) the skewness of uncertainty distribution. (Table 1 provides a typology of tournaments that should be open or feature restricted entry.) First, we show that an open tournament is optimal if an innovation problem involves a sufficiently large uncertainty-effort ratio. The intuition is as follows. More participants in the tournament can reduce agents' incentives to exert effort, yet they can help the organizer benefit from having a more diverse set of solutions from participants. For a sufficiently large uncertainty-effort ratio, the positive impact of having a

Table 1 Settings where open or restricted-entry tournaments are optimal.

	Low uncertainty-effort ratio and symmetric or right-skewed distribution	High uncertainty-effort ratio or left-skewed distribution
Small number of contributors	Restrict (e.g., architectural design tournaments)	Open (e.g., InnoCentive RTP challenges)
Large number of contributors	Open (e.g., Samsung Smart App Challenge)	Open (e.g., Arcelik Exploratory Testing Challenge)

diverse set of solutions outweighs the potentially negative incentive effect. Therefore, an organizer seeking solutions with a high uncertainty-effort ratio (e.g., innovative solutions) may benefit from an open tournament. Our result provides a plausible explanation for why a wide range of innovation tournaments featuring large uncertainty (e.g. InnoCentive RTP challenge) are open.

A tournament features a relatively low uncertainty-effort ratio when it involves low uncertainty (e.g., as in Arcelik Exploratory Testing Challenge) or the agents' effort plays a substantial role in their solution performance (e.g., app design or architectural design). For such a tournament, the benefit of having a diverse set of solutions is not large enough to offset a potentially negative incentive effect. In this case, our results indicate that there are two cases where an open tournament can still be optimal. The first case is when the organizer aims to utilize many solutions. For instance, the Samsung Smart App Challenge has a large number of estimated contributors and it is an open tournament. The second case where an open tournament can be optimal is when more participants encourage agents to exert more effort. We find that agents can increase effort with more participants when their uncertainty features a left-skewed distribution, as opposed to the prior literature that has argued that agents always reduce effort with more participants since increased competition lowers agents' probability of winning an award. In fact, we find that the driver behind how agents change their effort with more participants is a marginal change of the winning probability with additional effort rather than the winning probability itself. As the number of participants increases, the marginal change of an agent's winning probability may increase because additional effort helps the agent gain an edge against more competitors. Thus, more participants can encourage agents to exert higher efforts under left-skewed distributions. This result not only helps explain why some Topcoder coding challenges with small uncertainty-effort ratios are open, but also is consistent with observations in the laboratory experiments conducted by List et al. (2020).

While explaining the frequent use of open tournaments in practice, our paper also shows when it is optimal to restrict entry. Specifically, restricting entry is optimal in tournaments with a low uncertainty-effort ratio, a symmetric or right-skewed distribution, and a small number of contributors. This result may offer a plausible explanation for why architectural design contests often restrict entry. When taken together, our results can help explain mixed policies in practice that cannot be explained by the results in the prior literature. For instance, our results may help explain why some tournaments with low uncertainty-effort ratios are open (e.g., Samsung Smart App Challenge) whereas others choose restricted entry (e.g., architectural design contests). Our result also provides theoretical support for the empirical finding of Boudreau et al. (2011) which implies that a free-entry open tournament should be encouraged when problems are highly uncertain but restricted entry can be optimal when problems feature low uncertainty.

Previous work has provided mixed answers to when a tournament should be open. (We review the prior studies that are concerned with our research question, while referring readers to Ales et al. (2019) and Chen et al. (2020) for a comprehensive review of the literature on tournaments.[2]) Taylor (1995) and Fullerton and McAfee (1999) argue that an open tournament is *never* optimal because more intense competition hinders agents' incentives to exert effort. Terwiesch and Xu (2008) reach the same conclusion when the organizer aims to maximize the performance of the average solution. However, by assuming that the output uncertainty is sufficiently large, they conclude that an open tournament is *always* optimal if the organizer wants to maximize the performance of the best solution, because the organizer can benefit from a more diverse set of solutions.

Our contribution is to sharpen these mixed results in the prior literature and help explain mixed policies in practice by showing when an open tournament is optimal and when it is not. To achieve this goal, we consider a general log-concave distribution for the solution uncertainty instead of a specific distribution (e.g., Gumbel in Terwiesch and Xu 2008 or uniform in Mihm and Schlapp 2019) and a general number of contributors as opposed to focusing on the best solution (e.g., Taylor 1995, Mihm and Schlapp 2019) or all submitted solutions (e.g., Green and Stokey 1983, Kalra and Shi 2001). (Erat and Krishnan (2012) also consider a case where the organizer is interested in the best two solutions.) Our general model not only takes prior models as special cases, but also characterizes the role of contributors in an organizer's decision to hold an open tournament. As the closest study to ours, Terwiesch and Xu (2008) consider a weighted combination of the performance of the best solution and the average performance of all solutions, while noting that the explicit approach of considering the best K submitted solutions might be intractable. We conduct a tractable analysis of the explicit approach, and show that it leads to qualitatively different results. Our results show that whether an open tournament is optimal is more subtle than what prior studies show because it depends on the number of contributors as well as the variance and skewness of uncertainty.

[2] Broadly speaking, innovation tournaments can be used as a tool to outsource some or all stages of product development. We refer the reader to Krishnan and Ulrich (2001), Kalkanci et al. (2019), and Rahmani and Ramachandran (2020) for a detailed review of the broader product-development literature. Also, for recent developments in empirical research on crowdsourcing, we refer the reader to Hwang et al. (2019), Aggarwal et al. (2020) and references therein.

2. Model

Consider an innovation tournament in which a tournament organizer elicits solutions to an innovation-related problem from a set of agents. A tournament proceeds in the following sequence. By anticipating the number of solutions to utilize at the end of the tournament, the organizer announces whether the tournament is open to anyone who wishes to participate, and how participants of the tournament will be compensated. Then agents decide whether to participate in the tournament, and if they do, they exert effort to develop their solutions, and submit them to the organizer. Finally, the organizer evaluates the submitted solutions and compensates agents accordingly. Below, we formalize the model.

Agents. There are $\overline{N}$ (≥ 3) agents who can potentially participate in the tournament. Let N $(\in \{2, 3, ..., \overline{N}\})$ be the number of agents who participate. Each participating agent i $(\in \{1, 2, ..., N\})$ develops a solution to the problem posed by the organizer, and generates an output $y_i \in \mathcal{Y} \subseteq \mathbb{R} \cup \{-\infty, \infty\}$. The output y_i can be interpreted as the quality of a solution or its monetary benefit to the tournament organizer. The output y_i is determined by two components: (i) agent i's effort and (ii) a stochastic output shock. We elaborate on each of these components next.

Each agent can enhance the output by exerting effort $e_i \in \mathbb{R}_+$. Effort e_i leads to a deterministic improvement of the agent's output by $r(e_i)$, where r is a strictly concave, increasing, and twice continuously differentiable function. An agent who exerts effort e_i incurs cost $\psi(e_i)$, where ψ is a convex, increasing, and twice continuously differentiable function of effort with $\psi(0) = 0$. The cost of effort may represent the monetary investment required to exert effort e_i or the disutility that agent i incurs from this effort. For ease of illustration, we use the following forms for r and ψ in the main body while extending our results to general r and ψ throughout the Online Appendix.

Assumption 1. *Suppose that* $r(e) = \gamma + \theta \log(e)$, *and* $\psi(e) = ce^b$ *for* $c, \theta > 0$ *and* $b \geq 1$.

The effort function coefficient θ captures the impact of effort on an agent's output. The larger the value of θ, the larger is the impact of a unit effort on output. The parameter b captures how fast the cost of effort is increasing, so we interpret it as a measure of difficulty in improving the output. In Assumption 1, we utilize the logarithmic effort function r to make our results comparable with Terwiesch and Xu (2008) who use a special case of the setting in Assumption 1 where $b = 1$. The power function form that we use for the cost function ψ is also common in the literature (e.g., Mihm and Schlapp 2019, Korpeoglu et al. 2020, Candoğan et al. 2020).

In addition to effort, each agent i's output is subject to a stochastic output shock $\widetilde{\xi}_i$ due to uncertainty involved in innovation and evaluation processes. Following the literature, we assume that $\widetilde{\xi}_i$'s are independent and identically distributed (i.i.d.) random variables with $E[\widetilde{\xi}_i] = 0$. We consider a general class of distributions with log-concave or increasing density functions (e.g., normal, uniform, exponential, logistic, Weibull, and Gumbel distributions). Thus, the output shock $\widetilde{\xi}_i$ ($\in \Xi$) has a density function h where either $\log(h)$ is concave or h is increasing; a cumulative distribution H with $\Xi = [\underline{s}, \overline{s}]$. We make the following definitions related to the output shock $\widetilde{\xi}_i$. Let $\widetilde{\xi}^N_{(j)}$ be a random variable with cumulative distribution $H^N_{(j)}$ and density $h^N_{(j)}$ that represents the j-th highest value among N i.i.d. output shocks. Since $\widetilde{\xi}^N_{(j)}$ corresponds to the $(N-j+1)$-st order statistic among N random variables, we have $h^N_{(j)}(s) = \frac{N!}{(j-1)!(N-j)!}(1-H(s))^{j-1} H(s)^{N-j} h(s)$. To measure the variance of uncertainty for a general distribution H, we use the notion of a scale transformation (e.g., Rothschild and Stiglitz 1970).

Definition 1. Two distribution functions $\widehat{H}(\cdot)$ and $H(\cdot)$ differ by a scale transformation if there exists parameter α such that $\widehat{H}(s) = H(s/\alpha)$ (with density $\widehat{h}(s) = h(s/\alpha)/\alpha$) for all $s \in \Xi$.

The scale transformation of the output shock $\widetilde{\xi}_i$ with scale parameter α preserves the mean of zero while multiplying its variance by α^2. Thus, the variance of uncertainty is captured by α.

Given agent i's effort e_i and output shock $\widetilde{\xi}_i$, agent i's output is determined as

$$y(e_i, \widetilde{\xi}_i) = r(e_i) + \widetilde{\xi}_i. \tag{1}$$

The utility of agent i, $U_a(e_i, x_i): \mathbb{R}^2_+ \to \mathbb{R}$, is defined over the agent's effort e_i and the monetary compensation x_i that the agent receives from the organizer. The utility of the agent takes the following form: $U_a(e_i, x_i) = x_i - \psi(e_i)$. We refer to the agent who produces the best output as the winner of the tournament. As is common in the literature (e.g., Taylor 1995, Fullerton and McAfee 1999, Terwiesch and Xu 2008, Körpeoğlu et al. 2018, Hu and Wang 2019, Candoğan et al. 2020), we focus on "winner-takes-all" tournaments in which the organizer gives an award A (>0) only the winner of the tournament. It turns out that when the output shock $\widetilde{\xi}_i$ follows a log-concave or increasing density function, the winner-takes-all award scheme is optimal (Ales et al. 2017). Thus, each agent i receives $x_i = A$ if the agent wins the tournament, or $x_i = 0$ otherwise. In §EC.3 of the Online Appendix, we extend our results to the case in which the organizer offers multiple awards.

