Some results on stability and canonical metrics in Kähler geometry

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Declaration

I, Yoshinori Hashimoto confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

. . . .

Abstract

In this thesis, we prove various results on canonical metrics in Kähler geometry, such as extremal metrics or constant scalar curvature Kähler (cscK) metrics, and discuss connections to the notions of algebro-geometric stability of the underlying manifold.

After reviewing the background materials in Chapter 1, we discuss in Chapter 2 the extension of Donaldson's quantisation to the case where the automorphism group is no longer discrete. This is achieved by considering a new equation $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}\rho_k(\omega) = 0$; the $(1,0)$-part of the gradient of the Bergman function is a holomorphic vector field. The main result of this thesis is the existence of a solution to this equation for all large enough k, assuming the existence of extremal metrics. We also prove that the sequence $\{\omega_k\}_k$ of these solutions approximates the extremal metric for $k \gg 1$, and that the solvability of the equation implies that a polarised Kähler manifold admitting an extremal metric is asymptotically weakly Chow polystable relative to any maximal torus in the automorphism group; this stability result was originally proved by Mabuchi using a different method.

In Chapter 3 we discuss Kähler metrics with cone singularities along a divisor. We provide the first supporting evidence for the log Donaldson–Tian–Yau conjecture for general polarisations, and study various properties of the log Donaldson–Futaki invariant computed with respect to conically singular metrics.

In Chapter 4 we discuss canonical metrics on the blow-up of manifolds with canonical metrics. This problem is well-understood when we blow up points, but few examples are known when we blow up higher dimensional submanifolds. We prove that the projective spaces blown up along a line, $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$, cannot admit cscK metrics in any polarisations, but admit an extremal metric in each Kähler class that is close to the pullback of the Fubini–Study class, with an explicit formula in action-angle coordinates.

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Chapter 1

Introduction

1.1 Canonical metrics on a Kähler manifold and Calabi's proposal

The existence of a "canonical" Riemannian metric, as in the uniformisation theorem for Riemann surfaces, is a central problem in differential geometry. Since this problem usually takes the form of a nonlinear PDE problem in Riemannian metrics, it is extremely difficult to solve on a general Riemannian manifold. However, we have a significant simplification of the problem on *Kähler manifolds* by virtue of the existence of "potential functions"; if (X,ω) is a compact Kähler manifold[1], the set of Kähler metrics in the cohomology class $[\omega] \in H^2(X,\mathbb{R})$ can be identified with $\{\phi \in C^\infty(X,\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$. Moreover, we will often assume that X admits an ample line bundle L, often called **polarisation**, and focus on the Kähler metrics in $c_1(L) \in H^2(X,\mathbb{Z})$. The motivation for this will be explained in §1.2. Throughout in what follows, we shall consider the pair (X,L) as a primary object of study, and call it a **polarised Kähler manifold**.

In 1982, Calabi [23] posed the following question.

Question 1.1.1. (Calabi's proposal [23]) Given a cohomology class[2] $\kappa \in H^2(X,\mathbb{Z})$ which contains a Kähler metric, can one find a Kähler metric $\omega \in \kappa$ which (locally)

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[1] We shall often identify the Kähler form ω with its associated Riemannian metric $g = \omega(\cdot, J\cdot)$, where we write J throughout to denote the complex structure on X.

[2] Although we assume in this thesis that κ is in the integral cohomology class, Calabi's proposal makes sense for $\kappa \in H^0(X,\mathbb{R})$ in general.

minimises the **Calabi energy**

$$\mathrm{Cal}(\omega) := \int_X S(\omega)^2 \frac{\omega^n}{n!},$$

where $S(\omega)$ is the scalar curvature of ω?

The Euler–Lagrange equation of $\mathrm{Cal}(\omega)$ is known [23] to be equal to

$$\bar{\partial}\mathrm{grad}^{1,0}_{\omega} S(\omega) = 0,$$

where $\mathrm{grad}^{1,0}_{\omega} S(\omega)$ denotes the (1,0)-part of the gradient vector field $\mathrm{grad}_{\omega} S(\omega)$ and $\bar{\partial}$ is the (0,1)-part of the Chern connection on TX defined by ω. The Kähler metrics satisfying the above equation, called **extremal metrics**, will be the central theme of this thesis. It is important to note some special subclasses of extremal metrics. If there exists no nontrivial holomorphic vector field[3] on X, we necessarily have $\mathrm{grad}^{1,0}_{\omega} S(\omega) = 0$ which implies $\mathrm{grad}_{\omega} S(\omega) = 0$ by taking the real part. Hence we get $d(S(\omega)) = 0$, which is equivalent to $S(\omega) = \mathrm{const}$. A metric ω with $S(\omega) = \mathrm{const}$ will be called a **constant scalar curvature Kähler** metric, and abbreviated as a **cscK** metric. Further special cases are when $\omega \in c_1(K_X)$, $\omega \in -c_1(K_X)$, or $c_1(K_X) = 0$, where K_X is the canonical bundle of X; in these cases, basic Hodge theory shows that ω being cscK is equivalent to ω being **Kähler–Einstein**, i.e. satisfies $\mathrm{Ric}(\omega) = \lambda \omega$ for some constant λ which is -1 if $c_1(K_X) > 0$, $+1$ if $c_1(K_X) < 0$, and 0 if $c_1(K_X) = 0$. We summarise the above as follows.

Definition 1.1.2. A Kähler metric ω is called **extremal** if its scalar curvature $S(\omega)$ satisfies $\bar{\partial}\mathrm{grad}^{1,0}_{\omega} S(\omega) = 0$. It is called **cscK** if it satisfies $S(\omega) = \mathrm{const}$. A cscK metric is called **Kähler–Einstein** if it satisfies $\mathrm{Ric}(\omega) = \lambda \omega$ for some constant λ.

We thus see that the class of extremal metrics subsumes the classes of cscK and Kähler–Einstein metrics, and that Calabi's proposal can be regarded as providing a unifying framework for working with these classes of "canonical" Kähler metrics, if they exist.[4]

[3] This hypothesis can be slightly weakened; see Lemma 1.4.5.

[4] It is known that there does exist a Kähler manifold (e.g. certain iterated blow-ups of $\mathbb{P}^2$ [71]) which does not admit any extremal metric.

Remark 1.1.3. It is well-known that these canonical metrics are unique in each Kähler class, up to automorphisms [10, 12, 14, 22, 40, 83, 127].

Remark 1.1.4. Recall now that, in their celebrated work, Aubin [10] and Yau [127] (resp. Yau [127]) have proved the existence of Kähler–Einstein metrics on a compact Kähler manifold for the case $c_1(K_X) > 0$ (resp. $c_1(K_X) = 0$). The case $c_1(K_X) < 0$ lead to a deep conjecture which was only recently solved [28, 29, 30]; this will be discussed in §1.2 (see in particular Theorem 1.2.10).

Remark 1.1.5. Since $\omega \in c_1(L)$ and $S(\omega)\omega^n = n\mathrm{Ric}(\omega) \wedge \omega^{n-1} \in -nc_1(K_X)c_1(L)^{n-1}$ by Chern–Weil theory, the average $\bar{S}$ of $S(\omega)$ is determined as

$$\bar{S} = \frac{\int_X S(\omega)\frac{\omega^n}{n!}}{\int_X \frac{\omega^n}{n!}} = \frac{-n\int_X c_1(K_X)c_1(L)^{n-1}}{\int_X c_1(L)^n}.$$

If we write $S(\omega)$ in terms of local holomorphic coordinates $(z_1, \dots z_n)$ (where $n = \dim_{\mathbb{C}} X$), we get

$$S(\omega) = -\sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log\det(g_{k\bar{l}}),$$

and hence the csck equation $S(\omega) = \text{const}$ is a fully nonlinear fourth order PDE in the Kähler potential ϕ, with respect to which the metric tensor $g_{k\bar{l}}$ can be locally written as $g_{k\bar{l}} = \frac{\partial^2 \phi}{\partial z_k \partial \bar{z}_l}$. This means that the extremal equation $\bar{\partial}\mathrm{grad}^{1,0}_{\omega} S(\omega) = 0$ is a fully nonlinear sixth order PDE. Thus, finding a cscK or extremal metrics is equivalent to solving a fourth or sixth order fully nonlinear PDE, which is a very difficult problem in full generality. However, it is conjectured, and in some important cases proved, that the existence of these metrics are in fact equivalent to the *stability* of the underlying manifold, as we discuss in §1.2 (see in particular Conjecture 1.2.6); a difficult nonlinear PDE problem as discussed above can be translated into a purely algebro-geometric one, which is potentially more tractable.

1.2 *K*-stability and Donaldson–Tian–Yau conjecture

1.2.1 Statement of the conjecture

Inspired by the Kobayashi–Hitchin correspondence for vector bundles, Yau [128] conjectured that the existence of Kähler–Einstein metrics should be related to a notion of "stability" in algebraic geometry. Later, Tian [125] introduced the notion of K-stability as an appropriate stability condition for this problem. This was later refined by Donaldson [41], who also extended its scope to include cscK metrics and not just Kähler–Einstein metrics.

We first recall the notion of test configurations, in order to define K-stability in Definition 1.2.5.

Definition 1.2.1. A **test configuration** for a polarised projective scheme (X, L) with exponent $r \in \mathbb{N}$ is a projective scheme $\mathscr{X}$ together with a relatively ample line bundle $\mathscr{L}$ over $\mathscr{X}$ and a flat morphism $\pi : \mathscr{X} \to \mathbb{C}$ with a $\mathbb{C}^*$-action on $\mathscr{X}$, which covers the usual multiplication in $\mathbb{C}$ and lifts to $\mathscr{L}$ in an equivariant manner, such that the fibre $\pi^{-1}(1)$ is isomorphic to $(X, L^{\otimes r})$.

Remark 1.2.2. We recall the following important and well known observations.

1. By virtue of the (equivariant) $\mathbb{C}^*$-action on $\mathscr{X}$, all non-central fibres $\mathscr{X}_t := \pi^{-1}(t)$ $(t \in \mathbb{C}^*)$ are isomorphic and the central fibre $\mathscr{X}_0 := \pi^{-1}(0)$ is naturally acted on by $\mathbb{C}^*$.

2. We will exclusively focus on the case when X is a smooth manifold, but we remark that, even when the noncentral fibres are smooth, the central fibre $\mathscr{X}_0$ of a test configuration is usually not smooth. In fact, $\mathscr{X}_0$ is a priori just a scheme and not even a variety.

3. A test configuration $(\mathscr{X}, \mathscr{L})$ is called **product** if $\mathscr{X}$ is isomorphic $X \times \mathbb{C}$. Note that this isomorphism is not necessarily equivariant, so X may have a nontrivial $\mathbb{C}^*$-action (cf. Remark 1.3.4). $(\mathscr{X}, \mathscr{L})$ is called **trivial** if $\mathscr{X}$ is equivariantly isomorphic to $X \times \mathbb{C}$, i.e. with trivial $\mathbb{C}^*$-action on X.

Remark 1.2.3. A well-known pathology found by Li and Xu [72] means that we may have to assume that $\mathscr{X}$ is a normal variety when $(\mathscr{X}, \mathscr{L})$ is not product or trivial. Alternatively, we may have to assume that the L^2-norm of the test configuration (as introduced by Donaldson [42]) is non-zero to define the non-triviality of the test configuration, as proposed by Székelyhidi [119, 121]. See also [16, 37, 113].

Let $(\mathscr{X}_t, \mathscr{L}_t)$ be any fibre of a test configuration $(\mathscr{X}, \mathscr{L})$ with the polarisation given by $\mathscr{L}_t := \mathscr{L}|_{\mathscr{X}_t}$. If $t \neq 0$, we can use the Hirzebruch–Riemann–Roch formula and Kodaira–Serre vanishing to show

$$\begin{aligned}\dim H^0(\mathscr{X}_t, \mathscr{L}_t^{\otimes k}) &= \int_X \mathrm{ch}(L^{\otimes rk})\mathrm{Td}_X \\ &= \frac{k^n}{n!}\int_X c_1(L^{\otimes r})^n - \frac{k^{n-1}}{2(n-1)!}\int_X c_1(K_X)c_1(L^{\otimes r})^{n-1} + O(k^{n-2})\end{aligned}$$

for $k \gg 1$, where ch is the Chern character and Td_X is the Todd class of T_X. We define $a_0, a_1 \in \mathbb{Q}$ as $a_0 := \frac{1}{n!}\int_X c_1(L^{\otimes r})^n$ and $a_1 := -\frac{1}{2(n-1)!}\int_X c_1(K_X)c_1(L^{\otimes r})^{n-1}$. Observe that the flatness condition implies that

$$d_k := \dim H^0(\mathscr{X}_t, \mathscr{L}_t^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})$$

does not depend on t.

On the other hand, the $\mathbb{C}^*$-action on the central fibre $(\mathscr{X}_0, \mathscr{L}_0)$ induces a representation $\mathbb{C}^* \curvearrowright H^0(\mathscr{X}_0, \mathscr{L}_0^{\otimes k})$. Let w_k be the weight of the representation $\mathbb{C}^* \curvearrowright \bigwedge^{\max} H^0(\mathscr{X}_0, \mathscr{L}_0^{\otimes k})$. Equivariant Riemann–Roch theorem (cf. [41]) shows that

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})$$

with $b_0, b_1 \in \mathbb{Q}$. Now expand

$$\frac{w_k}{k d_k} = \frac{b_0}{a_0} + \frac{a_0 b_1 - a_1 b_0}{a_0^2} k^{-1} + O(k^{-2}).$$

Definition 1.2.4. The **Donaldson–Futaki invariant** $DF(\mathscr{X}, \mathscr{L})$ of a test configuration $(\mathscr{X}, \mathscr{L})$ is a rational number defined by $DF(\mathscr{X}, \mathscr{L}) = (a_0 b_1 - a_1 b_0)/a_0$.

Definition 1.2.5. A polarised projective scheme (X, L) is ***K*-semistable** if $DF(\mathscr{X}, \mathscr{L}) \geq 0$ for any test configuration $(\mathscr{X}, \mathscr{L})$ for (X, L). (X, L) is ***K*-polystable** if $DF(\mathscr{X}, \mathscr{L}) \geq 0$ with equality if and only if $(\mathscr{X}, \mathscr{L})$ is product, and is ***K*-stable** if $DF(\mathscr{X}, \mathscr{L}) \geq 0$ with equality if and only if $(\mathscr{X}, \mathscr{L})$ is trivial.

We see that the sign of $DF(\mathscr{X}, \mathscr{L})$ is unchanged when we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes r}$. Therefore, once $\mathscr{X}$ is fixed, we may assume that the exponent of the test configuration is always 1 with L being very ample.

We can now state the following conjecture, usually referred to as the **Donaldson–Tian–Yau conjecture**, which has been a central problem in Kähler geometry for many years.

Conjecture 1.2.6. (Donaldson [41], Tian [125], Yau [128]) (X,L) admits a cscK metric in $c_1(L)$ if and only if it is K-polystable.

Remark 1.2.7. There is also a "relative" version of this conjecture which is more suited to extremal metrics; see [116].

1.2.2 Brief review of some known results

We now briefly review some results concerning Conjecture 1.2.6. This is *by no means exhaustive*, and we will only mention the results that will be referred to later, and several other results that are closely related to them.

By considering a lower bound of $\mathrm{Cal}(\omega)$, Donaldson [42] proved the following foundational result.

Theorem 1.2.8. (Donaldson [42]) *(X,L) is K-semistable if it admits a cscK metric in $c_1(L)$.*

This theorem was later improved by Stoppa [111] as follows, establishing one direction of Conjecture 1.2.6. Let $\mathrm{Aut}_0(X,L)$ be the identity component of the group of holomorphic transformations of X which lifts to the total space of the line bundle L (cf. §1.3).

Theorem 1.2.9. (Stoppa [111]) *Suppose that $\mathrm{Aut}_0(X,L)$ is trivial, which holds e.g. if (X,L) has no nontrivial holomorphic vector fields. Then, (X,L) is K-stable if it admits a cscK metric in $c_1(L)$.*

A sharper version of this theorem is available for Fano manifolds, with $L=-K_X$, as proved by Berman [13]. See also the work of Mabuchi [78, 79].

Although the above theorems provide sufficient conditions for K-stability, proving K-(poly)stability of a given variety (X,L) is in general tremendously difficult; as of now, we do not even know how to prove $\mathbb{P}^2$ is K-polystable without using the fact that it admits a Kähler–Einstein metric. However, proving K-*in*stability is sometimes possible thanks to the weaker and more explicitly computable stability notion called **slope stability**, introduced by Ross and Thomas [102]. This will be recalled in §4.2.1, to which the reader is referred for more details. Being a weaker notion of K-stability, we have slope instability implying K-instability (cf. Theorem 4.2.4), and hence in some cases we can prove the non-existence of a cscK metric by showing slope instability, thanks to Theorem 1.2.8.

In general, not much is known about the other direction of Conjecture 1.2.6, i.e. whether K-stability implies the existence of cscK metrics. However, there are several important cases where this direction is also established. Perhaps the most important result is the following theorem for Fano manifolds that was recently proved by Chen, Donaldson, and Sun [28, 29, 30].

Theorem 1.2.10. (Chen–Donaldson–Sun [28, 29, 30]) *Let X be a Fano manifold. If $(X, -K_X)$ is K-polystable, then X admits a Kähler–Einstein metric $\omega \in -c_1(K_X)$.*

Conjecture 1.2.6 is also known for toric surfaces [41, 46], and on certain iterated blow-ups of ruled surfaces [101, 100].

1.3 Automorphism groups of polarised Kähler manifolds and product test configurations

We discuss product test configurations and the automorphism group of (X, L) in detail in this section. In this case, the Donaldson–Futaki invariant admits a differential-geometric formula as given in Theorem 1.3.5, which is called the (classical) Futaki invariant. We first briefly review the automorphism group of (X, L); the reader is referred to [53, 67, 70] for more details on what is discussed here.

We write $\mathrm{Aut}(X)$ for the group of holomorphic transformations of X, consisting of diffeomorphisms of X which preserve the complex structure J, and $\mathrm{Aut}_0(X)$ for the connected component of $\mathrm{Aut}(X)$ containing the identity.

Definition 1.3.1. A vector field v on X is called **real holomorphic** if it preserves the complex structure, i.e. $L_v J = 0$ where L_v is the Lie derivative along v. A vector field Ξ is called **holomorphic** if it is a global section of the holomorphic tangent sheaf T_X, i.e. $\Xi \in H^0(X, T_X)$.

Remark 1.3.2. Observe that $\mathfrak{aut}(X) := \mathrm{LieAut}_0(X)$ is exactly the set of all real holomorphic vector fields. Recall also that $v \in \mathfrak{aut}(X)$ if and only if $Jv \in \mathfrak{aut}(X)$ (as J is integrable, cf. Proposition 2.10, Chapter IX, [68]).

Remark 1.3.3. It is well-known (cf. Proposition 2.11, Chapter IX, [68]) that there exists a one-to-one correspondence between the elements in $\mathfrak{aut}(X)$ and $H^0(X,T_X)$; the map $f^{1,0} : \mathfrak{aut}(X) \ni v \mapsto v^{1,0} \in H^0(X,T_X)$ defined by taking the $(1,0)$-part and the map $f^{\mathrm{Re}} : H^0(X,T_X) \ni \Xi \mapsto \mathrm{Re}(\Xi) \in \mathfrak{aut}(X)$ defined by taking the real part are the inverses of each other.

We now write $\mathrm{Aut}(X,L)$ for the subgroup of $\mathrm{Aut}(X)$ consisting of the elements whose action lifts to an automorphism of the total space of the line bundle L, and write $\mathrm{Aut}_0(X,L)$ for the connected component of $\mathrm{Aut}(X,L)$ containing the identity. $\mathrm{Aut}_0(X,L)$ is in fact equal to the maximal connected linear algebraic subgroup in $\mathrm{Aut}_0(X)$, and it is equal to the kernel of the Jacobi homomorphism from $\mathrm{Aut}_0(X)$ to the Albanese torus [53], and that the Lie algebra of $\mathrm{Aut}_0(X,L)$ is the set of all real holomorphic vector fields on X that have a zero. Moreover, it is also known that for any $v \in \mathrm{LieAut}_0(X,L)$ and Kähler metric ω on X there exists $f \in C^\infty(X,\mathbb{C})$ such that

$$\iota(v^{1,0})\omega = -\bar{\partial} f,$$

called the **holomorphy potential** of $v^{1,0}$ with respect to ω, where ι denotes the interior product. Conversely, if a holomorphic vector field $\Xi \in H^0(X,T_X)$ admits a holomorphy potential, its real part $\mathrm{Re}(\Xi)$ lies in $\mathrm{LieAut}_0(X,L)$. The reader is referred to Theorem 1 in [70], and Theorems 9.4 and 9.7 in [67] for more details.

Remark 1.3.4. It is immediate that a (nontrivial) product test configuration for (X,L) is exactly a choice of 1-parameter subgroup $\mathbb{C}^*$ in $\mathrm{Aut}_0(X,L)$, where we recall that the $\mathbb{C}^*$-action has to lift to the total space of the line bundle L to define a test configuration (cf. Definition 1.2.1). If we write $v \in \mathrm{LieAut}_0(X,L)$ for the generator of this subgroup $\mathbb{C}^* \leq \mathrm{Aut}_0(X,L)$, the above argument shows that $v^{1,0} \in H^0(X,T_X)$ admits a holomorphy potential, and that conversely $\Xi \in H^0(X,T_X)$ admitting a holomorphy potential defines a 1-parameter subgroup $\mathbb{C}^* \leq \mathrm{Aut}_0(X,L)$ under the correspondence in Remark 1.3.3. To summarise, *a product test configuration is exactly a choice of* $\Xi \in H^0(X,T_X)$ *which admits a holomorphy potential.*

Finally, we recall the following theorem.

Theorem 1.3.5. (Donaldson [41], Futaki [54]) *Let $f \in C^\infty(X,\mathbb{C})$ be the holomorphy potential of a holomorphic vector field Ξ_f on X with respect to a Kähler metric $\omega \in c_1(L)$. If $(\mathscr{X},\mathscr{L})$ is the product test configuration generated by Ξ_f, the Donaldson–Futaki invariant can be written as*

$$DF(\mathscr{X},\mathscr{L}) = \frac{1}{4\pi}\int_X f(S(\omega)-\bar{S})\frac{\omega^n}{n!},$$

where $S(\omega)$ is the scalar curvature of ω and $\bar{S}$ is the average of $S(\omega)$ over X. The integral in the right hand side

$$\mathrm{Fut}(\Xi_f,[\omega]) := \int_X f(S(\omega)-\bar{S})\frac{\omega^n}{n!},$$

*called the **Futaki invariant** or **classical Futaki invariant**, does not depend on the specific choice of Kähler metric ω, i.e. is an invariant of the cohomology class $[\omega]$.*

1.4 Extremal metrics and the Lichnerowicz operator

We define an operator $\mathfrak{D}_\omega : C^\infty(X,\mathbb{C}) \to C^\infty(T^{1,0}X \otimes \Omega^{0,1}(X))$ by

$$\mathfrak{D}_\omega\phi := \bar{\partial}(\mathrm{grad}^{1,0}_\omega\phi)$$

where $\mathrm{grad}^{1,0}_\omega\phi$ is the $T^{1,0}X$-component of the gradient vector field $\mathrm{grad}_\omega\phi$ of ϕ with respect to ω and $\bar{\partial}$ is the (0,1)-part of the Chern connection on TX. Thus $\mathfrak{D}_\omega\phi = 0$ if and only if $\mathrm{grad}^{1,0}_\omega\phi$ is a holomorphic vector field. Writing $\mathfrak{D}^*_\omega$ for the formal adjoint of $\mathfrak{D}_\omega$ with respect to ω, we have the following formula (cf. [70])

$$\mathfrak{D}^*_\omega\mathfrak{D}_\omega\phi = \Delta^2_\omega\phi + (\mathrm{Ric}(\omega),\sqrt{-1}\partial\bar{\partial}\phi)_\omega + (\partial S(\omega),\bar{\partial}\phi)_\omega, \tag{1.1}$$

where $(,)_\omega$ stands for the pointwise inner product on the space of differential forms defined by ω, and Δ_ω is the negative $\bar{\partial}$-Laplacian $-\bar{\partial}\bar{\partial}^* - \bar{\partial}^*\bar{\partial}$. Note that this is a fourth order self-adjoint elliptic operator, but may *not* be a real operator; $\mathfrak{D}^*_\omega\mathfrak{D}_\omega\phi$ may be a $\mathbb{C}$-valued function even when ϕ is a real function, due to the third term $(\partial S(\omega),\bar{\partial}\phi)_\omega$ of $\mathfrak{D}^*_\omega\mathfrak{D}_\omega$. On the other hand, note the obvious $\ker\mathfrak{D}^*_\omega\mathfrak{D}_\omega = \ker\mathfrak{D}_\omega$, since X is compact.

We define another operator $\mathscr{D}^*_\omega \mathscr{D}_\omega : C^\infty(X,\mathbb{R}) \to C^\infty(X,\mathbb{R})$ by

$$\mathscr{D}^*_\omega \mathscr{D}_\omega \phi = \Delta^2_\omega \phi + (\mathrm{Ric}(\omega), \sqrt{-1}\partial\bar{\partial}\phi)_\omega + \frac{1}{2}(dS(\omega), d\phi)_\omega. \tag{1.2}$$

This is a 4-th order self-adjoint elliptic operator, which we call the **Lichnerowicz operator**.

We observe that we can write $\mathscr{D}^*_\omega \mathscr{D}_\omega = \frac{1}{2}(\mathfrak{D}^*_\omega \mathfrak{D}_\omega + \overline{\mathfrak{D}^*_\omega \mathfrak{D}_\omega})$, where the operator $\overline{\mathfrak{D}^*_\omega \mathfrak{D}_\omega}$ is defined by $\overline{\mathfrak{D}^*_\omega \mathfrak{D}_\omega}\phi = \Delta^2_\omega \phi + (\mathrm{Ric}(\omega), \sqrt{-1}\partial\bar{\partial}\phi)_\omega + (\bar{\partial}S(\omega), \partial\phi)_\omega$. Thus the kernels of $\mathscr{D}^*_\omega \mathscr{D}_\omega$ and $\mathfrak{D}^*_\omega \mathfrak{D}_\omega$ may not have anything to do with each other when we consider $\mathbb{C}$-valued functions in general, but we have the following well-known lemma for the real functions.

Lemma 1.4.1. *A real function $\phi \in C^\infty(X,\mathbb{R})$ satisfies $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi = 0$ if and only if $\mathfrak{D}_\omega \phi = 0$.*

Proof. We first observe that, since ϕ is real, we have

$$\mathscr{D}^*_\omega \mathscr{D}_\omega \phi = \frac{1}{2}(\mathfrak{D}^*_\omega \mathfrak{D}_\omega \phi + \overline{\mathfrak{D}^*_\omega \mathfrak{D}_\omega}\phi) = \frac{1}{2}(\mathfrak{D}^*_\omega \mathfrak{D}_\omega \phi + \overline{\mathfrak{D}^*_\omega \mathfrak{D}_\omega \phi}).$$

Thus, $\mathfrak{D}_\omega \phi = 0$ implies $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi = 0$. Observe also that we have

$$\int_X \phi \mathscr{D}^*_\omega \mathscr{D}_\omega \phi \frac{\omega^n}{n!} = \frac{1}{2}\left(\int_X |\mathfrak{D}_\omega \phi|^2_\omega \frac{\omega^n}{n!} + \overline{\int_X |\mathfrak{D}_\omega \phi|^2_\omega \frac{\omega^n}{n!}} \right),$$

and hence $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi = 0$ implies $\mathfrak{D}_\omega \phi = 0$. □

Suppose now that we consider a Hamiltonian vector field v_ϕ generated by $\phi \in C^\infty(X,\mathbb{R})$ with respect to ω. We use the sign convention $\iota(v_\phi)\omega = -d\phi$ for the Hamiltonian. We observe that we can write

$$\mathrm{grad}_\omega \phi = -Jv_\phi, \tag{1.3}$$

where J is the complex structure on TX. Recall that $\mathrm{grad}^{1,0}_\omega \phi$ being a holomorphic vector field is equivalent to $\mathrm{grad}_\omega \phi$ being a real holomorphic vector field (cf. Remark 1.3.3), and that a vector field v_ϕ is real holomorphic if and only if Jv_ϕ is real holomorphic (cf. Remark 1.3.2). We thus get the following well-known result.

Lemma 1.4.2. *Suppose that* $\phi \in C^{\infty}(X,\mathbb{R})$ *satisfies* $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}\phi = 0$ *(or equivalently* $\mathscr{D}^*_{\omega}\mathscr{D}_{\omega}\phi = 0$*). Then the Hamiltonian vector field* v_{ϕ} *generated by* ϕ *with respect to* ω *is a real holomorphic vector field. Conversely, if the Hamiltonian vector field* v_{ϕ} *is real holomorphic, we need to have* $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}\phi = 0$ *(or equivalently* $\mathscr{D}^*_{\omega}\mathscr{D}_{\omega}\phi = 0$*).*

Remark 1.4.3. Note that, since ω is Kähler, a Hamiltonian real holomorphic vector field must preserve the associated Riemannian metric $g = \omega(\cdot, J\cdot)$, and hence is necessarily a Hamiltonian Killing vector field with respect to g.

Suppose now that ω is an extremal metric, so that $\mathrm{grad}^{1,0}_{\omega}S(\omega)$ is a holomorphic vector field. By the above argument and the equation (1.3), $J\mathrm{grad}_{\omega}S(\omega)$ is a real holomorphic vector field equal to to the Hamiltonian vector field v_s generated by $S(\omega)$.

Definition 1.4.4. The Hamiltonian real holomorphic vector field v_s generated by the scalar curvature $S(\omega)$ of an extremal metric ω, satisfying $\iota(v_s)\omega = -dS(\omega)$, is called an **extremal vector field**.

By taking the $(0,1)$-component of the equation $\iota(v_s)\omega = -dS(\omega)$, we have $\iota(v^{1,0}_s)\omega = -\bar{\partial}S(\omega)$, i.e. $S(\omega)$ is the holomorphy potential of $v^{1,0}_s$, and hence $v_s \in \mathrm{LieAut}_0(X,L)$ by the argument given in §1.3. This implies that if $\mathrm{Aut}_0(X,L)$ is trivial, we have $v_s = 0$ and hence an extremal metric is necessarily a cscK metric. Also, Calabi [24] proved that an extremal metric is cscK if and only if the Futaki invariant is 0. We summarise these observations in the following.

Lemma 1.4.5. (cf. [70, 24]) *Suppose that* ω *is an extremal metric. Then*

1. ω *is cscK if* $\mathrm{Aut}_0(X,L)$ *is trivial,*

2. ω *is cscK if and only if the Futaki invariant evaluated against the* $(1,0)$*-part of the extremal vector field* v_s *is zero, i.e.* $\mathrm{Fut}(v^{1,0}_s, [\omega]) = 0$.

If $\mathrm{Aut}_0(X,L)$ is not trivial, an extremal metric need not be cscK. Indeed, Calabi [23] explicitly constructed a non-cscK extremal metric on the total space of a projectivised bundle $\mathbb{P}(\mathscr{O}_{\mathbb{P}^{n-1}}(-m) \oplus \mathbb{C})$ over $\mathbb{P}^{n-1}$ for all $n,m \in \mathbb{N}$ in every Kähler class, as we shall see in §4.1.3.1 (cf. Theorem 4.1.7).

Chapter 2

Quantisation of extremal Kähler metrics

2.1 Introduction

2.1.1 Donaldson's quantisation

Donaldson's work on the constant scalar curvature Kähler (cscK) metrics and the projective embeddings [40, 43] is undoubtedly one of the most important results in Kähler geometry in the last few decades. It states that, if the automorphism group $\mathrm{Aut}(X,L)$ of a polarised compact Kähler manifold (X,L) is discrete (cf. §2.2.1.1) and (X,L) admits a cscK metric $\omega \in c_1(L)$, then for all large enough k there exists a balanced metric at the level k (cf. Definition 2.2.13). Our starting point is a naive re-interpretation of the cscK metric as satisfying $\bar{\partial} S(\omega) = 0$ and the balanced metric as satisfying $\bar{\partial} \rho_k(\omega) = 0$, where $\rho_k(\omega)$ is the Bergman function (cf. Definition 2.2.10). We also observe that $\mathrm{Aut}(X,L)$ being discrete is equivalent to the connected component $\mathrm{Aut}_0(X,L)$ containing the identity of $\mathrm{Aut}(X,L)$ being trivial, where we note that $\mathrm{Aut}_0(X,L)$ will be used more frequently in what follows. We record Donaldson's theorem in this form here.

Theorem 2.1.1. (Donaldson [40]) *Suppose that the connected component of the automorphism group* $\mathrm{Aut}_0(X,L)$ *of a polarised Kähler manifold* (X,L) *is trivial and* (X,L) *admits a Kähler metric* $\omega \in c_1(L)$ *satisfying* $\bar{\partial} S(\omega) = 0$*. Then for any large enough* k *there exists a Kähler metric* $\omega_k \in c_1(L)$ *satisfying* $\bar{\partial} \rho_k(\omega_k) = 0$ *and* $\omega_k \to \omega$ *in* C^∞ *as* $k \to \infty$.

Theorem 2.1.2. (Donaldson [40]) *If a sequence of Kähler metrics* $\{\omega_k\}_k$*, each of which* *satisfies* $\bar{\partial}\rho_k(\omega_k) = 0$*, converges to a Kähler metric* $\omega_\infty \in c_1(L)$ *in* C^∞*, then the limit* ω_∞ *satisfies* $\bar{\partial}S(\omega_\infty) = 0$.

We note that Theorem 2.1.2 does not assume the existence of a cscK metric or the triviality of $\mathrm{Aut}_0(X,L)$, unlike Theorem 2.1.1. The importance of Donaldson's theorem, in one direction, is that Theorem 2.1.1 provides the first general result on the existence of cscK metric implying algebro-geometric "stability", along the line conjectured by Yau [128], Tian [125], and Donaldson [39], and also extending the previous works of Tian [125] on Kähler–Einstein metrics to the cscK metrics. Namely, we have the following corollary.

Corollary 2.1.3. (Zhang [131], Luo [76], Donaldson [40]) *If a polarised Kähler manifold* (X,L) *with trivial* $\mathrm{Aut}_0(X,L)$ *admits a cscK metric* $\omega \in c_1(L)$*, it is asymptotically Chow stable.*

This follows from the theorem of Luo [76] and Zhang [131] stating that (X,L) is Chow stable at the level k if and only if L admits a balanced metric at the level k (cf. Theorem 2.6.2), combined with the above Theorem 2.1.1, where the reader is referred to Definition 2.6.1 for the definition of (asymptotic) Chow stability.

In another direction, Theorem 2.1.1 provides an approximation scheme for the cscK metrics. Recall now that the existence of many cscK metrics (e.g. Calabi–Yau metrics on compact Kähler manifolds) is guaranteed only by abstract existence theorems and explicit formulae for these metrics are in general extremely difficult to obtain. However, we can in fact find a numerical algorithm for finding a balanced metric as explained in [43] and [106], and hence it is (in principle) possible to numerically approximate a cscK metric. Various mathematicians have used this method to attack this problem of "explicitly" approximating a cscK metric, and there already seems to be a substantial accumulation of research. We only mention here [18, 19, 47, 49, 66], which actually implemented the above algorithm.

That such a numerical algorithm should exist could be seen intuitively from the following fact. Suppose that we choose a basis $\{Z_i\}$ for $H^0(X,L^k)$ (for large enough k) so as to have an isomorphism $H^0(X,L^k) \xrightarrow{\sim} \mathbb{C}^{N_k}$ and an embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*) \cong \mathbb{P}^{N_k-1}$. We then consider the moment map for the $U(N_k)$-action on $\mathbb{P}^{N_k-1}$, and integrate it over the image $\iota(X)$ of X to get the centre of mass $\bar{\mu}_X$ (see §2.2.2

for the details); namely $\bar{\mu}_X$ is defined as

$$\bar{\mu}_X := \int_{\iota(X)} \frac{h_{FS}(Z_i, Z_j)}{\sum_{l=1}^{N_k} |Z_l|^2_{FS}} \frac{\omega^n_{FS}}{n!} \in \sqrt{-1}\mathfrak{u}(N_k)$$

where h_{FS} is the Fubini–Study metric on $\mathscr{O}_{\mathbb{P}^{N_k-1}}(1)$. We can move the image $\iota(X)$ by an $SL(N_k,\mathbb{C})$-action on $\mathbb{P}^{N_k-1}$, and we write $\bar{\mu}_X(g)$ for the new centre of mass when we move $\iota(X)$ by $g \in SL(N_k,\mathbb{C})$ to $\iota_g(X)$, say. It is well-known (cf. [76, 131], see also Theorem 2.2.19) that there exists a balanced metric at the level k if and only if there exists $g \in SL(N_k,\mathbb{C})$ such that $\bar{\mu}_X(g)$ is equal to a constant multiple of the identity. Thus, the seemingly intractable PDE problem $\bar{\partial}\rho_k(\omega) = 0$ can in fact be reduced to a *finite dimensional* problem on the vector space $H^0(X,L^k)$.

Given the above, we may interpret Theorem 2.1.1 as associating an essentially finite dimensional problem on $\mathbb{P}(H^0(X,L^k)^*)$ to a differential-geometric problem of solving $\bar{\partial}S(\omega) = 0$ on (X,L), with an "error" which goes to 0 as $k \to \infty$ (cf. Theorem 2.3.7). This is often called **quantisation**, by regarding $H^0(X,L^k)$ as a set of quantum-mechanical wave functions and $\sqrt{k}$ as the inverse of Planck's constant, so that the limit $k \to \infty$ corresponds to the semiclassical limit.

Remark 2.1.4. We now recall that the hypothesis of $\mathrm{Aut}_0(X,L)$ being trivial is essential in Theorem 2.1.1. Indeed, Della Vedova and Zuddas [32] showed (Example 4.3, [32]) that $\mathbb{P}^2$ blown up at 4 points, all but one aligned, is Chow *unstable* at the level k for *all* large enough k with respect to an appropriate polarisation, although a well-known theorem of Arezzo and Pacard [7] (see in particular Example 7.3 in [7]) shows that it admits a cscK metric in that polarisation. We also recall that Ono, Sano, and Yotsutani [93] showed that there exists a toric Kähler–Einstein Fano manifold that are asymptotically Chow unstable (with respect to the anticanonical polarisation, even after replacing K_X^{-1} by a higher tensor power).

2.1.2 Statement of the results

Our aim is to find how Theorems 2.1.1, 2.1.2, and Corollary 2.1.3 can extend to the case where $\mathrm{Aut}_0(X,L)$ is no longer trivial. Since Theorem 2.1.1 (and hence Corollary 2.1.3) does fail to hold when $\mathrm{Aut}_0(X,L)$ is nontrivial (cf. Remark 2.1.4), we need a new ingredient. Suppose now that we replace $\bar{\partial}$ by an operator $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}$ (cf. §1.4) and consider the equation $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}S(\omega) = 0$, i.e. ω is an extremal metric, which can be regarded as a "generalisation" of cscK metrics when $\mathrm{Aut}_0(X,L)$ is no longer trivial (cf. §1.4).

Now, when we change $\bar{\partial}S(\omega) = 0$ to $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}S(\omega) = 0$, the corresponding equation $\bar{\partial}\rho_k(\omega_k) = 0$ changes to

$$\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0, \tag{2.1}$$

and this seems to suggest that this is the equation which "quantises" the extremal metric, when $\mathrm{Aut}_0(X,L)$ is no longer trivial; observe that when $\mathrm{Aut}_0(X,L)$ is trivial and hence (X,L) admits no nontrivial holomorphic vector field, the above equation implies $\rho_k(\omega_k) = \mathrm{const}$ and hence we recover the balanced metric.

The aim of this chapter is to establish an "extremal" analogue of Theorems 2.1.1 and 2.1.2 by using the equation (2.1). First of all, an analogue of Theorem 2.1.2 can be established as follows.

Theorem 2.1.5. *If a sequence of Kähler metrics $\{\omega_k\}_k$ in $c_1(L)$, each of which satisfies $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$, converges to a Kähler metric $\omega_\infty \in c_1(L)$ in C^∞, then the limit ω_∞ satisfies $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_\infty}S(\omega_\infty) = 0$, i.e. is an extremal metric.*

Proof. By recalling the well-known expansion[1] of the Bergman function (Theorem 2.3.7), we have $0 = \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}4\pi k\rho_k(\omega_k) = \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}(S(\omega_k) + O(1/k))$. Since $\{\omega_k\}_k$ converges to ω_∞ in C^∞ as $k \to \infty$, we have $S(\omega_k) \to S(\omega_\infty)$ in C^∞ and $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}F \to \bar{\partial}\mathrm{grad}^{1,0}_{\omega_\infty}F$ in C^∞ for any fixed smooth function F. Thus

$$0 = \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}4\pi k\rho_k(\omega_k) = \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}(S(\omega_k) - S(\omega_\infty)) + \bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}S(\omega_\infty) + O(1/k),$$

and hence we get $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_\infty}S(\omega_\infty) = \lim_{k\to\infty}\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}S(\omega_\infty) = 0$.

□

An important aspect of the equation (2.1) is that, similarly to the case when $\mathrm{Aut}_0(X,L)$ is trivial, we can find an equivalent characterisation in terms of the centre of mass $\bar{\mu}_X$, so that solving the equation (2.1) can be reduced to an essentially finite

[1] Note in particular that the expansion is uniform when the metric varies in a family of uniformly equivalent metrics which is compact with respect to the C^∞-topology (Theorem 2.3.7).

dimensional problem (cf. §2.4); we shall see (Proposition 2.4.5 and Corollary 2.4.16) that the equation (2.1) holds if and only if there exists $g \in SL(N_k, \mathbb{C})$ such that[2] $\bar{\mu}_X(g)^{-1}$ generates a holomorphic vector field on $\mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{P}^{N_k - 1}$ that is tangential to the image $\iota(X)$ of X.

Let $K := \mathrm{Isom}(\omega) \cap \mathrm{Aut}_0(X, L)$, where $\mathrm{Isom}(\omega)$ is the isometry group of the extremal metric ω (cf. §2.2.1.2). We now state our main result as follows; it is an analogue of Theorem 2.1.1 when $\mathrm{Aut}_0(X, L)$ is nontrivial.

Theorem 2.1.6. *Suppose that (X, L) admits an extremal metric $\omega \in c_1(L)$. Replacing L by L^r for a large but fixed $r \in \mathbb{N}$ if necessary, for each $l \in \mathbb{N}$, there exists $k_l \in \mathbb{N}$ such that for all $k \geq k_l$ there exists a smooth K-invariant Kähler metric $\omega_{k,l} \in c_1(L)$ which satisfies $\bar{\partial} \mathrm{grad}^{1,0}_{\omega_{k,l}} \rho_k(\omega_{k,l}) = 0$ and converges to ω in C^l as $k \to \infty$.*

The reader is referred to Remark 2.5.18 for comments on the dependence on l, and the possibility of the convergence $\omega_{k,l} \to \omega$ in C^∞. Combined with Theorem 2.6.10 proved by Mabuchi [82, 86], we obtain an alternative proof of the following result that was first obtained by Mabuchi.

Corollary 2.1.7. (cf. Mabuchi [82, 84, 85]) *Suppose that (X, L) admits an extremal metric in $c_1(L)$. Replacing L by L^r for a large but fixed $r \in \mathbb{N}$ if necessary, (X, L) is asymptotically weakly Chow polystable relative to any maximal torus in $K \leq \mathrm{Aut}_0(X, L)$.*

As explained in Remark 2.6.13, Corollary 2.1.7 does *not* imply Theorem 2.1.6. The reader is referred to §2.6.2 for the discussion on (weak) Chow stability relative to a torus, as well as the proof for how Corollary 2.1.7 follows from Theorem 2.1.6.

Remark 2.1.8. That we have to replace L by a large enough tensor power is a new phenomenon which did not appear in the case where $\mathrm{Aut}_0(X, L)$ is discrete [40, 43]. This essentially comes from the need to linearise $\mathrm{Aut}_0(X, L)$-action on X to the total space of L, which may not be possible unless we raise L to a higher tensor power (cf. Lemma 2.2.1, Remark 2.2.2).

Finally, recalling the characterisation of the equation (2.1) in terms of the centre of mass (Proposition 2.4.5), we hope that Theorem 2.1.6 may potentially provide a numerical approximation to the extremal metrics, as in the cscK case.

[2] See Lemma 2.2.20 for the perhaps surprising appearance of the inverse sign in $\bar{\mu}_X(g)^{-1}$.

2.1.3 Comparison to previously known results

We recall that, in fact, the problem of "quantising" the extremal metrics has been considered by several mathematicians[3], notably by Mabuchi [82, 84, 85, 86], Sano–Tipler [107]. The work of Apostolov–Huang [5] is also related, and contains a neat survey of Mabuchi's work. These notions of "quantised" extremal metrics will be reviewed in §2.6.4.

An important special case of Theorem 2.1.6 is when $\mathrm{Aut}_0(X,L)$ is nontrivial but the centre $Z(K)$ of K is discrete. As is well-known, if ω is extremal, the Hamiltonian vector field v_s generated by $S(\omega)$ has to belong to the centre $\mathfrak{z} := \mathrm{Lie}(Z(K))$ of the Lie algebra $\mathfrak{k} := \mathrm{Lie}(K)$ (cf. Lemma 2.3.4). Thus, $Z(K)$ being discrete implies $v_s = 0$, and hence ω is cscK. On the other hand, if $Z(K)$ is discrete and a K-invariant Kähler metric ω_k satisfies $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$, then Lemmas 2.2.21 and 2.3.4 show that the Hamiltonian vector field v generated by $\rho_k(\omega_k)$ has to lie in $\mathfrak{z}$; thus $Z(K)$ being discrete and Theorem 2.1.6 implies that $\rho_k(\omega_k)$ has to be constant, i.e. ω_k is a balanced metric for all large (and divisible) k, and hence by a theorem of Zhang [131], (X,L) is asymptotically Chow semistable (cf. Remark 2.6.3).

This is in fact an easy consequence of the results proved by Futaki [55] and Mabuchi [81, 84], which we now recall. If (X,L) is cscK, Mabuchi [81] proved that there exists an obstruction for (X,L) being asymptotically Chow polystable when $\mathrm{Aut}_0(X,L)$ is nontrivial, and also showed that the vanishing of these obstructions is sufficient for a cscK (X,L) to be asymptotically Chow polystable [84]. Futaki [55] proved that the vanishing of Mabuchi's obstructions is equivalent to the vanishing of a series of integral invariants, which may be called "higher Futaki invariants". We can show that they all vanish when (X,L) is cscK and $Z(K)$ is discrete as follows; since the higher Futaki invariants are Lie algebra *characters* defined on $\mathrm{LieAut}_0(X,L) = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$ (by Matsushima–Lichnerowicz theorem, cf. Theorem 4.1.5), the centre of $\mathfrak{k}$ being trivial implies that these higher Futaki invariants are all equal to 0, and hence that (X,L) is indeed asymptotically Chow polystable, which in particular implies that (X,L) is asymptotically Chow semistable.

We saw in Remark 2.1.4 the example of cscK, or even Kähler–Einstein, manifolds

[3] We also mention the work of Bunch and Donaldson [21] for the toric case, and also note that Berman and Witt Nyström [15] and Takahashi [122] treat similar problems in the context of Kähler–Ricci solitons.

that are asymptotically Chow unstable even after replacing L by a large enough tensor power. However, Theorem 2.1.6 and Corollary 2.1.7 imply that it is still possible to find a Kähler metric ω_k with $\bar{\partial}\text{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$ on these manifolds, and hence they are asymptotically Chow stable *relative to any maximal torus* in K.

Finally, we recall the theorem of Stoppa and Székelyhidi [114], which states that the existence of extremal metrics implies the K-polystability relative to a maximal torus in the automorphism group, where the notion of relative K-stability was introduced by Székelyhidi [116].

Remark 2.1.9. Recalling Corollary 5 of [40], it is natural to expect that Theorem 2.1.6 implies the uniqueness of extremal metrics in $c_1(L)$ up to $\text{Aut}_0(X,L)$-action. Indeed, we set up the problem of finding the solution to (2.1) as a variational problem of finding the critical point of the modified balancing energy $\mathscr{Z}^A$ on a finite dimensional manifold $\mathscr{B}_k^K$, where A is essentially equal to $\text{grad}_{\omega_k}\rho_k(\omega_k)$; see §2.4 and §2.5 for more details. It is clear from the convexity (cf. Remark 2.5.2, Theorem 2.5.3) of $\mathscr{Z}^A$ that the critical point of $\mathscr{Z}^A$ is unique up to $\text{Aut}_0(X,L)$-action for each *fixed* A. However, the problem is that we do not know whether there exist two metrics ω_1 and ω_2 in $c_1(L)$, both satisfying $\bar{\partial}\text{grad}^{1,0}_{\omega_1}\rho_k(\omega_1) = 0$ and $\bar{\partial}\text{grad}^{1,0}_{\omega_2}\rho_k(\omega_2) = 0$, but with $\text{grad}_{\omega_1}\rho_k(\omega_1) \neq \text{grad}_{\omega_2}\rho_k(\omega_2)$. The existence of such ω_1 and ω_2 would imply that we cannot prove the uniqueness of the "quantised" approximant (as in Theorem 1, [40]) of extremal metrics, and hence the uniqueness of extremal metric itself. On the other hand, the uniqueness of extremal metrics itself was established by Mabuchi [83], Berman and Berndtsson [14].

2.1.4 Organisation of the chapter

The strategy of the proof of Theorem 2.1.6, which occupies most of what follows, is essentially the same as in [40]; we construct an approximate solution to $\bar{\partial}\text{grad}^{1,0}_{\omega_h}\rho_k(\omega_h) = 0$, reduce the problem to a finite dimensional one, and use the gradient flow on a finite dimensional manifold to perturb the approximate solution to the genuine one.

After reviewing in §2.2 some well-known results on the automorphism group of polarised Kähler manifolds and Donaldson's theory of quantisation, we construct approximate solutions in §2.3; after some preliminary work in §2.3.1, we establish the main technical result Proposition 2.3.13 and its consequence Corollary 2.3.15. We establish in §2.4 the characterisation of the equation $\bar{\partial}\text{grad}^{1,0}_{\omega_h}\rho_k(\omega_h) = 0$ in terms of the centre of mass $\bar{\mu}_X$, so as to reduce the problem to a finite dimensional one; the main results of the section are Proposition 2.4.5, Corollaries 2.4.11 and 2.4.16. We set up the problem as a variational one in §2.5.1 by introducing the "modified" balancing energy $\mathscr{Z}^A$, so that the solution of $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_h}\rho_k(\omega_h) = 0$ can be obtained by finding the critical point of $\mathscr{Z}^A$. By recalling the well-known estimates on the Hessian of the balancing energy in §2.5.2, we run the gradient flow (2.41) in §2.5.3 driven by $\mathscr{Z}^A$. Unfortunately, the nontrivial automorphism group $\mathrm{Aut}_0(X,L)$ means that the limit of the gradient flow does not achieve the critical point of $\mathscr{Z}^A$ (cf. Proposition 2.5.13). However, in §2.5.4 we set up an inductive procedure to (exponentially) decrease $\delta\mathscr{Z}^A$, which is shown to converge, so as to give the critical point of $\mathscr{Z}^A$ (Proposition 2.5.15); the trick is in fact to perturb the auxiliary parameter A to decrease $\delta\mathscr{Z}^A$.

