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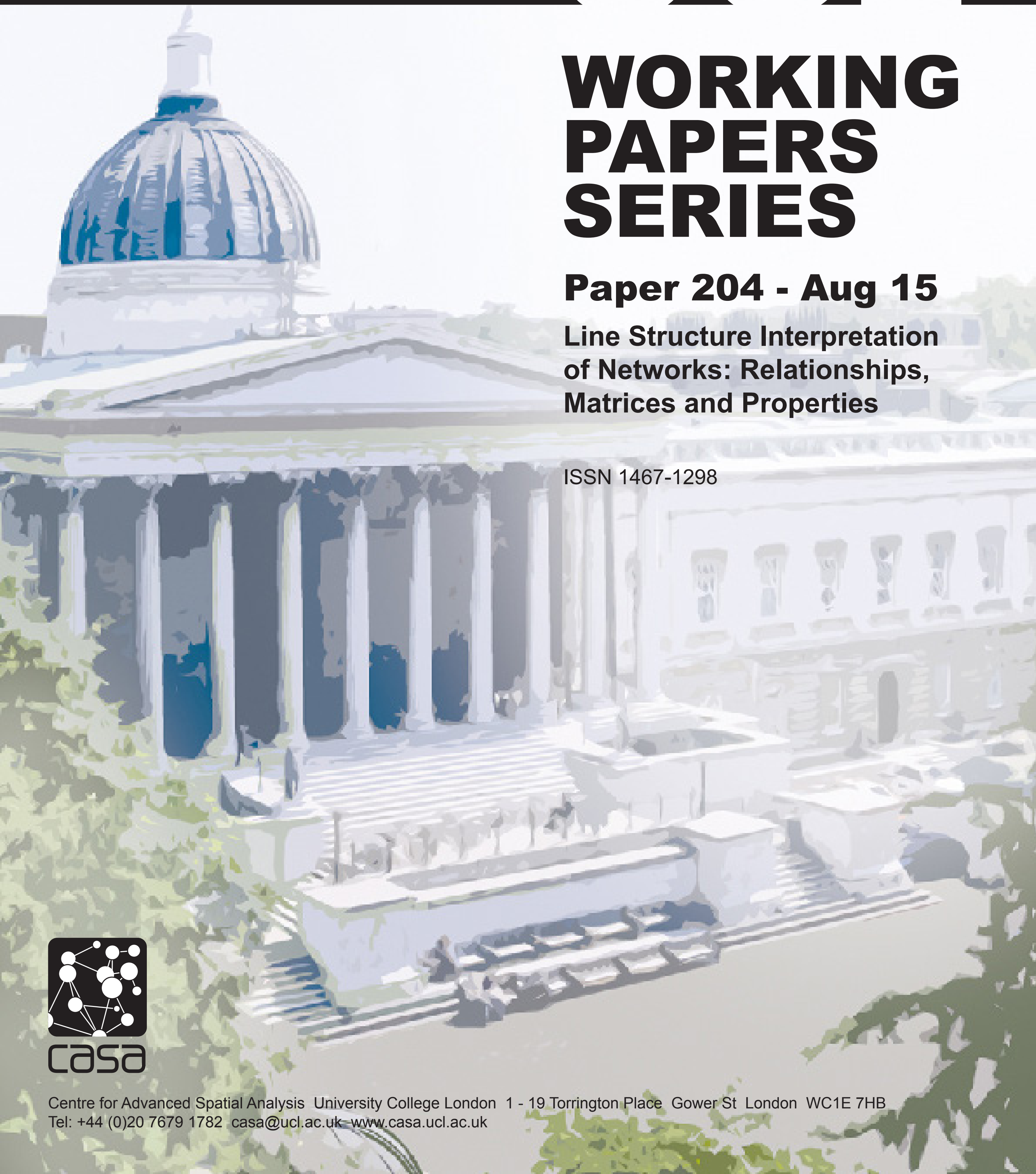

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Line Structure Interpretation of Networks: Relationships, Matrices and Properties

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Line Structure Interpretation of Networks: Relationships, Matrices and Properties

Working paper

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Abstract. This working paper sets out an interpretation of the line structure as a device for representing network structure. First, the line structure $S^{\#}$ is briefly introduced with reference to Euclidean and Cartesian geometry. Then its relation to the primal graph (G') and dual graph (G'') are set out, demonstrated using an intermediate form, the hypergraph $H^{\#}$. A series of point matrices $\dot{X}$ and line matrices $\bar{X}$ for specifying the line structure are set out, from which line structural properties such as continuity and connectivity are derived.

1. Introduction

Euclidean geometry deals with abstract elements such as points, lines and areas, while graphs deal with arcs and vertices. The two systems can map on to each other in various ways: a Euclidean rectangle could be represented as a graph, while the diagram of a graph on a page could be interpreted in geometric terms as a set of curves and points. Another form of representation, which relates to both, is the *line structure*, which is interpretable as a linearly ordered incidence structure, and which has been demonstrated in the context of representing road networks (Marshall, 2016).

In mathematical terms, an incidence structure is a partial linear space (assuming every line is incident with at least two points and if every two distinct points are incident with at most one line); more concretely, a point-line incidence structure is a triple S=(P, L, I) comprising a set of incidence relations I between points P and lines L (e.g. de Bruyn, 2006:1). An incidence structure can also be interpreted as a hypergraph, or generalisation of a graph where relationships can be between more than two elements.

A line structure $S^{\#}$ is taken as a linearly ordered incidence structure featuring a linear series of elements which may be a discrete ordered series of points, referred to as an *ordinal line structure* ($S_O^{\#}$), or a continuous series of points, referred to as a *parametric line structure* ($S_P^{\#}$). In its parametric form, a line structure goes beyond a conventional incidence structure or hypergraph by featuring lines comprising continuous series of points, not just discrete nodal points (Marshall, 2016).

The present paper sets out an interpretation of line structures and their properties, from an abstract point of view, that is, independently of application to the context of road network analysis. It could apply to any context for spatial analysis where structured sets of linear elements are significant objects of scrutiny. The paper first sets out an interpretation of line structures (section 2); then demonstrates the attributes of line structures, in particular in relation to primal and dual graphs (G' and G''), partly via an intermediate form, the hypergraph $H^{\#}$ (section 3). The paper sets out a series of matrices as a way of specifying these structures, and how we can derive a series of line structural properties, such as continuity and connectivity, obtainable not only by visual inspection from the line structure diagrams, but directly from the line structure matrices (section 4). Overall, the paper primarily represents a consolidation and elaboration of the 'internal workings' of the line structure, and as such may be regarded as complementary to the existing treatment (Marshall, 2016).

2. The line structure

This section introduces two types of structure, the parametric line structure and the ordinal structure, and shows how these relate to coordinate geometry and graph theory.

2.1 Euclidean and Cartesian geometry

Consider the rectangle PQRS as found in elementary Euclidean geometry (Figure 1).

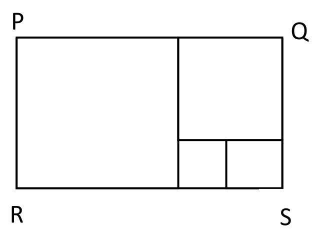


Figure 1. A rectangle and subdivisions, as Euclidean geometry.

This can be expressed in co-ordinate geometry as shown in Figure 2.

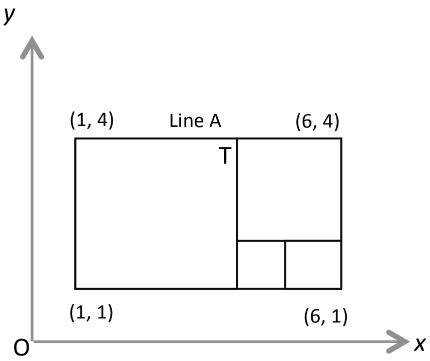


Figure 2. A rectangle and subdivisions, as co-ordinate geometry.

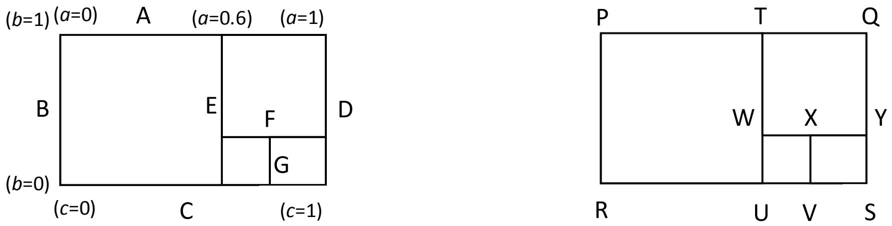
Each line has an equation that can be expressed in terms of x and y. For example the line A has the equation y=4 (1≤x≤6).

2.2 Parametric format

We could also express the line A in the form of a *parametric equation* in a, thus: x=5a+1; y=4; 0≤a≤1. Here the parameter a has a geometric significance in terms of the proportionate distance along the line A, from its left-hand end at P (1,4) where a=0 to its right-hand end at Q (6,4) where a=1. In fact, line A can be considered a *line segment*, i.e. just that part of the line y=4 (going from x = −∞ to x= ∞) lying between x=1 and x=6 inclusive; similarly, the parameter a could be considered to extend from a geometrically hypothetical −∞ to ∞, where the particular line segment A is defined by the range 0≤a≤1. The significance of the parametric form is that it demonstrates a localisation of line A; any point on line A can be expressed purely in terms of a (i.e. rather than involving x or y). So for example the point T (4, 4) is specified uniquely by the position a=0.6. Locally, this designation is more concise than (x,y) co-ordinates, as it needs only one parameter (a); but also the local 'meaning' of the parameter (0.6) is also naturally clear, being six tenths of the way along the line (between P and Q).

2.3 The parametric line structure

In fact we can 'parameterise' the whole figure, by giving each line (A, B, C, …) its own parameter (a, b, c, …), signifying the interval between 0 and 1 inclusive. This means that 0 and 1 will always signify the ends of a line segment, and a fractional number will signify an intermediate position (point) along the line segment. The parametric value a of a line A is equivalent to the abscissa value x along the X-axis, but parametric values are applied to any line, each as it were its own axis. Hence we get the following figure, forming a *parametric line structure*, $S_P^{\#}$ (Figure 3). The convention adopted for this figure is that a line's parametric value will go from 0 to 1 from left to right, and from bottom to top (cf. *x* and *y* in Cartesian axes). The lines are given by upper case Roman letters, and the parametric values are the corresponding lower case letters.



(a) (b)

Figure 3. Parametric line structure, $S_P^{\#}$. (a) with 7 lines labelled (A–G) and selected points with their parametric values; in each case, parametric values increase from 0 to 1 from left to right, or bottom to top. (b) The same structure with 10 points labelled (P–Y).

This parametric line structure forms a local 'co-ordinate system' of its own, by which any point may be specified in terms of the line or lines that it lies on. Hence:

- The point P is (*a*=0, *b*=1)
- The point T is (*a*=0.6, *e*=1)
- The point Q is (*a*=1, *d*=1), etc.

These can be generalised to relate to parameters for all lines (*a*, *b*, *c*, *d*, *e*, *f*, *g*), thus:

- The point P is (0, 1, –, –, –, –, –)
- The point T is (0.6, –, –, –, 1, –, –)
- The point Q is (1, –, –, 1, –, –, –), etc.

Note that in fact one label would be sufficient to specify any point: for example, the point T can be uniquely specified by *a*=0.6. Unlike the Cartesian system, where there is an indefinitely large number of points on the plane either side of the X-axis or Y-axis, for the parametric line structure each line is its own 'axis'. The term 'co-ordinate system' is in quote marks because it is not the familiar Cartesian system which represents a plane. Here, *only* the lines exist. The blank interstitial spaces between the lines contain no points. In other words we can extract the rectangular figure – the set of lines {A, B, C, D, E, F, G} – from the Cartesian plane. We are left with a line structure. This line structure can be used to represent geometric figures in general.

A Cartesian system is of course devised precisely so that any possible position on a plane or in space can be represented, with respect to reference axes. However, here, we are not concerned with the existence of any points that are *not* on a line; any such points are not part of the system (just as in a planar co-ordinate system we ignore the possible existence of space above or below the plane). For example, the Cartesian position (2,3) – that is, (x=2, y=3) – does not exist in the framework of the line structure of Figure 3. Indeed the only point at which both x and y are present is the point O (x=0, y=0) in Figure 2; x and y do not feature at all in Figure 3.