The Organizer. The organizer's utility $\widehat{U}_o(Y, A)$ is defined over the output vector $Y = (y_1, y_2, ..., y_N)$ and the award A. We consider the case where the organizer benefits from K best outputs (where $1 \le K \le \overline{N}$), and refer to those agents who produce the K best outputs as *contributors*. Formally, we have the following definition:

Definition 2. Let $Y^{(K)} = \{y_{(1)}[Y], \ldots, y_{(K)}[Y]\}$ where $y_{(j)}[Y]$ represents the j-th highest output in Y - for ease of notation, we use $y_{(j)}$ in short for any $j = 1, 2, .., K$. The organizer's utility has K contributors if for all $Y \in \mathcal{Y}^N$,

(i) There exists a continuously differentiable function U_o so that $\widehat{U}_o(Y, A) = U_o(Y^{(K)}, A)$;

(ii) For all $j = 1, 2, \ldots, K$, $\frac{\partial U_o(Y^{(K)}, A)}{\partial y_{(j)}} > 0$.

In §3, we use the following linear utility function for the organizer with K contributors:

$$U_o(Y^{(K)}, A) = E[\sum_{j=1}^{K} y_{(j)}] - A, \qquad \forall\, Y \in \mathcal{Y}. \tag{2}$$

We consider a more general utility function in §EC.2 of the Online Appendix. We note that Ales et al. (2017) also use a K contributor setup in their model, while focusing on deriving an optimal award scheme. That paper does not study when it is optimal to hold an open tournament as we do in this paper. Our model as well as theirs takes K given exogenously. In practice, the organizer should have an estimated value of K (e.g., K=150 in Samsung Smart App Challenge described in §1) before conducting a tournament because K affects its optimal decision on tournament rules. Our model thus allows us to isolate the impact of K on the organizer's and agents' decisions, while generalizing several prior papers that assume $K = 1$ or N (see §1). In §4, we also discuss alternative models in which the organizer determines K endogenously ex-ante or ex-post.

The organizer chooses the number of agents who participate N (where $K \leq N \leq \overline{N}$) and the award A that maximize its utility. A tournament where the organizer allows entry of all agents who can potentially participate (i.e., chooses $N = \overline{N}$) is called an *open tournament*.

We consider a static model where N agents simultaneously participate in the tournament and N is common knowledge. This modeling approach is common in the tournament literature and seems suitable for tournaments at platforms such as InnoCentive for two reasons. First, our interview with a business development manager at InnoCentive reveals that each agent at platforms is notified by e-mail when a new tournament is posted, so the number of participants becomes stable within a short period of time. Thus, it may be reasonable to assume that all agents participate at once. Second, at platforms, the number of participants N is shared with agents, so agents have a fairly good idea about N.

The Equilibrium. As is standard in the tournament literature, we focus on a symmetric pure-strategy Nash equilibrium. Let e^* denote the agent's equilibrium effort, and $P^N[e_i, e^*]$ denote the probability that agent i is the winner of the tournament when agent i exerts effort e_i and all other $(N-1)$ agents exert the equilibrium effort e^*. We can compute this probability as

$$P^N[e_i, e^*] = \int_{s \in \Xi} H(s + r(e_i) - r(e^*))^{N-1} h(s) ds. \tag{3}$$

Each agent i's problem is to choose effort e_i that maximizes the agent's expected award $AP^N[e_i, e^*]$ less the agent's cost of exerting effort e_i, $\psi(e_i)$, by solving

$$\max_{e_i \in \mathbb{R}_+} A \int_{s \in \Xi} H\left(r\left(e_i\right) - r(e^*) + s\right)^{N-1} h\left(s\right) ds - \psi(e_i). \tag{4}$$

In Lemmas EC.1-3 of the Online Appendix, we show the existence of a unique symmetric pure-strategy Nash equilibrium effort e^* that solves (4) under specified conditions on the effort function r, cost function ψ, and output shock $\widetilde{\xi}_i$. Throughout the paper, we assume that at least one of these conditions is satisfied for all N up to $\overline{N}$. Under these conditions, the agent's equilibrium effort e^* satisfies the following first-order condition of (4) evaluated at $e_i = e^*$:

$$\frac{\psi'(e^*)}{r'(e^*)} = AI_N \text{ where } I_N \equiv \int_{s\in\Xi} (N-1)\, H(s)^{N-2}\, h(s)^2\, ds. \tag{5}$$

The I_N term in (5) is related to the marginal impact of additional effort on the winning probability. The left-hand side of (5) is increasing in e^* because $(\psi'(e^*)/r'(e^*))' = \frac{\psi''(e^*)}{r'(e^*)} - \frac{\psi'(e^*)r''(e^*)}{(r'(e^*))^2} > 0$, so e^* is increasing I_N. The dependence of e^* on I_N is important as it indicates the possibility that e^* increases with the number of participants N. We will expand on this observation in our analysis in §3 after we present our main result related to when an open tournament is optimal.

In equilibrium, $e_i = e^*$, so each agent's probability of winning is $1/N$, and each agent i's utility from the tournament is $U_a = \frac{A}{N} - \psi(e^*)$. Consistent with the innovation contest literature, we assume that each agent has zero outside option. Then, under the assumption that an e^* that solves (4) exists, agents obtain higher utility by exerting effort e^* than they do by exerting zero effort (which is equivalent to not participating), so agents always find it beneficial to participate with effort e^* (i.e., $U_a \geq 0$).

When each agent exerts effort e^*, the j-th highest output can be written as $y_{(j)} = r(e^*) + \widetilde{\xi}^N_{(j)}$. Therefore, the organizer chooses N (where $K \leq N \leq \overline{N}$) and A that maximize its expected utility

$$U_o = Kr(e^*) + E\left[\sum_{j=1}^{K} \widetilde{\xi}^N_{(j)}\right] - A. \tag{6}$$

3. Analysis

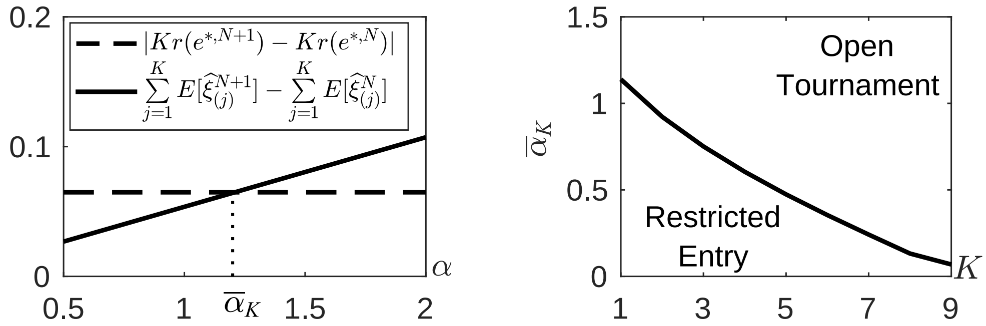
Our primary goal is to determine when the organizer benefits from an open tournament (i.e., choose $N = \overline{N}$) as opposed to restricting entry of participants (i.e., choose $N < \overline{N}$). To answer this question, we examine how the number of participants (N) affects the organizer's utility: $U_o = Kr(e^{*,N}) + E[\sum_{j=1}^{K} \widetilde{\xi}^N_{(j)}] - A^*$, where A^* is the optimal award and superscript N in $e^{*,N}$ denotes the number of participants. When U_o is maximized under $N = \overline{N}$, it is optimal for the organizer to choose an open tournament. The first term in U_o, $Kr(e^{*,N})$, increases (resp., decreases) with N if the agent's equilibrium effort $e^{*,N}$ increases (resp., decreases) with N. The second term in U_o, $E[\sum_{j=1}^{K} \widetilde{\xi}^N_{(j)}]$, represents the expected value of the best K outcomes from N i.i.d. random variables. It is easy to see that this term increases with N for any K; in other words, a more diverse set of solutions increases the expected value of the best K outputs. The last term in U_o, A^*, does not depend on N under Assumption 1 (which is relaxed throughout the Online Appendix including Corollary EC.5 that extends Theorem 1). Therefore, whether an open tournament is optimal depends on how the first two terms change with N. Theorem 1 captures this tradeoff and characterizes when an open tournament is optimal. All proofs are presented in the Appendix.

THEOREM 1. *Consider a scale transformation of the output shock $\widetilde{\xi}_i$ with scale parameter $\alpha > 0$.*
(a) For any number of contributors K and any number of potential participants $\overline{N}$, there exists $\overline{\alpha}_K$ such that U_o is maximized at $N = \overline{N}$ if and only if $\alpha \geq \overline{\alpha}_K$, where $\overline{\alpha}_K$ is non-decreasing in the effort function coefficient θ.
(b) $\overline{\alpha}_K$ is non-increasing in the number of contributors K and the cost function parameter b.
(c) If I_N in (5) is increasing in N up to some N^* $(\geq \overline{N})$, then $\overline{\alpha}_K = 0$.

Theorem 1 shows that whether an open tournament is optimal depends on output uncertainty (α), the number of contributors (K), cost function parameter b, the effort function coefficient θ, and how the equilibrium effort e^* changes with N. Theorem 1(a) shows that an open tournament is optimal if and only if α is above threshold $\overline{\alpha}_K$ which decreases with the effort coefficient θ.[3] Figure 1(a) illustrates the underlying mechanisms of Theorem 1(a) under fixed θ. In the setting of this figure, additional participants lead to a reduction in the equilibrium effort $e^{*,N}$, and hence $Kr(e^{*,N})$ in the organizer's utility U_o decreases with N, whereas $E[\sum_{j=1}^{K} \widetilde{\xi}_{(j)}^{N}]$ in U_o increases with N. Thus, it is not obvious how U_o changes with N. Figure 1(a) displays that a larger output uncertainty (α) raises $(\sum_{j=1}^{K} E[\widehat{\xi}_{(j)}^{N+1}] - \sum_{j=1}^{K} E[\widehat{\xi}_{(j)}^{N}])$, which captures the contribution of an additional participant to the organizer's utility from having a more diverse set of solutions. This is intuitive. On the other hand, a larger output uncertainty α does not change $|Kr(e^{*,N+1}) - Kr(e^{*,N})|$, which captures the negative impact of an additional participant on the organizer's utility due to agents' reduced effort. Although the latter result might also appear intuitive, it is not necessarily true for a general effort function r. Nevertheless, Corollary EC.5 in the Online Appendix shows that when the variance of the output shock is sufficiently large for any general distribution, the benefit of having a more diverse set of solutions dominates the potentially negative incentive effect as well as its impact on the optimal award. Thus, the benefit of having a more diverse set of solutions from a larger number of participants dominates its potentially negative incentive effect, only when α is sufficiently large (relative to θ since $\overline{\alpha}_K$ decreases in θ).

Theorem 1(a) has important implications for both tournament theory and practice. Prior literature in economics has shown that when the organizer wants to maximize the best output (i.e., $K = 1$), an open tournament is *never* optimal (e.g., Taylor 1995, Fullerton and McAfee 1999) because a larger number of participants has a negative incentive effect on agents' effort. Terwiesch and Xu (2008) argue that an open tournament is *always* optimal because the benefit of having

[3] Theorem 1(a) is derived under the organizer's objective of maximizing the best K outputs (see §2). While this objective is suitable for innovation tournaments, there are other types of tournaments in which the organizer is purely interested in the agents' effort (e.g., Tullock 1980).