Finally, we consider the connection to the stability of (X,L) in §2.6, where we also discuss the relationship to the previously known results, particularly by Sano–Tipler [107] and Mabuchi [82, 84, 85, 86]; in particular, we provide the proof of Corollary 2.1.7 at the end of §2.6.2.

Notation 2.1.10. In this chapter, we shall consistently write $N = N_k$ for $\dim_{\mathbb{C}} H^0(X,L^k)$, and V for $\int_X c_1(L)^n/n!$.

2.2 Background

2.2.1 Further properties of automorphism groups of polarised Kähler manifolds

2.2.1.1 Linearisation of the automorphism group

This section is a review of well-known results, and the reader is referred to [53, 67, 70, 89] for more details on what is discussed here. Let (X,L) be a polarised Kähler manifold i.e. a Kähler manifold X with an ample line bundle L over X. By taking $r \in \mathbb{N}$ to be large enough, we may assume that L^r is very ample and also have the surjection $\bigotimes_{i=1}^m H^0(X,L^r) \twoheadrightarrow H^0(X,L^{rm})$ for any $m \geq 1$. We now have the embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X,L^r)^*)$. We write $\mathrm{Aut}(X)$ for the group of holomorphic transformations of X, and $\mathrm{Aut}_0(X)$ for the connected component of $\mathrm{Aut}(X)$ containing the identity. We also write $\mathrm{Aut}(X,L^r)$ for the subgroup of $\mathrm{Aut}(X)$ consisting of the elements whose action lifts to an automorphism of the total space of the line bundle L^r, and write $\mathrm{Aut}_0(X,L^r)$

for the connected component of $\mathrm{Aut}(X, L^r)$ containing the identity. We now recall the well-known fact that $\mathrm{Aut}_0(X, L^r)$ is equal to the maximal connected linear algebraic subgroup in $\mathrm{Aut}_0(X)$, equal to the kernel of Jacobi homomorphism from $\mathrm{Aut}_0(X)$ to the Albanese torus [53], and that the Lie algebra of $\mathrm{Aut}_0(X, L^r)$ is the set of all holomorphic vector fields on X that have a zero (cf. Theorem 1 of [70]). Given these remarks, we shall (abusively) write $\mathrm{Aut}_0(X, L)$ for $\mathrm{Aut}_0(X, L^r)$, for any $r > 0$.

Suppose that we write $\tilde{f}$ for the automorphism of the total space of the line bundle L^r obtained by lifting $f \in \mathrm{Aut}_0(X, L)$, where we note that such $\tilde{f}$ is well-defined only up to an overall constant multiple (acting as a fibrewise multiplication). Thus, the lift $f \mapsto \tilde{f}$ gives a map $\mathrm{Aut}_0(X, L) \to GL(H^0(X, L^r))$, acting by pull-back, which is well-defined only up to an overall constant multiple, since we only have $\tilde{f}_1 \circ \tilde{f}_2 = a\widetilde{f_1 \circ f_2}$ for some constant $a \in \mathbb{C}^*$ (cf. proof of Theorem 9.2 in [67]). In other words, the lift $f \mapsto \tilde{f}$ gives a well-defined homomorphism $\theta : \mathrm{Aut}_0(X, L) \to PGL(H^0(X, L^r))$. Considering the action $PGL(H^0(X, L^r)) \curvearrowright \mathbb{P}(H^0(X, L^r)^*)$ given by the dual representation, it is easy to see that $\theta(f)$, $f \in \mathrm{Aut}_0(X, L)$, defines an element in $PGL(H^0(X, L^r))$ which fixes the image $\iota(X)$ of X under the Kodaira embedding.

Conversely, a well-known theorem (Theorem 9.4 of [67]) asserts that, for every element f of $\mathrm{Aut}_0(X, L)$, there exists a unique projective linear transformation $g \in PGL(H^0(X, L^r))$ which fixes the image $\iota(X)$ of X under the Kodaira embedding, such that f is the restriction of the action of g on $\mathbb{P}(H^0(X, L^r)^*)$ to the image $\iota(X)$ of X in $\mathbb{P}(H^0(X, L^r)^*)$; in other words, we have $g \circ \iota = \iota \circ f$ as an equality between maps $X \to \mathbb{P}(H^0(X, L^r)^*)$ (cf. p84, [67]). Note also that ι being an embedding means that θ is injective. Summarising the argument as above, we now have an injective homomorphism $\theta : \mathrm{Aut}_0(X, L) \to PGL(H^0(X, L^r))$ which satisfies $\theta(f) \circ \iota = \iota \circ f$.

However, we will often need $\theta(f)$ to be a "genuine" linear transformation rather than a projective linear transformation. It is well-known (cf. Proposition 9.3 [67]) that, by replacing L by L^{rR} where $R := \dim_{\mathbb{C}} H^0(X, L^r)$, this representation θ can indeed be "lifted" to a linear transformation on the ambient vector space; namely there exists a faithful representation $\theta : \mathrm{Aut}_0(X, L^{rR}) \to SL(H^0(X, L^{rR}))$, which we still denote by θ by abuse of notation. From now on, we replace L by L^{rR} in the above. Summarising the above argument and also recalling the surjection $\bigotimes_{i=1}^m H^0(X, L^r) \twoheadrightarrow H^0(X, L^{rm})$, we obtain the following well-known result.

Lemma 2.2.1. *By replacing L by a large tensor power if necessary, we have a unique faithful group representation*

$$\theta : \mathrm{Aut}_0(X,L) \to SL(H^0(X,L^k))$$

for all $k \in \mathbb{N}$, which satisfies

$$\theta(f) \circ \iota = \iota \circ f \tag{2.2}$$

for the Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*)$.

Proof. Since the existence follows from the above discussion, we only have to show the uniqueness. Suppose that we have two faithful representations θ and θ', both satisfying (2.2). Observe that we have $\theta(g) \circ \theta'(g)^{-1} \circ \iota = \theta(g) \circ \iota \circ g^{-1} = \iota \circ (gg^{-1}) = \iota$ for all $g \in \mathrm{Aut}_0(X,L)$ by (2.2). Since the image $\iota(X)$ of X cannot be contained in any linear subspace of $\mathbb{P}(H^0(X,L^k)^*)$, the above equation implies that $\theta(g) \circ \theta'(g)^{-1} = \nu_N(g)I$, where $\nu_N(g)$ is an N-th root of unity (which may depend on g) and I is the identity in $SL(H^0(X,L^k))$. Since $\mathrm{Aut}_0(X,L)$ is connected and $\theta(e) = \theta'(e)$ for the identity $e \in \mathrm{Aut}_0(X,L)$, we get $\nu_N(g) = 1$ for all $g \in \mathrm{Aut}_0(X,L)$, i.e. $\theta(g) = \theta'(g)$ for all $g \in \mathrm{Aut}_0(X,L)$. □

Remark 2.2.2. Recalling that $\mathrm{Aut}_0(X,L)$ is the maximal connected linear algebraic subgroup in $\mathrm{Aut}_0(X)$, Lemma 2.2.1 is simply re-stating the well-known fact that, for any connected linear algebraic group G acting on X, L admits a G-linearisation after raising it to a higher tensor power, say L^r, if necessary (cf. Corollary 1.6, [89]). In other words, having θ as above in Lemma 2.2.1 is equivalent to fixing an $\mathrm{Aut}_0(X,L)$-linearisation of the line bundle L, by replacing L by L^r if necessary. It is well-known that we cannot always take $r = 1$ (§3, [89]). It is also well-known that a linearisation of a G-action on a projective variety X is unique up to the fibrewise $\mathbb{C}^*$-action (cf. pp105-106 in [38], Proposition 1.4 in [89]).

2.2.1.2 Automorphism groups of extremal Kähler manifolds

Now, suppose that (X,L) contains an extremal Kähler metric ω. As we remarked in §1.4, we have $\mathrm{grad}_\omega S(\omega) = -Jv_s$, where v_s is the Hamiltonian vector field generated by $S(\omega)$ with respect to ω. The vector field v_s is called the **extremal vector field**.

Lemmas 1.4.1 and 1.4.2 (and also Remark 1.4.3) imply that v_s is a Hamiltonian Killing vector field of ω. On the other hand, a well-known theorem of Calabi [24] asserts that the identity component of the isometry group $\mathrm{Isom}(\omega)$ of an extremal metric ω is a maximal compact subgroup of $\mathrm{Aut}_0(X)$. We now set and fix $K := \mathrm{Isom}(\omega) \cap \mathrm{Aut}_0(X,L)$ once and for all[4] as the (connected) maximal compact subgroup of $\mathrm{Aut}_0(X,L)$. The above discussion means that we have $v_s \in \mathfrak{k} := \mathrm{Lie}(K)$. In fact, v_s lies in the centre of $\mathfrak{k}$ by Lemma 2.3.4, which means, in particular, that the identity component $Z(K)_0$ of the centre $Z(K)$ of K must be nontrivial if X admits a non-cscK extremal metric.

Recall that we can write $\mathrm{Aut}_0(X,L) = K^{\mathbb{C}} \ltimes R_u$ as a semidirect product of the complexification $K^{\mathbb{C}}$ of K and the unipotent radical R_u of $\mathrm{Aut}_0(X,L)$ (recalling that it is a linear algebraic group, cf. [53, 57]).

Notation 2.2.3. We summarise our notational convention as follows.

1. $G := \mathrm{Aut}_0(X,L)$ and $\theta : G \to SL(H^0(X,L^k))$ is the faithful representation of G as defined in Lemma 2.2.1, and we write $\theta_* : \mathrm{Lie}(G) \to \mathfrak{sl}(H^0(X,L^k))$ for the induced (injective) Lie algebra homomorphism,

2. $K \leq G$ is the group of isometries of the extremal Kähler metric ω inside G; $K := \mathrm{Isom}(\omega) \cap G$. This is a maximal compact subgroup of G and we write $G = K^{\mathbb{C}} \ltimes R_u$ as a semidirect product of the complexification $K^{\mathbb{C}}$ of K and the unipotent radical R_u of G,

3. $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{k} := \mathrm{Lie}(K)$, and $\mathfrak{z} := \mathrm{Lie}(Z(K))$; we may also write $\mathfrak{sl}$ for $\mathfrak{sl}(H^0(X,L^k))$.

In what follows, we occasionally confuse G with $\theta(G) \leq SL(H^0(X,L^k))$, and $\mathfrak{g}$ with $\theta_*(\mathfrak{g}) \leq \mathfrak{sl}(H^0(X,L^k))$.

2.2.1.3 Some technical remarks

Let K be a maximal compact subgroup of $\mathrm{Aut}_0(X,L)$. By Lemma 2.2.1, we can consider the action of K on $H^0(X,L^k)$ afforded by θ, and hence it makes sense to consider K-invariant (or more precisely $\theta(K)$-invariant) hermitian forms on $H^0(X,L^k)$. Observe now the following lemma.

[4] Some results (e.g. the ones in §2.2.1.3 or §2.2.3), however, will hold for any fixed choice of maximal compact subgroup K in $\mathrm{Aut}_0(X,L)$. Still, it may be convenient to have a specific choice of K in mind.

Lemma 2.2.4. *If $f \in K$, $\theta(f)$ is unitary with respect to any K-invariant positive hermitian form on $H^0(X,L^k)$, and $A \in \theta_*(\sqrt{-1}\mathfrak{k})$ is a hermitian endomorphism with respect to any K-invariant positive hermitian form on $H^0(X,L^k)$. Conversely, if $A \in \theta_*(\mathfrak{k} \oplus \sqrt{-1}\mathfrak{k})$ is hermitian with respect to a K-invariant hermitian form, then $A \in \theta_*(\sqrt{-1}\mathfrak{k})$.*

In what follows, we shall confuse a positive definite hermitian form $\langle , \rangle_H$ with a positive definite hermitian endomorphism H, by fixing a reference $\langle , \rangle_{H_0}$. It is convenient in what follows to use a $\langle , \rangle_{H_0}$-orthonormal basis as a "reference" basis for $H^0(X,L^k)$. Although it is simply a matter of convention, this certainly enables us to fix a "reference" once and for all.

Notation 2.2.5. In what follows, we shall write $\mathscr{B}_k$ for the set of all positive definite hermitian forms on $H^0(X,L^k)$. Observe $\mathscr{B}_k \cong GL(N,\mathbb{C})/U(N)$ and that the tangent space of $\mathscr{B}_k$ at a point is the set $\mathrm{Herm}(H^0(X,L^k))$ of all hermitian endomorphisms on $H^0(X,L^k)$. We shall also write $\mathscr{B}_k^K$ for the $\theta(K)$-invariant elements in $\mathscr{B}_k$, and $\mathrm{Herm}(H^0(X,L^k))^K$ for the tangent space at a point in $\mathscr{B}_k^K$, which is the set of all hermitian endomorphisms on $H^0(X,L^k)$ commuting with the elements in $\theta(K)$.

Finally, since the action of G on X is holomorphic, observe

$$v \in \mathfrak{k} \Rightarrow Jv \in \sqrt{-1}\mathfrak{k}. \tag{2.3}$$

2.2.2 Review of Donaldson's quantisation

We now recall the details of Donaldson's quantisation, namely the maps $Hilb$ ("quantising map") and FS ("dequantising map"), following the exposition given in [43]. Heuristically, it aims to associate the projective geometry of $\mathbb{P}(H^0(X,L^k)^*)$ to the differential geometry of (X,L^k), up to an error which decreases as $k \to \infty$ ("semiclassical limit"), thereby hoping that a difficult PDE problem in differential geometry (e.g. $\bar{\partial}S(\omega) = 0$ or $\bar{\partial}\mathrm{grad}^{1,0}_{\omega}S(\omega) = 0$) can be reduced to a finite dimensional problem on $H^0(X,L^k)$ up to an error of order k^{-1}, say (cf. Theorem 2.3.7). Let $\mathscr{H}(X,L)$ be the space of all positively curved hermitian metrics on L, which is the same as the set of all Kähler potentials $\mathscr{K} = \{\phi \in C^\infty(X,\mathbb{R}) \mid \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$ in $c_1(L)$ (where $\omega_0 \in c_1(L)$ is a reference metric). We may confuse $h \in \mathscr{H}(X,L)$ with the associated Kähler metric $\omega_h \in \mathscr{K}$ when it seems appropriate.

Definition 2.2.6. The map $Hilb : \mathscr{H}(X,L) \to \mathscr{B}_k$, where $\mathscr{B}_k$ is the set of all positive definite hermitian forms on $H^0(X,L^k)$, is defined by

$$Hilb(h) := \frac{N}{V}\int_X h^k(,)\frac{\omega_h^n}{n!}$$

(recalling Notation 2.1.10), and the map $FS : \mathscr{B}_k \to \mathscr{H}(X,L)$ is defined by the equation

$$\sum_{i=1}^N |s_i|^2_{FS(H)^k} = 1 \tag{2.4}$$

where $\{s_i\}$ is an H-orthonormal basis for $H^0(X,L^k)$. $FS(H)$ may also be written as $h_{FS(H)}$. Observe that, fixing a reference hermitian metric h_0 on L and writing $FS(H) = e^{-\varphi}h_0$, the equation (2.4) implies $\varphi = \frac{1}{k}\log\left(\sum_{i=1}^N |s_i|^2_{h_0^k}\right)$. Thus, the equation (2.4) uniquely defines a hermitian metric $h_{FS(H)}$ on L, and hence the map FS is well-defined.

Remark 2.2.7. The reader is referred to §5.2.2, Chapter 5, [77] for the proof of the well-known fact that $h^k_{FS(H)}$ agrees with the pullback by the Kodaira embedding $X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*)$ of the hermitian metric $\tilde{h}_{FS(H)}$ on $\mathscr{O}_{\mathbb{P}(H^0(X,L^k)^*)}(1)$ defined by $H \in \mathscr{B}_k$.

The author believes that some of the following results (Lemmas 2.2.8 and 2.2.9) should be well-known to the experts, although he could not find an explicitly written proof in the existing literature.

Lemma 2.2.8. *Suppose that L^k is very ample. Then $Hilb : \mathscr{H}(X,L) \to \mathscr{B}_k$ is surjective.*

Proof. The main line of the argument presented below is almost identical to §2 in the paper by Bourguignon, Li, and Yau [17].

Since L^k is very ample, we have the Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*) \xrightarrow{\sim} \mathbb{P}^{N-1}$. First of all pick homogeneous coordinates $\{Z_i\}$ on $\mathbb{P}^{N-1}$; all matrices appearing in what follows will be with respect to this basis $\{Z_i\}$. This then defines a hermitian metric $\tilde{h} := \tilde{h}_{FS(I)}$ on $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ and the Fubini–Study metric $\omega_{\widetilde{FS}(I)}$ on $\mathbb{P}^{N-1}$. Suppose that we write $d\mu_Z$ for the volume form on $\mathbb{P}^{N-1}$ defined by $\omega_{\widetilde{FS}(I)}$, and $d\mu_{BZ}$ for the one defined by $\omega_{\widetilde{FS}(H)}$ where $H := \overline{(B^{-1})^t}B^{-1}$ and $B \in GL(N,\mathbb{C})$ (cf. Remark 2.2.17).

Suppose that we write

$$\mathscr{J}^{\circ} := \{B \in GL(N,\mathbb{C}) \mid B = B^*,\ B > 0\}/\{B \sim \alpha B \mid \alpha > 0\},$$

which we compactify to $\mathscr{J}$ by adding a topological boundary $\partial \mathscr{J} := \{B \in GL(N,\mathbb{C}) \mid B = B^*,\ B \geq 0,\ \mathrm{rank} B \leq N-1\}/\{\alpha > 0\}$. We also write

$$\mathscr{H} := \{N \times N \text{ positive semi-definite hermitian matrices with trace } 1\},$$

with the interior $\mathscr{H}^{\circ}$ consisting of positive definite ones, and the boundary $\partial \mathscr{H}$ consisting of those with rank $\leq N-1$. Note $\dim_{\mathbb{R}} \mathscr{J}^{\circ} = \dim_{\mathbb{R}} \mathscr{H}^{\circ} = N^2 - 1$ and that $\mathscr{J}^{\circ}$ and $\mathscr{H}^{\circ}$ can be identified with a connected bounded open subset in $\mathbb{R}^{N^2-1}$.

Now, noting $d\mu_{(\alpha B)Z} = d\mu_{BZ}$, consider a map $\Psi_0 : \mathscr{J}^{\circ} \to \mathscr{H}^{\circ}$ defined by

$$\Psi_0(B)_{ij} := \left(\int_{\mathbb{P}^{N-1}} \frac{\sum_l |Z_l|^2_{\tilde{h}}}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} d\mu_{BZ} \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} d\mu_{BZ},$$

where $\Psi_0(B)_{ij}$ stands for the (i,j)-th entry of $\Psi_0(B)$. Writing $\xi_B : \mathbb{P}^{N-1} \xrightarrow{\sim} \mathbb{P}^{N-1}$ for the biholomorphic map induced from $B \in \mathscr{J}^{\circ}$, we note

$$\begin{aligned}
\Psi_0(I)_{ij} &= \left(\int_{\mathbb{P}^{N-1}} d\mu_Z \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\tilde{h}(Z_i, Z_j)}{\sum_l |Z_l|^2_{\tilde{h}}} d\mu_Z \\
&= \left(\int_{\mathbb{P}^{N-1}} (\xi_B^* d\mu_Z) \right)^{-1} \int_{\mathbb{P}^{N-1}} \xi_B^* \left(\frac{\tilde{h}(Z_i, Z_j)}{\sum_l |Z_l|^2_{\tilde{h}}} \right) (\xi_B^* d\mu_Z) \\
&= \left(\int_{\mathbb{P}^{N-1}} d\mu_{BZ} \right)^{-1} \int_{\mathbb{P}^{N-1}} \frac{\sum_{l,m} \tilde{h}(B_{il} Z_l, B_{jm} Z_m)}{\sum_l |\sum_m B_{lm} Z_m|^2_{\tilde{h}}} d\mu_{BZ}
\end{aligned}$$

and hence, recalling $\mathrm{tr}(\Psi_0(B)) = 1$ and writing B^t for the transpose of B, we get

$$\Psi_0(B) = \frac{(B^t)^{-1} \Psi_0(I) (B^t)^{-1}}{\mathrm{tr}((B^t)^{-1} \Psi_0(I) (B^t)^{-1})}.$$

We claim that it defines a diffeomorphism between $\mathscr{J}^{\circ}$ and $\mathscr{H}^{\circ}$. It is easy to check that Ψ_0 is a smooth bijective map from $\mathscr{J}^{\circ}$ to $\mathscr{H}^{\circ}$. Its linearisation $\delta\Psi_0|_B$ at B can be computed as

$$\begin{aligned}&\delta\Psi_0|_B(A)\\&=-(B^t)^{-1}A^t\Psi_0(B)-\Psi_0(B)A^t(B^t)^{-1}+\mathrm{tr}((B^t)^{-1}A^t\Psi_0(B)+\Psi_0(B)A^t(B^t)^{-1})\Psi_0(B)\end{aligned}$$

where A is a hermitian matrix which is not a constant multiple of B. Observe that $\delta\Psi_0|_B(A)=0$ holds if and only if

$$f_B(A):=-(B^t)^{-1}A^t\Psi_0(B)-\Psi_0(B)A^t(B^t)^{-1}$$

is a constant multiple of $\Psi_0(B)$. Noting that $\Psi_0(B)$ is a positive definite hermitian matrix, we can show by direct computation that $f_B(A)$ cannot be a constant multiple of $\Psi_0(B)$ unless A is a constant multiple of B. Thus the linearisation of Ψ_0 is nondegenerate at each point in $\mathscr{J}^\circ$, and hence Ψ_0 defines a diffeomorphism between $\mathscr{J}^\circ$ and $\mathscr{H}^\circ$ with a nontrivial degree at every point in $\mathscr{H}^\circ$. We also see that, using $\Psi_0(B)=\Psi_0(\alpha B)$ for $\alpha>0$, Ψ_0 extends continuously to the boundary, mapping elements of $\partial\mathscr{J}$ into $\partial\mathscr{H}$, such that the degree of the map $\Psi_0:\partial\mathscr{J}\to\partial\mathscr{H}$ is nontrivial.

Now suppose that we write $\iota_X^*(d\mu_{BZ})$ for the measure induced from $d\mu_{BZ}$ which is supported only on $\iota(X)\subset\mathbb{P}^{N-1}$, and consider a continuous map $\Psi:\mathscr{J}^\circ\to\mathscr{H}^\circ$ defined by

$$\Psi(B)_{ij}:=\left(\int_{\mathbb{P}^{N-1}}\frac{\sum_l|Z_l|^2_{\tilde{h}}}{\sum_l|\sum_m B_{lm}Z_m|^2_{\tilde{h}}}\iota_X^*(d\mu_{BZ})\right)^{-1}\int_{\mathbb{P}^{N-1}}\frac{\tilde{h}(Z_i,Z_j)}{\sum_l|\sum_m B_{lm}Z_m|^2_{\tilde{h}}}\iota_X^*(d\mu_{BZ}).$$

We first show that Ψ extends continuously to the boundary. Recall that $\iota_X^*(d\mu_{BZ})$ is, as a measure on X, equal to $\iota^*(\omega^n_{\widetilde{FS}(H)}/n!)$, and observe

$$\begin{aligned}\int_{\mathbb{P}^{N-1}}\frac{\tilde{h}(Z_i,Z_j)}{\sum_l|\sum_m B_{lm}Z_m|^2_{\tilde{h}}}\iota_X^*(d\mu_{BZ})&=\int_{\iota(X)\subset\mathbb{P}^{N-1}}\frac{\tilde{h}(Z_i,Z_j)}{\sum_l|\sum_m B_{lm}Z_m|^2_{\tilde{h}}}\frac{\omega^n_{\widetilde{FS}(H)}}{n!}\\&=\sum_{r,s}(B^*)^{-1}_{ri}B^{-1}_{js}\sum_{p,q}\int_{\iota(X)\subset\mathbb{P}^{N-1}}\frac{\tilde{h}(B_{rp}Z_p,B_{sq}Z_q)}{\sum_l|\sum_m B_{lm}Z_m|^2_{\tilde{h}}}\frac{\omega^n_{\widetilde{FS}(H)}}{n!}\\&=\sum_{r,s}(B^*)^{-1}_{ri}B^{-1}_{js}\int_{\xi_B\circ\iota(X)\subset\mathbb{P}^{N-1}}\frac{\tilde{h}(Z_r,Z_s)}{\sum_l|Z_l|^2_{\tilde{h}}}\frac{\omega^n_{\widetilde{FS}(I)}}{n!},\end{aligned}$$

since $(\xi_B \circ \iota)^*(Z_i) = \sum_p B_{ip}\iota^*(Z_p)$. Writing $\Phi(B)$ for the matrix defined by

$$\Phi(B)_{rs} := \int_{\xi_B \circ \iota(X) \subset \mathbb{P}^{N-1}} \frac{\tilde{h}(Z_r, Z_s)}{\sum_l |Z_l|^2_{\tilde{h}}} \frac{\omega^n_{\widetilde{FS(I)}}}{n!},$$

we have $\Psi(B) = (B^t)^{-1}\Phi(B)(B^t)^{-1}/\mathrm{tr}((B^t)^{-1}\Phi(B)(B^t)^{-1})$. If $\{B_\nu\}$ is any sequence in $\mathscr{J}^\circ$ converging to a point in $\partial\mathscr{J}$, we immediately see $\Psi(\lim_\nu B_\nu) = \lim_\nu \Psi(B_\nu)$ since $\Phi(\lim_\nu B_\nu) = \lim_\nu \Phi(B_\nu)$. As $\Phi(\lim_\nu B_\nu)$ is positive semi-definite, the formula $\Psi(B) = (B^t)^{-1}\Phi(B)(B^t)^{-1}/\mathrm{tr}((B^t)^{-1}\Phi(B)(B^t)^{-1})$ also proves that Ψ maps a sequence $\{B_\nu\}$ in $\mathscr{J}^\circ$ approaching $\partial\mathscr{J}$ to a sequence which accumulates at a point in $\partial\mathscr{H}$.

We can now define a 1-parameter family of continuous maps $\Psi_t := \mathscr{J} \to \mathscr{H}$ by $\Psi_t(B) := t\Psi(B) + (1-t)\Psi_0(B)$ (this can be viewed as using a measure $t\iota_X^*(d\mu_{ZB}) + (1-t)d\mu_{BZ}$ in the integrals above). By what we have established above, Ψ_t is a continuous 1-parameter family of maps between $\mathscr{J}$ and $\mathscr{H}$ which maps $\partial\mathscr{J}$ into $\partial\mathscr{H}$. Since Ψ_0 is a diffeomorphism between $\mathscr{J}^\circ$ and $\mathscr{H}^\circ$ and has a nontrivial degree on the boundary and Ψ maps sequences approaching $\partial\mathscr{J}$ to sequences accumulating at points in $\partial\mathscr{H}$, $\Psi : \partial\mathscr{J} \to \partial\mathscr{H}$ has a nontrivial degree. We thus see that Ψ is surjective since the degree of a continuous map is a homotopy invariant (cf. Theorems 12.10 and 12.11, [4]).

Finally, we recall that $\iota_X^*(d\mu_{BZ}) = \iota^*(\omega^n_{\widetilde{FS(H)}}/n!)$ is equal to $k^n\omega_{FS(H)}/n!$. Note also that, writing h^k for $\iota^*\tilde{h}$, we have

$$\Psi(B)_{ij} = \left(\int_X \frac{\sum_l |s_l|^2_{h^k}}{\sum_l |\sum_m B_{lm}s_m|^2_{h^k}} \frac{\omega^n_{FS(H)}}{n!}\right)^{-1} \int_X \frac{h^k(s_i,s_j)}{\sum_l |\sum_m B_{lm}s_m|^2_{h^k}} \frac{\omega^n_{FS(H)}}{n!}.$$

where we wrote $s_i := \iota^* Z_i$. Observe also that there exists $\beta \in C^\infty(X,\mathbb{R})$ such that $\omega^n_{FS(H)} = e^\beta \omega^n_h$. We have thus proved that, fixing a basis $\{s_i\}$ for $H^0(X,L^k)$, for any positive definite hermitian matrix G there exists a function $\phi \in C^\infty(X,\mathbb{R})$ such that

$$\frac{N}{V}\int_X e^{\beta+\phi} h^k(s_i,s_j)\frac{\omega_h^n}{n!} = G_{ij}.$$

We thus aim to find a function $f \in C^\infty(X,\mathbb{R})$, such that $e^{-f}h^k$ is positively curved and $Hilb(e^{-f}h^k)(s_i,s_j) = \frac{N}{V}\int_X e^{\beta+\phi}h^k(s_i,s_j)\frac{\omega_h^n}{n!}$, to finally establish the claim. For this, it is sufficient to solve for f the following nonlinear PDE:

$$\left(\omega_h + \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}f\right)^n = e^{f+\beta+\phi}\omega_h^n,$$

which is solvable by the Aubin–Yau theorem (cf. Theorem 4, p383 [127]).

□

Lemma 2.2.9. *Suppose that we choose k to be large enough, and that $H, H' \in \mathscr{B}_k$ satisfy $FS(H)^k = (1+f)FS(H')^k$ with $\sup_X |f| \leq \varepsilon$ for $\varepsilon \geq 0$ satisfying $N^{\frac{3}{2}}\varepsilon \leq 1/4$. Then we have $||H - H'||_{op} \leq 2N^2\varepsilon$, where $||\cdot||_{op}$ is the operator norm, i.e. the maximum of the moduli of the eigenvalues (cf. §2.2.1.3). In particular, considering the case $\varepsilon = 0$, we see that FS is injective for all large enough k.*

Proof. We now pick an H-orthonormal basis $\{s_i\}$ and represent H (resp. H') as a matrix H_{ij} (resp. H'_{ij}) with respect to the basis $\{s_i\}$. H_{ij} is the identity matrix, and replacing $\{s_i\}$ by an H-unitarily equivalent basis if necessary, we may further assume $H'_{ij} = \mathrm{diag}(d_1^2, \dots, d_N^2)$ for some $d_i > 0$. Recall that the equation (2.4) implies that we can write $FS(H')^k = e^{-\varphi}FS(H)^k$ with $\varphi = \log\left(\sum_{i=1}^N d_i^{-2}|s_i|^2_{FS(H)^k}\right)$. Thus the equation $FS(H)^k = (1+f)FS(H')^k$ implies $1 + f = \sum_i d_i^{-2}|s_i|^2_{FS(H)^k}$, and hence, by recalling (2.4),

$$(1+f)\sum_i |s_i|^2_{h^k} = \sum_i d_i^{-2}|s_i|^2_{h^k}, \tag{2.5}$$

with respect to *any* hermitian metric h on L, by noting that we may multiply both sides of (2.5) by any strictly positive function $e^{k\phi}$. We now fix this basis $\{s_i\}$, and the operator norm or the Hilbert–Schmidt norm used in this proof will all be computed with respect to this basis.

We now choose N hermitian metrics $h_1, \dots, h_N$ on L^k as follows. Recall now that, by Lemma 2.2.8, for any N-tuple of strictly positive numbers $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ there exists $\phi_{\vec{\lambda}} \in C^\infty(X, \mathbb{R})$ such that the hermitian metric $h' := \exp(\phi_{\vec{\lambda}})h$ satisfies $\int_X |s_i|^2_{(h')^k}\frac{\omega^n_{h'}}{n!} = \lambda_i$. We thus take $\vec{\lambda}_i = (e^{-k}, \dots, e^{-k}, 1, e^{-k}, \dots, e^{-k})$ with 1 in the i-th place, and choose $\phi_i \in C^\infty(X, \mathbb{R})$ appropriately (cf. Lemma 2.2.8) so that $h_i := \exp(\phi_i)h$ satisfies

$$\vec{\lambda}_i = \left(\int_X |s_1|^2_{h_i^k}\frac{\omega^n_{h_i}}{n!}, \int_X |s_2|^2_{h_i^k}\frac{\omega^n_{h_i}}{n!}, \dots, \int_X |s_N|^2_{h_i^k}\frac{\omega^n_{h_i}}{n!}\right).$$

Now consider the matrix

$$\Lambda := \begin{pmatrix} \vec{\lambda}_1 \\ \vdots \\ \vec{\lambda}_N \end{pmatrix}$$

and observe that the modulus of each entry is at most 1, and that $||\Lambda||_{op} \leq 2$ and $||\Lambda^{-1}||_{op} \leq 2$ if k is large enough. Then, multiplying both sides of (2.5) by $\exp(k\phi_i)$ and integrating over X with respect to the measure $\omega^n_{h_i}/n!$, we get the following system of linear equations

$$(\Lambda + F)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \Lambda \begin{pmatrix} d_1^{-2} \\ \vdots \\ d_N^{-2} \end{pmatrix},$$

where F is a matrix defined by

$$F_{ij} := \int_X f|s_j|^2_{h_i^k} \frac{\omega^n_{h_i}}{n!}$$

whose max norm (i.e. the maximum of the moduli of its entries) satisfies $||F||_{max} \leq \sup_X |f| \leq \varepsilon$ since the modulus of each entry of Λ is at most 1. We thus get

$$\begin{pmatrix} d_1^{-2} - 1 \\ \vdots \\ d_N^{-2} - 1 \end{pmatrix} = \Lambda^{-1} F \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus, noting $||\Lambda^{-1}F||_{op} \leq ||\Lambda^{-1}||_{op}||F||_{op} \leq 2||F||_{HS} \leq 2N||F||_{max} \leq 2N\varepsilon$, we get

$$|d_i^{-2} - 1| \leq \sqrt{\sum_i |d_i^{-2} - 1|^2} \leq 2N^{1+\frac{1}{2}}\varepsilon.$$

Thus we get $1 - 2N^{\frac{3}{2}}\varepsilon \leq d_i^{-2} \leq 1 + 2N^{\frac{3}{2}}\varepsilon$, and by the assumption $N^{\frac{3}{2}}\varepsilon \leq 1/4$ we have

$$1 - 2N^2\varepsilon < 1 - \frac{2N^{\frac{3}{2}}\varepsilon}{1 + 2N^{\frac{3}{2}}\varepsilon} \leq d_i^2 \leq 1 + \frac{2N^{\frac{3}{2}}\varepsilon}{1 - 2N^{\frac{3}{2}}\varepsilon} < 1 + 2N^2\varepsilon$$

as required. □

In order to describe the map $FS \circ Hilb : \mathscr{H}(X,L) \to \mathscr{H}(X,L)$ (cf. Theorem 2.2.11), we introduce the following function which is important in complex geometry and complex analysis.

Definition 2.2.10. Let $h \in \mathscr{H}(X,L)$, and let $\{s_i\}$ be a $\int_X h^k(,)\frac{\omega_h^n}{n!}$-orthonormal basis for $H^0(X,L^k)$. The **Bergman function** or the **density of states function** $\rho_k(\omega_h)$ is defined as

$$\rho_k(\omega_h) := \sum_{i=1}^{N} |s_i|^2_{h^k}.$$

We will also use a scaled version of $\rho_k(\omega_h)$ defined as

$$\bar{\rho}_k(\omega_h) := \frac{V}{N}\rho_k(\omega_h),$$

where the scaling is made so that the average of $\bar{\rho}_k(\omega_h)$ over X is 1.

It is easy to see that $\rho_k(\omega_h)$ depends only on the Kähler metric ω_h rather than h itself, i.e. is invariant under the scaling $h \mapsto e^c h$ for any $c \in \mathbb{R}$. Recall now the following theorem, which easily follows from the definition (2.4) of FS.

Theorem 2.2.11. (Rawnsley [98]) $FS(Hilb(h)) = (\rho_k(\omega_h)V/N)^{-1/k}h$ *for any* $h \in \mathscr{H}(X,L)$ *and large enough* $k > 0$ *such that* L^k *is very ample.*

Remark 2.2.12. Suppose in general that we are given an embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ of X (not necessarily defined by sections of an ample line bundle) such that $\iota(X)$ is not contained in any hyperplane. It is possible to define the Bergman function in this situation by using Theorem 2.2.11.

An obvious corollary of Theorem 2.2.11 is that $FS(Hilb(h)) = h$ if and only if $\rho_k(\omega_h) = \text{const} = N/V$, and $h \in \mathscr{H}(X,L)$ satisfying this is called balanced.

Definition 2.2.13. A hermitian metric $h \in \mathscr{H}(X,L)$ is called **balanced** at the level k if it satisfies the following two equivalent conditions.

1. $\rho_k(\omega_h) = N/V$ or $\bar{\rho}_k(\omega_h) = 1$,

2. $FS(Hilb(h)) = h$.

An important point is that we have an "extrinsic" characterisation of balanced metrics, in terms of the Kodaira embedding. For this, we fix some basis $\{Z_i\}$ for $H^0(X,L^k)$, which may be called a **reference basis**.[5] With this choice of basis, it is possible to identify $H^0(X,L^k)$ with its dual, and also with $\mathbb{C}^N$, and hence $\mathbb{P}(H^0(X,L^k)^*) \cong \mathbb{P}^{N-1}$. Note then that the Kodaira embedding ι can be written as

$$\iota : X \ni x \mapsto [\mathrm{ev}_x Z_1 : \cdots : \mathrm{ev}_x Z_N] \in \mathbb{P}^{N-1}$$

where ev_x is the evaluation map at x. This embedding may be called a **reference embedding**, and will always be denoted by ι from now on. It is important to fix *some* reference basis for the identification $\mathbb{P}(H^0(X,L^k)^*) \cong \mathbb{P}^{N-1}$, but a different choice of reference basis will only result in moving (the image of) X inside $\mathbb{P}^{N-1}$ by an $SL(N,\mathbb{C})$-action (cf. Remark 2.2.17).

Definition 2.2.14. Defining a standard Euclidean metric on $\mathbb{C}^N$ which we write as the identity matrix I, we define the **centre of mass** as

$$\bar{\mu}_X := \int_{\iota(X)} \frac{\tilde{h}_{FS}(Z_i,Z_j)}{\sum_l |Z_l|^2_{\widetilde{FS}}} \frac{\omega^n_{\widetilde{FS}}}{n!} = \int_X \frac{h^k_{FS}(s_i,s_j)}{\sum_l |s_l|^2_{FS^k}} \frac{k^n \omega^n_{FS}}{n!} \in \sqrt{-1}\mathfrak{u}(N)$$

where h^k_{FS} is (the pullback by the Kodaira embedding of) the Fubini–Study metric $\tilde{h}_{FS}$ on $\mathbb{P}^{N-1}$ induced from I on $\mathbb{C}^N$ covering $\mathbb{P}^{N-1}$ (see also Notation 2.2.16 below).

Remark 2.2.15. Note that the equation (2.4) implies that we in fact have $\bar{\mu}_X = \int_X h^k_{FS}(s_i,s_j) \frac{k^n \omega^n_{FS}}{n!}$.

Notation 2.2.16. As a matter of notation, we will often write $\{Z_i\}$ for a basis for $H^0(X,L^k)$ when we see it as an abstract vector space and $\{s_i\}$ when we see it as a space of holomorphic sections on X; thus we can write $\iota^* Z_i = s_i$ by using the Kodaira embedding ι. We also write $\tilde{h}_{FS}$ for the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ induced from I on $\mathbb{C}^N$ covering $\mathbb{P}^{N-1}$, and write $\omega_{\widetilde{FS}}$ for the corresponding Kähler metric on $\mathbb{P}^{N-1}$.

We can now move the image of X in $\mathbb{P}^{N-1}$ by the $SL(N,\mathbb{C})$-action on $\mathbb{P}^{N-1}$ (or rather on the $\mathbb{C}^N$ covering it). Writing $\xi_g : \mathbb{P}^{N-1} \xrightarrow{\sim} \mathbb{P}^{N-1}$ for the biholomorphic map induced from $g \in SL(N,\mathbb{C})$, note that moving the image $\iota(X)$ of X by $g \in SL(N,\mathbb{C})$ is equivalent to considering the embedding $\iota_g := \xi_g \circ \iota : X \hookrightarrow \mathbb{P}^{N-1}$, and the effect of ξ_g is such that Z_i changes to $Z'_i := \sum_j g_{ij} Z_j$, where g_{ij} is the matrix for g represented with

[5] We may take this to be an orthonormal basis for the reference $\langle , \rangle_{H_0}$ in §2.2.1.3.

respect to the basis $\{Z_i\}$. Thus, the Fubini–Study metric $\omega_{FS} = \iota^* \frac{\sqrt{-1}}{2\pi k} \partial\bar{\partial} \log(\sum |Z_i|^2)$ changes to $(\xi_g \circ \iota)^* \frac{\sqrt{-1}}{2\pi k} \partial\bar{\partial} \log(\sum |Z_i|^2) = \iota^* \frac{\sqrt{-1}}{2\pi k} \partial\bar{\partial} \log(\sum |Z'_i|^2)$, which we can see is equal to $\omega_{FS(H)}$, i.e. (the pullback by ι of) the Fubini–Study metric on $\mathbb{P}^{N-1}$ induced from the hermitian form $H := \overline{(g^{-1})^t} g^{-1}$ on $\mathbb{C}^N$.

Thus, writing $\bar{\mu}_X(g)$ for the new centre of mass after moving the image of X by g, namely the centre of mass of X with respect to the embedding $\iota_g = \xi_g \circ \iota$, we have

$$\begin{aligned}
\bar{\mu}_X(g) &= \int_{\iota_g(X)} \frac{\tilde{h}_{FS}(Z_i, Z_j)}{\sum_l |Z_l|^2_{\widetilde{FS}}} \frac{\omega^n_{\widetilde{FS}}}{n!} \\
&= \int_{\iota(X)} \frac{\tilde{h}_{FS(H)}(Z'_i, Z'_j)}{\sum_l |Z'_l|^2_{\widetilde{FS(H)}}} \frac{\omega^n_{\widetilde{FS(H)}}}{n!} = \int_X \frac{h^k_{FS(H)}(s'_i, s'_j)}{\sum_l |s'_l|^2_{FS(H)^k}} \frac{k^n \omega^n_{FS(H)}}{n!}.
\end{aligned}$$

Remark 2.2.17. Suppose that we have another choice of reference basis, say $\{Z'_i\}$, to compute the centre of mass, say $\bar{\mu}'_X$. Since we can write $Z'_i = \sum_j g_{ij} Z_j$ for some $g \in SL(N, \mathbb{C})$, we see that choosing a new reference basis is simply moving the image of X inside $\mathbb{P}^{N-1}$ (with respect to the old reference basis) by $g \in SL(N, \mathbb{C})$; namely $\bar{\mu}'_X = \bar{\mu}_X(g)$.

Observe that the new basis $\{Z'_i\}$ is an H-orthonormal basis where the hermitian form H is defined by $H = \overline{(g^{-1})^t} g^{-1}$.

Definition 2.2.18. The Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ is called **balanced** if there exists $g \in SL(N, \mathbb{C})$ such that $\bar{\mu}_X(g)$ is a multiple of the identity in $\sqrt{-1}\mathfrak{u}(N)$; equivalently, $\bar{\mu}_X(g)$ is in the kernel of the natural projection $\sqrt{-1}\mathfrak{u}(N) \twoheadrightarrow \sqrt{-1}\mathfrak{su}(N)$.

Note that the definition of being balanced does not depend on the choice of reference basis that we chose to have $\mathbb{P}(H^0(X, L^k)^*) \xrightarrow{\sim} \mathbb{P}^{N-1}$, by Remark 2.2.17.

A fundamental result is the following, which easily follows from Lemma 2.2.9, Definition 2.2.10, and Remark 2.2.15.

Theorem 2.2.19. (Luo [76], Zhang [131]) *Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X, L^k)^*) \cong \mathbb{P}^{N-1}$ is balanced if and only if L admits a balanced metric at the level k.*

The reader is referred to §2.6.1 (in particular Theorem 2.6.2), for its connection to a stability notion in algebraic geometry.

Finally, we prove the following general lemma, which also applies to the case of general embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ as discussed in Remark 2.2.12. The author thanks Joel Fine for pointing it out to him.

Lemma 2.2.20. *Let F be the Hamiltonian for the vector field on $\mathbb{P}^{N-1}$ generated by $\sqrt{-1}k^n\bar{\mu}_X(g)^{-1}$ with respect to the Kähler metric $\omega_{\widetilde{FS(H)}}$, where $H = \overline{(g^{-1})^t}g^{-1}$. Then, $\rho_k(\omega_{FS(H)}) = \iota^*F$.*

Proof. Let $\{s_i\}$ be an H-orthonormal basis and $\{s'_i\}$ be a $\int_X h^k_{FS(H)}(,)\frac{\omega^n_{FS(H)}}{n!}$-orthonormal basis. Let P be the change of basis matrix from $\{s_i\}$ to $\{s'_i\}$. This implies

$$\sum_{l,q} P^*_{li}P_{jq}(\bar{\mu}'_X)_{lq} = \sum_{l,q} P^*_{li}P_{jq}\int_X h^k_{FS(H)}(s_l,s_q)\frac{k^n\omega^n_{FS(H)}}{n!} = k^n\delta_{ij},$$

which implies $\bar{\mu}'_X = k^n(P^*P)^{-1}$, where $\bar{\mu}'_X$ is the centre of mass defined with respect to the basis $\{s_i\}$. Note $\bar{\mu}'_X = \bar{\mu}_X(g)$ by Remark 2.2.17. Note also that

$$\rho_k(\omega_{FS(H)}) = \sum_i |s'_i|^2_{FS(H)^k} = \sum_{i,q,l} P^*_{li}P_{iq}h^k_{FS(H)}(s_l,s_q) = \sum_{q,l}(P^*P)_{lq}h^k_{FS(H)}(s_l,s_q),$$

and hence we get

$$\rho_k(\omega_{FS(H)}) = \sum_{i,j}(k^n\bar{\mu}_X(g)^{-1})_{ij}h^k_{FS(H)}(s_i,s_j). \tag{2.6}$$

Now, using the homogeneous coordinates $\{Z_i\}$ on $\mathbb{P}^{N-1}$ corresponding to $\{s_i\}$, i.e. $\iota^*Z_i = s_i$, we have

$$\sum_{i,j}(\bar{\mu}_X(g)^{-1})_{ij}h^k_{FS(H)}(s_i,s_j) = \iota^*\left(\sum_{i,j}(\bar{\mu}_X(g)^{-1})_{ij}\frac{Z_i\bar{Z}_j}{\sum_l|Z_l|^2}\right). \tag{2.7}$$

Recall that for $A \in \mathfrak{u}(N)$ regarded as a Hamiltonian vector field on $\mathbb{P}^{N-1}$, the Hamiltonian F_A for A with respect to $\omega_{\widetilde{FS(H)}}$ is given by (cf. p88, [119])

$$F_A = -\sqrt{-1}\sum_{ij}A_{ij}\frac{Z_i\bar{Z}_j}{\sum_l|Z_l|^2}. \tag{2.8}$$

Thus, taking $A = \sqrt{-1}k^n\bar{\mu}_X(g)^{-1} \in \mathfrak{u}(N)$, we get the claimed statement from the equations (2.6), (2.7), and (2.8).

□

2.2.3 A general lemma and its consequences

We prove the following general lemma.

Lemma 2.2.21. *For any* $f \in \mathrm{Aut}_0(X,L)$*,*

1. $f^*\rho_k(\omega_h) = \rho_k(f^*\omega_h)$,
2. $Hilb(f^*h) = \theta(f^{-1})^* Hilb(h)\theta(f^{-1})$,
3. $f^*FS(H) = FS(\theta(f^{-1})^* H\theta(f^{-1}))$.

Remark 2.2.22. We recall now that we have $\theta : \mathrm{Aut}_0(X,L) \to SL(H^0(X,L^k))$ as in Lemma 2.2.1 (by replacing L by a large enough tensor power if necessary) which implies that we have a "consistent" choice of the lift $\tilde{f}$ of $f \in \mathrm{Aut}_0(X,L)$ to the automorphism of the total space of the bundle L so that $\tilde{f}_1 \circ \tilde{f}_2 = \widetilde{f_1 \circ f_2}$ (i.e. fixed linearisation of the action; see Remark 2.2.2). For a hermitian metric h on L, f^*h in the above statement is meant to be $\tilde{f}^*h$ for this choice of $\tilde{f}$.

Proof. Note the elementary

$$\int_X h^k(s,s')\frac{\omega_h^n}{n!} = \int_X f^*(h^k(s,s'))\frac{f^*\omega_h^n}{n!} = \int_X (f^*h^k)(\theta(f)s,\theta(f)s')\frac{f^*\omega_h^n}{n!}$$

for any two sections s and s', by recalling (2.2). This means that, if $\{s_i\}$ is a $Hilb(h)$-orthonormal basis, then $\{\sum_j \theta(f)_{ij}s_j\}$ is a $Hilb(f^*h^k)$-orthonormal basis where $\theta(f)_{ij}$ is the matrix for $\theta(f)$ represented with respect to $\{s_i\}$. We thus have $f^*\rho_k(\omega_h) = \sum_i |\sum_j \theta(f)_{ij}s_j|^2_{f^*h^k} = \rho_k(f^*\omega_h)$.

For the second part of the lemma, we just recall that $\{\sum_j \theta(f)_{ij}s_j\}$ is a $Hilb(f^*h)$-orthonormal basis to see $Hilb(f^*h) = \theta(f^{-1})^* Hilb(h)\theta(f^{-1})$.

For the third part of the lemma, apply f^* to the defining equation $\sum |s_i|^2_{FS(H)^k} = 1$ for $FS(H)$ (equation (2.4)), where $\{s_i\}$ is an H-orthonormal basis. We then get $\sum_i |\sum_j \theta(f)_{ij}s_j|^2_{f^*FS(H)^k} = 1$, which means that $f^*(FS(H)) = FS(H')$ with H' having $\{\sum_j \theta(f)_{ij}s_j\}$ as its orthonormal basis, i.e. $H' = \theta(f^{-1})^* H\theta(f^{-1})$. Thus $f^*FS(H) = FS(\theta(f^{-1})^* H\theta(f^{-1}))$.

□

Lemma 2.2.21 proves the following observation of Sano and Tipler.