The above may be generalised as follows: (1) Each line X_i has a parameter x_i indicating position along the line; (2) let this parameter x_i be a real number, being 0 at one end of the line and 1 at the other. Hence any line $X_i = \{x_i \in \mathbb{R} \mid 0 \leq x_i \leq 1\}$. Any point on a line X can be specified by the value of parameter x; therefore the set of points (P) in $S_P^\#$ is 'internalised' in the parametric definitions of the lines (L). Furthermore, an intersection point can be specified by the two (or more) lines intersecting ($P=I \subseteq L \times L$). Any point can be expressed as a combination of the parametric values of any or *every* line in the set. So a point Y can be expressed as Y (x_{1Y}, x_{2Y}, …, x_{nY}), where x_{iY} is the parametric value of the point Y on line x_i. A parametric line structure $S_P^\#$ can hence therefore be specified as follows: $S_P^\# = (L, P)$; $L = \{X_1, X_2, X_3, \ldots X_n\}$; $X_n = \{x_i \in \mathbb{R} \mid 0 \leq x_i \leq 1\}$ for each of n lines; $P = \{Y_1, Y_2, Y_3, \ldots Y_m\}$; Y_m (x_{1Y}, x_{2Y}, …, x_{mY}) for m intersections (Marshall, 2016).

Hence in Figure 3, the line structure $S_P^\#$ is given by: $S_P^\# = (L, P)$; $L = \{A, B, C, D, E, F, G\}$; $A=\{a \in \mathbb{R} \mid 0 \leq a \leq 1\}$; $B=\{b \in \mathbb{R} \mid 0 \leq b \leq 1\}$; $C=\{c \in \mathbb{R} \mid 0 \leq c \leq 1\}$; $D=\{d \in \mathbb{R} \mid 0 \leq d \leq 1\}$; $E=\{e \in \mathbb{R} \mid 0 \leq e \leq 1\}$; $F=\{f \in \mathbb{R} \mid 0 \leq f \leq 1\}$; $G=\{g \in \mathbb{R} \mid 0 \leq g \leq 1\}$; P={P, Q, R, S, T, U, V, W, X, Y}; and P (0, 1, –, –, –, –, –); Q (1, –, –, 1, –, –, –); R (–, 0, 0, –, –, –, –); S (–,–, 1, 0, –, –, –); T (0.6,–, –, –, 1, –, –); U (–,–, 0.6, –, 0, –, –); V (–,–, 0.8, –, –, –, 0); W (–,–, –, –, 0.333, 0, –); X (–,–, –, –, –, 0.5, 0.333); Y (–,–, –, 0.333, –, 1, –).

2.3 The ordinal line structure

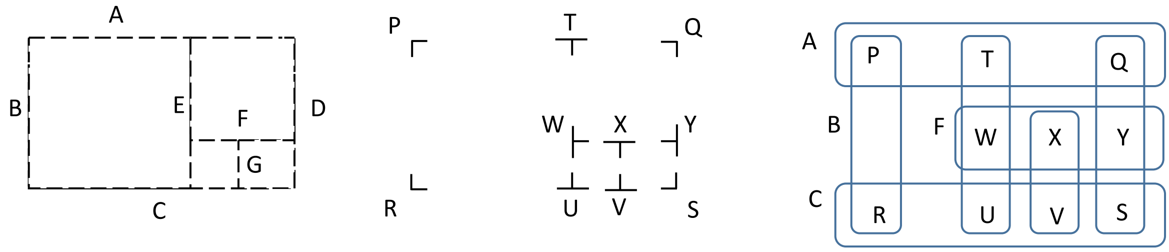
We could simplify this line structure further, by removing all the intermediate points along each line (and hence the intermediate parametric values). This converts it into an *ordinal structure*. This is a topological structure where there is only a set of lines (linear relations or ordered sets) and their intersection points. Here, there are no 'intermediate points' between intersection points. There is simply a unit distance between points, equivalent to unit distance from one vertex to another in a graph. Here, we express lines in terms of the points they go through:

- The line A {a=0, …, a=0.6, …, a=1} becomes simply A {P, T, Q};
- The line B {b=0, …, b=1} becomes B {P, R};
- The line C {c=0, …, c=0.6, …, c=0.8, …, c=1} becomes C {R, U, V, S};

… and so on.

Put formally, let $S_0^{\#}$ be the set of lines L and points P; where $L = \{X_1, X_2, \ldots X_3\}$ and where any line X_i comprises a linearly ordered set of n elements, $\{x_1, x_2, \ldots, x_n\}$, being the linearly ordered set of discretely identified points along the line. Since order matters, $\{x_i, x_j, x_k\} \neq \{x_i, x_k, x_j\}$ (Marshall, 2016). In Figure 4, L={A, B, C, D, E, F, G}; P={P, Q, R, S, T, U, V, W, X, Y}; A={P, T, Q}; B={R, P}; C={R, U, V, S}; D={S, Y, Q}; E={U, W, T}; F={W, X, Y}; G={V, X}. This set of information completely specifies $S_0^{\#}$.

The ordinal line structure may have the same graphic form as the parametric one, though we could distinguish the two by using dashed lines (Figure 4a), or leaving a gap between intersection points (to indicate that there is nothing between those points, Figure 4 b). Note that an ordinal line structure need not involve 'lines' as such (Figure 4 c), but could comprise ordered strings of letters rather than points. Arguably an ordinal line structure could be simply referred to as an ordinal structure – but the term ordinal line structure is retained here to cement the connection with the parametric line structure. An ordinal line structure can be regarded as a kind of hypergraph, where each line can be seen as being composed of individual line segments (e.g. PT, TQ) corresponding to the edges of a graph. In other words, a line in an ordinal line structure is equivalent to the hyperedge in a hypergraph.[1]



(a) (b) (c)

Figure 4. Ordinal structure, $S_0^{\#}$, shown using alternative graphical formats. (a) dashed lines; (b) intersection points; (c) Venn diagram format.

2.5 Comparison with equivalent graphs

We can express the original figure in the form of a graph – sometimes referred to as a primal graph, G' (Figure 5). This has 10 vertices and 13 edges: V=10; E=13. Clearly, the vertices correspond with the intersection points on the line structure $S^{\#}$, while the edges represent line segments or parts of the lines on $S^{\#}$. In effect, the 7 lines on $S^{\#}$ are

[1] Note that in this sense, the parametric line structure is *not* a hypergraph, since although a hypergraph allows elements that are continuous through a series of points, the elements themselves are not continuously constituted by points, while in a parametric line structure lines comprise any number of points.

broken into 13 discontinuous edges; or more strictly, 4 of the lines on $S^{\#}$ (A, D, E, F) comprising two line segments are each broken into two edges, the line (C) with three line segments is broken into 3 individual edges, and two lines (B and G) are single edges in the first place.

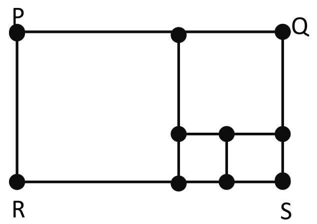


Figure 5. The primal graph G' corresponding to Figure 1 (cf. Figures 3 and 4).

Alternatively we can create another kind of graph, sometimes referred to as the dual graph, or *G''* (Figure 6). Here, each vertex on the graph corresponds with a line on the line structure $S^{\#}$; each edge corresponds with an intersection between lines on $S^{\#}$.

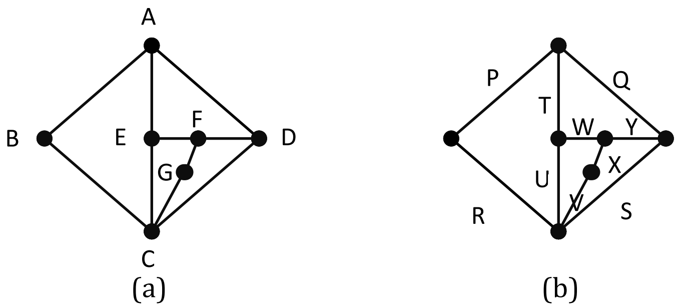


Figure 6. The dual graph G'' corresponding to Figure 3 (cf. Figures 1, 4). Here, there are 7 vertices and 10 edges (corresponding with the 7 lines and 10 intersection points in $S^{\#}$).

Overall, then, we have moved from a geometric figure, expressed in elemental form and coordinate geometric form (Figures 1, 2), to a parametric line structure (Figure 3), to an ordinal line structure (Figure 4), to graph formats (Figures 5, 6).

- The Cartesian axes, extracted from their embedment in the Cartesian plane, can be represented as parametric lines. Hence a line structure can be seen as a special kind of plane, comprising only those points constituting a set of discrete lines;
- An ordinal line structure can be seen as a special kind of line structure, where there are no intermediate points between intersection points and/or pendant ends;
- A graph can be seen as a special kind of ordinal structure, where all lines are discontinuous, each simply spanning between a pair of points.

3. Relations between line structures and graph representations

It is possible to demonstrate relations between line structures and graphs in a systematic way, through a new kind of representation, the **hypergraph *H*#**, that contains the same information as a line structure $S^{\#}$. First, we introduce the hypergraph $H^{\#}$ and demonstrate its properties and relation to $S^{\#}$. Then we demonstrate the relation between the hypergraph $H^{\#}$ and *G'*; then between $H^{\#}$ and *G''*; hence also the relation between *G'* and *G''*; and then 'missing' components, present in $H^{\#}$ but neither *G'* nor *G''*, which can be identified by a **line set $S^{=}$**. The remainder of the paper is based on the consideration of an example line structure given in Figure 7, comprising 5 lines (labelled A–E) and 9 nodal points, comprising 5 pendant ends (labelled 3, 5, 6, 8, 9) and 4 intersection points (1, 2, 4, 7). This structure is also subsequently expressed variously as a hypergraph $H^{\#}$, line set $S^{=}$, primal graph *G'* or dual graph *G''*.

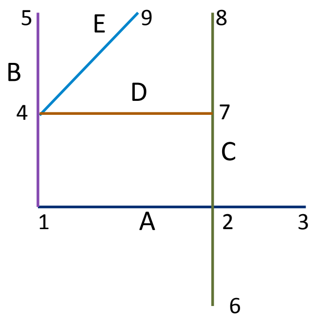


Figure 7. Example line structure, $S^{\#}$.

3.1 The hypergraph $H^{\#}$

Here we introduce a special kind of hypergraph $H^{\#}$ devised for the purposes of capturing the properties of the line structure, and relating it to *G'* and *G''*. The line structure represented in Figure 7 can be represented as a hypergraph $H^{\#}$ shown in Figure 8 (where the # superscript indicates the full expression of a connected line structure, as opposed to a set of individual lines). That is to say, this hypergraph $H^{\#}$ is the hypergraph-equivalent of the line structure $S^{\#}$. It takes this degree of complexity of (hyper)graph to contain all the information in the (graphically simpler) line structure.

$H^{\#}$ is a special kind of hypergraph in that it contains *two different kinds of edge*: **links** and **ties.** The links are shown solid and horizontal (and in colour); the ties are shown dashed and vertical or diagonal (black). Moreover, the links may be assembled linearly into **chains**, and the ties may form connected complexes that can be termed **knots,** that are complete (fully connected) sub-graphs of the vertices concerned.[2]

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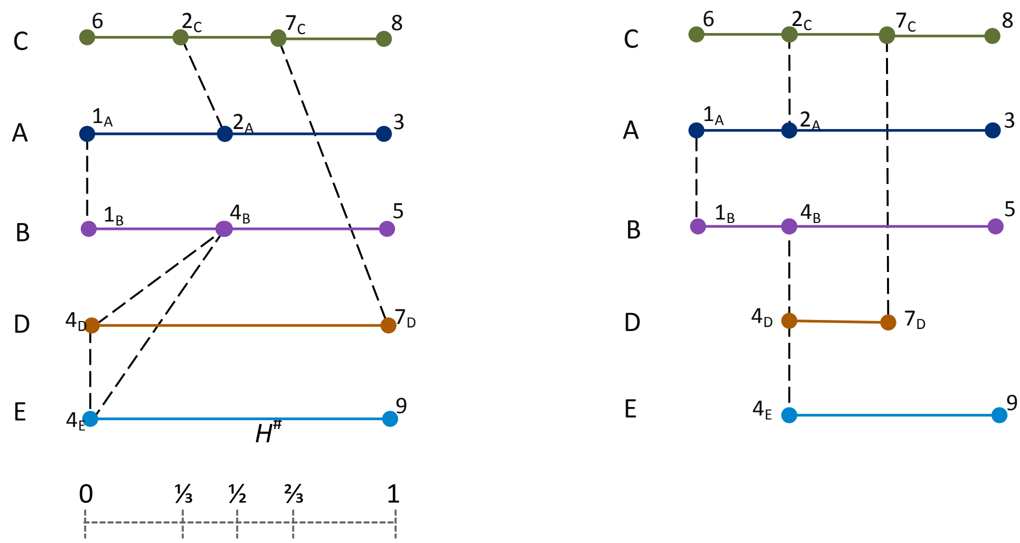
[2] The figure in Figure 8 could also be interpreted in principle as a special kind of graph $G^{\#}$, with two kinds of edge, namely links and ties. Here, there would be no hyperedges; strictly there would be no recognition of chains nor knots; a chain would simply be a set of links in series; and



(a) $H^{\#}$ (b) $H^{\#}{}_{(alt)}$

Figure 8. The hypergraph $H^{\#}$, that is equivalent to the line structure $S^{\#}$ in Figure 7. (a) $H^{\#}$ drawn with a parametric scale added, to indicate the relation with the parametric line structure $S_P{}^{\#}$. In this case the triangular nature of the knot 4 is explicit. (b) Alternative format $H^{\#}{}_{(alt)}$ drawn to emphasise vertices that represent a coincident point on $S^{\#}$. Note however that the existence of a tie 4_B–4_E is not explicitly distinguishable and must be inferred by convention. Note also that here, all knots are vertical. The whole diagram becomes a set of horizontal or vertical lines; ties can be distinguished from links purely by being vertical as opposed to horzontal, rather than needing to be shown as dashed. The whole diagram could be considered reminiscent of a metro system diagram, where the links/ chains are metro lines and the ties/ knots are interchange connections.