(a) Impact of increased N when $K=1$.

(b) $\overline{\alpha}$ as a function of K.

Figure 1 (a) The impact of an additional participant on the contributors' total effort (i.e., $Kr(e^{*,N+1}) - Kr(e^{*,N+1})$) and shock (i.e., $E[\sum_{j=1}^{K}\widehat{\xi}_{(j)}^{N+1} - \sum_{j=1}^{K}\widehat{\xi}_{(j)}^{N}]$) as a function of scale parameter α; (b) Minimum scale parameter $\overline{\alpha}_K$ for an open tournament. Parameters used: $\widetilde{\xi}_i \backsim$ Normal(0,1); $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$; $N=10$; $r(e)=log(e)$ and $\psi(e)=e$.

a more diverse set of solutions outweighs the negative incentive effect. They derive this result under the assumption that the output shock follows a Gumbel distribution with a sufficiently large scale parameter μ. Our result sharpens existing theories by showing that the benefit of having a diverse set of solutions outweighs the potentially negative incentive effect if and only if the variance of the output shock (captured by α) relative to the impact of effort (captured by θ), i.e., the "uncertainty-effort ratio," is sufficiently large. Our result is corroborated with empirical evidence, and seems consistent with practice. Specifically, Boudreau et al. (2011), who empirically analyze 9,661 software tournaments at Topcoder, conclude that free entry should be encouraged in contests for which problems are highly uncertain. In practice, this may be the case when a tournament features large uncertainty (e.g., InnoCentive RTP challenges).

Theorem 1(b) states that the threshold on the level of uncertainty over which an open tournament is optimal (i.e., $\overline{\alpha}_K$) decreases as the organizer anticipates utilizing a larger number of solutions (i.e., larger K). This result suggests that even when a tournament features a low uncertainty-effort ratio, an open tournament may still be optimal if the number of contributors K is sufficiently large; see Figure 1(b). Our result provides a plausible explanation to some industry examples. For example, the Samsung Smart App Challenge was conducted as an open tournament because a large number of contributors were anticipated. On the other hand, an architectural design tournament often features restricted entry. Although the latter tournament may involve a similar uncertainty-effort ratio to the former tournament, it seeks a single contributor, so an open tournament is less desirable. Theorem 1(b) further shows the threshold on the level of uncertainty over which an open tournament is optimal (i.e., $\overline{\alpha}_K$) decreases as the cost function parameter b increases. This shows that an open tournament is more desirable in settings where the agent's cost of effort increases faster; for instance, when seeking solutions to difficult problems, improving solution quality requires a significant increase in the agent's cost of effort.

We note that our finding related to the number of contributors K contrasts sharply with the result in the literature. To capture cases where the organizer aims to utilize multiple solutions, Terwiesch and Xu (2008) consider a weighted combination of the performance of the best solution and the average performance of all solutions, while noting on page 1534 that "[i]t seems plausible that the seeker might be interested in the best K submitted solutions. These cases lead to qualitatively similar results, yet are analytically intractable." We complement Terwiesch and Xu (2008) by modeling the organizer's utility as an explicit function of "the best K submitted solutions" and still conducting a tractable analysis of this model. We show that a larger number of contributors reinforces the diversity effect and increases the value of an open tournament. This is *qualitatively different* from the result of Terwiesch and Xu (2008) that an open tournament is less likely to be optimal when the organizer's weight on the best solution decreases, or equivalently, when the weight on the average solution increases. A primary reason for these seemingly opposite results is that their model approximates "the best K submitted solutions" because the average performance is computed by averaging the performance of all solutions including *poor* solutions (which do not belong to the best K submitted solutions).

Theorem 1(c) shows that an open tournament is optimal when I_N in (5) is increasing in N (which means e^* is increasing in N as discussed below (5)) up to some N^* ($\geq \overline{N}$). In this case, more participants to the tournament not only provide a more diverse set of solutions to the organizer, but also induce higher effort from participants. Thus, an organizer can benefit from an open tournament even when the output uncertainty is so low that there is little diversity among agents' solutions. This is also true when the organizer's objective is to maximize the *average* output of all agents, where the impact of diversity disappears completely.

Corollary 1. *Suppose that I_N in (5) is increasing in N up to some N^* ($\geq \overline{N}$). When the organizer maximizes the average output of all agents, an open tournament is optimal.*

Our results also have implications about when it is optimal to restrict entry to a tournament. Specifically, Theorem 1 shows that there are two conditions for restricting entry. First, the threshold $\overline{\alpha}_K$ should be positive. This is guaranteed when I_N in (5) is decreasing in N for all N ($\leq \overline{N}$). Second, the uncertainty-effort ratio and the number of contributors should be sufficiently small (i.e., $\alpha < \overline{\alpha}_K$). In this case, as Theorem 1 formally shows and Figure 1 illustrates, the organizer may choose to restrict entry. The following corollary formally presents the two conditions for the optimality of restricted entry.[4]

[4] When the organizer restricts entry (i.e., $N < \overline{N}$), there exist multiple equilibria where N agents participate and ($\overline{N} - N$) agents do not. The analysis of any of these equilibria yields the same insights, because the organizer's utility is the same under any of these equilibria.

COROLLARY 2. *Suppose that I_N in (5) is decreasing for all N $(\leq \overline{N})$. Then, $\overline{\alpha}_K > 0$, and for any scale transformation of the output shock $\widetilde{\xi}_i$ with scale parameter $\alpha \in (0, \overline{\alpha}_K)$, restricted entry is optimal.*

We next analyze how the equilibrium effort e^* changes with the number of participants N. This analysis will help us better understand the conditions given in Theorem 1(c), Corollary 1, and Corollary 2. As discussed earlier below (5), whether $e^{*,N}$ increases or decreases with N depends on whether I_N defined in (5) increases or decreases with N. How I_N changes with N depends on the distribution of agent's uncertainty. For instance, when the output shock $\widetilde{\xi}_i$ follows a Gumbel distribution with mean 0 and scale parameter μ, $I_N = \frac{N-1}{\mu N^2}$ is decreasing in N, and so is e^*. In contrast, when $\widetilde{\xi}_i$ follows a Weibull distribution with mean 0, shape parameter $\beta = 1$, and scale parameter μ (i.e., $h(s) = \frac{1}{\mu} \exp\left\{-(\frac{\mu - s}{\mu})\right\}$ as in the literature on extreme-value distributions and new product development (e.g., Dahan and Mendelson 2001)), $I_N = \frac{N-1}{\mu N}$ as well as e^* is increasing in N.[5] This example illustrates a counter-intuitive result that more participants can induce larger effort from agents. The reason is as follows. From (4), the agent's marginal benefit of increasing effort is $A \left.\frac{\partial P^N[e_i, e^*]}{\partial e_i}\right|_{e_i = e^*} = A r'(e^*) I_N$. For any given award A, this increases with $\left.\frac{\partial P^N[e_i, e^*]}{\partial e_i}\right|_{e_i = e^*} = r'(e^*) I_N$, which represents the marginal impact of additional effort on the winning probability. When $I_{N+1} > I_N$ (i.e., I_N increases with N), $\left.\frac{\partial P^{N+1}[e_i, e^*]}{\partial e_i}\right|_{e_i = e^*} > \left.\frac{\partial P^N[e_i, e^*]}{\partial e_i}\right|_{e_i = e^*}$, implying that one unit of effort increases the winning probability more when there are $(N+1)$ participants than when there are N participants; consequently, agents exert larger effort with $(N+1)$ participants than with N participants. Thus, although more participants always lower the probability of winning for agents under *any* distribution of the output shock $\widetilde{\xi}_i$, more participants do *not always* lead agents to reduce their effort.[6]

Building on this observation, Proposition 1(a) presents a necessary and sufficient condition on the output shock $\widetilde{\xi}_i$ under which more participants induce (weakly) lower effort, and Proposition 1(b) presents sufficient conditions under which more participants induce higher effort.

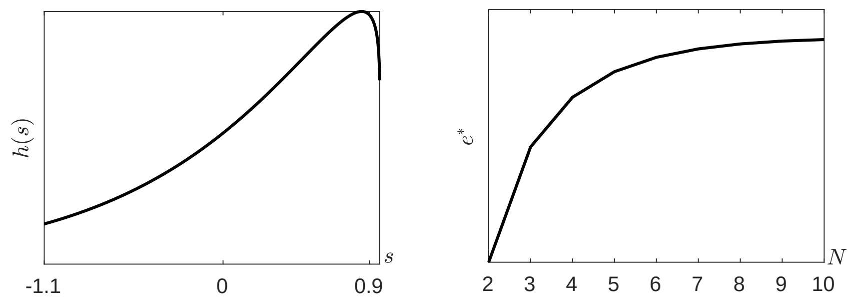
PROPOSITION 1. *(a) The equilibrium effort e^* is non-increasing for any $N \geq 2$ if and only if the density $h(s)$ of the output shock $\widetilde{\xi}_i$ satisfies*

$$\int_{s \in \Xi} (1 - H(s)) H(s) h'(s) ds \leq 0. \tag{7}$$

When the inequality in (7) is satisfied strictly, e^* is decreasing for any $N \geq 2$.

[5] The Weibull distribution has an alternative version with a density function $h(s) = (\beta/\mu)(s/\mu)^{\beta-1} exp\{-(s/\mu)^\beta\}$. Under this alternative Weibull distribution, I_N is decreasing in N.

[6] When the organizer maximizes the average output and the shock $\widetilde{\xi}_i$ follows a Gumbel distribution, Terwiesch and Xu (2008) show that an open tournament is *always* suboptimal (i.e., restricted entry is always optimal). Corollary 1 together with Proposition 1 indicates that this result may not hold under a general distribution of $\widetilde{\xi}_i$.



(a) Density function $h(s)$ (b) e^* as a function of N

Figure 2 The density function $h(s)$ and the equilibrium effort e^* when the output shock $\widetilde{\xi}_i$ follows a Weibull distribution with mean 0, scale parameter 1, and shape parameter 1.1. Parameters used: $\theta = b = c = 1$.

(b) e^* is increasing up to some N^* if $E[(h'/h)(\widetilde{\xi}_{(1)}^{N^*})] \geq 0$ (where $N^* = \infty$ if h is increasing) or if h is a symmetric function of some density function h_r with respect to y-axis (i.e., $h(s) \equiv h_r(-s)$ for all s) where h_r satisfies (7) strictly.

Condition (7) is satisfied by any symmetric log-concave density (e.g., normal, logistic) as well as Gumbel and exponential densities (see Remark EC.2 in the Online Appendix). This implies that when agents have roughly symmetric or right-skewed distributions for output uncertainty, they tend to decrease effort with more participants.

