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Low-Reynolds-number flow past a cylinder with uniform blowing or sucking

Q1 **C. A. Klettner**[1,†] **and I. Eames**[1]

Q2 [1]University College London, Torrington Place, London WC1E 6BT, UK

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We analyse the low-Reynolds-number flow generated by a cylinder (of radius a) in a stream (of velocity U_∞) which has a uniform through-surface blowing component (of velocity U_b). The flow is characterized in terms of the Reynolds number Re ($=2aU_\infty/\nu$, where ν is the kinematic viscosity of the fluid) and the dimensionless blow velocity $\varLambda$ ($=U_b/U_\infty$). We seek the leading-order symmetric solution of the vorticity field which satisfies the near- and far-field boundary conditions. The drag coefficient is then determined from the vorticity field. For the no-blow case Lamb's (*Phil. Mag.*, vol. 21, 1911, pp. 112–121) expression is retrieved for $Re\rightarrow 0$. For the strong-sucking case, the asymptotic limit, $C_D\sim -2\pi\varLambda$, is confirmed. For blowing, the
Q3 limit of validity is ~~$\beta<1$ or~~ $\varLambda<4/Re$, after which the flow is unsymmetrical about $\theta=\pi/2$. The analytical results are compared with full numerical solutions for the drag coefficient C_D and the fraction of drag due to viscous stresses. The predictions show good agreement for $Re=0.1$ and $\varLambda=-5, 0, 5$.

Key words: low-Reynolds-number flows, Stokesian dynamics

1. Introduction

The modification of the flow past a body due to a uniform blowing or sucking component is of fundamental importance in many areas of engineering. A through-surface flux is introduced to cool turbine blades or modify the force acting on lifting surfaces or can be generated by a phase change (e.g. evaporation).

Dukowicz (1982) derived a closed-form expression for the drag force acting on a blowing/sucking sphere at low Reynolds numbers which retrieves Stokes' (1851) solution for $\varLambda=0$. For strongly sucking flows, the flow is irrotational in the far field and the drag force reduces to what is expected by a global momentum analysis. At a Reynolds number of $Re=1$, the difference between the full numerical results and Dukowicz's solution is approximately 10 % in the drag coefficient for blowing flows
Q4 (Cliffe & Lever 1985).

† Email address for correspondence: ucemkle@ucl.ac.uk

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For the case of a cylinder, the complexity of the analysis is increased by the requirement of a far-field or Oseen correction (see the discussion by Stokes (1851)). The study of the force on a cylinder at low *Re* has been developed over the last 100 years, and it is worth pointing out some of the historical elements, as they provide a guide to the different ways in which we could treat the problem in this paper. Later editions of Lamb's book 'Hydrodynamics' include a discussion of the flow past a cylinder at low *Re* (Lamb 1932, p. 614, 1911). The technique Lamb employed attracted criticism in the 1960s because it was not a rigorous asymptotic analysis. The construction technique that Lamb employed is reasonably accurate, giving predictions for the drag coefficient up to $Re = 1$ that are within 5 % of the full solution. Lamb's (1911) technique follows that of Oseen, introducing a correction (advective) term to account for the far-field flow, but it is simpler as it makes use of a transformational split that is not extendable to the problem in this paper. While a matched asymptotic solution is mathematically rigorous and can account for the full inertia term, the series expansion method by Dennis & Shimshoni (1965) is just as powerful and accurate, though far less elegant mathematically.

The purpose of this paper is to examine the low-Reynolds-number flow past a cylinder which has a through-surface component and to develop an understanding of the influence of *Re* and Λ on the drag force. The leading-order solution is calculated using a construction technique, which has the advantage of being simple. The fidelity of this approach is tested with comparisons against full numerical solutions. The mathematical model is described in § 2. Approximate solutions are developed in the limit of $\Lambda = 0$ and strongly sucking flows and described in § 3. A comparison between predictions and numerical solutions is shown in § 4.