Mathematically, $H^{\#}$ is a quintuple (V, L, T, C, K) comprising a set of vertices (V), a set of links (L), a set of ties (T), a set of chains (C) and a set of knots (K). The set of vertices may be partitioned in two ways (Figure 9). First, there are the subsets of end vertices (V_E) and intermediate vertices (V_M), such that $V_E \subset V$; $V_M \subset V$; $V_E \cap V_M = \emptyset$; $V_E \cup V_M = V$. In Figures 8 and 9, $V_E = \{1_A, 1_B, 3, 4_D, 4_E, 5, 6, 7_D, 8, 9\}$; $V_M = \{2_A, 2_C, 4_B, 7_C\}$. Secondly, there are the mutually exclusive subsets of pendant vertices (V_P) and intersection vertices (V_I), such that $V_P \subset V$; $V_I \subset V$; $V_P \cap V_I = \emptyset$; $V_P \cup V_I = V$. In Figures 8 and 9, $V_P = \{3, 5, 6, 8, 9\}$; $V_I = \{1_A, 1_B, 2_A, 2_C, 4_B, 4_D, 4_E, 7_C, 7_D\}$. In fact, as is evident from Figure 9, all pendant vertices are

a tie complex simply a complete (fully connected) sub-graph. In practice, however, the value of $H^{\#}$ rests on equating lines with chains and intersection points with knots; and the simplest way of doing this is to recognise chains and knots as necessarily distinct kinds of hyperedge, rather than arbitrarily selected sub-graphs.

necessarily end vertices ($V_P \subset V_E$) while all intermediate vertices are intersection vertices ($V_M \subset V_I$). Meanwhile, we can recognise the set of 'anchoring' vertices (V_A) where a line or lines terminate at an intersection with another line or lines. This gives another partitioning, into three distinct kinds of vertex: (i) pendant end vertices ($V_E \cap V_P = V_P$), (ii) intermediate intersecting ($V_M \cap V_I = V_M$) and (iii) 'anchoring' (end-intersecting) vertices ($V_E \cap V_I = V_A$).[3] In Figures 8 and 9, these are respectively the sets {3, 5, 6, 8, 9}, {2_A, 2_C, 4_B, 7_C} and {1_A, 1_B, 4_D, 4_E, 7_D}.

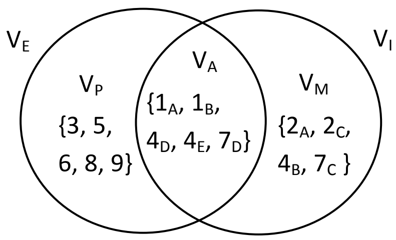


Figure 9. Venn diagram representation of V_E, V_P, V_A, V_M and *V_I.*

Links may be joined linearly 'in series' to form chains. For each link, $|V|=2$. Each link can be considered part of a chain; $C \cap L = L$. A chain (C) is a set of links (L) with their associated vertices (V_L), necessarily connected in series (else C itself would not be a linear element). For each chain, $|L| \geq 1$; $|V_E|=2$; $|V_M| \geq 0$; hence $|V_L| \geq 2$. By convention, a chain does not have any links or vertices in common with another chain; $C_i(L_i, V_{Li}) \cap C_j(L_j, V_{Lj}) = \emptyset$, $i \neq j$. There are no vertices not belonging to a chain: $C \cap V = V$; meanwhile, the set of vertices associated with links (and hence chains) is the set of all vertices: $V_L = V$. By this convention in $H^\#$ we split vertices on a graph G (e.g. vertex 1) into different vertices on $H^\#$ (e.g. vertices 1_A and 1_B; i.e. $1_A \neq 1_B$). Also by this convention a discrete link is regarded as a single link chain; we can also identify a 'proper chain' where L>1.

Ties always connect between two different chains. For each tie, $|V|=2$. Each tie can be considered part of a knot; $K \cap T = T$. A knot (K) is a set of ties (T) with their associated set of vertices (V_T), such that the set of vertices V_T is a complete (fully connected) subgraph (since all lines on $S^\#$ connecting at a point are deemed to connect with all others). For each knot, $|T| \geq 1$ and $|V_T| \geq 2$. We can recognise a tie that does not connect with any other tie as a single tie knot, while a 'proper knot' is where $|T|>1$. A knot does not have any ties or vertices in common with any other knot; $K_i(T_i, V_{Ti}) \cap K_j(T_j, V_{Tj}) = \emptyset$, $i \neq j$. All vertices associated with ties are intersection vertices, and vice versa: $V_T = V_I$.

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[3] A hypothetical further kind of vertex is ruled out: a point cannot be both pendant and intermediate; $V_M \cap V_P = \emptyset$.

A line structure $S^{\#}$ must comprise at least one line, implying at least two line-ends; hence for any $H^{\#}$, $|L|\geq1$; $|V|\geq2$; $|C|\leq|L|$; $|T|\geq0$; $|K|\leq T$. In the example structure (Figure 8), $|L|$=9 (the number of edges on G'); $|V|$=14; $|C|$=5 (the number of vertices on G''); $|T|$=6 (the number of distinct line intersections, or edges on G''); $|K|$=4 (the number of intersection points on $S^{\#}$).

When we say that $H^{\#}$ is equivalent to $S^{\#}$, this means that these convey the same structural information about the entity (e.g. road layout) they are representing, even though they express this in different ways: for example, $H^{\#}$ includes entities such as vertices and chains and knots that do not feature in a line structure that by its nature contains only lines and points. The hypergraph splits point 1 into separate vertices 1_A and 1_B; this is equivalent to the ordinal line structure labels A_1 and B_1, or parameters a=0 and b=0, which refer to the same point.

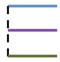
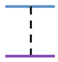
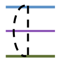
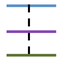
The equivalences between a hypergraph $H^{\#}$ and line structure $S^{\#}$ are shown in Table 1. Note that the whole set of vertices in a given knot on $H^{\#}$ refer to a *single* intersection point on $S^{\#}$. Hence in Figure 8, where precisely two lines meet in $S^{\#}$ (at A∩B or C∩A or C∩D), we have a single tie linking a pair of points on $H^{\#}$ (respectively 1_A1_B, 2_A2_C, 7_C7_D); where three lines meet in $S^{\#}$ (at B∩D∩E), we have a triangle of ties ($4_B4_D4_E$). In general, where r lines meet on $S^{\#}$, we obtain a knot on $H^{\#}$ that is a complete (sub)graph of r vertices joined by (r(r–1)/2) ties.

Table 1. Correspondence between elements of the hypergraph $H^{\#}$ and line structure $S^{\#}$.

Case	$H^{\#}$	$S^{\#}$	Example	No.
L	Link	Line segment	$A_{(1\text{-}2)}$, etc.	9
C	Chain	Line	A, B, etc.	5
V	Vertex	Nodal point referent	1_A, 1_B, etc. A_1, B_1, etc.	14
T	Tie	Line intersection	A∩C, etc.	6
$V_E \cap V_E$	Tie between end vertices	Mutually terminating lines	A∩B	2
$V_M \cap V_M$	Tie between intermediate vertices	Mutually intersecting lines	A∩C	1
$V_E \cap V_M$	Tie between end and intermediate vertices	Line yielding on another	D≤C, D≤B, E≤B	3
K	Knot	Intersection point	1, 2, 4, 7	4
V_P	Pendant vertex	Pendant line-end	3, 5, 6, etc.	5

We can also recognise intersection types graphically (Table 2). Note that the K type of intersection incorporates L and T; Y incorporates L; and '*' incorporates X.

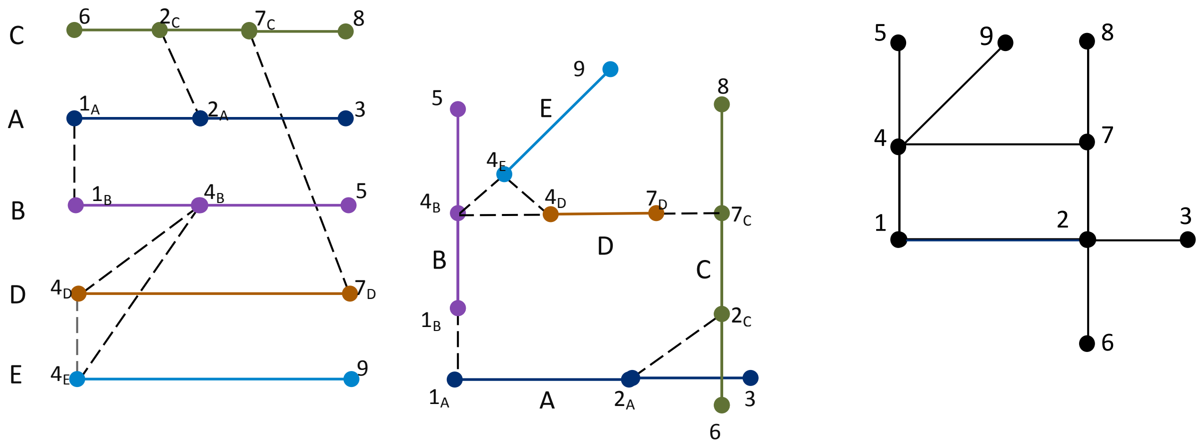
Table 2. A series of intersection types represented in $S^{\#}$ and $H^{\#}$.

Intersection type	$S^{\#}$	$H^{\#}$	$H^{\#}_{(alt)}$
L (2 lines)			
T (2 lines)			
Y (3 lines)			
X (2 lines)			
K (3 lines)			
***** (3 lines)			

Hence $H^{\#}$ contains all the information embodied in $S^{\#}$: (i) what each line is made up of (which points it passes through, in which order, and the degree of the nodes, the number of line segments, hence continuity); (ii) the intersections (and specific points of intersection) between lines; hence the connectivity of each line; and (iii) continuity and termination conditions (CTC); and hence which lines yield to other lines; and junction type (L, T, Y, X, K, etc.). Since $H^{\#}$ is the hypergraph equivalent of line structure $S^{\#}$, when performing analysis on a line structure we do not actually need to use $H^{\#}$ as well as $S^{\#}$. The purpose here is demonstrative; not least to elaborate all the complexity that is contained in the deceptively simple looking line structure $S^{\#}$. And we can also employ it usefully to demonstrate how it links to G' and G''.

3.2 Relation between $H^{\#}$ and G'.

Figure 10 shows how $H^{\#}$ can be converted to the primal graph G'. This involves 'collapsing' the knots down to individual points, that become the vertices of G'. Put another way, $H^{\#}$ can be read as an 'assembly diagram' (Figure 10 b), where the chains are the elements to be assembled, and the ties are indications of which bits plug in where. In effect, what happens is that the links and nodes comprising the chains (represented by solid lines and vertices in $H^{\#}$) are retained in G' (but not the chains themselves as entities); while the ties (dashed lines on $H^{\#}$) disappear (as these are embodied within the nodes). With the loss of C and T, we lose the identity of the lines and their intersections.



(a) $H^{\#}$ (b) $H^{\#}$ (c) G'

Figure 10. Converting $H^{\#}$ to G'. (a) The hypergraph $H^{\#}$, ‘packed flat’; (b) in ‘assembly diagram’ format; hence (c) the associated primal graph G'.

3.3 Relation between $H^{\#}$ and G''

We can also relate hypergraph $H^{\#}$ to the dual graph G'' via a set of intermediate steps, by creating two intermediate hypergraphs $H^{\ddagger}$ and $H^{\dagger}$ (Figure 11).

First we can create a hypergraph $H^{\ddagger}$ comprising a set of horizontal chains (representing lines on $S^{\#}$) each comprising either precisely one link and two vertices, or else two links and three vertices, where each end vertex (V_E) represents a separate line-end (on $S^{\#}$) while the intermediate vertex (V_M) on $H^{\ddagger}$ (if present) represents the set of all intermediate intersections (on the corresponding line on $S^{\#}$). In the example, vertices 2_C and 7_C on $H^{\#}$ (Figure 11 a) collapse to a single vertex on $H^{\ddagger}$ (Figure 11 b).