Whenever the necessary and sufficient condition given in (7) is violated, the equilibrium effort e^* is increasing in N up to some N^*. Proposition 1(b) shows that this condition is violated by any density with $E[(h'/h)(\widetilde{\xi}_{(1)}^{N})] \geq 0$ or any density $h(s)$ of which the symmetric function with respect to y-axis, $h(-s)$, satisfies (7) strictly. For example, when the output shock has an increasing density function such as the Weibull distribution in the above example (which satisfies the former condition for any N) or a left-skewed density function as in Figure 2 (which satisfies the latter condition), agents' uncertainty is likely to contribute a positive value to their solutions, so the equilibrium effort e^* may increase with more participants. The intuition is as follows. The equilibrium effort e^* depends on the marginal impact of effort on winning an award, and more participants have two opposing effects on the marginal impact of effort. When the number of participants increases, additional effort gives the agent an edge against more competitors, pushing the marginal impact of effort up; yet the overall probability of winning decreases, pulling the marginal impact of effort down. When the agent's uncertainty is likely to contribute a positive value to the agent's solution, the agent is likely to receive a favorable output, so more participants decrease the agent's probability of winning slowly. Thus, the agent increases effort to gain an edge against more competitors, and in this case, by Theorem 1, an open tournament is optimal for the organizer.

Our results indicate that when agents' output uncertainty is likely to contribute a positive value to their solutions, more participants may induce agents to increase effort. We may examine the problem in a tournament to see if the output uncertainty has this property or not. For instance, as discussed in §1, a left-skewed density function such as the Weibull distribution is suitable for modeling innovation processes where the upside potential for a solution is limited (Dahan and Mendelson 2001). In practice, this property can be satisfied by Topcoder coding challenges such as the Arcelik Exploratory Testing Challenge where the upside potential of agents' output is limited. Our result may offer a plausible explanation for why such Topcoder coding challenges are open tournaments.

Our findings not only help explain some open tournaments in practice, but also are supported by experimental results. Specifically, List et al. (2020) observed that participants increased their effort level when the number of participants in a tournament increased from 2 to 4, and participants knew that they had a high probability of receiving a good draw. List et al. (2020) interpret skewness of the density function as an indicator for agents' beliefs of good outcomes in their experiment. This insight is in line with our findings. (For a detailed review of other experimental studies, we refer the reader to Dechenaux et al. (2015).) List et al. (2020) also have an analytical result under a linear effort function and an output shock with a monotonic density function over a symmetric finite support. They show that when the density function has a positive (resp., negative) slope in the entire support, more participants induce higher (resp., lower) effort from agents. (Gerchak and He (2003) also show the same analytical result. They further show that when the density function is symmetric, more participants induce lower effort from agents.) Our Proposition 1 generalizes their analysis to a general class of distributions, and highlights how the outcomes observed in their experiments are not anomalies but the outcome of rational decision-making.

4. Conclusion

In this paper, we study tournaments in which a tournament organizer seeks solutions to an innovation-related problem from a group of agents. The organizer faces a key tradeoff concerning the number of participants to admit in a tournament. Running an open tournament, which allows anyone who wishes to participate to do so, increases the diversity of solutions, but might also induce agents to reduce their effort. Possibly for that reason, we observe mixed policies in practice, where some tournaments are open and others restrict entry.

Our modeling approach is quite general allowing for a general class of distributions (with either a log-concave or increasing density function) to describe the uncertainty faced by participants. We also allow the utility of the organizer to depend on a general number of contributors. The generality of our model is key as our main finding highlights the importance of the level of uncertainty relative to the impact of effort (i.e., uncertainty-effort ratio), the skewness of uncertainty, and the number of contributors in determining whether to run an open tournament. Specifically, we find that a tournament should be open when an innovation problem involves a large uncertainty-effort ratio, when the tournament features a small uncertainty-effort ratio but many contributors, or when agents increase effort with more participants in the tournament. We show that agents may increase effort with more participants when they face a high likelihood that their uncertainty contributes a positive value to their solutions (i.e., their uncertainty has a left-skewed distribution). We further show that restricted entry is optimal when a tournament features a low uncertainty-effort ratio, a small number of contributors, and a symmetric or right-skewed distribution of uncertainty. This result may help explain why some tournaments restrict entry in practice. Taken together, our results have a clear implication for practitioners: in designing a tournament, organizers should take into account the level and type of agents' uncertainty and the number of contributors.

Our paper may lead to several interesting future research directions. First, our paper considers the number of contributors as exogenous, and as a future research avenue, one may consider a different case in which the number of contributors is determined endogenously either before or after the tournament. In one approach, the organizer determines the optimal number of contributors ex-ante before conducting a tournament. This approach can be handled by extending our current model: The organizer can choose ex-ante the optimal value of contributors that results in the highest expected utility. As an alternative approach, the organizer may choose a rule about how to select contributors before conducting a tournament, and determine the number of contributors ex-post after collecting all solutions. Second, while our paper focuses on when it is optimal for the organizer to run an open tournament or to restrict entry, one may extend it further by examining specific approaches to restricting entry. For example, the organizer may (i) restrict the number of participants to a certain number and accept participants in a first-come-first-served basis, (ii) invite only a certain group of agents to participate, (iii) restrict participants to a certain geographical region, or (iv) apply some preselection mechanism with possibly a noisy performance threshold. The first three approaches can be directly captured in our current model and analysis; the fourth one may require a different model and analysis, so we leave it for future research. Third, while our paper assumes homogenous agents to tease out the impact of output uncertainty on agents' effort and the organizer's incentive to hold an open tournament, there are some papers in the literature that analyze the impact of agents' heterogeneity by assuming that heterogeneous agents produce deterministic outputs (e.g., Körpeoğlu and Cho 2018). Recently, Ales et al. (2019) develop a framework that integrates both agent heterogeneity and uncertainty into a general form. Yet, characterizing equilibrium in such a general model remains challenging and such an endeavor can be an important future research direction.

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Appendix. Proofs

Proof of Theorem 1. *(a)* To prove that an open tournament is optimal, we show that for any finite N and D $(>N)$, there exists a scale transformation such that the organizer's utility with D participants is higher than that with N participants. Thus, we need

$$U_o^{D-N} \equiv \left(Kr(e^{*,D}) + \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{D}] - A^{*,D}\right) - \left(Kr(e^{*,N}) + \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{N}] - A^{*,N}\right) \geq 0, \tag{8}$$

where $e^{*,N}$ is the equilibrium effort when there are N participants and the winner award is optimally chosen as $A^{*,N}$. Under Assumption 1, for any number of participants N, we can show that $A^{*,N} = \frac{K\theta}{b}$ and $e^{*,N} = \left(\frac{K\theta^2 I_N}{cb^2}\right)^{\frac{1}{b}}$. Thus, for some scale transformation $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ of the output shock $\widetilde{\xi}_i$ with scale parameter α, (8) can be written as

$$U_o^{D-N}(\alpha) = \frac{K\theta}{b}\log\left(\frac{I_D}{I_N}\right) + \alpha\sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{D} - \widetilde{\xi}_{(j)}^{N}] \geq 0, \tag{9}$$

which is satisfied if $\alpha \geq \frac{K\theta}{b}\log(I_N/I_D) / \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{D} - \widetilde{\xi}_{(j)}^{N}]$. Thus, the organizer's utility is maximized at $N = \overline{N}$ if and only if $\alpha \geq \overline{\alpha}_K$, where

$$\overline{\alpha}_K \equiv \max\left\{\frac{K\theta}{b} \max_{N\in\{K,K+1,\ldots,\overline{N}\}} \left\{\log(I_N/I_{\overline{N}}) / \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{\overline{N}} - \widetilde{\xi}_{(j)}^{N}]\right\}, 0\right\}. \tag{10}$$

(b) From (10), we see that $\overline{\alpha}_K$ is non-decreasing in θ and non-increasing in b. To show that $\overline{\alpha}_K$ is non-increasing in K, it suffices to prove that for any scale parameter α such that an open tournament is optimal for K $(<N)$ contributors, an open tournament is also optimal for $(K+1)$ contributors. Suppose that an open tournament is optimal for K contributors and for some scale transformation $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ of the output shock $\widetilde{\xi}_i$. Then, from (9), we obtain that for all $N < \overline{N}$,

$$U_o^{\overline{N}-N}[K] = \frac{K\theta}{b}\log\left(\frac{I_{\overline{N}}}{I_N}\right) + \sum_{j=1}^{K} E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}] \geq 0, \tag{11}$$

where $U_o^{\overline{N}-N}[K]$ is the difference in the organizer's utility with K contributors when the number of participants increases from N to $\overline{N}$. Furthermore, for $K+1$ contributors,

$$U_o^{\overline{N}-N}[K+1] = \frac{(K+1)\theta}{b}\log\left(\frac{I_{\overline{N}}}{I_N}\right) + \sum_{j=1}^{K+1} E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}] = \frac{\theta}{b}\log\left(\frac{I_{\overline{N}}}{I_N}\right) + E[\widehat{\xi}_{(K+1)}^{\overline{N}} - \widehat{\xi}_{(K+1)}^{N}] + U_o^{\overline{N}-N}[K].$$

By Lemma EC.4 in the Online Appendix, $E[\widehat{\xi}_{(K+1)}^{\overline{N}} - \widehat{\xi}_{(K+1)}^{N}] > E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}]$ for any $j < K+1$; so,

$$E[\widehat{\xi}_{(K+1)}^{\overline{N}} - \widehat{\xi}_{(K+1)}^{N}] > \frac{1}{K}\sum_{j=1}^{K} E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}] \geq -\frac{\theta}{b}\log\left(\frac{I_{\overline{N}}}{I_N}\right), \tag{12}$$

where the last inequality follows from (11). The combination of (11) and (12) yields the desired result that $U_o^{\overline{N}-N}[K+1] > 0$ for any N.

(c) Suppose I_N is increasing in N up to some N^* $(\geq \overline{N})$. Then $\log(I_N/I_{\overline{N}}) \leq 0$ for all $N \in \{K, K+1, \ldots, \overline{N}\}$. We also have $E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}] > 0$. Thus, from (10), $\overline{\alpha}_K = 0$. ■

Proof of Corollary 1. A sufficient condition for an open tournament to be optimal is that the organizer's utility, which can be written as $U_o = (1/N)\sum_{i=1}^{N} y_i - A = r(e^{*,N}) - A$, is increasing in N up to $\overline{N}$. Under the stated assumptions on r and ψ, for any number of participants N ($< \overline{N}$), it is easy to show that the optimal award is $A^{*,N} = \theta$ and the equilibrium effort is $e^{*,N} = \frac{\theta^2 I_N}{c}$. If the organizer maximizes the average output of all agents, the change in the organizer utility when the number of participants increases from N to $N+1$ can be written as $U_o^{(N+1)-N} \equiv \theta \log\left(\frac{I_{N+1}}{I_N}\right)$. By definition of N^*, for all $N < N^*$, $I_{N+1} > I_N$, and hence $U_o^{(N+1)-N} > 0$. Thus, since $\overline{N} \leq N^*$, an open tournament is optimal. ■

Proof of Proposition 1. Recall from §3 that equilibrium effort e^* satisfies $\frac{\psi'(e^*)}{r'(e^*)} = AI_N$, and that e^* is decreasing (resp., increasing) in N if I_N is decreasing (resp., increasing) in N.