Lemma 2.2.23. (Sano–Tipler, §2.2.1 of [107]) *If H is $\theta(K)$-invariant, then $FS(H)$ is K-invariant. Conversely, if the Kähler metric ω_h is K-invariant, then $Hilb(h)$ defines a $\theta(K)$-invariant hermitian form on $H^0(X,L^k)$.*

Proof. The first statement is an obvious consequence of the third item of Lemma 2.2.21. To show the second statement, observe first that ω_h being K-invariant means that we have $f^*h^k = e^c h^k$ with some $c = c(f) \in \mathbb{R}$ for any $f \in K$ (with respect to the fixed linearisation of the action, as we saw in Remark 2.2.22). Recall also

$$f^*(h^k(s,s')) = (f^*h^k)(\theta(f)s, \theta(f)s') = e^c h^k(\theta(f)s, \theta(f)s')$$

for any two $s,s' \in H^0(X,L^k)$. Since $f^*\omega_h = \omega_h$, we thus have $Hilb(h) = e^c \theta(f)^* Hilb(h) \theta(f)$ by noting

$$Hilb(h)(s,s') = \frac{N}{V}\int_X h^k(s,s')\frac{\omega_h^n}{n!} = \frac{N}{V}\int_X f^*(h^k(s,s'))\frac{f^*\omega_h^n}{n!} = e^c\frac{N}{V}\int_X h^k(\theta(f)s,\theta(f)s')\frac{\omega_h^n}{n!}.$$

We now take the determinant of both sides of the equation $Hilb(h) = e^c\theta(f)^* Hilb(h)\theta(f)$ to conclude $c = 0$, by recalling $\theta(f) \in SL(H^0(X,L^k))$. We then have $Hilb(h) = \theta(f)^* Hilb(h)\theta(f)$, i.e. $Hilb(h)$ is $\theta(K)$-invariant. □

2.3 Construction of approximate solutions to $\bar\partial \mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$

2.3.1 Preliminaries

For the sake of convenience, we decide to have the following naming convention.

Definition 2.3.1. We say that ω_ϕ is **ρ-balanced** if it satisfies $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\rho_k(\omega_\phi) = 0$.

Suppose now that (X,L) admits an extremal metric $\omega \in c_1(L)$, and that K stands for $\mathrm{Isom}(\omega) \cap \mathrm{Aut}_0(X,L)$ from now on. ω being extremal, its scalar curvature $S(\omega)$ generates a Hamiltonian Killing vector field $v_s \in \mathfrak{k}$. The first step of the proof of Theorem 2.1.6 is to construct a metric ω' which "approximately" satisfies $\bar\partial \mathrm{grad}^{1,0}_{\omega'}\rho_k(\omega') = 0$. We thus consider the following problem.

Problem 2.3.2. Starting with an extremal metric ω satisfying $\mathscr{D}^*_\omega\mathscr{D}_\omega S(\omega)=0$, can one find for each $m\in\mathbb{N}$ a sequence $\{H_m(k)\}_k$ with $H_m(k)\in\mathscr{B}^K_k$ so that $\omega_{(m)}:=\omega_{FS(H_m(k))}$ satisfies $||\omega_{(m)}-\omega||_{C^l,\omega}<c_{m,l}/k$ for some constant $c_{m,l}>0$ for each $l\in\mathbb{N}$ and all large enough k, and also

$$\left|\left|\mathscr{D}^*_{\omega_{(m)}}\mathscr{D}_{\omega_{(m)}}\bar{\rho}_k(\omega_{(m)})\right|\right|_{C^l}<C_{m,l}(\omega)k^{-m-2}$$

for each $l\in\mathbb{N}$, with a constant $C_{m,l}(\omega)$ that depends only on m,l and ω?

As in the usual cscK case, the construction of approximately ρ-balanced metrics will crucially depend on the well-known asymptotic expansion of the Bergman function (cf. Theorem 2.3.7), so that $\mathscr{D}^*_{\omega_{(m)}}\mathscr{D}_{\omega_{(m)}}\bar{\rho}_k(\omega_{(m)})$ is going to be zero "order by order" in the powers of k^{-1}. For this purpose, it turns out that it is easier to work with a pair of equations (cf. (2.10)) that is equivalent to $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\bar{\rho}_k(\omega_\phi)=0$, which we discuss shortly.

Before doing so, we briefly recall the explicit formula for describing how the Hamiltonian for the extremal vector field v_s changes when we change the Kähler metric from ω to $\omega_\phi:=\omega+\sqrt{-1}\partial\bar{\partial}\phi$. We have a general lemma as follows.

Lemma 2.3.3. (cf. Lemma 4.0.1 of [8]) *Suppose that* $v\in\mathfrak{k}=\mathrm{Lie}(K)$ *is a Hamiltonian Killing vector field, with Hamiltonian* $\tilde{F}$ *with respect to* ω*. Suppose also that the Lie derivative of* $\phi\in C^\infty(X,\mathbb{R})$ *along* v *is zero. Then* $\tilde{F}+\frac{1}{2}(d\tilde{F},d\phi)_\omega$ *is the Hamiltonian of* v *with respect to* $\omega+\sqrt{-1}\partial\bar{\partial}\phi$*. Namely,*

$$\iota(v)(\omega+\sqrt{-1}\partial\bar{\partial}\phi)=-d\left(\tilde{F}+\frac{1}{2}(d\tilde{F},d\phi)_\omega\right).$$

Proof. Since the complex structure J is K-invariant (since $K\leq\mathrm{Aut}_0(X,L)$), we have $L_v(Jd\phi)=0$, where L_v is the Lie derivative along a vector field $v\in\mathfrak{k}=\mathrm{Lie}(K)$. In other words, $\iota(v)(dJd\phi)=-d(\iota(v)Jd\phi)=d(Jv(\phi))$, where we recall that J acts on a 1-form α by $J\alpha(\xi)=-\alpha(J\xi)$ for any vector field ξ, which also implies

$$J\iota(v)\omega(\xi)=-\omega(v,J\xi)=\omega(Jv,\xi)=\iota(Jv)\omega(\xi) \tag{2.9}$$

for any vector field ξ. Recall also that $dJd\phi=2\sqrt{-1}\partial\bar{\partial}\phi$ (cf. §3.1, [8]). We thus have

$\iota(v)\sqrt{-1}\partial\bar{\partial}\phi = \frac{1}{2}d(Jv(\phi))$. Note also that, when v is generated by $\tilde{F}$, we have $-Jv = \mathrm{grad}_\omega \tilde{F}$, and hence $Jv(\phi) = -\mathrm{grad}_\omega \tilde{F}(\phi) = -(d\tilde{F}, d\phi)_\omega$ where $(,)_\omega$ is the pointwise norm on the space of 1-forms defined by the metric ω.

□

In what follows, we shall apply the above lemma to the case where v is the extremal vector field v_s and $\tilde{F} = S(\omega)$. Observe also the following well-known fact.

Lemma 2.3.4. *Suppose that ω_h is K-invariant. Then the Hamiltonian vector field v generated by a K-invariant function $\tilde{F}$ commutes with the action of any element in K. In particular, if v is a Hamiltonian Killing vector field with respect to ω_h, v lies in the centre $\mathfrak{z}$ of $\mathfrak{k}$. In particular, the extremal vector field v_s lies in $\mathfrak{z}$.*

Proof. Applying $f \in K \leq G$ to the equation $\iota(v)\omega_h = -d\tilde{F}$, we have $\iota((f^{-1})_* v) f^* \omega_h = -df^*\tilde{F}$. Since ω_h and $\tilde{F}$ are K-invariant, this yields $\iota((f^{-1})_* v)\omega_h = \iota(v)\omega_h$. Since ω_h is non-degenerate, we have $(f^{-1})_* v = v$, which is equivalent to saying that the 1-parameter subgroup generated by v commutes with the action of any element in K. □

We now consider a pair of equations

$$\begin{cases} S(\omega) + \frac{1}{2}(dS(\omega), d\phi)_\omega = 4\pi k \bar{\rho}_k(\omega_\phi) + f \\ \mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi} f = 0 \end{cases}$$

to be solved for a pair of K-invariant functions (ϕ, f), which we will be concerned with from now on. The following lemma shows that solving this equation is equivalent to having a ρ-balanced metric.

Lemma 2.3.5. *Suppose that ω is an extremal metric and write $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. There exists a K-invariant function ϕ which satisfies $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\bar{\rho}_k(\omega_\phi) = 0$ if and only if we can find a pair of K-invariant functions (ϕ, f) which satisfies*

$$\begin{cases} S(\omega) + \frac{1}{2}(dS(\omega), d\phi)_\omega = 4\pi k \bar{\rho}_k(\omega_\phi) + f \\ \mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi} f = 0. \end{cases} \tag{2.10}$$

Remark 2.3.6. It is important to note that, in (2.10), we need $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi} f = 0$ and *not* $\mathscr{D}^*_{\omega}\mathscr{D}_{\omega} f = 0$. This will cause an extra complication in the construction of approximately ρ-balanced metric, which did not happen in the cscK case (cf. Remark 2.3.14).

Note that ω being an extremal metric is essential in the above lemma. We also remark that the real holomorphic vector field v_f, generated by the Hamiltonian f in the above, precisely represents the discrepancy between the vector field v_s generated by $S(\omega)$ and the vector field v generated by $4\pi k\bar{\rho}_k(\omega_\phi)$, as we shall see in the proof.

Proof. Suppose that we can find a pair (ϕ, f) of K-invariant functions satisfying (2.10). Then, recalling Lemma 2.3.3, we have

$$\iota(v_s)(\omega+\sqrt{-1}\partial\bar{\partial}\phi) = -d\left(S(\omega)+\frac{1}{2}(dS(\omega),d\phi)_\omega\right) = -d\left(4\pi k\bar{\rho}_k(\omega_\phi)+f\right).$$

Since f satisfies $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi} f = 0$, there exists a real holomorphic vector field v_f such that $\iota(v_f)(\omega+\sqrt{-1}\partial\bar{\partial}\phi) = -df$. Thus we get $\iota(v_s - v_f)(\omega+\sqrt{-1}\partial\bar{\partial}\phi) = -d\left(4\pi k\bar{\rho}_k(\omega_\phi)\right)$. Since $v_s - v_f$ is a real holomorphic vector field, we have $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\bar{\rho}_k(\omega_\phi) = 0$ by Lemma 1.4.2.

Conversely suppose $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\bar{\rho}_k(\omega_\phi) = 0$. Then there exists a real holomorphic vector field v such that $\iota(v)\omega_\phi = -d4\pi k\bar{\rho}_k(\omega_\phi)$. Then, writing $v = (v - v_s) + v_s$, we have

$$\begin{aligned} -d\left(4\pi k\bar{\rho}_k(\omega_\phi)\right) &= \iota(v_s)\omega_\phi + \iota(v-v_s)\omega_\phi \\ &= -d\left(S(\omega)+\frac{1}{2}(dS(\omega),d\phi)_\omega\right) - df \end{aligned}$$

where we put $f := 4\pi k\bar{\rho}_k(\omega_\phi) - S(\omega) - \frac{1}{2}(dS(\omega),d\phi)_\omega$ in the third line. Note that Lemmas 1.4.2 and 2.3.3 imply $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}(S(\omega)+\frac{1}{2}(dS(\omega),d\phi)_\omega) = 0$. Recalling $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi}\bar{\rho}_k(\omega_\phi) = 0$ in our assumption, we thus have $\mathscr{D}^*_{\omega_\phi}\mathscr{D}_{\omega_\phi} f = 0$. Note that $f = 4\pi k\bar{\rho}_k(\omega_\phi) - S(\omega) - \frac{1}{2}(dS(\omega),d\phi)_\omega$ is K-invariant if ϕ is K-invariant by Lemma 2.2.21. This gives us an equation $4\pi k\bar{\rho}_k(\omega_\phi) = S(\omega)+\frac{1}{2}(dS(\omega),d\phi)_\omega + f + \text{const}$. Replacing $f + \text{const}$ by f, we get the equation (2.10).

□

2.3.2 Perturbative construction by using the asymptotic expansion

We now recall the following famous theorem, which will be of fundamental importance for us. We refer to [77] (in particular to Theorem 4.1.2) for more detailed discussions.

Theorem 2.3.7. (Tian [123], Yau [129], Ruan [104], Zelditch [130], Catlin [27], Lu [75], Ma–Marinescu [77]) *The Bergman function $\rho_k(\omega_\phi)$ admits the following asymptotic expansion in k^{-1}*

$$\rho_k(\omega_\phi) = k^n + k^{n-1}b_1 + k^{n-2}b_2 + \cdots$$

with $b_1 = \frac{1}{4\pi}S(\omega_\phi)$, and each coefficient $b_i = b_i(\omega_\phi)$ can be written as a polynomial in the curvature $\mathrm{Riem}(\omega_\phi)$ *of ω_ϕ and its derivatives of order $\leq 2i-2$, and the metric contraction by ω_ϕ.*

More precisely, there exist smooth functions b_i such that, for any $m,l \in \mathbb{N}$ there exists a constant $C_{m,l}$ such that for any $k \in \mathbb{N}$ we have

$$\left|\left|\rho_k(\omega_\phi) - k^n - \sum_{i=1}^{m} b_i k^{n-i}\right|\right|_{C^l} < C_{m,l}k^{n-m-1}.$$

Moreover, the constant $C_{m,l}$ can be chosen independently of ω_ϕ provided it varies in a family of uniformly equivalent metrics which is compact with respect the C^∞-topology.

Remark 2.3.8. In what follows, we shall often use the standard shorthand notation for the asymptotic expansion to write $\rho_k(\omega_\phi) = k^n + k^{n-1}b_1 + k^{n-2}b_2 + O(k^{n-3})$ to mean the above statement.

Remark 2.3.9. Since ϕ is K-invariant and K acts as an isometry of ω, each coefficient b_i appearing in this expansion is K-invariant.

Remark 2.3.10. Theorem 2.3.7 and the Riemann–Roch theorem immediately implies the asymptotic expansion $\bar{\rho}_k(\omega_\phi) = 1 + \frac{1}{k}(b_1 - \bar{b}_1) + \frac{1}{k^2}(b_2 - \bar{b}_2) + \cdots$, where $\bar{b}_i$ is the average of b_i over X with respect to ω_ϕ, which is determined by the Chern–Weil theory (and hence depends only on (X,L) and not on the specific choice of the metric). For notational convenience, we will often use this in the form

$$\bar{\rho}_k(\omega_\phi) = c_1 + \frac{1}{k}b_1 + \sum_{i=2}^{m} \frac{1}{k^i}(b_i - \bar{b}_i) + O(k^{-m-1}),$$

with the constant $c_1 := 1 - \bar{b}_1/k$, in what follows.

We now recall $S(\omega_\phi) = S(\omega) + \mathbb{L}_\omega\phi + O(\phi^2)$ where $\mathbb{L}_\omega$ is an operator defined by

$$\mathbb{L}_\omega\phi := -\Delta^2_\omega\phi - (\mathrm{Ric}(\omega), \sqrt{-1}\partial\bar{\partial}\phi)_\omega$$

with Δ_ω being the negative $\bar{\partial}$-Laplacian $-\bar{\partial}\bar{\partial}^* - \bar{\partial}^*\bar{\partial}$. Recall the well-known identity (equation (2.2) in [70])

$$\mathscr{D}^*_\omega\mathscr{D}_\omega\phi = -\mathbb{L}_\omega\phi + \frac{1}{2}(dS(\omega), d\phi)_\omega.$$

Given these remarks, we now study how the equation (2.10) will be perturbed when we perturb the metric ω to $\omega_{(1)} := \omega + \sqrt{-1}\partial\bar{\partial}\phi_1/k$. First of all we expand $S(\omega_{(1)})$ in ϕ_1/k, which leads to the following asymptotic expansion

$$S(\omega_{(1)}) = S(\omega) + \frac{1}{k}\mathbb{L}_\omega\phi_1 + O(k^{-2}) \tag{2.11}$$

in k^{-1}.

Note also that each coefficient b_i in the asymptotic expansion of the Bergman function changes as

$$b_i(\omega_{(1)}) = b_i(\omega) + O(1/k), \tag{2.12}$$

noting that $b_i(\omega_{(1)})$ can be written as a polynomial in the curvature $\mathrm{Riem}(\omega_{(1)})$ and its derivatives, with the metric contraction by $\omega_{(1)}$.

Remark 2.3.11. Note that this also implies that each coefficient of the powers of k^{-1} in the above expansions (2.11) and (2.12) is K-invariant, if we can choose ϕ_1 to be K-invariant.

Thus we have $\rho_k(\omega_{(1)}) = k^n + \frac{k^{n-1}}{4\pi}S(\omega) + \frac{k^{n-2}}{4\pi}(\mathbb{L}_\omega\phi_1 + 4\pi b_2(\omega)) + O(k^{n-3})$, which means

$$4\pi k\bar{\rho}_k(\omega_{(1)}) = 4\pi k c_1 + S(\omega) + \frac{1}{k}\left(\mathbb{L}_\omega\phi_1 + 4\pi b_2(\omega) - 4\pi\bar{b}_2\right) + O(k^{-2}).$$

Note that, for any fixed ϕ', we have $\mathscr{D}^*_{\omega_{(1)}}\mathscr{D}_{\omega_{(1)}}\phi' = \mathscr{D}^*_\omega\mathscr{D}_\omega\phi' + O(1/k)$ by recalling the formula (1.2) and expanding it in $1/k$. Note also that the second $O(1/k)$ term can be estimated by $C(\phi'; \omega, \phi_1)/k$, where $C(\phi'; \omega, \phi_1)$ is a constant which depends only on the C^4-norm of ϕ' and the C^∞-norm of ω and ϕ_1. In what follows, we shall (rather abusively) refer to this fact by saying that "we have $\mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} = \mathscr{D}^*_\omega \mathscr{D}_\omega + \frac{1}{k} D$, where D is some differential operator of order at most 4 which depends on ω and ϕ_1".

Thus the equation (2.10) to be solved becomes, up to the order $1/k$,

$$\begin{cases} S(\omega) + \frac{1}{2k}(dS(\omega), d\phi_1)_\omega \stackrel{?}{=} 4\pi k c_1 + S(\omega) + \frac{1}{k}\left(\mathbb{L}_\omega \phi_1 + 4\pi b_2(\omega) - 4\pi \bar{b}_2\right) + f + O(k^{-2}) \\ \mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} f \stackrel{?}{=} 0. \end{cases}$$

We write $f_0 := -4\pi k c_1$ and decide to find f that is of order $1/k$, i.e. decide to find f_1 independent of k such that $f = f_1/k$. Namely, we re-write the above equation as

$$\begin{cases} \frac{1}{2k}(dS(\omega), d\phi_1)_\omega \stackrel{?}{=} 4\pi k c_1 + f_0 + \frac{1}{k}\left(\mathbb{L}_\omega \phi_1 + 4\pi b_2(\omega) + f_1 - 4\pi \bar{b}_2\right) + O(k^{-2}) \\ \qquad = \frac{1}{k}\left(\mathbb{L}_\omega \phi_1 + 4\pi b_2(\omega) + f_1 - 4\pi \bar{b}_2\right) + O(k^{-2}) \\ \mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} (f_0 + f_1/k) = \frac{1}{k} \mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} f_1 \stackrel{?}{=} 0, \end{cases}$$

by noting that constant functions generate a trivial holomorphic vector field. We note that by Remark 2.3.11, each coefficient of the powers of k^{-1} in the above asymptotic expansion is K-invariant, if we choose ϕ_1 to be K-invariant.

We now wish to solve this equation up to the leading order, i.e. the order $O(1/k)$. Namely, we wish to find a K-invariant ϕ_1 such that

$$-\mathbb{L}_\omega \phi_1 + \frac{1}{2}(dS(\omega), d\phi_1)_\omega = 4\pi b_2(\omega) - 4\pi \bar{b}_2 + f_1$$

for some f_1 which satisfies $\mathscr{D}^*_\omega \mathscr{D}_\omega f_1 = 0$ and is K-invariant. Recalling that the left-hand side of this equation is equal to $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi_1$ (cf. (1.2)), we are now in place to apply some well-known results concerning the Lichnerowicz operator, collected in the appendix. By applying Lemma A.0.12, we can certainly have a pair (ϕ'_1, f'_1) of C^∞-functions on X which satisfies

$$\begin{cases} \mathscr{D}^*_\omega \mathscr{D}_\omega \phi'_1 = 4\pi b_2(\omega) - 4\pi \bar{b}_2 + f'_1 \\ \mathscr{D}^*_\omega \mathscr{D}_\omega f'_1 = 0. \end{cases}$$

It remains to prove that ϕ'_1 and f'_1 are both K-invariant. We now recall that ω is invariant under K, and hence $\mathscr{D}^*_\omega \mathscr{D}_\omega$ and $b_2(\omega)$ are both invariant under K. Thus, we may take the average over K of the above equation as

$$\begin{cases} \mathscr{D}^*_\omega \mathscr{D}_\omega \int_K g^* \phi'_1 d\mu = 4\pi b_2(\omega) - 4\pi \bar{b}_2 + \int_K g^* f'_1 d\mu \\ \mathscr{D}^*_\omega \mathscr{D}_\omega \int_K g^* f'_1 d\mu = 0. \end{cases} \tag{2.13}$$

where $g \in K$, and $d\mu$ is the normalised Haar measure on the compact Lie group K. Thus, setting $\phi_1 := \int_K g^* \phi'_1 d\mu$ and $f_1 := \int_K g^* f'_1 d\mu$, we find a pair (ϕ_1, f_1) of K-invariant functions which satisfies $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi_1 = 4\pi b_2(\omega) - 4\pi \bar{b}_2 + f_1$ and $\mathscr{D}^*_\omega \mathscr{D}_\omega f_1 = 0$. Note that ϕ_1 and f_1 as constructed above are independent of k.

This means that, going back to the equation (2.10), we have found a metric $\omega_{(1)} = \omega + \sqrt{-1}\partial\bar{\partial}\phi_1/k$ and f_1 such that

$$\begin{cases} S(\omega) + \frac{1}{2k}(dS(\omega), d\phi_1)_\omega = 4\pi k \bar{\rho}_k(\omega_{(1)}) + f_0 + f_1/k + O(k^{-2}) \\ \mathscr{D}^*_\omega \mathscr{D}_\omega (f_0 + f_1/k) = 0. \end{cases}$$

where we recall $f_0 = -4\pi k c_1$. Note that, knowing that ϕ_1 is K-invariant means that $\omega_{(1)}$ is K-invariant, and hence each coefficient of the powers of k^{-1} in the above asymptotic expansion is K-invariant.

It is important to note that we only have $\mathscr{D}^*_\omega \mathscr{D}_\omega f_1 = 0$ and *not* $\mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} f_1 = 0$ (cf. Remarks 2.3.6 and 2.3.14). However, noting $\mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} = \mathscr{D}^*_\omega \mathscr{D}_\omega + \frac{1}{k} D$ with some differential operator D of order at most 4 which depends only on ω and ϕ_1, we still have

$$\mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} (f_0 + f_1/k) = \frac{1}{k} \mathscr{D}^*_{\omega_{(1)}} \mathscr{D}_{\omega_{(1)}} f_1 = O(k^{-2})$$

and the main point of what we prove in the following (Proposition 2.3.13 and Corollary 2.3.15) is that this is enough for solving Problem 2.3.1 by an inductive argument.

Our aim now is to repeat this procedure inductively to get an improved estimate. We thus wish to find a sequence of K-invariant smooth functions $(\phi_{1,k}, \dots, \phi_{m,k})$ such that the metric $\omega_{(m)} := \omega + \sqrt{-1}\partial\bar{\partial}(\sum_{i=1}^m \phi_{i,k}/k^i)$ is approximately ρ-balanced. Unlike the cscK case, we will not be able to have each $\phi_{i,k}$ independently of k (see Remark 2.3.14 below), and we will only be able to show[6] that each $\phi_{i,k}$ converges to some $\phi_{i,\infty}$

[6] Recall on the other hand that we could certainly choose (ϕ_1, f_1) independently of k.

in C^∞ as $k \to \infty$, if $i \geq 2$. This convergence property is obviously of crucial importance in ensuring that $\omega_{(m)}$ converges to ω as $k \to \infty$ (in C^∞-topology), from which it also follows that we can apply Theorem 2.3.7 for each of $\{\omega_{(m)}\}_m$ as they only vary within a compact subset with respect to the C^∞-topology, when k is large enough.

Remark 2.3.12. In what follows, we allow each coefficient $B_i = B_{i,k}$ of the asymptotic expansion to depend on k *as long as it converges to some* $B_{i,\infty}$ *in* C^∞ *as* $k \to \infty$.

For notational convenience, we decide to write $\phi_{(m)} := \sum_{i=1}^m \phi_{i,k}/k^i$ for a sequence of K-invariant functions $(\phi_{1,k}, \dots, \phi_{m,k})$, each $\phi_{i,k}$ converging to some $\phi_{i,\infty}$ in C^∞, and $\omega_{(m)} := \omega + \sqrt{-1}\partial\bar{\partial}\phi_{(m)}$. We also write $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ for the Lichnerowicz operator $\mathscr{D}^*_{\omega_{(m)}}\mathscr{D}_{\omega_{(m)}}$ with respect to $\omega_{(m)}$. Given all these remarks, the main technical result of this section can be stated as follows.

Proposition 2.3.13. *Suppose that for* $m \geq 1$ *there exist sequences* $(\phi_{1,k}, \dots, \phi_{m,k})$ *and* $(f_{1,k}, \dots, f_{m,k})$ *of K-invariant real functions with the following properties: each* $\phi_{i,k}$ *(resp.* $f_{i,k}$*) converges to some* $\phi_{i,\infty}$ *(resp.* $f_{i,\infty}$*) in* C^∞ *as* $k \to \infty$*, and the pair* $(\phi_{(m)}, f_{(m)})$*, with* $\phi_{(m)} = \sum_{i=1}^m \phi_{i,k}/k^i$ *and* $f_{(m)} = \sum_{i=1}^m f_{i,k}/k^i$ *satisfies*

$$\begin{cases} S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega = 4\pi k \bar{\rho}_k(\omega_{(m)}) + f_0 + f_{(m)} + O(k^{-(m+1)}) \\ \mathscr{D}^*_{(m-1)}\mathscr{D}_{(m-1)} f_{(m)} = 0 \end{cases}$$

such that each coefficient of the powers of k^{-1} *in the asymptotic expansion is K-invariant and converges in* C^∞ *as* $k \to \infty$*, with* $f_0 = -4\pi k c_1 = -4\pi k(1 - \bar{b}_1/k)$ *being a constant. Then we can find a pair of K-invariant real functions* $(\phi_{m+1,k}, f_{m+1,k})$*, each converging to some* $(\phi_{i,\infty}, f_{i,\infty})$ *in* C^∞ *as* $k \to \infty$ *such that the pair* $\phi_{(m+1)} = \sum_{i=1}^{m+1} \phi_{i,k}/k^i$ *and* $f_{(m+1)} = \sum_{i=1}^{m+1} f_{i,k}/k^i$ *satisfies*

$$\begin{cases} S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m+1)})_\omega = 4\pi k \bar{\rho}_k(\omega_{(m+1)}) + f_0 + f_{(m+1)} + O(k^{-(m+2)}) \\ \mathscr{D}^*_{(m)}\mathscr{D}_{(m)} f_{(m+1)} = 0 \end{cases}$$

such that each coefficient of the powers of k^{-1} *in the asymptotic expansion is K-invariant and converges in* C^∞ *as* $k \to \infty$.

Remark 2.3.14. We note that $\phi_{i,k}$ and $f_{i,k}$ ($i \geq 2$) cannot be chosen to be independent of k, and can only prove the existence of families of functions $\{\phi_{i,k}\}_k$, $\{f_{i,k}\}_k$ converging to some smooth functions $\phi_{i,\infty}$ and $f_{i,\infty}$ in C^∞ as $k \to \infty$. In particular, $\phi_{i,k}$'s and $f_{i,k}$'s vary in a bounded subset of $C^\infty(X,\mathbb{R})$ for all large enough k. This is an important part of the induction hypothesis, where we also note that it was certainly satisfied in the base case $m = 1$, where ϕ_1 and f_1 could be chosen independent of k.

This is inevitable, since when we solve the equation $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_i = B'_i$ for some B'_i as we will do in the proof, the solution $\phi_i = \phi_{i,k}$ will depend on k as $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ depends on k (even when we have B'_i independently of k).

We note that this problem did not happen in the cscK case [40], where we could solve $\mathscr{D}^*_\omega\mathscr{D}_\omega\phi = \text{const}$ at each order to get an approximately balanced metric, with respect to the *fixed* (cscK) metric ω. This should be fundamentally related to the fact that $\bar{\partial}$ in the cscK condition $\bar{\partial}S(\omega) = 0$ (or the corresponding "quantised" equation $\bar{\partial}\bar{\rho}_k(\omega) = 0$) is independent of the metric ω, whereas $\bar{\partial}\mathrm{grad}^{1,0}_\omega$ in $\bar{\partial}\mathrm{grad}^{1,0}_\omega S(\omega) = 0$ (or the corresponding $\bar{\partial}\mathrm{grad}^{1,0}_\omega\bar{\rho}_k(\omega) = 0$) does depend on ω.

Before we start the proof, we see the consequence of it.

Corollary 2.3.15. *Problem 2.3.2 can be solved affirmatively.*

Proof. Proceeding by induction on m, where we recall that we have established the base case $m = 1$ at the beginning of this section, we find $\omega_{(m)}$ for each $m \in \mathbb{N}$ which satisfies the properties claimed in Proposition 2.3.13. We now compute $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\bar{\rho}_k(\omega_{(m)})$. Note that $S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega$ is the Hamiltonian of the real holomorphic vector field v_s with respect to $\omega_{(m)}$ so $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega) = 0$. Since $\mathscr{D}^*_{(m-1)}\mathscr{D}_{(m-1)}f_{(m)} = 0$, $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)} = \mathscr{D}^*_{(m-1)}\mathscr{D}_{(m-1)} + O(k^{-m})$, and $f_{(m)} = O(k^{-1})$, we have $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f_{(m)} = O(k^{-m-1})$. This means $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\bar{\rho}_k(\omega_{(m)}) = O(k^{-m-2})$, and $\omega_{(m)}$ is K-invariant since each of $(\phi_{1,k}, \dots, \phi_{m,k})$ is K-invariant.

Now, arguing as in the appendix of [47], for any $\nu \in \mathbb{N}$ there exists some $H = H_{\nu,m} \in \mathscr{B}_k$ such that $\omega_{(m)} = \omega_{FS(H)} + O(k^{-\nu})$. Note that $\omega_{(m)}$ being K-invariant implies that each coefficient in the expansion of $\rho_k(\omega_{(m)})$ is K-invariant. Thus, using Lemma 2.2.23 in applying the argument in the appendix of [47], we see that H is in fact $\theta(K)$-invariant, i.e. $H \in \mathscr{B}^K_k$. Thus, taking $\nu = 2m$ for example, we have $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)} = \mathscr{D}^*_{\omega_{FS(H)}}\mathscr{D}_{\omega_{FS(H)}} + O(k^{-2m})$ and $\bar{\rho}_k(\omega_{(m)}) = \bar{\rho}_k(\omega_{FS(H)}) + O(k^{-2m})$, and hence

$$\mathscr{D}^*_{\omega_{FS(H)}} \mathscr{D}_{\omega_{FS(H)}} \bar{\rho}_k(\omega_{FS(H)}) = O(k^{-m-2}). \tag{2.14}$$

This means that, without loss of generality, we may assume in what follows that $\omega_{(m)}$ is of the form $\omega_{FS(H)}$ for some $H \in \mathscr{B}_k^K$.

□

We now prove Proposition 2.3.13. Some technical results, which are used in the proof, about the Lichnerowicz operator are collected in the appendix.

Proof of Proposition 2.3.13. We invoke Theorem 2.3.7 to have the asymptotic expansion

$$\begin{aligned}
&4\pi k \bar{\rho}_k(\omega_{(m+1)}) + f_0 \\
&= S(\omega_{(m+1)}) + \frac{4\pi}{k}\left(b_2(\omega_{(m+1)}) - \bar{b}_2\right) + \cdots + \frac{4\pi}{k^i}\left(b_{i+1}(\omega_{(m+1)}) - \bar{b}_{i+1}\right) + \cdots + O(k^{-(m+2)})
\end{aligned}$$

where $i \leq m+1$, which is valid as long as each of $\phi_{1,k}, \dots, \phi_{m+1,k}$ varies in a bounded subset of $C^\infty(X, \mathbb{R})$, as ensured by the induction hypothesis. We then expand each coefficient $S(\omega_{(m+1)})$ and $b_i(\omega_{(m+1)})$ $(2 \leq i \leq k+1)$ in $\phi_{m+1,k}/k^{m+1}$. They are of the form

$$S(\omega_{(m+1)}) = S(\omega_{(m)}) + k^{-(m+1)} \mathbb{L}_{\omega_{(m)}} \phi_{m+1,k} + O(k^{-2(m+1)})$$

and

$$b_i(\omega_{(m+1)}) = b_i(\omega_{(m)}) + O(k^{-(m+1)}).$$

Since ω and $(\phi_{1,k}, \dots, \phi_{m,k})$ are K-invariant, the same argument as in Remark 2.3.11 implies that each coefficient of the powers of k^{-1} in the above expansions is K-invariant and converges to some smooth function in C^∞ as $k \to \infty$ once we know that $\phi_{m+1,k}$ is K-invariant and converges to some $\phi_{m+1,\infty}$ in C^∞ as $k \to \infty$.

We can thus write

$$\begin{aligned}
4\pi k \bar{\rho}_k(\omega_{(m+1)}) + f_0 &= S(\omega_{(m)}) + \frac{1}{k^{m+1}} \mathbb{L}_{\omega_{(m)}} \phi_{m+1,k} + \frac{4\pi}{k}\left(b_2(\omega_{(m)}) - \bar{b}_2\right) + \cdots \\
&\qquad + \frac{4\pi}{k^i}\left(b_{i+1}(\omega_{(m)}) - \bar{b}_{i+1}\right) + \cdots + O(k^{-(m+2)}) \\
&= 4\pi k \bar{\rho}_k(\omega_{(m)}) + f_0 + \frac{1}{k^{m+1}} \mathbb{L}_{\omega_{(m)}} \phi_{m+1,k} + O(k^{-(m+2)}). \qquad (2.15)
\end{aligned}$$

By the induction hypothesis,

$$\left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega\right) = 4\pi k\bar{\rho}_k(\omega_{(m)}) + f_0 + f_{(m)} + O(k^{-(m+1)}),$$

and there exists a family of K-invariant functions $\{B_{m+1,k}\}_k$, converging to some K-invariant function $B_{m+1,\infty}$ in C^∞ as $k \to \infty$, such that

$$\left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega\right) = 4\pi k\bar{\rho}_k(\omega_{(m)}) + f_0 + f_{(m)} + k^{-(m+1)}B_{m+1,k} + O(k^{-(m+2)}). \tag{2.16}$$

Thus, combining (2.15) and (2.16),

$$\begin{aligned}
&4\pi k\bar{\rho}_k(\omega_{(m+1)})\\
&= \left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m)})_\omega\right) - f_{(m)} + \frac{1}{k^{m+1}}(\mathbb{L}_{\omega_{(m)}}\phi_{m+1,k} - B_{m+1,k}) + O(k^{-(m+2)})\\
&= \left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m+1)})_\omega\right) - f_{(m)}\\
&\qquad + \frac{1}{k^{m+1}}(\mathbb{L}_{\omega_{(m)}}\phi_{m+1,k} - \frac{1}{2}(dS(\omega), d\phi_{m+1,k})_\omega - B_{m+1,k}) + O(k^{-(m+2)}).
\end{aligned}$$

Note that, since $\phi_{(m)} = \sum_{i=1}^m \phi_{i,k}/k^i = O(1/k)$ implies $(d\mathbb{L}_\omega\phi_{(m)}, d\phi_{m+1,k})_\omega = O(1/k)$ and $(dS(\omega_{(m)}), d\phi_{m+1,k})_{\omega_{(m)}} = (dS(\omega_{(m)}), d\phi_{m+1,k})_\omega + O(1/k)$, we have

$$\begin{aligned}
&\mathbb{L}_{\omega_{(m)}}\phi_{m+1,k} - \frac{1}{2}(dS(\omega), d\phi_{m+1,k})_\omega\\
&= \mathbb{L}_{\omega_{(m)}}\phi_{m+1,k} - \frac{1}{2}(dS(\omega_{(m)}), d\phi_{m+1,k})_\omega + \frac{1}{2}(d\mathbb{L}_\omega\phi_{(m)}, d\phi_{m+1,k})_\omega + O(k^{-2})\\
&= \mathbb{L}_{\omega_{(m)}}\phi_{m+1,k} - \frac{1}{2}(dS(\omega_{(m)}), d\phi_{m+1,k})_{\omega_{(m)}} + O(k^{-1})\\
&= -\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k} + O(k^{-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
4\pi k\bar{\rho}_k(\omega_{(m+1)}) &= \left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m+1)})_\omega\right) - f_{(m)}\\
&\quad + \frac{1}{k^{m+1}}(-\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k} - B_{m+1,k}) + O(k^{-(m+2)}).
\end{aligned} \tag{2.17}$$

Observe that, in all the expansions above, each coefficient of the powers of k^{-1} is K-invariant and converges to some smooth function in C^∞ as $k \to \infty$ once we know that

$\phi_{m+1,k}$ is K-invariant and converges to some $\phi_{m+1,\infty}$ in C^∞ as $k \to \infty$.

Our aim now is to find K-invariant functions $\phi_{m+1,k}$ and $f_{m+1,k}$ such that

$$\begin{cases} \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k} + B_{m+1,k} = f_{m+1,k} \\ \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f_{(m+1)} = 0. \end{cases}$$

Recall first that we have $\mathscr{D}^*_{(m-1)}\mathscr{D}_{(m-1)}f_{(m)} = 0$ by the induction hypothesis. Since $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)} = \mathscr{D}^*_{(m-1)}\mathscr{D}_{(m-1)} + \frac{1}{k^m}D$ with some differential operator D of order at most 4 which depends only on ω and $(\phi_{1,k},\dots,\phi_{m,k})$, and recalling $f_{(m)} = \sum_{i=1}^m f_{i,k}/k^i = O(1/k)$, we have $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f_{(m)} = O(k^{-(m+1)})$. We first aim to find $\{f'_{m+1,k}\}_k$ which satisfies $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(f_{(m)} + f'_{m+1,k}/k^{m+1}) = 0$ and also converges to a smooth function in C^∞ as $k \to \infty$.

Let $F_k := k^{m+1}(f_{(m)} - \mathrm{pr}_{(m)}f_{(m)})$ where $\mathrm{pr}_{(m)} : C^\infty(X,\mathbb{R}) \twoheadrightarrow \ker \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ is the projection onto the kernel of $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ in terms of the L^2-orthogonal direct sum decomposition $C^\infty(X,\mathbb{R}) = \ker \mathscr{D}^*_{(m)}\mathscr{D}_{(m)} \oplus \ker \mathscr{D}^*_{(m)}\mathscr{D}^{\perp}_{(m)}$. We write $G_k := \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_k$. Since $f_{i,k}$ $(1 \leq i \leq m)$ converges to a smooth function in C^∞ as $k \to \infty$, $\omega_{(m)} \to \omega$ in C^∞ as $k \to \infty$, and $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f_{(m)} = O(k^{-m-1})$, G_k converges to a smooth function, say G_∞, in C^∞ as $k \to \infty$. Now we observe that F_k is the solution to the equation $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_k = G_k$ with the minimum L^2-norm.

We now aim to show that F_k converges in C^∞ as $k \to \infty$. We aim to use Lemma A.0.14 in the end, but we first have to establish $G_\infty \in \mathrm{im}\mathscr{D}^*_\omega\mathscr{D}_\omega$ to do so. By using the L^2-orthogonal direct sum decomposition $C^\infty(X,\mathbb{R}) = \ker \mathscr{D}^*_{(m)}\mathscr{D}_{(m)} \oplus \ker \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}{}^{\perp}$ and recalling the standard elliptic regularity, for each $p \in \mathbb{N}$ there exists a constant $C_p(\omega,\{\phi_{i,k}\})$ which depends only on ω and $\phi_{i,k}$ $(1 \leq i \leq m)$ such that $||F_k||_{L^2_{p+4}} < C_p(\omega,\{\phi_{i,k}\})||G_k||_{L^2_p}$, with the Sobolev norm $||\cdot||_{L^2_p}$ for a fixed but large enough p. By noting that each ϕ_i converges in C^∞ as $k \to \infty$ and G_k converges in C^∞ as $k \to \infty$, we can find a uniform bound on $||F_k||_{L^2_{p+1}}$ as $||F_k||_{L^2_{p+4}} < C'_p(\omega,\{\phi_{i,\infty}\})||G_\infty||_{L^2_p}$ which is independent of k. Thus there exists a subsequence $\{F_{k_l}\}$ of $\{F_k\}$ which converges in the Sobolev space L^2_{p+3} by Rellich compactness. Let F_∞ be its limit in L^2_{p+3}. Now, recalling $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_{k_l} = G_{k_l}$, consider

$$\mathscr{D}^*_\omega\mathscr{D}_\omega F_{k_l} - G_{k_l} = \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_{k_l} - G_{k_l} - \frac{1}{k_l}D(F_{k_l}) = -\frac{1}{k_l}D(F_{k_l})$$

where we wrote $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}=\mathscr{D}^*_\omega\mathscr{D}_\omega+\frac{1}{k_l}D$ with some differential operator D of order at most 4, which depends only on ω and ϕ_{i,k_l} $(1\le i\le m)$. Thus, since each ϕ_{i,k_l} $(1\le i\le m)$ converges in C^∞ by the induction hypothesis, we have

$$||\mathscr{D}^*_\omega\mathscr{D}_\omega F_{k_l}-G_{k_l}||_{L^2_{p-1}}\le\frac{C''_p(\omega,\{\phi_{i,\infty}\})}{k_l}||F_{k_l}||_{L^2_{p-5}},$$

and hence by taking the limit $k_l\to\infty$, we have the equation $\mathscr{D}^*_\omega\mathscr{D}_\omega F_\infty=G_\infty$ in L^2_{p-1}. Since $G_\infty\in C^\infty(X,\mathbb{R})$ we have $F_\infty\in C^\infty(X,\mathbb{R})$ by elliptic regularity, and hence $\mathscr{D}^*_\omega\mathscr{D}_\omega F_\infty=G_\infty$ in $C^\infty(X,\mathbb{R})$ shows $G_\infty\in\mathrm{im}\mathscr{D}^*_\omega\mathscr{D}_\omega$. Lemma A.0.13 shows $\mathrm{pr}_\omega F_\infty=0$, and hence F_∞ is the L^2-minimum solution to the equation $\mathscr{D}^*_\omega\mathscr{D}_\omega F_\infty=G_\infty$. We can now apply Lemma A.0.14 to conclude that $\{F_k\}$ converges to F_∞ in C^∞ (we note that the convergence holds for the whole sequence $\{F_k\}$ and not just for the subsequence $\{F_{k_l}\}$ that we used).

Now setting $f'_{m+1,k}:=-F_k=-k^{m+1}(f_{(m)}-\mathrm{pr}_{(m)}f_{(m)})$, we have $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(f_{(m)}+f'_{m+1,k}/k^{m+1})=0$ and that $f'_{m+1,k}$ converges in C^∞ as $k\to\infty$. Also, we note that $f'_{m+1,k}$ is K-invariant because $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ and $f_{(m)}$ are both K-invariant.

For this choice of $f'_{m+1,k}$, we solve

$$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k}+B_{m+1,k}=f'_{m+1,k}$$

modulo some function $f''_{m+1,k}$ with $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f''_{m+1,k}=0$, i.e. we solve for $\phi_{m+1,k}$ the equation

$$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k}+B_{m+1,k}=f'_{m+1,k}+f''_{m+1,k}$$

for some $f''_{m+1,k}$ with $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}f''_{m+1,k}=0$. This is possible by Lemma A.0.12, where we also recall that $f''_{m+1,k}$ is in fact $-\mathrm{pr}_{(m)}(f'_{m+1,k}-B_{m+1,k})$, where $\mathrm{pr}_{(m)}:C^\infty(X,\mathbb{R})\twoheadrightarrow\ker\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$. Thus we have

$$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k}=(f'_{m+1,k}-B_{m+1,k})-\mathrm{pr}_{(m)}(f'_{m+1,k}-B_{m+1,k}).$$

$f'_{m+1,k}$ converges in C^∞ as we saw above, and $B_{m+1,k}$ converges in C^∞ since $\phi_{i,k}$ $(1\le i\le m)$ converges in C^∞ and by the induction hypothesis. Thus, by Lemma A.0.13, the right hand side of the above equation is a smooth function, with parameter k, that converges, say to G_∞, in C^∞ as $k \to \infty$, which is in the image of $\mathscr{D}^*_\omega \mathscr{D}_\omega$. Thus, writing $\phi_{m+1,\infty}$ for the solution to the equation $\mathscr{D}^*_\omega \mathscr{D}_\omega \phi_{m+1,\infty} = G_\infty$, we can apply Lemma A.0.14 for $G_k := (f'_{m+1,k} - B_{m+1,k}) - \mathrm{pr}_{(m)}(f'_{m+1,k} - B_{m+1,k})$ to conclude that $\phi_{m+1,k}$ converges to $\phi_{m+1,\infty}$ in C^∞ (since without loss of generality we may assume that they are all L^2-minimum solutions).

Noting that $f''_{m+1,k} := -\mathrm{pr}_{(m)}(f'_{m+1,k} - B_{m+1,k})$ is K-invariant since $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$, $B_{m+1,k}$, and $f'_{m+1,k}$ are all K-invariant, we can take the average over K of $\phi_{m+1,k}$ as we did in (2.13). Thus $\phi_{m+1,k}$ can be chosen to be K-invariant.

We then note that $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(f_{(m)} + (f'_{m+1,k} + f''_{m+1,k})/k^{m+1}) = 0$ so we define $f_{m+1,k} := f'_{m+1,k} + f''_{m+1,k}$ which is K-invariant and converges to some smooth function as $k \to \infty$. For this choice of $f_{m+1,k}$, we thus have two K-invariant functions $\phi_{m+1,k}$ and $f_{m+1,k}$ with $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)} f_{(m+1)} = 0$ and $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}\phi_{m+1,k} + B_{m+1,k} = f_{m+1,k}$ each converging to a smooth function in C^∞ as $k \to \infty$.

Now, going back to the equation (2.17), we find

$$\begin{aligned} 4\pi k \bar{\rho}_k(\omega_{(m+1)}) &= \left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m+1)})_\omega\right) - f_{(m)} - \frac{1}{k^{m+1}} f_{m+1,k} + O(k^{-(m+2)}) \\ &= \left(S(\omega) + \frac{1}{2}(dS(\omega), d\phi_{(m+1)})_\omega\right) - f_{(m+1)} + O(k^{-(m+2)}) \end{aligned}$$

with $f_{(m+1)}$ satisfying $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)} f_{(m+1)} = 0$. As remarked just after the equation (2.17), each coefficient of the powers of k^{-1} in the above asymptotic expansion is K-invariant and converges to some smooth function in C^∞ as $k \to \infty$ since $\phi_{m+1,k}$ is K-invariant and converges to some $\phi_{m+1,\infty}$ in C^∞ as $k \to \infty$. Since $\phi_{m+1,k}$ and $f_{m+1,k}$ are both K-invariant functions that converge to smooth functions in C^∞ as $k \to \infty$, the sequences $(\phi_{1,k}, \dots, \phi_{m+1,k})$ and $(f_{1,k}, \dots, f_{m+1,k})$ satisfy all the requirements stated in the induction hypothesis in Proposition 2.3.13 for the induction to continue.

□

It is immediate that for each $l \in \mathbb{N}$ there exists $C_{l,m} > 0$ so that $||\omega_{(m)} - \omega||_{C^l,\omega} < C_{l,m}/k$. The equation (2.14) and the standard elliptic regularity mean that we can find $F_{m,k} \in C^\infty(X, \mathbb{R})$, for each k, such that

$$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(4\pi k \bar{\rho}_k(\omega_{(m)}) + F_{m,k}/k^{m+1}) = 0$$

and that for each $p \in \mathbb{N}$ the L^2_p-norm of $\{F_{m,k}\}_k$ is bounded uniformly of k. In particular, observe that $\sup_X |F_{m,k}|$ is bounded uniformly of k. Moreover, since $\omega_{(m)}$ and $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}$ are both K-invariant, we may choose $F_{m,k}$ to be K-invariant, as we did in (2.13). This means that the vector field $v_{(m)}$ defined by

$$\iota(v_{(m)})\omega_{(m)} = -d\left(\bar{\rho}_k(\omega_{(m)}) + \frac{F_{m,k}}{4\pi k^{m+2}}\right) \tag{2.18}$$

is real holomorphic and lies in the centre $\mathfrak{z}$ of $\mathfrak{k}$ by Lemmas 2.3.4 (recall also that Lemma 2.2.21 shows that $\bar{\rho}_k(\omega_h)$ is indeed K-invariant if $K \leq \mathrm{Isom}(\omega_h)$).

Remark 2.3.16. Note further that $d\left(\bar{\rho}_k(\omega_{(m)})\right) = \frac{1}{4\pi k}dS(\omega_{(m)}) + O(k^{-2}) = \frac{1}{4\pi k}dS(\omega) + O(k^{-2})$, and that $F_{m,k}$ is of order 1 in k^{-1} imply that $4\pi k v_{(m)}$ converges to the extremal vector field v_s generated by $S(\omega)$. In particular, if we use the (pointwise) norm $|\cdot|_{k\omega}$ on TX defined by $k\omega$, we have $\sup_X |Jv_{(m)}|^2_{k\omega} \leq \mathrm{const}/k$. This fact will be important in Lemma 2.4.8.

2.4 Reduction to a finite dimensional problem

Recall that the equation $\bar{\partial}\rho_k(\omega_h) = 0$ (or equivalently $\rho_k(\omega_h) = \mathrm{const}$) is equivalent to finding a balanced embedding, i.e. the embedding where $\bar{\mu}_X(g)$ is a constant multiple of the identity (cf. Theorem 2.2.19). This means that the seemingly intractable PDE problem $\bar{\partial}\rho_k(\omega_h) = 0$ can be reduced to a finite dimensional problem of finding the balanced embedding. The main result (Corollary 2.4.16, see also Proposition 2.4.5) of this section is to establish this reduction in the "relative" setting, namely to establish a connection between the equation $\mathscr{D}^*_{\omega_h}\mathscr{D}_{\omega_h}\rho_k(\omega_h) = 0$ and the projective embedding in terms of the centre of mass $\bar{\mu}_X$.

In what follows, we shall be mostly focused on the Kähler metrics of the form $\omega_{FS(H)}$ with $H \in \mathscr{B}_k$ or $H \in \mathscr{B}_k^K$. To simplify the notation, we will often write as follows.