We can then create a hypergraph $H^{\dagger}$ comprising a set of rows (representing lines on $S^{\#}$) where each row comprises precisely one vertex or else one horizontal link and two vertices, where each left-hand vertex on $H^{\dagger}$ represents the set of end points (V_E), i.e. points where the given line (on $S^{\#}$) terminates, while the right-hand vertex (if present) on $H^{\dagger}$ represents the set of intermediate intersection points points (V_M), i.e. where the given line (on $S^{\#}$) is continuous. Hence in the example, in Figure 11 (c), A, B and C are links each representing lines on $S^{\#}$ which have intermediate intersections; while D or E are vertices (featuring only on the left-hand side of $H^{\dagger}$) representing lines (on $S^{\#}$) that have no intermediate intersections. Note that the left-hand vertex represents up to a maximum of two points (representing line-ends that are intersection points) on $S^{\#}$; while the right-hand vertix (if present) can represent any number of points (greater than one) through which the line on $S^{\#}$ is continuous.

Hence Figure 11 shows how $H^{\#}$ can be converted to the dual graph G'', by progressive 'collapsing'. First, we collapse the mid-line points V_M together, to get graph $H^{\ddagger}$ (b); this loses the 'continuity' of the line, the number and degree of intermediate nodes and the order of intersection points. Then, we collapse the end points V_E together, to get $H^{\dagger}$ (c). This loses information about pendant nodes, but $H^{\dagger}$ still has sufficient infomation on continuity and termination conditions to infer hierarchy. Finally we collapse to the dual graph G'' (d), which loses the hierarchical information. Note that throughout, the integrity and connectivity of lines is maintained.

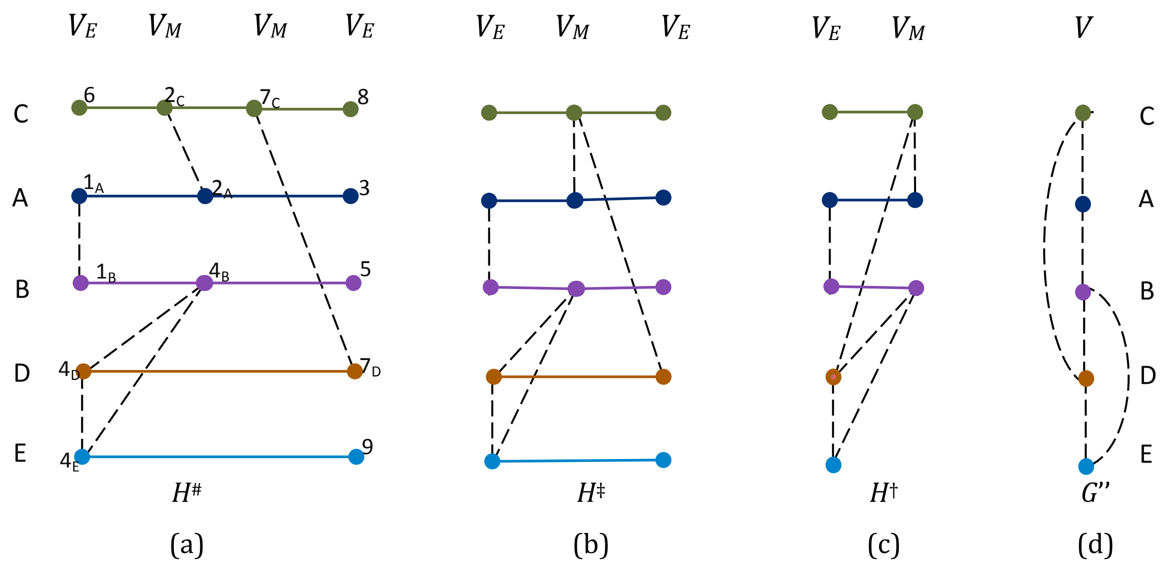


Figure 11 The 'collapse' of hypergraph $H^{\#}$ into the dual graph G''. In intermediate case (b), the presence of a line's pendant end(s) (V_P) is retained, while in case (c), we retain only information about the set of end points V_E (left-hand vertex) and, if present, the set of intermediate intersection points V_M (right-hand vertex),

3.4 Relation between *G'* and *G''*.

The foregoing analysis demonstrates that the primal (G') and dual (G'') graphs have no elements in common: $G' \cap G'' = \emptyset$. In effect, from $H^{\#}$, the solid horizontal lines (L) with their associated vertices (V_L) become the links with their associated nodes in G', while the dashed vertical or diagonal ties (T) and associated vertices (V_T) become the edges and vertices in G''. While the vertices correspond with the same points on the line structure, and while the lines and links constitute the same lines or line segments on the line structure, these have different identities.[4] Although G' and G'' are complementary,

[4] This identity disjunction has common-sense significance. Let R be the set of Roman Roads in Britain {Watling Street, Ermine Street, Via Devana…} and A be the set of 'A' roads {A1, A2, A3, …}. A given stretch of road (say, Edgware Road in London) could coincide with both Watling Street and the A5; but A5 is not an element of R; and no analysis of the set of R, of itself, will

their union $G' \cup G''$ is not enough to specify $H^{\#}$, because neither G' nor G'' can tell us the 'continuity and termination conditions': how many line segments each line is constituted by, or whether line X terminates upon line Y, or *vice versa*. This invites consideration of what is this 'missing' information.

3.5 Continuity and termination information: $S^{=}$.

We can define a 'line set' $S^{=}$ as the set of individual lines making up a line structure (here, the superscript $^{=}$ denotes the lines considered individually), together with their local continuity and termination conditions (CTC), but not including information about which *particular* lines they connect to or what happens to those other lines beyond their intersection. By defining $S^{=}$ this way, we aim to capture information in $H^{\#}$ that is complementary to $G' \cup G''$; that is, information over and above what is contained in G' or G'' but without specifying the full structure $S^{\#}$ (which would happen if we defined the intersection points, pendant ends and identity of lines in relation to each other).

Let $S^{=}$ comprise the set of lines L $\{X_1, X_2, \ldots, X_n\}$ and the set of continuity and termination conditions (CTC) for each line. In general, for line X, with n nodal points (intersections or pendant ends; $n \geq 2$), the continuity and termination conditions are $\{[q_{01}, q_{11}], [q_{02}, q_{12}], \ldots, [q_{0n}, q_{1n}]\}$, where q_{0i} is the total number of lines terminating at point i (where $q_{0i} \geq 0$), and q_{1i} is the total number of lines continuing at point i (where $q_{1i} \geq 0$). For the network in Figures 5, 6 and 7, this gives $S^{=}$=(L, CTC); L={A, B, C, D, E}; CTC: A={[2,0], [0,2], [1,0]}; B={[2,0], [2,1], [1,0]}; C= {[1,0], [0,2], [1,1], [1,0]}; D={[2,1], [1,1]}; E={[2,1], [1,0]} (Figure 12).

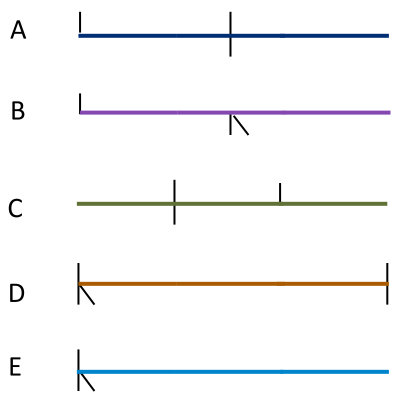


Figure 12. The line set $S^{=}$ of continuity and termination conditions for the lines in the line structure $S^{\#}$ of Figure 7.

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yield information about the set of A roads, even if they contain stretches of actual road in common (Marshall, 2016).

Figure 12 is not a graph but a series of line structure parts, where each bold (colured) horizontal line represents the line in question, and the fine vertical or diagonal line stubs represent parts of other lines. In effect, this information equates with the specification of individual routes in route structure analysis (Marshall, 2005) (therein defined graphically but not explicitly in terms of continuity and termination conditions).

3.6 Relations between $S^{\#}$, $H^{\#}$, G', G'' and $S^{=}$

We can set out fully the information contained in $S^{\#}$, $H^{\#}$, G', G'' *and* $S^{=}$ (Table 3). From the foregoing it can be seen that $H^{\#}$, which is the hypergraph corresponding to $S^{\#}$, contains all the information in $S^{\#}$, which amounts to the sum of information contained in G', G'' and $S^{=}$, i.e.

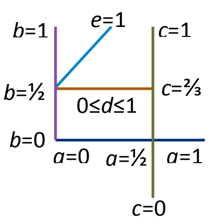
$$S_P^{\#} \equiv S_0^{\#} \equiv H^{\#} = G' \cup G'' \cup S^{=}$$

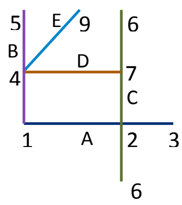
This is an interesting and significant finding, as it demonstrates the tightly fit relationship between the three kinds of structure: (1) the primal G' and dual G'' graphs are mutually exclusive or complementary; (2) they are integrated within $H^{\#}$; (3) yet none of G', G'' nor $G' \cup G''$ are enough to obtain $H^{\#}$ or $S^{\#}$; (4) the 'missing' element is 'continuity and termination conditions', this is supplied by $S^{=}$; (5) together these three make up $H^{\#}$, a hypergraph corresponding to $S^{\#}$. It should be reiterated here that $H^{\#}$ is 'equivalent' to $S^{\#}$ in the sense that it contains the equivalent structural information in $S^{\#}$ but expresses this using different devices (e.g. additional vertices, knots, etc.).[5]

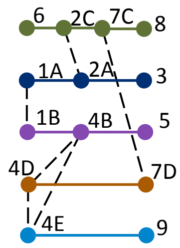
---

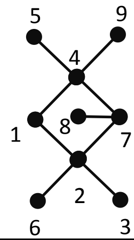
[5] The graphs G' and G'' can be regarded as just sets of numbers; so are $S_0^{\#}$ and $S_P^{\#}$. But in the case of G' and G'', these are pairs of numbers, where each pairing represents an edge (link). Although a link may correspond to a line or line segment in a line structure, the link itself does not have intermediate points; it is merely an ordered pair. This means that while for convenience we may equate the link {1, 2} with the interval $0 \le a \le ⅓$, link {1, 2} is really rather just the pair of points (i.e. 1, 2)(or a=0; a=⅓), whereas the line segment on the parametric line structure *really is a line* including all the intermediate points ($0 \le a \le ⅓$) as well. Strictly speaking, then, the parametric line structure is different from the ordinal line structure in that the former contains a continuum of points between nodal points.

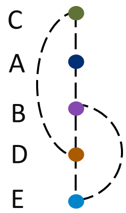
Table 3. Relationships between the elements of line structures and graphs

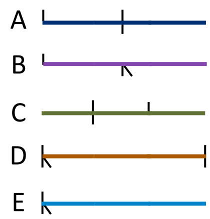












Line structure		Hypergraph	Primal graph	Dual graph	Line set with CTC ($S^{=}$)
Parametric ($S_P^{\#}$)	Ordinal ($S_O^{\#}$)	($H^{\#}$)	(G')	(G'')	
5 lines		5 chains	–	5 vertices	5 lines
A: 0≤*a*≤1	A{1, 2, 3}	A={1_A,2_A,3}		A	A{a_1, a_2, a_3}
B: 0≤*b*≤1	B{1, 4, 5}	B={1_B,4_B,5}		B	B{b_1, b_2, b_3}
C: 0≤*c*≤1	C{6, 2, 7, 8}	C={6,2_C,7_C,8}		C	C{c_1, c_2, c_3, c_4}
D: 0≤*d*≤1	D{4, 7}	D={4_D,7_D}		D	D{d_1, d_2}
E: 0≤*e*≤1	E{4, 9}	E={4_E,9}		E	E{e_1, e_2}
9 line segments		9 links	9 links	–	9 line segments
0≤*a*≤½	A_1{1, 2}	A_1{1, 2}	{1, 2}		A_1{a_1–a_2}
½≤*a*≤1	A_2{2, 3}	A_2{2, 3}	{2, 3}		A_2{a_2–a_3}
0≤*b*≤½, etc.	B_1{1, 4}, etc.	B_1{1, 4}, etc.	{1, 4}, etc.		B_1{b_1–b_2}, etc.
14 nodal point referents		14 vertices	–	–	14 nodal point referents
{*a*=0, *a*=½, *a*=1, *b*=0, *b*=½, …, *e*=0; *e*=1}	{A_1, A_2, A_3, B_1, B_2, …, E_2}	{1_A,1_B,2_A,2_C,3,4_B, 4_D,4_E,5,6,7_C,7_D,8,9}			{a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2, c_3,c_4,d_1,d_2,e_1,e_2}
9 nodal points		9 knots or pendant ends	9 nodes	–	9 nodal points
{*a*=0, *a*=½, *a*=1, *b*=½, *b*=1, …, *e*=1}	{1, 2, 3, 4, 5, 6, 7, 8, 9}	$K \cup V_P$	{1, 2, 3, 4, 5, 6, 7, 8, 9}		4 intersection pts. 5 pednant ends
4 Intersection points		4 knots	4 intersection nodes	–	4 intersection pts.
{*a*=0, *a*=½, *b*=½, *c*=⅔}	{1, 2, 4, 7}	{1, 2, 4, 7}	{1, 2, 4, 7}		3 between 2 lines 1 between 3 lines
6 line intersections		6 ties	–	6 Edges	6 line intersections
(*a*=0, *b*=0) (*a*=½, *c*=⅓) (*b*=½, *d*=0), etc.	A∩B A∩C B∩D, etc.	A∩B A∩C B∩D, etc.		AB AC BD, etc.	3 at 2-line points 3 at 3-line points
3 yields		3 yields	–	–	3 yields
d=0 @ *b*=½ *d*=1 @ *c*=⅔ *e*=0 @ *b*=½	{C_1D_0, B_1D_0, B_1E_0}	{C_1D_0, B_1D_0, B_1E_0}			Dx2; E
2 mutual terminations		2 mutual termin.			2 mutual termin.
a=0 @ *b*=0 *d*=0 @ e=0	{A_0B_0, D_0E_0}	{A_0B_0, D_0E_0}			A, B
1 mutual continuation		1 mutual contin.			1 mutual contin.
a=½ @ *c*=⅓	{A_1C_1}	{A_1C_1}			A, C
5 intersection types		5 intersection types	–	–	5 intersection types
L [2,0]@1 **X** [0,2]@2, etc.		**L** [2,0]@1 **X** [0,2]@2, etc.			{[0,2], [1,0], [1,1], [2,0], [2,1]}

