A Weak Convergence Criterion for Constructing Changes of Measure*

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Abstract

Based on a weak convergence argument, we provide a necessary and sufficient condition that guarantees that a nonnegative local martingale is indeed a martingale. Typically, conditions of this sort are expressed in terms of integrability conditions (such as the well-known Novikov condition). The weak convergence approach that we propose allows to replace integrability conditions by a suitable tightness condition. We then provide several applications of this approach ranging from simplified proofs of classical results to characterizations of processes conditioned on first passage time events and changes of measures for jump processes.

1 Introduction

Changing the probability measure is a powerful tool in modern probability. Changes of measure arise in areas of wide applicability such as in mathematical finance, in the setting of so-called equivalent pricing measures. A change of probability measure often relies on the specification of a nonnegative martingale process which in turn yields the underlying Radon-Nikodym derivative behind the change of measure. The key step in the typical construction of changes of measure involves showing the martingale property of a process of putative Radon-Nikodym derivatives. In order to verify this martingale property one often starts by defining a process that easily can be seen to be a local martingale via Itô's formula. The difficult part then involves ensuring that the local martingale is actually a martingale.

Since the distinction between local martingales and martingales involves verification of integrability properties (the ones behind the strict definition of a martingale), it is most natural to search for a criterion based on integrability of the underlying local martingale. This is the basis, for instance, of the so-called Novikov condition, which is a well-known criterion used to verify the martingale property of an exponential local martingale in the diffusion setting. Nevertheless, if ultimately one has the existence of a new probability measure, then one has a martingale defined by the corresponding change of measure. Thus, it appears that lifting the local martingale property for a nonnegative stochastic process to a bona-fide martingale property has more to do with the fact that the induced probability measure is indeed well-defined.

Our contribution in this note consists in putting into focus the aspect of tightness when proving the martingale property of a nonnegative local martingale. Connecting tightness with the verification of the martingale property is an almost trivial exercise, formulated in Theorem 1.1 below. Although only a very

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simple observation, this point of view is powerful as the applications in Section 3 illustrate. In particular, we illustrate our result in the context of the following four applications:

1. We provide a new proof of the result by Beneš (1971) on the existence of weak solutions to certain stochastic differential equations. (Subsection 3.1)
2. We prove the equivalence of weak solutions to stochastic differential equations that involve compound Poisson processes, whose intensity may depend on the current state of the system. (Subsection 3.2)
3. We weaken the assumptions of Giesecke and Zhu (2013) that yield the martingale property of certain local martingales involving counting processes. (Subsection 3.3)
4. We provide a new representation for conditional expectations of an Ornstein-Uhlenbeck process conditioned to hit a large level before hitting zero. (Subsection 3.4)

In all of these applications we avoid direct estimates of expectations and instead apply sample path arguments or weak convergence techniques under a sequence of approximating changes of measure.

For the sake of clear notation, for a sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$, each defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$, and a random variable Y, defined on a probability space $(\Omega, \mathcal{F}, P)$, we write

$$(P_n, Y_n) \overset{\mathfrak{w}}{\Longrightarrow} (P, Y) \ (n \uparrow \infty) \qquad \text{if} \qquad \lim_{n\uparrow\infty} P_n\left(Y_n \leq x\right) = P\left(Y \leq x\right) \text{ for each continuity point } x \text{ of } P\left(Y \leq \cdot\right).$$

Throughout we denote the corresponding expectation operators by E and E_n, respectively.

The proof of the following theorem is very simple and only relies on the definition of tightness; it is given in Section 2.

Theorem 1.1. *The following two statements hold:*

1. *Let M denote a nonnegative sub- or supermartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and let $\{M_n\}_{n\in\mathbb{N}}$ denote a sequence of nonnegative martingales, each defined on a filtered probability space $(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, P_n)$ such that $M_n(0) = 1$. Fix any sequence of (deterministic) times $\{t_m\}_{m\in\mathbb{N}}$ with $t_1 = 0$ and $\lim_{m\uparrow\infty} t_m = \infty$ and assume that $(P_n, M_n(t_m)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(t_m))$ $(n \uparrow \infty)$ for each $m \in \mathbb{N}$. Define a family $\{Q_n^m\}_{n,m\in\mathbb{N}}$ of probability measures via $\mathrm{d}Q_n^m = M_n(t_m)\mathrm{d}P_n$. Then M is a true martingale with $M(0) = 1$ if and only if*
$$\sup_{n\in\mathbb{N}} Q_n^m(M_n(t_m) > \kappa) \to 0 \qquad (\kappa \uparrow \infty) \tag{1}$$
for each $m \in \mathbb{N}$. That is, M is a true martingale if and only if $\{M_n(t_m)\}_{n\in\mathbb{N}}$ is tight under the sequence of measures $\{Q_n^m\}_{n\in\mathbb{N}}$ for each $m \in \mathbb{N}$.
2. *Let $M(\infty)$ denote a nonnegative random variable on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{M_n(\infty)\}_{n\in\mathbb{N}}$ denote a sequence of nonnegative random variables, each defined on a probability space $(\Omega_n, \mathcal{F}_n, P_n)$ such that $\lim_{n\uparrow\infty} E_n[M_n(\infty)] = 1$. Assume that $(P_n, M_n(\infty)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(\infty))$ $(n \uparrow \infty)$. Define a family $\{Q_n\}_{n\in\mathbb{N}}$ of probability measures via $\mathrm{d}Q_n = M_n(\infty)\mathrm{d}P_n$. Then $\mathbb{E}[M(\infty)] = 1$ holds if and only if*
$$\sup_{n\in\mathbb{N}} Q_n(M_n(\infty) > \kappa) \to 0 \qquad (\kappa \uparrow \infty).$$

It is important to note that showing the martingale property of the underlying nonnegative local martingale becomes an exercise in tightness in a very weak topology. Given the enormous literature on weak convergence analysis of stochastic processes, we feel that our test of the martingale property would be a useful one. For example, in order to show tightness of a sequence of random variables $\{A_n\}_{n\in\mathbb{N}}$ of the form $A_n = \exp(B_n + C_n)$ it is sufficient to show tightness for the sequences of random variables $\{B_n\}_{n\in\mathbb{N}}$ and $\{C_n\}_{n\in\mathbb{N}}$ separately; a task that is often easy, as we shall illustrate in Section 3.

Relevant literature

The standard way to show the martingale property of a nonnegative local martingale is to check some standard integrability condition; see for example Novikov (1972), Kazamaki and Sekiguchi (1983), or Ruf (2013). If the local martingale dynamics include jumps, a case that we explicitly allow here, then integrability conditions exist but they might not be trivial to check; see Lépingle and Mémin (1978), Protter and Shimbo (2008), Sokol (2013), Larsson and Ruf (2014), and Ruf (2015b) for such conditions and related literature.

Under additional assumptions on the local martingale, such as the assumption that it is constructed via an underlying Markovian process, further sufficient (and sometimes also necessary) criteria can be derived. Here we only provide the reader with some pointers to this vast literature. The following papers develop conditions different from Novikov-type conditions by utilizing the (assumed) Markovian structure of some underlying stochastic process, and contain a far more complete list of references: Cheridito et al. (2005), Cheridito et al. (2007), Blei and Engelbert (2009), Mijatović and Urusov (2012), and Ruf (2015a). Kallsen and Muhle-Karbe (2010) study the martingale property of stochastic exponentials of affine processes; their approach via the explicit construction of a candidate measure and the use of a simple lemma in Jacod and Shiryaev (2003) is close in spirit to our approach.

The weak existence of solutions to stochastic differential equations is often proven by means of changing the probability measure, see for example Portenko (1975), Engelbert and Schmidt (1984), Yan (1988), or Stummer (1993). This strategy for proving the weak existence of solutions requires the true martingale property of the putative Radon-Nikodym density. Our approach to prove the martingale property of such a density is in the spirit of the reverse direction: The tightness condition that implies the martingale property of a putative Radon-Nikodym density by Theorem 1.1 corresponds basically to the asserted existence of a certain probability measure — often corresponding to the existence of a solution to a stochastic differential equation.