2. Mathematical model

We consider a cylinder of radius a fixed at the origin and set in a uniform flow. To account for the far field at low Reynolds numbers, the Oseen approximation is applied which uses a linear approximation for the inertia term so that $\boldsymbol{u} \cdot \nabla \boldsymbol{u} \approx U_\infty \partial \boldsymbol{u}/\partial x$. Q5 We are interested in examining the flow past a cylinder with a through-surface flow so that the radial blow component is included, and therefore seek to determine the leading-order solution to

$$\rho \left(U_\infty \frac{\partial \boldsymbol{u}}{\partial x} + \frac{U_b a}{r} \frac{\partial \boldsymbol{u}}{\partial r} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u}, \tag{2.1}$$

where μ is the dynamic viscosity, ρ is the density, U_b is the blow velocity and U_∞ is the far-field velocity. The boundary conditions imposed on the flow are

$$(u_r, u_\theta) = (U_b, 0) \tag{2.2}$$

at the surface of the cylinder (at $r = a$) and

$$(u_r, u_\theta) \to (U_\infty \cos\theta, -U_\infty \sin\theta) \tag{2.3}$$

in the far field (as $r \to \infty$).

2.1. *Defining equations*

A vorticity–stream function (ω–ψ) method of solution is employed (see Batchelor 1967, appendix 2), where the velocity and vorticity fields are defined by

$$u_r = \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \quad \omega = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}. \tag{2.4a–c}$$

The boundary conditions (2.2) impose significant constraints on the velocity near the boundary, mainly because $\partial u_\theta/\partial\theta = \partial u_r/\partial\theta = 0$. This means that

$$\omega = \left.\frac{\partial u_\theta}{\partial r}\right|_{r=a} \tag{2.5}$$

and

$$\left.\frac{\partial u_r}{\partial r}\right|_{r=a} = -\frac{U_b}{a} \tag{2.6}$$

(where mass conservation was used in (2.6)). From (2.1), the vorticity equation is

$$U_\infty \frac{\partial \omega}{\partial x} + \frac{U_b a}{r}\frac{\partial \omega}{\partial r} = \nu \nabla^2 \omega. \tag{2.7}$$

Our starting point is quite similar to that of Lamb (1932) and involves expressing the vorticity field $\tilde{\omega}$ $(= 2a\omega/U_\infty)$ as

$$\tilde{\omega} = \mathrm{e}^{(Re/4)\tilde{r}\cos\theta} P, \tag{2.8}$$

giving

$$\frac{Re^2}{8}\Lambda\frac{1}{\tilde{r}}\cos\theta P + \frac{Re}{2}\frac{\Lambda}{\tilde{r}}\frac{\partial P}{\partial \tilde{r}} + \left(\frac{Re}{4}\right)^2 P = \tilde{\nabla}^2 P, \tag{2.9}$$

where $Re = 2aU_\infty/\nu$ and $\tilde{r} = r/a$. Following Dukowicz (1982), we seek the leading-order symmetric solution, which is of the form

$$P = P_1(\tilde{r})\sin\theta, \tag{2.10}$$

where P_1 satisfies

$$P_1'' + \frac{P_1'}{\tilde{r}}(1-2\beta) - \left(\left(\frac{Re}{4}\right)^2 + \frac{1}{\tilde{r}^2}\right)P_1 = 0, \tag{2.11}$$

where $\beta = Re\,\Lambda/4$. This is valid provided that $Re^2|\Lambda| \ll 1$ and for $\beta \to -\infty$ because the flow is symmetric about $\theta = \pi/2$, but not for $\beta \to \infty$. The solution that satisfies $P_1 \to 0$ as $\tilde{r} \to \infty$ is

$$P_1 = C_1 \tilde{r}^\beta K_{(1+\beta^2)^{1/2}}(Re\,\tilde{r}/4). \tag{2.12}$$

The stream function, ψ, can be constructed by writing it as the sum of the known blowing and free-stream components, together with a component to be determined. As such we write

$$\psi = U_\infty a(\Lambda\theta + \tilde{r}\sin\theta + f_1\sin\theta). \tag{2.13}$$