4. Line Structure Matrices

Just as any graph G can be specified using an incidence matrix or adjacency matrix, a line structure $S^{\#}$ can be specified using matrices of different kinds. Here we set out a series of matrices for specifying line structures. There are two basic formats treated here: the *point matrix* (Section 4.1) and the *line matrix*, (section 4.2), each of which has a number of different variants. In each case, entries in the matrices – referring to individual attributes of points or lines – are of the form $\dot{x}_{ij}$ (referring to their position in row i and column j of a point matrix $\dot{X}$) or $\bar{x}_{ij}$ (referring to their position in row i and column j of a line matrix $\bar{X}$). As before, attributes of an individual element such as a line or point (albeit represented by a *set* of entries in the matrix) are denoted by lower case letters, while the attributes of a whole structure (line structure or graph) are denoted by upper case letters. All matrices from here on refer to an example structure shown in Figure 13 as (a) a line structure $S^{\#}$, (b) primal graph G' and (c) dual graph G''.

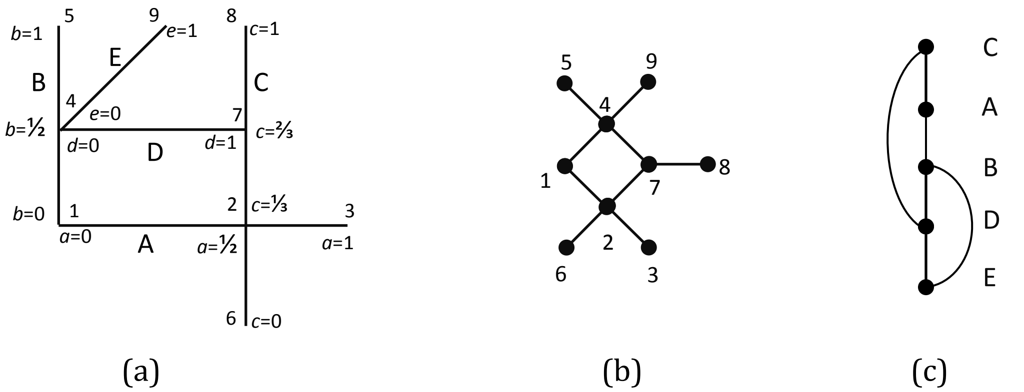


Figure 13. Example structure: (a) line structure $S^{\#}$; (b) corresponding primal graph (G'); (c) corresponding dual graph (G'').

4.1 Point matrices

In a point matrix $\dot{X}$, each row i of entries equates with a point, while each column j of entries equates with a line.

4.1.1 Ordinal point matrix, $\dot{O}$

In the ordinal point matrix $\dot{O}$, each numerical entry $\dot{o}_{ij}$ indicates the order of a point I (row i) on a line J (column j). Where I lies on J, the order is numbered from 1 to n, i.e. $1 \leq \dot{o}_{ij} \leq n$; n, $\dot{o}_{ij} \in \mathbb{N}$. Each line has a minimum of two points, i.e. $n \geq 2$. If point I does not lie on J, a dash '–' is indicated (i.e. no numerical entry $\dot{o}_{ij}$ appears in position ij). Put another way, a line J is given by the set of points $\dot{o}_{ij}$, $\{1 \leq \dot{o}_{ij} \leq n \mid \dot{o}_{ij}, n \in \mathbb{N}\}$, where $n = |J|$. The number of numerical entries equals the number of vertices on $H^{\#}$.

$\dot{O}$	a	b	c	d	e
1	1	1	–	–	–
2	2	–	2	–	–
3	3	–	–	–	–
4	–	2	–	1	1
5	–	3	–	–	–
6	–	–	1	–	–
7	–	–	3	2	–
8	–	–	4	–	–
9	–	–	–	–	2

n	3	3	4	2	2	
l	2	2	3	1	1	**9**
m	1	1	2	0	0	

Note that columns of numbered entries on $\dot{O}$ equate with lines on $S^{\#}$, or the (horizontally indicated) chains in $H^{\#}$. Where there are two or more values in a column, this equates with a line on $S^{\#}$ (vertex on *G''*) – whereas a single value in a column would imply an entity that was a single point rather than a line. Any pair of consecutive natural numbers in a column indicates a line segment directly connecting the points concerned. Hence we can infer from $\dot{O}$ that, since column *c* contains consecutive numbers 2 and 3 (rows 2 and 7), there is a line segment joining the corresponding points 2 and 7 on $S^{\#}$. From the set of such line segments, we can obtain *G'*.

The rows of numbered entries on $\dot{O}$ equate with intersections on $S^{\#}$, or the (vertically or diagonally indicated) knots on $H^{\#}$. Where a row contains two or more values, this equates with a single intersection node on *G'*; hence we can infer A∩B (at point 1), A∩C (at point 2), etc., and hence obtain the edges on *G''*. A single numerical entry in a row indicates a pendant end, e.g. entry 3 in column B indicates that point 5 is pendant.

For a given line (column *j*), the highest value of $\dot{o}_{ij}$ indicates the number of nodal points (intersection points or pendant end points) a line joins, i.e. n (where $n \geq 2$). The continuity of a line (the number of line segments is then $l=n-1$ (where $l \geq 1$). The number of intermediate intersection points is $m=n-2$ (where $m \geq 0$). Across the whole line structure, the total number of line segments (equivalent to the number of links in the corresponding graph *G'*) is $L=\Sigma l$. The total number of nodal points, N, is the number of rows in the matrix (but $N \neq \Sigma n$, since that would involve double counting of nodal points). In the example, for line A, the highest value in column A is 3, so line A comprises 3 nodal points (n=3) and two line segments (l=2), and with one intermediate intersection (m=1); overall $L=\Sigma l$ =9.

4.1.2 *Parametric point matrix, $\dot{P}$*

In the parametric point matrix, $\dot{P}$, each numerical entry $\dot{p}_{ij}$ indicates the parametric value *j*, along line *J*, of a point *I*. For example, $\dot{p}_{2,1}$ (a=½) here implies that point 2 is half-way along line A. In a parametric line structure, the parametric value $\dot{p}_{ij}$ is numbered from 0 to 1, i.e. $0 \leq \dot{p}_{ij} \leq 1$, $\dot{p}_{ij} \in \mathbb{R}$. Each line has a minimum of two identified points (namely the two end points), hence each column has precisely one value of '0' and one value of '1'. If point *I* does not lie on line *J*, a dash '–' is indicated (i.e. there is no numerical entry $\dot{p}_{ij}$ appearing in position *ij*). Put another way, a line X, defined by parameter *x*, is given by the set of points $\dot{p}_{ix}$, $\{0 \leq \dot{p}_{ix} \leq 1 \mid \dot{p}_{ix} \in \mathbb{R}\}$. As with $\dot{O}$, the number of numerical entries equals the number of vertices on $H^{\#}$.

$\dot{P}$	*a*	*b*	*c*	*d*	*e*
1	0	0	–	–	–
2	½	–	⅓	–	–
3	1	–	–	–	–
4	–	½	–	0	0
5	–	1	–	–	–
6	–	–	0	–	–
7	–	–	⅔	1	–
8	–	–	1	–	–
9	–	–	–	–	1

m	1	1	2	0	0
l	2	2	3	1	1
n	3	3	4	2	2

Here, as with $\dot{O}$, *n* is the total number of nodal points (intersection points or pendant end points) associated with a particular line on $S^{\#}$, which is the number of numerical values (i.e. entries between 0 and 1 inclusive) in a given column ($n \geq 2$); *l* is the number of line segments constituting a particular line, which is equal to $n-1$ (hence $l \geq 1$); and *m* is the number of intermediate intersection points, which is the number of fractional values of $\dot{p}_{ij}$ (i.e. $0 < \dot{p}_{ij} < 1$); $m = n-2$ (hence $m \geq 0$). As with $\dot{O}$, we can infer from $\dot{P}$ all the ordered nodes and links on the equivalent graph *G'* and all the vertices and edges on *G''*. In fact, since it is known that every line has a point with $\dot{p}_{ij}=0$ and another with $\dot{p}_{ij}=1$, it is not even necessary to specify pendant ends explicitly; these may simply be inferred. Hence in our case, we need not include points 3, 5, 6, 8 or 9 in the matrix since those points (i.e. the existence of points at which $\dot{p}_{ij}=0$ and $\dot{p}_{ij}=1$) are inferred from the parametric specification. Hence $\dot{P}$ can be collapsed down to a concise version:

$\dot{P}$	a	b	c	d	e
1	0	0	–	–	–
2	½	–	⅓	–	–
4	–	½	–	0	0
7	–	–	⅔	1	–

This 4×5 matrix $\dot{P}$, together with the range values for $\dot{p}_{ij}$ (i.e. $0 \le \dot{p}_{ij} \le 1$), completely specifies $S_P{}^{\#}$. Going from $\dot{O}$ to $\dot{P}$ and vice versa:

- From $\dot{O}$ to $\dot{P}$, for a given column, relabel the first node ($\dot{o}_{ij}=1$) as $\dot{p}_{ij}=0$, and the last (nth) node ($\dot{o}_{ij}=n$) as $\dot{p}_{ij}=1$, and distribute the intermediate points with fractional parametric values in numerical order.
- From $\dot{P}$ to $\dot{O}$, for a given column, let the point $\dot{p}_{ij}=0$ become $\dot{o}_{ij}=1$, and relabel intermediate intersection points in order via natural numbers, until the final point $\dot{p}_{ij}=1$ becomes $\dot{o}_{ij}=n$. (If the starting point is the concise version of $\dot{P}$, then a first step would be to create additional rows to accommodate explicitly any additional points representing $\dot{p}_{ij}=0$ or 1.)

4.1.3 *Continuity point matrix,* $\dot{Q}$

In the continuity point matrix, $\dot{Q}$, a numerical entry $\dot{q}_{ij}$ indicates if a line J is continuous through point I or not: if line J is continuous through point I, then $\dot{q}_{ij}=1$; if line J terminates at point I, then $\dot{q}_{ij}=0$. If point I is not incident on line J, then we indicate a dash ('–'). We can express $\dot{q}_{ij}$ in relation to $\dot{p}_{ij}$ as follows: for any line J, $\dot{q}_{ij}=0$ where $\dot{p}_{ij}=0$ or $\dot{p}_{ij}=1$; $\dot{q}_{ij}=1$ where $\dot{p}_{ij}$ is fractional, i.e. where $0<\dot{p}_{ij}<1$. Hence it is possible to obtain $\dot{Q}$ directly from $\dot{P}$.

$\dot{Q}$	a	b	c	d	e		(q_0, q_1)	r	d	Node type	Hierarchy relations
1	0	0	–	–	–		(2,0)	2	2	L	$h(A) = h(B)$
2	1	–	1	–	–		(0,2)	2	4	X	$h(A) = h(C)$
3	0	–	–	–	–		(1,0)	1	1	P	–
4	–	1	–	0	0		(2,1)	3	4	K	$h(B) \ge h(D), h(E)$
5	–	0	–	–	–		(1,0)	1	1	P	–
6	–	–	0	–	–		(1,0)	1	1	P	–
7	–	–	1	0	–		(1,1)	2	3	T	$h(B) \ge h(C)$
8	–	–	0	–	–		(1,0)	1	1	P	–
9	–	–	–	–	0		(1,0)	1	1	P	–
m	**1**	**1**	**2**	**0**	**0**						
l	2	2	3	1	1	**9**					

Note that each column contains exactly two '0' entries, indicating that each line has two distinct end points. Each column therefore must have ≥2 numerical entries (i.e. two or more entries where $\dot{q}$=0 or $\dot{q}$=1).