Notation 2.4.1. We will often write ω_H for $\omega_{FS(H)}$, and $\mathscr{D}^*_H\mathscr{D}_H$ for $\mathscr{D}^*_{\omega_{FS(H)}}\mathscr{D}_{\omega_{FS(H)}}$.

2.4.1 General lemmas and their consequences

We start with the following general lemmas.

Lemma 2.4.2. (cf. equation (5.3) in [97]) *For any $f \in \mathrm{Aut}_0(X,L)$ and any $H \in \mathscr{B}_k$, we have*

$$f^* \omega_H = \omega_H + \frac{\sqrt{-1}}{2\pi k} \partial \bar{\partial} \log(\sum_i | \sum_j \theta(f)_{ij} s_j |^2_{FS(H)^k}), \tag{2.19}$$

where $\{s_j\}$ is an H-orthonormal basis for $H^0(X,L^k)$.

Proof. Suppose that we write (at first) $\{Z'_i\}$ for an H-orthonormal basis for $H^0(X,L^k)$, giving an isomorphism $H^0(X,L^k) \cong \mathbb{C}^N$ and hence defining an embedding $\iota' : X \hookrightarrow \mathbb{P}^{N-1}$. By recalling $\theta(f) \circ \iota' = \iota' \circ f$ (the equation (2.2)), we have

$$\begin{aligned} f^* \omega_H &= f^* \iota'^* \frac{\sqrt{-1}}{2\pi k} \partial \bar{\partial} \log \left(\sum_i |Z'_i|^2 \right) \\ &= \iota'^* \frac{\sqrt{-1}}{2\pi k} \partial \bar{\partial} \log \left(\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2 \right) \end{aligned}$$

where $\theta(f)_{ij}$ is the matrix for $\theta(f)$ represented with respect to $\{Z'_i\}$. We can write the above as (cf. equation (5.3) in [97])

$$f^* \omega_H = \omega_H + \iota'^* \frac{\sqrt{-1}}{2\pi k} \partial \bar{\partial} \log \left(\frac{\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2}{\sum_i |Z'_i|^2} \right),$$

where we note that $\frac{\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2}{\sum_i |Z'_i|^2}$ is a well-defined function on $\mathbb{P}^{N-1}$ as it is a ratio of two homogeneous polynomials in the homogeneous coordinates. Pick now any hermitian metric $\tilde{h}$ on $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$. We now observe that, by choosing a local trivialisation of $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ and writing $\tilde{h} = e^{-\phi}$ locally, multiplying both the denominator and the numerator by $e^{-\phi}$ yields

$$\frac{\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2}{\sum_i |Z'_i|^2} = \frac{\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2_{\tilde{h}}}{\sum_i |Z'_i|^2_{\tilde{h}}},$$

by noting that any ambiguity in choosing the local trivialisation in the denominator is cancelled by the one in the numerator. Thus, choosing $\tilde{h}$ to be the hermitian metric $\tilde{h}_{FS(H)}$ on $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ induced from H (so that $\iota'^* \tilde{h}_{FS(H)} = h^k_{FS(H)}$) and writing $s_i := \iota'^* Z'_i$, we have

$$\iota'^* \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\sum_i | \sum_j \theta(f)_{ij} Z'_j |^2}{\sum_i |Z'_i|^2} \right) = \sqrt{-1} \partial \bar{\partial} \log \left(\frac{\sum_i | \sum_j \theta(f)_{ij} s_j |^2_{FS(H)^k}}{\sum_i |s_i|^2_{FS(H)^k}} \right).$$

On the other hand, $\sum_i |s_i|^2_{FS(H)^k}$ is constantly equal to 1 since $\{s_i\}$ is an H-orthonormal basis, by the definition (2.4) of $FS(H)$. Thus we finally have (2.19). □

Lemma 2.4.3. *Suppose that ψ is a Hamiltonian of the Killing vector field $v \in \mathfrak{k}$ with respect to ω_H, $H \in \mathscr{B}_k^K$, so that we have $\iota(v)\omega_H = -d\psi$ and $\mathscr{D}_H^* \mathscr{D}_H \psi = 0$.*

Suppose also that we write (cf. (2.3)) $A := \theta_*(Jv) \in \theta_*(\sqrt{-1}\mathfrak{k})$ for the real holomorphic vector field Jv and the (injective) Lie algebra homomorphism $\theta_* : \mathfrak{g} = \mathrm{LieAut}_0(X,L) \to \mathfrak{sl}(N,\mathbb{C})$. Then, writing $\{s_i\}$ for an H-orthonormal basis for $H^0(X,L^k)$, we have

$$\psi = -\frac{1}{2\pi k}\sum_{i,j} A_{ij} h^k_{FS(H)}(s_i,s_j) + \mathrm{const} \tag{2.20}$$

where A_{ij} is the matrix for A represented with respect to $\{s_i\}$.

Proof. Take the 1-parameter subgroup $\{\sigma(t)\} \leq \mathrm{Aut}_0(X,L)$ generated by Jv. Then, by Lemma 2.4.2, we have

$$\sigma(t)^*\omega_H = \omega_H + \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}\log\left(\sum_i |\sum_j \theta(\sigma(t))_{ij} s_i|^2_{FS(H)^k}\right)$$

for an H-orthonormal basis $\{s_i\}$. We observe $\theta(\sigma(t)) = e^{tA}$ by the definition $A = \theta_*(Jv)$. We now see

$$\begin{aligned}
L_{Jv}\omega_H &= \lim_{t\to 0}\frac{\sigma(t)^*\omega_{FS(H)} - \omega_{FS(H)}}{t} \\
&= \lim_{t\to 0}\frac{\sqrt{-1}}{2\pi k}\frac{\partial\bar{\partial}\log\left(\sum_i |\sum_j \theta(\sigma(t))_{ij} s_j|^2_{FS(H)^k}\right)}{t} \\
&= \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}\left(\left.\frac{d}{dt}\right|_{t=0}\log(\sum_i |\sum_j (e^{tA})_{ij} s_j|^2_{FS(H)^k})\right) \\
&= \frac{\sqrt{-1}}{\pi k}\partial\bar{\partial}\left(\sum_{i,j} A_{ij} h^k_{FS(H)}(s_i,s_j)\right),
\end{aligned}$$

by noting that A is hermitian since H is $\theta(K)$-invariant (cf. Lemma 2.2.4).

Note on the other hand that, since ψ is the Hamiltonian for v, we have, by using the Cartan homotopy formula,

$$L_{Jv}\omega_H = d\iota(Jv)\omega_H = dJ\iota(v)\omega_H = -dJd\psi = -2\sqrt{-1}\partial\bar{\partial}\psi$$

where we used (2.9) in the second equality. We thus have $\psi = -\frac{1}{2\pi k}\sum_{i,j} A_{ij} h^k_{FS(H)}(s_i, s_j) +$ const as claimed.

□

Remark 2.4.4. Conversely, given $A \in \theta_*(\sqrt{-1}\mathfrak{k})$, it immediately follows that ψ as defined in (2.20) satisfies $\mathscr{D}^*_H \mathscr{D}_H \psi = 0$, generating a real holomorphic vector field $v := J^{-1}\theta_*^{-1}(A)$.

Suppose now that $H \in \mathscr{B}^K_k$ satisfies $\mathscr{D}^*_H \mathscr{D}_H \bar{\rho}_k(\omega_H) = 0$. Then Lemma 2.4.3 and (2.4) implies

$$\bar{\rho}_k(\omega_H) = C - \frac{1}{2\pi k}\sum_{i,j} A_{ij} h^k_{FS(H)}(s_i, s_j) = \sum_{i,j}\left(CI - \frac{1}{2\pi k}A\right)_{ij} h^k_{FS(H)}(s_i, s_j) \qquad (2.21)$$

for $A := \theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_H)) \in \theta_*(\sqrt{-1}\mathfrak{k})$ and some constant $C \in \mathbb{R}$ which can be determined by integrating both sides of the equation, so that the average over X of both sides is 1. We now have the following proposition, pointed out to the author by Joel Fine, which distills the essential point of our main result Corollary 2.4.16 to be proved later.

Proposition 2.4.5. *The equation $\mathscr{D}^*_H \mathscr{D}_H \bar{\rho}_k(\omega_H) = 0$ (or equivalently $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_H}\rho_k(\omega_H) = 0$) holds if and only if $\bar{\mu}_X(g)^{-1}$ generates a holomorphic vector field on $\mathbb{P}^{N-1}$ that is tangential to $\iota(X) \subset \mathbb{P}^{N-1}$, where $H = \overline{(g^{-1})^t} g^{-1}$.*

This proposition also applies to the embedding $\iota : X \hookrightarrow \mathbb{P}^{N-1}$ in general, as discussed in Remark 2.2.12.

Proof. Lemma 2.4.3 and Remark 2.4.4 imply that $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_H}\rho_k(\omega_H) = 0$ is satisfied if and only if $\rho_k(\omega_H) = \frac{N}{V}\left(CI - \frac{1}{2\pi k}A\right)_{ij} h^k_{FS(H)}(s_i, s_j)$, where $C \in \mathbb{R}$ is some constant and $A = \theta_*(-\mathrm{grad}\rho_k(\omega_H)) \in \theta_*(\sqrt{-1}\mathfrak{k})$. Combined with (2.6), we see that $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_H}\rho_k(\omega_H) = 0$ holds if and only if

$$\sum_{i,j}\left(k^n \bar{\mu}_X(g)^{-1} - \frac{N}{V}\left(CI - \frac{1}{2\pi k}A\right)\right)_{ij} h^k_{FS(H)}(s_i, s_j) = 0,$$

which holds if and only if $\bar{\mu}_X(g)^{-1} = \frac{N}{Vk^n}\left(CI - \frac{1}{2\pi k}A\right)$ by arguing as in the proof of Lemma 2.2.9.

We are thus reduced to proving the following: if $\bar{\mu}_X(g)^{-1}$ generates a holomorphic vector field on $\mathbb{P}^{N-1}$ that is tangential to $\iota(X) \subset \mathbb{P}^{N-1}$, then its trace-free part

$(\bar{\mu}_X(g)^{-1})_0$ must lie in $\theta_*(\sqrt{-1}\mathfrak{k})$. Observe first that by Lemma 2.2.1, $\bar{\mu}_X(g)^{-1}$ generates a holomorphic vector field on $\mathbb{P}^{N-1}$ that is tangential to $\iota(X) \subset \mathbb{P}^{N-1}$ if and only if $(\bar{\mu}_X(g)^{-1})_0 \in \theta_*(\mathfrak{g})$. Write $\mathfrak{g} = (\mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}) \oplus_\pi \mathfrak{n}$, where $\mathfrak{n} := \mathrm{Lie}(R_u)$ is a nilpotent Lie algebra and $\oplus_\pi$ is the semidirect product in the Lie algebra corresponding to $G = K^{\mathbb{C}} \ltimes R_u$ (cf. Notation 2.2.3). Then, noting that $\bar{\mu}_X(g)^{-1}$ is hermitian, the $\mathfrak{n}$-component of $(\bar{\mu}_X(g)^{-1})_0$ must be zero by the Jordan–Chevalley decomposition, and Lemma 2.2.4 implies that the trace-free part of $\bar{\mu}_X(g)^{-1}$ must lie in $\theta_*(\sqrt{-1}\mathfrak{k})$.

□

Note that the above proof implies that $CI - \frac{1}{2\pi k}A$ is always positive definite, and in particular invertible. However, for the later argument (cf. Remark 2.4.7), it will be necessary to have more precise estimates on the operator norm $||A||_{op}$ of A (i.e. the maximum of the moduli of the eigenvalues of A) and $|C|$. In particular, we shall need to focus on the case where $||A||_{op}$ is bounded uniformly of k. First of all, we see that $|C|$ can be bounded in terms of $||A||_{op}$ as follows. Note from (2.4) that we have

$$\left| C - \frac{1}{2\pi k}||A||_{op} \right| \le \sum_{i,j} \left(CI - \frac{1}{2\pi k}A \right)_{ij} h^k_{FS(H)}(s_i, s_j) \le C + \frac{1}{2\pi k}||A||_{op}. \tag{2.22}$$

By assuming that $||A||_{op}$ is bounded uniformly for all large enough k, we have $C - \frac{1}{2\pi k}||A||_{op} > 0$ for all large enough k. We now take the average of (2.22) over X with respect to ω_H to get $1 - \frac{1}{2\pi k}||A||_{op} \le C \le 1 + \frac{1}{2\pi k}||A||_{op}$, and get the following.

Proposition 2.4.6. *Suppose now that $H \in \mathscr{B}^K_k$ satisfies $\mathscr{D}^*_H \mathscr{D}_H \bar{\rho}_k(\omega_H) = 0$ and the operator norm of $A := \theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_H))$ is bounded uniformly of k. Then we can write*

$$\bar{\rho}_k(\omega_H) = \sum_{i,j} \left(I + C_A I - \frac{1}{2\pi k}A \right)_{ij} h^k_{FS(H)}(s_i, s_j),$$

where C_A is a constant which satisfies $-\frac{1}{2\pi k}||A||_{op} \le C_A \le \frac{1}{2\pi k}||A||_{op}$, and hence is of order $1/k$, in particular.

Remark 2.4.7. The uniform bound for $||A||_{op}$ will be crucially important in §2.4.2 and §2.5. In what follows, we shall discuss some sufficient conditions under which we can assume the bound $||A||_{op} <$ const uniformly of k. It turns out that these conditions are always satisfied for our purpose (cf. Corollary 2.4.11 and (2.31)).

We now discuss the operator norm of A. In what follows, we occasionally write $H(k) \in \mathscr{B}_k^K$ for $H \in \mathscr{B}_k^K$, just in order to make clear its dependence on k.

Lemma 2.4.8. *Suppose that we have a real holomorphic vector field $v \in \sqrt{-1}\mathfrak{k}$ on X and a sequence $\{H(k)\}_k$ with $H(k) \in \mathscr{B}_k^K$, which satisfy $|v|^2_{k\omega_{H(k)}} = O(1/k)$, where $|\cdot|_{k\omega_{H(k)}}$ is a pointwise norm on TX defined by $k\omega_{H(k)}$. Then $||\theta_*(v)||_{op} \leq \mathrm{const}$ uniformly for all large enough k.*

Proof. Recall now the equation (2.2) so that we can write

$$e^{\theta_*(v)} \circ \iota = \iota \circ e^v. \tag{2.23}$$

Since ι is an isometry if we choose the metrics $k\omega_{H(k)}$ on X and $H(k)$ on $\mathbb{C}^N \cong H^0(X, L^k)$ covering $\mathbb{P}^{N-1}$, the assumption $|v|^2_{k\omega_{H(k)}} = O(1/k)$ implies $|\iota_* \circ v|^2_{H(k)} = |v|^2_{k\omega_{H(k)}} = O(1/k)$, where $|\cdot|_{k\omega_{H(k)}}$ (resp. $|\cdot|_{H(k)}$) is a pointwise norm on TX given by $k\omega_{H(k)}$ (resp. on $T\mathbb{P}^{N-1}$ by $H(k)$). This means that for all point $p \in X$ we have

$$\mathrm{dist}_{H(k)}(\iota(e^v(p)), \iota(p)) \to 0 \tag{2.24}$$

as $k \to \infty$, where $\mathrm{dist}_{H(k)}$ is the distance in $\mathbb{P}^{N-1}$ given by the Fubini–Study metric defined by $H(k)$.

Suppose now that $||\theta_*(v)||_{op} \to +\infty$ as $k \to +\infty$ (by taking a subsequence if necessary) and aim for a contradiction. Then for each (large enough) k there exists a vector w_k in $\mathbb{C}^N \cong H^0(X, L^k)$ such that

$$\frac{||\theta_*(v) w_k||_{H(k)}}{||w_k||_{H(k)}} \to +\infty,$$

where $||\cdot||_{H(k)}$ is the norm on $H^0(X, L^k)$ defined by $H(k)$. Since X is not contained in any proper linear subspace of $\mathbb{P}^{N-1}$, this means that there exists a constant $\delta > 0$ such that for all large enough k there exists a point $q_k \in X$ with $\mathrm{dist}_{H(k)}(e^{\theta_*(v)} \circ \iota(q_k), \iota(q_k)) > \delta$. Recalling the equation (2.24), this contradicts (2.23).

□

We apply Lemma 2.4.8 to prove the following.

Lemma 2.4.9. *Suppose that we have a reference metric ω_0 and a sequence $\{H(k)\}_k$ with $H(k) \in \mathscr{B}_k^K$ which satisfies*

$$\sup_X |k\omega_{H(k)} - k\omega_0|_{\omega_0} < R' \tag{2.25}$$

for some constant $R' > 0$ uniformly of k, and $A = \theta_*(v)$ for $v \in \sqrt{-1}\mathfrak{k}$ such that $|v|^2_{k\omega_0} = O(1/k)$ where $|\cdot|_{k\omega_0}$ is a (pointwise) norm on TX defined by $k\omega_0$. Then $||A||_{op} < C(R')$ for some constant $C(R') > 0$ which depends only on R' and is independent of k.

Remark 2.4.10. Note that the hypothesis (2.25) of the lemma is slightly different from each $H(k)$ having R'-bounded geometry (as defined in Definition 2.5.5).

Proof. Note that $\sup_X |k\omega_{H(k)} - k\omega_0|_{\omega_0} < R'$ uniformly of k, combined with $|v|^2_{k\omega_0} = O(1/k)$, implies $|v|^2_{k\omega_{H(k)}} = O(1/k)$. Thus we can just apply Lemma 2.4.8. □

In what follows, we take the reference metric ω_0 to be the extremal metric ω. Recalling Remark 2.3.16, we thus obtain the following corollary of Proposition 2.4.6.

Corollary 2.4.11. *Suppose that we have a sequence $\{H(k)\}_k$ with $H(k) \in \mathscr{B}_k^K$, each of which satisfies $\mathscr{D}^*_{H(k)}\mathscr{D}_{H(k)}\bar{\rho}_k(\omega_{H(k)}) = 0$ and $\sup_X |k\omega_{H(k)} - k\omega|_\omega < R'$ for some constant $R' > 0$ uniformly of k. Then we can write*

$$\bar{\rho}_k(\omega_{H(k)}) = \sum_{i,j} \left(I + C_A I - \frac{1}{2\pi k} A \right)_{ij} h^k_{FS(H(k))}(s_i, s_j), \tag{2.26}$$

where $A := \theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_{H(k)})) \in \theta_*(\sqrt{-1}\mathfrak{k})$ satisfies $||A||_{op} < C(R')$ uniformly of k, and C_A is a constant which satisfies $-\frac{1}{2\pi k}||A||_{op} \leq C_A \leq \frac{1}{2\pi k}||A||_{op}$, so that the average over X of both sides of (2.26) is 1.

Remark 2.4.12. Note that Lemmas 2.2.21 and 2.3.4 imply that A in fact lies in $\theta_*(\sqrt{-1}\mathfrak{z}) \leq \theta_*(\sqrt{-1}\mathfrak{k})$.

Remark 2.4.13. Recalling that the centre of mass $\bar{\mu}'_X$ with respect to the basis $\{s_i\}$ is given by $(\bar{\mu}'_X)_{ij} := k^n \int_X h^k_{FS(H(k))}(s_i, s_j) \frac{\omega^n_{H(k)}}{n!}$, we integrate both sides of the equation (2.26) to find $k^n V = \sum_{i,j} \left(I + C_A I - \frac{1}{2\pi k} A \right)_{ij} (\bar{\mu}'_X)_{ij}$. Noting $\mathrm{tr}(\bar{\mu}'_X) = k^n V$ which follows from (2.4), we thus get C_A explicitly as $C_A = \frac{1}{2\pi k^{n+1} V} \mathrm{tr}(A\bar{\mu}'_X)$.

By replacing $\{s_i\}$ by an H-unitarily equivalent basis if necessary, we may assume that $I+C_AI-\frac{A}{2\pi k}$ is diagonal: $I+C_AI-\frac{A}{2\pi k}=\mathrm{diag}(a_1,\dots,a_N)$ with each $a_i\in\mathbb{R}$ satisfying $a_i=1+O(1/k)$, by Corollary 2.4.11, thus $a_i>0$ for $k\gg 1$. This implies

$$\bar{\rho}_k(\omega_H)=\sum_i a_i|s_i|^2_{FS(H)^k}=\sum_i|\sqrt{a_i}s_i|^2_{FS(H)^k}. \tag{2.27}$$

Writing H' for the hermitian form $\int_X h^k_{FS(H)}(,)\frac{\omega_H^n}{n!}$ on $H^0(X,L^k)$ and $\{s'_i\}$ for an H'-orthonormal basis, we thus have

$$\bar{\rho}_k(\omega_H)=\frac{V}{N}\sum_i|s'_i|^2_{FS(H)^k}=\sum_i|\sqrt{a_i}s_i|^2_{FS(H)^k},$$

where the first equality is the definition of $\bar{\rho}_k$ (cf. Definition 2.2.10) and the second equality is provided by (2.27). This means that the basis $\{\sqrt{a_i}s_i\}$ must be H'-unitarily equivalent to $\{\sqrt{V/N}s'_i\}$ by the following lemma.

Lemma 2.4.14. *Suppose that we write H' for the hermitian form $\int_X h^k_{FS(H)}(,)\frac{\omega^n_{FS(H)}}{n!}$ on $H^0(X,L^k)$ and that $\{s'_i\}$ is a H'-orthonormal basis. If we have $\rho_k(\omega_H)=\sum_i|s'_i|^2_{FS(H)^k}=\sum_i|\tilde{s}_i|^2_{FS(H)^k}$ for another basis $\{\tilde{s}_i\}$, then $\{\tilde{s}_i\}$ is H'-unitarily equivalent to $\{s'_i\}$.*

Proof. We now write $h_{FS(H)}=e^{\phi}h_{FS(H')}$ for some $\phi\in C^\infty(X,\mathbb{R})$. Multiplying both sides of the equation $\sum_i|s'_i|^2_{FS(H)^k}=\sum_i|\tilde{s}_i|^2_{FS(H)^k}$ by $e^{-k\phi}$, we get $1=\sum_i|s'_i|^2_{FS(H')^k}=\sum_i|\tilde{s}_i|^2_{FS(H')^k}$ since $\{s'_i\}$ is an H'-orthonormal basis (cf. the equation (2.4)). Since FS is injective (Lemma 2.2.9), this means that $\{\tilde{s}_i\}$ must be H'-unitarily equivalent to $\{s'_i\}$. □

We thus obtain the following (cf. Remark 2.4.4).

Proposition 2.4.15. *Suppose that we have a sequence $\{H(k)\}_k$ with $H(k)\in\mathscr{B}^K_k$, each of which satisfies $\mathscr{D}^*_{H(k)}\mathscr{D}_{H(k)}\bar{\rho}_k(\omega_{H(k)})=0$ and $\sup_X|k\omega_{H(k)}-k\omega|_\omega<R'$ uniformly of k. Then, writing $\{s_i\}$ for an $H(k)$-orthonormal basis and $A:=\theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_{H(k)}))\in\theta_*(\sqrt{-1}\mathfrak{k})$, the basis $\{s'_i\}$ defined by*

$$s'_i:=\sqrt{\frac{N}{V}}\left(I+C_AI-\frac{A}{2\pi k}\right)^{1/2}_{ij}s_j, \tag{2.28}$$

is a $\int_X h^k_{FS(H(k))}(,)\frac{\omega^n_{H(k)}}{n!}$-orthonormal basis, where C_A is some constant of order $1/k$

and $\left(I+C_AI-\frac{A}{2\pi k}\right)_{ij}$ *is the matrix for* $I+C_AI-\frac{A}{2\pi k}$ *represented with respect to* $\{s_i\}$.

Conversely, if the basis $\{s_i'\}$ *as defined in (2.28) is a* $\int_X h^k_{FS(H(k))}(,)\frac{\omega^n_{H(k)}}{n!}$*-orthonormal basis, then* $H(k)$ *satisfies* $\mathscr{D}^*_{H(k)}\mathscr{D}_{H(k)}\bar{\rho}_k(\omega_{H(k)})=0$.

In particular, we get the following results (cf. Remark 2.4.12) that improve Proposition 2.4.5.

Corollary 2.4.16.

1. *Suppose that we have a sequence* $\{H(k)\}_k$ *with* $H(k)\in\mathscr{B}_k^K$*, each of which satisfies* $\mathscr{D}^*_{H(k)}\mathscr{D}_{H(k)}\bar{\rho}_k(\omega_{H(k)})=0$ *and* $\sup_X|k\omega_{H(k)}-k\omega|_\omega<R'$ *for some constant* $R'>0$ *uniformly of k. Then there exists* $g\in SL(N,\mathbb{C})$ *such that*

$$\bar{\mu}_X(g)=\frac{Vk^n}{N}\left(I+C_AI-\frac{A}{2\pi k}\right)^{-1}, \tag{2.29}$$

   where $A:=\theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_{H(k)}))\in\theta_*(\sqrt{-1}\mathfrak{z})$ *satisfies* $||A||_{op}<C(R')$ *uniformly of k and* $C_A\in\mathbb{R}$ *is some constant which satisfies* $C_A=\frac{1}{2\pi k^{n+1}V}\mathrm{tr}(A\bar{\mu}_X(g))$.

2. *Conversely, if there exist a basis* $\{s_i'\}$ *for* $H^0(X,L^k)$ *defining a* $\theta(K)$*-invariant hermitian form* $H(k)\in\mathscr{B}_k^K$, $A\in\theta_*(\sqrt{-1}\mathfrak{z})$*, and some constant* C_A*, which satisfy* $\bar{\mu}_X'=\frac{Vk^n}{N}\left(I+C_AI-\frac{A}{2\pi k}\right)^{-1}$*, then* $H(k)\in\mathscr{B}_k^K$ *satisfies*

$$\mathscr{D}^*_{H(k)}\mathscr{D}_{H(k)}\bar{\rho}_k(\omega_{H(k)})=0$$

   with $\theta_*(-\mathrm{grad}\bar{\rho}_k(\omega_{H(k)}))=A$.

Remark 2.4.17. Suppose that we have $\bar{\mu}_X(g)=\frac{Vk^n}{N}(I+C_AI-\frac{A}{2\pi k})^{-1}$ for some constant C_A. Then multiplying both sides by $\bar{\mu}_X(g)^{-1}$ and taking the inverse, we have $\frac{Vk^n}{N}I=\bar{\mu}_X(g)+C_A\bar{\mu}_X(g)-\frac{\bar{\mu}_X(g)A}{2\pi k}$, and hence by taking the trace, we have $C_A=\frac{1}{2\pi k^{n+1}V}\mathrm{tr}(A\bar{\mu}_X(g))$, by noting $\mathrm{tr}(\bar{\mu}_X(g))=k^nV$ for any g. Thus, recalling Remark 2.4.13, C_A for which the trace is consistent in (2.29) is the same as the one for which the averages are consistent in (2.26).

The appearance of the inverse on the right hand side of (2.29) may look surprising, but this essentially comes from the one in Lemma 2.2.20; see also Proposition 2.4.5. On the other hand, this inverse will be the essential obstruction for proving the relative asymptotic Chow polystability of (X,L) admitting an extremal metric (cf. Conjecture 2.6.16). We shall discuss some stability notions that are appropriate to (X,L) with an extremal metric in §2.6, and the reader is in particular referred to §2.6.4 for more discussions on this issue, and also to Question 2.6.15.

2.4.2 Approximate solutions to $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$ in terms of the centre of mass

Now take the approximately ρ-balanced metric $\omega_{(m)}$, as obtained in Corollary 2.3.15, which satisfies (2.18). By Lemma 2.4.3, we have

$$\bar{\rho}_k(\omega_{(m)}) + \frac{F_{m,k}}{4\pi k^{m+2}} = -\frac{1}{2\pi k}\sum_{ij} A_{ij} h^k_{(m)}(s_i,s_j) + \mathrm{const} \tag{2.30}$$

where $A := \theta_*(Jv_{(m)})$ and we recall that $h_{(m)}$ is of the form $FS(H)$ and that $\{s_i\}$ in the above formula is an H-orthonormal basis. Noting that $\omega_{(m)}$ satisfies (for all large enough k)

$$\sup_X |k\omega_{(m)} - k\omega|_{\omega} < \mathrm{const.}|\partial\bar{\partial}\phi_1|_{\omega} < R', \tag{2.31}$$

say, and recalling Remark 2.3.16, we find that $||A||_{op} < C(R')$ by Lemma 2.4.9 (by taking ω_0 to be the extremal metric ω).

Remark 2.4.18. In this section, the Hilbert–Schmidt norm $||\cdot||_{HS}$ will be with respect to H which defines $\omega_{(m)}$ by ω_H, as obtained in Corollary 2.3.15.

Suppose that we write P for the change of basis matrix from $\{s_i\}$ to a $\int_X h^k_{FS(H)}(,)\frac{\omega_H^n}{n!}$-orthonormal basis $\{s'_i\}$, so that we have $\bar{\mu}'_X = k^n(P^*P)^{-1}$, where $\bar{\mu}'_X$ is the centre of mass defined with respect to the basis $\{s_i\}$ (cf. the proof of Lemma 2.2.20).

Now re-write the equation (2.30) as

$$\sum_{i,j}\left(\frac{V}{N}P^*P - C_0 I + \frac{A}{2\pi k}\right)_{ij} h^k_{(m)}(s_i,s_j) = 1 - \frac{F_{m,k}}{4\pi k^{m+2}},$$

where the constant C_0 can be determined by taking the average of both sides; namely C_0 can be determined by the equation $1 - \frac{1}{V}\sum_{i,j}\left(C_0 I - \frac{A}{2\pi k}\right)_{ij}\int_X h^k_{(m)}(s_i,s_j)\frac{\omega_{(m)}}{n!} = 1 +$

$O(k^{-m-2})$. Arguing as in (2.22), we see that C_0 can be estimated as

$$|C_0| \leq \frac{||A||_{op}+1}{2\pi k} \tag{2.32}$$

for all sufficiently large k, which is of order $1/k$ since $||A||_{op}$ is uniformly bounded by (2.31). We thus have, by noting $\sum_i |s_i|^2_{h^k_{(m)}} = 1$,

$$\sum_{i,j} \left(\frac{V}{N} P^* P - C_0 I + \frac{A}{2\pi k} \right)_{ij} h^k_{(m)}(s_i, s_j) - \left(1 - \frac{F_{m,k}}{4\pi k^{m+2}} \right) \sum_i |s_i|^2_{h^k_{(m)}} = 0, \tag{2.33}$$

with some constant C_0 that is of order $1/k$. Since $\frac{V}{N} P^* P - C_0 I + \frac{A}{2\pi k}$ is a hermitian matrix, we can replace $\{s_i\}$ by an H-unitarily equivalent basis so that $\frac{V}{N} P^* P - C_0 I + \frac{A}{2\pi k} = \mathrm{diag}(d_1, \dots, d_N)$, with $d_i \in \mathbb{R}$, with respect to the basis $\{s_i\}$. We can thus re-write the equation (2.33) as

$$\sum_i \left(d_i - \left(1 - \frac{F_{m,k}}{4\pi k^{m+2}} \right) \right) |s_i|^2_{h^k_{(m)}} = 0.$$

Hence, arguing exactly as we did in the proof of Lemma 2.2.9, we find

$$|d_i - 1| \leq \frac{1}{2\pi} N^2 \sup_X |F_{m,k}| k^{-m-2} = O(k^{2n-m-2}),$$

by recalling that $\sup_X |F_k|$ is bounded uniformly of k (see the discussion preceding the equation (2.18)).

We thus see that there exists a hermitian matrix E with $||E||_{op} = O(k^{2n-m-2})$ such that $\frac{V}{N} P^* P = I + C_0 I - \frac{A}{2\pi k} + E$, or

$$\bar{\mu}_X = k^n (P^* P)^{-1} = \frac{V k^n}{N} \left(I + C_0 I - \frac{A}{2\pi k} + E \right)^{-1}.$$

Define

$$E' := \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1} E,$$

which has $||E'||_{op} = O(k^{2n-m-2})$ by $||A||_{op} < C(R')$ and (2.32). We may take m and k

to be large enough so that $||E'||_{op} < 1/2$, say. We thus have

$$(\bar{\mu}'_X)_{ij} = \int_X h^k_{(m)}(s_i, s_j) \frac{(k\omega_{(m)})^n}{n!} = \frac{Vk^n}{N} \left[(I - E' + (E')^2 + \cdots) \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1} \right]_{ij}$$
$$= \frac{Vk^n}{N} \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1}_{ij} + (E'')_{ij}$$

where $E'' \in T_H \mathscr{B}_k^K$ is a hermitian matrix defined by

$$E'' := \frac{Vk^n}{N} \left(-E' + (E')^2 + \cdots \right) \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1}$$

which satisfies $||E''||_{op} = O(k^{2n-m-2})$ (by $||A||_{op} < C(R')$ and (2.32)). Since m could be any positive integer, and recalling $||E''||_{HS} = \mathrm{tr}(E''E'') \leq \sqrt{N}||E''||_{op}$, we may replace m by $m + 2n + n/2$ so as to have $||E''||_{HS} = O(k^{-m})$ (for notational convenience).

We now show that by perturbing C_0 slightly, we can assume that $\mathrm{tr}(E'') = 0$. More precisely, we have the following.

Lemma 2.4.19. *Suppose $||E''||_{HS} = O(k^{-m})$, $||A||_{op} \leq \mathrm{const}$, and $C_0 = O(1/k)$. Then there exists a constant $\delta \in \mathbb{R}$ with $|\delta| < 4N^{-1/2}||E''||_{HS} = O(k^{-m-n/2})$ such that*

$$\frac{Vk^n}{N} \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1} + E'' - \frac{Vk^n}{N} \left(I + (C_0 + \delta) I - \frac{A}{2\pi k} \right)^{-1}$$

is a trace free hermitian endomorphism which has the Hilbert-Schmidt norm of order k^{-m}; more precisely, we have

$$\left|\left| \frac{Vk^n}{N} \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1} + E'' - \frac{Vk^n}{N} \left(I + (C_0 + \delta) I - \frac{A}{2\pi k} \right)^{-1} \right|\right|_{HS}$$
$$\leq \left(1 + 8 \left|\left| \left(I + C_0 I - \frac{A}{2\pi k} \right)^{-1} \right|\right|_{op} \right) ||E''||_{HS} = O(k^{-m}). \quad (2.34)$$

Proof. We show that the map $U : \mathbb{R} \to \mathbb{R}$ defined by

$$U(\delta) := \mathrm{tr} \left(\left(I + (C_0 + \delta) I - \frac{A}{2\pi k} \right)^{-1} \right)$$

is a local diffeomorphism with a particular lower bound on its linearisation. Writing

$\left(I+C_0I-\frac{A}{2\pi k}\right)=\mathrm{diag}(a_1,\dots,a_N)$ by unitarily diagonalising it, where $a_i=1+O(1/k)$ by $||A||_{op}<C(R')$ and (2.32), we have $U(\delta)=\sum_i(a_i+\delta)^{-1}=\sum_i a_i^{-1}(1+\delta/a_i)^{-1}$, whose linearisation at 0 is $DU|_0(\delta)=-\delta\sum_i a_i^{-2}$.

Since $a_i=1+O(1/k)$ implies $|DU|_0|>N/2$ if k is sufficiently large, we see that U is indeed a local diffeomorphism whose linearisation can be bounded from below by $N/2$.

Thus, by using the quantitative version of the inverse function theorem (see e.g. Theorem 5.3 in [50]), we can show that there exists some $\delta\in\mathbb{R}$ so that we have

$$\mathrm{tr}\left(\frac{Vk^n}{N}\left(I+(C_0+\delta)I-\frac{A}{2\pi k}\right)^{-1}\right)=\mathrm{tr}\left(\frac{Vk^n}{N}\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right)+\mathrm{tr}(E''),$$

which satisfies

$$|\delta|<4\frac{|\mathrm{tr}(E'')|}{N}\leq 4N^{-1/2}||E''||_{HS}=O(k^{-m-\frac{n}{2}}) \tag{2.35}$$

since $|\mathrm{tr}(E'')|\leq\sqrt{N}||E''||_{HS}$.

We now estimate the Hilbert–Schmidt norm of the trace free hermitian matrix

$$\begin{aligned}&\frac{Vk^n}{N}\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}+E''-\frac{Vk^n}{N}\left(I+(C_0+\delta)I-\frac{A}{2\pi k}\right)^{-1}\\&=E''+\frac{Vk^n}{N}\delta\left(I+C_0I-\frac{A}{2\pi k}\right)^{-2}-\frac{Vk^n}{N}\delta^2\left(I+C_0I-\frac{A}{2\pi k}\right)^{-3}+\cdots.\end{aligned}$$

Recalling $\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\leq$ const independently of k, we find

$$\left|\left|\delta\left(I+C_0I-\frac{A}{2\pi k}\right)^{-2}-\frac{Vk^n}{N}\delta^2\left(I+C_0I-\frac{A}{2\pi k}\right)^{-3}+\cdots\right|\right|_{op}=O(k^{-m-\frac{n}{2}})$$

for all sufficiently large k, and hence has Hilbert–Schmidt norm of order k^{-m}. Recalling

$||E''||_{HS} = O(k^{-m})$, we finally see

$$\begin{aligned}
&\left|\left|\frac{Vk^n}{N}\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}+E''-\frac{Vk^n}{N}\left(I+(C_0+\delta)I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{HS} \\
&\leq ||E''||_{HS}+\sqrt{N}\left|\left|\delta\left(I+C_0I-\frac{A}{2\pi k}\right)^{-2}-\frac{Vk^n}{N}\delta^2\left(I+C_0I-\frac{A}{2\pi k}\right)^{-3}+\cdots\right|\right|_{op} \\
&\leq ||E''||_{HS}+2\sqrt{N}\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}|\delta| \\
&\leq \left(1+8\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)||E''||_{HS}=O(k^{-m}),
\end{aligned}$$

where we used $||\cdot||_{HS} \leq \sqrt{N}||\cdot||_{op}$ in the first inequality and (2.35) in the last inequality.

□

Summarising the argument above, we obtain the following.

Corollary 2.4.20. *For any $m \geq 1$ and any large enough $k \gg 1$ there exists a $\theta(K)$-invariant hermitian form $H = H_m(k) \in \mathscr{B}_k^K$ and a traceless hermitian $\theta(K)$-invariant endomorphism $\tilde{E} = \tilde{E}_m(k)$ on $H^0(X,L^k)$ which satisfy the following: there exists an element $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ with $||A||_{op} < \mathrm{const}$ uniformly of k and a constant $C_0 \in \mathbb{R}$ which is of order $1/k$ such that the equation*

$$(\bar{\mu}'_X)_{ij} = \int_X h^k_{FS(H)}(s_i,s_j)\frac{(k\omega_{FS(H)})^n}{n!} = \frac{Vk^n}{N}\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}_{ij}+(\tilde{E})_{ij}$$

holds with respect to an H-orthonormal basis $\{s_i\}$, with $||\tilde{E}||_{HS} \leq \mathrm{const}.k^{-m}$ where the Hilbert–Schmidt norm $||\cdot||_{HS}$ is defined with respect to H.

Remark 2.4.21. Since H is $\theta(K)$-invariant, $\bar{\mu}'_X$ is $\theta(K)$-invariant. This means that $\tilde{E}$ is $\theta(K)$-invariant and hermitian, since $\bar{\mu}'_X - \frac{Vk^n}{N}(I+C_0I-\frac{A}{2\pi k})^{-1}$ is.

Henceforth we write H_0 for $H_m(k)$ above, and $\tilde{E}_0$ for $\tilde{E}_m(k)$ above.

2.5 Gradient flow

2.5.1 Modified balancing energy $\mathscr{Z}^A$

Recall first of all that in the cscK case, i.e. when $\mathrm{Aut}_0(X,L)$ is trivial, balanced metrics are precisely the critical points of a functional $\mathscr{Z} : \mathscr{B}_k \to \mathbb{R}$ called the **bal-**

ancing energy, defined for a geodesic $\{H(t)\}$ in $\mathscr{B}_k$, where $H(t) = e^{tB}H(0)$ with $B \in T_{H(0)}\mathscr{B}_k \cong \mathrm{Herm}(H^0(X,L^k))$, as

$$\mathscr{Z}(H(t)) := I \circ FS(H(t)) + \frac{Vk^n}{N}\mathrm{tr}(\log H(t)).$$

In the above, $I : \mathscr{H}(X,L) \to \mathbb{R}$ is defined for a path $\{e^{\phi_t}h\}$ in $\mathscr{H}(X,L)$ by

$$I(e^{\phi_t}h) := -k^{n+1}\int_X \phi_t \sum_{i=1}^{n}(\omega_h - \sqrt{-1}\partial\bar{\partial}\phi_t)^i \wedge \omega_h^{n-i},$$

where h is some reference metric, and changing the reference metric h will only result in an overall additive constant.

The original argument for finding the balanced metric (in the cscK case) in [40] was to find an approximately balanced metric, which is very close to attaining the minimum of $\mathscr{Z}$, and then perturb it to a genuinely balanced metric (i.e. the minimum of $\mathscr{Z}$) by driving it along the gradient flow of $\mathscr{Z}$ to attain the global minimum. The reader is referred to [40] for the details. The crucial point is that $\mathscr{Z}$ is *convex* along geodesics in $\mathscr{B}_k$ (with respect to the bi-invariant metric), as we recall in Theorem 2.5.3.

We now consider the following functional, which is more appropriate for our purpose of finding ρ-balanced metrics.

Definition 2.5.1. We define a functional $\mathscr{Z}^A : \mathscr{B}_k^K \to \mathbb{R}$ by

$$\mathscr{Z}^A(H(t)) := I \circ FS(H(t)) + \frac{Vk^n}{N}\mathrm{tr}\left(\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}\log H(t)\right),$$

for some fixed $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ and some fixed constant $C_A \in \mathbb{R}$. We call $\mathscr{Z}^A$ the **modified balancing energy**.

Remark 2.5.2. Note that the Hessian[7] of $\mathscr{Z}$ is equal to the Hessian of $\mathscr{Z}^A$, since their difference

$$\mathscr{Z}(H(t)) - \mathscr{Z}^A(H(t)) = \frac{Vk^n}{N}\mathrm{tr}(\log H(t)) - \frac{Vk^n}{N}\mathrm{tr}\left(\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}\log H(t)\right),$$

with $H(t) = e^{tB}H(0)$, is linear in t. Thus, we see that $\mathscr{Z}^A$ is *convex* along geodesics in

[7] More precisely, the Hessian of $\mathscr{Z}|_{\mathscr{B}_k^K}$.

$\mathscr{B}_k^K$ (cf. Theorem 2.5.3).

Similarly to the usual balanced case (cf. Lemma 3 of [43]), the first variation of $\mathscr{Z}^A$ can be computed as follows

$$\delta \mathscr{Z}^A(H(t)) = -\int_X h^k_{FS(H(t))}(s_i^{H(t)}, s_j^{H(t)}) \frac{k^n \omega^n_{FS(H(t))}}{n!} + \frac{Vk^n}{N}\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}_{ij},$$

where $\{s_i^{H(t)}\}$ is an $H(t)$-orthonormal basis and, $\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}_{ij}$ in the above is the hermitian endomorphism $\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}$ represented with respect to $\{s_i^{H(t)}\}$. This implies that $\delta \mathscr{Z}^A(H(t)) = 0$ if and only if $\{s_i^{H(t)}\}$ defines an embedding with $\bar{\mu}'_X = \frac{Vk^n}{N}\left(I + C_A I - \frac{A}{2\pi k}\right)^{-1}$. Summarising the discussion above, the solution of the equation $\bar{\mu}'_X = \frac{Vk^n}{N}(I + C_A I - \frac{A}{2\pi k})^{-1}$ can be characterised as the critical point of the functional $\mathscr{Z}^A$, which is convex along geodesics in $\mathscr{B}_k^K$.

2.5.2 Hessian of the balancing energy

We now recall the Hessian of the (usual) balancing energy $\mathscr{Z}$, following the exposition given in [51, 52]. Fixing $H(t) \in \mathscr{B}_k^K$ for the moment, consider now the orthogonal decomposition $\iota^* T\mathbb{P}^{N-1} = TX \oplus \mathscr{N}_t$ (as C^∞-vector bundles on X) with respect to the Fubini–Study metric $\omega_{\widetilde{FS}(H(t))}$ (or more precisely $\iota^*\omega_{\widetilde{FS}(H(t))}$) on $\mathbb{P}^{N-1}$ induced from $H(t)$ (cf. p703, [96]). Given a hermitian endomorphism $\xi \in T_{H(t)}\mathscr{B}_k^K \cong \mathrm{Herm}(H^0(X, L^k))^K$, we write X_ξ for the corresponding holomorphic vector field on $\mathbb{P}^{N-1}$. Write $\pi_{\mathscr{N}_t}(X_\xi)$ for the projection of X_ξ on the $\mathscr{N}_t$-factor in $\iota^* T\mathbb{P}^{N-1} = TX \oplus \mathscr{N}_t$, and $\pi_T(X_\xi)$ for the one on the TX-factor. We thus get a map $P : T_{H(t)}\mathscr{B}_k^K \to C^\infty(\mathscr{N}_t)$ defined by $P(\xi) := \pi_{\mathscr{N}_t}(X_\xi)$. Write P^* for the adjoint of P defined with respect to the inner product $\mathrm{tr}(\xi_1\xi_2)$ on $T_{H(t)}\mathscr{B}_k^K$ and the L^2-metric defined by $\omega_{\widetilde{FS}(H(t))}$ on the fibres and $k\omega_{H(t)}$ on the base. Note that the inner product $\mathrm{tr}(\xi_1\xi_2)$ is nothing but the Hilbert–Schmidt inner product defined with respect to $H(t)$, since $\xi_1, \xi_2 \in T_{H(t)}\mathscr{B}_k^K$.

Theorem 2.5.3. (Lemma 17, Fine [51]) *Writing $P : T_{H(t)}\mathscr{B}_k^K \to C^\infty(\mathscr{N}_t)$ defined by $P(\xi) := \pi_{\mathscr{N}_t}(X_\xi)$, as above, we have* $\mathrm{Hess}(\mathscr{Z}(H(t))) = P^*P$. *In particular,*

$$\begin{aligned} \mathrm{tr}(\xi_1 \mathrm{Hess}(\mathscr{Z}(H(t))\xi_2) &= (\pi_{\mathscr{N}_t}(X_{\xi_1}), \pi_{\mathscr{N}_t}(X_{\xi_2}))_{L^2(t)} \\ &= \int_X \mathrm{Re}(\pi_{\mathscr{N}_t}(X_{\xi_1}), \pi_{\mathscr{N}_t}(X_{\xi_2}))_{\iota^*\omega_{\widetilde{FS}(H(t))}} \frac{k^n \omega^n_{H(t)}}{n!}. \end{aligned}$$

Remark 2.5.4. The "diagonal" elements of $\mathrm{Hess}(\mathscr{Z}(H(t)))$ are in fact computed by Phong and Sturm [95, 96] and implicitly by Donaldson [40]. We will later need to know some of the off-diagonal terms of $\mathrm{Hess}(\mathscr{Z}(H(t))) = \mathrm{Hess}(\mathscr{Z}^A(H(t)))$.

We now wish to estimate $||\pi_{\mathscr{N}_t}(X_\xi)||^2_{L^2(t)}$. This was done originally by Donaldson [40] and improved by Phong and Sturm [96] when the connected component $\mathrm{Aut}_0(X,L)$ of the automorphism group was trivial. In our situation we cannot assume this hypothesis, but we now invoke the following trick used by Mabuchi [84, 85]. Recall that, by Lemma 2.2.1, $\mathrm{Aut}_0(X,L)$ (with the Lie algebra $\mathfrak{g}$) is a subgroup of $SL(H^0(X,L^k))$ (with Lie algebra $\mathfrak{sl} = \mathfrak{sl}(H^0(X,L^k))$), and hence we have $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^\perp$, where $\mathfrak{g}_t^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{sl}$ with respect to the L^2-inner product defined by the Fubini–Study metric on $\mathbb{P}^{N-1}$ given by $H(t)$, i.e. with respect to the metric $(,)$ defined by $(\xi_1,\xi_2) := (X_{\xi_1}, X_{\xi_2})_{L^2(t)}$, where the L^2-product is defined by $\omega_{\widetilde{FS(H(t))}}$ on the fibres and $k\omega_{FS(H(t))}$ on the base, as we mentioned above. Note that this L^2-product does define a metric on $\mathfrak{sl}$ since X is not contained in any proper linear subspace of $\mathbb{P}^{N-1}$.

Writing $\xi = \alpha + \beta$ where $\alpha \in \mathfrak{g}$ and $\beta \in \mathfrak{g}_t^\perp$, we obviously have $\pi_{\mathscr{N}_t}(X_\xi) = \pi_{\mathscr{N}_t}(X_\beta)$. An intuitive idea is that, if $\xi \in \mathfrak{sl}$ is contained in the $\mathfrak{g}_t^\perp$-factor, we can apply the well-known estimate (Theorem 2.5.6) due to Donaldson, Phong–Sturm, and Fine, to get the lower bound of the eigenvalues of the Hessian of $\mathscr{Z}^A(H(t))$ (restricted to $\mathfrak{g}_t^\perp$) so that we can run the downward gradient flow on the space of positive definite K-invariant hermitian matrices $\mathscr{B}_k^K$ driven by $\mathrm{pr}_{\perp,t}(\delta\mathscr{Z}^A(H(t)))$; see §2.5.3 for the details.

We now recall the following notion from [40].

Definition 2.5.5. A metric $\tilde{\omega} \in kc_1(L)$ has R**-bounded geometry** if it satisfies the following conditions: fixing an integer $l \geq 4$ and a reference metric $\omega_0 \in c_1(L)$, $\tilde{\omega}$ satisfies $\tilde{\omega} > R^{-1}k\omega_0$ and $||\tilde{\omega} - k\omega_0||_{C^l,k\omega_0} < R$ where $||\cdot||_{C^l,k\omega_0}$ is the C^l-norm on the space of 2-forms defined with respect to the metric $k\omega_0$. The basis $\{s_i\}$ is said to have R**-bounded geometry** if the hermitian endomorphism $H(t)$ which has $\{s_i\}$ as its orthonormal basis has R-bounded geometry.

With these preparations, we can now state the following theorem (cf. Theorem 2 in [96]).