Substitution of (2.13) into (2.4) gives

$$f_1'' + \frac{f_1'}{r} - \frac{f_1}{r^2} = -\frac{1}{\pi}\int_0^{\pi} \sin\theta\,\tilde{\omega}\,\mathrm{d}\theta = -\frac{1}{2\pi}P_1(\tilde{r})\int_0^{2\pi} \mathrm{e}^{(Re\,\tilde{r}/4)\cos\theta}\sin^2\theta\,\mathrm{d}\theta. \tag{2.14}$$

The boundary conditions for f_1 are

$$f_1(1) = -1, \quad f_1'(1) = -1, \quad \lim_{\tilde{r}\to\infty} f_1(\tilde{r}) = 0. \tag{2.15a–c}$$

The right-hand side of (2.14) is defined as $C_1 p_1(\tilde{r})$, where

$$p_1(\tilde{r}) = -\tilde{r}^{\beta}\frac{J_1(\mathrm{i}Re\,\tilde{r}/4)}{\mathrm{i}Re\,\tilde{r}/4}K_{(1+\beta^2)^{1/2}}(Re\,\tilde{r}/4). \tag{2.16}$$

We can solve (2.14) exactly by writing $f_1 = \tilde{r}g_1$, which transforms the ordinary differential equation to

$$\frac{\mathrm{d}(\tilde{r}^3 g_1')}{\mathrm{d}\tilde{r}} = C_1 p_1(\tilde{r})\tilde{r}^2. \tag{2.17}$$

The boundary conditions on g_1 are $g_1(1) = -1$, $g_1'(1) = 0$ and $g_1 \to 0$ as $\tilde{r} \to \infty$. Integrating twice, we find two results. The first is that

$$f_1 = C_1\tilde{r}\left(-\frac{1}{2\tilde{r}^2}G(1) - \int_{\infty}^{\tilde{r}} G(\tilde{r})\tilde{r}^{-3}\,\mathrm{d}\tilde{r}\right), \tag{2.18}$$

where

$$G(\tilde{r}) = \int_{\tilde{r}}^{\infty} p_1(\tilde{r})\tilde{r}^2\,\mathrm{d}\tilde{r}. \tag{2.19}$$

The second result (which ensures that the far-field boundary condition is satisfied) is

$$C_1 = \frac{2}{\displaystyle\int_1^{\infty} p_1(\tilde{r})\,\mathrm{d}\tilde{r}}. \tag{2.20}$$

The integrand scales as $\tilde{r}^{\beta-2}$ in the far field (using $K_n(z) \sim \exp(-z)/z^{1/2}$ and $J_1(\mathrm{i}z) \sim \exp(z)/z^{1/2}$) and so the integral converges when $-\infty < \beta < 1$.

2.2. *Diagnostics*

The pressure and viscous drag coefficients for characterizing the force on a cylinder are

$$C_P = \int_0^{2\pi}\left(\frac{1}{Re}\frac{\partial\tilde{\omega}}{\partial\tilde{r}} - \frac{1}{2}\Lambda\tilde{\omega}\right)\sin\theta\,\mathrm{d}\theta, \quad C_v = -\frac{1}{Re}\int_0^{2\pi}\tilde{\omega}\sin\theta\,\mathrm{d}\theta, \tag{2.21a,b}$$

which is an extension of the relationship given by Dennis & Shimshoni (1965) to include a through-surface flow. On substituting (2.10) into (2.21),

$$\left.\begin{aligned} C_P &= -C_1\frac{\pi}{Re}\left((\beta + (1+\beta^2)^{1/2})K_{(1+\beta^2)^{1/2}}(Re/4) + \frac{Re}{4}K_{(1+\beta^2)^{1/2}-1}(Re/4)\right), \\ C_v &= -C_1\frac{\pi}{Re}K_{(1+\beta^2)^{1/2}}(Re/4). \end{aligned}\right\} \tag{2.22}$$

The ratio of the viscous drag to the total drag coefficient is

$$\frac{C_v}{C_D} = \frac{1}{\beta + (1+\beta^2)^{1/2} + 1 + \dfrac{Re}{4}\dfrac{K_{(1+\beta^2)^{1/2}-1}(Re/4)}{K_{(1+\beta^2)^{1/2}}(Re/4)}}, \tag{2.23}$$

where $C_D = C_P + C_v$. Since Re is small, C_v/C_D is effectively a function of $Re\,\Lambda$. It should be noted that (2.23) does not depend on C_1.