(a) Suppose that (7) holds. We will show that $I_{N+1} \leq I_N$ for any $N \geq 2$. Applying integration by parts on (7) yields the following difference equation:

$$I_{N+1} - I_N = \int_{\underline{s}}^{\overline{s}} (1 - H(s))\, H(s)^{N-1}\, h'(s)\, ds, \quad \forall N \geq 2. \tag{13}$$

Since both $H(s)$ and $(1 - H(s))$ are positive, (13) implies that when $h(s)$ is decreasing, constant or increasing, I_N is decreasing, constant or increasing in N, respectively. (This also proves the result about increasing density $h(s)$ in part (b).) Thus, we will prove part (a) when h is non-monotonic and log-concave, which implies that there exists $s_0 \in (\underline{s}, \overline{s})$, such that $h' \geq 0$ for $s < s_0$, and $h' \leq 0$ for $s > s_0$ (i.e., h is unimodal; see, e.g., Cule et al. 2010). When $N \geq 2$,

$$\begin{aligned} I_{N+1} - I_N &= \int_{\underline{s}}^{s_0} (1 - H(s))\, H(s)^{N-1}\, h'(s)\, ds + \int_{s_0}^{\overline{s}} (1 - H(s))\, H(s)^{N-1}\, h'(s)\, ds \\ &\leq \int_{\underline{s}}^{s_0} (1 - H(s))\, H(s) H(s_0)^{N-2}\, h'(s)\, ds + \int_{s_0}^{\overline{s}} (1 - H(s))\, H(s) H(s_0)^{N-2}\, h'(s)\, ds \\ &= H(s_0)^{N-2} \int_{\underline{s}}^{\overline{s}} (1 - H(s))\, H(s) h'(s)\, ds \leq 0, \end{aligned}$$

where the first inequality holds because density h is unimodal and non-monotonic, and the last inequality holds from (7).

Suppose that the effort e^* is non-increasing for any $N \geq 2$. Then, (13) is non-positive for all $N \geq 2$. The right-hand side of (13) is the same as the left-hand side of (7) for $N = 2$, so (7) holds.

(b) Suppose that $E[(h'/h)(\widetilde{\xi}_{(j)}^{N^*})] > 0$ for some N^*. Note that when $h(s)$ is increasing, $E[(h'/h)(\widetilde{\xi}_{(j)}^{N^*})] > 0$ is always satisfied so $N^* = +\infty$. In this case, as we prove in part (a), e^* is increasing. Suppose h is log-concave. Using integration by parts, we can write I_N as follows:

$$\begin{aligned} I_N &= \int_{\underline{s}}^{\overline{s}} (N-1) H(s)^{N-2} h(s)^2 ds = H(s)^{N-1} h(s)\Big|_{\underline{s}}^{\overline{s}} - \int_{\underline{s}}^{\overline{s}} H(s)^{N-1} h'(s) ds \\ &= \lim_{s \to \overline{s}} h(s) - \frac{1}{N} \int_{\underline{s}}^{\overline{s}} N H(s)^{N-1} h(s) \frac{h'(s)}{h(s)} ds = \lim_{s \to \overline{s}} h(s) - \frac{1}{N} E\left[\frac{h'}{h}(\widetilde{\xi}_{(1)}^{N})\right]. \end{aligned}$$

Then, for any N, we can write the following difference equation:

$$I_{N+1} - I_N = \frac{1}{N} E\left[\frac{h'}{h}(\widetilde{\xi}^{N}_{(1)})\right] - \frac{1}{N+1} E\left[\frac{h'}{h}(\widetilde{\xi}^{N+1}_{(1)})\right]. \tag{14}$$

Note that (h'/h) is decreasing because h is log-concave. Thus, because $\widetilde{\xi}^{N+1}_{(1)}$ first-order stochastically dominates $\widetilde{\xi}^{N}_{(1)}$ and not vice versa, by Theorem 1.A.3 of Shaked and Shanthikumar (2007), $E\left[\frac{h'}{h}(\widetilde{\xi}^{N}_{(1)})\right] > E\left[\frac{h'}{h}(\widetilde{\xi}^{N+1}_{(1)})\right]$. Then, from (14), whenever $E\left[\frac{h'}{h}(\widetilde{\xi}^{N+1}_{(j)})\right] \geq 0$, we have $I_{N+1} > I_N$. Similarly, when $E\left[\frac{h'}{h}(\widetilde{\xi}^{N^*}_{(j)})\right] \geq 0$, we have $E\left[\frac{h'}{h}(\widetilde{\xi}^{2}_{(j)})\right] > E\left[\frac{h'}{h}(\widetilde{\xi}^{3}_{(j)})\right] > \cdots > E\left[\frac{h'}{h}(\widetilde{\xi}^{N^*}_{(j)})\right] \geq 0$, which implies from (14) that $I_{N^*} > I_{N^*-1} > \cdots > I_2$. Therefore, e^* is increasing up to N^*.

Let the density function h_r be the symmetric function of h with respect to y-axis; i.e., $h_r(s) = h(-s)$ for all s. Let $H(s)$ and $H_r(s)$ be the corresponding distribution functions and $\Xi = [\underline{s}, \overline{s}]$ and $\Xi_r = [\underline{s}_r, \overline{s}_r]$ be the supports for $h(s)$ and $h_r(s)$, respectively. By definition, we have $1 - H(-s) = H_r(s)$, $-h'(-s) = h'_r(s)$, $\overline{s} = -\underline{s}_r$, and $\underline{s} = -\overline{s}_r$. Suppose that h_r satisfies (7) strictly; i.e.,

$$\int_{\underline{s}_r}^{\overline{s}_r} (1 - H_r(s))H_r(s)h'_r(s)ds < 0. \tag{15}$$

Using symmetry of h_r and h, (15) can be written as:

$$\int_{\underline{s}_r}^{\overline{s}_r} -H(-s)(1 - H(-s))h'(-s)ds < 0. \tag{16}$$

Making a change of variables as $t = -s$, and noting that $ds = -dt$, (16) becomes

$$\int_{-\underline{s}_r}^{-\overline{s}_r} H(t)(1 - H(t))h'(t)dt = -\int_{-\overline{s}_r}^{-\underline{s}_r} H(t)(1 - H(t))h'(t)dt < 0. \tag{17}$$

Thus, $h(s)$ violates (7) because (17) can be rewritten as $\int_{\underline{s}}^{\overline{s}} (1 - H(t))H(t)h'(t)dt > 0$. Because the left-hand side of (7) equals $I_3 - I_2$, I_N as well as e^* is increasing up to some $N^* \geq 3$. ■

Online Appendix

EC.1. Existence of Equilibrium

In this section, we prove the concavity of the agent's utility function and the existence of a pure strategy Nash equilibrium in two lemmas.

LEMMA EC.1. *Suppose that the output shock $\widetilde{\xi}_i$ is transformed to $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ via a scale transformation with $\alpha > 0$. For all $N \in \{2, ..., \overline{N}\}$, ceteris paribus, when $-r''/(r')^2$, ψ'', and α are sufficiently large, agent's utility function $U_a(e_i) = AP^N[e_i, e^*] - \psi(e_i)$ is concave in agent's effort e_i.*

Proof. Take an arbitrary N. We will show sufficient conditions for concavity of $U_a(e_i)$ using its second derivative, $U_a''(e_i) = A\frac{\partial^2 P^N[e_i,e^*]}{\partial e_i^2} - \psi''(e_i)$. After a scale transformation of the output shock to $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ with α, $P^N[e_i, e^*] = \int_{s\in\Xi} H\left(s + \frac{r(e_i)-r(e^*)}{\alpha}\right)^{N-1} h(s)ds = E\left[H_{(1)}^{N-1}\left(\widetilde{\xi}_i + \frac{r(e_i)-r(e^*)}{\alpha}\right)\right]$. The derivative of $P^N[e_i, e^*]$ with respect to e_i is $\frac{\partial P^N[e_i,e^*]}{\partial e_i} = \frac{r'(e_i)}{\alpha} E\left[h_{(1)}^{N-1}\left(\widetilde{\xi}_i + \frac{r(e_i)-r(e^*)}{\alpha}\right)\right]$. Then, the second derivative of $P^N[e_i, e^*]$ with respect to e_i is (where $r^* = r(e^*)$)

$$\frac{\partial^2 P^N[e_i, e^*]}{\partial e_i^2} = \left(\frac{r'(e_i)}{\alpha}\right)^2 E\left[\left(h_{(1)}^{N-1}\right)'\left(\widetilde{\xi}_i + \frac{r(e_i) - r^*}{\alpha}\right)\right] + \frac{r''(e_i)}{\alpha} E\left[h_{(1)}^{N-1}\left(\widetilde{\xi}_i + \frac{r(e_i) - r^*}{\alpha}\right)\right]. \quad \text{(EC.1)}$$

Noting that $U_a''(e_i) = A\frac{\partial^2 P^N[e_i,e^*]}{\partial e_i^2} - \psi''(e_i)$, where $\psi''(e_i) \geq 0$ due to convexity of ψ, there are three sufficient conditions that make $U_a'' < 0$. First, as α gets large, both expectation terms in (EC.1) converge, and $E\left[h_{(1)}^{N-1}\left(\widetilde{\xi}_i + \frac{r(e_i)-r^*}{\alpha}\right)\right]$ converges to a positive constant. Furthermore, when r is strictly concave, $(r'(e_i)/\alpha)^2$ (> 0) approaches 0 faster than $r''(e_i)/\alpha$ (< 0). Thus, for sufficiently large α, U_a'' as well as $\partial^2 P^N[e_i, e^*]/\partial e_i^2$ becomes negative, and hence U_a becomes concave. (Note that Lazaer and Rosen (1981) also mention this condition as a sufficient condition for the existence of equilibrium in their footnote 2.) Second, $P^N[e_i, e^*]$ is concave when $-r''/(r')^2$ is sufficiently large. Third, regardless of $\partial^2 P^N[e_i, e^*]/\partial e_i^2$, the utility U_a is concave for sufficiently convex $\psi(e_i)$, i.e., when $\psi''(e_i)$ is sufficiently large. ■

REMARK EC.1. Although it is not possible to represent the above sufficient conditions as a closed-form function of exogenous parameters in our general model, it is possible to do so for specific cases of output shock $\widetilde{\xi}_i$. For instance, when $\widetilde{\xi}_i$ follows a Gumbel distribution with scale parameter μ under general effort function r and cost function ψ, the condition $r'' + (r')^2/\mu \leq 0$ (which is guaranteed under the conditions in Lemma EC.1) is sufficient for the existence of equilibrium for all N. This condition is satisfied, for example, when $r(e) = \theta\log(e)$ and $\psi(e) = ce$ as in Terwiesch and Xu (2008) or when $r(e) = 1 - \theta\exp(-e)$ (i.e., constant absolute risk-aversion (CARA) function) and $\psi(e) = ce^b$ $(b > 1)$ under $\mu \geq \theta$.

The following lemma proves the existence of equilibrium.