2 Martingale property and tightness

In this section, we prove Theorem 1.1 and make some related observations.

Proof of Theorem 1.1. We start by showing the second statement. To simplify notation, we set $Y = M(\infty)$ and $Y_n = M_n(\infty)$ for all $n \in \mathbb{N}$. Assume first that $E[Y] = 1$. Then, for fixed $\kappa > 1$ and for a continuous function $f : [0,\infty] \to [0,\kappa]$ with $f(x) \leq x$ for all $x \geq 0$, $f(x) = x$ for all $x \in [0, \kappa - 1]$ and $f(x) = 0$ for all $x \in [\kappa, \infty)$, we compute that

$$\begin{aligned} Q_n(Y_n > \kappa) = E_n[Y_n \mathbf{1}_{\{Y_n > \kappa\}}] &= E_n[Y_n] - E_n[Y_n \mathbf{1}_{\{Y_n \leq \kappa\}}] \leq E_n[Y_n] - E_n[f(Y_n)] \\ &\to 1 - E[f(Y)] \leq 1 - E[Y \mathbf{1}_{\{Y \leq \kappa - 1\}}] = E[Y \mathbf{1}_{\{Y > \kappa - 1\}}] \qquad (n \uparrow \infty). \end{aligned}$$

Since Y is integrable, $\lim_{\kappa\uparrow\infty} E[Y \mathbf{1}_{\{Y > \kappa-1\}}] = 0$, we obtain one direction of the statement.

For the other direction, fix $\epsilon > 0$ and the continuous, bounded function $f : [0,\infty] \to \mathbb{R}$ with $f(x) = x \wedge \kappa$ for all $x \geq 0$. Then

$$E[Y] \geq E[f(Y)] = \lim_{n\uparrow\infty} E_n[f(Y_n)] \geq \liminf_{n\uparrow\infty} E_n[Y_n \mathbf{1}_{\{Y_n \leq \kappa\}}] = \liminf_{n\uparrow\infty} \left(E_n[Y_n] - E_n[Y_n \mathbf{1}_{\{Y_n > \kappa\}}]\right) \geq 1 - \epsilon$$

for κ large enough. This yields $E[Y] \geq 1$. Similarly, we have

$$E[f(Y)] = \lim_{n\uparrow\infty} E_n[f(Y_n)] \leq \lim_{n\uparrow\infty} E_n[Y_n] = 1$$

and letting κ tend to ∞, monotone convergence yields $E[Y] \leq 1$. This concludes the proof of the second statement in the theorem.

For the first statement, note that $[0,\infty) \ni t \mapsto E[M(t)]$ is monotone thanks to the sub- or supermartingale property of M, respectively. The statement then follows exactly as above, by using $Y = M(t_m)$ and $Y_n = M_n(t_m)$, for all $m, n \in \mathbb{N}$. □

Remark 2.1. We now make a few comments concerning Theorem 1.1 and its proof.

- The first statement of Theorem 1.1 can be further generalized since, for each t_m, a different approximating sequence of martingales might be used.
- Theorem 1.1 can also be proven by embedding all random variables in a common probability space, via Skorohod's representation theorem; see Billingsley (1999). The result then follows from a characterization of uniform integrability.
- An alternative way[1] to prove the second statement in Theorem 1.1 might proceed by using directly the fact that $E[M(\infty)] = 1$ if and only if the nondecreasing function $G : [0,\infty) \to [0,1]$, $z \mapsto E[\min\{M(\infty), z\}]$ is a distribution function, that is, satisfies $\lim_{z\uparrow\infty} G(z) = 1$. This approach, it turns out, yields that the condition that $(P_n, M_n(\infty)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(\infty))$ $(n \uparrow \infty)$ can be replaced by

$$\lim_{n\uparrow\infty} \int_0^z P_n\left(M_n(\infty) \leq x\right) \mathrm{d}x = \int_0^z P\left(M(\infty) \leq x\right) \mathrm{d}x \qquad \text{for each } z \in [0,\infty), \tag{2}$$

which appears to be weaker. Nevertheless, it is not difficult to see that $(P_n, M_n(\infty)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(\infty))$ $(n \uparrow \infty)$ if and only if (2) holds. The necessity follows from dominated convergence. Sufficiency follows from the fact that if a sequence of convex functions $\{G_n\}_{n\in\mathbb{N}}$ satisfies $\lim_{n\uparrow\infty} G_n(z) = G(z)$ for some convex function G, for all $z \in \mathbb{R}$, then the corresponding right derivatives $D^+G_n(x)$ converge to the derivative $DG(x)$, at any point $x \in \mathbb{R}$ at which G is differentiable. To see this , fix $x \in \mathbb{R}$ such that the derivative $DG(x)$ exists. Then, for all $\varepsilon > 0$, there exists an $h > 0$ such that $(G(x+h) - G(x))/h < DG(x) + \varepsilon$. Moreover, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $D^+G_n(x) \leq (G_n(x+h) - G_n(x))/h < DG(x)+\varepsilon$; thus $\lim_{n\uparrow\infty} D^+G_n(x) \leq DG(x)$. The other inequality follows from similar computations for the left derivatives which yields $\lim_{n\uparrow\infty} D^-G_n(x) \geq DG(x)$. □

The following corollary can be interpreted as a generalization of Theorem 1.3.5 in Stroock and Varadhan (2006) to processes with jumps. See also Lemma III.3.3 in Jacod and Shiryaev (2003) for a similar statement where a certain candidate measure Q is assumed to exist. We remark that the sequence of stopping times in the statement could, but need not, be a localization sequence of a local martingale. For example, suppose that the underlying local martingale M is continuous at its first hitting time τ of zero; then it suffices that the sequence of stopping times in Corollary 2.2 converge to τ.

Corollary 2.2. *Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a sequence of stopping times and $M_n = M^{\tau_n}$ the stopped versions of a nonnegative local martingale M with $M(0) = 1$. Assume that $\lim_{n\uparrow\infty} M_n(t) = M(t)$ P–almost surely for each $t \geq 0$ and that, for each fixed n, M_n is a martingale. Further, for each $t \geq 0$, define $\mathrm{d}Q_n^t = M_n(t)\mathrm{d}P$. Then M is a martingale if $\lim_{n\uparrow\infty} Q_n^t(\tau_n \leq t) = 0$ for all $t > 0$. Further, under the additional assumption that $\lim_{n\uparrow\infty} \tau_n = \infty$ P–almost surely, the converse also holds; that is, if M is a martingale then $\lim_{n\uparrow\infty} Q_n^t(\tau_n \leq t) = 0$ for all $t > 0$.*

Proof. Fix $t \geq 0$ and $\kappa > 0$ and observe that $(P, M_n(t)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(t))$ $(n \uparrow \infty)$. Also note that for each $n \in \mathbb{N}$,

$$Q_n^t\left(M_n\left(t\right) > \kappa\right) \leq Q_n^t\left(\tau_n \leq t\right) + E\left[M_n(t)\mathbf{1}_{\{\tau_n > t\}\bigcap\{M_n(t)>\kappa\}}\right] \leq Q_n^t\left(\tau_n \leq t\right) + E\left[M(t)\mathbf{1}_{\{M(t)>\kappa\}}\right]$$

because $M_n(t)\mathbf{1}_{\{\tau_n>t\}} = M(t)\mathbf{1}_{\{\tau_n>t\}}$. Since by assumption we can make the first term on the right-hand side arbitrarily small by increasing n, the martingale property of M follows directly from dominated convergence and Theorem 1.1. For the reverse direction, assume that M is a martingale and that $\lim_{n\uparrow\infty} \tau_n = \infty$ P–almost surely. Then,

$$\lim_{n\uparrow\infty} Q_n^t(\tau_n \leq t) = \lim_{n\uparrow\infty} E[M_n(t)\mathbf{1}_{\{\tau_n\leq t\}}] = \lim_{n\uparrow\infty} E[M(t)\mathbf{1}_{\{\tau_n\leq t\}}] = 0$$

by optional sampling and dominated convergence. □

The next result is of course well-known and only a very special case of, for instance, the theory of BMO martingales; see for example Kazamaki (1994). However, as we shall use the result below and as we would like to make this note self-contained, we provide a proof based on the observations we have made here before:

[1] We thank one of the referees for pointing us to this alternative proof technique.