3. Approximate solutions

We present a leading-order solution to (2.14) which will then be used to understand two limits: (*a*) the no-blow case where the result reduces to Lamb's (1911) original Q6 solution and (*b*) strongly sucking flows.

3.1. *No-blow case* ($\Lambda = 0$)

The purpose here is to retrieve Lamb's solution for the no-blow case. When $\Lambda = 0$, p_1 can be expressed exactly as

$$p_1 = -K_1(Re\,\tilde{r}/4)\frac{J_1(\mathrm{i}Re\,\tilde{r}/4)}{\mathrm{i}Re\,\tilde{r}/4}. \tag{3.1}$$

Using the substitution $z = Re\,\tilde{r}/4$, the integral in (2.20) can be written as

$$\int_1^\infty p_1\,\mathrm{d}\tilde{r} = \frac{4}{Re}\int_{Re/4}^\infty K_1(z)\frac{J_1(\mathrm{i}z)}{\mathrm{i}z}\,\mathrm{d}z = \frac{4}{Re}\int_{Re/4}^\infty K_1(z)\left(\frac{1}{2} + \sum_{n=1}^\infty \frac{z^{2n}}{2^{2n+1}n!(n+1)!}\right)\mathrm{d}z, \tag{3.2}$$

such that

$$\int_1^\infty p_1\,\mathrm{d}\tilde{r} = \frac{2}{Re}\left(K_0(Re/4) + \sum_{n=1}^\infty \int_{Re/4}^\infty \frac{z^{2n}K_1(z)}{2^{2n}n!(n+1)!}\,\mathrm{d}z\right). \tag{3.3}$$

In the limit of $Re \ll 1$, the lower limit is close to zero and it can be shown, using (A 4), that

$$\sum_{n=1}^\infty \int_0^\infty \frac{z^{2n}K_1(z)}{2^{2n}n!(n+1)!}\,\mathrm{d}z = \frac{1}{2}, \tag{3.4}$$

such that

$$C_1 = \frac{Re}{\frac{1}{2} + K_0(Re/4)}. \tag{3.5}$$

The drag coefficient corresponding to (3.5) is

$$C_D = -\frac{2\pi}{\frac{1}{2} + K_0(Re/4)}\left(K_1(Re/4) + \frac{Re}{8}K_0(Re/4)\right) \cong -\frac{8\pi}{Re(\frac{1}{2} + K_0(Re/4))}, \tag{3.6}$$

where use was made of (A 2). Now, Lamb (1932, p. 616) derived the following expression for vorticity:

$$\omega = C\mathrm{e}^{(\tilde{r}Re/4)\cos\theta}\frac{\partial}{\partial y}K_0(Re\,\tilde{r}/4) = -\frac{C}{a}\frac{Re}{4}\mathrm{e}^{(\tilde{r}Re/4)\cos\theta}K_1(Re\,\tilde{r}/4)\sin\theta, \tag{3.7}$$

which is the same expression as that derived here from the vorticity equation. (It should be noted that there is a typographical error in Lamb's vorticity expression, where e^{kz} should read e^{kx}; the correct analysis is given in Lamb (1911) except for the typographical error of 'sphere' which should be 'cylinder' after equation (54).) A higher-order expansion of the stream function was determined,

$$C = \frac{2U_\infty}{\frac{1}{2} + K_0(Re/4)}, \tag{3.8}$$

or equivalently

$$C_D = -\frac{8\pi}{Re\left(\frac{1}{2} + K_0(Re/4)\right)}. \tag{3.9}$$

Therefore, the general expression for the drag coefficient agrees exactly with Lamb's expression as $Re \to 0$ (the difference for $Re = 1$ is less than 1 %).