LEMMA EC.2. *Suppose that either* $\lim_{e\to 0} r'(e) = +\infty$ *or* $\lim_{e\to 0} \psi'(e) = 0$. *Suppose also that* $U_a(e_i)$ *is pseudo-concave in* e_i *and* $\lim_{e\to+\infty} \psi'(e) = +\infty$. *Then there exists* e^* *that solves (4), and* e^* *satisfies (5). Furthermore, the solution to (4) is the unique symmetric Nash equilibrium.*

Proof. From Theorem 1.2 of Fudenberg and Tirole (1991), a pure strategy Nash equilibrium exists if each agent i's action set (i.e., set of possible effort levels) is non-empty, convex, and compact subset of the Euclidean space, and the agent's utility U_a is quasi-concave in effort e_i. Recall that $U_a(e_i) = AP^N[e_i, e^*] - \psi(e_i)$. Because agent's expected payment $AP^N[e_i, e^*]$ is bounded by A but agent's cost of effort is unbounded, there exists $\bar{e}$ such that $U_a(e_i) < 0$ for all $e_i > \bar{e}$. Because $U_a(0) = 0$, without loss of optimality, each agent i's action set can be restricted to $[0, \bar{e}]$, which is non-empty, convex, and compact. Because U_a is pseudo-concave, it is also quasi-concave. Therefore, a pure strategy Nash equilibrium exists. To show that there exists e^* that solves (4) and satisfies (5), we take the first-order conditions of the agent's problem evaluated at $e_i = e^*$,

$$Ar'(e_i)E\left[h_{(1)}^{N-1}\left(\widetilde{\xi}_i + r(e_i) - r(e^*)\right)\right] - \psi'(e_i)\bigg|_{e_i=e^*} = Ar'(e^*)E\left[h_{(1)}^{N-1}(\widetilde{\xi}_i)\right] - \psi'^*) = 0, \qquad \text{(EC.2)}$$

which is sufficient for optimality due to pseudo-concavity of U_a. Therefore, if a solution to (EC.2) exists, then it is a pure strategy Nash equilibrium. Let $\Omega(e) = Ar'(e)E\left[h_{(1)}^{N-1}(\widetilde{\xi}_i)\right] - \psi'(e)$. Clearly $\Omega(e)$ is continuous in e because r' and and ψ' are continuous. Furthermore, $\lim_{e^*\to 0} \Omega(e) > 0$ because, by assumption, either $\lim_{e\to 0} r'(e) = +\infty$ or $\lim_{e\to 0} \psi'(e) = 0$. Also, $\lim_{e^*\to+\infty} \Omega(e) < 0$ because, by assumption, $\lim_{e\to+\infty} \psi'(e) = +\infty$. Thus, the Intermediate Value Theorem dictates that there exists e^* such that $\Omega(e^*) = 0$. Furthermore, because $\Omega'(e) < 0$, only a unique e^* can satisfy $\Omega(e^*) = 0$. Therefore, e^* that solves (4) is the unique symmetric Nash equilibrium. ■

EC.2. Extension to General Utility Function Form

In this section, we consider the following general utility function for the organizer: $U_o = u_o\left(Y^{(K)}\right) - \psi_o(A)$, where $Y^{(K)} = (y_{(1)}, y_{(2)} ..., y_{(K)})$, ψ_o is an increasing, continuously differentiable, and convex (including linear) function, and u_o is a non-decreasing, continuous, and almost everywhere differentiable function that satisfies $\frac{\partial u_o(Y^{(K)})}{\partial y_{(j)}} > 0$ when $y_{(j)} > 0$ for $j = 1, 2, \ldots, K$.[7] This utility function not only generalizes the utility function defined in §2, but also includes the following three interesting and commonly-used forms:

(1) Product complementarity: Product complementarity models in economics (e.g., Constant Elasticity of Substitution (CES) form $u_o(Y^{(K)}) = (\sum_{j=1}^{K} \omega_{(j)} y_{(j)}^{\rho})^{\frac{1}{\rho}}$ for $\omega_{(j)}, \rho > 0$) and marketing/operations (e.g., CES form with $\rho = 1$ in Desai et al. 2001) are useful when the outputs of contributors complement each other. For example, this model may be used to analyze innovative

[7] In case when the organizer cares only about agents' outputs that contribute positive utility to the organizer, one may use a utility function that is defined over $\mathbb{R}_+^K$ and let $u_o(Y^K) = u_o\left(\max\{y_{(1)}, 0\}, \max\{y_{(2)}, 0\}, ..., \max\left\{y_{(K)}, 0\right\}\right)$. As long as u_o satisfies the same assumptions as mentioned above, the results continue to hold.

ideas on how to produce, ship, and store vaccines for neglected tropical diseases in Grand Challenges Explorations (a tournament which solicit innovative solutions to fight global health challenges).

(2) Additive-separable risk aversion: Additively separable risk-aversion models in economics (e.g., Constant Relative Risk Aversion (CRRA) form $u_o\left(Y^{(K)}\right)=\sum_{j=1}^{K}\omega_{(j)}\frac{y_{(j)}^{1-a}}{1-a}$ for $\omega_{(j)}>0$, $a\in(0,1)$) can be used when the organizer is risk-averse and the outputs of contributors are non-monetary and non-complementary. In this model, CRRA parameter a determines the organizer's level of risk aversion with respect to the output of each contributor. When a is large, the organizer is highly risk-averse against the adverse outcome of each contributor.

(3) Portfolio management: Portfolio management models in finance (e.g., $u_o\left(Y^{(K)}\right)=\left(\sum_{j=1}^{K}\omega_{(j)}y_{(j)}\right)^a$ for $\omega_{(j)}>0$, $a\in(0,1]$ in Müller and Stoyan 2002) may be appropriate for the situation in which the organizer maximizes total value from the outputs of contributors having different weight $\omega_{(j)}$ for different rank $j=1,2,...,K$; for example, top rankers may receive better marketing and financial support in commercializing innovative product ideas from Staples Invention Quest (a tournament in which ordinary people submit innovative ideas for office products). This model with a small parameter a signals a high level of risk aversion on the overall outcome of projects. Such a model may be suitable when the tournament's outcome entails significant downside risk.

We examine the results in §3 under the general utility function U_o. Note from (4) that the agent's equilibrium effort e^* does not depend on a specific form of U_o, so Proposition 1 continues to hold. We next extend Theorem 1 to the general U_o. We make the following two assumptions:

Assumption EC.1. *u_o is homogenous of degree d; i.e., for any $Y^{(K)}$ and $\alpha>0$, $\alpha^d u_o\left(Y^{(K)}\right)=u_o\left(\alpha Y^{(K)}\right)$.*

Assumption EC.2. *For any $\sigma>0$, $\lim_{\alpha\to\infty} r\left(\left(\frac{r'}{\psi'}\right)^{-1}(\sigma\alpha)\right)/\alpha=r_0\in\mathbb{R}$.*

Assumption EC.1 is satisfied by all of the special cases of U_o discussed above. Assumption EC.2 concerns the agent's effort function r and cost function ψ, and it requires that $r((r'/\psi')^{-1})$ does not diverge to negative infinity faster than linearly. The assumption is trivially satisfied when r is bounded from below. It is also satisfied by the setting in Assumption 1.

Corollary EC.1. *Suppose that Assumptions EC.1 and EC.2 hold. For any fixed award A, any general utility function U_o, and any distribution H of output shock $\widetilde{\xi}_i$, there exists $\overline{\alpha}$ such that under a scale transformation of $\widetilde{\xi}_i$ with $\alpha\geq\overline{\alpha}$, an open tournament with unrestricted entry is optimal.*

Proof. Let $y_{(j)}^N=r(e^{*,N})+\widetilde{\xi}_{(j)}^N$, where $e^{*,N}$ is the equilibrium effort when there are N participants and the winner award is A. To prove that an open tournament is optimal, we will show that for any finite N and D $(>N)$, there exists a scale transformation such that the organizer's utility with D participants is higher than that with N participants. Thus, we want to show

$$U_o^{D-N}\equiv E\left[u_o\left(y_{(1)}^D,\ldots,y_{(K)}^D\right)-\psi_o(A)\right]-E\left[u_o\left(y_{(1)}^N,\ldots,y_{(K)}^N\right)-\psi_o(A)\right]\geq 0. \qquad \text{(EC.3)}$$

We will show that there exists a scale transformation $\alpha\widetilde{\xi}_i$ $(\alpha > 1)$ of the output shock $\widetilde{\xi}_i$ for which condition (EC.3) is satisfied. After the scale transformation and simplifications, (EC.3) becomes

$$U_o^{D-N}(\alpha) \equiv E\left[u_o\left(r(e^{*,D}) + \alpha\widetilde{\xi}^D_{(1)}, ..., r(e^{*,D}) + \alpha\widetilde{\xi}^D_{(K)}\right)\right] - E\left[u_o\left(r(e^{*,N}) + \alpha\widetilde{\xi}^N_{(1)}, ..., r(e^{*,N}) + \alpha\widetilde{\xi}^N_{(K)}\right)\right].$$

u_o is homogenous of degree d by Assumption EC.1. Thus, if we divide both sides by α^d, take the limit as α approaches infinity, and note that $\lim_{\alpha\to\infty} r(e^{*,N})/\alpha = \lim_{\alpha\to\infty} r\left(\left(\frac{r'}{\psi'}\right)^{-1}\left(\frac{\alpha}{AI_N}\right)\right)/\alpha = r_0 \in \mathbb{R}$ for all N by Assumption EC.2, we obtain the following relationship:

$$\lim_{\alpha\to\infty} U_o^{D-N}(\alpha)/\alpha^d = E\left[u_o\left(r_0 + \widetilde{\xi}^D_{(1)}, ..., r_0 + \widetilde{\xi}^D_{(K)}\right)\right] - E\left[u_o\left(r_0 + \widetilde{\xi}^N_{(1)}, ..., r_0 + \widetilde{\xi}^N_{(K)}\right)\right]. \quad \text{(EC.4)}$$

Let $\leq_{st}$ denote the first order stochastic dominance. By definition of order statistics, we have

$$\widetilde{\xi}^N_{(j)} \leq_{st} \widetilde{\xi}^D_{(j)} \Longrightarrow \widetilde{\xi}^N_{(j)} + r_0 \leq_{st} \widetilde{\xi}^D_{(j)} + r_0.$$

Since the output shock has a continuous density function and u_o is non-decreasing (and increasing for $y_{(j)} > 0$), (EC.4) is positive by Theorem 6.B.19 of Shaked and Shanthikumar (2007) (assuming that $E[u_o(r_0 + \widetilde{\xi}^N_{(1)}, ..., r_0 + \widetilde{\xi}^N_{(K)})] \in \mathbb{R}$ for all N). By continuity of u_o, there exists $\overline{\alpha}$ such that $U_o^{D-N}(\alpha)/\alpha^d > 0$ for all $\alpha > \overline{\alpha}$. Thus, we can use the steps in the proof of Theorem 1 to show that an open tournament with unrestricted entry is optimal for any $\alpha > \overline{\alpha}$. ■

EC.3. Extension to Multiple Awards

In this section, we generalize our results to a case in which the organizer offers multiple awards. Let $A_{(j)}$ be the award given for the j-th highest output where $A = \sum_{j=1}^N A_{(j)}$. Suppose that the organizer offers an award scheme with a set of $L \geq 1$ awards, $A^{(N)} = (A_{(1)}, ..., A_{(N)})$, such that $A_{(1)} \geq ... \geq A_{(L)} > 0$ and $A_{L+1} = ... = A_N = 0$. Let $P^N_{(j)}[e_i, e^*]$ be the probability that the output of agent i is the j-th highest output when agent i exerts effort e_i and all other $(N-1)$ agents exert the equilibrium effort e^*. We can compute this probability as

$$P^N_{(j)}[e_i, e^*] = \int_{s\in\Xi} \binom{N-1}{j-1} H(s + r(e_i) - r(e^*))^{N-j}(1 - H(s + r(e_i) - r(e^*)))^{j-1} h(s) ds. \quad \text{(EC.5)}$$

We start by characterizing the agent's equilibrium effort under the general award scheme. The following lemma generalizes (5).