Corollary 2.3. *Let L denote a continuous local martingale with $L(0) = 0$. Assume there exists some nondecreasing (deterministic) function $c : [0, \infty) \to \mathbb{R}$ such that $\min\{L(t), \langle L\rangle(t)\} \leq c(t)$ for all $t \geq 0$ almost surely. Then, $M = \mathcal{E}(L) = \exp(L - \langle L\rangle/2)$ is a martingale.*

Proof. For each $n \in \mathbb{N}$ let τ_n denote the crossing time of level n by M and fix $t > 0$. Obviously, $M_n = M^{\tau_n}$ satisfies $\lim_{n\uparrow\infty} M_n(t) = M(t)$ P–almost surely. Define the probability measures $\{Q_n^t\}_{n\in\mathbb{N}}$ as in Corollary 2.2 and observe that $\{Q_n^t(\tau_n \leq t)\}_{n\in\mathbb{N}}$ is a decreasing sequence since

$$Q_{n+1}^t(\tau_{n+1} \leq t) \leq Q_{n+1}^t(\tau_n \leq t) = Q_n^t(\tau_n \leq t)$$

for all $n \in \mathbb{N}$.

Now, fix $\epsilon \in (0, 1)$ and some $n \in \mathbb{N}$ with $n > \exp(c(t))/\epsilon$ and observe that

$$\{M^{\tau_n}(t) \geq n\} \subset \{L^{\tau_n}(t) > c(t)\} \subset \{\langle L\rangle(t \wedge \tau_n) \leq c(t)\}$$

holds P–almost surely. Therefore, we have

$$\{\tau_n \leq t\} = \{M^{\tau_n}(t) \geq n\} = \{M^{\tau_n}(t) \geq n\} \cap \{\langle L\rangle(t \wedge \tau_n) \leq c(t)\} \subset \left\{\widetilde{M}^{\tau_n}(t) > \frac{1}{\epsilon}\right\}$$

P–almost surely and thus Q_m^t–almost surely, where $\widetilde{M}(\cdot) = M^{\tau_n}(\cdot)/\exp(\langle L\rangle(\cdot \wedge t \wedge \tau_n))$ is a bounded, nonnegative Q_n^t–martingale by Girsanov's theorem. Markov's inequality then implies that $Q_n^t(\tau_n \leq t) \leq \epsilon$ and an application of Corollary 2.2 concludes. □

The next observation is useful when applying Theorem 1.1 in a continuous setup:

Lemma 2.4. *Let $\{L_n\}_{n\in\mathbb{N}}$ denote a sequence of continuous local martingales, each defined on a filtered probability space $(\Omega_n, \mathcal{F}_n, \mathbb{F}_n, Q_n)$ such that $L_n(0) = 0$. Assume that the sequence $\{\langle L_n\rangle(t)\}_{n\in\mathbb{N}}$ is tight along the sequence $\{Q_n\}_{n\in\mathbb{N}}$ of probability measures for some $t \in [0, \infty]$. Then also the sequence $\{L_n(t)\}_{n\in\mathbb{N}}$ is tight along $\{Q_n\}_{n\in\mathbb{N}}$.*

Proof. Fix $n \in \mathbb{N}$, let ρ_κ denote the first time that $\langle L_n\rangle$ crosses some level $\kappa > 0$ and observe that

$$\begin{aligned} Q_n\left(L_n(t) > \kappa\right) &\leq Q_n\left(L_n(t \wedge \rho_\kappa) > \kappa\right) + Q_n\left(\rho_\kappa < t\right) \leq \frac{E_n\left[L_n^2(t \wedge \rho_\kappa)\right]}{\kappa^2} + Q_n\left(\rho_\kappa < t\right) \\ &\leq \frac{1}{\kappa} + Q_n\left(\langle L_n\rangle(t) > \kappa\right) \end{aligned}$$

by Chebyshev's inequality and the fact that $E_n[L_n^2(t \wedge \rho_\kappa)] \leq E_n[\langle L_n\rangle(t \wedge \rho_\kappa)] \leq \kappa$; those last inequalities follow from the observation that the process $L_n^2(\cdot \wedge \rho_\kappa) - \langle L_n\rangle(\cdot \wedge \rho_\kappa)$ is a local martingale, bounded from below by $-\kappa$. □

3 Applications

Our goal here is to show that our approach could have advantages in terms of its relative simplicity. We shall write $\|\cdot\|$ for the Euclidean L_2–norm on $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We denote the space of cadlag paths $\omega : [0, t) \to \mathbb{R}^d$ for some $d \in \mathbb{N}$ and $t \in (0, \infty]$, endowed with the standard Skorokhod topology, by $D_{[0,t)}(\mathbb{R}^d)$. For sake of brevity, we shall use $D_{[0,\infty)} = D_{[0,\infty)}(\mathbb{R}^1)$.

3.1 Continuous processes: linear growth of drift

We begin by proving an extension of the well-known result by Beneš (1971) on the existence of weak solutions to a certain stochastic differential equation.

Theorem 3.1. *Let W be a d-dimensional Brownian motion and W^* the running maximum of its vector norm; to wit, $W^*(t) = \max_{s\in[0,t]}\{\|W(s)\|\}$. Let Y be a nonnegative supermartingale (under the same filtration) with cadlag paths such that $[Y, W_i] = 0$ for all $i \in \{1, \ldots, d\}$. Furthermore, let Y^* denote its maximum process, that is, $Y^*(t) = \max_{s\in[0,t]}\{Y(s)\}$ for each $t \geq 0$. Moreover, suppose that μ is a progressively measurable process satisfying*

$$\|\mu(t)\| \leq c(t, Y^*(t))\,(1 + W^*(t))$$

for all $t \geq 0$ and some function $c : [0,\infty) \times [0,\infty) \to [0,\infty)$ that is nondecreasing in both arguments. Then the local martingale M, defined as

$$M(\cdot) = \exp\left(\int_0^{\cdot} \mu(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^{\cdot} \|\mu(s)\|^2\,\mathrm{d}s\right)$$

is a martingale.

Proof. First, since the function c is nondecreasing in both arguments and thus

$$\int_0^t \|\mu(s)\|^2\,\mathrm{d}s \leq tc(t, Y^*(t))^2(1 + W^*(t))^2 < \infty$$

for all $t \geq 0$, the stochastic integral $\int_0^{\cdot} \mu(s)\mathrm{d}W(s)$ exists and M is well-defined.

Let us now define the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of progressively measurable processes by $\mu_n(\cdot) = (\mu(\cdot)\wedge n)\vee(-n)$, where the minimum and maximum are taken coordinate-wise. It follows easily, for example by applying the definition of the stochastic integral, that the sequence of local martingales M_n, defined as

$$M_n(\cdot) = \exp\left(\int_0^{\cdot} \mu_n(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^{\cdot} \|\mu_n(s)\|^2\,\mathrm{d}s\right)$$

satisfies $(P, M_n(t)) \overset{\mathfrak{w}}{\Longrightarrow} (P, M(t))$ $(n \uparrow \infty)$ for all $t \geq 0$. By Corollary 2.3, the local martingale M_n is a true martingale since μ_n is bounded. Now, observe that

$$B_n(\cdot) = W(\cdot) - \int_0^{\cdot\wedge T} \mu_n(s)\mathrm{d}s$$

is a Brownian motion under the probability measure Q_n, induced by M_n via $\mathrm{d}Q_n = M_n(T)\mathrm{d}P$ for some $T \geq 0$, and that

$$M_n(t) = \exp\left(\int_0^t \mu_n(s)\mathrm{d}B_n(s) + \frac{1}{2}\int_0^t \|\mu_n(s)\|^2\,\mathrm{d}s\right), \qquad 0 \leq t \leq T.$$

Moreover, Y is a nonnegative $\mathbb{Q}_n$–supermartingale since $[Y, W_i] = 0$ for all $i \in \{1, \ldots, d\}$ by assumption.