3.2. *Strongly sucking flow* $(-\Lambda \gg 1)$

For strongly sucking flows where $|\beta| \gg 1$ and $\beta < 0$, we can write

$$\frac{\int_1^\infty p_1\,\mathrm{d}\tilde{r}}{K_{(1+\beta^2)^{1/2}}(Re/4)} = -\frac{1}{2}\left(\frac{4}{Re}\right)^{\beta+1}\int_{Re/4}^\infty z^\beta \frac{K_{(1+\beta^2)^{1/2}}(z)}{K_{(1+\beta^2)^{1/2}}(Re/4)}\,\mathrm{d}z \cong \frac{1}{2(\beta - (1+\beta^2)^{1/2} + 1)}, \tag{3.10}$$

which can be substituted into (2.22), giving a drag coefficient of

$$C_D \approx -\frac{4\pi}{Re}(\beta + (1+\beta^2)^{1/2} + 1)(\beta - (1+\beta^2)^{1/2} + 1). \tag{3.11}$$

This reduces to

$$C_D \approx -2\pi\Lambda. \tag{3.12}$$

This approximation is appropriate when $|\beta| \geqslant 1$ or $\Lambda < -4/Re$ (so that the asymptotic approximation is valid). Equation (3.12) agrees with a global momentum analysis when the far-field downstream flow is irrotational, which was derived by Pankhurst & Thwaites (1953, appendix I) for high-Re flows. This is to be expected because in Q7
both cases, the boundary layer is thin compared with the size of the cylinder.

4. Numerical results

4.1. *Solution technique*

The solution for $\tilde{\omega}$ contains an unknown, C_1, which is determined from (2.20). The numerical solution to the Navier–Stokes equation was solved using a finite-element method that employs a characteristic-based split (CBS) methodology (see Zienkiewicz, Taylor & Nithiarasu 2005). The ACEsim code has been validated for two-dimensional flows (e.g. Nicolle & Eames 2011; Klettner & Eames 2012). For low Re, White (1945) suggests a domain width of $2000a$ for the no-blow case to be unaffected by boundedness. As the influence of boundedness is increased for strong blowing/sucking, the domain width was increased to $20\,000a$ for the two cases of $|\Lambda| = 5$.

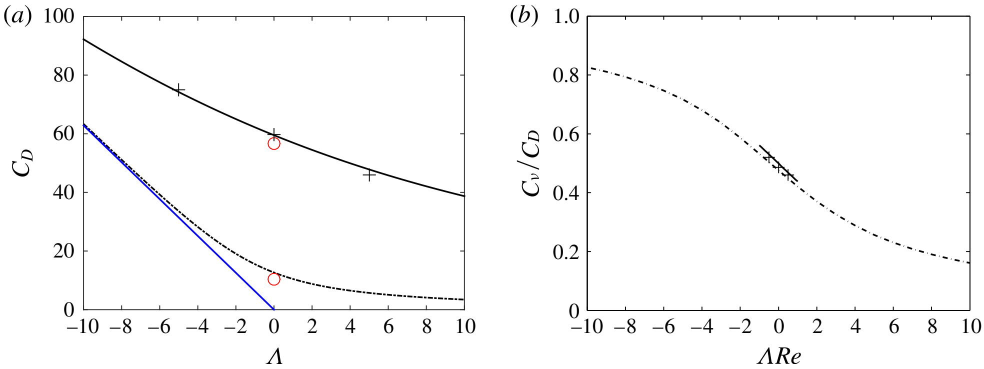

FIGURE 1. A comparison between the theoretical predictions and full numerical calculations of (*a*) C_D and (*b*) C_v/C_D as functions of Λ. In (*a*,*b*) the dot-dashed and full curves correspond to the predictions (2.22), (2.23) for $Re=1$, 0.1 respectively and the full numerical simulations for $Re=0.1$ are represented by crosses. The numerical results of Dennis & Shimshoni (1965) are plotted as red circles for the no-blow case. The blue line is the strongly sucking solution, $C_D=-2\pi\Lambda$, given in Pankhurst & Thwaites (1953, appendix I).