Considering each column of $\dot{Q}$, we can denote the sum of the $\dot{q}$ values by m, i.e. $m=\Sigma\dot{q}_j$. The value m indicates the number of intermediate intersections along a line j. We can obtain the structural property of continuity (l) – which is the number of line segments a line is constituted by – as $l=m+1$. The sum of continuity values $L=\Sigma l=9$ is equivalent to the number of links (edges) in the equivalent graph G'. Meanwhile the number of rows gives N, the total number of nodal points on the line structure $S^{\#}$, or nodes on the equivalent graph G'; here N=9.

We can also learn useful information from the rows. Let the 'continuity ordinates' of a given point be (q_0, q_1) where q_0 is the number of terminating lines (zeroes in the continuity matrix $\dot{Q}$, i.e. where $\dot{q}_{ij}$=0) and q_1 is the number of continuing lines (ones in the continuity matrix $\dot{Q}$, i.e. where $\dot{q}_{ij}$=1). From this we can obtain the value of r, the number of lines at a point, $r=q_0+q_1$; and the degree of the nodal point corresponding to the point, $d=q_0+2q_1$. We can also infer node type: (1, 0) is a pendant end (labelled 'P' in the matrix above); (1, 1) is a T-intersection; (2, 0) is an L intersection; (0, 2) is an X intersection; (2, 1) is a K intersection. Furthermore, we can also infer hierarchical relations: at point 1, two lines are mutually terminating; at point 2, the lines are mutually intersecting; while at point 7, one line terminates on another, and at point 4, two lines terminate on one continuing line. Hence, where we have a line X and a line Y, and hierarchical values h(X) and h(Y), then where $\dot{q}_{ix}>\dot{q}_{iy}$ at their point of intersection i, $h(X)\geq h(Y)$, where a higher numerical value of h represents a higher hierarchical status.[6]

The continuity point matrix $\dot{Q}$ is therefore able to tell us information about the kind of intersection (e.g. not just two lines meeting, but whether these two lines meeting at an L or a T or an X junction) and encapsulate the information on continuity, hierarchy and junction type present in $S^{\#}$ that is missing from G' or G''.

4.1.4 Connectivity point matrix, $\dot{K}$

In the connectivity point matrix, $\dot{K}$, a numerical entry $\dot{k}_{ij}$ represents the number of other lines that a line J intersects with at point I. This value will be the same for each line at point I, therefore all numerical entries in row i will have a constant value ($\dot{k}_i$) for all

[6] The inequality symbol '≥' is used here (as per Marshall, 2016) to allow flexibility of interpretation according to different possible indicators of hierarchy; and allows the possibility of two roads of the same status to meet where one continues and the other terminates; but disallows the possibility that the through route is of lower status than the terminating route.

applicable values of j. Since $\dot{k}_i$ is the number of *other* lines intersecting at point I, $\dot{k}_i$ will equal the total number of lines r incident at the point, less one, i.e. $\dot{k}_i=r_i-1$. As such, the values $\dot{k}_i$ can be obtained directly from the continuity matrix $\dot{Q}$. For example, each row of $\dot{K}$ has a numerical value (or set of numerical entries with the same value) $\dot{k}_i$ that is equal to one less than the number of lines incident, r (the value in the 'r' column to the right of matrix $\dot{Q}$ above). For example, for point 4, r=3 so the entries for row 4 in $\dot{K}$ will be $\dot{k}_4=3-1=2$. Note that at a pendant line-end, $r_i=1$, hence $\dot{k}_i=0$. The expression $\dot{k}_{ij}=0$ means line J connects with no other line at point I (not that no line is incident on the point I).

$\dot{K}$	a	b	c	d	e	Σ
1	1	1	–	–	–	2
2	1	–	1	–	–	2
3	0	–	–	–	–	0
4	–	2	–	2	2	6
5	–	0	–	–	–	0
6	–	–	0	–	–	0
7	–	–	1	1	–	2
8	–	–	0	–	–	0
9	–	–	–	–	0	0
Σ	2	3	2	3	2	**12** $= C$

Connectivity factor (k) values of each point

Connectivity (c) values of each line

Considering each column of $\dot{K}$, the sum of the $\dot{k}_j$ values is the connectivity c_j of the line J, i.e. in the context of the matrix $\dot{K}$, $\Sigma\dot{k}_j=c_j$. The sum of connectivity values for all lines (Σc_j), that is, the sum of the Σ row at the bottom of the matrix $\dot{K}$, gives the aggregate connectivity value C for the whole structure, i.e. $C=\Sigma c_j$ (as per route structure analysis). Meanwhile, considering each row of $\dot{K}$, the sum of $\dot{k}_i$ values is a number called the 'connectivity factor' (denoted k in route structure analysis[7]; $k=r^2-r$), hence for any point I, the connectivity factor $k=\Sigma\dot{k}_i$. (i.e. the sum of row i). The sum of these k values also equals the total network connectivity, i.e. $C=\Sigma k$. Hence $C=\Sigma k=\Sigma c_j$. In the example

[7] The connectivity factor k should not be confused with the property cardinality, also denoted k (Marshall, 2016), nor the connectivity matrix entries ($\dot{k}_{ij}$ or $\bar{k}_{ij}$),

network, $C=\Sigma k=\Sigma c_j=(2+2+6+2)=(2+3+2+3+2)=12$. The value C is also the sum of the degree values of the vertices in G'' (Figure 13 c).

We can also relate $\dot{K}$ to $H^{\#}$, G' and G''. A pair of ones in a row of $\dot{K}$ equates with a single tie in $H^{\#}$ (e.g. vertices 1_A and 1_B, Figure 8). A triple of twos equates with a knot, that is, a triple (or triangle) of ties in $H^{\#}$ (e.g. vertices 4_B, 4_D and 4_E, Figure 8); and so on. Indeed each positive value of $\dot{k}$ (i.e. $\dot{k}>0$) indicates the degree of the vertex in a sub-graph comprising only a tie or knot on $H^{\#}$ (this sub-graph excluding links and chains) (e.g. vertex 1_A on $H^{\#}$ has a degree of 1; vertex 4_B on $H^{\#}$ has a degree of 2; Figure 8a). If a $\dot{k}$ value is zero, this indicates a pendant node on $H^{\#}$, not part of a knot on $H^{\#}$.

4.1.5 Intersection point matrix, $\dot{J}$

In the intersection point matrix, $\dot{J}$, a numerical entry $j_{ij}=1$ represents an intersection point on line J, i.e. where line J intersects with at least one other line. The case of $j_{ij}=0$ represents a point on a line not on an intersection, i.e. a pendant end. We can obtain $\dot{J}$ from $\dot{K}$, by setting $j_{ij}=1$ if $\dot{k}_{ij}\geq 1$; $j_{ij}=0$ if $\dot{k}_{ij}=0$; or j_{ij}='–' if $\dot{k}_{ij}$='–'.[8]

$\dot{J}$	*a*	*b*	*c*	*d*	*e*	Σ
1	1	1	–	–	–	2
2	1	–	1	–	–	2
3	0	–	–	–	–	0
4	–	1	–	1	1	3
5	–	0	–	–	–	0
6	–	–	0	–	–	0
7	–	–	1	1	–	2
8	–	–	0	–	–	0
9	–	–	–	–	0	0
Σ	2	2	2	2	1	**9**

r_i=no. of lines at each point I

i=no. of intersection points on each line J

Considering each column of $\dot{J}$, the sum of the entries Σj_j values is the number of intersection points (non pendant nodes) i on a line, $i=\Sigma j_j$. Considering each row (for

[8] $\dot{J}$ can be obtained from $\dot{P}$ or $\dot{Q}$ as well as $\dot{K}$. From $\dot{P}$ for a given numerical entry ($0\leq\dot{p}_{ij}\leq 1$), if there is at least one *other* numerical entry ($0\leq\dot{p}_{ij}\leq 1$) in the row, then $j_{ij}=1$; if $\dot{p}_{ij}$ is the only numerical entry in the row, then $j_{ij}=0$; if $\dot{p}$='–', then j= '–'. Likewise from $\dot{Q}$ for a given numerical entry ($\dot{q}$= 0 or 1), if there is at least one *other* numerical entry ($\dot{q}$= 0 or 1) in the row, then $j_{ij}=1$; if $\dot{q}_{ij}$ is the only numerical entry in the row, then $j_{ij}=0$.

which, incidentally, each numerical entry j_i, will be constant, either a single '0' or a set of '1' values) the sum of the entries Σj_i is the number of lines intersecting at a given point, i.e. $r_i=\Sigma j_i$. As is clear from the matrix, $\Sigma i=\Sigma r_i$. The significance of this matrix is really in identifying intermediate points of intersection along a line. For the ordinal structure, this means an entry exists if and only if j (or i or $\dot{k}$) has a numerical value (0 or 1).

4.1.6 Incidence point matrix, $\dot{I}$

In the incidence point matrix, $\dot{I}$, an entry i_{ij} equals 1 if the point I lies on line J, otherwise is 0. This may be obtained:

- from $\dot{Q}$ by setting $i_{ij}=1$ if $\dot{q}_{ij}=0$ or 1; $i_{ij}=0$ if $\dot{q}_{ij}=$ '–';
- from $\dot{K}$ by setting $i_{ij}=1$ if $\dot{k}_{ij}$ is any numerical entry ($\dot{k}_{ij}\geq 0$); else $i_{ij}=0$ if $\dot{k}_{ij}=$ '–'; or
- from $\dot{J}$ by setting $i_{ij}=1$ if j_{ij} is 0 or 1; else $i_{ij}=0$ if $j_{ij}=$ '–'.

Hence:

$\dot{I}$	a	b	c	d	e		r
1	1	1	0	0	0		2
2	1	0	1	0	0		2
3	1	0	0	0	0		1
4	0	1	0	1	1		3
5	0	1	0	0	0		1
6	0	0	1	0	0		1
7	0	0	1	1	0		2
8	0	0	1	0	0		1
9	0	0	0	0	1		1
n	**3**	**3**	**4**	**2**	**2**		
l	2	2	3	1	1	**9**	

The sum of entries (i_i values) in a given row i gives the number of lines (r) incident at the corresponding point, i.e. $r=\Sigma i_i$. For example, 3 lines meet at point 4. Meanwhile the sum of entries (i_j values) in a given column equates with the number of nodal points (n) incident on a line j, i.e. $n_j=\Sigma i_j$. Hence from $\dot{I}$, we can also obtain the property of continuity (the number of line segments), $l=n-1$. As before, the total number of line segments in the line structure, $L=\Sigma l=\Sigma(n-1)$, in this case, $L=9$.

From the foregoing we can see that from $\dot{P}$ we can obtain $\dot{Q}$; from $\dot{P}$ or $\dot{Q}$ we can obtain $\dot{K}$; from $\dot{K}$ (or $\dot{P}$ or $\dot{Q}$) we can obtain $\dot{J}$, and from $\dot{J}$ (or from $\dot{P}$, $\dot{Q}$ or $\dot{K}$) we can obtain $\dot{I}$. We can also obtain $\dot{J}$ or $\dot{K}$ from $\dot{I}$.

4.2 Line matrices

In a line matrix $\bar{X}$, both rows and columns refer to *lines*; the matrix is of the form $R \times R$, where R is the number of lines in the line structure. The interesting thing about the line matrix is that it is defined purely by lines while the points are in a sense derivative, merely defined by the intersection of lines, or inferred along lines; or put another way, the proportionate or fractional lengths along lines.

4.2.1 *Ordinal line matrix,* $\bar{O}$

In the ordinal line matrix, $\bar{O}$, each entry $\bar{o}_{ij}$ indicates the order of a point along line J (column j) at its intersection with line I (row i). The order of points on a line J is numbered from 1 to n, i.e. $1 \leq \bar{o}_{ij} \leq n$; $n, \bar{o}_{ij} \in \mathbb{N}$. Each line has a minimum of two points, i.e. $n \geq 2$. A dash '–' indicates a case where line I does not intersect with line J, or where $i=j$. Hence any line X is given by the set of points $\bar{o}_x$, $\{1 \leq \bar{o}_x \leq n \mid \bar{o}_x \in \mathbb{N}\}$, where $n=|X|$, or the number of explicitly identified nodal points along the line. For example, B={1, 4, 5}; $n=|B|=3$. Note that the pattern of positions of numerical entries (but – in general – not also their numerical values) is symmetrical about the primary diagonal. The number of rows equals the number of columns equals the number of lines, hence R=5.

$\bar{O}$	A	B	C	D	E
A	–	1	2	–	–
B	1	–	–	1	1
C	2	–	–	2	–
D	–	2	3	–	1
E	–	2	–	1	–

This matrix $\bar{O}$ of itself does not completely specify the line structure, since it omits reference to pendant ends, which need to be specified separately. We may do this by adding a row '0' at the top to accommodate any pendant first point, and a row 'R+1' at the bottom, to accommodate any pendant last point, hence creating an *extended ordinal line matrix* $\bar{O}^+$.