We now note that

$$\|W(t)\| \leq \|B_n(t)\| + \int_0^t c(s, Y^*(s))\,(1 + W^*(s))\,\mathrm{d}s \leq B_n^*(t) + c(t, Y^*(t))t + c(t, Y^*(t))\int_0^t W^*(s)\mathrm{d}s$$

for all $t \geq 0$, where B_n^* is defined in the same way as W^*. An application of Gronwall's inequality then yields

$$W^*(t) \leq (B_n^*(t) + c(t, Y^*(t))t)\exp(c(t, Y^*(t))t)$$

for all $t \geq 0$. Since $\{B_n^*(t)\}_{n\in\mathbb{N}}$ and $\{Y^*(t)\}_{n\in\mathbb{N}}$ are tight along $\{Q_n\}_{n\in\mathbb{N}}$, the later as the maximum process of a nonnnegative supermartingale, we also obtain that $\{W^*(t)\}_{n\in\mathbb{N}}$ is tight along $\{Q_n\}_{n\in\mathbb{N}}$, for all $t \geq 0$. This guarantees that $\{\int_0^t \|\mu_n(s)\|^2\,\mathrm{d}s\}_{n\in\mathbb{N}}$ is tight as well and Lemma 2.4 then yields the tightness of $\{\int_0^t \mu_n(s)\mathrm{d}B_n(s)\}_{n\in\mathbb{N}}$ for all $t \geq 0$. Thus, $\{M_n(t)\}_{n\in\mathbb{N}}$ is tight along $\{Q_n\}_{n\in\mathbb{N}}$ for each $t \geq 0$ and M is a true P–martingale by Theorem 1.1. $\square$

To recover the result by Beneš (1971), suppose that $\widetilde{\mu} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is measurable and satisfies

$$\|\widetilde{\mu}\,(t, x)\| \leq \widetilde{c}(t)\,(1 + \|x\|)$$

for all $t \geq 0$, $x \in \mathbb{R}^d$, and some nondecreasing function $\widetilde{c} : [0, \infty) \to [0, \infty)$. Then, for any $T > 0$, with $\mu(t) = \widetilde{\mu}(t, W(t))$ in the last proposition, the above computations show the weak existence of a solution to the stochastic differential equation

$$X(t) = \int_0^t \widetilde{\mu}\,(s, X(s))\,\mathrm{d}s + B(t), \qquad 0 \leq t \leq T,$$

where B denotes a Brownian motion. For an alternative proof of this statement, using the Novikov condition along with "salami tactics," see Proposition 5.3.6 in Karatzas and Shreve (1991).

The more general assertion of Theorem 3.1 cannot be proven via this "salami tactics." For example, if Y is a nonnegative pure-jump supermartingale, then the quadratic covariation processes of Y and the components of W are zero, even if the jump sizes of Y depend, in a nonanticipative way, on the paths of W.

3.2 Compound Poisson processes

We continue with an application of Theorem 1.1 to a class of stochastic differential equations (SDEs) involving jumps. Towards this end, for any $\omega \in D_{[0,\infty)}(\mathbb{R}^d)$, we shall write $\Delta\omega(t) = \omega(t) - \omega(t-)$ for all $t > 0$. We call a function g with domain $[0, \infty) \times D_{[0,\infty)}(\mathbb{R}^d)$ *predictable* if g is measurable and satisfies $g(s, \omega) = g(s, \varpi)$ for all $s \leq t$ and for all $\omega, \varpi \in D_{[0,\infty)}(\mathbb{R}^d)$ with $\omega(r) = \varpi(r)$ for all $r < t$, for all $t \geq 0$.

Let F denote the distribution of an $\mathbb{R}^d \setminus \{0\}$–valued random variable for some $d \in \mathbb{N}$ and fix $x_0 \in \mathbb{R}^d$ and a predictable function $g : [0, \infty) \times D \to [0, \infty)$. Define $\Psi_g(t, \omega) = \int_0^t g(s, \omega)\mathrm{d}s$ for all $t \geq 0$ and $\omega \in D_{[0,\infty)}(\mathbb{R}^d)$. We say that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ along with an adapted process X with cadlag paths in $\mathbb{R}^d$ is a weak solution to the SDE

$$X(\cdot) = x_0 + \sum_{j=1}^{N_g(\cdot)} Z_j^F, \tag{3}$$

if $X(0) = x_0$, the jumps $\{\Delta X \mid \Delta X \neq 0\}$ of X are independent and identically distributed according to F, and $L_g\,(\cdot) = N_g\,(\cdot) - \Psi_g(\cdot, X)$ is a P–local martingale up to the first hitting time of infinity by N_g, where $N_g\,(\cdot) = \sum_{s \leq \cdot} \mathbf{1}_{\{\Delta X(s) \neq 0\}}$ is the sum of jumps. Theorem 3.6 in Jacod (1975) yields the existence and uniqueness of a weak solution to (3); however, such solution might be explosive in the sense that $N_g\,(t) = \infty$ for some $t \in (0, \infty)$ with positive probability. Below, in Lemma 3.2, we will provide sufficient conditions to ensure a non-explosive solution.

Any non-explosive solution $(\Omega, \mathcal{F}, \mathbb{F}, P), X$ of (3) corresponds to a compound Poisson process with jumps distributed according to F such that its instantaneous intensity to jump at time t equals $g(t, X)$; more precisely

$$\sum_{s \leq \cdot} \mathbf{1}_{\{\Delta X(s) \neq 0\}} = N\left(\Psi_g(\cdot, X)\right)$$

for some Poisson process N with unit rate.

Such a non-explosive solution exists, for example, if $g(t, X) = \mathfrak{g}(t)$ only depends on time and $\int_0^t \mathfrak{g}(s)\mathrm{d}s < \infty$ for all $t \geq 0$. The following lemma yields another existence result:

Lemma 3.2. *Fix $x_0 \in \mathbb{R}^d$ and let F denote the distribution of an $\mathbb{R}^d \setminus \{0\}$–valued random variable whose components have finite expected value. Let $\mathfrak{g} : \mathbb{R}^d \to \mathbb{R}$ be measurable, such that there exists $c > 0$ with $\mathfrak{g}(y) \leq c(1 + \|y\|)$ for all $y \in \mathbb{R}^d$. Then there exists a non-explosive weak solution to* (3) *with $g(t, \omega) = \mathfrak{g}(\omega(t-))$ for all $(t, \omega) \in [0, \infty) \times D$.*

Proof. Let N denote a Poisson process with unit rate and $Z = \{Z_j^F\}_{j \geq 1}$ a sequence of independent F–distributed random variables independent of N. First observe that the process

$$J\,(\cdot) = x_0 + \sum_{j=1}^{N(\cdot)} Z_j^F$$

always exists and that the process

$$\Gamma(\cdot) = \int_0^{\cdot} \frac{1}{\mathfrak{g}(J(s))} \mathrm{d}s \geq \frac{1}{c} \int_0^{\cdot} \frac{1}{1 + \|J(s)\|} \mathrm{d}s$$

is strictly increasing (before hitting infinity) and satisfies $\lim_{t \uparrow \infty} \Gamma(t) = \infty$ since there exists $K(\omega) \in (0, \infty)$ such that $\|J(t)\| \leq K(\omega)(1+t)$ by the strong law of large numbers, for each $t \geq 0$. Thus, Γ yields a valid time change. Now, consider the non-explosive process $X(\cdot) = J(\Gamma^{-1}(\cdot))$ and observe that $D\Gamma^{-1}(\cdot) = 1/D\Gamma(\Gamma^{-1}(\cdot))$, which implies

$$\Gamma^{-1}(\cdot) = \int_0^{\cdot} D\Gamma^{-1}(s) ds = \int_0^{\cdot} \mathfrak{g}(J(\Gamma^{-1}(s))) \mathrm{d}s = \int_0^{\cdot} \mathfrak{g}(X(s)) \mathrm{d}s = \psi_g(\cdot, X),$$

which in turn verifies that X is a non-explosive solution to (3). □

The next theorem provides a sufficient condition that guarantees that the intensity in Poisson processes can be changed without changing the nullsets of the underlying probability measure. For example, any compound Poisson process whose jump intensity is strictly positive, can be changed, via an equivalent change of measure, to a compound Poisson process with unit intensity (set $g_2 = 1$ below).