4.2. *Results*

Figure 1(*a*) shows the drag coefficient as a function of Λ for $Re=0.1$. Good agreement is found between the analytical results and full numerical simulations. The asymptotic limit for strongly sucking flows ($C_D\sim-2\pi\Lambda$) is confirmed for $Re=1$. Figure 1(*b*) shows the fraction of the total force due to viscous stresses for $Re=0.1$. For small $|\Lambda|$ ($\ll 1$), the influence of blowing and sucking is symmetric on the drag force. For strongly sucking flows, the drag force increases linearly with $|\Lambda|$ because the viscous stresses near the wall scale as $\mu|\Lambda|U_\infty/a$. For strongly blowing flows, $C_D\rightarrow 0$ because the vorticity is blown off the surface of the cylinder. Therefore, the influence of the through-surface flow is asymmetric on the drag force at large Λ. For $\beta>0$, the range of validity of the analysis was determined to be $\beta<1$ or $\Lambda<4/Re$ using a scaling analysis.

5. Conclusion

We identified the gap of low-Reynolds-number flow past a cylinder with a through-surface flow, and studied this problem using a analytical technique that identifies the leading-order component to the vorticity field. For the case of $\Lambda=0$ and $Re\rightarrow 0$, we retrieve Lamb's (1911) result for the drag force. For strongly sucking flows, where the flow is irrotational outside the thin boundary layer, the asymptotic result $C_D=-2\pi\Lambda$ is recovered. The agreement between the analytical results and the full numerical solutions is good for $Re=0.1$.

Appendix A. Useful relationships for K_n

We list the recurrent and asymptotic relationships that are used in this paper.

$$\frac{\mathrm{d}K_n}{\mathrm{d}z}=-K_{n-1}(z)-\frac{n}{z}K_n(z). \qquad \text{(A 1)}$$

The expansion for K_1 is

$$K_1(z) = \frac{1}{z} + \frac{z}{2}\log\left(\frac{z}{2}\right) + \cdots . \tag{A 2}$$

When the argument $n \gg 1$,

$$K_n(z) \cong \frac{\Gamma(n)}{2}\left(\frac{z}{2}\right)^{-n} . \tag{A 3}$$

Another useful formula is

$$\int_0^\infty z^m K_n(z)\,\mathrm{d}z = 2^{m-1}\Gamma\left(\frac{n+m+1}{2}\right)\Gamma\left(\frac{m+1-n}{2}\right). \tag{A 4}$$

References

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*, 1st edn. Cambridge University Press.

CLIFFE, K. A. & LEVER, D. A. 1985 Isothermal flow past a blowing sphere. *Intl J. Numer. Meth. Fluids* **5**, 709–725.

DENNIS, S. C. R. & SHIMSHONI, M. 1965 The steady flow of a viscous fluid past a circular cylinder. *Ministry of Aviation – Aeronautical Research Council* **797**, 1–48.

DUKOWICZ, J. K. 1982 An exact solution for the drag of a sphere in low Reynolds number flow with strong uniform suction or blowing. *Phys. Fluids* **25**, 1117–1118.

KLETTNER, C. A. & EAMES, I. 2012 The laminar free surface boundary layer of a solitary wave. *J. Fluid Mech.* **696**, 423–433.

LAMB, H. 1911 On the uniform motion of a sphere through a viscous fluid. *Phil. Mag.* **21**, 112–121.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

NICOLLE, A. & EAMES, I. 2011 Numerical study of flow through and around a circular array of cylinders. *J. Fluid Mech.* **679**, 1–31.

PANKHURST, R. C. & THWAITES, B. 1953 Experiments on the flow past a porous cylinder fitted with a Thwaites flap. *Aero. Res. Council Reports and Memoranda*. R. & M. 2787 (A.R.C. Technical Report). 29pp.

STOKES, G. G. 1851 On the effect of the internal friction of fluids on the motion of pendulums. *Camb. Phil. Soc.* **9**, 8–106.

WHITE, C. M. 1945 The drag of cylinders in fluids at low speeds. *Proc. R. Soc. Lond.* A **186**, 472–479.

ZIENKIEWICZ, O. C., TAYLOR, R. L. & NITHIARASU, P. 2005 *The Finite Element Method for Fluid Dynamics*, 6th edn. Butterworth-Heinemann.

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