Theorem 2.5.6. (Donaldson [40], Phong–Sturm [96], Fine [52]) *Suppose that* $\mathrm{Aut}_0(X,L)$ *is trivial. Suppose also that we have a basis* $\{s_i\}$ *with respect to which* $\bar{\mu}'_X = D_k + E_k$*, where* D_k *is a scalar matrix with* $D_k \to I$ *as* $k \to \infty$*. For any* $R > 0$ *there exists a positive constant* C_R *depending only on* R *and* $\varepsilon < 1/10$ *such that, for any* k*, if the basis* $\{s_i\}$ *for* $H^0(X,L^k)$ *has* R*-bounded geometry and if* $||E_k||_{op} < \varepsilon$*, then*

$$||\pi_{\mathscr{N}_t}(X_\xi)||^2_{L^2(t)} > C_R k^{-2} ||\xi||^2_{HS(t)},$$

where the L^2*-metric* $||\cdot||_{L^2(t)}$ *on the vector fields on* X *is defined by the Fubini-Study metric of the hermitian form* $H(t)$ *which has* $\{s_i\}$ *as its orthonormal basis, and the Hilbert-Schmidt norm* $||\cdot||_{HS(t)}$ *is defined by the hermitian form* $H(t)$ *which has* $\{s_i\}$ *as its orthonormal basis.*

Remark 2.5.7. The hypothesis $\bar{\mu}'_X = D_k + E_k$ is satisfied when we have $\bar{\mu}'_X = \frac{Vk^n}{N}\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1} + \tilde{E}$ with $||\tilde{E}||_{HS} = O(k^{-m})$, as in Corollary 2.4.20, by noting that we can define $D_k := \frac{1}{N}\mathrm{tr}\left(\frac{Vk^n}{N}\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}\right) I$, which does converge to I as $k \to \infty$, and that the operator norm of $E_k := \bar{\mu}'_X - D_k$ is of order $1/k$, since $||A||_{op} <$ const and $C_0 = O(1/k)$ by (2.32).

We now recall the proof of this theorem, where we closely follow the exposition given in pp702-710 [96]. The theorem is a consequence of the following three estimates:

$$||\xi||^2_{HS(t)} \leq C'_R k ||X_\xi||^2_{L^2(t)} \tag{2.36}$$

$$||X_\xi||^2_{L^2(t)} = ||\pi_T(X_\xi)||^2_{L^2(t)} + ||\pi_{\mathscr{N}_t}(X_\xi)||^2_{L^2(t)} \tag{2.37}$$

$$C_R ||\pi_T(X_\xi)||^2_{L^2(t)} \leq k ||\pi_{\mathscr{N}_t}(X_\xi)||^2_{L^2(t)} \tag{2.38}$$

The second equality (2.37) is an obvious consequence of the orthogonal decomposition $\iota^* T\mathbb{P}^{N-1} = TX \oplus \mathscr{N}_t$ with respect to $\omega_{\widetilde{FS}(H(t))}$, and the first inequality (2.36) does not use the hypothesis that $\mathrm{Aut}_0(X,L)$ is trivial, and hence carries over word by word to the case when $\mathrm{Aut}_0(X,L)$ is not trivial.

The hypothesis of $\mathrm{Aut}_0(X,L)$ being trivial was crucially used in the third estimate (2.38), which relies on the following estimate ((5.12) in [96]) for an arbitrary smooth vector field W on X

$$||W||^2_{L^2(t)} \leq \text{const.}||\bar{\partial}(W)||^2_{L^2(t)} \tag{2.39}$$

which is true if and only if $\mathrm{Aut}(X)$ is discrete[8]. Phong–Sturm's argument was to apply this inequality to $W = \pi_T(X_\xi)$ and combine it with the estimate ((5.15) in [96])

$$||\pi_{\mathcal{N}_t}(\tilde{V})||^2_{L^2(t)} \geq C_R||\bar{\partial}(\pi_{\mathcal{N}_t}(\tilde{V}))||^2_{L^2(t)}$$

which holds for any holomorphic vector field $\tilde{V}$ on $\mathbb{P}^{N-1}$ (which we take to be X_ξ), irrespective of whether $\mathrm{Aut}(X)$ is discrete or not. Observe that $\bar{\partial}\tilde{V} = 0 = \bar{\partial}(\pi_T(\tilde{V})) + \bar{\partial}(\pi_{\mathcal{N}_t}(\tilde{V}))$ implies $c_R||\bar{\partial}(\pi_T(\tilde{V}))||_{L^2(t)} \leq ||\pi_{\mathcal{N}_t}(\tilde{V})||_{L^2(t)}$. Thus, by applying this and the estimate (2.39) applied to $W = \pi_T(X_\xi)$, we get (2.38).

Thus, the only hindrance to extending Phong–Sturm's theorem to the case where $\mathrm{Aut}_0(X,L)$ is not trivial is the lack of (2.39), which is substantial. However, the decomposition $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^\perp$ means that the estimate (2.39) holds for the (smooth) vector fields of the form $\pi_T(X_\beta)$ where $\beta \in \mathfrak{g}^\perp$, since the elements $\alpha \in \mathfrak{g}$ are precisely the ones that generate X_α with $\bar{\partial}(\pi_T(X_\alpha)) = 0$, i.e. the kernel $\ker\bar{\partial}$ is precisely the image $\{X_\alpha | \alpha \in \mathfrak{g}\}$ of $\mathfrak{g}$. Since the image $\{X_\beta | \beta \in \mathfrak{g}_t^\perp\}$ of $\mathfrak{g}_t^\perp$ is precisely the L^2-orthogonal complement of $\ker\bar{\partial}$ in $\mathfrak{sl}$, recalling that $\mathfrak{g}_t^\perp$ is defined as an orthogonal complement of $\mathfrak{g}$ with respect to the L^2 metric induced from $\omega_{\widetilde{FS}(H(t))}$, $\bar{\partial}$ is invertible on the set of vector fields $\pi_T(X_\beta)$ with $\beta \in \mathfrak{g}_t^\perp$, with the estimate (2.39).

Thus we have the following estimate.

Lemma 2.5.8. (cf. Mabuchi; p235 in [84], p130 in [85]) *Suppose that we have the same hypotheses as in Theorem 2.5.6, apart from that* $\mathrm{Aut}_0(X,L)$ *is no longer trivial. We have*

$$||\pi_{\mathcal{N}_t}(X_\beta)||^2_{L^2(t)} \geq C_R k^{-2}||\beta||^2_{HS(t)} \tag{2.40}$$

for any $\beta \in \mathfrak{g}_t^\perp$.

2.5.3 Gradient flow

Let H_0 be the approximately ρ-balanced matrix as obtained in Corollary 2.4.20. We now aim to perturb this matrix to a genuinely ρ-balanced one by using a geometric

[8] It is possible to modify the argument for the case $\mathrm{Aut}(X)$ being discrete to the case where we only know $\mathrm{Aut}(X,L)$ is discrete, as done by Phong and Sturm [96]

flow on a finite dimensional manifold $\mathscr{B}_k^K$. In this section, we show that such flow does converge, but also show that $\mathrm{Aut}_0(X,L)$ being nontrivial implies that the limit of the flow is not quite the (genuine) ρ-balanced metric that we seek (cf. Proposition 2.5.13); it will be obtained in Proposition 2.5.15, §2.5.4, by an iterative construction.

Recall the decomposition $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^{\perp}$ with respect to $H(t) \in \mathscr{B}_k^K$, as introduced in §2.5.2. Suppose that we write $\mathrm{pr}_{\mathfrak{g}} : \mathfrak{sl} \twoheadrightarrow \mathfrak{g}$ for the projection onto $\mathfrak{g}$ and $\mathrm{pr}_{\perp,t} : \mathfrak{sl} \twoheadrightarrow \mathfrak{g}_t^{\perp}$ for the projection onto $\mathfrak{g}_t^{\perp}$. We consider the following ODE

$$\frac{dH(t)}{dt} = -\mathrm{pr}_{\perp,t}\left(\delta \mathscr{Z}^A(H(t))\right) \tag{2.41}$$

on the finite dimensional symmetric space $\mathscr{B}_k^K$, with the initial condition $H(0) = H_0$. This is well-defined, since at $t = 0$, $\delta \mathscr{Z}^A(H_0)$ is K-invariant and hermitian by Corollary 2.4.20, and hence $\mathrm{pr}_{\perp,t}\left(\delta \mathscr{Z}^A(H(0))\right)$ is indeed K-invariant (since K acts on $\mathfrak{g}$ and hence preserves $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^{\perp}$, by noting that the orthogonality is defined by a K-invariant metric $\widetilde{FS}(H(t))$) and hermitian, defining a vector in $T_{H_0}\mathscr{B}_k^K$. By exactly the same argument, along the flow (2.41), $\mathrm{pr}_{\perp,t}\left(\delta \mathscr{Z}^A(H(t))\right)$ remains K-invariant and hermitian for $t > 0$ since $H(t) \in \mathscr{B}_k^K$.

Moreover, we can multiply the right hand side of the equation (2.41) by a cutoff function that is supported on a compact neighbourhood of radius 1 around H_0 without changing the flow; this will be justified in (2.44) and (2.45), as they state that the the flow is contained in this neighbourhood for all time if we start from H_0. Then the vector field on the right hand side of (2.41) is compactly supported, and the flow can be extended indefinitely by the standard ODE theory, i.e. the solution to (2.41) exists for all time.

Note

$$\frac{d}{dt}\left(\delta \mathscr{Z}^A(H(t))\right) = \mathrm{Hess}(\mathscr{Z}^A(H(t))) \cdot \frac{dH(t)}{dt} = -\mathrm{Hess}(\mathscr{Z}^A(H(t))) \cdot \mathrm{pr}_{\perp,t}\left(\delta \mathscr{Z}^A(H(t))\right).$$

and recall that the Hessian of $\mathscr{Z}^A$ is exactly the same as that of $\mathscr{Z}$, the usual balancing energy (cf. Remark 2.5.2), and that the Hessian of $\mathscr{Z}$ is degenerate along the $\mathfrak{g}$-direction, as we saw in Theorem 2.5.3. This means that we have a block diagonal decomposition of $\mathrm{Hess}(\mathscr{Z}^A(H(t)))$ as

$$\mathrm{Hess}(\mathscr{Z}^A(H(t))) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{P}_t \end{pmatrix},$$

according to the decomposition $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^{\perp}$, where $\tilde{P}_t$ is a positive definite matrix whose lowest eigenvalue can be estimated as in (2.40). In particular, we obtain the following.

Lemma 2.5.9.

$$\frac{d}{dt}\mathrm{pr}_{\mathfrak{g}}\left(\delta \mathscr{Z}^A(H(t))\right) = 0$$

along the flow $H(t)$ defined by (2.41).

Suppose that we write $\mathscr{G}^A(t)$ for $\mathrm{pr}_{\perp,t}\left(\delta \mathscr{Z}^A(H(t))\right)$ in order to simplify the notation. We then have

$$\frac{1}{2}\frac{d}{dt}||\mathscr{G}^A(t)||^2_{HS(t)} = \frac{1}{2}\frac{d}{dt}\mathrm{tr}(\mathscr{G}^A(t)\mathscr{G}^A(t)) = -\mathrm{tr}\left(\mathscr{G}^A(t)\cdot \mathrm{Hess}(\mathscr{Z}^A(H(t)))\cdot \mathscr{G}^A(t)\right),$$

by recalling that $\mathrm{tr}(\mathscr{G}^A(t)\mathscr{G}^A(t))$ is equal to $||\mathscr{G}^A(t)||^2_{HS(t)}$. Recall (cf. Remark 2.5.2) that $\mathscr{Z}^A(H(t))$ is convex along geodesics for all t. Thus, the above equation means that, along the flow, $||\mathscr{G}^A(t)||_{HS(t)}$ is monotonically decreasing. Combined with Lemma 2.5.9 and Lemma 2.5.14 to be proved later, this means that the hypotheses in Theorem 2.5.6 are always satisfied along the flow. Thus we can apply the estimate given by Theorem 2.5.6 along the flow for all $t > 0$. Theorem 2.5.3 and the estimate (2.40) imply that we have

$$\frac{1}{2}\frac{d}{dt}||\mathscr{G}^A(t)||^2_{HS(t)} \leq -\lambda_1 ||\mathscr{G}^A(t)||^2_{HS(t)},$$

where we wrote

$$\lambda_1 := C_R k^{-2} > 0 \tag{2.42}$$

for the lowest eigenvalue of $\mathrm{Hess}(\mathscr{Z}^A(H(t)))$ restricted to $\mathfrak{g}_t^{\perp}$, as estimated in (2.40). It easily follows that we have

$$||\mathscr{G}^A(t)||_{HS(t)} \leq e^{-\lambda_1 t}||\mathscr{G}^A(0)||_{HS(0)}. \tag{2.43}$$

We now evaluate the length of the path $\{H(t)\}$ with respect to the bi-invariant metric. Namely, we compute $\mathrm{dist}(H(t_1),H(t_2)) := \int_{t_1}^{t_2} ||H'(s)||_{HS(s)} ds$ for $t_1 > t_2$. Observe first of all that

$$\begin{aligned} \int_{t_1}^{t_2} ||H'(s)||_{HS(s)} ds &= \int_{t_1}^{t_2} ||\mathscr{G}^A(s)||_{HS(s)} ds \\ &\leq \frac{1}{\lambda_1}(e^{-\lambda_1 t_1} - e^{-\lambda_1 t_2})||\mathscr{G}^A(0)||_{HS(0)}, \end{aligned} \tag{2.44}$$

where we used $H'(t) = -\mathscr{G}^A(t)$, which is just (2.41), and the estimate (2.43). Thus, given an increasing sequence $\{t_i\}_i$ of positive real numbers, we see that the sequence $\{H(t_i)\}_i$ is Cauchy in $\mathscr{B}_k^K$ with respect to the bi-invariant metric. Thus the limit exists in $\mathscr{B}_k^K$, and the distance from the initial metric H_0 to the limit can be estimated as

$$\begin{aligned} \mathrm{dist}(H(\infty),H(0)) = \int_0^\infty ||H'(s)||_{HS(s)} ds &= \int_0^\infty ||\mathscr{G}^A(s)||_{HS(s)} ds \\ &\leq \frac{1}{\lambda_1}||\mathscr{G}^A(0)||_{HS(0)} = O(k^{-m+2}). \end{aligned} \tag{2.45}$$

Remark 2.5.10. Observe that (2.45) implies that we can write $H(\infty) = e^{\xi} H(0)$ with $\xi \in T_{H(0)}\mathscr{B}_k^K$ satisfying $||\xi||_{HS(0)} \leq ||\mathscr{G}^A(0)||_{HS(0)}/\lambda_1 = O(k^{-m+2})$. We thus get

$$\frac{1}{2}||\cdot||_{HS(0)} \leq ||\cdot||_{HS(\infty)} \leq 2||\cdot||_{HS(0)} \tag{2.46}$$

for all large enough k.

In particular, since the limit $H(\infty)$ exists, we get $\lim_{t\to\infty}\mathscr{G}^A(t) = 0$ from (2.43). Thus, combined with Lemma 2.5.9, we get the following.

Lemma 2.5.11. *The limit $H_1 := H(\infty)$ of the gradient flow (2.41) exists and satisfies $\mathrm{pr}_{\perp,\infty}(\delta\mathscr{Z}^A(H_1)) = 0$ and $\mathrm{pr}_{\mathfrak{g}}(\delta\mathscr{Z}^A(H_1)) = \mathrm{pr}_{\mathfrak{g}}(\delta Z^A(H(0)))$. In other words, the flow (2.41) annihilates the $\mathfrak{g}_t^\perp$-component of $\delta\mathscr{Z}^A(H(t))$.*

This means $\delta(\mathscr{Z}^A(H_1)) \in \theta_*(\mathfrak{g})$, but we can prove the following more precise result.

Lemma 2.5.12. *We have $\delta\mathscr{Z}^A(H_1) \in \theta_*(\sqrt{-1}\mathfrak{z})$ at the limit of the flow $H(t)$.*

Proof. Write G for $\mathrm{Aut}_0(X,L)$ and $\mathfrak{g}$ for its Lie algebra. By Lemma 2.5.11, we have $\delta\mathscr{Z}^A(H_1) \in \theta_*(\mathfrak{g})$ at the limit H_1 of the gradient flow (2.41). Suppose that $\delta\mathscr{Z}^A(H_1) =$

$\tilde{A}_1 \in \theta_*(\mathfrak{g})$. Since $\delta\mathscr{Z}^A(H_1)$ is a K-invariant hermitian matrix (as $H_1 \in \mathscr{B}_k^K$), $\tilde{A}_1$ must be a $\theta(K)$-invariant hermitian matrix in $\theta_*(\mathfrak{g})$. This means that $\theta(f)^*\tilde{A}_1\theta(f) = \tilde{A}_1$ for any $f \in K$, and hence $\tilde{A}_1$ commutes with any element in $\theta_*(\mathfrak{k})$. Thus $\tilde{A}_1$ is contained in the Lie algebra $\mathrm{Lie}(Z_G(K))$ of the centraliser $Z_G(K)$ of K in G. If G is reductive, we see that $Z_G(K)$ is equal to the complexification of the centre $Z(K)$ of K. Thus $\tilde{A}_1 \in \theta_*(\mathfrak{z} \oplus \sqrt{-1}\mathfrak{z})$, but $\tilde{A}_1$ being hermitian implies $\tilde{A}_1 \in \theta_*(\sqrt{-1}\mathfrak{z})$ by Lemma 2.2.4. If G is not reductive, we write $\mathfrak{g} = \left(\mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}\right) \oplus_\pi \mathfrak{n}$ where $\mathfrak{n} := \mathrm{Lie}(R_u)$ is a nilpotent Lie algebra and $\oplus_\pi$ is the semidirect product in the Lie algebra corresponding to $G = K^{\mathbb{C}} \ltimes R_u$ (cf. Notation 2.2.3). Since $\tilde{A}_1 = \delta\mathscr{Z}^A(H_1)$ is hermitian, Jordan–Chevalley decomposition immediately tells us that the $\mathfrak{n}$-component of $\tilde{A}_1$ is zero, and hence $\tilde{A}_1 \in \theta_*(\mathfrak{k} \oplus \sqrt{-1}\mathfrak{k})$. Thus, exactly as in the case when G is reductive, $\tilde{A}_1$ commuting with any element in $\theta_*(\mathfrak{k})$ and $\tilde{A}_1$ being hermitian implies $\tilde{A}_1 \in \theta_*(\sqrt{-1}\mathfrak{z})$ by Lemma 2.2.4.

□

Summarising these results, we get the following.

Proposition 2.5.13. *At the limit H_1 of the gradient flow (2.41), we have*

$$\int_X h^k_{FS(H_1)}(s_i,s_j)\frac{(k\omega_{FS(H_1)})^n}{n!} = \frac{Vk^n}{N}\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}_{ij} + \frac{Vk^n}{N}(\tilde{A}_1)_{ij}$$

where $\frac{Vk^n}{N}\tilde{A}_1 \in \theta_*(\sqrt{-1}\mathfrak{z})$ is equal to $-\mathrm{pr}_{\mathfrak{g}}\left(\delta\mathscr{Z}^A(H(0))\right)$, and $\{s_i\}$ is an H_1-orthonormal basis.

2.5.4 Iterative construction and the completion of the proof of Theorem 2.1.6

Although Proposition 2.5.13 does not provide us with the ρ-balanced metric that we seek, we can use it to construct an iterative procedure which converges to one, as we discuss in the following.

We first need to estimate the Hilbert–Schmidt norm of $\tilde{A}_1$ (in Proposition 2.5.13) in terms of the one of $\tilde{E}_0$.

Lemma 2.5.14. *There exists a constant $C'(R,\varepsilon)$ which depends only on R and ε as in*

Theorem 2.5.6 such that

$$||\tilde{A}_1||_{HS(t)} \leq C'(R,\varepsilon)k^{1/2}||\tilde{E}_0||_{HS(t)},$$

where $||\cdot||_{HS(t)}$ *is defined in terms of* $H(t)$.

Proof. The equation (5.10) in [96], together with the hypothesis $\bar{\mu}'_X = D_k + E_k$, D_k being a scalar matrix with $D_k \to I$ as $k \to \infty$ and $||E||_{op} < \varepsilon$ (cf. Remark 2.5.7), implies

$$||X_\xi||^2_{L^2(t)} \leq C(R,\varepsilon)||\xi||^2_{HS(t)}$$

for any $\xi \in \mathfrak{sl}$ in general. On the other hand, the estimate (2.36) (cf. (5.7) of [96]) implies

$$||\xi||^2_{HS(t)} \leq C'_R k||X_\xi||^2_{L^2(t)} \tag{2.47}$$

for any $\xi \in \mathfrak{sl}$ in general.

Since $\tilde{E}_0 = -\delta\mathscr{Z}^A(H(0))$ and $\frac{Vk^n}{N}\tilde{A}_1 = -\mathrm{pr}_{\mathfrak{g}}\left(\delta\mathscr{Z}^A(H(0))\right)$, it is sufficient to bound $||\alpha||_{HS(t)}$ in the decomposition $\xi = \alpha + \beta$ (according to $\mathfrak{sl} = \mathfrak{g} \oplus \mathfrak{g}_t^\perp$) in terms of $||\xi||_{HS(t)}$. Then, noting

$$||X_{\alpha+\beta}||^2_{L^2(t)} = ||X_\alpha + X_\beta||^2_{L^2(t)} = ||X_\alpha||^2_{L^2(t)} + ||X_\beta||^2_{L^2(t)}$$

since $\mathfrak{g}_t^\perp$ is defined with respect to the L^2-metric induced from $H(t)$ (cf. §2.5.2), we have

$$||X_\alpha||^2_{L^2(t)} + ||X_\beta||^2_{L^2(t)} \leq C(R,\varepsilon)||\xi||^2_{HS(t)}. \tag{2.48}$$

Thus, by (2.47) and (2.48), there exists a constant $C'(R,\varepsilon) > 0$ such that

$$\frac{1}{C'(R,\varepsilon)k}\left(||\alpha||^2_{HS(t)} + ||\beta||^2_{HS(t)}\right) \leq ||\xi||^2_{HS(t)} = ||\alpha+\beta||^2_{HS(t)}.$$

which implies $||\alpha||^2_{HS(t)} \leq C'(R,\varepsilon)k||\alpha+\beta||^2_{HS(t)} \leq C'(R,\varepsilon)k||\xi||^2_{HS(t)}$ as required.

□

In what follows, we write $||\cdot||_{HS,0}$ for the Hilbert–Schmidt norm defined with respect to H_0 and $||\cdot||_{HS,1}$ for the one with respect to H_1 (which is equal to the limit $H(\infty)$ of the flow (2.41)).

In particular, Lemma 2.5.14 and (2.46) imply that we have

$$||\tilde{A}_1||_{HS,1} \leq 2||\tilde{A}_1||_{HS,0} \leq 2C'(R,\varepsilon)k^{1/2}||\tilde{E}_0||_{HS,0} = O(k^{-m+\frac{1}{2}}). \tag{2.49}$$

Now, writing $A_1 = A + 2\pi k\tilde{A}_1$, we observe

$$\frac{Vk^n}{N}\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1} + \frac{Vk^n}{N}\tilde{A}_1 = \frac{Vk^n}{N}\left(\left(I + C_0 I - \frac{A_1}{2\pi k} + \tilde{A}_1\right)^{-1} + \tilde{A}_1\right)$$

and noting that all the matrices appearing here commute (as $A, \tilde{A}_1 \in \theta_*(\sqrt{-1}\mathfrak{z})$), we have

$$\begin{aligned}\left(I + C_0 I - \frac{A_1}{2\pi k} + \tilde{A}_1\right)^{-1} + \tilde{A}_1 &= \left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1} - \left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-2}\tilde{A}_1 + \tilde{A}_1 \\ &\quad + \text{ terms at least quadratic in } \tilde{A}_1 \\ &= \left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1} + 2\left(C_0\tilde{A}_1 - \frac{A_1\tilde{A}_1}{2\pi k}\right) \\ &\quad + \text{ higher order terms in } k\end{aligned}$$

by recalling (2.32), $||A||_{op} \leq \text{const}$, and (2.49). Now the Hilbert–Schmidt norm of

$$\begin{aligned}\tilde{E}_1' &:= \left(I + C_0 I - \frac{A_1}{2\pi k} + \tilde{A}_1\right)^{-1} + \tilde{A}_1 - \left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1} \\ &= 2\left(C_0\tilde{A}_1 - \frac{A_1\tilde{A}_1}{2\pi k}\right) + \text{ higher order terms in } k\end{aligned}$$

with respect to H_1 can be estimated as

$$||\tilde{E}_1'||_{HS,1} \leq 4\left|\left|C_0\tilde{A}_1 - \frac{A_1\tilde{A}_1}{2\pi k}\right|\right|_{HS,1} \tag{2.50}$$

$$\begin{aligned}&\leq 8\left|\left|C_0 I - \frac{A}{2\pi k}\right|\right|_{op}||\tilde{A}_1||_{HS,1} \leq 8C'(R,\varepsilon)k^{1/2}\left|\left|C_0 I - \frac{A}{2\pi k}\right|\right|_{op}||\tilde{E}_0||_{HS,1} \\ &\leq 8C'(R,\varepsilon)\frac{||A||_{op}+1}{k^{1/2}}||\tilde{E}_0||_{HS,0} = O(k^{-m-\frac{1}{2}}),\end{aligned} \tag{2.51}$$

for all large enough k, by recalling the estimate (2.32), $||A||_{op} \leq \text{const}$, and (2.46); we also used (cf. Proposition 2.5.13) $||\tilde{A}_1||^2_{op} \leq \sqrt{\mathrm{tr}(\tilde{A}_1\tilde{A}_1)} = ||\tilde{A}_1||_{HS,1}$. We then modify the constant C_0 to make $\tilde{E}_1$ term trace free, by arguing as we did in Lemma 2.4.19. This will change C_0 by a constant of order $k^{-m-\frac{1}{2}-\frac{n}{2}}$, to C_1 say, satisfying the bound

$$|C_0 - C_1| < 4N^{-1/2}||\tilde{E}_1'||_{HS,1} \leq 8N^{-1/2}||\tilde{E}_1'||_{HS,0} \tag{2.52}$$

as in (2.35). Hence there exists a trace free hermitian matrix $\tilde{E}_1$ which satisfies

$$\frac{Vk^n}{N}\left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1} + \frac{Vk^n}{N}\tilde{E}_1' = \frac{Vk^n}{N}\left(I + C_1 I - \frac{A_1}{2\pi k}\right)^{-1} + \tilde{E}_1$$

and $||\tilde{E}_1||_{HS,1}$ can be bounded by

$$\begin{aligned}
||\tilde{E}_1||_{HS,1} &\leq \left(1 + 8\left|\left|\left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\frac{Vk^n}{N}||\tilde{E}_1'||_{HS,1} && (2.53)\\
&\leq 4\left(1 + 8\left|\left|\left(I + C_0 I - \frac{A_1}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\left|\left|C_0 I - \frac{A_1}{2\pi k}\right|\right|_{op}||\tilde{A}_1||_{HS,1} && (2.54)\\
&\leq 16C'(R,\varepsilon)k^{1/2}\left(1 + 8\left|\left|\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\left|\left|C_0 I - \frac{A}{2\pi k}\right|\right|_{op}||\tilde{E}_0||_{HS,1} \\
&\leq 32C'(R,\varepsilon)k^{1/2}\left(1 + 8\left|\left|\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\frac{||A||_{op}+1}{k}||\tilde{E}_0||_{HS,0} && (2.55)\\
&= O(k^{-m-\frac{1}{2}}) && (2.56)
\end{aligned}$$

where we used (2.34) in the first line, (2.50) in the second line, Lemma 2.5.14 and (2.49) in the third line, and (2.32) in the fourth line.

Recalling Proposition 2.5.13, the above calculations mean that we get

$$\int_X h^k_{FS(H_1)}(s_i^{H_1}, s_j^{H_1})\frac{(k\omega_{FS(H_1)})^n}{n!} - \frac{Vk^n}{N}\left(I + C_1 I - \frac{A_1}{2\pi k}\right)^{-1}_{ij} = (\tilde{E}_1)_{ij}$$

where $\tilde{E}_1 \in T_{H_1}\mathscr{B}_k^K$ is a trace free hermitian matrix which satisfies $||\tilde{E}_1||_{HS,1} = O(k^{-m-\frac{1}{2}})$ by (2.56). We now return to the gradient flow (2.41), starting at H_1, apart from that it is now driven by $\mathrm{pr}_{\perp,t}(\delta\mathscr{Z}^{A_1}(H(t)))$, replacing A by A_1; namely we run the new gradient flow

$$\frac{dH^{(1)}(t)}{dt} = -\mathrm{pr}_{\perp,t}\left(\delta\mathscr{Z}^{A_1}(H^{(1)}(t))\right) \tag{2.57}$$

starting at $H(\infty) := H_1$, where we observe that the error term $\tilde{E}_1$ (at $t = 0$) has now been improved to $||\tilde{E}_1||_{HS,1} = O(k^{-m-\frac{1}{2}})$ by (2.56), as opposed to $||\tilde{E}_0||_{HS,0} = O(k^{-m})$ that we initially had in Corollary 2.4.20. Also note that now the projection $\mathrm{pr}_{\perp,t} : \mathfrak{sl} \twoheadrightarrow \mathfrak{g}_t^{\perp}$ is onto the L^2-orthogonal complement of $\mathfrak{g}$ in $\mathfrak{sl}$ with respect to the Fubini–Study metric induced from $H^{(1)}(t)$.

We summarise what we have achieved as follows. We started with an approximately ρ-balanced metric $H_0 \in \mathscr{B}_k^K$, obtained in Corollary 2.4.20 which satisfies $\delta\mathscr{Z}^A(H_0) = \tilde{E}_0$ with $||\tilde{E}_0||_{HS,0} \leq \text{const.}k^{-m}$; ran the gradient flow (2.41) to annihilate $\mathrm{pr}_{\perp,t}(\delta\mathscr{Z}^A)$, so that at the limit $H_1 \in \mathscr{B}_k^K$ of the flow we have $\mathrm{pr}_{\perp,\infty}(\delta\mathscr{Z}^A(H_1)) = 0$; set $\tilde{A}_1 := -\frac{N}{Vk^n}\mathrm{pr}_{\mathfrak{g}}(\delta\mathscr{Z}^A(H_1)) \in \theta_*(\sqrt{-1}\mathfrak{z})$ and replaced A by $A_1 := A + 2\pi k\tilde{A}_1$, to consider the functional $\mathscr{Z}^{A_1}$ with a new constant C_1, which differs from C_0 by $O(k^{-m-\frac{1}{2}-\frac{n}{2}})$; wrote $\tilde{E}_1 := -\delta\mathscr{Z}^{A_1}(H_1)$ with $\tilde{E}_1$ satisfying $||\tilde{E}_1||_{HS,1} \leq \text{const.}k^{-1/2}||\tilde{E}_0||_{HS,0} = O(k^{-m-\frac{1}{2}})$ as given in (2.55), i.e. H_1 is an approximately ρ-balanced metric of order $k^{-m-\frac{1}{2}}$. We then go back to the first step, by replacing H_0 with H_1. We repeat the above process inductively, as in the following proposition.

Proposition 2.5.15. *Suppose that we run the iterative procedure, starting with $i = 0$, to find ρ-balanced metrics as follows:*

Step 1 *start with an approximately ρ-balanced metric $H_i \in \mathscr{B}_k^K$ of order $k^{-m-i/2}$;*

Step 2 *run the gradient flow*

$$\frac{dH^{(i)}(t)}{dt} = -\mathrm{pr}_{\perp,t}\left(\delta\mathscr{Z}^{A_i}(H^{(i)}(t))\right)$$

to annihilate $\mathrm{pr}_{\perp,t}(\delta\mathscr{Z}^{A_i})$, so that at the limit $H^{(i)}(\infty) =: H_{i+1} \in \mathscr{B}_k^K$ of the flow we have $\mathrm{pr}_{\perp,\infty}(\delta\mathscr{Z}^{A_i}(H_{i+1})) = 0$;

Step 3 *set $\tilde{A}_{i+1} := -\frac{N}{Vk^n}\mathrm{pr}_{\mathfrak{g}}(\delta\mathscr{Z}^{A_i}(H_{i+1})) \in \theta_*(\sqrt{-1}\mathfrak{z})$ and replace A_i by $A_{i+1} := A_i + 2\pi k\tilde{A}_{i+1}$, to consider the functional $\mathscr{Z}^{A_{i+1}}$ with a new constant C_{i+1}, which differs from C_i by $O(k^{-m-(n+i)/2})$;*

Step 4 *observe that H_{i+1} satisfies $||\delta\mathscr{Z}^{A_{i+1}}(H_{i+1})||_{HS,i+1} = O(k^{-m-(i+1)/2})$, where $||\cdot||_{HS,i+1}$ is the Hilbert–Schmidt norm defined with respect to H_{i+1} i.e. H_{i+1} is an approximately ρ-balanced metric of order $k^{-m-(i+1)/2}$;*

Step 5 *go back to the step 1, with an improved error term (i.e. the approximately ρ-balanced metric H_{i+1} now has order $k^{-m-(i+1)/2}$);*

so that, by repeating these steps, we get a sequence $\{(A_i, C_i, H_i)\}_i$ in $\theta_*(\sqrt{-1}\mathfrak{z}) \times \mathbb{R} \times \mathscr{B}_k^K$.

Then, as $i \to \infty$, A_i, C_i, and H_i converges to $A_\infty \in \theta_*(\sqrt{-1}\mathfrak{z})$, $C_\infty \in \mathbb{R}$, and $H_\infty \in \mathscr{B}_k^K$, respectively.

The proof is given in the following two lemmas, which rely on the estimates that we have established so far. We first prove the existence of A_∞ and C_∞.

Lemma 2.5.16. *$A/k + 2\pi\tilde{A}_1 + 2\pi\tilde{A}_2 + \cdots$ converges, and hence $A_\infty := A + 2\pi k\tilde{A}_1 + 2\pi k\tilde{A}_2 + \cdots$ exists. Also C_∞ exists.*

Proof. We first claim that there exist some constants $\gamma_1, \gamma_2 > 0$ such that $||\tilde{A}_i||_{HS,i} \leq k^{-m+1}(k^{-1/2}\gamma_1)^i$ and $|C_i - C_{i-1}| \leq N^{-1/2}k^{-m}(k^{-1/2}\gamma_2)^i$. Observe that $||\tilde{A}_i||_{HS,i} \leq k^{-m+1}(k^{-1/2}\gamma_1)^i$ implies $||\tilde{A}_i||_{HS,0} \leq k^{-m+1}(2k^{-1/2}\gamma_1)^i$ by inductively using (2.46).

Note that these estimates are satisfied when $i = 1$; more specifically, Lemma 2.5.14, (2.46), and $||\tilde{E}_0||_{HS,0} = O(k^{-m})$ imply that there exists a constant $\gamma > 0$ such that

$$||\tilde{A}_1||_{HS,1} \leq 2C'(R,\varepsilon)k^{1/2}||\tilde{E}_0||_{HS,0} \leq \gamma C'(R,\varepsilon)k^{-m+\frac{1}{2}},$$

and (2.51), (2.52) imply

$$|C_0 - C_1| \leq 4N^{-1/2}8C'(R,\varepsilon)\frac{||A||_{op}+1}{k^{1/2}}||\tilde{E}_0||_{HS,0} \leq 32N^{-1/2}\gamma C'(R,\varepsilon)(||A||_{op}+1)k^{-m-\frac{1}{2}}.$$

In what follows, we assume $C'(R,\varepsilon) \geq 1$ and $\gamma \geq 1$ without loss of generality.

We argue by induction; suppose that the statement holds at the $(i-1)$-th step. Combined with Lemma 2.5.14 and (2.46), the argument in (2.54) at the i-th step implies

$$\begin{aligned} ||\tilde{A}_i||_{HS,i} &\leq C'(R,\varepsilon)k^{1/2}||\tilde{E}_{i-1}||_{HS,i} \\ &\leq 8C'(R,\varepsilon)k^{1/2}\left(1 + 8\left|\left|\left(I + C_{i-2}I - \frac{A_{i-1}}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\left|\left|C_{i-2}I - \frac{A_{i-1}}{2\pi k}\right|\right|_{op}||\tilde{A}_{i-1}||_{HS,i-1} \end{aligned}$$

for all $i \geq 2$. Then the induction hypothesis and (2.32) imply $1 + 8\left|\left|\left(I + C_{i-2}I - \frac{A_{i-1}}{2\pi k}\right)^{-1}\right|\right|_{op} \leq$

$2\left(1+8\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)$ and $\left|\left|C_{i-2}I-\frac{A_{i-1}}{2\pi k}\right|\right|_{op}\leq\frac{||A||_{op}+1}{k}$ (cf. (2.55)). Thus

$$||\tilde{A}_i||_{HS,i}\leq 16C'(R,\varepsilon)\left(1+8\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\frac{||A||_{op}+1}{k^{1/2}}||\tilde{A}_{i-1}||_{HS,i-1} \tag{2.58}$$

for all large enough k. We can thus take

$$\gamma_1:=\max\left\{16C'(R,\varepsilon)\left(1+8\left|\left|\left(I+C_0I-\frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)(||A||_{op}+1),\ \ \gamma C'(R,\varepsilon)\right\}.$$

We also have

$$|C_i-C_{i-1}|\leq 4N^{-1/2}||\tilde{E}_i'||_{HS,i}\leq 16N^{-1/2}\left|\left|C_{i-1}-\frac{A_i}{2\pi k}\right|\right|_{op}||\tilde{A}_i||_{HS,i}$$

by arguing as in (2.50) and (2.52). The induction hypothesis and (2.58) imply that

$$|C_i-C_{i-1}|\leq 32N^{-1/2}\left|\left|C_0-\frac{A}{2\pi k}\right|\right|_{op}\gamma_1^ik^{-m+1-\frac{i}{2}}\leq 16N^{-1/2}(||A||_{op}+1)\gamma_1^ik^{-m-\frac{i}{2}}$$

where we used (2.32) and $||A||_{op}\leq$ const, and hence we can take $\gamma_2:=16C'(R,\varepsilon)(||A||_{op}+1)\gamma_1$, by noting $16C'(R,\varepsilon)(||A||_{op}+1)\gamma_1^i<\gamma_2^i$.

Having established the claim as above, we thus have

$$||A/k+2\pi\tilde{A}_1+2\pi\tilde{A}_2+\cdots||_{HS,0}\leq\left(\frac{\gamma_1}{k}+k^{-m+1}(2\gamma_1k^{-1/2})+k^{-m+1}(2\gamma_1k^{-1/2})^2+\cdots\right)<\infty$$

for all large enough k, and

$$|C_\infty|\leq\left(\frac{\gamma_2}{k}+N^{-1/2}k^{-m}(\gamma_2k^{-1/2})+N^{-1/2}k^{-m}(\gamma_2k^{-1/2})^2+\cdots\right)<\infty.$$

□

We now prove the existence of H_∞.

Lemma 2.5.17. *Repeating the procedure as given in Proposition 2.5.15 infinitely many times moves H_0 by a finite distance in $\mathscr{B}_k^K$ with respect to the bi-invariant metric, i.e.* $\mathrm{dist}(H_\infty,H_0)<\infty$.

Proof. Consider first the case $i = 1$. Recall that we use the limit $H_1 = H(\infty)$ of the first gradient flow (2.41) as the initial condition for the second gradient flow (2.57). By proceeding as we did in (2.45), we get

$$\mathrm{dist}(H_2, H_0) \leq \frac{1}{\lambda_1}\left(||\mathscr{G}^A(0)||_{HS,0} + ||\mathscr{G}^{A_1}(0)||_{HS,1}\right).$$

Recalling $\mathscr{G}^A(0) = \tilde{E}_0$ and $\mathscr{G}^{A_1}(0) = \tilde{E}_1$, we get $\mathrm{dist}(H_2, H_0) \leq \frac{1}{\lambda_1}\left(||\tilde{E}_0||_{HS,0} + ||\tilde{E}_1||_{HS,1}\right)$. Inductively continuing as described in Proposition 2.5.15, we have $\mathrm{dist}(H_{i+1}, H_j) \leq \frac{1}{\lambda_1}\left(||\tilde{E}_j||_{HS,j} + \cdots + ||\tilde{E}_i||_{HS,i}\right)$ for $i > j$, and also

$$\mathrm{dist}(H_{i+1}, H_0) \leq \frac{1}{\lambda_1}\left(||\tilde{E}_0||_{HS,0} + ||\tilde{E}_1||_{HS,1} + \cdots + ||\tilde{E}_i||_{HS,i}\right). \tag{2.59}$$

Now the estimates as in (2.53)-(2.55) at the i-th step (and also Lemma 2.5.16) implies that we have

$$||\tilde{E}_i||_{HS,i} \leq 32C'(R,\varepsilon)\left(1 + 8\left|\left|\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\frac{||A||_{op}+1}{k^{1/2}}||\tilde{E}_{i-1}||_{HS,i-1}$$

and hence

$$||\tilde{E}_i||_{HS,i} \leq \left(32C'(R,\varepsilon)\left(1 + 8\left|\left|\left(I + C_0 I - \frac{A}{2\pi k}\right)^{-1}\right|\right|_{op}\right)\frac{||A||_{op}+1}{k^{1/2}}\right)^i ||\tilde{E}_0||_{HS,0}. \tag{2.60}$$

Thus we find that there exists a constant $c > 0$, independent of k, such that $||\tilde{E}_i||_{HS,i} \leq (ck^{-1/2})^{i+1}||\tilde{E}_0||_{HS,0}$, and hence we get

$$\mathrm{dist}(H_{i+1}, H_j) \leq k^2||\tilde{E}_0||_{HS,0}\left((ck^{-1/2})^j + \cdots + (ck^{-1/2})^i\right)$$

for $i > j$, and

$$\begin{aligned}
&\mathrm{dist}(H_{i+1}, H_0) \\
&\leq \frac{1}{\lambda_1}\left(||\tilde{E}_0||_{HS,0} + ck^{-1/2}||\tilde{E}_0||_{HS,0} + c^2k^{-1}||\tilde{E}_0||_{HS,0} + \cdots + +(ck^{-1/2})^i||\tilde{E}_0||_{HS,0}\right) \\
&= O(k^{-m+2})
\end{aligned} \tag{2.61}$$

for all large enough i, where we recall $||\tilde{E}_0||_{HS,0} = O(k^{-m})$ (cf. Corollary 2.4.20) and (2.42). Thus the sequence $\{H_i\}_i$ is Cauchy in $\mathscr{B}_k^K$ with respect to the bi-invariant metric, and hence the limit $H_\infty \in \mathscr{B}_k^K$ exists.

□

We finally see that (2.61) implies $||H_\infty - H_0||_{HS,0} = O(k^{-m+2})$ (cf. Remark 2.5.10). We claim $||\omega_{H_\infty} - \omega_{(m)}||_{C^l,\omega} = O(k^{-m+n+l+1})$, recalling the definitional $\omega_{H_0} = \omega_{(m)}$. To make explicit the dependence on k and m, we write $H_\infty(k,m)$ for $H_\infty \in \mathscr{B}_k^K$ and $H_0(k,m)$ for $H_0 \in \mathscr{B}_k^K$. By taking a suitable $H_0(k,m)$-orthonormal basis $\{s_i\}$, we may assume that $H_0(k,m)$ is the identity matrix and $H_\infty(k,m)$ is given by $\mathrm{diag}(d_1^2, \dots, d_N^2)$. $||H_\infty(k,m) - H_0(k,m)||_{HS,0} = O(k^{-m+2})$ implies that we have $d_i^2 - 1 = O(k^{-m+2})$, which in turn implies $d_i^{-2} - 1 = O(k^{-m+2})$. Observe that we can write

$$\omega_{H_\infty(k,m)} = \omega_{(m)} + \frac{\sqrt{-1}}{2\pi k} \partial\bar{\partial} \log \left(\sum_i d_i^{-2} |s_i|^2_{FS(H_0(k,m))^k} \right).$$

We may choose local coordinates $(z_1, \dots, z_n)$ and reduce to local computation. The equation (2.4) and $d_i^{-2} - 1 = O(k^{-m+2})$ imply that we have $\sum_i d_i^{-2} |s_i|^2_{FS(H_0(k,m))^k} = 1 + O(k^{-m+n+2})$, and hence it suffices to evaluate its derivatives.

We fix a local trivialisation of the line bundle L to write $h_{FS(H_0(k,m))} = e^{-\phi_{m,k}}$, and regard each s_i as a holomorphic function. Observe that (2.4) implies $\sum_i |s_i|^2 = e^{k\phi_{m,k}}$. We then apply $\frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ on both sides to find

$$\sum_i e^{-k\phi_{m,k}} \left| \frac{\partial}{\partial z_j} s_i \right|^2 \leq k^2 C_1(\phi_{m,k}),$$

for a constant $C_1(\phi_{m,k})$ which depends only on (first and second derivatives of) $\phi_{m,k}$. Higher order derivatives can be similarly bounded in terms of C^l-norms of $\phi_{m,k}$; namely we get $\sum_i e^{-k\phi_{m,k}} \left| \frac{\partial^l}{\partial z_{j_1} \cdots \partial z_{j_l}} s_i \right|^2 \leq k^{2l} C_2(\phi_{m,k}, l)$ for a constant $C_2(\phi_{m,k}, l)$ which depends only on the C^{2l}-norm of $\phi_{m,k}$. In particular, we have

$$e^{-k\phi_{m,k}/2} \left| \frac{\partial^l}{\partial z_{j_1} \cdots \partial z_{j_l}} s_i \right| \leq k^l C_3(\phi_{m,k}, l)$$

for each $i = 1, \dots, N$ and $j_1, \dots, j_l \in \{1, \dots, n\}$.

Observe that (2.4) implies $e^{-k\phi_{m,k}/2}|s_i| \le 1$. Thus we get, again using (2.4),

$$\begin{aligned}\frac{\partial}{\partial z_j}\sum_i d_i^{-2}|s_i|^2_{FS(H_0(k,m))^k} &= \frac{\partial}{\partial z_j}\sum_i (d_i^{-2}-1)e^{-k\phi_{m,k}}|s_i|^2 \\ &= -k\frac{\partial \phi_{m,k}}{\partial z_j}\sum_i (d_i^{-2}-1)e^{-k\phi_{m,k}}|s_i|^2 + \sum_i (d_i^{-2}-1)e^{-k\phi_{m,k}}\bar{s}_i\frac{\partial s_i}{\partial z_j},\end{aligned}$$

and hence

$$\left|\frac{\partial}{\partial z_j}\sum_i d_i^{-2}|s_i|^2_{FS(H_0(k,m))^k}\right| \le k^{-m+n+3}C_4(\phi_{m,k}).$$

Thus, inductively continuing, we get

$$\left|\frac{\partial^r}{\partial \bar{z}_{j_1}\cdots\partial \bar{z}_{j_r}}\frac{\partial^l}{\partial z_{j_1}\cdots\partial z_{j_l}}\sum_i d_i^{-2}|s_i|^2_{FS(H_0(k,m))^k}\right| \le k^{-m+n+l+r+2}C_5(\phi_{m,k},l+r).$$

Thus we get $||\omega_{H_\infty}-\omega_{(m)}||_{C^l,\omega_{H_0(k,m)}} \le C_6(\phi_{m,k},l)k^{-m+n+(l+2)-1}$.

Writing $h = e^{-\phi}$ for the hermitian metric corresponding to the extremal metric ω (i.e. $\omega = -\sqrt{-1}\partial\bar{\partial}\log h$), we have $\phi_{m,k} \to \phi$ in C^∞ as $k \to \infty$ (cf. the proof of Corollary 2.3.15). Thus we get $||\omega_{H_\infty}-\omega_{(m)}||_{C^l,\omega_{H_0(k,m)}} \le C_l k^{-m+n+(l+2)-1}$ for a constant C_l which depends only on l, as claimed.

We thus get

$$\begin{aligned}||\omega_{H_\infty(k,m)}-\omega||_{C^l,\omega} &\le 2||\omega_{H_\infty(k,m)}-\omega_{H_0(k,m)}||_{C^l,\omega_{H_0(k,m)}} + ||\omega_{H_0(k,m)}-\omega||_{C^l,\omega} \\ &\le \tilde{C}_l(k^{-m+n+l+1}+k^{-1}).\end{aligned}$$

Thus, given $l \in \mathbb{N}$, we can choose m to be large enough so that the sequence $\{\omega_{H_\infty(k,m)}\}_k$ converges to ω in C^l, establishing all the statements claimed in Theorem 2.1.6.

Remark 2.5.18. It is tempting to say that, given such $\omega_{H_\infty(k,m)}$'s, there exists a sequence $\{\omega_k\}_k$ which converges to ω in C^∞ by diagonal argument. However, k must be chosen to be large enough for $\omega_{H_\infty(k,m)}$ to be well-defined, and how large k must be depends on m (cf. §2.3), and hence on l. Thus, by diagonal argument, we can only claim the existence of ω_k's (with $\omega_k \to \omega$ in C^∞) satisfying $\bar{\partial}\mathrm{grad}^{1,0}_{\omega_k}\rho_k(\omega_k) = 0$ for *infinitely many* k's rather than for *all sufficiently large* k's.

We finally note that, if we have the uniqueness theorem as mentioned in Remark 2.1.9, it follows that $\omega_{H_\infty(k,m)} = \omega_{H_\infty(k,m')}$ for all m and m', and hence we can say that the sequence converges in C^∞ (cf. §4.2 of [40]).

2.6 Stability of (X,L)

2.6.1 Chow stability

This is a review of classical theory, and we refer the reader to §1.16 of Mumford's paper [90] and §2 of Futaki's survey [56] for the details on the materials presented here. Consider a polarised Kähler manifold (X,L) with $\dim_{\mathbb{C}} X = n$ and degree $d_k := \int_X c_1(L^k)^n$, and the Kodaira embedding $\iota : X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*)$. Writing $V_k := H^0(X,L^k)$, observe that $n+1$ points $H_1,\dots,H_{n+1}$ in $\mathbb{P}(V_k)$ determines $n+1$ divisors in $\mathbb{P}(V_k^*)$, and that

$$\{(H_1,\dots,H_{n+1}) \in \mathbb{P}(V_k)\times\cdots\times\mathbb{P}(V_k) \mid H_1\cap\cdots\cap H_{n+1}\cap\iota(X)\neq\emptyset \text{ in } \mathbb{P}(V_k^*)\}$$

is a divisor in $\mathbb{P}(V_k)\times\cdots\times\mathbb{P}(V_k)$. The polynomial $\Phi_{X,k} \in (\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$ defining this divisor, or the point $[\Phi_{X,k}]$ in $\mathbb{P}((\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)})$ is called the **Chow form** of $X \hookrightarrow \mathbb{P}(H^0(X,L^k)^*)$. It is a classical fact [90, 105] that $[\Phi_{X,k}]$ corresponds bijectively to a subvariety in $\mathbb{P}(H^0(X,L^k)^*)$ of dimension n and degree d_k.

Chow stability of (X,L) is nothing but the GIT stability of the point $[\Phi_{X,k}] \in \mathbb{P}((\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)})$ with respect to the $SL(V_k^*)$-action on $(\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$. More precisely, it can be defined as follows.