Theorem 3.3. *Fix $x_0 \in \mathbb{R}^d$ and let F denote the distribution of an $\mathbb{R}^d \setminus \{0\}$–valued random variable. Moreover, let $g_1, g_2 : [0, \infty) \times D_{[0,\infty)}(\mathbb{R}^d) \to (0, \infty)$ be strictly positive, predictable functions and denote the corresponding weak solutions of (3) with $g = g_1$ and $g = g_2$ by X_1, and X_2. Assume that X_2 is non-explosive. Then, the process M, defined by*

$$M(\cdot) = \exp\left(\int_0^{\cdot} (\log g_2(s, X_1) - \log g_1(s, X_1)) \mathrm{d}L_{g_1}(s) - \int_0^{\cdot} (g_2(s, X_1) - g_1(s, X_1)) \mathrm{d}s \right),$$

is a true martingale. Furthermore, for each $t \geq 0$, the distribution of X_1^t under Q^t, defined by $\mathrm{d}Q^t = M(t)\mathrm{d}P$, equals the distribution of X_2^t.

Proof. Theorem VI.2 in Brémaud (1981) yields that M is a local martingale. If M is a true martingale, then Theorem VI.3 in Brémaud (1981) yields the assertion on the distribution of X_1^t under the probability measure Q^t, for each $t \geq 0$. Define the approximating sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of stopping times via

$$\tau_n = \inf \left\{ t \geq 0 : M(t) > n \text{ or } M(t) < \frac{1}{n} \right\}$$

and note that we can write those stopping times as functions of the jump process X_1; to wit, $\tau_n = \tau_n(X_1)$. We have included the lower bound to deal with the case in which X_1 is explosive; in such case, M will hit zero at the time of the explosion. Note that such explosion, if it ever occurs, cannot occur at the time of a jump; thus the local martingale $M_n = M^{\tau_n}$ is strictly positive.

Next, fix $t > 0$. By Theorem 1.1 it is now sufficient to show that $\{M_n(t)\}_{n \in \mathbb{N}}$ is tight along the sequence of probability measures $\{Q_n\}_{n \in \mathbb{N}}$, defined via $\mathrm{d}Q_n = M_n(t)\mathrm{d}P$ to obtain the martingale property of M. For $i = 1, 2$, we shall see that

$$\left\{ \int_0^{\tau_n \wedge t} |\log g_i(s, X_1)| \, \mathrm{d}L_{g_1}(s) \right\}_{n \in \mathbb{N}} \text{ and } \left\{ \int_0^{\tau_n \wedge t} g_i(s, X_1) \mathrm{d}s \right\}_{n \in \mathbb{N}}$$

are tight along $\{Q_n\}_{n \in \mathbb{N}}$, which then proves the statement. Towards this end, Theorem VI.3 in Brémaud (1981) again yields that under Q_n the process $X_1(\cdot \wedge \tau_n)$ solves the martingale problem induced by (3) with $g(s, \omega) = g_2(s, \omega)\mathbf{1}_{\{\tau_n(\omega) > s\}}$. On the other hand, it is immediate that $X_2(\cdot \wedge \tau_n)$ also satisfies (3) with $g(s, \omega) = g_2(s, \omega)\mathbf{1}_{\{\tau_n(\omega) > s\}}$. By the uniqueness of solutions implied by Theorem 3.6 in Jacod (1975) we have that up to the stopping time τ_n, the Q_n–dynamics of X_1 coincide with the dynamics of X_2. Thus, it is sufficient to observe that

$$\begin{aligned} Q_n \left(\int_0^{\tau_n \wedge t} |\log g_i(s, X_1)| \, \mathrm{d}L_{g_1}(s) > \kappa \right) &= P \left(\int_0^{\tau_n \wedge t} |\log g_i(s, X_2)| \, \mathrm{d}L_{g_2}(s) > \kappa \right) \\ &\leq P \left(\int_0^{t} |\log g_i(s, X_2)| \, \mathrm{d}L_{g_2}(s) > \kappa \right) \end{aligned}$$

for all $\kappa > 0$ and $i = 1, 2$, where the right-hand side does not depend on n and tends to zero as κ increases (because X_2 is assumed to be non-explosive). The same observations hold for the other terms of the local martingale M. □

3.3 Counting processes

In this application of Theorem 1.1, we generalize a result by Giesecke and Zhu (2013) concerning the martingale property of a local martingale involving a counting process.

Theorem 3.4. *Let L denote a non-explosive counting process with compensator A and assume that A is continuous, that is, the jumps of L are totally inaccessible. Fix a measurable, deterministic function $u : [0, \infty) \to [-c, c]$ for some $c > 0$. Then the process M, given by*

$$M(\cdot) = \exp\left(-\int_0^{\cdot} u(s)\mathrm{d}L(s) - \int_0^{\cdot} (\exp(-u(s)) - 1)\,\mathrm{d}A(s)\right),$$

is a martingale.

Before we provide the proof of this result we note that Theorem 3.4 generalizes Proposition 3.1 in Giesecke and Zhu (2013) in two ways. First, it does not assume that the function u is constant. Second, no integrability assumption on A is made, such as $E[\exp(A_t)] < \infty$ for some $t > 0$. However, for sake of simplicity, we consider here only the one-dimensional setup with unit jumps.

Proof of Theorem 3.4. First, observe that there exists a counting process $\widehat{L}$, possible on an extension of the probability space, with compensator $\widehat{c}A$, where $\widehat{c}$ is the smallest integer greater than or equal to $\exp(c)$. For example, the process $\widehat{L}$ can by constructed by adding $\widehat{c}$ independent versions of L, exploiting the fact that the jumps of L are totally inaccessible. A standard thinning argument implies that there also exists a counting process L^u with compensator $A^u(\cdot) = \int_0^{\cdot} \exp(-u(s))\mathrm{d}A(s)$. Moreover, by Jacod (1975) and by using the minimal filtration, if two counting processes L^u and $\widehat{L}^u$ have the same compensator then they follow the same probability law.

Simple computations yield that M is a local martingale. Let $\{\tau_n\}_{n\in\mathbb{N}}$ denote a localization sequence, set $M_n = M^{\tau_n}$, fix $t > 0$, and define the probability measures Q_n by $\mathrm{d}Q_n = M_n(t)\mathrm{d}P$. It is sufficient to prove that $\{M_n(t)\}_{n\in\mathbb{N}}$ is tight along the sequence $\{Q_n\}_{n\in\mathbb{N}}$. First, observe that

$$Q_n\left(\exp\left(-\int_0^{\tau_n} u(s)\mathrm{d}L(s)\right) > \kappa\right) \le Q_n\left(\exp\left(cL(\tau_n)\right) > \kappa\right) \le P\left(\exp(c\widehat{L}(t)) > \kappa\right)$$

for all $n \in \mathbb{N}$ and $\kappa > 0$. For later use, note from the second inequality in the previous display that $\{L^{\tau_n}(t)\}_{n\in\mathbb{N}}$ is tight along $\{Q_n\}_{n\in\mathbb{N}}$. Tightness of $\{M_n(t)\}_{n\in\mathbb{N}}$ now follows as soon as we have shown the tightness of $\{A^{\tau_n}(t)\}_{n\in\mathbb{N}}$; that is, $\sup_{n\in\mathbb{N}} Q_n(A^{\tau_n}(t) > \kappa) \to 0$ as $\kappa \uparrow \infty$. However, we can write

$$A^{\tau_n}(t) = \int_0^{t\wedge\tau_n} \exp(u(s))\mathrm{d}A^u(s) \le \exp(c)A^{u,\tau_n}(t)$$

and thus, with $\kappa_c = \exp(-c)\kappa$ and $N = A^{u,\tau_n} - L^{\tau_n}$,

$$\begin{aligned} Q_n\left(A^{\tau_n}(t) > \kappa\right) &\le Q_n\left(A^{u,\tau_n}(t) > \kappa_c\right) = Q_n\left(N(t) + L^{\tau_n}(t) > \kappa_c\right) \\ &\le Q_n\left(L^{\tau_n}(t) > [\sqrt{\kappa_c}]\right) + Q_n\left(N(t\wedge\rho) + [\sqrt{\kappa_c}] + 1 > \kappa_c + 1\right), \end{aligned}$$

where ρ is the first crossing time of $[\sqrt{\kappa_c}]$ by L. The tightness of $\{L^{\tau_n}(t)\}_{n\in\mathbb{N}}$ and Markov's inequality applied to the nonnegative Q_n–supermartingale $N^\rho + [\sqrt{\kappa_c}] + 1$ then yield the tightness of $\{A^{\tau_n}(t)\}_{n\in\mathbb{N}}$ and Theorem 1.1 yields the statement. □