LEMMA EC.3. *The equilibrium effort e^* under multiple awards is the unique solution of*

$$\frac{\psi'(e^*)}{r'(e^*)} = \sum_{j=1}^{N-1} \left[A_{(j)} - A_{(j+1)}\right] E\left[h(\widetilde{\xi}^{N-1}_{(j)})\right]. \quad \text{(EC.6)}$$

Proof. Under $(A_{(1)},...,A_{(N)})$, the agent's maximizes $U_a = \max_{e\in\mathbb{R}_+} \sum_{j=1}^N P^N_{(j)}[e, e^*] A_{(j)} - \psi(e)$. Substituting $e = e^*$ into the first-order condition of the agent's problem yields

$$\psi'(e^*) = \sum_{j=1}^N A_{(j)} \frac{\partial P^N_{(j)}}{\partial e}[e^*], \quad \text{(EC.7)}$$

where $\frac{\partial P_{(j)}^N}{\partial e}[e^*]$ is the partial derivative of $P_{(j)}^N[e_i, e^*]$ with respect to e_i evaluated at $e_i = e^*$. It is not difficult to show: $\frac{\partial P_{(1)}^N}{\partial e}[e^*] = r'(e^*)E\left[h(\widetilde{\xi}_{(1)}^{N-1})\right]$, $\frac{\partial P_{(j)}^N}{\partial e}[e^*] = r'(e^*)\left(E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right] - E\left[h(\widetilde{\xi}_{(j-1)}^{N-1})\right]\right)$ for $j \in \{2, ..., N-1\}$, and $\frac{\partial P_{(N)}^N}{\partial e}[e^*] = -r'(e^*)E[h(\widetilde{\xi}_{(N-1)}^{N-1})]$. Thus, (EC.7) can be written as $\psi'(e^*) = \sum_{j=1}^{N-1}\left[A_{(j)} - A_{(j+1)}\right] r'(e^*)E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]$. Therefore, the equilibrium effort satisfies (EC.6). ■

The corollary below generalizes our Proposition 1 to multiple awards and generalizes Proposition 1 of List et al. (2020) to a general award scheme.

COROLLARY EC.2. *(a) Proposition 1 extends to the case with multiple awards where $A_{(1)} = A_1$ and $A_{(j)} = A_2$ $(A_1 > A_2)$ for all $j > 1$.*
(b) For a general set of awards $\left(A_{(1)}, ..., A_{(N)}\right)$, such that $A_{(1)} \geq ... \geq A_{(L)} > 0$ and $A_{L+1} = ... = A_N = 0$, when the density $h(s)$ of the output shock $\widetilde{\xi}_i$ is increasing, decreasing, and constant in s, the equilibrium effort e^* is increasing, decreasing, and constant in N, respectively.

Proof. (a) Suppose that $A_{(1)} = A_1$ and $A_{(j)} = A_2$ $(A_1 > A_2)$ for all $j > 1$. Then by Lemma EC.3, $\frac{\psi'(e^*)}{r'(e^*)} = (A_1 - A_2)E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right] = (A_1 - A_2)I_N$. Thus, Proposition 1 holds.
(b) For a general set of awards $\left(A_{(1)}, ..., A_{(N)}\right)$, such that $A_{(1)} \geq ... \geq A_{(L)} > 0$ and $A_{L+1} = ... = A_N = 0$, when h is increasing, by Theorem 1.A.3 of Shaked and Shanthikumar (2007), $E\left[h(\widetilde{\xi}_{(j)}^{N})\right] > E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]$ because $\widetilde{\xi}_{(j)}^{N}$ first order stochastically dominates $\widetilde{\xi}_{(j)}^{N-1}$(and not vice versa). Thus,

$$\begin{aligned}\frac{\psi'(e^{*,N+1})}{r'(e^{*,N+1})} &= \sum_{j=1}^{N}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N})\right] = \sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N})\right] \\ &> \sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right] = \sum_{j=1}^{N}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right] = \frac{\psi'(e^{*,N})}{r'(e^{*,N})}\end{aligned}$$

Since $\frac{\psi'}{r'}$ is increasing, $e^{*,N+1} \geq e^{*,N}$. When h is decreasing (resp., constant), by Theorem 1.A.3 of Shaked and Shanthikumar (2007), $E\left[h(\widetilde{\xi}_{(j)}^{N})\right] < E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]$ (resp., $E\left[h(\widetilde{\xi}_{(j)}^{N})\right] = E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]$). Therefore, the result follows in a similar way as above. ■

Second, in the following corollary, we show that an open tournament is optimal if the output shock $\widetilde{\xi}_i$ is sufficiently spread out, generalizing Theorem 1(a) and Corollary EC.1.

COROLLARY EC.3. *Suppose that U_o takes the general form given in §EC.2. For any fixed set of awards $\left(A_{(1)}, ..., A_{(N)}\right)$, such that $A_{(1)} \geq ... \geq A_{(L)} > 0$ and $A_{L+1} = ... = A_N = 0$, Theorem 1(a) holds.*

Proof. As in Theorem 1 and Corollary EC.1, we want to show

$$U_o^{D-N} \equiv E\left[\sum_{j=1}^{K} u_{o,(j)}\left(r(e^{*,D}) + \widetilde{\xi}_{(j)}^{D}\right) - \psi_o(\sum_{j=1}^{N} A_{(j)})\right] - E\left[\sum_{j=1}^{K} u_{o,(j)}\left(r(e^{*,N}) + \widetilde{\xi}_{(j)}^{N}\right) - \psi_o(\sum_{j=1}^{N} A_{(j)})\right] \geq 0, \quad \text{(EC.8)}$$

where $e^{*,N}$ is the equilibrium effort when there are N participants and the set of awards is $A_{(1)}, ..., A_{(N)}$. Under the same set of awards, a sufficient condition for (EC.8) to hold is

$$\delta_{(j)} \equiv E\left[u_{o,(j)}\left(r(e^{*,D}) + \widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[u_{o,(j)}\left(r(e^{*,N}) + \widetilde{\xi}_{(j)}^{N}\right)\right] \geq 0 \text{ for all } j. \quad \text{(EC.9)}$$

We will show that there exists a scale transformation under which Condition (EC.9) is satisfied. Consider a scale transformation of the output shock to $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$. After the scale transformation,

$$\delta_{(j)}(\alpha) \equiv E\left[\sum_{j=1}^{K} u_{o,(j)}\left(r(e^{*,D}) + \alpha\widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[\sum_{j=1}^{K} u_{o,(j)}\left(r(e^{*,N}) + \alpha\widetilde{\xi}_{(j)}^{N}\right)\right]$$
$$= f_{(j)}(\alpha)\left(E\left[\widehat{u}_{o,(j)}\left(\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[\widehat{u}_{o,(j)}\left(\left(\frac{r(e^{*,N})}{\alpha} - \frac{r(e^{*,D})}{\alpha}\right) + \frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{N}\right)\right]\right).$$

By Assumption EC.1,

$$\lim_{\alpha\to\infty}\frac{\delta_{(j)}(\alpha)}{f_{(j)}(\alpha)} = \lim_{\alpha\to\infty}\left(E\left[\widehat{u}_{o,(j)}\left(\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[\widehat{u}_{o,(j)}\left(\left(\frac{r(e^{*,N})}{\alpha} - \frac{r(e^{*,D})}{\alpha}\right) + \frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{N}\right)\right]\right)$$
$$= \left(E\left[\widehat{u}_{o,(j)}\left(\lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[\widehat{u}_{o,(j)}\left(\lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{N}\right)\right]\right),$$

Let $\leq_{st}$ denote the first order stochastic dominance. By definition of order statistics, we have

$$\widetilde{\xi}_{(j)}^{N} \leq_{st} \widetilde{\xi}_{(j)}^{D} \Longrightarrow \widetilde{\xi}_{(j)}^{N} + \lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} \leq_{st} \widetilde{\xi}_{(j)}^{D} + \lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha},$$

because $\lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} = \lim_{\alpha\to\infty}\frac{r\left(\left(\frac{r'}{\psi'}\right)^{-1}\left(\alpha/\sum_{j=1}^{N-1}\left[A_{(j)}-A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]\right)\right)}{\alpha} \in \mathbb{R}$ by Assumption EC.2. Since the output shock has a continuous density function and $u_{o,(j)}$ is strictly increasing,

$$\lim_{\alpha\to\infty}\frac{\delta_{(j)}(\alpha)}{f_{(j)}(\alpha)} = E\left[\widehat{u}_{o,(j)}\left(\lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{D}\right)\right] - E\left[\widehat{u}_{o,(j)}\left(\lim_{\alpha\to\infty}\frac{r(e^{*,D})}{\alpha} + \widetilde{\xi}_{(j)}^{N}\right)\right] > 0,$$

by Theorem 1.A.3 of Shaked and Shanthikumar (2007). By continuity of $u_{o,(j)}$, and hence $\delta_{(j)}(\alpha)$, there exists $\overline{\alpha}$ such that $\frac{\delta_{(j)}(\alpha)}{\alpha^{d_{(j)}}} > 0$ for all $\alpha > \overline{\alpha}$ and $j \in \{1, ..., N\}$. Therefore, an open tournament is optimal for any $\alpha > \overline{\alpha}$. ∎

Finally, in the following corollary, we extend Theorem 1(b) to multiple awards.

Corollary EC.4. *For any fixed set of awards $\left(A_{(1)}, ..., A_{(N)}\right)$, such that $A_{(1)} \geq ... \geq A_{(L)} > 0$ and $A_{L+1} = ... = A_N = 0$, Theorem 1(b) holds.*

Proof. We prove that whenever unrestricted entry is optimal for K $(< N)$ contributors, it is also optimal for $K+1$ contributors. For any number of participants N $(< \overline{N})$, from Lemma EC.3, we have that $e^{*,N} = \left(\frac{\theta}{cb}\sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]\right)^{\frac{1}{b}}$, where $A_{(L+1)} = 0$. Suppose that unrestricted entry is optimal for K contributors and for some scale transformation $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ of the output shock $\widetilde{\xi}_i$. Then, adjusting (9), we obtain that for all $N < \overline{N}$,

$$U_o^{\overline{N}-N}[K] = \frac{K\theta}{b}\log\left(\frac{\sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{\overline{N}-1})\right]}{\sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]}\right) + \sum_{j=1}^{K}E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}] \geq 0, \tag{EC.10}$$

where $U_o^{\overline{N}-N}[K]$ is the difference in U_o with K contributors when the number of participants increases from N to $\overline{N}$. Furthermore, for $K+1$ contributors,

$$U_o^{\overline{N}-N}[K+1] = \frac{(K+1)\theta}{b}\log\left(\frac{\sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{\overline{N}-1})\right]}{\sum_{j=1}^{L}\left[A_{(j)} - A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]}\right) + \sum_{j=1}^{K+1}E[\widehat{\xi}_{(j)}^{\overline{N}} - \widehat{\xi}_{(j)}^{N}]$$

$$= \frac{\theta}{b}\log\left(\frac{\sum_{j=1}^{L}\left[A_{(j)}-A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{\overline{N}-1})\right]}{\sum_{j=1}^{L}\left[A_{(j)}-A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]}\right)+E[\widehat{\xi}_{(K+1)}^{\overline{N}}-\widehat{\xi}_{(K+1)}^{N}]+U_o^{\overline{N}-N}[K].$$