Definition 2.6.1. A polarised Kähler manifold (X,L) is said to be:

1. **Chow polystable at the level** k if the $SL(V_k^*)$-orbit of $\Phi_{X,k}$ is closed in $(\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$,
2. **Chow stable at the level** k if it is Chow polystable and $\Phi_{X,k}$ has finite isotropy,
3. **Chow semistable at the level** k if the $SL(V_k^*)$-orbit of $\Phi_{X,k}$ does not contain $0 \in (\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$,
4. **Chow unstable at the level** k if it is not Chow semistable,
5. **asymptotically Chow stable** (resp. **polystable**, **semistable**) if there exists $k_0 \in \mathbb{N}$ such that it is Chow stable (resp. polystable, semistable) at the level k for all $k \geq k_0$.

We recall the following fundamental theorem.

Theorem 2.6.2. (Luo [76], Zhang [131]) *Suppose that* $\mathrm{Aut}_0(X,L)$ *is trivial. Then* (X,L) *is Chow stable at the level k if and only if it admits a balanced metric at the level k.*

Remark 2.6.3. Zhang [131] also proved that, even when $\mathrm{Aut}_0(X,L)$ is nontrivial, the existence of balanced metric at the level k implies that (X,L) is Chow semistable at the level k (cf. Theorem 3.2, [131]).

Remark 2.6.4. It is well-known that asymptotic Chow stability is closely related to K-stability. For example, it is known that asymptotic Chow stability implies K-semistability [103]. More intuitively, asymptotic Chow stability can be seen as a version for varieties of Gieseker stability, and K-stability as corresponding to Mumford stability[9] [103].

2.6.2 Chow polystability relative to a torus

We now review the version of Chow stability which is "relative" to the automorphism group $G = \mathrm{Aut}_0(X,L)$, as introduced by Mabuchi [82]. The reader is referred to the survey given in Apostolov–Huang [5] for further discussions. Since we have the G-linearisation of L (or θ in Lemma 2.2.1, cf. Remark 2.2.2), choosing a real torus T in $K = \mathrm{Isom}(\omega)$, we can consider the representation $\theta|_{T^c} : T^c \curvearrowright H^0(X,L^k)$ where T^c is the complexification of T. We then consider a subspace

$$V_k(\chi) := \{s \in H^0(X,L^k) \mid \theta(t)\cdot s = \chi(t)s \text{ for all } t \in T^c\}$$

of $H^0(X,L^k)$, where $\chi \in \mathrm{Hom}(T^c,\mathbb{C}^*)$ is a character. We then have a decomposition

$$H^0(X,L^k) = \bigoplus_{\nu=1}^{r} V_k(\chi_\nu) \tag{2.62}$$

for mutually distinct characters $\chi_1,\dots,\chi_r \in \mathrm{Hom}(T^c,\mathbb{C}^*)$. We then define

$$G^c_T := \left\{ \mathrm{diag}(A_1,\dots,A_r) \in \prod_{\nu=1}^{r} GL(V_k(\chi_\nu)) \,\middle|\, \prod_{\nu=1}^{r} \det(A_\nu) = 1 \right\}$$

[9] The reader is referred to e.g. [103] for the definition of Gieseker stability and Mumford stability (called *slope stability* in [103]), defined for vector bundles.

for the "elements in $SL(H^0(X,L^k))$ that commute with the T^c-action", and

$$G^c_{T^\perp} := \left\{ \mathrm{diag}(A_1,\dots,A_r) \in \prod_{\nu=1}^{r} GL(V_k(\chi_\nu)) \,\middle|\, \prod_{\nu=1}^{r} \det(A_\nu)^{1+\log|\chi_\nu(t)|} = 1 \text{ for all } t \in T^c \right\}$$

for the "subgroup of G^c_T that is orthogonal to the T^c-action". See §1.3 of [115] for the motivation for these definitions. We then define the relative Chow stability as follows.

Definition 2.6.5. A polarised Kähler manifold (X,L) is said to be **Chow polystable at the level** k **relative to** T if the $G^c_{T^\perp}$-orbit of $\Phi_{X,k}$ is closed in $(\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$.

On the other hand, we can consider an action of a smaller group $\widetilde{G^c_{T^\perp}} := \prod_{\nu=1}^r SL(V_k(\chi_\nu))$; observe $\widetilde{G^c_{T^\perp}} \le G^c_{T^\perp}$. This leads to the notion of "weak" stability as follows (cf. [82, 5]).

Definition 2.6.6. A polarised Kähler manifold (X,L) is said to be **weakly Chow polystable at the level** k **relative to** T if the $\widetilde{G^c_{T^\perp}}$-orbit of $\Phi_{X,k}$ is closed in $(\mathrm{Sym}^{d_k}(V_k^*))^{\otimes(n+1)}$.

Recall that in the case $\mathrm{Aut}_0(X,L)$ is trivial, Chow stability corresponds to the existence of balanced metrics, as proved by Luo [76] and Zhang [131] (cf. Corollary 2.1.3). The notion of "balanced" metrics in the relative setting was proposed by Mabuchi [82] as follows.

Definition 2.6.7. A hermitian metric $h \in \mathscr{H}(X,L)$ is said to be **balanced at the level** k **relative to** T if $Hilb(h)$ is T-invariant and satisfies the following property: writing $\{s_{\nu,i}\}_{\nu,i}$ for a $Hilb(h)$-orthonormal basis for $H^0(X,L^k)$, where each $\{s_{\nu,i}\}_i$ is a $Hilb(h)$-orthonormal basis for $V_k(\chi_\nu)$, there exist positive constants $(b_1,\dots,b_r)$, $b_\nu > 0$, such that

$$\sum_{\nu,i} b_\nu |s_{\nu,i}|^2_{h^k} = 1.$$

A fundamental theorem is the following.

Theorem 2.6.8. (Mabuchi [82, 86]; see also Theorems 2 and 4 of Apostolov–Huang's paper [5]) (X,L) *is Chow polystable at the level* k *relative to* T *if and only if it admits* *a hermitian metric balanced relative to T with each b_ν satisfying*[10]

$$b_\nu = 1 + \log|\chi_\nu(t)| \tag{2.63}$$

for some $t \in T^c$, i.e. b_ν's are the eigenvalues of $I + A$ for some $A \in \theta_*(\mathrm{Lie}(T^c))$.

Corollary 2.6.9. (cf. §2 of Apostolov–Huang [5]) *(X,L) is Chow polystable at the level k relative to T if and only if there exists $g \in SL(N)$ which commutes*[11] *with the elements in $\theta(T)$ such that*

$$\bar{\mu}_X(g) = \frac{V}{N} I + A$$

for some $A \in \theta_*(\sqrt{-1}\mathrm{Lie}(T))$. In other words, the trace free part of $\bar{\mu}_X(g)$ generates a holomorphic automorphism of $\mathbb{P}^{N-1}$ which preserves the image of X under the Kodaira embedding.

Proof. Suppose that we have a metric balanced at the level k relative to T, satisfying $\sum_{\nu,i} b_\nu |s_{\nu,i}|^2_{h^k} = 1$ with b_ν's satisfying (2.63). We then see that h can be written as $h = FS(H)$ with H having $\{\sqrt{b_\nu} s_{\nu,i}\}_{\nu,i}$ as its orthonormal basis (cf. equation (2.4)), and that H is T-invariant (cf. Definition 1 of [5] and the argument that follows; see also Lemma 2.2.21). Then, the centre of mass $\bar{\mu}'_X$ with respect to this basis can be computed as

$$\begin{aligned} \bar{\mu}'_X &= \frac{V}{N} I + \frac{V}{N} \mathrm{diag}(\log|\chi_1(t)| \mathrm{id}_{V_k(\chi_1)}, \dots, \log|\chi_r(t)| \mathrm{id}_{V_k(\chi_r)}) \\ &= \frac{V}{N} I + \frac{V}{N} \log\theta(t), \end{aligned}$$

and we simply define $A := \frac{V}{N} \log\theta(t) \in \theta_*(\sqrt{-1}\mathrm{Lie}(T))$.

Conversely, writing $A = \frac{V}{N} \log\theta(t)$ for some $t \in T^c/T$, suppose that we have $\bar{\mu}'_X = \frac{V}{N} I + \frac{V}{N} \log\theta(t)$. Diagonalising $\log\theta(t)$, and defining b_ν's as in (2.63), we see that $\{\sqrt{b_\nu^{-1}} s'_{\nu,i}\}_{\nu,i}$ is a $Hilb(h)$-orthonormal basis, when $\{s'_{\nu,i}\}_{\nu,i}$ is an H-orthonormal basis. We thus get

$$1 = \sum_{\nu,i} |s'_{\nu,i}|^2_{h^k} = \sum_{\nu,i} b_\nu \left| \sqrt{b_\nu^{-1}} s'_{\nu,i} \right|^2_{h^k}$$

as required. □

[10] Note that in our setting, $\mathrm{diag}(\log|\chi_1(t)| \mathrm{id}_{V_k(\chi_1)}, \dots, \log|\chi_r(t)| \mathrm{id}_{V_k(\chi_r)})$ will be a trace free matrix, and hence $1 + \sum_{\nu=1}^r \dim V_k(\chi_\nu) \log|\chi_\nu(t)|/N = 1$.

[11] This corresponds to the $\theta(K)$-invariance of the hermitian matrix $H = \overline{(g^{-1})^t} g^{-1}$; see Remark 2.2.17.

2.6.3 Proof of Corollary 2.1.7

We now recall the following "weak" version of Theorem 2.6.8.

Theorem 2.6.10. (Mabuchi [82, 86]; see also the discussion preceding Definition 5 of [5]) *(X,L) is weakly Chow polystable at the level k relative to T if and only if it admits a hermitian metric balanced relative to T with some $b_\nu > 0$, not necessarily satisfying (2.63).*

Corollary 2.6.11. *(X,L) is weakly Chow polystable at the level k relative to T if and only if there exists $g \in SL(N)$ which commutes with $\theta(K)$-action such that*

$$\bar{\mu}_X(g) = \mathrm{diag}(b_1 \mathrm{id}_{V_k(\chi_1)}, \dots, b_r \mathrm{id}_{V_k(\chi_r)})$$

with respect to the decomposition $H^0(X,L^k) = \bigoplus_{\nu=1}^r V_k(\chi_\nu)$, for some $b_\nu > 0$ (not necessarily satisfying (2.63)).

Proposition 2.6.12. *If $FS(H)$, $H \in \mathscr{B}_k^K$, satisfies $\mathscr{D}_H^* \mathscr{D}_H \rho_k(\omega_H) = 0$, which is equivalent (by Lemma 2.4.3 and also (2.21)) to $\bar{\rho}_k(\omega_H) = \sum_{i,j} \left(CI - \frac{1}{2\pi k} A\right)_{ij} h^k_{FS(H)}(s_i, s_j)$, and if $CI - \frac{1}{2\pi k} A$ is positive definite, then $FS(H)$ is balanced at the level k relative to any maximal torus in K for some $b_\nu > 0$ (not necessarily satisfying (2.63)).*

Proof. By Proposition 2.4.15, writing $\{s_i\}$ for an H-orthonormal basis, we see that

$$s'_i := \sqrt{\frac{N}{V}} \left(CI - \frac{A}{2\pi k}\right)^{1/2}_{ij} s_j, \tag{2.64}$$

where $\left(CI - \frac{A}{2\pi k}\right)_{ij}$ is the matrix for $CI - \frac{A}{2\pi k}$ represented with respect to $\{s_i\}$, is a $\int_X h^k_{FS(H)}(,) \frac{\omega_H^n}{n!}$-orthonormal basis. Moreover, by replacing $\{s_i\}$ by an H-unitarily equivalent basis if necessary, we may assume that A is diagonal. For notational convenience, we write $\{s_{\nu,i}\}_{\nu,i}$ (resp. $\{s'_{\nu,i}\}_{\nu,i}$) for $\{s_i\}_i$ (resp. $\{s'_i\}_i$) for the rest of the proof, according to the decomposition (2.62), just to make explicit which sector $V_k(\chi_\nu)$ each basis element s_i belongs to. $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ implies that we may write

$$A_{ij} = \mathrm{diag}(a_1 \mathrm{id}_{V_k(\chi_1)}, \dots, a_r \mathrm{id}_{V_k(\chi_r)}),$$

since the centre Z of K is contained in any maximal torus of K. Thus we can write

$$\left(CI - \frac{A}{2\pi k}\right)_{ij} = \mathrm{diag}(b_1^{-1}\mathrm{id}_{V_k(\chi_1)}, \dots, b_r^{-1}\mathrm{id}_{V_k(\chi_r)})$$

for some $b_\nu > 0$. In particular, (2.64) can be re-written as $s'_{i,\nu} = \sqrt{\frac{N}{V}} b_\nu^{-1/2} s_{i,\nu}$. This means that we can write

$$\sum_{\nu,i} b_\nu |s'_{\nu,i}|^2_{FS(H)^k} = \frac{N}{V} \sum_{\nu,i} |s_{\nu,i}|^2_{FS(H)^k} = \text{const} \tag{2.65}$$

by the equation (2.4), as required. Observe also that these b_ν's in the above equation are the eigenvalues of $(CI - \frac{A}{2\pi k})^{-1}$, and *not* of $CI - \frac{A}{2\pi k}$, so a priori does not satisfy the equation (2.63). □

Remark 2.6.13. The proof above in fact shows that ω_H satisfies $\mathscr{D}_H^* \mathscr{D}_H \rho_k(\omega_H) = 0$ if and only if it satisfies the equation (2.65) with b_ν's being the eigenvalues of $(CI - \frac{A}{2\pi k})^{-1}$ for some $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ (cf. Theorem 2.6.8).

Recalling that Z is contained in any maximal torus in K, we have the following.

Corollary 2.6.14. *If there exists a sequence of hermitian metrics* $\{FS(H(k))\}_k$, $H(k) \in \mathscr{B}_k^K$, *on* (X, L^k) *which satisfies* $\mathscr{D}_{H(k)}^* \mathscr{D}_{H(k)} \rho_k(\omega_{H(k)}) = 0$ *with the bound* $||\theta_*(\mathrm{grad}\bar{\rho}_k(\omega_{H(k)}))||_{op} < \text{const}$ *uniformly of k, then* (X, L) *is asymptotically weakly Chow polystable relative to any maximal torus.*

We finally prove Corollary 2.1.7.

Proof of Corollary 2.1.7. This follows from Theorem 2.1.6, Lemma 2.4.9, and Corollary 2.6.14. □

2.6.4 Relationship to previously known results and further questions

Suppose now that we can answer the following question in the affirmative.

Question 2.6.15. For any $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ and any positive constant $C > 0$ so that $CI + A$ is positive definite, does there exist $A' \in \theta_*(\sqrt{-1}\mathfrak{z})$ and a positive constant $C' > 0$ such that

$$CI + A = C' e^{A'} ?$$

Conversely, for any $A' \in \theta_*(\sqrt{-1}\mathfrak{z})$ and any positive constant $C' > 0$, does there exist $A \in \theta_*(\sqrt{-1}\mathfrak{z})$ and a positive constant $C > 0$ such that

$$C'e^{A'} = CI + A?$$

Setting $C := 1 + C_A$ in (2.29), we would then have

$$\frac{N}{Vk^n}\bar{\mu}_X(g) = \left(CI - \frac{A}{2\pi k}\right)^{-1} = (C'e^{A'})^{-1} = (C')^{-1}e^{-A'} = C'' + A''$$

for some constant $C'' > 0$ and some $A'' \in \theta_*(\sqrt{-1}\mathfrak{z})$, and hence would show that (X,L) is relatively Chow polystable (by recalling Corollary 2.6.9), rather than weakly relatively Chow polystable. In other words, the affirmative resolution of Question 2.6.15 would prove the following conjecture (cf. Conjecture 1, [5]).

Conjecture 2.6.16. The existence of extremal metrics in $c_1(L)$ implies asymptotic Chow polystability of (X,L) relative to any maximal torus.

The affirmative resolution of Question 2.6.15 would have another consequence that we discuss now. Recall the following notion proposed by Sano and Tipler [107].

Definition 2.6.17. A Kähler metric ω_h is said to be **σ-balanced** if there exists $\sigma \in \mathrm{Aut}_0(X,L)$ such that $\omega_{FS(Hilb(h))} = \sigma^*\omega_h$.

By Lemma 2.4.2 and Theorem 2.2.11, we have

$$\begin{aligned}(\sigma^{-1})^*\omega_{FS(Hilb(h))} &= \omega_{FS(Hilb(h))} + \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}\log\left(\sum_i |\sum_j \theta(\sigma^{-1})_{ij}s_i|_{FS(Hilb(h))^k}\right) \\ &= \omega_h + \frac{\sqrt{-1}}{2\pi k}\partial\bar{\partial}\log\left(\sum_i |\sum_j \theta(\sigma^{-1})_{ij}s_i|_{h^k}\right).\end{aligned}$$

for a $Hilb(h)$-orthonormal basis $\{s_i\}$. Thus, being σ-balanced is equivalent to $\sum_i |\sum_j \theta(\sigma^{-1})_{ij}s_i|_{h^k}$ being constant for a $Hilb(h)$-orthonormal basis $\{s_i\}$. Arguing as in the proof of Proposition 2.6.12, we see that this is equivalent to h being balanced relative to a torus containing σ^{-1}, at the level k, with the index b_ν being the eigenvalues of $\theta(\sigma^{-1})^*\theta(\sigma^{-1})$. If the answer to Question 2.6.15 is affirmative, it would thus imply that a Kähler metric $\omega_{FS(H)}$, $H \in \mathscr{B}_k$, is σ-balanced in the sense of Sano–Tipler if and only if it satisfies $\mathscr{D}^*_{\omega_h}\mathscr{D}_{\omega_h}\rho_k(\omega_h) = 0$.

Remark 2.6.18. Note also that if ω_h is σ-balanced, Lemma 2.2.21 implies $\omega_h = (\sigma^{-1})^* \omega_{FS(Hilb(h))} = \omega_{FS(Hilb((\sigma^{-1})^* h))}$, and hence h must be necessarily of the form $FS(H')$ for some $H' \in \mathscr{B}_k$. Given the above argument, it seems natural to expect the following: if $\omega \in c_1(L)$ satisfies $\mathscr{D}^*_{\omega}\mathscr{D}_{\omega}\rho_k(\omega) = 0$, then it is necessarily of the form $\omega = \omega_{FS(H)}$ for some $H \in \mathscr{B}_k$.

Remark 2.6.19. For the toric case, Bunch and Donaldson [21] introduced the notion of "balanced" metrics for toric manifolds. It seems natural to expect that either of the above conditions, $\omega_{FS(Hilb(h))} = \sigma^* \omega_h$ or $\mathscr{D}^*_{\omega_h}\mathscr{D}_{\omega_h}\rho_k(\omega_h) = 0$, should be equivalent to their notion of balanced metrics when X is a toric Kähler manifold.

Chapter 3

Scalar curvature and Futaki invariant of Kähler metrics with cone singularities along a divisor

3.1 Introduction and the statement of the results

3.1.1 Kähler metrics with cone singularities along a divisor and log K-stability

Let D be a smooth effective divisor on a polarised Kähler manifold (X,L) of dimension n. Our aim is to study Kähler metrics that have cone singularities along D, which can be defined as follows (cf. §2 of [64]).

Definition 3.1.1. A **Kähler metric with cone singularities along D with cone angle** $2\pi\beta$ is a smooth Kähler metric on $X \setminus D$ which satisfies the following conditions when we write $\omega_{sing} = \sum_{i,j} g_{i\bar{j}} \sqrt{-1} dz_i \wedge d\bar{z}_j$ in terms of the local holomorphic coordinates $(z_1, \dots, z_n)$ on a neighbourhood $U \subset X$ with $D \cap U = \{z_1 = 0\}$:

1. $g_{1\bar{1}} = F|z_1|^{2\beta-2}$ for some strictly positive smooth bounded function F on $X \setminus D$,

2. $g_{1\bar{j}} = g_{i\bar{1}} = O(|z_1|^{2\beta-1})$,

3. $g_{i\bar{j}} = O(1)$ for $i, j \neq 1$.

Although this definition makes sense for any $\beta \in \mathbb{R}$, we are primarily interested in the case $0 < \beta < 1$ (cf. [48]). On the other hand, we sometimes need to consider the case $\beta > 1$ (cf. Remark 3.3.5), while some results (e.g. Theorem 3.1.15) will hold only for $0 < \beta < 3/4$. We thus set our convention as follows: we shall assume $0 < \beta < 1$ in what follows, and specifically point out when this assumption is violated.

Remark 3.1.2. We recall that the usual (cf. [28, 64, 110] amongst many others) definition of the conically singular Kähler metric ω_{sing} is that ω_{sing} is a smooth Kähler metric on $X \setminus D$ which is asymptotically quasi-isometric to the model cone metric $|z_1|^{2\beta-2}\sqrt{-1}dz_1 \wedge d\bar{z}_1 + \sum_{i=2}^n \sqrt{-1}dz_i \wedge d\bar{z}_i$ around D, with coordinates $(z_1, \dots, z_n)$ as above. The above definition is more restrictive than this usual definition, but will include all the cases that we shall treat in this chapter (cf. Definition 3.1.10).

Remark 3.1.3. We can regard a conically singular metric ω_{sing} as a $(1,1)$-current on X, and hence can make sense of its cohomology class $[\omega_{sing}] \in H^2(X, \mathbb{R})$.

Kähler–Einstein metrics that have cone singularities along a divisor were studied on Riemann surfaces by McOwen [88] and Troyanov [126], and on general Kähler manifolds by Tian [124] and Jeffres [63]. They have attracted renewed interest since the foundational work of Donaldson [48] on the linear theory of Kähler–Einstein metrics with cone singularities along a divisor, and since then, there has already been a huge accumulation of research on such metrics. Precisely, a conically singular metric ω_h is said to be **Kähler–Einstein with cone singularities along** $D \in |-\lambda K_X|$ **with cone angle** $2\pi\beta$, where $\lambda \in \mathbb{N}$ is some fixed integer, if it satisfies the following complex Monge–Ampère equation

$$\omega_h^n = |s|_{h^\lambda}^{2\beta-2}\Omega_h$$

on $X \setminus D$, where a hermitian metric h on $-K_X$ defines the Kähler metric ω_h and the volume form Ω_h on X, and s is a section of $-\lambda K_X$ which defines D by $\{s = 0\}$.

We now recall the **log** K**-stability**, which was introduced by Donaldson [48] and played a crucially important role in proving the Donaldson–Tian–Yau conjecture (Conjecture 1.2.6) for Fano manifolds [28, 29, 30] (cf. Theorem 1.2.10); see also Remark 3.2.6. We first recall (cf. Theorem 1.3.5) that the notion of K-stability can be regarded as an "algebro-geometric generalisation" of the vanishing of the Futaki invariant[1]

$$\mathrm{Fut}(\Xi_f, [\omega]) = \int_X f(S(\omega) - \bar{S})\frac{\omega^n}{n!} = \int_X f\left(\mathrm{Ric}(\omega) - \frac{\bar{S}}{n}\omega\right) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

[1] In what follows, we prefer to use the second expression using the Ricci curvature.

in the sense that $\mathrm{Fut}(\Xi_f, [\omega]) = 0$ is equivalent to $DF(\mathscr{X}, \mathscr{L}) = 0$ for the product test configuration $(\mathscr{X}, \mathscr{L})$ generated by Ξ_f (cf. Remark 1.3.4). Looking at the product *log* test configurations, we have an analogue of the Futaki invariant in the log case, which was first introduced by Donaldson [48]. It is written as

$$\mathrm{Fut}_{D,\beta}(\Xi_f, [\omega]) = \frac{1}{2\pi} \int_X f(S(\omega) - \bar{S}) \frac{\omega^n}{n!} - (1-\beta) \left(\int_D f \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)} \int_X f \frac{\omega^n}{n!} \right),$$

and may be called the **log Futaki invariant** (cf. §3.2, particularly Theorem 3.2.7). As in the case of the (classical) Futaki invariant, $\mathrm{Fut}_{D,\beta}$ is expected to vanish on Kähler classes which contain a Kähler–Einstein or constant scalar curvature Kähler metric with cone singularities along D with cone angle $2\pi\beta$.[2]

Now, in view of the work of Donaldson [39, 40, 41], we are naturally led to the idea of replacing the ample $-K_X$ by an arbitrary ample line bundle L, on a manifold X that is not necessarily Fano, and consider the constant scalar curvature Kähler metrics in $c_1(L)$ with cone singularities along a smooth effective divisor D (cf. Remark 3.1.3). Conically singular metrics having the constant scalar curvature can be defined as follows.

Definition 3.1.4. A Kähler metric ω_{sing} with cone singularities along D with cone angle $2\pi\beta$ is said to be of **constant scalar curvature Kähler** or **cscK** if its scalar curvature $S(\omega_{sing})$, which is a well-defined smooth function on $X \setminus D$, satisfies $S(\omega_{sing}) = \mathrm{const}$ on $X \setminus D$.

Remark 3.1.5. We now note that all the results on the conically singular Kähler metrics mentioned above are about Kähler–Einstein metrics with the anticanonical polarisation, and there seem to be very few results concerning the conically singular metrics along a divisor in a general polarisation. To the best of the author's knowledge, we only have [36, 65, 73, 92] treating general polarisations.

An important point, unlike in the Fano case where $D \in |-\lambda K_X|$ for some $\lambda \in \mathbb{N}$ was natural, is that D and L can be chosen completely independently; D can be any smooth effective divisor in X and the corresponding line bundle $\mathscr{O}_X(D)$ does not even have to be ample.

[2]This certainly holds for Kähler–Einstein metrics on Fano manifolds; see Theorem 2.1, [110] and also Theorem 7, [30].

Remark 3.1.6. In general, if ω_{sing} is a metric with cone singularities along D (as in Remark 3.1.2), then it follows that any $f \in C^{\infty}(X,\mathbb{R})$ is integrable with respect to the measure ω^n_{sing} on any open set $U \subset X \setminus D$; this is because there exist positive constants C_1, C_2 such that

$$C_1 |z_1|^{2\beta-2} \prod_{i=1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i \leq \omega^n_{sing} \leq C_2 |z_1|^{2\beta-2} \prod_{i=1}^{n} \sqrt{-1} dz_i \wedge d\bar{z}_i$$

locally around D, and $|z_1|^{2\beta-2}\sqrt{-1}dz_1 \wedge d\bar{z}_1 = 2r^{2\beta-1}drd\theta$, $z_1 = re^{\sqrt{-1}\theta}$, is integrable over the punctured unit disk in $\mathbb{C}$. This fact will be used many times in what follows.

In particular, the volume $\int_{X \setminus D} \omega^n_{sing}$ of $X \setminus D$ is finite. By regarding ω^n_{sing} as an absolutely continuous measure on the whole of X, we shall write $\mathrm{Vol}(X, \omega_{sing}) := \int_{X \setminus D} \omega^n_{sing}$ in what follows.

3.1.2 Momentum-constructed metrics and log Futaki invariant

The study of cscK metrics is considered to be much harder than that of Kähler–Einstein metrics, since there is no analogue of the complex Monge–Ampère equation which reduces the fourth order fully nonlinear PDE to a second order fully nonlinear PDE. However, when the space X is endowed with some symmetry, it is often possible to simplify the PDE by exploiting the symmetry of the space X. One such example, which we shall treat in detail in what follows, is the **momentum construction** introduced by Hwang [61] and generalised by Hwang–Singer [62], which works, for example, when X is the projective completion $\mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ of a pluricanonical bundle $\mathscr{F}$ over a product of Kähler–Einstein manifolds (see §3.3.1 for details). The point is that this theory converts the cscK equation to a *second order linear ODE*, as we recall in §3.3.1.

Moreover, it is also possible to describe the cone singularities in terms of the boundary value of the function called momentum profile; a detailed discussion on this can be found in §3.3.2. This means that we have on $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ a particular class of conically singular metrics, which we may call **momentum-constructed conically singular metrics**, whose scalar curvature is easy to handle.

By using the above theory of momentum construction, we obtain the following main result of this chapter. Suppose that (M, ω_M) is a product of Kähler–Einstein Fano manifolds (M_i, ω_i), $i = 1, \dots, r$, each with $b_2(M_i) = 1$, and of dimension n_i so that

$n-1=\sum_{i=1}^{r} n_i$. Let $\mathscr{F} := \bigotimes_{i=1}^{r} p_i^* K_i^{\otimes l_i}$, $l_i \in \mathbb{Z}$, K_i be the canonical bundle of M_i, and $p_i : M \twoheadrightarrow M_i$ be the obvious projection. The statement is as follows.

Theorem 3.1.7. *Let $X := \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$, and write D for the ∞-section of $\mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ and Ξ for the generator of the fibrewise $\mathbb{C}^*$-action. Then, each Kähler class $[\omega] \in H^2(X,\mathbb{R})$ of X admits a momentum-constructed cscK metric with cone singularities along D with cone angle $2\pi\beta \in [0,\infty)$ if and only if* $\mathrm{Fut}_{D,\beta}(\Xi,[\omega]) = 0$.

The reader is referred to §3.3.1 for more details on this statement, including where the various hypotheses on X came from. Simple examples to which the above theorem applies are given in Remark 3.3.5.

Remark 3.1.8. Note that the value of β for which this happens is unique in each Kähler class $[\omega] \in H^2(X,\mathbb{R})$, given by the equation $\mathrm{Fut}_{D,\beta}(\Xi,[\omega]) = 0$ which we can re-write as

$$\beta = 1 - \mathrm{Fut}(\Xi,[\omega]) \left(\int_D f \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)} \int_X f \frac{\omega^n}{n!} \right)^{-1},$$

where f is the holomorphy potential of Ξ; the denominator in the second term is equal to $Q(b)(b-B/A)$ in the notation of (3.25), which is strictly positive. We also need to note that we do *not* necessarily have $0<\beta<1$; although we can show $\beta \geq 0$, there are examples where $\beta > 1$. See Remark 3.3.5 for more details.

Remark 3.1.9. A naive re-phrasing of the above result is that each rational Kähler class (or polarisation) of $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ admits a momentum-constructed cscK metric with cone singularities along D with cone angle $2\pi\beta$ if and only if it is log K-polystable with cone angle $2\pi\beta$ with respect to the product log test configuration generated by the fibrewise $\mathbb{C}^*$-action on X. As far as the author is aware, this is the first supporting evidence for the log Donaldson–Tian–Yau conjecture (Conjecture 3.2.5) for the polarisations that are not anticanonical.

3.1.3 Log Futaki invariant computed with respect to the conically singular metrics

Although the log Futaki invariant is conjectured to be related to the existence of conically singular cscK metrics, the log Futaki invariant itself is computed with respect to a *smooth* Kähler metric in $c_1(L)$. We now consider the following question: what is the value of the log Futaki invariant if we compute it with respect to a conically singular Kähler metric?[3] Namely, we wish to compute the following quantity

$$\begin{aligned}\mathrm{Fut}_{D,\beta}(\Xi_f,\omega_{sing}) = &\int_X f\left(\mathrm{Ric}(\omega_{sing}) - \frac{\overline{S}(\omega_{sing})}{n}\omega_{sing}\right)\wedge\frac{\omega_{sing}^{n-1}}{(n-1)!} \\ &-2\pi(1-\beta)\left(\int_D f\frac{\omega_{sing}^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega_{sing})}{\mathrm{Vol}(X,\omega_{sing})}\int_X f\frac{\omega_{sing}^{n}}{n!}\right),\end{aligned}$$

where $\overline{S}(\omega_{sing}) := \frac{1}{\mathrm{Vol}(X,\omega_{sing})}\int_X \mathrm{Ric}(\omega_{sing})\wedge\frac{\omega_{sing}^{n-1}}{(n-1)!}$. However, this is not a priori well-defined for any conically singular metric ω_{sing}; first of all $\int_D f\frac{\omega_{sing}^{n-1}}{(n-1)!}$ does not naively make sense as ω_{sing} is not well-defined on D, and it is not obvious that the integral $\int_X \mathrm{Ric}(\omega_{sing})\wedge\frac{\omega_{sing}^{n-1}}{(n-1)!}$ or $\int_X f\mathrm{Ric}(\omega_{sing})\wedge\frac{\omega_{sing}^{n-1}}{(n-1)!}$ makes sense.[4]

In what follows, we do not claim any result on this problem that is true for *all* conically singular metrics, and restrict our attention to the case where the conically singular metric ω_{sing} has some "preferable" form. By this, we mean that ω_{sing} is either of the following types.

Definition 3.1.10.

1. Let $\mathscr{O}_X(D)$ be the line bundle associated to D and s be a global section that defines D by $\{s=0\}$. Giving a hermitian metric h on $\mathscr{O}_X(D)$, we define $\hat{\omega} := \omega + \lambda\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$ which is indeed a Kähler metric if $\lambda > 0$ is chosen to be sufficiently small. Metrics of such form have been studied in many papers ([20, 25, 48, 64] amongst others), but, due to the apparent lack of the naming convention in the existing literature[5], we decide to call such a metric $\hat{\omega}$ a conically singular metric **of elementary form**.

2. When X is a projective completion $\mathbb{P}(\mathscr{F}\oplus\mathbb{C})$ of a line bundle $\mathscr{F}$ over a Kähler manifold M, with the projection map $p:\mathscr{F}\to M$, we can consider a **momentum-constructed** metric (as we mentioned in §3.1.2; see also §3.3.1 for the details). We have an explicit description of cone singularities, as we shall see in §3.3.2.

[3] Auvray [11] established an analogous result for the Poincaré type metric, which can be regarded as the $\beta = 0$ case.

[4] Note that $\mathrm{Vol}(X,\omega_{sing})$ does make sense by Remark 3.1.6.

[5] Calamai and Zheng [25] in fact call it a *model metric*, but we decide not to use this terminology in order to avoid confusion with the *model cone metric* that appeared in Remark 3.1.2.

What is common in these two classes of metrics is that they can be written as a sum of a smooth differential form on X and a term of order $O(|z_1|^{2\beta})$, together with some more explicit estimates on the second $O(|z_1|^{2\beta})$ term, which will be important for us in proving that these metrics enjoy some nice estimates on the Ricci (and scalar) curvature (cf. §3.3.2, §3.4.1); see also Remark 3.4.8.

For these types of metrics, $\hat{\omega}$ and ω_φ, we first show that $\mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1}$ and $\mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}$ define a current that is well-defined on the whole of X. In fact, we can even show that they are well-defined as a current on any open subset Ω in X, as stated in the following. They are the main technical results that are used in what follows to compute the log Futaki invariant.

Theorem 3.1.11. *Let $\hat{\omega}$ be a conically singular Kähler metric of elementary form $\hat{\omega} = \omega + \lambda \sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$ with $0 < \beta < 1$. Then the following equation*

$$\int_\Omega f \mathrm{Ric}(\hat{\omega}) \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_{\Omega \setminus D} f S(\hat{\omega}) \frac{\hat{\omega}^n}{n!} + 2\pi(1-\beta) \int_{\Omega \cap D} f \frac{\omega^{n-1}}{(n-1)!}$$

holds for any open set $\Omega \subset X$ and any $f \in C^\infty(X, \mathbb{R})$, and all the integrals are finite.

Theorem 3.1.12. *Let $p : \mathscr{F} \to M$ be a holomorphic line bundle with hermitian metric $h_{\mathscr{F}}$ over a Kähler manifold (M, ω_M), and ω_φ be a momentum-constructed conically singular Kähler metric on $X := \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ with a real analytic momentum profile φ and $0 < \beta < 1$. Then the following equation*

$$\int_\Omega f \mathrm{Ric}(\omega_\varphi) \wedge \frac{\omega_\varphi^{n-1}}{(n-1)!} = \int_{\Omega \setminus D} f S(\omega_\varphi) \frac{\omega_\varphi^n}{n!} + 2\pi(1-\beta) \int_{\Omega \cap D} f \frac{p^* \omega_M(b)^{n-1}}{(n-1)!}$$

holds for any open set $\Omega \subset X$ and any $f \in C^\infty(X, \mathbb{R})$, and all the integrals are finite, where $\omega_M(b)$ is as defined in (3.3).

Remark 3.1.13. We note that Theorems 3.1.11 and 3.1.12 bear some similarities to the equation (4.60) in Proposition 4.2, proved by Song and Wang [110]. The main difference is that our theorems show that $\mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1}$ (resp. $\mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}$) is a current well-defined over any open subset Ω in X, as opposed to just computing $\int_X \mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1}$ (resp. $\int_X \mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}$); indeed our proof is quite different to theirs, although we have in common the basic strategy of doing the integration by parts "correctly".

Recalling (cf. Theorem 3.2.7) that the log Futaki invariant $\mathrm{Fut}_{D,\beta}$ is defined as a sum of the classical Futaki invariant (cf. Theorem 1.3.5) and a "correction" term, we first compute the classical Futaki invariant with respect to the conically singular metrics, of elementary form and momentum-constructed, as follows. Theorem 3.1.11 enables us to make sense[6] of the following quantity

$$\mathrm{Fut}(\Xi,\hat{\omega}) := \int_X \hat{H}\left(\mathrm{Ric}(\hat{\omega}) - \frac{\bar{S}(\hat{\omega})}{n}\right) \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!},$$

where $\hat{H}$ is the holomorphy potential of Ξ with respect to $\hat{\omega}$. Similarly, Theorem 3.1.12 enables us to make sense of $\mathrm{Fut}(\Xi,\omega_\varphi)$ computed with respect to the momentum-constructed conically singular metric ω_φ with real analytic momentum profile φ. The result that we obtain is as follows.

Corollary 3.1.14.

1. *Suppose that Ξ is a holomorphic vector field on X which preserves D. Write H for the holomorphy potential of Ξ with respect to ω, and $\hat{H}$ for the one with respect to a conically singular metric of elementary form $\hat{\omega}$ with $0<\beta<1$. Then we have*

$$\begin{aligned}\mathrm{Fut}(\Xi,\hat{\omega}) = &\int_{X\setminus D} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega}))\frac{\hat{\omega}^n}{n!}\\ &+2\pi(1-\beta)\left(\int_D H\frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\hat{\omega})}\int_X \hat{H}\frac{\hat{\omega}^n}{n!}\right),\end{aligned}$$

   where $\underline{S}(\hat{\omega})$ is the average of $S(\hat{\omega})$ over $X\setminus D$ and all the integrals are finite.

2. *Writing Ξ for the generator of the fibrewise $\mathbb{C}^*$-action on $X=\mathbb{P}(\mathscr{F}\oplus\mathbb{C})$, and τ for the holomorphy potential with respect to a momentum-constructed conically singular metric ω_φ with $0<\beta<1$, we have*

$$\begin{aligned}\mathrm{Fut}(\Xi,\omega_\varphi) = &\int_{X\setminus D} \tau(S(\omega_\varphi) - \underline{S}(\omega_\varphi))\frac{\omega_\varphi^n}{n!}\\ &+2\pi(1-\beta)\left(b\mathrm{Vol}(M,\omega_M(b)) - \frac{\mathrm{Vol}(M,\omega_M(b))}{\mathrm{Vol}(X,\omega_\varphi)}\int_X \tau\frac{\omega_\varphi^n}{n!}\right),\end{aligned}$$

[6] In fact, there is also a subtlety involving the asymptotic behaviour of the holomorphy potential $\hat{H}$, cf. §3.4.3.1 and §3.4.3.2.

where D is the ∞-section defined by $\tau = b$, and $\omega_M(b)$ is as defined in (3.3); see §3.3.1. All the integrals in the above are finite.

We finally compute the log Futaki invariant, as stated in the following theorem; a key result is that the "distributional" term in $\mathrm{Fut}(\Xi, \hat{\omega})$ (resp. $\mathrm{Fut}(\Xi, \omega_\varphi)$) exactly cancels the "correction" term in the log Futaki invariant (cf. Corollary 3.5.3 (resp. Corollary 3.5.7)). We also prove a partial invariance result for the Futaki invariant, when it is computed with respect to these classes of conically singular metrics. For the smooth metrics, that the Futaki invariant depends only on the Kähler class is a well-known theorem of Futaki [54] (cf. Theorem 1.3.5), where the proof crucially relies on the integration by parts. When we compute it with respect to conically singular metrics, we are essentially on the noncompact manifold $X \setminus D$, and hence cannot naively apply the integration by parts. Still, we can claim the following result.

Theorem 3.1.15. *Suppose* $0 < \beta < 3/4$.

1. *The log Futaki invariant computed with respect to a conically singular metric of elementary form $\hat{\omega}$, evaluated against a holomorphic vector field Ξ which preserves D and with the holomorphy potential $\hat{H}$, is given by*

$$\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega}) = \frac{1}{2\pi} \int_{X \setminus D} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!},$$

and it is invariant under the change $\hat{\omega} \mapsto \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi$ for any smooth function $\psi \in C^\infty(X, \mathbb{R})$ with $\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0$ on $X \setminus D$, i.e.

$$\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi) = \mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega}) = \frac{1}{2\pi} \int_{X \setminus D} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!}.$$

In particular, if $\hat{\omega}$ is cscK, $\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi) = 0$ for any $\psi \in C^\infty(X, \mathbb{R})$ with $\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\psi > 0$ on $X \setminus D$.

2. *Suppose that the σ-constancy hypothesis (cf. Definition 3.3.1) is satisfied for our data, and let D be the ∞-section of $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$. Then the log Futaki invariant computed with respect to a momentum-constructed conically singular metric ω_φ,*

evaluated against the generator Ξ of fibrewise $\mathbb{C}^*$-action, is given by

$$\mathrm{Fut}_{D,\beta}(\Xi,\omega_\varphi)=\int_{X\setminus D}\tau(S(\omega_\varphi)-\underline{S}(\omega_\varphi))\frac{\omega_\varphi^n}{n!},$$

and it is invariant under the change $\omega_\varphi \mapsto \omega_\varphi+\sqrt{-1}\partial\bar{\partial}\psi$ for any smooth function $\psi \in C^\infty(X,\mathbb{R})$ with $\omega_\varphi+\sqrt{-1}\partial\bar{\partial}\psi>0$ on $X\setminus D$.

Remark 3.1.16. The author conjectures that the result should be true for $0<\beta<1$ in general.

3.1.4 Organisation of the chapter

We first review the basics on log K-stability and log Futaki invariant in §3.2.

§3.3 discusses in detail the momentum-constructed conically singular metrics and log Futaki invariant, in particular our main result Theorem 3.1.7; §3.3.1 is a general introduction, and §3.3.2 discusses some basic properties of momentum-constructed metrics that have cone singularities. §3.3.3 is devoted to the proof of Theorem 3.1.7.

§3.4 and §3.5 discuss in detail the log Futaki invariant computed with respect to conically singular metrics, as presented in §3.1.3. After collecting some basic estimates on conically singular metrics of elementary form in §3.4.1, we prove in §3.4.2 that the current $\mathrm{Ric}(\hat{\omega})\wedge\hat{\omega}^{n-1}$ (and $\mathrm{Ric}(\omega_\varphi)\wedge\omega_\varphi^{n-1}$) is well-defined on the whole of X, as stated in Theorems 3.1.11 and 3.1.12. Corollary 3.1.14 is proved in §3.4.3.

§3.5 is concerned with the proof of Theorem 3.1.15; the main result of §3.5.1 is Corollary 3.5.3 (see also Remark 3.5.4), which reduces the claim (for the conically singular metrics of elementary form) to the computations that we do in §3.5.2 along the line of proving the invariance of the classical Futaki invariant (i.e. the smooth case). §3.5.3 establishes the claim for the momentum-constructed conically singular metrics.

3.2 Log Futaki invariant and log K-stability

Donaldson [48] introduced the notion of log K-stability, in the attempt to solve Conjecture 1.2.6 for the Fano manifolds; see also Remark 3.2.6. This is a variant of K-stability that is expected to be more suited to conically singular cscK metrics. We refer to [48, 92] for a general introduction.

This purely algebro-geometric notion can be defined for an n-dimensional polarised normal variety (X,L) together with an effective integral reduced divisor $D \subset X$, but we will throughout assume that (X,L) is a polarised Kähler manifold and $D \subset X$ is a smooth effective divisor as this is the case we will be exclusively interested in. We write $((X,D);L)$ for these data.

Suppose now that we have a test configuration $(\mathscr{X},\mathscr{L})$ for (X,L). As in §1.2.1, the equivariant $\mathbb{C}^*$-action on $\mathscr{X}$ induces an action on the central fibre $\mathscr{X}_0$, and hence an action on $H^0(\mathscr{X}_0,\mathscr{L}^{\otimes k}|_{\mathscr{X}_0})$ for any $k \in \mathbb{N}$. We write d_k for $\dim H^0(\mathscr{X}_0,\mathscr{L}^{\otimes k}|_{\mathscr{X}_0})$ and w_k for the weight of the $\mathbb{C}^*$-action on $\bigwedge^{\max} H^0(\mathscr{X}_0,\mathscr{L}^{\otimes k}|_{\mathscr{X}_0})$. As we saw in §1.2.1, these admit an expansion in $k \gg 1$ as

$$\begin{aligned} d_k &= a_0 k^n + a_1 k^{n-1} + \cdots \\ w_k &= b_0 k^{n+1} + b_1 k^n + \cdots \end{aligned}$$

where a_i, b_i are some rational numbers.

The $\mathbb{C}^*$-action on $\mathscr{X}$ naturally induces a test configuration $(\mathscr{D},\mathscr{L}|_{\mathscr{D}})$ of $(D,L|_D)$ by supplementing the orbit of D (under the $\mathbb{C}^*$-action) with the flat limit. Similarly to the above, writing $\mathscr{D}_0$ for the central fibre, we write $\tilde{d}_k$ for $\dim H^0(\mathscr{D}_0,\mathscr{L}^{\otimes k}|_{\mathscr{D}_0})$ and $\tilde{w}_k$ for the weight of the $\mathbb{C}^*$-action on $\bigwedge^{\max} H^0(\mathscr{D}_0,\mathscr{L}^{\otimes k}|_{\mathscr{D}_0})$. We have the expansion

$$\begin{aligned} \tilde{d}_k &= \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + \cdots \\ \tilde{w}_k &= \tilde{b}_0 k^n + \tilde{b}_1 k^{n-1} + \cdots \end{aligned}$$

exactly as above, where $\tilde{a}_i$, $\tilde{b}_i$ are some rational numbers.

Thus a test configuration $(\mathscr{X},\mathscr{L})$ and a choice of divisor $D \subset X$ gives us two test configurations $(\mathscr{X},\mathscr{L})$ and $(\mathscr{D},\mathscr{L}|_{\mathscr{D}})$. We call the pair $(\mathscr{X},\mathscr{L})$ and $(\mathscr{D},\mathscr{L}|_{\mathscr{D}})$ constructed as above a **log test configuration** for the pair $((X,D);L)$, and write $((\mathscr{X},\mathscr{D});\mathscr{L})$ to denote these data. We now define the **log Donaldson–Futaki invariant**

$$DF(\mathscr{X},\mathscr{D},\mathscr{L},\beta) := \frac{2(a_0 b_1 - a_1 b_0)}{a_0} - (1-\beta)\left(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0\right), \tag{3.1}$$

analogously to Definition 1.2.4.

We now consider a special case where the log test configuration $((\mathscr{X},\mathscr{D});\mathscr{L})$ is given by a $\mathbb{C}^*$-action on X which lifts to L and preserves D. We then have isomorphisms $\mathscr{X} \cong X \times \mathbb{C}$ and $\mathscr{D} \cong D \times \mathbb{C}$, and in particular the central fibre $\mathscr{X}_0$ (resp. $\mathscr{D}_0$) is isomorphic to X (resp. D). Note that the above isomorphisms are not necessarily equivariant, and hence the central fibres $\mathscr{X}_0 \cong X$ and $\mathscr{D}_0 \cong D$ could have a nontrivial $\mathbb{C}^*$-action. In this case the log test configuration $((\mathscr{X},\mathscr{D});\mathscr{L})$ is called **product**. In the more restrictive case where the above isomorphisms are equivariant, i.e. when $\mathbb{C}^*$-action acts trivially on the central fibres $\mathscr{X}_0 \cong X$ and $\mathscr{D}_0 \cong D$, the log test configurations is called **trivial**.

Remark 3.2.1. As in Remark 1.3.4, a product log test configuration is exactly a choice of $\Xi \in H^0(X, T_X)$ that admits a holomorphy potential and preserves D (i.e. is tangential to D).

With these preparations, the log K-stability can now be defined as follows.

Definition 3.2.2. A pair $((X,D);L)$ is called **log K-semistable with cone angle** $2\pi\beta$ if $DF(\mathscr{X},\mathscr{D},\mathscr{L},\beta) \geq 0$ for any log test configuration $((\mathscr{X},\mathscr{D});\mathscr{L})$ for $((X,D);L)$. It is called **log K-polystable with cone angle** $2\pi\beta$ if it is log K-semistable with cone angle $2\pi\beta$ and $DF(\mathscr{X},\mathscr{D},\mathscr{L},\beta) = 0$ if and only if $((\mathscr{X},\mathscr{D});\mathscr{L})$ is product. It is called **log K-stable with cone angle** $2\pi\beta$ if it is log K-semistable with cone angle $2\pi\beta$ and $DF(\mathscr{X},\mathscr{D},\mathscr{L},\beta) = 0$ if and only if $((\mathscr{X},\mathscr{D});\mathscr{L})$ is trivial.

Remark 3.2.3. We need some restriction on the singularities of $\mathscr{X}$ and $\mathscr{D}$ to define log K-stability (cf. Remark 1.2.3), when the log test configuration is not product or trivial (cf. [92]), but we do not discuss this issue since only the product log test configurations will be important for us later.

Remark 3.2.4. While we shall see later (cf. Corollary 3.5.3 and Remark 3.5.4 that follows) in differential-geometric context how the "extra" terms $(1-\beta)\left(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0\right)$ in (3.1) (or the corresponding terms in (3.2)) come out, they come out naturally in the blow-up formalism [92] in algebraic geometry (cf. Theorem 3.7, [92]).

The following may be called the **log Donaldson–Tian–Yau conjecture**. This seems to be a folklore conjecture in the field, and is mentioned in e.g. [36, 65].

Conjecture 3.2.5. $((X,D);L)$ is log K-polystable with cone angle $2\pi\beta$ if and only if X admits a cscK metric in $c_1(L)$ with cone singularities along D with cone angle $2\pi\beta$.

Remark 3.2.6. When X is Fano with $L = -\lambda K_X$ (for some $\lambda \in \mathbb{N}$) and $D \in |-\lambda K_X|$, this conjecture was affirmatively solved by Berman [13] and Chen–Donaldson–Sun [28, 29, 30]. Berman [13] first proved that the existence of conically singular Kähler–Einstein metric with cone angle $2\pi\beta$ implies log K-stability of $((X,D);-\lambda K_X)$ with cone angle $2\pi\beta$, and Chen–Donaldson–Sun [28, 29, 30] proved that the log K-stability with cone angle $2\pi\beta$ implies the existence of the conically singular Kähler–Einstein metric with cone angle $2\pi\beta$, in the course of proving the "ordinary" version of the Donaldson–Tian–Yau conjecture (Conjecture 1.2.6) for Fano manifolds.