3.4 Ornstein-Uhlenbeck process conditioned on first passage time events

In this application, we are given an Ornstein-Uhlenbeck process X, started at $X(0) = 1$ and mean-reverting to the origin. We are interested in finding a representation for conditional expectations that can be used to design simulation estimators involving the rare event that X hits a large level $N \in \mathbb{N}$ before hitting 0. We achieve such a representation by relating the Ornstein-Uhlenbeck process to the time-reversal of a three-dimensional Bessel process. Although the probability of the conditioning event that X hits N before 0 decreases *exponentially* in the threshold parameter N, the representation provided here can be used to design estimators that run in *linear* time as a function of N. To obtain this representation, we approximate X by a sequence of irreducible and positive recurrent discrete-time Markov chains, apply a result in Blanchet (2013), and then use the second part of Theorem 1.1 to conclude.

Our goal in this section is to use Theorem 1.1 in order to obtain a suitable analogue of Proposition A.1 in the appendix for continuous processes. We will not provide full details of an extension in general, but will focusing on proving a tractable representation for an Ornstein-Uhlenbeck process conditioned on reaching a high level before returning to the origin. Tractable means that the representation should be directly applicable for the purposes of sampling.

Theorem 3.5. *Fix $N \in \mathbb{N}$ with $N > 1$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting four independent Brownian motions B_i for $i = 0, \ldots, 3$. Let X denote an Ornstein-Uhlenbeck process of the form*

$$X(\cdot) = 1 - \int_0^{\cdot} X(s)\,\mathrm{d}s + B_0(\cdot),$$

and X' be given by

$$X'(\cdot) = N - \left(B_1^2(\cdot) + B_2^2(\cdot) + B_3^2(\cdot)\right)^{1/2}.$$

For all $x \in \mathbb{R}$, let $T_x = \inf\{t \geq 0 : X(t) = x\}$ and define T'_x similarly. Define a random variable $M'(\infty)$ by

$$M'(\infty) = \exp\left(\frac{1}{2}\left(N^2 + T'_0 - \int_0^{T'_0} X'(s)^2\,\mathrm{d}s\right)\right) \tag{4}$$

Then the random variable $M'(\infty)$ has finite expectation under P and

$$E\left[f\left(X(s) : 0 \leq s \leq T_N\right) | \{T_N < T_0\}\right] = E\left[f\left(X'(\xi'_1 - s) : 0 \leq s \leq \xi'_1\right)\frac{M'(\infty)}{E[M'(\infty)]}\right],$$

where $\xi'_1 = \max\{0 \leq t \leq T_0 : X'(t) = 1\}$, for all continuous and bounded functions $f : D_{[0,\infty)} \to \mathbb{R}$.

Remark 3.6. The previous result can be used to efficiently estimate conditional expectations involving Ornstein-Uhlenbeck processes, conditioned on $\{T_N < T_0\}$ when N is large in a way that is analogous to the methods described in Blanchet (2013). This then leads to algorithms that have linear running time uniformly as $N \uparrow \infty$. This approach will be studied in future work. □

Remark 3.7. It is well known that X', the modified three-dimensional Bessel process in Theorem 3.5, satisfies the stochastic differential equation

$$X'(\cdot) = N - \int_0^{\cdot} \frac{1}{N - X'(t)}\mathrm{d}t + B(\cdot) \tag{5}$$

for some Brownian motion B.

Also, note that

$$\begin{aligned} M'(\infty) &= \exp\left(-\int_0^{T'_0} X'(s)\mathrm{d}X'(s) - \frac{1}{2}\int_0^{T'_0} X'(s)^2\,\mathrm{d}s\right) \\ &= \exp\left(-\int_0^{T'_0} X'(s)\mathrm{d}B(s) - \frac{1}{2}\int_0^{T'_0} X'(s)^2\,\mathrm{d}s\right)\exp\left(\int_0^{T'_0}\frac{X'(s)}{N - X'(s)}\mathrm{d}s\right), \end{aligned}$$

where the first equality follows from an application of Itô's lemma and the last equality from (5). □

Proof of Theorem 3.5. We will consider a suitably defined class of discrete processes that approximate X and then apply Proposition A.1. We proceed in several steps.

Step 1: We start by constructing a sequence of stochastic processes $\{X_n\}_{n\in\mathbb{N}}$, each taking values in the state space $\mathcal{S}_n = \{k\Delta_n^{1/2}\}_{k\in\mathbb{N}_0}$, where $\Delta_n = 2^{-2n}$. For the moment, we fix $n \in \mathbb{N}$. We now let X_n evolve as a pure jump process, which jumps only at times $\{k\Delta_n\}_{k\in\mathbb{N}}$ with jump size $\Delta X_n^k \in \{-\Delta_n^{1/2}, 0, \Delta_n^{1/2}\}$. That is, using the notation $X_n^k = X_n(k\Delta_n)$, for all $k \in \mathbb{N}$, we assume that

$$\begin{aligned} \Delta X_n^k &= X_n^k - X_n^{k-1} \in \{-\Delta_n^{1/2}, 0, \Delta_n^{1/2}\}, && \text{on } \{X_n^{k-1} > 0\}; \\ \Delta X_n^k &= X_n^k - X_n^{k-1} \in \{0, \Delta_n^{1/2}\}, && \text{on } \{X_n^{k-1} = 0\} \end{aligned}$$

for all $k \in \mathbb{N}$. We also introduce the stopping times

$$T_0^n = \inf\{t \geq \Delta_n : X_n(t) = 0\}; \qquad T_N^n = \inf\{t \geq \Delta_n : X_n(t) = N\},$$

both taking values in $\{k\Delta_n\}_{k\in\mathbb{N}}$. We note that we assumed, in particular, that those stopping times cannot take the value zero.

Step 2: We now introduce two probability measures P_n and $\widehat{P}_n$, on the canonical path space such that X_n is Markov under both measures. We start by defining, on $\{X_n^{k-1} > 0\}$,

$$\begin{aligned} P_n(\Delta X_n^k = -\Delta_n^{1/2} | X_n^{k-1}) &= \frac{1 + \Delta_n^{1/2}(X_n^{k-1} \wedge n)}{2}; & P_n(\Delta X_n^k = \Delta_n^{1/2} | X_n^{k-1}) &= \frac{1 - \Delta_n^{1/2}(X_n^{k-1} \wedge n)}{2}; \\ \widehat{P}_n(\Delta X_n^k = -\Delta_n^{1/2} | X_n^{k-1}) &= \frac{1}{2}; & \widehat{P}_n(\Delta X_n^k = \Delta_n^{1/2} | X_n^{k-1}) &= \frac{1}{2}; \end{aligned}$$

whereas,

$$P_n(\Delta X_n^k = \Delta_n^{1/2} | \{X_n^{k-1} = 0\}) = \widehat{P}_n(\Delta X_n^k = \Delta_n^{1/2} | \{X_n^{k-1} = 0\}) = 1$$

for all $k \in \mathbb{N}$.