$\bar{O}$	A	B	C	D	E	
0	–	–	1	–	–	
A	–	1	2	–	–	
B	1	–	–	1	1	
C	2	–	–	2	–	
D	–	2	3	–	1	
E	–	2	–	1	–	
$R+1$	3	3	4	–	2	
n	3	3	4	2	2	
l	2	2	3	1	1	**9**

The example specifies, in row 0, the value of 1, indicating the first point on line C. Meanwhile, in row 'R+1', the value of 4 indicates the final point on line C. Similarly, row 'R+1' contains the end points of lines A, B and E. For a given column in the extended matrix $\bar{O}^{+}$, the number of nodal points a line comprises, n, is indicated by the highest value of $\bar{o}_j$ in column j. The continuity of a line (the number of line segments), is then $l=n-1$. Hence $L=\Sigma l=9$.

As with the corresponding point matrix, $\dot{O}$, we can infer from matrix $\bar{O}$ that in this example A∩B (at the first point on A and the first on B), C∩D (at the third point on C and the second on D), etc. The pattern of linkage of line segments is not so directly evident in the line matrix, because a given point is not represented by a single entry or row, but must be inferred from the complete set of entries, symmetrically across the primary diagonal, e.g. the third point on C is the 2nd point on D (because the pair $\bar{o}_{\mathrm{CD}}$ and $\bar{o}_{\mathrm{DC}}$ will represent the same point on $S^{\#}$, i.e. where C∩D=D∩C). In general, for an intersection between two lines X and Y, this is a pair of entries $\bar{o}_{\mathrm{XY}}$, $\bar{o}_{\mathrm{YX}}$ (e.g. $\bar{o}_{\mathrm{AB}}$, $\bar{o}_{\mathrm{BA}}$). For an intersection between three lines, this is a set of three pairs of entries, $\bar{o}_{\mathrm{XY}}$, $\bar{o}_{\mathrm{YX}}$, $\bar{o}_{\mathrm{XZ}}$, $\bar{o}_{\mathrm{ZX}}$, $\bar{o}_{\mathrm{YZ}}$, $\bar{o}_{\mathrm{ZY}}$ (e.g. $\bar{o}_{\mathrm{BD}}$, $\bar{o}_{\mathrm{DB}}$, $\bar{o}_{\mathrm{BE}}$, $\bar{o}_{\mathrm{EB}}$, $\bar{o}_{\mathrm{DE}}$, $\bar{o}_{\mathrm{ED}}$). In general, for an intersection between r lines, there will be $k=r^2-r$ entries, or $k/2$ pairs of entries.

4.2.2 *Parametric line matrix, $\bar{P}$*

Recall that each position on a line X is indicated by a parametric value x where $0\le x\le 1$, $x\in\mathbb{R}$. In the parametric line matrix, $\bar{P}$, an entry $\bar{p}_{ij}$ ($0\le\bar{p}_{ij}\le 1$; $\bar{p}_{ij}\in\mathbb{R}$) indicates the parametric value j along a line J (column j) of its point of intersection with a line I (row i). A dash in the matrix indicates a case where line I does not intersect with line J, or where $i=j$.[9] In matrix $\bar{P}$, for a connected structure (ie. where each line connects to at least one other line), each row and each column has a minimum of one intersection point (i.e. one numerical entry, i.e. $0\le\bar{p}_{ij}\le 1$). As with $\bar{O}$, the pattern of numerical entries is symmetrical about the primary diagonal.

$\bar{P}$	a	b	c	d	e
A	–	0	⅓	–	–
B	0	–	–	0	0
C	½	–	–	1	–
D	–	½	⅔	–	0
E	–	½	–	0	–

[9] That is, where i and j refer to the same line. In this paper we disallow a line to end on itself or otherwise connect with itself, so the primary diagonal will always be composed of dashes.

This $R \times R$ matrix $\bar{P}$, together with the range values for $\bar{p}_{ij}$ (i.e. $0 \le \bar{p}_{ij} \le 1$) completely specifies $S_P^{\#}$. As with the equivalent point matrix, $\dot{P}$, it is known that every line X has a point with x=0 and another with x=1, and so it is not necessary to specify pendant ends in matrix $\bar{P}$: pendant ends may be inferred (which by definition do not relate to – fall upon – any other line; therefore they do not, need not feature in the matrix). In the example, the points a=1, b=1, c=0, c=1 and e=1 are not specified but their existence on $S^{\#}$ is inferred (because every line must have two ends hence have values 0 and 1).

To get:

- From the extended matrix $\bar{O}^+$ to $\bar{P}$: going down each of the column entries, convert any entry corresponding to the first end ($\bar{o}_{ij}$ =1) to $\bar{p}_{ij}$ =0, and convert any entry corresponding to the other end ($\bar{o}_{ij}=n$) to $\bar{p}_{ij}$=1, and distribute the intermediate points with suitable fractional parametric values. For example, going down column d, entries with $\bar{o}_{ij}$ =1 become $\bar{p}_{ij}$=0; while entry $\bar{o}_{ij}=2=n$ becomes $\bar{p}_{ij}$ =1.
- From $\bar{P}$ to $\bar{O}$, for a given column, let any entry $\bar{p}_{ij}$ =0 become $\bar{o}_{ij}$ =1, and let any entry $\bar{p}_{ij}$ =1 become $\bar{o}_{ij}$ $=n$; and relabel intermediate intersection points (fractional values, $0<\bar{p}_{ij}<1$) in order via natural numbers. If $\bar{p}_{ij}$=0 does not exist in the matrix $\bar{P}$, then add a row '0' to $\bar{O}$ to get $\bar{O}^+$ with an entry $\bar{o}_{ij}$=1; if $\bar{p}_{ij}$=1 does not exist in the matrix, then add a row 'R+1' to $\bar{O}$ to get $\bar{O}^+$ with an entry $\bar{o}_{ij}=n$.

4.2.3 Continuity line matrix $\bar{Q}$

In the continuity line matrix, $\bar{Q}$, a numerical entry $\bar{q}_{ij}$=0 or 1 indicates if a line J is continuous through its point of intersection with line I or not: if line J is continuous through the intersection with line I, then $\bar{q}_{ij}$=1; if line J terminates at line I, then $\bar{q}_{ij}$ =0. If line I does not intersect with line J, or if $i=j$, then we indicate a dash (–). We can express entry $\bar{q}$ in relation to the parameter $\bar{p}$ as follows: for any line j, $\bar{q}_{ij}$=0 where $\bar{p}_{ij}$=0 or $\bar{p}_{ij}$=1; $\bar{q}_{ij}$=1 where $\bar{p}_{ij}$ is fractional, i.e. $0<\bar{p}_{ij}<1$; and $\bar{q}_{ij}$='–' where $\bar{p}_{ij}$='–'. Hence it is possible to obtain $\bar{Q}$ directly from $\bar{P}$.

$\bar{Q}$	a	b	c	d	e
A	–	0	1	–	–
B	0	–	–	0	0
C	1	–	–	0	–
D	–	1	1	–	0
E	–	1	–	0	–

Comparing $\bar{q}_{ij}$ and $\bar{q}_{ji}$ gives us indication of hierarchical relation: if $\bar{q}_{ij}>\bar{q}_{ji}$, then line I is continuous while J is discontinuous, i.e. $\bar{q}_{ij}$=1, $\bar{q}_{ji}$=0; then $h(I) \ge h(J)$, where for any line X, $h(X)$ is the hierarchical status of line X (and where the higher value of h means higher in the hierarchy). For example, $\bar{q}_{Db}$=1; $\bar{q}_{Bd}$=0; if so then the hierarchical status h(B)≥h(D). Hence the line matrix can give us both continuity and hierarchy not found in *G'* or *G''* (although it does not give us any further useful information not contained in the equivalent point matrix). Basically, if starting out from $\bar{P}$, then we can get $\bar{Q}$ and associated properties (but if starting from $\dot{P}$, there is no need to invoke $\bar{P}$ or $\bar{Q}$ to get those properties).

4.2.4 Connectivity line matrix, $\bar{K}$

In the connectivity line matrix, $\bar{K}$, a numerical entry $\bar{k}_{ij}$ represents the number of lines that a line *J* intersects with at its point of intersection with line *I*.[10] Each entry $\bar{k}_{ij}$ is a natural number ($\bar{k}_{ij}$∈ℕ) or else a dash '–'.

$\bar{K}$	*a*	*b*	*c*	*d*	*e*	Σ
A	–	1	1	–	–	2
B	1	–	–	2	2	5
C	1	–	–	1	–	2
D	–	2	1	–	2	5
E	–	2	–	2	–	4
Σ	2	5	2	5	4	18

Here, pairs of numbers reflected across the primary diagonal tells us about the nature of the intersections between lines. In each case, $\bar{k}_{ij}=\bar{k}_{ji}$ ($i≠j$). If $\bar{k}_{ij}=\bar{k}_{ji}$=1, this corresponds with a single tie on $H^{\#}$, indicating two lines intersecting at a point (e.g. vertices 1_A and 1_B on $H^{\#}$, Figure 8, corresponding with point 1 on $S^{\#}$, Figure 13 a) – seen here in the matrix as a pair of entries across the primary diagonal. If $\bar{k}_{ij}=\bar{k}_{ji}$=2, this corresponds with a tie on $H^{\#}$ that is part of a 'triangular' knot, indicating two lines intersecting at a point at which a third line is also present, and hence we expect to find elsewhere in the matrix two further instances of $\bar{k}_{ij}=\bar{k}_{ji}$=2. In other words, the signature of a 3-line intersection on $S_P^{\#}$ or $H^{\#}$ is 6 entries of '2' in $\bar{K}$. From this information, in the example matrix we can infer that the line structure has three intersection points where two lines meet (A∩B, A∩C and C∩D), and one intersection point where three lines meet, involving three pairs of line intersections (B∩D, B∩E and D∩E). Hence information about intersection points may be gained from the line matrix even though the points are not explicitly labelled.

[10] Note that the sum of the rows and columns does not equal the aggregate connectivity *C*, since this would involve double counting.

4.2.5 *Intersection line matrix, $\bar{J}$*

We can also recognise an intersection line matrix $\bar{J}$, where an entry $\bar{j}_{ij}$=1 indicates an intersection between line I and line J; an entry $\bar{j}_{ij}$ =0 indicates there is no intersection between line I and line J. The intersection line matrix $\bar{J}$ is obtained as follows:

- From $\bar{P}$: if $\bar{p}_{ij}$ is any numerical value ($0 \leq \bar{p}_{ij} \leq 1$) then $\bar{j}_{ij}$=1; otherwise $\bar{j}_{ij}$=0.
- From $\bar{Q}$: if $\bar{q}_{ij}$ is any numerical value ($\bar{q}_{ij}$=0 or $\bar{q}_{ij}$=1) then $\bar{j}_{ij}$=1; otherwise j_{ij}=0.
- From $\bar{K}$: if $\bar{k}_{ij}$ is any numerical value ($\bar{k}_{ij} \geq 1$) then $\bar{j}_{ij}$=1; otherwise $\bar{j}_{ij}$=0.

$\bar{J}$	*a*	*b*	*c*	*d*	*e*	*Σ*
A	0	1	1	0	0	2
B	1	0	0	1	1	3
C	1	0	0	1	0	2
D	0	1	1	0	1	3
E	0	1	0	1	0	2
Σ	2	3	2	3	2	12

Considering each column of $\bar{J}$, the sum of the $\bar{j}$ values is the connectivity (c) of the line, i.e. $\Sigma\bar{j}_j=c_j$. Similarly, considering each row of $\bar{J}$, the sum of the $\bar{j}$ values is the connectivity (c) of the line, i.e. $\Sigma\bar{j}_i=c_i$. Of course, the matrix is symmetric about the primary diagonal; $c_j=c_i$. The sum of connectivity values for all lines gives the overall connectivity value for the network, i.e. $C=\Sigma c$. As such, the intersection line matrix $\bar{J}$ gives the same aggregate result of connectivity as the connectivity point matrix $\dot{K}$; though it does not give the connectivity factor (k) values for individual points; it gives less information because it is symmetrical and repeats information on continuity. So again the line matrix (in this case $\bar{J}$) is not necessary if we have the point matrix (in this case, $\dot{J}$) available; but if the starting point is $\bar{J}$ then we can use $\bar{J}$ to get C values, etc.

4.2.6 *Incidence line matrix, $\bar{I}$*

In the incidence line matrix $\bar{I}$, a numerical entry $\bar{\imath}_{ij}$ equals 1 if the line I intersects with line J, otherwise $\bar{\imath}_{ij}$=0. The incidence line matrix $\bar{I}$ is obtained from the parametric or continuity line matrices as follows:

- From $\bar{P}$: if $\bar{p}_{ij}$ is any numerical value ($0 \leq \bar{p}_{ij} \leq 1$) then $\bar{\imath}_{ij}$=1; otherwise $\bar{\imath}_{ij}$=0.
- From $\bar{Q}$: if $\bar{q}_{ij}$ is any numerical value ($\bar{q}_{ij}$=0 or $\bar{q}_{ij}$=1) then $\bar{\imath}_{ij}$=1; otherwise $\bar{\imath}_{ij}$=0.
- From $\bar{K}$: if $\bar{k}_{ij}$ is any numerical value ($\bar{k}_{ij} \geq 1$) then $\bar{\imath}_{ij}$=1; otherwise $\bar{\imath}_{ij}$=0.