Let $f \in C^{\infty}(X,\mathbb{C})$ be the holomorphy potential, with respect to ω, of the holomorphic vector field Ξ_f on X which preserves D. Recall that we use the sign convention $\iota(\Xi_f)\omega = -\bar{\partial} f$ for the holomorphy potential. Let $((\mathscr{X},\mathscr{D});\mathscr{L})$ be the product log test configuration defined by Ξ_f (cf. Remark 3.2.1). In this case, a straightforward adaptation of the argument in §2 of [41] shows the following.

Theorem 3.2.7. (Donaldson [41, 48]) *The log Donaldson-Futaki invariant reduces to the following differential-geometric formula*

$$
\begin{aligned}
DF(\mathscr{X},\mathscr{D},\mathscr{L},\beta) &= \mathrm{Fut}_{D,\beta}(\Xi_f,[\omega]) \\
&:= \frac{1}{2\pi}\mathrm{Fut}(\Xi_f,[\omega]) - (1-\beta)\left(\int_D f\frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)}\int_X f\frac{\omega^n}{n!}\right),
\end{aligned} \tag{3.2}
$$

defined for some (in fact any) smooth Kähler metric $\omega \in c_1(L)$, when the log test configuration $((\mathscr{X},\mathscr{D});\mathscr{L})$ is product, defined the holomorphic vector field Ξ_f on X which preserves D. In the formula above, $\mathrm{Vol}(D,\omega) := \int_D \frac{\omega^{n-1}}{(n-1)!}$ *and* $\mathrm{Vol}(X,\omega) := \int_X \frac{\omega^n}{n!}$ *are the volumes given by the smooth Kähler metric $\omega \in c_1(L)$.*

We may call the above $\mathrm{Fut}_{D,\beta}$ the **log Futaki invariant**, where the fact that $\mathrm{Fut}_{D,\beta}(\Xi_f,[\omega])$ depends only on the Kähler class $[\omega]$ (and not on the specific choice of the metric) can be shown exactly as the classical case; see e.g. §4.2, [119].

3.3 Momentum-constructed cscK metrics with cone singularities along a divisor

3.3.1 Background and overview

Consider a Kähler manifold (M, ω_M) of complex dimension $n-1$ together with a holomorphic line bundle $p : \mathscr{F} \to M$, endowed with a hermitian metric $h_{\mathscr{F}}$ with curvature form $\gamma := -\sqrt{-1}\partial\bar{\partial}\log h_{\mathscr{F}}$. We first consider Kähler metrics on the total space of $\mathscr{F}$, which can be regarded as an open dense subset of $X := \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$; we shall later impose some "boundary conditions" for these metrics to extend to X. Consider a Kähler metric on the total space of $\mathscr{F}$ of the form[7] $p^*\omega_M + dd^c f(t)$, where f is a function of t, and t is the log of the fibrewise norm function defined by $h_{\mathscr{F}}$ serving as a fibrewise radial coordinate. A Kähler metric of this form is said to satisfy the **Calabi ansatz**.

This setting was studied by Hwang [61] and Hwang–Singer [62], in terms of the moment map associated to the fibrewise $U(1)$-action on the total space of $\mathscr{F}$. Suppose that we write $\frac{\partial}{\partial\theta}$ for the generator of this $U(1)$-action, normalised so that $\exp(2\pi\frac{\partial}{\partial\theta}) = 1$, and τ for the corresponding moment map with respect to the Kähler form $\omega_f := p^*\omega_M + dd^c f(t)$. An observation of Hwang and Singer [62] was that the function $||\frac{\partial}{\partial\theta}||^2_{\omega_f}$ is constant on each level set of τ, and hence we have a function $\varphi : I \to \mathbb{R}_{\geq 0}$, defined on the range $I \subset \mathbb{R}$ of the moment map τ, given by

$$\varphi(\tau) := \left|\left|\frac{\partial}{\partial\theta}\right|\right|^2_{\omega_f}$$

which is called the **momentum profile** in [62].

An important point of this theory is that we can in fact "reverse" the above construction as follows. We start with some interval $I \subset \mathbb{R}$ (called **momentum interval** in [62]) and $\tau \in I$ such that

$$\omega_M(\tau) := \omega_M - \tau\gamma > 0, \tag{3.3}$$

and write $\{p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M), I\}$ for this collection of data. We now consider a function φ which is smooth on I and positive on the interior of I. Proposition 1.4 (and

[7] We shall use the convention $d^c := \sqrt{-1}(\bar{\partial} - \partial)$.

also §2.1) of [62] shows that the Kähler metric on $\mathscr{F}$ defined by

$$\omega_\varphi := p^*\omega_M - \tau p^*\gamma + \frac{1}{\varphi} d\tau \wedge d^c\tau = p^*\omega_M(\tau) + \frac{1}{\varphi} d\tau \wedge d^c\tau \tag{3.4}$$

is equal to $\omega_f = p^*\omega_M + dd^c f(t)$ satisfying the Calabi ansatz, where (f,t) and (φ,τ) are related in the way as described in (2.2) and (2.3) of [62].

We now come back to the projective completion $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ of $\mathscr{F}$, and suppose that $\omega_f = p^*\omega_M + dd^c f(t)$ extends to a well-defined Kähler metric on X. In this case, without loss of generality we may write $I = [-b, b]$ for some $b > 0$; $\tau = b$ (resp. $\tau = -b$) corresponds to the ∞-section (resp. 0-section) of $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$, cf. §2.1, [62]. Hwang [61] proved[8] that the condition for ω_φ defined by (3.4) to extend to a well-defined Kähler metric on X is given by the following boundary conditions for φ at ∂I: $\varphi(\pm b) = 0$ and $\varphi'(\pm b) = \mp 2$. We can thus construct a Kähler metric ω_φ on X from the data $\{p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M), I\}$, and such ω_φ is said to be **momentum-constructed**.

We recall the following notion.

Definition 3.3.1. The data $\{p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M), I\}$ are said to be **σ-constant** if the curvature endomorphism $\omega_M^{-1}\gamma$ has constant eigenvalues on M, and the Kähler metric $\omega_M(\tau)$ (on M) has constant scalar curvature for each $\tau \in I$.

The advantage of assuming the σ-constancy is that the scalar curvature $S(\omega_\varphi)$ of ω_φ can be written as

$$S(\omega_\varphi) = R(\tau) - \frac{1}{2Q} \frac{\partial^2}{\partial \tau^2} (\varphi Q)(\tau) \tag{3.5}$$

in terms of τ, where

$$Q(\tau) := \frac{\omega_M(\tau)^{n-1}}{\omega_M^{n-1}} \tag{3.6}$$

and

$$R(\tau) := \mathrm{tr}_{\omega_M(\tau)} \mathrm{Ric}(\omega_M) \tag{3.7}$$

are both functions of τ by virtue of the σ-constancy hypothesis. Note that (3.5) means that the cscK equation $S(\omega_\varphi) = \text{const}$ is now a second order linear ODE.

In what follows, we assume that (M, ω_M) is a product of Kähler–Einstein manifolds (M_i, ω_i), and $\mathscr{F} := \bigotimes_{i=1}^r p_i^* K_i^{\otimes l_i}$, where $l_i \in \mathbb{Z}$, $p_i : M \twoheadrightarrow M_i$ is the obvious projec-

[8] See also Proposition 1.4 and §2.1 of [62]. The boundary condition of φ at $\partial I = \{\pm b\}$ will be discussed later in detail.

tion, and K_i is the canonical bundle of M_i (we can in fact assume $l_i \in \mathbb{Q}$ as long as $K_i^{\otimes l_i}$ is a genuine line bundle, rather than a $\mathbb{Q}$-line bundle). It is easy to see that this satisfies the σ-constancy. We also assume that each M_i is Fano, as in [61]; this hypothesis is needed in the Appendix A of [61], which will also be used in §3.3.3.1.

We now recall the work of Hwang (cf. Theorem 1, [61]), who constructed an extremal metric on $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ in every Kähler class.

Theorem 3.3.2. (Hwang [61], Corollary 1.2 and Theorem 2) *The projective completion $\mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ of a line bundle $\mathscr{F} := \bigotimes_{i=1}^{r} p_i^* K_i^{\otimes l_i}$, over a product of Kähler–Einstein Fano manifolds, each with the second Betti number* 1*, admits an extremal metric in each Kähler class.*

Remark 3.3.3. We also recall that the scalar curvature of these extremal metrics can be written as $S(\omega_\varphi) = \sigma_0 + \lambda\tau$ where σ_0 and λ are constants (cf. Lemma 3.2 [61]).

Whether this extremal metric is in fact cscK depends on the (classical) Futaki invariant, by recalling Lemma 1.4.5. Hwang's argument, however, gives the following alternative viewpoint on this problem. The above formula $S(\omega_\varphi) = \sigma_0 + \lambda\tau$ for the scalar curvature of the extremal metric of course implies that ω_φ is cscK if and only if $\lambda = 0$, and hence the question reduces to whether there exists a well-defined extremal Kähler metric ω_φ such that $S(\omega_\varphi)$ has $\lambda = 0$. As Hwang [61] shows, the obstruction for achieving this is the following boundary conditions for φ at $\partial I = \{-b, +b\}$: $\varphi(\pm b) = 0$ and $\varphi'(\pm b) = \mp 2$. They are the conditions that must be satisfied for ω_φ to be a well-defined smooth metric on X; $\varphi(\pm b) = 0$ means that the fibres "close up", and $\varphi'(\pm b) = \mp 2$ means that the metric is smooth along the ∞-section (resp. 0-section).

It is not possible to achieve $\lambda = 0$, $\varphi(\pm b) = 0$, $\varphi'(\pm b) = \mp 2$ all at the same time if the Futaki invariant is not zero. On the other hand, however, we can brutally set $\lambda = 0$ and try to see what happens to $\varphi(\pm b)$ and $\varphi'(\pm b)$. In fact, it is possible to set $\lambda = 0$, $\varphi(\pm b) = 0$, and $\varphi(-b) = 2$ all at the same time[9], as discussed in §3.2 [61] and recalled in §3.3.3.1 below. Thus, we should have $\varphi'(b) \neq -2$ if the Futaki invariant is not zero. A crucially important point for us is that the value $-\pi\varphi'(b) = 2\pi\beta$ is the angle of the

[9]It is possible to set $\varphi(b) = -2$ instead of $\varphi(-b) = 2$ in here, and in this case ω_φ will be smooth along the ∞-section with cone singularities along the 0-section; this is purely a matter of convention. However, just to simplify the argument, we will assume henceforth that ω_φ is always smooth along the 0-section with the cone singularities forming along the ∞-section.

cone singularities that the metric develops along the ∞-section, if φ is real analytic on I. This point is briefly mentioned in p2299 of [62] and seems to be well-known to the experts (cf. Lemma 2.3 of [73]). However, as the author could not find an explicitly written proof in the literature, the proof of this fact is provided in Lemma 3.3.6, §3.3.2, where the author thanks Michael Singer for the instructions on how to prove it.

What we prove in §3.3.3.1 is that it is indeed possible to run the argument as above, namely it is indeed possible to have a cscK metric on X in each Kähler class, at the cost of introducing cone singularities along the ∞-section. An important point here is that the cone angle $2\pi\beta$ is *uniquely determined in each Kähler class*; we can even obtain an explicit formula (equation (3.22)) for the cone angle.

We compute in §3.3.3.2 the log Futaki invariant. The point is that the computation becomes straightforward by using the extremal metric, afforded by Theorem 3.3.2. It turns out that the vanishing of the log Futaki invariant gives an equation for β to satisfy (equation (3.26)); in other words, there is a unique value of β for which the log Futaki invariant vanishes. The content of our main result, Theorem 3.1.7, is that this value of β agrees with the one for which there exists a momentum-constructed conically singular cscK metric with cone angle $2\pi\beta$ (equation (3.22)).

Remark 3.3.4. The hypothesis $b_2(M_i) = 1$ in Theorem 3.1.7 is to ensure that each Kähler class of X can be represented by a momentum-constructed metric, as we now explain. Observe first that $b_2(M_i) = 1$ implies $H^2(M,\mathbb{R}) = \bigoplus_i \mathbb{R}[p_i^*\omega_i]$, by recalling that every Fano manifold is simply connected (cf. [31]). Thus recalling the Leray–Hirsch theorem, we have

$$H^2(X,\mathbb{R}) = p^*H^2(M,\mathbb{R}) \oplus \mathbb{R}c_1(\xi) = p^*\left(\bigoplus_i \mathbb{R}[p_i^*\omega_i]\right) \oplus \mathbb{R}c_1(\xi),$$

i.e. each Kähler class on X can be written as $\sum_{i=1}^r \alpha_i p^*[p_i^*\omega_i] + \alpha_{r+1}c_1(\xi)$ for some $\alpha_i > 0$, where ξ is the dual of the tautological bundle on X. We can now prove (cf. Lemma 4.2, [61]) that each Kähler class can be represented by a momentum-constructed metric $\omega_\phi = p^*\omega_M - \tau p^*\gamma + \frac{1}{\phi} d\tau \wedge d^c\tau$ as follows. Observe now that the form $-\tau p^*\gamma + \frac{1}{\phi} d\tau \wedge$

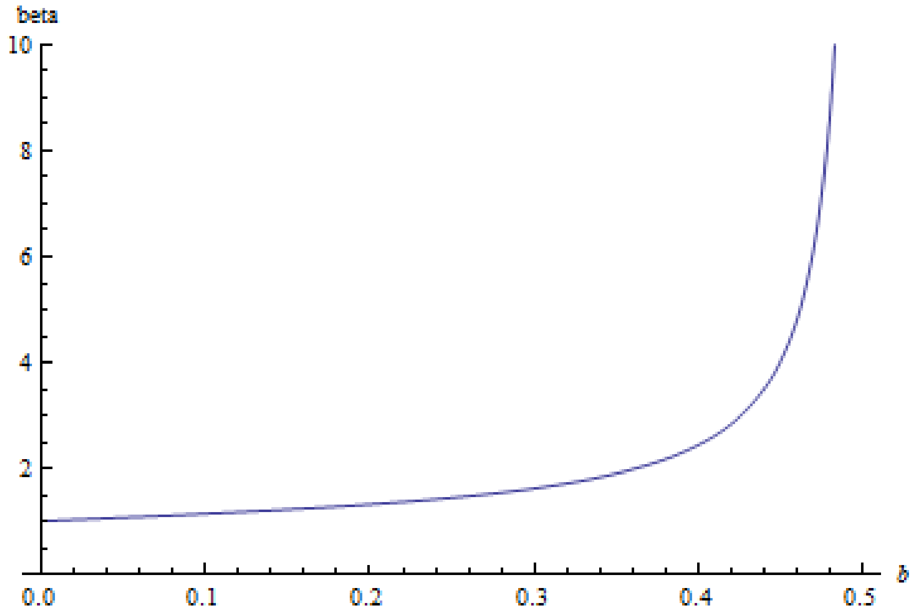

Figure 3.1: Graph of β as a function of b for $\mathscr{F} = p_1^*(K_{\mathbb{P}^1}^{-1}) \otimes p_2^*(K_{\mathbb{P}^1}^2)$ on $M = \mathbb{P}^1 \times \mathbb{P}^1$.

$d^c\tau$ is closed. Thus its cohomology class can be written as

$$\left[-\tau p^*\gamma + \frac{1}{\phi} d\tau \wedge d^c\tau\right] = \sum_{i=1}^{r} \alpha_i' p^*[p_i^*\omega_i] + \alpha_{r+1}' c_1(\xi)$$

for some $\alpha_i' > 0$. We shall prove in Lemma 3.3.9 that any momentum-constructed metric with the momentum interval $I = [-b, b]$ has fibrewise volume $4\pi b$. This proves $\alpha_{r+1}' = 4\pi b$. Thus, writing $\omega_M = \sum_{i=1}^r \tilde{\alpha}_i \omega_i$, we see that $[\omega_\phi] = \sum_{i=1}^r (\alpha_i' + \tilde{\alpha}_i) p^*[p_i^*\omega_i] + 4\pi b c_1(\xi)$. Thus, given any Kähler class in $\kappa \in H^2(X, \mathbb{R})$, we can choose $\tilde{\alpha}_i$ and b appropriately so that $[\omega_\phi] = \kappa$.

Remark 3.3.5. We do not necessarily have $0 < \beta < 1$ in Theorem 3.1.7; although $\beta \geq 0$ always holds, as we prove in §3.3.3.1, there are examples where $\beta > 1$. Indeed, when we take $M = \mathbb{P}^1 \times \mathbb{P}^1$, $\omega_M = p_1^*\omega_{KE} + p_2^*\omega_{KE}$ for the Kähler–Einstein metric $\omega_{KE} \in 2\pi c_1(-K_{\mathbb{P}^1})$ and $\mathscr{F} = p_1^*(-K_{\mathbb{P}^1}) \otimes p_2^*(2K_{\mathbb{P}^1})$, we always have $\beta > 1$ as shown in Figure 3.1, by noting that $0 < b < 0.5$ gives a well-defined momentum interval.

On the other hand, as shown in Figure 3.2, $\mathscr{F} = p_1^*(-2K_{\mathbb{P}^1}) \otimes p_2^*(K_{\mathbb{P}^1})$ with M and ω_M as above, $0 < b < 0.5$ implies $0.3 \lesssim \beta < 1$; in particular Theorem 3.1.7 is not vacuous even if we impose an extra condition $0 < \beta < 1$.

The author could not find an example where $\beta = 0$ is achieved.

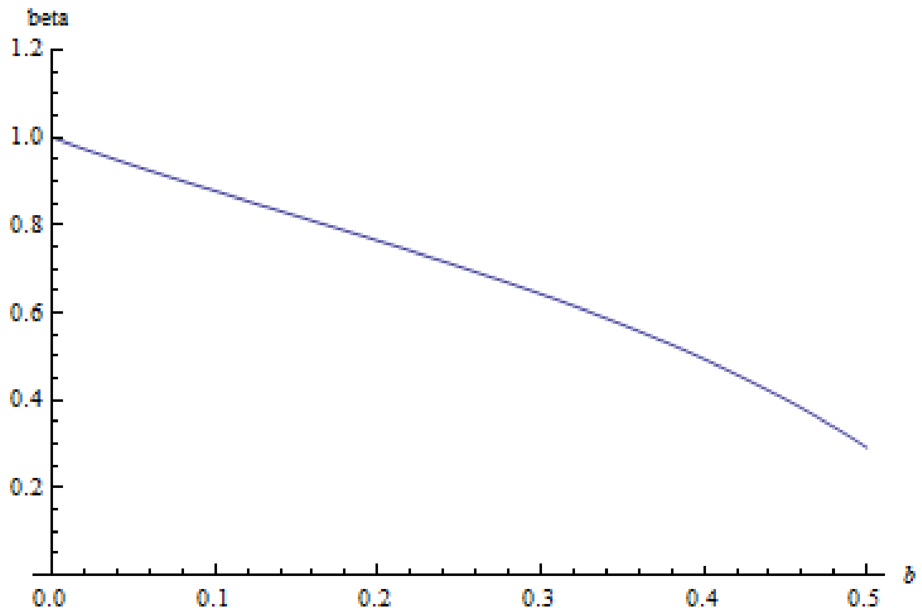

Figure 3.2: Graph of β as a function of b for $\mathscr{F} = p_1^*(K_{\mathbb{P}^1}^{-2}) \otimes p_2^*(K_{\mathbb{P}^1})$ on $M = \mathbb{P}^1 \times \mathbb{P}^1$.

3.3.2 Some properties of momentum-constructed metrics with $\varphi'(b) = -2\beta$

We do *not* assume in this section that the σ-constancy hypothesis (cf. Definition 3.3.1) is necessarily satisfied, but *do* assume that φ is real analytic.

We first prove that $\varphi'(b) = -2\beta$ does indeed define a Kähler metric that is conically singular along the ∞-section. The author thanks Michael Singer for the instructions on the proof of the following lemma.

Lemma 3.3.6. (Singer [109]; see also Li [73], Lemma 2.3) *Suppose that ω_φ is a momentum-constructed Kähler metric on $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ with the momentum interval $I = [-b, b]$ and the momentum profile φ that is real analytic on I with $\varphi(\pm b) = 0$, $\varphi'(-b) = 2$, and $\varphi'(b) = -2\beta$. Then ω_φ is smooth on $X \setminus D$, where $D = \{\tau = b\}$ is the ∞-section, and has cone singularities along D with cone angle $2\pi\beta$. Moreover, choosing the local coordinate system $(z_1, \dots, z_n)$ on X so that $D = \{z_1 = 0\}$ and that $(z_2, \dots, z_n)$ defines a local coordinate system on the base M, $b - \tau$ can be written as a locally uniformly convergent power series*

$$b - \tau = A_0 |z_1|^{2\beta} \left(1 + \sum_{i=1}^{\infty} A_i |z_1|^{2\beta i} \right)$$

around $D = \{\tau = b\} = \{z_1 = 0\}$, where A_i's are smooth functions which depend only on the local coordinates $(z_2, \dots, z_n)$ on M, and $A_0 > 0$ is in addition bounded away from 0.

Thus $\varphi(\tau)$ can be written as a locally uniformly convergent power series around D

$$\varphi(\tau) = 2\beta A'_1 |z_1|^{2\beta} + \sum_{i=2}^{\infty} A'_i |z_1|^{2\beta i}, \tag{3.8}$$

where A'_i's are smooth functions which depend only on the local coordinates $(z_2, \dots, z_n)$ on M, and $A'_1 > 0$ is in addition bounded away from 0. This means that the metric g_φ corresponding to ω_φ satisfies the following estimates around D:

1. $(g_\varphi)_{1\bar{1}} = O(|z_1|^{2\beta-2})$,
2. $(g_\varphi)_{1\bar{j}} = O(|z_1|^{2\beta-1})$ *($j \neq 1$)*,
3. $(g_\varphi)_{i\bar{j}} = O(1)$ *($i, j \neq 1$)*,

i.e. ω_φ is a Kähler metric with cone singularities along D with cone angle $2\pi\beta$ (cf. Definition 3.1.1).

Proof. Since Lemma 2.5 and Proposition 2.1 in [61] imply that ω_φ is smooth on $X \setminus D$, we only have to check that the condition $\varphi'(b) = -2\beta$ implies that ω_φ has cone singularities along D with cone angle $2\pi\beta$.

Writing t for the log of the fibrewise length measured by $h_{\mathscr{F}}$, we have

$$dt = \frac{d\tau}{\varphi(\tau)}, \tag{3.9}$$

by recalling the equation (2.2) in [62]. We now write φ as a convergent power series in $b - \tau$ around $\tau = b$ as

$$\varphi(\tau) = 2\beta(b-\tau) + \sum_{i=2}^{\infty} a'_i (b-\tau)^i, \tag{3.10}$$

since we assumed that φ is real analytic, where a'_i's are real numbers. Note that the coefficient of the first term is fixed by the boundary condition $\varphi'(b) = -2\beta$. This gives

$$t = \frac{1}{2} \log h_{\mathscr{F}}(\zeta, \zeta) = -\frac{1}{2\beta} \log(b-\tau) + \sum_{i=2}^{\infty} a''_i (b-\tau)^{i-1} + \text{const}$$

with some real numbers a''_i, where ζ is a fibrewise coordinate on $\mathscr{F} \to M$.

On the other hand, since ζ is a fibrewise coordinate on $\mathscr{F} \to M$, it gives a fibrewise local coordinate of $\mathbb{P}(\mathscr{F} \oplus \mathbb{C}) \to M$ around the 0-section; in other words, at each point $p \in M$, ζ gives a local coordinate on each fibre $\mathbb{P}^1$ in the neighbourhood containing $0 = [0:1] \in \mathbb{P}^1$. Since $\tau = b$ defines the ∞-section of $\mathbb{P}(L \oplus \mathbb{C}) \to M$, it is better to pass to the local coordinates on $\mathbb{P}^1$ in the neighbourhood containing $\infty = [1:0] \in \mathbb{P}^1$ in order to evaluate the asymptotics as $\tau \to b$. The coordinate change is of course given by $\zeta \mapsto 1/\zeta =: z_1$, and hence we have

$$\frac{1}{2} \log h_{\mathscr{F}}(\zeta, \zeta) = \frac{1}{2} \phi_{\mathscr{F}} - \frac{1}{2} \log |z_1|^2 = -\frac{1}{2\beta} \log(b - \tau) + \sum_{i=2}^{\infty} a''_i (b-\tau)^{i-1} + \text{const}$$

by writing $h_{\mathscr{F}} = e^{\phi_{\mathscr{F}}}$ locally around a point $p \in M$. This means that there exists a smooth function $A = A(z_2, \dots, z_n)$ which is bounded away from 0 and depends only on the coordinates $(z_2, \dots, z_n)$ on M such that

$$|z_1|^2 = A(b-\tau)^{\frac{1}{\beta}} \left(1 + \sum_{i=1}^{\infty} a'''_i (b-\tau) \right),$$

with some real numbers a'''_i and hence, by raising both sides of the equation to the power of β and applying the inverse function theorem, we have

$$b - \tau = A_0 |z_1|^{2\beta} \left(1 + \sum_{i=1}^{\infty} A_i |z_1|^{2\beta i} \right) \tag{3.11}$$

as a locally uniformly convergent power series around $D = \{\tau = b\} = \{z_1 = 0\}$, where each $A_i = A_i(z_2, \dots, z_n)$ is a smooth function which depends only on the coordinates $(z_2, \dots, z_n)$ on M, and $A_0 > 0$ is in addition bounded away from 0. In particular, we have $b - \tau = O(|z_1|^{2\beta})$, and combined with the equation (3.10), we thus get the result (3.8) that we claimed.

We now evaluate $\frac{1}{\varphi} d\tau \wedge d^c \tau$ in $\omega_\varphi = p^* \omega_M - \tau p^* \gamma + \frac{1}{\varphi} d\tau \wedge d^c \tau$. The above equation (3.11) means

$$\partial (b - \tau) = A_0 \beta |z_1|^{2\beta - 2} \bar{z}_1 B_1 dz_1 + |z_1|^{2\beta} \sum_{i=2}^{n} B_{2,i} dz_i$$

and

$$\bar{\partial}(b-\tau) = A_0\beta|z_1|^{2\beta-2}z_1B_1d\bar{z}_1 + |z_1|^{2\beta}\sum_{i=2}^{n}\overline{B_{2,i}}d\bar{z}_i,$$

where we wrote $B_1 := 1+\sum_{i=1}^{\infty} iA_i|z_1|^{2\beta i}$ and $B_{2,i} := \frac{\partial}{\partial z_i}\left(A_0 + A_0\sum_{j=1}^{\infty}A_i|z_1|^{2\beta j}\right)$. We thus have

$$\begin{aligned} d\tau\wedge d^c\tau &= d(b-\tau)\wedge d^c(b-\tau) \\ &= 2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}\sqrt{-1}dz_1\wedge d\bar{z}_1 + 2\beta|z_1|^{4\beta-2}\bar{z}_1A_0B_1\sum_{i=2}^{n}\overline{B_{2,i}}\sqrt{-1}dz_1\wedge d\bar{z}_i \\ &\qquad + c.c. + O(|z_1|^{4\beta}). \end{aligned} \tag{3.12}$$

where $O(|z_1|^{4\beta})$ stands for a term of the form

$$\begin{aligned} &|z_1|^{4\beta}\times(\text{smooth function in }(z_2,\dots,z_n)) \\ &\qquad\times(\text{locally uniformly convergent power series in }|z_1|^{2\beta}). \end{aligned}$$

We now estimate the behaviour of each component $(g_\varphi)_{i\bar{j}}$ of the Kähler metric $\omega_\varphi = \sum_{i,j=1}^{n}(g_\varphi)_{i\bar{j}}\sqrt{-1}dz_i\wedge d\bar{z}_j$ in terms of the local holomorphic coordinates $(z_1,z_2,\dots,z_n)$ on X. The above computation with $\varphi(\tau) = O(|z_1|^{2\beta})$ means that $(g_\varphi)_{1\bar{1}} = O(|z_1|^{2\beta-2})$, $(g_\varphi)_{1\bar{j}} = O(|z_1|^{2\beta-1})$ $(j\neq 1)$, $(g_\varphi)_{i\bar{j}} = O(1)$ $(i,j\neq 1)$ as it approaches the ∞-section, proving that ω_φ has cone singularities of cone angle $2\pi\beta$ along D.

□

We also see that the above means that the inverse matrix $(g_\varphi)^{i\bar{j}}$ satisfies the following estimates.

Lemma 3.3.7. *Suppose that g_φ is a momentum-constructed conically singular Kähler metric with cone angle $2\pi\beta$ along $D = \{z_1 = 0\}$, with the real analytic momentum profile φ. Then, around D,*

1. $(g_\varphi)^{1\bar{1}} = O(|z_1|^{2-2\beta})$,
2. $(g_\varphi)^{1\bar{j}} = O(|z_1|)$ *if* $j\neq 1$,

3. $(g_{\varphi})^{i\bar{j}} = O(1)$ *if* $i, j \neq 1$.

Thus, $\Delta_{\omega_\varphi} f = \sum_{i,j=1}^{n} (g_{\varphi})^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} f$ *is bounded if* f *is a smooth function on* X*. Also, if* f' *is a smooth function on* $X \setminus D$ *that is of order* $|z_1|^{2\beta}$ *around* D*, then* $\Delta_{\omega_\varphi} f' = O(1) + O(|z_1|^{2\beta})$*. In particular,* $\Delta_{\omega_\varphi} f'$ *remains bounded on* $X \setminus D$*.*

We now prove the following estimates on the Ricci curvature and the scalar curvature of ω_φ around the ∞-section, i.e. when $\tau \to b$.

Lemma 3.3.8. *Choosing a local coordinate system* $(z_1, \dots, z_n)$ *on* X *so that* z_1 *is the fibrewise coordinate which locally defines the* ∞*-section* D *by* $z_1 = 0$ *and that* $(z_2, \dots, z_n)$ *defines a local coordinate system on the base* M*, we have, around* D*,*

1. $\mathrm{Ric}(\omega_\varphi)_{1\bar{1}} = O(1) + O(|z_1|^{2\beta-2})$*,*

2. $\mathrm{Ric}(\omega_\varphi)_{1\bar{j}} = O(1) + O(|z_1|^{2\beta-1})$ *(*$j \neq 1$*),*

3. $\mathrm{Ric}(\omega_\varphi)_{i\bar{j}} = O(1) + O(|z_1|^{2\beta})$ *(*$i, j \neq 1$*),*

for a momentum-constructed metric ω_φ *with smooth* φ *and* $\varphi'(b) = -2\beta$*. In particular, combined with Lemma 3.3.7, we see that* $S(\omega_\varphi)$ *is bounded on* $X \setminus D$ *if* $0 < \beta < 1$*.*

Proof. First note that (cf. Lemma 3.3.6, the equation (3.4), and p2296 in [62]) $\omega_\varphi^n = \frac{n}{\varphi} p^* \omega_M(\tau)^{n-1} \wedge d\tau \wedge d^c \tau$ is of order

$$\omega_\varphi^n = |z_1|^{2\beta-2} F p^* \omega_M(\tau)^{n-1} \wedge \sqrt{-1} dz_1 \wedge d\bar{z}_1,$$

where F stands for some locally uniformly convergent power series in $|z_1|^{2\beta}$ that is bounded from above and away from 0 on $X \setminus D$ (this follows from Lemma 3.3.6).

Writing $\omega_0 := p^* \omega_M + \delta \omega_{FS}$ for a reference Kähler form on $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$, where ω_{FS} is a fibrewise Fubini-Study metric and $\delta > 0$ is chosen to be small enough so that $\omega_0 > 0$, we thus have

$$\frac{\omega_\varphi^n}{\omega_0^n} = \frac{p^* \omega_M(\tau)^{n-1}}{p^* \omega_M^{n-1}} |z_1|^{2\beta-2} F'$$

with another locally uniformly convergent power series F' in $|z_1|^{2\beta}$ on $X \setminus D$, which is bounded from above and away from 0 (note also that the derivatives of F' in the z_1-direction are not necessarily bounded on $X \setminus D$ due to the dependence on $|z_1|^{2\beta}$;

they may have a pole of fractional order along D). Recalling (3.3), we see that $p^*\omega_M(\tau)^{n-1}/\omega_M^{n-1}$ depends polynomially on τ. We thus have a locally uniformly convergent power series

$$\frac{\omega_\varphi^n}{\omega_0^n} = |z_1|^{2\beta-2}\left(F_0 + \sum_{j=1}^{\infty} F_j |z_1|^{2\beta j}\right) \tag{3.13}$$

with some smooth functions F_j depending only on the coordinates $(z_2, \dots, z_n)$ on M, where F_0 is also bounded away from 0.

Choosing a local coordinate system $(z_1, \dots, z_n)$ on X so that $D = \{z_1 = 0\}$ and that $(z_2, \dots, z_n)$ defines a local coordinate system on the base M, we evaluate the order of each component of the Ricci curvature $\mathrm{Ric}(\omega_\varphi) = -\sqrt{-1}\partial\bar{\partial}\log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)$ around the ∞-section, i.e. as $\tau \to b$. Writing $\mathrm{Ric}(\omega_\varphi)_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\left(\frac{\omega_\varphi^n}{\omega_0^n}\right)$ and noting $\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log|z_1|^2 = 0$ on $X \setminus D$ for all i, j, we see that $\mathrm{Ric}(\omega_\varphi)_{1\bar{1}} = O(1) + O(|z_1|^{2\beta-2})$, $\mathrm{Ric}(\omega_\varphi)_{1\bar{j}} = O(1) + O(|z_1|^{2\beta-1})$ $(j \neq 1)$, and $\mathrm{Ric}(\omega_\varphi)_{i\bar{j}} = O(1) + O(|z_1|^{2\beta})$ $(i, j \neq 1)$. In particular, we see that $S(\omega_\varphi)$ is bounded if $0 < \beta < 1$.

□

3.3.3 Proof of Theorem 3.1.7

3.3.3.1 Construction of conically singular cscK metrics on $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$

We start from recalling the materials in §3.2 of [61], particularly Propositions 3.1 and 3.2. We first define a function

$$\phi(\tau) := \frac{1}{Q(\tau)}\left(2(\tau+b)Q(-b) - 2\int_{-b}^{\tau}(\sigma_0 + \lambda x - R(x))(\tau - x)Q(x)dx\right) \tag{3.14}$$

where $Q(\tau)$, $R(\tau)$ are defined as in (3.6) and (3.7). These being functions of τ follows from σ-constancy (Definition 3.3.1). We re-write this as

$$(\phi Q)(\tau) = 2(\tau+b)Q(-b) - 2\int_{-b}^{\tau}(\sigma_0 + \lambda x - R(x))(\tau - x)Q(x)dx, \tag{3.15}$$

and differentiate both sides of (3.15) twice, to get

$$R(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial \tau^2}(\phi Q)(\tau) = \sigma_0 + \lambda \tau. \tag{3.16}$$

We can show, as in Proposition 3.1 of [61], that there exist constants σ_0 and λ such that ϕ satisfies $\phi(\pm b) = 0$, $\phi'(\pm b) = \mp 2$, and $\phi(\tau) > 0$ if $\tau \in (-b, b)$; namely that ϕ defines a smooth momentum-constructed metric ω_ϕ. We thus have $S(\omega_\phi) = \sigma_0 + \lambda\tau$, by recalling (3.5) and (3.16), so that ω_ϕ is extremal.

Roughly speaking, our strategy is to "brutally substitute $\lambda = 0$" in the above to get a cscK metric with cone singularities along the ∞-section. More precisely, we aim to solve the equation

$$R(\tau) - \frac{1}{2Q}\frac{\partial^2}{\partial \tau^2}(\varphi Q)(\tau) = \sigma_0' \tag{3.17}$$

with some constant σ_0', for a profile φ that is strictly positive on the interior $(-b, b)$ of I with boundary conditions $\varphi(b) = \varphi(-b) = 0$ and $\varphi'(-b) = -2$. The value $\varphi'(b)$ has more to do with the cone singularities of the metric ω_φ, and we shall see at the end that the metric ω_φ associated to such φ defines a Kähler metric with cone singularities along the ∞-section with cone angle $-\pi\varphi'(b) = 2\pi\beta$.

Since

$$\varphi(\tau) := \frac{1}{Q(\tau)}\left(2(\tau+b)Q(-b) - 2\int_{-b}^{\tau}(\sigma_0' - R(x))(\tau - x)Q(x)dx\right)$$

certainly satisfies the equation (3.17), we are reduced to checking the boundary conditions at ∂I and the positivity of φ on the interior of I. Note first that the equality

$$(\varphi Q)(\tau) = 2(\tau+b)Q(-b) - 2\int_{-b}^{\tau}(\sigma_0' - R(x))(\tau - x)Q(x)dx \tag{3.18}$$

immediately implies that $\varphi(-b) = 0$ and $\varphi'(-b) = 2$ are always satisfied. Imposing $\varphi(b) = 0$, we get

$$0 = 2bQ(-b) - \int_{-b}^{b}(\sigma_0' - R(x))(b - x)Q(x)dx \tag{3.19}$$

from (3.18), which in turn determines σ_0'. Differentiating both sides of (3.18) and evaluating at b, we also get

$$\varphi'(b)Q(b) = 2Q(-b) - 2\int_{-b}^{b} (\sigma_0' - R(x))Q(x)dx. \tag{3.20}$$

Writing $A := \int_{-b}^{b} Q(x)dx$ and $B := \int_{-b}^{b} xQ(x)dx$ we can re-write (3.19), (3.20) as

$$\begin{pmatrix} A\sigma_0' \\ B\sigma_0' \end{pmatrix} = \begin{pmatrix} Q(-b) - \varphi'(b)Q(b)/2 + \int_{-b}^{b} R(x)Q(x)dx \\ -bQ(-b) - b\varphi'(b)Q(b)/2 + \int_{-b}^{b} xR(x)Q(x)dx \end{pmatrix}, \tag{3.21}$$

which can be regarded as an analogue of the equations (26) and (27) in [61]. The consistency condition $B(A\sigma_0') = A(B\sigma_0')$ gives an equation for $\varphi'(b)$, which can be written as

$$\begin{aligned} &-\frac{\varphi'(b)}{2} \\ &= -\frac{Q(-b)\int_{-b}^{b}(b+x)Q(x)dx - \int_{-b}^{b}Q(x)dx\int_{-b}^{b}xR(x)Q(x)dx + \int_{-b}^{b}xQ(x)dx\int_{-b}^{b}R(x)Q(x)dx}{Q(b)\int_{-b}^{b}(b-x)Q(x)dx} \\ &= \frac{Q(-b)(bA+B) - A\int_{-b}^{b}xR(x)Q(x)dx + B\int_{-b}^{b}R(x)Q(x)dx}{Q(b)(bA-B)}. \end{aligned} \tag{3.22}$$

Summarising the above argument, we have now obtained a profile function φ which solves (3.17) with boundary conditions $\varphi(b) = \varphi(-b) = 0$, $\varphi'(-b) = -2$, and $\varphi'(b)$ as specified by (3.22). Now, Hwang's argument (Appendix A of [61]) applies word by word to show that φ is strictly positive on the interior of I, and hence it now remains to show that the Kähler metric ω_φ has cone singularities along the ∞-section. Since $Q(\tau)$ is a polynomial in τ and $R(\tau)$ is a rational function in τ (with no poles when $\tau \in [-b,b]$), we see from (3.17) that φ is real analytic on $I = [-b,b]$ by the standard ODE theory. Thus the value $-\pi\varphi'(b) = 2\pi\beta$ is the angle of the cone singularities that ω_φ develops along the ∞-section of $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$, by Lemma 3.3.6. This completes the construction of the momentum-constructed conically singular metric ω_φ, with cone angle $-\pi\varphi'(b) = 2\pi\beta$ as specified by (3.22).

We also see $\varphi'(b) \leq 0$ since otherwise $\varphi'(-b) > 0$, $\varphi'(b) > 0$, and $\varphi(\pm b) = 0$ imply that φ has to have a zero in $(-b,b)$, contradicting the positivity $\varphi > 0$ on $(-b,b)$. Hence $\beta \geq 0$.

Finally, we identify the Kähler class $[\omega_\varphi] \in H^2(X,\mathbb{R})$ of the momentum-constructed conically singular cscK metric ω_φ. We first show that the restriction $\omega_\varphi|_{\text{fibre}}$ of ω_φ to each fibre has (fibrewise) volume $4\pi b$. This is well-known when the metric is smooth, but we reproduce the proof here to demonstrate that the same argument works even when ω_φ has cone singularities. Related discussions can also be found in §3.5.3 (see Lemma 3.5.6 in particular).

Lemma 3.3.9. (§4 in [61], or §2.1 in [62]) *Suppose that ω_φ is a (possibly conically singular) momentum-constructed metric with the momentum profile $\varphi : [-b, b] \to \mathbb{R}_{\geq 0}$. Then the fibrewise volume of ω_φ is given by $4\pi b$.*

Proof. The equation (3.9) means that the restriction of ω_φ at each fibre (which is isomorphic to $\mathbb{P}^1$) is given by (cf. equation (2.5) in [62])

$$\omega_\varphi|_{\text{fibre}} = \frac{1}{2}\varphi(\tau)|\zeta|^{-2}\sqrt{-1}d\zeta \wedge d\bar{\zeta} = \varphi(\tau)r^{-2}rdr \wedge d\theta$$

where $\zeta = re^{\sqrt{-1}\theta}$ is a holomorphic coordinate on each fibre ($|\cdot|$ denotes the fibrewise Euclidean norm defined by $h_{\mathscr{F}}$; see §2.1 of [62] for more details). By using (3.9), we can re-write this as

$$\omega_\varphi|_{\text{fibre}} = \frac{d\tau}{dt}r^{-1}dr \wedge d\theta = \frac{d\tau}{dr}dr \wedge d\theta \tag{3.23}$$

since $t = \log r$. Integrating this over the fibre, we get

$$\int_{\text{fibre}} \omega_\varphi = 2\pi \int_0^\infty \frac{d\tau}{dr}dr = 2\pi \int_{-b}^{b} d\tau = 4\pi b$$

since $\tau = b$ corresponds to $\infty \in \mathbb{P}^1$ and $\tau = -b$ to $0 \in \mathbb{P}^1$. □

Thus we can write $[\omega_\varphi] = \sum_{i=1}^r \alpha_i p^*[p_i^*\omega_i] + 4\pi b c_1(\xi)$ for some $\alpha_i > 0$, in the notation used in Remark 3.3.4. Since the same proof applies to the smooth metric ω_ϕ, we also have $[\omega_\phi] = \sum_{i=1}^r \tilde{\alpha}_i p^*[p_i^*\omega_i] + 4\pi b c_1(\xi)$ for some $\tilde{\alpha}_i > 0$. On the other hand, since $\omega_\varphi|_M = \omega_M(b) = \omega_\phi|_M$ (where M is identified with the 0-section), it immediately follows that $\alpha_i = \tilde{\alpha}_i$ for all i, i.e. $[\omega_\varphi] = [\omega_\phi]$.

3.3.3.2 Computation of the log Futaki invariant

We again take the (smooth) momentum-constructed extremal metric ω_ϕ, with ϕ defined as in (3.14), and write $S(\omega_\phi) = \sigma_0 + \lambda\tau$ for its scalar curvature.

Recall that the generator v_f of the fibrewise $U(1)$-action has $a\tau$ as its Hamiltonian function with respect to ω_ϕ (cf. §2.1, [62]), with some $a \in \mathbb{R}$ up to an additive constant which does not change v_f. This means that $a\tau$ (up to an additive constant) is the holomorphy potential for the holomorphic vector field $\Xi_f := v_f^{1,0}$ (cf. Remark 1.3.3) which generates the complexification of the fibrewise $U(1)$-action, i.e. the fibrewise $\mathbb{C}^*$-action. Thus we can take $f = a(\tau - \bar{\tau})$, with $\bar{\tau}$ being the average of τ over X with respect to ω_ϕ, for the holomorphy potential f in the formula (3.2). Then, noting that $S(\omega_\phi) - \bar{S} = \lambda(\tau - \bar{\tau})$, we compute the (classical) Futaki invariant as

$$\mathrm{Fut}(\Xi_f, [\omega_\phi]) = \int_X a\lambda(\tau - \bar{\tau})^2 \frac{\omega_\phi^n}{n!} = 2\pi a\lambda \mathrm{Vol}(M, \omega_M) \int_{-b}^{b} (\tau - \bar{\tau})^2 Q(\tau) d\tau$$

with $\mathrm{Vol}(M, \omega_M) := \int_M \frac{\omega_M^{n-1}}{(n-1)!}$, by Lemma 2.8 of [61]. Recalling $D = \{\tau = b\}$, the second term in the log Futaki invariant can be obtained by computing

$$\begin{aligned}\int_D f \frac{\omega_\phi^{n-1}}{(n-1)!} = \int_D a(\tau - \bar{\tau}) \frac{\omega_\phi^{n-1}}{(n-1)!} &= \int_D a(b - \bar{\tau}) \frac{p^* \omega_M(b)^{n-1}}{(n-1)!} \\ &= a(b - \bar{\tau}) Q(b) \int_M \frac{\omega_M^{n-1}}{(n-1)!} \\ &= a(b - \bar{\tau}) Q(b) \mathrm{Vol}(M, \omega_M)\end{aligned}$$

where we used

$$\omega_\phi^{n-1} = p^* \omega_M(\tau)^{n-1} + \frac{n-1}{\phi} p^* \omega_M(\tau)^{n-2} d\tau \wedge d^c \tau \tag{3.24}$$

which was proved in p2296 in [62], and the definitional $Q(b) = \omega_M(b)^{n-1}/\omega_M^{n-1}$ (cf. equation (3.9)). We also note the trivial equality $\int_X f \frac{\omega_\phi^n}{n!} = \int_X \lambda(\tau - \bar{\tau}) \frac{\omega_\phi^n}{n!} = 0$ to see that the third term of the log Futaki invariant is 0. Collecting these calculations together, the log Futaki invariant evaluated against Ξ_f is given by

$$\mathrm{Fut}_{D,\beta}(\Xi_f, [\omega_\phi]) = a\lambda \mathrm{Vol}(M, \omega_M) \int_{-b}^{b} (\tau - \bar{\tau})^2 Q(\tau) d\tau - (1-\beta) a(b - \bar{\tau}) Q(b) \mathrm{Vol}(M, \omega_M).$$

Thus, writing $A := \int_{-b}^{b} Q(\tau) d\tau$, $B := \int_{-b}^{b} \tau Q(\tau) d\tau$, and $C := \int_{-b}^{b} \tau^2 Q(\tau) d\tau$ and noting

$\bar{\tau} = B/A$, setting $\mathrm{Fut}_{D,\beta}(\Xi_f, [\omega_\phi]) = 0$ gives an equation for the cone angle β as

$$\begin{aligned} \beta &= 1 - \frac{\lambda \int_{-b}^{b} (\tau - \bar{\tau})^2 Q(\tau) d\tau}{(b - \bar{\tau}) Q(b)} \\ &= \frac{Q(b)(bA - B) - \lambda \left(AC - B^2\right)}{Q(b)(bA - B)} \end{aligned} \tag{3.25}$$

Applying (3.19) and (3.20) to the case of smooth extremal metric ω_ϕ, i.e. with $\phi'(b) = -2$, we get the equations (26) and (27) in [61] which can be re-written as

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \sigma_0 \\ \lambda \end{pmatrix} = \begin{pmatrix} Q(-b) + Q(b) + \int_{-b}^{b} R(x) Q(x) dx \\ -bQ(-b) + bQ(b) + \int_{-b}^{b} x R(x) Q(x) dx \end{pmatrix},$$

and hence, noting $AC - B^2 > 0$ by Cauchy–Schwarz (where we regard $Q(\tau)d\tau$ as a measure on $I = [-b, b]$), we get

$$\lambda = \frac{-B \left(Q(-b) + Q(b) + \int_{-b}^{b} R(x) Q(x) dx \right) + A \left(-bQ(-b) + bQ(b) + \int_{-b}^{b} x R(x) Q(x) dx \right)}{AC - B^2},$$

and hence

$$\begin{aligned} \beta &= \frac{Q(b)(bA - B) - \lambda \left(AC - B^2\right)}{Q(b)(bA - B)} \\ &= \frac{Q(-b)(bA + B) + B \int_{-b}^{b} R(x) Q(x) dx - A \int_{-b}^{b} x R(x) Q(x) dx}{Q(b)(bA - B)} \end{aligned} \tag{3.26}$$

which agrees with (3.22). This is precisely what was claimed in Theorem 3.1.7.

3.4 Log Futaki invariant computed with respect to the conically singular metrics

3.4.1 Some estimates for the conically singular metrics of elementary form

We now consider conically singular metrics of elementary form $\hat{\omega} = \omega + \lambda \sqrt{-1} \partial \bar{\partial} |s|_h^{2\beta}$, as defined in Definition 3.1.10. We collect here some estimates that we need later.

Remark 3.4.1. What we discuss in here is just a review of well-known results, and in fact for the most part, is nothing more than a repetition of §2 in the paper of Jeffres–Mazzeo–Rubinstein [64] or §3 in the paper of Brendle [20].

Pick a local coordinate system $(z_1, \dots, z_n)$ around a point in X so that D is locally given by $\{z_1 = 0\}$. We then write

$$\hat{\omega} = \sum_{i,j} \hat{g}_{i\bar{j}} \sqrt{-1} dz_i \wedge d\bar{z}_j = \sum_{i,j} g_{i\bar{j}} \sqrt{-1} dz_i \wedge d\bar{z}_j + \lambda \sum_{i,j} \frac{\partial^2 |s|_h^{2\beta}}{\partial z_i \partial \bar{z}_j} \sqrt{-1} dz_i \wedge d\bar{z}_j$$

which means

$$(\hat{g}_{i\bar{j}})_{i\bar{j}} = \begin{pmatrix} g_{1\bar{1}} + O(|z_1|^{2\beta-2}) & g_{1\bar{2}} + O(|z_1|^{2\beta-1}) & \dots & g_{1\bar{n}} + O(|z_1|^{2\beta-1}) \\ g_{2\bar{1}} + O(|z_1|^{2\beta-1}) & g_{2\bar{2}} + O(|z_1|^{2\beta}) & \dots & g_{2\bar{n}} + O(|z_1|^{2\beta}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n\bar{1}} + O(|z_1|^{2\beta-1}) & g_{n\bar{2}} + O(|z_1|^{2\beta}) & \dots & g_{n\bar{n}} + O(|z_1|^{2\beta}) \end{pmatrix}.$$

Thus, writing $\hat{g}$ for the metric corresponding to $\hat{\omega}$, we have (cf. Definition 3.1.1)

1. $\hat{g}_{1\bar{1}} = O(|z_1|^{2\beta-2})$,
2. $\hat{g}_{1\bar{j}} = O(|z_1|^{2\beta-1})$ if $j \neq 1$,
3. $\hat{g}_{i\bar{j}} = O(1)$ if $i, j \neq 1$.