Next, let us introduce the random variable

$$M_n'(\infty) = \prod_{k=1}^{T_0^n/\Delta_n} \left(1 - \Delta X_n^k (X_n^{k-1} \wedge n)\right).$$

It is clear that

$$\widehat{E}_n \left[1 - \Delta X_n^k (X_n^{k-1} \wedge n) \mid X_n^{k-1}\right] = 1$$

on $\{X_n^{k-1} > 0\}$ for all $k \in \mathbb{N}$ and that $\widehat{P}_n(T_0^n < \infty) = P_n(T_0^n < \infty) = 1$. Let us now fix a random variable $H = f(X)$, where $f : D_{[0,\infty)} \to [0,\infty)$ satisfies $f(X) = f(X^{T_0^n})$. Observe that the two measures P_n and $\widehat{P}_n$ satisfy

$$\begin{aligned} \widehat{E}_n \left[M_n'(\infty) H | X_n^0\right] &= \sum_{j=1}^{\infty} \widehat{E}_n \left[\prod_{k=1}^{j/\Delta_n} \left(1 - \Delta X_n^k (X_n^{k-1} \wedge n)\right) H \mathbf{1}_{\{T_0^n = j\}} \middle| X_n^0 \right] = \sum_{j=1}^{\infty} E_n \left[H \mathbf{1}_{\{T_0^n = j\}} | X_n^0\right] \\ &= E_n[H | X_n^0]. \end{aligned} \tag{6}$$

Step 3: Next, we consider the measure $\widehat{P}_n^h = \widehat{P}_n(\cdot | \{X_n^0 = N\} \cap \{T_0^n < T_N^n\})$. To compute the corresponding transition probabilities, we note that $\widehat{P}_n^h$ is Doob's h–transform of $\widehat{P}_n$ (up to time T_0^n) for the function $h_n : [0, N] \cap \mathcal{S}_n \to [0, 1]$, given by $h_n(x) = 1 - x/N$ and satisfying $h_n(X_n^0) = \widehat{P}_n(T_0^n < T_N^n | X_n^0)$ on $\{X_n^0 \in (0, N)\}$.

Thus, we get, on $\{X_n^{k-1} \in (0,N)\} \cap \{T_0^n \geq k\}$,

$$\widehat{P}_n^h(\Delta X_n^k = -\Delta_n^{1/2} | X_n^{k-1}) = \frac{h_n(X_n^{k-1} - \Delta_n^{1/2})}{2h_n(X_n^{k-1})} = \frac{1}{2}\left(1 + \frac{\Delta_n^{1/2}}{N - X_n^{k-1}}\right);$$

$$\widehat{P}_n^h(\Delta X_n^k = \Delta_n^{1/2} | X_n^{k-1}) = \frac{h_n(X_n^{k-1} + \Delta_n^{1/2})}{2h_n(X_n^{k-1})} = \frac{1}{2}\left(1 - \frac{\Delta_n^{1/2}}{N - X_n^{k-1}}\right);$$

and, on $\{X_n^{k-1} = N\} \cap \{T_0^n \geq k\}$, $\widehat{P}_n^h(\Delta X_n^k = -\Delta_n^{1/2} | X_n^{k-1}) = 1$ for all $k \in \mathbb{N}$.

As above, we fix again a random variable $H = f(X)$, where $f : D_{[0,\infty)} \to [0,\infty)$ satisfies $f(X) = f(X^{T_0^n})$. Now, (6) yields

$$E_n\left[H | \{T_0^n < T_N^n\} \cap \{X_n^0 = N\}\right] = \frac{E_n\left[H\mathbf{1}_{\{T_0^n < T_N^n\}} | \{X_n^0 = N\}\right]}{P_n(T_0^n < T_N^n | \{X_n^0 = N\})} = \frac{\widehat{E}_n\left[H\mathbf{1}_{\{T_0^n < T_N^n\}} M_n'(\infty) | \{X_n^0 = N\}\right]}{\widehat{E}_n\left[\mathbf{1}_{\{T_0^n < T_N^n\}} M_n'(\infty) | \{X_n^0 = N\}\right]}$$
$$= \widehat{E}_n^h\left[H M_n(\infty)\right], \tag{7}$$

where $M_n(\infty) = M_n'(\infty)/\widehat{E}_n^h[M_n'(\infty)]$.

<u>Step 4</u>: We now study the random variable $M_n'(\infty)$. The Taylor series expanision $1-x = \exp\left(-x - x^2/2 + O\left(x^3\right)\right)$ as $x \downarrow 0$ implies that

$$M_n'(\infty) = \prod_{k=1}^{T_0^n/\Delta_n} \exp\left(-\Delta X_n^k (X_n^{k-1} \wedge n) - \frac{(\Delta X_n^k)^2 (X_n^{k-1} \wedge n))^2}{2} + O(\Delta_n^{3/2})\right)$$

as $n \uparrow \infty$, where the term $O(\Delta_n^{3/2})$ is actually uniform in X_n^{k-1} for all $k \leq T_N^n$ on $\{X_n^0 \leq N\}$. Note that, if $n > N$, on the event $\{X_n^0 = N\} \cap \{T_0^n < T_N^n\}$,

$$\frac{1}{2}N^2 = \frac{1}{2}\left(X_n^{T_0^n} - X_n^0\right)^2 = \frac{1}{2}\sum_{k=1}^{T_0^n/\Delta_n} (\Delta X_n^k)^2 + \sum_{k=1}^{T_0^n/\Delta_n}\sum_{j=1}^{k-1} \Delta X_n^k \Delta X_n^j = \frac{1}{2}T_0^n + \sum_{k=1}^{T_0^n/\Delta_n} \Delta X_n^k (X_n^{k-1} - N)$$
$$= \frac{1}{2}T_0^n + N^2 + \sum_{k=1}^{T_0^n/\Delta_n} \Delta X_n^k X_n^{k-1}.$$

Therefore, if $n > N$, on the event $\{X_n^0 = N\} \cap \{T_0^n < T_N^n\}$,

$$M_n'(\infty) = \exp\left(\frac{N^2}{2} + \frac{T_0^n}{2} - \frac{\Delta_n}{2}\sum_{k=1}^{T_0^n/\Delta_n} (X_n^{k-1})^2 + T_0^n O(\Delta_n^{1/2})\right). \tag{8}$$

<u>Step 5</u>: We now apply Proposition A.1. Towards this end, we introduce

$$\xi_1^n = \max\{0 \leq t \leq T_0^n : X_n(t) = 1\} \in \{k\Delta_n\}_{k \in \mathbb{N}_0}.$$

We note that the process X_n (or, more precisely, a rescaled version) satisfies the assumption of Proposition A.1, which yields, for each continuous bounded function $f : D_{[0,\infty)} \to [0,\infty)$, that

$$E_n\left[f(X_n(s) : 0 \leq s \leq T_N) | \{X_n^0 = 1\} \cap \{T_N^n < T_0^n\}\right]$$
$$= E_n\left[f\left(X_n(\xi_1^n - s) : 0 \leq s \leq \xi_1^n\right) | \{X_n^0 = N\} \cap \{T_0^n < T_N^n\}\right] = \widehat{E}_n^h\left[f\left(X_n(\xi_1^n - s) : 0 \leq s \leq \xi_1^n\right) M_n(\infty)\right],$$

where the last equality is an application of (7).

Step 6: We now link the Markov chains $\{X_n\}_{n\in\mathbb{N}}$ with the Ornstein-Uhlenbeck process X and the three-dimensional Bessel process X' of the statement. Towards this end, it is not difficult to verify using the method of weak convergence of generators in Ethier and Kurtz (1986) that

$$\left(P_n(\cdot|\{X_n^0 = 1\}), X_n(\cdot \wedge T_N^n \wedge T_0^n)\right) \stackrel{\mathfrak{w}}{\Longrightarrow} (P, X(\cdot \wedge T_N \wedge T_0)) \ (n \uparrow \infty) \tag{9}$$

on $D_{[0,\infty)}$. Proposition 5.33 in Pitman (1975) implies that

$$\left(\widehat{P}_n^h, X_n(\cdot \wedge T_0^n)\right) \stackrel{\mathfrak{w}}{\Longrightarrow} (P, X'(\cdot \wedge T_0)) \ (n \uparrow \infty)$$

on $D_{[0,\infty)}$. The continuous mapping principle, applied with a standard extension to handle the stopping times $\{T_0^n\}_{n\in\mathbb{N}}$ to the representation in (8), yields the weak convergence result

$$\left(\widehat{P}_n^h, M_n'(\infty)\right) \stackrel{\mathfrak{w}}{\Longrightarrow} (P, M'(\infty)) \ (n \uparrow \infty),$$

where $M'(\infty)$ is defined in (4).