By Lemma EC.4 in the Online Appendix, $E[\widehat{\xi}_{(K+1)}^{\overline{N}}-\widehat{\xi}_{(K+1)}^{N}]>E[\widehat{\xi}_{(j)}^{\overline{N}}-\widehat{\xi}_{(j)}^{N}]$ for any $j<K+1$. So,

$$E[\widehat{\xi}_{(K+1)}^{\overline{N}}-\widehat{\xi}_{(K+1)}^{N}]>\frac{1}{K}\sum_{j=1}^{K}E[\widehat{\xi}_{(j)}^{\overline{N}}-\widehat{\xi}_{(j)}^{N}]\geq-\frac{\theta}{b}\log\left(\frac{\sum_{j=1}^{L}\left[A_{(j)}-A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{\overline{N}-1})\right]}{\sum_{j=1}^{L}\left[A_{(j)}-A_{(j+1)}\right]E\left[h(\widetilde{\xi}_{(j)}^{N-1})\right]}\right), \quad \text{(EC.11)}$$

where the last inequality follows from (EC.10). The combination of (EC.10) and (EC.11) yields the desired result that $U_o^{\overline{N}-N}[K+1]>0$ for any N. ■

EC.4. Additional Results

Remark EC.2. When h is symmetric around 0, $(1-H(s))=H(-s)$, $H(s)(1-H(s))=H(-s)(1-H(-s))$ and $h'(s)=-h'(s)$. So, $H(s)(1-H(s))h'(s)+H(-s)(1-H(-s))h'(-s)=0$ for any $s\in\Xi$. Thus, condition (7) is satisfied as an equality, so the equilibrium effort is always non-increasing under symmetric log-concave density functions such as Normal and Logistic densities. For instance, under Normal distribution, I_N does not change when N increases from 2 to 3 and is decreasing for any $N\geq 3$. For densities with a single peak or increasing failure rate, I_N can increase or decrease with N. For instence, under Gumbel with mean 0 and scale parameter μ, and exponential with parameter λ, the left-hand side of (7) is $-\frac{1}{36\mu}$ and $-\frac{\lambda}{6}$, respectively. Thus, they both satisfy (7) so I_N is decreasing in N. For reversed Gumbel with mean 0 and scale parameter μ and reversed Weibull with mean 0, shape parameter 1, and scale parameter μ, the left-hand side of (7) is $\frac{1}{36\mu}$ and $\frac{1}{6\mu}$, respectively. Thus, they both violate (7), so I_N is increasing up to some $N^*\geq 3$.

We will next present three lemmas that we use for the proof of Theorem 1 under general effort function r and cost function ψ.

Lemma EC.4. *Suppose that the density h is log-concave. Then $E[\widetilde{\xi}_{(j)}^{N+1}]-E[\widetilde{\xi}_{(j)}^{N}]<E[\widetilde{\xi}_{(j+1)}^{N+1}]-E[\widetilde{\xi}_{(j+1)}^{N}]$ for all $j\in\{1,...,N-1\}$.*

Proof. Let $\delta_{(j)}^{N}\equiv E[\widetilde{\xi}_{(j)}^{N}]-E[\widetilde{\xi}_{(j+1)}^{N}]$. We want to show that $\delta_{(j)}^{N}>\delta_{(j)}^{N+1}$ for all j. From Galton (1902),

$$\delta_{(j)}^{N}=\binom{N}{j}\int_{\underline{s}}^{\overline{s}}H(s)^{N-j}(1-H(s))^{j}ds.$$

Rewriting this equation in terms of density of the j-th highest output shock, $h_{(j)}^{N}(s)$, and integrating it by parts, we obtain the following relationship:

$$\delta_{(j)}^{N}=\frac{1}{j}\int_{\underline{s}}^{\overline{s}}h_{(j)}^{N}(s)\frac{(1-H(s))}{h(s)}ds=\frac{1}{j}H_{(j)}^{N}(s)\frac{(1-H(s))}{h(s)}\Big|_{\underline{s}}^{\overline{s}}-\frac{1}{j}\int_{\underline{s}}^{\overline{s}}H_{(j)}^{N}(s)\left(\frac{(1-H(s))}{h(s)}\right)'ds.$$

Using the equation above, we can derive $\delta_{(j)}^{N+1}-\delta_{(j)}^{N}$ as

$$\delta_{(j)}^{N+1}-\delta_{(j)}^{N}=E[\widetilde{\xi}_{(j)}^{N+1}]-E[\widetilde{\xi}_{(j)}^{N}]-(E[\widetilde{\xi}_{(j+1)}^{N+1}]-E[\widetilde{\xi}_{(j+1)}^{N}])=\int_{\underline{s}}^{\overline{s}}[H_{(j)}^{N}(s)-H_{(j)}^{N+1}(s)]\left(\frac{(1-H(s))}{h(s)}\right)'ds<0;$$

because $H_{(j)}^{N+1}(s) \le H_{(j)}^{N}(s)$ for all s (and $<$ for a measurable subset of Ξ), and log-concavity implies that $\left(\frac{(1-H(s))}{h(s)}\right)' < 0$ for all s (see Bergstrom and Bagnoli 2005, Theorem 4). ■

LEMMA EC.5. *If h is log-concave, then $NI_N = NE\left[h(\widetilde{\xi}_{(1)}^{N-1})\right]$ is increasing in N.*

Proof. We will prove that $(N+1)\,I_{N+1} > NI_N$ for all N. First, we have

$$NI_N = \int_{\underline{s}}^{\overline{s}} N\,(N-1)\,H(s)^{N-2}\,(1-H(s))\,\frac{h(s)^2}{1-H(s)}ds = \int_{\underline{s}}^{\overline{s}} h_{(2)}^{N}(s)\lambda(s)ds,$$

where $\lambda(s) = \frac{h(s)}{1-H(s)}$. Using integration by parts, after simplifications,

$$(N+1)\,I_{N+1} - NI_N = -\int_{\underline{s}}^{\overline{s}} (H_{(2)}^{N+1}(s) - H_{(2)}^{N}(s))\lambda'(s)ds. \tag{EC.12}$$

Note that $H_{(2)}^{N+1}(s) - H_{(2)}^{N}(s) \le 0$ for all s (and $<$ for a measurable subset of Ξ) because $\widetilde{\xi}_{(2)}^{N+1}$ first-order stochastically dominates $\widetilde{\xi}_{(2)}^{N}$. Moreover, log-concavity implies that $\lambda'(s) > 0$ for all s. Thus, $(N+1)\,I_{N+1} > NI_N$. ■

LEMMA EC.6. *Suppose that $r'(g(x))g'(x)$ is decreasing in x, and that the output shock $\widetilde{\xi}_i$ is transformed to $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ via a scale transformation with $\alpha > 0$. Then,* $\lim_{\alpha\to+\infty} \frac{A^*}{\alpha} = 0$.

Proof. Let $g = \left(\frac{\psi'}{r'}\right)^{-1}$. Under a scale transformation of $\widehat{\xi}_{im} = \alpha\widetilde{\xi}_{im}$, I_N is converted to $\widehat{I_N} = I_N/\alpha$. Note that the award $A^*[\alpha]$ satisfies

$$r'\left(g\left(\frac{A^*[\alpha]I_N}{\alpha}\right)\right)g'\left(\frac{A^*[\alpha]I_N}{\alpha}\right)\frac{I_N}{\alpha} - 1 = 0. \tag{EC.13}$$

Because $r'(g(x))g'(x)$ is decreasing in x, and I_N/α is decreasing in α, in order for A to satisfy (EC.13), $A^*[\alpha]/\alpha$ should be decreasing with α. Since $A^*[\alpha]/\alpha$ is decreasing in α, and $A^*[\alpha] \ge 0$, $A^*[\alpha]/\alpha$ converges. Furthermore, because $\lim_{\alpha\to+\infty} \frac{I_N}{\alpha} = 0$, to satisfy (EC.13), we need $\lim_{\alpha\to+\infty} \frac{A^*[\alpha]}{\alpha} = 0$. ■

The following corollary shows that an open tournament is optimal under high uncertainty without imposing Assumption 1.

COROLLARY EC.5. *For any distribution H of output shock $\widetilde{\xi}_i$, there exists $\overline{\alpha}$ such that under a scale transformation of $\widetilde{\xi}_i$ with $\alpha \ge \overline{\alpha}$, an open tournament with unrestricted entry is optimal for any number of contributors K.*

Proof. To prove that an open tournament is optimal, we will show that for any finite N and D ($> N$), there exists a scale transformation such that the organizer's utility with D participants is higher than that with N participants. We want to show

$$U_o^{D-N} \equiv \left(Kr(e^{*,D}) + \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{D}] - A^{*,D}\right) - \left(Kr(e^{*,N}) + \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^{N}] - A^{*,N}\right) \ge 0, \tag{EC.14}$$

where $e^{*,N}$ is the equilibrium effort when there are N participants and the winner award is optimally chosen as $A^{*,N}$. Notice from (5) that when there are D participants, and the organizer pays $\frac{I_N A^{*,N}}{I_D}$

to the winner, the equilibrium effort $e^{*,D}$ is the same as $e^{*,N}$. Also, due to optimality of $(e^{*,D}, A^{*,D})$, when there are D participants, $(e^{*,D}, A^{*,D})$ yields a weakly greater utility to the organizer than $(e^{*,N}, \frac{I_N A^{*,N}}{I_D})$. Thus,

$$U_o^{D-N} \geq \sum_{j=1}^{K} E[\widetilde{\xi}_{(j)}^D - \widetilde{\xi}_{(j)}^N] - \frac{I_N A^{*,N}}{I_D} + A^{*,N} \geq KE[\widetilde{\xi}_{(1)}^D - \widetilde{\xi}_{(1)}^N] + A^{*,N}\frac{I_D - I_N}{I_D}, \tag{EC.15}$$

where the last inequality follows from Lemma EC.4. As a final step, consider a scale transformation $\widehat{\xi}_i = \alpha\widetilde{\xi}_i$ of the output shock $\widetilde{\xi}_i$, which implies $E[\widehat{\xi}_{(1)}^N] = \alpha E[\widetilde{\xi}_{(1)}^N]$. From Lemma EC.5, we have $DI_D \geq NI_N$, so

$$U_o^{D-N} \geq K\alpha E[\widetilde{\xi}_{(1)}^D - \widetilde{\xi}_{(1)}^N] + A^{*,N}[\alpha]\frac{N-D}{N}. \tag{EC.16}$$

Since $E[\widetilde{\xi}_{(1)}^D - \widetilde{\xi}_{(1)}^N] > 0$ for all $N < D$, and $\lim_{\alpha\to+\infty} \frac{A^{*,N}[\alpha]}{\alpha} = 0$ by Lemma EC.6, as α increases, (EC.16) becomes positive. Thus, for any D and N, there exists a sufficiently large α such that $U_o^{D-N} > 0$. This result implies that for sufficiently large α, the organizer's utility increases with N for any $N < \overline{N}$. Thus, there exists $\overline{\alpha}$ such that an open tournament is optimal for any $\alpha > \overline{\alpha}$. ■

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