$\bar{I}$	a	b	c	d	e	Σ
A	0	1	1	0	0	2
B	1	0	0	1	1	3
C	1	0	0	1	0	2
D	0	1	1	0	1	3
E	0	1	0	1	0	2
Σ	2	3	2	3	2	**12**

In fact, as it happens, this turns out the same matrix as the intersection line matrix $\bar{J}$. (The reason that the corresponding point matrices $\dot{I}$ and $\dot{J}$ are *not* the same is that for a point matrix, there is a 1 if line J is incident with point I, which may be the case even if there is no intersection between lines at that point: at a pendant end, entries i and j will differ, whereas $\bar{\imath}$ and $\bar{\jmath}$ are the same.)

4.3 Conversion between point and line matrices

Here, we illustrate going from $\dot{P}$ to $\bar{P}$ and vice versa. In each case there is a stepwise procedure which can be applied manually or could readily be automated.

4.3.1 From $\dot{P}$ to $\bar{P}$

The parametric point matrix $\dot{P}$...

$\dot{P}$	a	b	c	d	e
1	0	0	–	–	–
2	½	–	⅓	–	–
4	–	½	–	0	0
7	–	–	⅔	1	–

…can be converted into a parametric line matrix $\bar{P}$ as follows:

- The entries {a=0, ½} in column a of $\dot{P}$ correspond with points 1 and 2 whose respective rows also have numerical entries for columns b and c, hence indicating points of intersection of line A with lines B and C;
- These values become entries {a=0, ½} in column a of $\bar{P}$, feeding rows B and C;
- The entry {a=1} in the fuller version of $\dot{P}$ is discarded as it is a pendant end;
- Meanwhile, from $\dot{P}$, we know from column b that Line B intersects with A at point 1 where b=0; while we know from column c that line C intersects with line A at point 2 where c=⅓;
- These generate entries {0, ⅓} which feed the content of row A of $\bar{P}$, as the first entries for columns b and c.

This creates the first row and first column in the line matrix $\bar{P}$ for A as follows:

	a	b	c	d	e
A	–	0	⅓	–	–
B	0				
C	½				
D	–				
E	–				

We can repeat the above procedure for lines B to E, to get the full line matrix of $\bar{P}$:

$\bar{P}$	a	b	c	d	e
A	–	0	⅓	–	–
B	0	–	–	0	0
C	½	–	–	1	–
D	–	½	⅔	–	0
E	–	½	–	0	–

4.3.2 From $\bar{P}$ to $\dot{P}$

Converting from $\bar{P}$ to $\dot{P}$ requires inventing labels for points of intersection, which are otherwise not explicit in $\bar{P}$ (and hence cannot possibly be reconstituted from $\bar{P}$); in this case we suggest for convenience P1, P2, P3, P4 (corresponding with points 1, 2, 4, 7). Then, the parametric line matrix $\bar{P}$ can be converted into a parametric point matrix $\dot{P}$ by identifying sets of pairings across the primary diagonal, as follows:

- The diagonal pairing ($\bar{p}_{\mathrm{A}b}$=0, $\bar{p}_{\mathrm{B}a}$=0) indicates the existence of a point at which a=0 and b=0. This can be designated P1 (0, 0, –, –, –) for a 5 line structure; this becomes the first row of the point matrix $\dot{P}$;
- The diagonal pairing (½, ⅓) indicates the existence of a point at which a=½ and c=⅓. This can be designated P2 (½,–, ⅓, –, –); this becomes the second row of $\dot{P}$;
- The diagonal pairing ($\bar{p}_{\mathrm{D}b}$=½, $\bar{p}_{\mathrm{B}d}$=0) indicates the existence of a point at which b=½ and d=0. But there is also a second diagonal pairing ($\bar{p}_{\mathrm{E}b}$=½, $\bar{p}_{\mathrm{B}e}$=0) involving b=½, indicating the same point, where (also) e=0. Finally, the diagonal pairing ($\bar{p}_{\mathrm{D}e}$=0, $\bar{p}_{\mathrm{B}d}$=0) also indicates the same point, where d=0 and e=0. This point can be designated P3 (–, ½, –, 0, 0), forming the third row of $\dot{P}$;
- Finally, the diagonal pairing (⅔, 1) indicates the existence of a point where c=⅔ and d=1. This can be designated P4 (–, –, ⅔, 1, –); this becomes the final row of $\dot{P}$.

Hence we arrive at $\dot{P}$, albeit with newly generated point labels:

$\dot{P}$	a	b	c	d	e
P1	0	0	–	–	–
P2	½	–	⅓	–	–
P3	–	½	–	0	0
P4	–	–	⅔	1	–

5. Conclusion

This paper has demonstrated the line structure and its associated matrices as a means of specifying network structure. The line structure can be seen as a bridge between Euclidean and Cartesian geometry on the one hand, and graph theory on the other, as seen via demonstration of how $S^{\#}$ relates to G' and G'', via hypergraphs $H^{\#}$, $H^{\ddagger}$ and $H^{\dagger}$. In fact, we have seen that the line structure $S^{\#}$ captures 'more that is structural' than either G' or G'' – in particular in identifying properties of continuity and hierarchy. All of the elements and properties of and G' and G'' are found within $S^{\#}$; similarly all properties of G' and G'' including nodal degree are possible to obtain from the matrices $\dot{P}$ or $\bar{P}$. We have seen how a series of point matrices ($\dot{O}$, $\dot{P}$, $\dot{Q}$, $\dot{K}$, $\dot{J}$, $\dot{I}$) and line matrices ($\bar{O}$, $\bar{P}$, $\bar{Q}$, $\bar{K}$, $\bar{J}$, $\bar{I}$) may be used to express line structure, and derive structural properties such as connectivity, continuity and hierarchy therefrom (e.g. c, l, n, h, k, C, L, N).

It is has been seen that we can get from the parametric point matrix $\dot{P}$ to the parametric line matrix $\bar{P}$ and vice versa. From the ordinal or parametric matrices we can obtain all information about $S^{\#}$. However, as we proceed beyond those to subsequent matrices, we lose some information, variously about the position of intermediate points along lines, and the number of routes at an intersection, and how lines connect (at their ends or at intermediate points) so that when we arrive at the intersection or incidence line matrix, we cannot tell information about continuity or hierarchical relationships between lines: only which line is incident upon which.

Overall, the set of line matrices $\bar{X}$ give less information directly (what is missing is explicit information on points, pendant end points, junction type) than the set of point matrices $\dot{X}$. In this sense, it is unnecessary to use line matrices if point matrices are available. However, if starting from the line matrix $\bar{P}$, it is straightforward to obtain the continuity line matrix $\bar{Q}$, and hence values of hierarchy, etc. Even if less practically functional than the point matrix, the conceptual elegance of the line matrix is that it is purely about lines (and fractional lengths along lines), rather than being about points. As such, they are least like graphs (which emphasise relations between nodal points). Then again, the information requirement for the line matrix may be less than that for the corresponding point matrix or graph G' if the number of lines is less than the number of nodal points (vertices in G'). In the case of the example structure, there are 9 nodal points but only 5 lines, so the line matrix $\bar{P}$ (5×5) will be more parsimonious than the full point matrix $\dot{P}$ (9×5) or for that matter the incidence matrix (9×9) for *G'*. The concise version of $\dot{P}$ may be yet more parsimonious, as in the example (4×5).

The paper makes explicit several relationships that are already implicit. As such, it is primarily and elaboration and consolidation of existing knowledge about properties, albeit set out by a systematic set of conventions for specifying information, such as via matrices, as well as identification of the hypergraph $H^{\#}$. The details of the hypergraph $H^{\#}$ (also, $H^{\ddagger}$ and $H^{\dagger}$) and the extended set of matrices (beyond consideration of a single matrix such as matrix $\dot{P}$ that could of itself be used to fully specify $S^{\#}$) are not primarily intended for onward operational use in network analysis, but in a sense are used here to show how or why the line structure $S^{\#}$ has the potential to capture more information, more concisely than graphs *G'* or *G''*. In principle, that is, the line structure can 'say more, with less'; however, in actual practical utility of this theoretical potential would depend on a number of factors, not least the data format of available information.

While the line structure may be interpreted as a mathematical retrofit or generalisation of the route structure (Marshall, 2005; 2015), the line structure itself, as a mathematical model, is not the sole preserve of route structure analysis, but can be applied to represent *any* kind of network based on linear elements – which could include named streets, natural streets, road centre-lines or axial lines (albeit with different interpretations of the meaning and functioning of those lines) – all of which could therefore utilise the properties such as continuity and cardinality in onward network analyses. Line structure could also be used to represent any other structures of linear elements not connected with transport networks, such as engineering structures.

References

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List of symbols

Properties of lines or points

a	parametric value of a point along a line A
b	parametric value of a point along a line B
c	parametric value of a point along a line C
c	connectivity of a line
d	degree of a nodal point in a line structure, or node in equivalent graph G
d	parametric value of a point along a line D
e	parametric value of a point along a line E
$h(X)$	hierarchical status of line X (higher value indicates higher status)
$_i$	parameter indicating position in row of a matrix
i	no. of intersection points along a line
$\dot{\imath}_{ij}$	numerical entry in incidence point matrix $\dot{I}$
$\bar{\imath}_{ij}$	numerical entry in incidence line matrix $\bar{I}$
$_j$	parameter indicating position in row of a matrix
$\dot{\jmath}_{ij}$	numerical entry in intersection point matrix $\dot{J}$
$\bar{\jmath}_{ij}$	numerical entry in intersection line matrix $\bar{J}$
k	connectivity factor for an intersection
$\dot{k}_{ij}$	numerical entry in parametric point matrix $\dot{K}$
$\bar{k}_{ij}$	numerical entry in parametric point matrix $\bar{K}$
l	continuity of a line; no. of route segments between junctions and/or ends
m	no. of intermediate junctions along a line
n	no. of nodal points along a line, equivalent to nodes in graph G
$\dot{o}_{ij}$	numerical entry in ordinal point matrix $\dot{O}$
$\bar{o}_{ij}$	numerical entry in ordinal line matrix $\bar{O}$
$\dot{p}_{ij}$	numerical entry in parametric point matrix $\dot{P}$
$\bar{p}_{ij}$	numerical entry in parametric point matrix $\bar{P}$
q_0	no. of terminating lines at a point
q_1	no. of continuing lines at a point
$\dot{q}_{ij}$	numerical entry in parametric point matrix $\dot{Q}$
$\bar{q}_{ij}$	numerical entry in parametric point matrix $\bar{Q}$
r	no. of lines connecting at an intersection point
x	any variable; or parametric value of a point along a line X (x-axis)
y	any variable; or parametric value of a point along a line Y (y-axis)

Properties of whole line structure, graphs, matrices, etc.

C	aggregate connectivity (of a line structure)
C	the set of chains in hypergraph $H^{\#}$
E	the set of edges in a graph G
G'	primal graph
G''	dual graph
$H^{\#}$	hypergraph corresponding to $S^{\#}$
$\dot{I}$	incidence point matrix
$\bar{I}$	incidence line matrix
$\dot{J}$	intersection point matrix
$\bar{J}$	intersection line matrix
$\dot{K}$	parametric point matrix
$\bar{K}$	parametric point matrix
L	aggregate continuity (of a line structure); no. of links in corresponding graph G'
L	the set of links in hypergraph $H^{\#}$
N	no. of nodal points (in a line structure); no. of nodes in corresponding graph G'
N_i	no. of nodal points or nodes of degree i
$\dot{O}$	ordinal point matrix
$\bar{O}$	ordinal line matrix
P	label used to indicate a pendant node
$\dot{P}$	parametric point matrix
$\bar{P}$	parametric line matrix
$\dot{Q}$	continuity point matrix
$\bar{Q}$	continuity line matrix
R	no. of lines (in a line structure)
$S^{\#}$	line structure
$S^{=}$	line set corresponding to $S^{\#}$
$S_O^{\#}$	line structure (ordinal)
$S_P^{\#}$	line structure (parametric)
T	the set of ties in in hypergraph $H^{\#}$
V	the set of vertices in a graph G, or hypergraph $H^{\#}$
V_A	the set of 'anchored' vertices in hypergraph $H^{\#}$
V_E	the set of end vertices in hypergraph $H^{\#}$
V_I	the set of intersection vertices in hypergraph $H^{\#}$
V_M	the set of intermediate vertices in hypergraph $H^{\#}$
V_P	the set of pendant vertices in hypergraph $H^{\#}$
X	any line X (set of points) on $S^{\#}$