Step 7: We next argue that for any subsequence $\{n_m\}_{m\in\mathbb{N}}$ there exist some $C \in (0,\infty]$ and a subsubsequence $\{n_{m_k}\}_{m\in\mathbb{N}}$ such that

$$\left(\widehat{P}_{n_{m_k}}^h, M_{n_{m_k}}(\infty)\right) \stackrel{\mathfrak{w}}{\Longrightarrow} \left(P, \frac{M'(\infty)}{C}\right) \ (k \uparrow \infty). \tag{10}$$

This follows from the Bolzano-Weierstrass property, which yields the existence of a subsubsequence $\{n_{m_k}\}_{m\in\mathbb{N}}$ such that $C = \lim_{k\uparrow\infty} \widehat{E}_{n_{m_k}}^h[M_{n_{m_k}}'(\infty)] \in [0,\infty]$. Moreover, an application of Fatou's lemma, in conjunction with a Skorokhod embedding argument also yields that $C > 0$.

Step 8: It remains to be argued that $\{M_n'(\infty)\}_{n\in\mathbb{N}}$ is uniformly integrable. Towards this end, let $\{n_{m_k}\}_{k\in\mathbb{N}}$ be a subsequence satisfying (10). Assume that we have argued that $\{M_{n_{m_k}}(\infty)\}_{k\in\mathbb{N}}$ is uniformly integrable. Then C in (10) must satisfy $C = E[M'(\infty)]$. Consequently, since $\{n_m\}$ was arbitrary, $\lim_{n\uparrow\infty} \widehat{E}_n^h[M_n'(\infty)] = E[M'(\infty)]$. This in conjunction with the weak convergence (and nonnegativity) of $\{M_n'(\infty)\}_{n\in\mathbb{N}}$ yields its uniform integrability.

Finally to argue that $\{M_{n_{m_k}}(\infty)\}_{k\in\mathbb{N}}$ is uniformly integrable, let us write $j = n_{m_k}$ for sake of notation. Thanks to Theorem 1.1, it suffices to show that $\{M_j(\infty)\}_j$, given in (7), is tight along the sequence $\{Q_j\}_j$ of probability measures, defined by $\mathrm{d}Q_j = M_j(\infty)\mathrm{d}\widehat{P}_j^h$. Thanks to (8), it is sufficient to show the tightness of $\{T_0^j\}_j$ under $\{Q_j\}_j$. We have

$$Q_j(T_0^j > \kappa) = \widehat{E}_j^h\left[\mathbf{1}_{\{T_0^j > \kappa\}} M_j(\infty)\right] = P_j\left(T_0^j > \kappa | \{X_j^0 = N\} \cap \{T_0^j < T_N^j\}\right) \tag{11}$$

$$\leq \frac{P_j\left(T_0^j > \kappa|\{X_j^0 = 1\}\right) P_j\left(T_1^j < T_N^j|\{X_j^0 = N\}\right)}{P_j\left(T_0^j < T_N^j|\{X_j^0 = N\}\right)} = \frac{P_j\left(T_0^j > \kappa|\{X_j^0 = 1\}\right)}{P_j\left(T_0^j < T_N^j|\{X_j^0 = 1\}\right)}, \tag{12}$$

where the second equality in (11) comes from (7). Recall (9); therefore

$$\lim_j P_j\left(T_0^j < T_N^j|\{X_j^0 = 1\}\right) = P(T_0 < T_N) > 0,$$

and $\{T_0^j\}$ is tight along $\{P_j\}_j$; thus, $\kappa > 0$ can be chosen so that the right hand side of (12) can be made as small as desired as $j \uparrow \infty$. This concludes the proof. □

Remark 3.8. We end our discussion by noting that the previous result illustrates the convenience of Theorem 1.1. A standard approach would involve verifying directly the uniform integrability of the process $\{M_n'(\infty)\}_{n\in\mathbb{N}}$ under $\{\widehat{P}_n^h\}_{n\in\mathbb{N}}$, and the expectation of the term $\exp(T_0^n/2)$ is difficult to handle. An application of Theorem 1.1 bypasses the need for this by a simple application of the strong Markov property as shown in (12). □

A Proposition 1 in Blanchet (2013)

We recall here a simplified version of a result in Blanchet (2013) that is used in the proof of Theorem 3.5.

Proposition A.1. *Let $X = \{X(k)\}_{k\in\mathbb{N}_0}$ denote an irreducible and positive recurrent discrete time Markov chain taking values in $\mathbb{N}_0$ with $X(k) - X(k-1) \in \{-1, 0, 1\}$ for all $k \in \mathbb{N}$. For each $x \in \mathbb{N}_0$, let P^x be the probability measure in the path space associated with X, conditioned on the event $\{X(0) = x\}$. For any $y \in \mathbb{N}_0$ define $T_y = \inf\{k \in \mathbb{N} : X(k) = y\}$. Fix $x, N \in \mathbb{N}_0$ with $x < N$. Then*

$$P^x\left((X(0), \ldots, X(T_N)) \in \cdot \,|\, \{T_N < T_0\}\right) = P^N\left((X(\xi_x)), \ldots, X(0)) \in \cdot \,|\, \{T_0 < T_N\}\right),$$

where $\xi_x = \max\{0 \le k \le T_0 : X(k) = x\}$.

Proposition A.1 states that we can sample $\{X(k)\}_{0\le k\le T_N}$ conditioned on the event $\{X(0) = x\} \cap \{T_N < T_0\}$ by sampling $\{X(k)\}_{0\le k\le T_0}$ conditioned on the event $\{X(0) = N\} \cap \{T_0 < T_N\}$, thereby obtaining $\{X(k)\}_{0\le k\le \xi_x}$, and finally letting $X(k) = X(\xi_x - k)$ for all $k \in \{0, ..., \xi_x\}$.

We do not provide a proof of Proposition A.1 here, but instead refer to Blanchet (2013). However, to provide some intuition, we give some computations here for the case $x = 0$, which indicate the validity of the result. In this case, we also have $\xi_0 = T_0$. Towards this end, let $\{K(y, z)\}_{y,z\in\mathbb{N}_0}$ denote the transition matrix of X. Recall, since X is time-reversible, that

$$K(z, y) = \frac{\pi(y)}{\pi(z)} K(y, z),$$

where π is the stationary distribution of X. Then, if $N \ge 2$, note that

$$\begin{aligned}
\pi(0)P^0(T_N < T_0) &= \sum_{k\in\mathbb{N}} \sum_{x_1,\ldots,x_k\in\mathbb{N}_0} \pi(0)K(0, x_1)\, K(x_1, x_2) \cdots K(x_k, N)\, \mathbf{1}_{0<x_1<N,\ldots,0<x_k<N} \\
&= \sum_{k\in\mathbb{N}} \sum_{x_1,\ldots,x_k\in\mathbb{N}} K(x_1, 0)\, \pi(x_1)\, K(x_1, x_2) \cdots K(x_k, N)\, \mathbf{1}_{x_1<N,\ldots,x_k<N} \\
&\cdots \\
&= \sum_{k\in\mathbb{N}} \sum_{x_1,\ldots,x_k\in\mathbb{N}} \pi(N)K(N, x_{k-1}) \cdots K(x_2, x_1)K(x_1, 0)\mathbf{1}_{x_{k-1}<N,\ldots,x_1<N} \\
&= \pi(N)P^N(T_0 < T_N).
\end{aligned}$$

The previous identities provide a representation for $P^0(T_N < T_0)$ (which is small when N is large) in terms of $P^N(T_0 < T_N)$. Blanchet (2013) shows that the contribution of $P^N(T_0 < T_N)$ remains bounded away from zero as N increases to ∞ for a significant class of processes of interest. Therefore, approximating probabilities for rare events of the form $\{T_N < T_0\}$ (which involve the whole sample path) can be reduced to approximating probabilities of rare events of the form $\{X(0) = N\}$ (which only involve the random variable $X(0)$ following the stationary distribution).

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