The above also means that the volume form $\hat{\omega}^n$ can be estimated as (cf. p10 of [20])

$$\hat{\omega}^n = \left(|z_1|^{2\beta-2} \sum_{j=0}^{n-1} a_j |z_1|^{2\beta j} + \sum_{j=0}^{n} b_j |z_1|^{2\beta j} \right) \omega_0^n$$

where ω_0 is a smooth reference Kähler form on X, a_j's and b_j's being smooth functions on X, and a_0 is also strictly positive. Thus we immediately have the following lemma.

Lemma 3.4.2. *We may write $\hat{\omega}^n = |z_1|^{2-2\beta} \alpha$ with some (n,n)-form α, which is smooth on $X \setminus D$ and bounded as we approach $D = \{z_1 = 0\}$, but whose derivatives (in z_1-direction) may not be bounded around D due to the dependence on the fractional power $|z_1|^{2\beta}$.*

We also see, analogously to Lemma 3.3.7, that the above means that the inverse matrix $\hat{g}^{i\bar{j}}$ satisfies the following estimates.

Lemma 3.4.3. *Suppose that $\hat{g}$ is a conically singular Kähler metric of elementary form with cone angle $2\pi\beta$ along $D = \{z_1 = 0\}$. Then, around D,*

1. $\hat{g}^{1\bar{1}} = O(|z_1|^{2-2\beta})$,
2. $\hat{g}^{1\bar{j}} = O(|z_1|)$ *if* $j \neq 1$,
3. $\hat{g}^{i\bar{j}} = O(1)$ *if* $i, j \neq 1$.

Thus, $\Delta_{\hat{\omega}} f = \sum_{i,j=1}^{n} \hat{g}^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} f$ is bounded if f is a smooth function on X. Also, if f' is a smooth function on $X \setminus D$ that is of order $|z_1|^{2\beta}$ around D, then $\Delta_{\hat{\omega}} f' = O(1) + O(|z_1|^{2\beta})$. In particular, $\Delta_{\hat{\omega}} f'$ remains bounded on $X \setminus D$.

We now evaluate the Ricci curvature of $\hat{\omega}$. In terms of the local coordinate system $(z_1, \dots, z_n)$ as above, we have

$$\mathrm{Ric}(\hat{\omega})_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(\frac{\hat{\omega}^n}{\omega_0^n} \right) = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(|z_1|^{2\beta-2} \sum_{j=0}^{n-1} a_j |z_1|^{2\beta j} + \sum_{j=0}^{n} b_j |z_1|^{2\beta j} \right).$$

Since $\partial\bar{\partial} \log |z_1|^2 = 0$ on $X \setminus D$, we have

$$\mathrm{Ric}(\hat{\omega})_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \left(\sum_{j=0}^{n-1} a_j |z_1|^{2\beta j} + \sum_{j=0}^{n} b_j |z_1|^{2-2\beta+2\beta j} \right).$$

Note now that we can write

$$\begin{aligned} \log \left(\sum_{j=0}^{n-1} a_j |z_1|^{2\beta j} + \sum_{j=0}^{n} b_j |z_1|^{2-2\beta+2\beta j} \right) &= F_0 + \log \Big(O(1) + O(|z_1|^{2-2\beta}) + O(|z_1|^{2\beta}) \Big) \\ &= O(1) + O(|z_1|^{2-2\beta}) + O(|z_1|^{2\beta}) \qquad (3.27) \end{aligned}$$

with some smooth function F_0, around the divisor D. We thus have $\mathrm{Ric}(\hat{\omega})_{1\bar{1}} = O(1) + O(|z_1|^{-2\beta}) + O(|z_1|^{2\beta-2})$, $\mathrm{Ric}(\hat{\omega})_{1\bar{j}} = O(1) + O(|z_1|^{1-2\beta}) + O(|z_1|^{2\beta-1})$ ($j \neq 1$), and $\mathrm{Ric}(\hat{\omega})_{j\bar{k}} = O(1)$ ($j,k \neq 1$). Together with Lemma 3.4.3, this means the following.

Lemma 3.4.4. *Suppose that $\hat{g}$ is a conically singular Kähler metric of elementary form with cone angle $2\pi\beta$ along D locally defined by $z_1 = 0$. Then*

1. $\mathrm{Ric}(\hat{\omega})_{1\bar{1}} = O(1) + O(|z_1|^{-2\beta}) + O(|z_1|^{2\beta-2})$,
2. $\mathrm{Ric}(\hat{\omega})_{1\bar{j}} = O(1) + O(|z_1|^{1-2\beta}) + O(|z_1|^{2\beta-1})$ *($j \neq 1$),*

3. $\mathrm{Ric}(\hat{\omega})_{j\bar{k}} = O(1)$ *($j,k \neq 1$).*

In particular, combined with Lemma 3.4.3, we see that the scalar curvature $S(\hat{\omega})$ can be estimated as $S(\hat{\omega}) = O(1) + O(|z_1|^{2-4\beta})$.

Remark 3.4.5. We observe that the above estimate implies

$$\left| \int_{\Omega \setminus D} S(\hat{\omega}) \frac{\hat{\omega}^n}{n!} \right| < \text{const.} \int_{\text{unit disk in } \mathbb{C}} (1 + |z_1|^{2-4\beta}) |z_1|^{2\beta - 2} \sqrt{-1} dz_1 \wedge d\bar{z}_1$$
$$< \text{const.} \int_0^1 (r^{2\beta-1} + r^{-2\beta+1}) dr < \infty$$

for any open set $\Omega \subset X$ with $\Omega \cap D \neq \emptyset$, as $0 < \beta < 1$.

3.4.2 Scalar curvature as a current

In order to compute the log Futaki invariant with respect to a conically singular metric ω_{sing}, we need to make sense of $\mathrm{Ric}(\omega_{sing}) \wedge \omega_{sing}^{n-1}$ globally on X. However, this is not well-defined for a general conically singular metric ω_{sing}, as we discuss in Remark 3.4.8. We thus restrict our attention to the case of conically singular metrics of elementary form $\hat{\omega}$ or the momentum-constructed cynically singular metrics ω_φ. Theorems 3.1.11 and 3.1.12 state that in these cases it is indeed possible to have a well-defined current $\mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1}$ or $\mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}$ on X, and this section is devoted to the proof of these results.

Remark 3.4.6. We decide to present the argument for the conically singular metric of elementary form $\hat{\omega}$ in parallel with the one for the momentum-constructed conically singular metric ω_φ, as they have much in common. From now on, when we write "momentum-constructed conically singular metric ω_φ on X", it is always assumed that X is of the form $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ over a base Kähler manifold (M, ω_M) with the projection $p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M)$. We do *not* necessarily assume that $p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M)$ satisfies σ-constancy (cf. Definition 3.3.1), but *do* need to assume that φ is real analytic; we will only rely on the results proved in §3.3.2, in which we did not assume σ-constancy but assumed that φ is real analytic.

On the other hand, when we consider the conically singular metrics of elementary form $\hat{\omega} = \omega + \lambda \sqrt{-1} \partial \bar{\partial} |s|_h^{2\beta}$, X can be any (polarised) Kähler manifold with some smooth effective divisor $D \subset X$.

Remark 3.4.7. Suppose that we write, for a conically singular metric of elementary form $\hat{\omega}$,

$$\bar{S}(\hat{\omega}) := \frac{1}{\mathrm{Vol}(X,\hat{\omega})} \int_X \mathrm{Ric}(\hat{\omega}) \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!}$$

for the "average of $S(\hat{\omega})$ on the whole of X", where we note $\mathrm{Vol}(X,\hat{\omega}) := \int_X \hat{\omega}^n/n! = \int_{X \setminus D} \hat{\omega}^n/n! < \infty$ (by recalling Remark 3.1.6). We then have, from Theorem 3.1.11,

$$\bar{S}(\hat{\omega}) = \underline{S}(\hat{\omega}) + 2\pi(1-\beta)\frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\hat{\omega})},$$

where $\underline{S}(\hat{\omega}) := \int_{X \setminus D} S(\hat{\omega}) \frac{\hat{\omega}^n}{n!} / \mathrm{Vol}(X,\hat{\omega})$ is the average of $S(\hat{\omega})$ over $X \setminus D$, which makes sense by Remark 3.4.5. Similarly, for a momentum-constructed conically singular metric ω_φ, we have (by recalling Theorem 3.1.12 and Lemma 3.3.8)

$$\begin{aligned}\bar{S}(\omega_\varphi) &= \underline{S}(\omega_\varphi) + 2\pi(1-\beta)\frac{\mathrm{Vol}(D, p^*\omega_M(b))}{\mathrm{Vol}(X,\omega_\varphi)} \\ &= \underline{S}(\omega_\varphi) + 2\pi(1-\beta)\frac{\mathrm{Vol}(M, \omega_M(b))}{\mathrm{Vol}(X,\omega_\varphi)}.\end{aligned}$$

The reader is warned that the average of the scalar curvature $\bar{S}(\hat{\omega})$ computed with respect to the conically singular metrics may *not* be a cohomological invariant since $\mathrm{Ric}(\hat{\omega})$ is not necessarily a de Rham representative of $c_1(L)$ due to the cone singularities of $\hat{\omega}$, whereas $\mathrm{Vol}(D,\omega) = \int_D c_1(L)^{n-1}/(n-1)!$ certainly is. Exactly the same remark of course applies to the momentum-constructed conically singular metric ω_φ. On the other hand, we can show $\mathrm{Vol}(X,\hat{\omega}) = \int_X c_1(L)^n/n!$ (cf. Lemma 3.5.1), and $\mathrm{Vol}(X,\omega_\varphi) = 4\pi b \mathrm{Vol}(M,\omega_M)$ (cf. Remark 3.3.4) for $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$.

Remark 3.4.8. We will use in the proof the estimates established in §3.3.2 and §3.4.1, and our proof will not apply to conically singular metrics in full generality. Most importantly, we do not know what the "distributional" component (i.e. the second term in Theorems 3.1.11 and 3.1.12) should be for a general conically singular metric ω_{sing}; the proof below shows that it should be equal to $[D] \wedge \omega_{sing}^{n-1}$, $[D]$ being a current of integration over D, but it is far from obvious that it is well-defined (particularly so since ω_{sing} is singular along D). Indeed, even for the case of conically singular metrics of elementary form $\hat{\omega}$, $[D] \wedge \hat{\omega}^{n-1}$ being well-defined as a current (Lemma 3.4.10) seems to be a new result.

Proof of Theorems 3.1.11 and 3.1.12. The proof is essentially a repetition of the usual proof of the Poincaré–Lelong formula (cf. [35]), with some modifications needed to take care of the cone singularities of $\hat{\omega}$ and ω_φ.

We first consider the case of the conically singular metric of elementary form $\hat{\omega}$. We first pick a C^∞-tubular neighbourhood D_0 around D with (small but fixed) radius ε_0, meaning that points in D_0 have distance less than ε_0 from D measured in the metric ω. We then write

$$\int_\Omega f\mathrm{Ric}(\hat{\omega})\wedge\frac{\hat{\omega}^{n-1}}{(n-1)!}=\int_{\Omega\setminus D_0} f\mathrm{Ric}(\hat{\omega})\wedge\frac{\hat{\omega}^{n-1}}{(n-1)!}+\int_{\Omega\cap D_0} f\mathrm{Ric}(\hat{\omega})\wedge\frac{\hat{\omega}^{n-1}}{(n-1)!}$$

and apply the partition of unity on the compact manifold $\overline{\Omega\cap D_0}$ (i.e. the closure of $\Omega\cap D_0$) to reduce to the local computation in a small open set $U\subset\Omega\cap D_0$ around the divisor D. Confusing $U\subset\Omega\cap D_0$ with an open set in $\mathbb{C}^n$, this means that we take an open set U in $\mathbb{C}^n$ (by abuse of notation) endowed with the Kähler metric ω, where we may also assume that U is biholomorphic to the polydisk $\{(z_1,\dots,z_n)\mid |z_1|_\omega<\varepsilon_0/2, |z_2|_\omega<\varepsilon_0/2,\dots,|z_n|_\omega<\varepsilon_0/2\}$, in which the divisor D is given by the local equation $z_1=0$. Thus our aim now is to show

$$\int_U f\mathrm{Ric}(\hat{\omega})\wedge\frac{\hat{\omega}^{n-1}}{(n-1)!}=\int_{U\setminus\{z_1=0\}} fS(\hat{\omega})\frac{\hat{\omega}^n}{n!}+2\pi(1-\beta)\int_{\{z_1=0\}} f\frac{\omega^{n-1}}{(n-1)!},$$

where we recall that the partition of unity allows us to assume that f is smooth and compactly supported on U.

Note that exactly the same argument applies to the momentum-constructed conically singular metric ω_φ, by using some reference smooth metric ω_0 on X (in place of ω) to define D_0. Hence our aim for the momentum-constructed conically singular metric ω_φ is to show

$$\int_U f\mathrm{Ric}(\omega_\varphi)\wedge\frac{\omega_\varphi^{n-1}}{(n-1)!}=\int_{U\setminus\{z_1=0\}} fS(\omega_\varphi)\frac{\omega_\varphi^n}{n!}+2\pi(1-\beta)\int_{\{z_1=0\}} f\frac{p^*\omega_M(b)^{n-1}}{(n-1)!},$$

for a smooth and compactly supported f.

For the conically singular metrics of elementary form $\hat{\omega}$, we recall Lemma 3.4.2 and write $\hat{\omega}^n=|z_1|^{2\beta-2}\alpha$ with some smooth bounded (n,n)-form α on $X\setminus D$, and hence have $\partial\bar{\partial}\log\det(\hat{\omega})=(\beta-1)\partial\bar{\partial}\log|z_1|^2+R$ where R is a 2-form which is smooth on $U \setminus \{z_1 = 0\}$ but may have a pole (of fractional order) along $\{z_1 = 0\}$. We thus write

$$\begin{aligned} \mathrm{Ric}(\hat{\omega}) \wedge \hat{\omega}^{n-1} &= -\sqrt{-1}\partial\bar{\partial} \log \det(\hat{\omega}) \wedge \hat{\omega}^{n-1} \\ &= (1-\beta)\sqrt{-1}\partial\bar{\partial} \log |z_1|^2 \wedge \hat{\omega}^{n-1} - \sqrt{-1} R \wedge \hat{\omega}^{n-1}. \end{aligned} \tag{3.28}$$

On the other hand, we can argue in exactly the same way, by using (3.13) in place of Lemma 3.4.2, to see that for a momentum-constructed conically singular metric ω_φ, we can write

$$\begin{aligned} \mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1} &= -\sqrt{-1}\partial\bar{\partial} \log \det(\omega_\varphi) \wedge \omega_\varphi^{n-1} \\ &= (1-\beta)\sqrt{-1}\partial\bar{\partial} \log |z_1|^2 \wedge \omega_\varphi^{n-1} - \sqrt{-1} R_\varphi \wedge \omega_\varphi^{n-1} \end{aligned} \tag{3.29}$$

for some 2-form R_φ that is smooth on $U \setminus \{z_1 = 0\}$ but may have a pole (of fractional order) along $\{z_1 = 0\}$.

We aim to show that these formulae (3.28) and (3.29) are well-defined in the weak sense. This means that we aim to show that

$$\int_U f \mathrm{Ric}(\hat{\omega}) \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_U f\sqrt{-1} R \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!} + (1-\beta) \int_U f\sqrt{-1}\partial\bar{\partial} \log |z_1|^2 \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!}$$

is well-defined and is equal to

$$\int_{U \setminus \{z_1=0\}} fS(\hat{\omega}) \frac{\hat{\omega}^n}{n!} + 2\pi(1-\beta) \int_{\{z_1=0\}} f \frac{\omega^{n-1}}{(n-1)!}$$

for any smooth function f with compact support in U. Theorem 3.1.11 obviously follows from this, and exactly the same argument applies to ω_φ to prove Theorem 3.1.12.

We prove these claims as follows. Let U_ε be a subset of U defined for sufficiently small $\varepsilon \ll \varepsilon_0$ by $U_\varepsilon := \{(z_1, \dots, z_n) \in U \mid 0 < \varepsilon < |z_1|\}$ (the norm in the inequality $\varepsilon < |z_1|$ is given by the Euclidean metric on $\mathbb{C}^n$). In Lemma 3.4.9, we shall prove that

$$-n \int_U f\sqrt{-1} R \wedge \hat{\omega}^{n-1} = -n \lim_{\varepsilon \to 0} \int_{U_\varepsilon} f\sqrt{-1} R \wedge \hat{\omega}^{n-1} = \int_{U \setminus \{z_1=0\}} fS(\hat{\omega})\hat{\omega}^n$$

for a conically singular metric of elementary form $\hat{\omega}$, and

$$-n\int_U f\sqrt{-1}R_\varphi \wedge \omega_\varphi^{n-1} = -n\lim_{\varepsilon\to 0}\int_{U_\varepsilon} f\sqrt{-1}R_\varphi \wedge \omega_\varphi^{n-1} = \int_{U\setminus\{z_1=0\}} fS(\omega_\varphi)\omega_\varphi^n$$

for a momentum-constructed conically singular metric ω_φ, and that both of these terms are finite if f is compactly supported on U,

In Lemma 3.4.10 we shall prove

$$\int_U f\sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge \hat{\omega}^{n-1} = 2\pi\int_{\{z_1=0\}} f\omega^{n-1},$$

and in Lemma 3.4.12 we shall prove

$$\int_U f\sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge \omega_\varphi^{n-1} = 2\pi\int_{\{z_1=0\}} fp^*\omega_M(b)^{n-1},$$

if f is smooth. Granted these lemmas, we complete the proof of Theorems 3.1.11 and 3.1.12. □

Lemma 3.4.9. *For a conically singular metric of elementary form $\hat{\omega}$, we have*

$$-n\int_U f\sqrt{-1}R\wedge\hat{\omega}^{n-1} = -n\lim_{\varepsilon\to 0}\int_{U_\varepsilon} f\sqrt{-1}R\wedge\hat{\omega}^{n-1} = \int_{U\setminus\{z_1=0\}} fS(\hat{\omega})\hat{\omega}^n$$

and the integral is well-defined for any smooth function f compactly supported on U, i.e. $\left|\int_U f\sqrt{-1}R\wedge\hat{\omega}^{n-1}\right| < \infty$.

For a momentum-constructed conically singular metric ω_φ, we have

$$-n\int_U f\sqrt{-1}R_\varphi \wedge \omega_\varphi^{n-1} = -n\lim_{\varepsilon\to 0}\int_{U_\varepsilon} f\sqrt{-1}R_\varphi \wedge \omega_\varphi^{n-1} = \int_{U\setminus\{z_1=0\}} fS(\omega_\varphi)\omega_\varphi^n$$

and the integral is well-defined for any smooth function f compactly supported on U.

Proof. We first consider the case of the conically singular metric of elementary form $\hat{\omega}$. Although R is not bounded on the whole of $U\setminus\{z_1=0\}$, Lemma 3.4.4 shows that the metric contraction of R with $\hat{\omega}$ (which is equal to $S(\hat{\omega})/n$ on $X\setminus D$) satisfies

$$|\Lambda_{\hat{\omega}}R| < \text{const.}(1+|z_1|^{2-4\beta}). \tag{3.30}$$

on $U \setminus \{z_1 = 0\}$, thus

$$|R \wedge \hat{\omega}^{n-1}|_{\omega} \leq \text{const.}(|z_1|^{2\beta-2} + |z_1|^{2-4\beta+2\beta-2}) = \text{const.}(|z_1|^{2\beta-2} + |z_1|^{-2\beta})$$

on $U \setminus \{z_1 = 0\}$. Since f is bounded on the whole of U, we see, by writing $r := |z_1|$ and choosing a large but fixed number A which depends only on U and ω, that

$$\begin{aligned}
\lim_{\varepsilon \to 0} \left| \int_{U_\varepsilon} f\sqrt{-1}R \wedge \hat{\omega}^{n-1} \right| &\leq \text{const.} \lim_{\varepsilon \to 0} \int_{U_\varepsilon} (|z_1|^{2\beta-2} + |z_1|^{-2\beta})\omega^n \\
&\leq \text{const.} \lim_{\varepsilon \to 0} \int_{\varepsilon < |z_1| < A} (|z_1|^{2\beta-2} + |z_1|^{-2\beta})\sqrt{-1}dz_1 \wedge d\bar{z}_1 \\
&\leq \text{const.} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{A} (r^{2\beta-2} + r^{-2\beta}) r dr < \infty
\end{aligned}$$

since $0 < \beta < 1$. In other words, the above shows that the signed measure defined by $\sqrt{-1}R \wedge \hat{\omega}^{n-1}$ on U is well-defined. Observe also

$$\begin{aligned}
\left| \int_{U \setminus U_\varepsilon} f\sqrt{-1}R \wedge \hat{\omega}^{n-1} \right| &\leq \text{const.} \int_{U \setminus U_\varepsilon} |f\sqrt{-1}R \wedge \hat{\omega}^{n-1}|_{\omega}\omega^n \\
&\leq \text{const.} \int_0^{\varepsilon} \sup_{|z_1|=r} |f\sqrt{-1}R \wedge \hat{\omega}^{n-1}|_{\omega} r dr \qquad (3.31)\\
&\leq \text{const.} \int_0^{\varepsilon} (r^{2\beta-1} + r^{1-2\beta}) dr \to 0
\end{aligned}$$

as $\varepsilon \to 0$, where we used the elementary $\int_0^{\varepsilon} = \int_{[0,\varepsilon]} = \int_{(0,\varepsilon]}$ in (3.31) to apply (3.30), by noting that $\sup_{|z_1|=r} |f\sqrt{-1}R \wedge \hat{\omega}^{n-1}|_{\omega}$ is continuous in $r \in (0, \varepsilon]$ and its only singularity is the pole of fractional order at $r = 0$. We thus have

$$\int_U f\sqrt{-1}R \wedge \hat{\omega}^{n-1} = \lim_{\varepsilon \to 0} \int_{U_\varepsilon} f\sqrt{-1}R \wedge \hat{\omega}^{n-1} = \int_{U \setminus \{z_1 = 0\}} f\sqrt{-1}R \wedge \hat{\omega}^{n-1}$$

and the above integrals are all finite.

On the other hand, we know that $\partial\bar{\partial} \log |z_1|^2 = 0$ on $U \setminus \{z_1 = 0\}$, and hence, recalling (3.28), $S(\hat{\omega})\hat{\omega}^n = -n\sqrt{-1}R \wedge \hat{\omega}^{n-1}$ on $U \setminus \{z_1 = 0\}$. Thus we can write

$$-n \int_U f\sqrt{-1}R \wedge \hat{\omega}^{n-1} = -n \lim_{\varepsilon \to 0} \int_{U_\varepsilon} f\sqrt{-1}R \wedge \hat{\omega}^{n-1} = \int_{U \setminus \{z_1 = 0\}} S(\hat{\omega})\hat{\omega}^n$$

as claimed.

For the case of momentum-constructed conically singular metric ω_φ, Lemma 3.3.8 shows that $|\Lambda_{\omega_\varphi} R_\varphi|$ is bounded on $U \setminus \{z_1 = 0\}$. Since this is better than the estimate (3.30), all the following argument applies word by word. We thus establish the claim for the momentum-constructed conically singular metric. □

Lemma 3.4.10. *For a conically singular metric of elementary form $\hat{\omega}$,*

$$\int_U f\sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge \hat{\omega}^{n-1} = 2\pi \int_{\{z_1=0\}} f\omega^{n-1},$$

if f is smooth and compactly supported in U.

Remark 3.4.11. Note that we cannot naively apply the usual Poincaré–Lelong formula, since the metric $\hat{\omega}$ is singular along $\{z_1 = 0\}$. Note also that the integral $\int_{\{z_1=0\}} f\omega^{n-1}$ is manifestly finite.

Proof. We start by re-writing

$$\begin{aligned}
&\int_U \sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge f\hat{\omega}^{n-1} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0}\int_{U\setminus U_\varepsilon} dd^c \log|z_1|^2 \wedge f\hat{\omega}^{n-1} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0}\int_{U\setminus U_\varepsilon} d\left(d^c\log|z_1|^2 \wedge f\hat{\omega}^{n-1}\right) + \frac{1}{2}\lim_{\varepsilon\to 0}\int_{U\setminus U_\varepsilon} d^c\log|z_1|^2 \wedge df \wedge \hat{\omega}^{n-1} \quad (3.32)
\end{aligned}$$

since $\partial\bar{\partial}\log|z_1|^2 = 0$ if $|z_1| \neq 0$, where we used $d = \partial + \bar{\partial}$ and $d^c = \sqrt{-1}(\bar{\partial} - \partial)$.

We first claim $\lim_{\varepsilon\to 0}\int_{U\setminus U_\varepsilon} d^c\log|z_1|^2 \wedge df \wedge \hat{\omega}^{n-1} = 0$. We start by observing that $\hat{\omega}^{n-1}$ cannot contain the term proportionate to $dz_1 \wedge d\bar{z}_1$ when we take the wedge product of it with $d^c\log|z_1|^2$ or $d\log|z_1|^2$, since it will be cancelled by them. Namely, writing $|s|_h^{2\beta} = e^\phi|z_1|^{2\beta}$ and defining

$$\begin{aligned}
\tilde{\omega} &:= \hat{\omega} - \lambda\sqrt{-1}\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1 \wedge d\bar{z}_1 \\
&= \omega + \lambda\sqrt{-1}\left(\sum_{j=2}^n \beta|z_1|^{2\beta-2}z_1(\partial_{\bar{j}}e^\phi)dz_1 \wedge d\bar{z}_j + c.c. + |z_1|^{2\beta}\eta'\right) \quad (3.33)
\end{aligned}$$

where $\eta' := \partial\bar{\partial}e^\phi - \frac{\partial^2 e^\phi}{\partial z_1\partial\bar{z}_1}dz_1 \wedge d\bar{z}_1$ is a smooth 2-form, we have $d^c\log|z_1|^2 \wedge \hat{\omega}^{n-1} = d^c\log|z_1|^2 \wedge \tilde{\omega}^{n-1}$ and $d\log|z_1|^2 \wedge \hat{\omega}^{n-1} = d\log|z_1|^2 \wedge \tilde{\omega}^{n-1}$. It should be stressed that $\tilde{\omega}$ is not necessarily closed; indeed $d\tilde{\omega} = -\lambda\sqrt{-1}d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1 \wedge d\bar{z}_1\right)$. Note also that $\tilde{\omega} \leq \text{const.}\hat{\omega}$.

Combined with the well-known equality $d^c \log |z_1|^2 \wedge df \wedge \hat{\omega}^{n-1} = -d \log |z_1|^2 \wedge d^c f \wedge \hat{\omega}^{n-1}$, we find

$$
\begin{aligned}
&\int_{V_\varepsilon} d^c \log |z_1|^2 \wedge df \wedge \hat{\omega}^{n-1} \\
&= -\int_{V_\varepsilon} d \log |z_1|^2 \wedge d^c f \wedge \tilde{\omega}^{n-1} \\
&= -\int_{V_\varepsilon} d \left(\log |z_1|^2 d^c f \wedge \tilde{\omega}^{n-1} \right) + \int_{V_\varepsilon} \log |z_1|^2 dd^c f \wedge \tilde{\omega}^{n-1} - \int_{V_\varepsilon} \log |z_1|^2 d^c f \wedge d\tilde{\omega}^{n-1}
\end{aligned}
\tag{3.34}
$$

where we decide to write $V_\varepsilon := U \setminus U_\varepsilon$.

We evaluate each term separately and show that all of them go to 0 as $\varepsilon \to 0$. To evaluate the first term of (3.34), we write $\int_{V_\varepsilon} d \left(\log |z_1|^2 d^c f \wedge \tilde{\omega}^{n-1} \right) = \int_{\partial V_\varepsilon} \log |z_1|^2 d^c f \wedge \tilde{\omega}^{n-1}$. Observe now that

$$
\tilde{\omega}|_{\partial V_\varepsilon} = \omega|_{\partial V_\varepsilon} + \lambda \sqrt{-1} \left(\sum \varepsilon^{2\beta} (\partial_{\bar{j}} e^{\phi}) \sqrt{-1} e^{\sqrt{-1}\theta} d\theta \wedge d\bar{z}_j + c.c. + \varepsilon^{2\beta} \eta' |_{\partial V_\varepsilon} \right)
\tag{3.35}
$$

where we wrote $z_1 = \varepsilon e^{\sqrt{-1}\theta}$ on $\partial V_\varepsilon = \{|z_1| = \varepsilon\}$. This means that

$$
\left| \int_{\partial V_\varepsilon} \log |z_1|^2 d^c f \wedge \tilde{\omega}^{n-1} \right| \leq \text{const.} \log \varepsilon \left| \int_{\partial V_\varepsilon} (\varepsilon^{2\beta} + \varepsilon) d\theta \wedge dz_2 \wedge d\bar{z}_2 \wedge \dots dz_n \wedge d\bar{z}_n \right|
\tag{3.36}
$$

$$
\leq \text{const.}\varepsilon^{2\beta} \log \varepsilon \to 0
$$

as $\varepsilon \to 0$, by noting that $dz_1 = \varepsilon \sqrt{-1} e^{\sqrt{-1}\theta} d\theta$ on ∂V_ε and f is smooth on U.

The second term of (3.34) can be evaluated as

$$
\begin{aligned}
\left| \int_{V_\varepsilon} \log |z_1|^2 dd^c f \wedge \tilde{\omega}^{n-1} \right| &\leq \text{const.} \left| \int_{V_\varepsilon} \log r^2 \Delta_{\hat{\omega}} f \hat{\omega}^n \right| \\
&\leq \text{const.} \left| \int_{V_\varepsilon} \log r^2 \hat{\omega}^n \right| \\
&\leq \text{const.} \left| \int_0^\varepsilon r^{2\beta - 1} \log r dr \right| \to 0
\end{aligned}
\tag{3.37}
$$

as $\varepsilon \to 0$, by noting that $\Delta_{\hat{\omega}} f$ is bounded since f is smooth on U (cf. Lemma 3.4.3).

In order to evaluate the third term of (3.34), we start by re-writing it as

$$\begin{aligned}&\int_{V_\varepsilon}\log|z_1|^2 d^c f\wedge d\tilde{\omega}^{n-1}\\&=-\lambda(n-1)\int_{V_\varepsilon}\log|z_1|^2 d^c f\wedge d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})\sqrt{-1}dz_1\wedge d\bar{z}_1\right)\wedge\tilde{\omega}^{n-2}. \quad (3.38)\end{aligned}$$

We have

$$\begin{aligned}&d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})\sqrt{-1}dz_1\wedge d\bar{z}_1\right)\\&=\sum_{j=2}^n\Big(\beta^2(\partial_j e^\phi)|z_1|^{2\beta-2}+\beta\partial_j((\partial_1\phi)e^\phi)|z_1|^{2\beta-2}z_1\\&\qquad+\beta\partial_j((\partial_{\bar{1}}\phi)e^\phi)|z_1|^{2\beta-2}\bar{z}_1+|z_1|^{2\beta}\frac{\partial^3 e^\phi}{\partial z_1\partial\bar{z}_1\partial z_j}\Big)\sqrt{-1}dz_1\wedge d\bar{z}_1\wedge dz_j+c.c.\end{aligned}$$

Since $\tilde{\omega}$ does not have any term proportionate to dz_1 or $d\bar{z}_1$ when wedged with $d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1\wedge d\bar{z}_1\right)$, we have, from (3.33),

$$\begin{aligned}&\left|d^c f\wedge d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1\wedge d\bar{z}_1\right)\wedge\tilde{\omega}^{n-2}\right|_\omega\\&\le\text{const.}\left|d^c f\wedge d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1\wedge d\bar{z}_1\right)\wedge\omega^{n-2}\right|_\omega\end{aligned}$$

and noting that f is smooth on U, we have

$$\left|d^c f\wedge d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1\wedge d\bar{z}_1\right)\wedge\omega^{n-2}\right|_\omega\le\text{const.}|z_1|^{2\beta-2}. \quad (3.39)$$

Thus

$$\begin{aligned}&\left|\int_{V_\varepsilon}\log|z_1|^2 d^c f\wedge d\tilde{\omega}^{n-1}\right|\\&=\left|\lambda(n-1)\int_{V_\varepsilon}\log|z_1|^2 d^c f\wedge d\left(\frac{\partial^2}{\partial z_1\partial\bar{z}_1}(e^\phi|z_1|^{2\beta})dz_1\wedge d\bar{z}_1\right)\wedge\tilde{\omega}^{n-2}\right|\\&\le\text{const.}\left|\int_{V_\varepsilon}r^{2\beta-2}\log r\,\omega^n\right|\le\text{const.}\left|\int_0^\varepsilon r^{2\beta-1}\log r\,dr\right|\to 0 \quad (3.40)\end{aligned}$$

as $\varepsilon\to 0$, finally establishing $\int_{V_\varepsilon}d^c\log|z_1|^2\wedge df\wedge\hat{\omega}^{n-1}\to 0$ as $\varepsilon\to 0$.

Going back to (3.32), we have thus shown $\lim_{\varepsilon\to 0}\int_{V_\varepsilon}\sqrt{-1}\partial\bar{\partial}\log|z_1|^2\wedge f\hat{\omega}^{n-1}=$

$\frac{1}{2}\lim_{\varepsilon\to 0}\int_{V_\varepsilon} d\left(d^c \log|z_1|^2 \wedge f\hat{\omega}^{n-1}\right)$, and hence are reduced to evaluating

$$\lim_{\varepsilon\to 0}\int_{V_\varepsilon} d\left(d^c \log|z_1|^2 \wedge f\hat{\omega}^{n-1}\right) = \lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon} d^c \log|z_1|^2 \wedge f\hat{\omega}^{n-1} = \lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon} d^c \log|z_1|^2 \wedge f\tilde{\omega}^{n-1}.$$

Recall that $d^c \log|z_1|^2 = 2d\theta$ on $\{|z_1| = \varepsilon\}$, and also that $\lim_{\varepsilon\to 0}\tilde{\omega}|_{\partial V_\varepsilon} = \omega|_{\{z_1=0\}}$, which follows from (3.33). We thus have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon} d^c \log|z_1|^2 \wedge f\tilde{\omega}^{n-1}\\ &= \lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon} 2d\theta \wedge f\tilde{\omega}^{n-1} = \int_0^{2\pi} 2d\theta \int_{\{z_1=0\}} f\omega^{n-1} = 4\pi\int_{\{z_1=0\}} f\omega^{n-1}.\end{aligned}$$

This means that

$$\lim_{\varepsilon\to 0}\int_{V_\varepsilon} \sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge f\hat{\omega}^{n-1} = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{V_\varepsilon} dd^c \log|z_1|^2 \wedge f\hat{\omega}^{n-1} = 2\pi\int_{\{z_1=0\}} f\omega^{n-1}$$

as claimed.

□

Lemma 3.4.12. *For a momentum-constructed conically singular metric ω_φ,*

$$\int_U f\sqrt{-1}\partial\bar{\partial}\log|z_1|^2 \wedge \omega_\varphi^{n-1} = 2\pi\int_{\{z_1=0\}} f p^*\omega_M(b)^{n-1},$$

if f is smooth and compactly supported in U.

Proof. The proof is essentially the same as the one for Lemma 3.4.10. We note that we can proceed almost word by word, except for the places where we used the explicit description of $\hat{\omega}$ and $\tilde{\omega}$: the estimates (3.36), (3.37), and in estimating (3.38).

We certainly need to define a differential form, say $\tilde{\omega}_\varphi$, which replaces $\tilde{\omega}$ in the proof of Lemma 3.4.10. We define it as $\tilde{\omega}_\varphi := \omega_\varphi - \frac{2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}}{\varphi}\sqrt{-1}dz_1\wedge d\bar{z}_1$, by recalling the estimate (3.12).

Note again that this is not necessarily closed, and also that $\tilde{\omega}_\varphi$ does not even define a metric, since it is degenerate in the $dz_1\wedge d\bar{z}_1$-component, whereas we certainly have $\tilde{\omega}_\varphi \leq \text{const.}\omega_\varphi$. Observe that (3.12) and $\varphi = O(|z_1|^{2\beta})$ (as proved in Lemma 3.3.6) imply that

$$\tilde{\omega}_{\varphi}|_{\partial V_{\varepsilon}} = \omega_{\varphi}|_{\partial V_{\varepsilon}} + \frac{1}{\varphi}\left(2\beta|z_1|^{4\beta-2}\bar{z}_1 A_0 B_1 \sum_{i=2}^{n} \overline{B_{2,i}}\sqrt{-1}dz_1 \wedge d\bar{z}_i + c.c. + O(|z_1|^{4\beta})\right)\Bigg|_{\partial V_{\varepsilon}}$$
$$= \omega_{\varphi}|_{\partial V_{\varepsilon}} + O(\varepsilon^{2\beta}), \tag{3.41}$$

which replaces (3.35) in the proof of Lemma 3.4.10. Note also that, by recalling (3.12),

$$\omega_{\varphi}|_{\partial V_{\varepsilon}} = \left(p^*\omega_M(\tau) + \frac{1}{\varphi} d\tau \wedge d^c\tau\right)\Bigg|_{\partial V_{\varepsilon}}$$
$$= p^*\omega_M(\tau)|_{\partial V_{\varepsilon}} + \frac{1}{\varphi}\left(2\beta|z_1|^{4\beta-2}\bar{z}_1 A_0 B_1 \sum_{i=2}^{n} \overline{B_{2,i}}\sqrt{-1}dz_1 \wedge d\bar{z}_i + c.c. + O(|z_1|^{4\beta})\right)\Bigg|_{\partial V_{\varepsilon}}$$
$$= p^*\omega_M(\tau)|_{\partial V_{\varepsilon}} + \frac{1}{\varphi}\left(-2\varepsilon^{4\beta}\beta A_0 B_1 \sum_{i=2}^{n} \overline{B_{2,i}} d\theta \wedge d\bar{z}_i + c.c. + O(\varepsilon^{4\beta})\right)\Bigg|_{\partial V_{\varepsilon}} \tag{3.42}$$

where we wrote $z_1 = \varepsilon e^{\sqrt{-1}\theta}$ on $\partial V_{\varepsilon} = \{|z_1| = \varepsilon\}$ and used $dz_1 = \varepsilon\sqrt{-1}e^{\sqrt{-1}\theta}d\theta$. Thus, recalling $\varphi = O(|z_1|^{2\beta})$, $\omega_M(\tau) \leq \text{const.}\omega_M$, and that ω_M depends only on $(z_2, \dots, z_n)$, i.e. the coordinates on the base M, we have the estimate

$$\omega_{\varphi}|_{\partial V_{\varepsilon}} \leq \text{const.}\left(\sum_{i,j\neq 1} \sqrt{-1}dz_i \wedge d\bar{z}_j + \varepsilon^{2\beta}\sum_{j=2}^{n} \sqrt{-1}d\theta \wedge dz_j + c.c.\right)\Bigg|_{\partial V_{\varepsilon}} \tag{3.43}$$

from which it follows that

$$\left|\int_{\partial V_{\varepsilon}} \log|z_1|^2 d^c f \wedge \omega_{\varphi}^{n-1}\right| \leq \text{const.}\log\varepsilon \left|\int_{\partial V_{\varepsilon}} (\varepsilon^{2\beta} + \varepsilon) d\theta \wedge dz_2 \wedge d\bar{z}_2 \wedge \dots dz_n \wedge d\bar{z}_n\right|$$
$$\leq \text{const.}\varepsilon^{2\beta}\log\varepsilon \to 0 \tag{3.44}$$

as $\varepsilon \to 0$, for any smooth $f \in C^{\infty}(X, \mathbb{R})$. This means that the estimate (3.36) in the proof of Lemma 3.4.10 is still valid for momentum-constructed metrics ω_{φ}.

Also, Lemma 3.3.7 and the estimate (3.13) (and also $\tilde{\omega}_{\varphi} \leq \text{const.}\omega_{\varphi}$) means that the estimate in (3.37) in the proof of Lemma 3.4.10 is still valid for momentum-constructed metrics ω_{φ}.

We are thus reduced to estimating (3.38), which is the third term of (3.34) in the proof of Lemma 3.4.10. We first note

$$d\left(\frac{2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}}{\varphi}\sqrt{-1}dz_1\wedge d\bar{z}_1\right)=\sum_{i=2}^{n}\frac{\partial}{\partial z_i}\left(\frac{2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}}{\varphi}\right)\sqrt{-1}dz_1\wedge d\bar{z}_1\wedge dz_i$$
$$+c.c.$$

Recalling the estimate (3.43) and $\tilde{\omega}_\varphi \leq \text{const.}\omega_\varphi$, we thus have, by using a smooth reference metric ω_0 on X,

$$\begin{aligned}&\left|d^c f\wedge d\left(\frac{2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}}{\varphi}\sqrt{-1}dz_1\wedge d\bar{z}_1\right)\wedge\tilde{\omega}_\varphi^{n-2}\right|_{\omega_0}\\&\leq \text{const.}\left|d^c f\wedge d\left(\frac{2A_0^2B_1^2\beta^2|z_1|^{4\beta-2}}{\varphi}\sqrt{-1}dz_1\wedge d\bar{z}_1\right)\wedge\left(\sum_{i,j\neq 1}\sqrt{-1}dz_i\wedge d\bar{z}_j\right)^{n-2}\right|_{\omega_0}\\&\leq \text{const.}|z_1|^{2\beta-2},\end{aligned}\tag{3.45}$$

where in the last estimate we used the fact that f is smooth and that φ is of order $O(|z_1|^{2\beta})$ (cf. Lemma 3.3.6). This replaces (3.39) in the proof of Lemma 3.4.10, and hence we see that the estimate (3.40) is still valid for the momentum-constructed metrics, establishing that the third term of (3.34) in the proof of Lemma 3.4.10 goes to 0 as $\varepsilon \to 0$. Since all the other arguments in the proof of Lemma 3.4.10 do not need the estimates that use the specific properties of $\hat{\omega}$, and hence applies word by word to the momentum-constructed case, we finally have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon} d^c\log|z_1|^2\wedge f\tilde{\omega}_\varphi^{n-1}\\&=\lim_{\varepsilon\to 0}\int_{\partial V_\varepsilon}2d\theta\wedge f\tilde{\omega}_\varphi^{n-1}=\int_0^{2\pi}2d\theta\int_{\{z_1=0\}}fp^*\omega_M(b)^{n-1}=4\pi\int_{\{z_1=0\}}fp^*\omega_M(b)^{n-1},\end{aligned}$$

where we used $\tilde{\omega}_\varphi^{n-1}|_D=\omega_\varphi^{n-1}|_D=p^*\omega_M(b)^{n-1}$ by recalling (3.41), (3.42) and $D=\{z_1=0\}=\{\tau=b\}$. We can thus conclude, as in Lemma 3.4.10, that

$$\begin{aligned}&\int_U\sqrt{-1}\partial\bar{\partial}\log|z_1|^2\wedge f\omega_\varphi^{n-1}\\&=\lim_{\varepsilon\to 0}\int_{V_\varepsilon}\sqrt{-1}\partial\bar{\partial}\log|z_1|^2\wedge f\omega_\varphi^{n-1}=2\pi\int_{\{z_1=0\}}fp^*\omega_M(b)^{n-1},\end{aligned}$$

to get the claimed result. □

3.4.3 Log Futaki invariant computed with respect to the conically singular metrics

3.4.3.1 Conically singular metrics of elementary form

We first consider the conically singular metric of elementary form $\hat{\omega} = \omega + \lambda\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$. Suppose now that Ξ is a holomorphic vector field with the holomorphy potential $H \in C^\infty(X,\mathbb{C})$, with respect to ω, so that $\iota(\Xi)\omega = -\bar{\partial}H$. The holomorphy potential of Ξ with respect to $\hat{\omega}$ is given by $H - \lambda\sqrt{-1}\Xi(|s|_h^{2\beta})$, since, writing $\Xi = \sum_{i=1}^n v^i \frac{\partial}{\partial z_i}$ with $\bar{\partial} v^i = 0$ in terms of local holomorphic coordinates $(z_1,\dots,z_n)$, we have (cf. Lemma 4.10, [119])

$$\iota\left(v^i \frac{\partial}{\partial z_i}\right)\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta} = \sqrt{-1}v^i \frac{\partial^2 |s|_h^{2\beta}}{\partial z^i \partial \bar{z}^j} d\bar{z}^j = \bar{\partial}\left(\sqrt{-1}v^i \frac{\partial |s|_h^{2\beta}}{\partial z^i}\right). \tag{3.46}$$

Suppose we write $|s|_h^{2\beta} = e^{\beta\phi}|z_1|^{2\beta}$ in local coordinates on U, where $h = e^\phi$ for some function ϕ that is smooth on the closure of U. We now wish to evaluate $\Xi(e^{\beta\phi}|z_1|^{2\beta})$. If we assume that Ξ preserves the divisor $D = \{z_1 = 0\}$, we need to have $\Xi|_D = \sum_{i=2}^n v^i \frac{\partial}{\partial z_i}$, and so v^1 has to be a holomorphic function that vanishes on $\{z_1 = 0\}$. This means that we can write $v^1 = z_1 v'$ for another holomorphic function v'. We thus see that $\Xi(e^{\beta\phi}|z_1|^{2\beta}) = \sum_{i=1}^n v^i \partial_i (e^{\beta\phi}|z_1|^{2\beta})$ is of order $|z_1|^{2\beta}$ near D. We thus obtain that, for a holomorphic vector field Ξ preserving D, there exists a ($\mathbb{C}$-valued) function H' that is smooth on $X \setminus D$ and is of order $|z_1|^{2\beta}$ near D and satisfies

$$\iota(\Xi)\hat{\omega} = -\bar{\partial}(H + H'), \tag{3.47}$$

i.e. $\hat{H} := H + H'$ is the holomorphy potential of Ξ with respect to $\hat{\omega}$.

We wish to extend Theorem 3.1.11 to the case when f is replaced by the holomorphy potential $\hat{H}$ of a holomorphic vector field Ξ with respect to $\hat{\omega}$. This means that we need to extend Theorem 3.1.11 to functions f' that are not necessarily smooth on the whole of X but merely smooth on $X \setminus D$ and are asymptotically of order $O(|z_1|^{2\beta})$ near D. Note that most of the proof carries over word by word when we replace f by such f', except for the place where we showed $\lim_{\epsilon \to 0} \int_{U \setminus U_\epsilon} d^c \log|z_1|^2 \wedge df \wedge \hat{\omega}^{n-1} = 0$ in the equation (3.32) when we proved Lemma 3.4.10. More specifically, the smoothness of f was crucial in the estimates (3.36), (3.37), and (3.39) but not anywhere else. Thus the Lemma 3.4.10 still applies to f' if we can prove the estimates used in (3.36), (3.37), and (3.39) for f'. Note that we may still assume that f' is compactly supported on U, since this is the property coming from applying the partition of unity.

For (3.36), note first that on ∂V_ε, $|d^c f'|_\omega \leq \text{const.} |\varepsilon(\partial_1 f)d\theta + \sum_{i=2}^n (\partial_i f') + c.c.|_\omega = O(\varepsilon^{2\beta})$ by noting that $dz_1 = \sqrt{-1}\varepsilon e^{\sqrt{-1}\theta} d\theta$ on ∂V_ε. Thus we have

$$\left| \int_{\partial V_\varepsilon} \log |z_1|^2 d^c f' \wedge \tilde{\omega}^{n-1} \right| \leq \text{const.} \varepsilon^{2\beta} \log \varepsilon \left| \int_{\partial V_\varepsilon} (\varepsilon^{2\beta} + \varepsilon) d\theta \wedge dz_2 \wedge d\bar{z}_2 \wedge \dots dz_n \wedge d\bar{z}_n \right|$$
$$\leq \text{const.} \varepsilon^{4\beta} \log \varepsilon \to 0 \tag{3.48}$$

in place of (3.36).

For (3.37), we need to estimate $\Delta_{\hat{\omega}} f'$, but we simply recall Lemma 3.4.3 and see that $\Delta_{\hat{\omega}} f'$ is bounded on the whole of U. Thus the estimate established in (3.37)

$$\left| \int_{V_\varepsilon} \log |z_1|^2 dd^c f' \wedge \tilde{\omega}^{n-1} \right| \leq \text{const.} \left| \int_{V_\varepsilon} \log r^2 \hat{\omega}^n \right| \tag{3.49}$$

still holds for f'.

We are left to verify that the estimate (3.39) holds for f'. We remark that, in computing (3.39), we may replace $d^c f$ with $\sqrt{-1}\sum_{j=2}^n (\partial_{\bar{j}} f d\bar{z}_j - \partial_j f dz_j)$, since any term proportionate to dz_1 or $d\bar{z}_1$ will vanish when wedged with $d\left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1}(e^\phi |z_1|^{2\beta}) dz_1 \wedge d\bar{z}_1\right)$. Thus, since $\partial_{\bar{j}} f'$ and $\partial_j f'$ $(2 \leq j \leq n)$ are of order $O(r^{2\beta})$, we have

$$\left| d^c f' \wedge d\left(\frac{\partial^2}{\partial z_1 \partial \bar{z}_1} (e^\phi |z_1|^{2\beta}) dz_1 \wedge d\bar{z}_1 \right) \wedge \omega^{n-2} \right|_\omega \leq \text{const.} |z_1|^{4\beta - 2} \tag{3.50}$$

in place of (3.39), so that the conclusion (3.40) still holds.

Thus the proof of Lemma 3.4.10 carries over to f'. Noting that f' vanishes on D, we have $\int_U f' \sqrt{-1}\partial\bar{\partial} \log |z_1|^2 \wedge \hat{\omega}^{n-1} = 0$. In particular, if Ξ is a holomorphic vector field on X that preserves D whose holomorphy potential with respect to ω (resp. $\hat{\omega}$) is H (resp. $\hat{H} := H + H'$), we get

$$\int_X \hat{H} \text{Ric}(\hat{\omega}) \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_{X \setminus D} \hat{H} S(\hat{\omega}) \frac{\hat{\omega}^n}{n!} + 2\pi(1-\beta) \int_D H \frac{\omega^{n-1}}{(n-1)!}.$$

Combined with Remark 3.4.7, we thus get the first item of Corollary 3.1.14.

3.4.3.2 Momentum-constructed conically singular metrics

We now consider the momentum-constructed conically singular metrics ω_φ and the generator Ξ of the fibrewise $\mathbb{C}^*$-action that has τ as its holomorphy potential (see the argument at the beginning of §3.3.3.2). Recalling that $\tau - b$ is of order $O(|z_1|^{2\beta})$, as we proved in Lemma 3.3.6, we are thus reduced to establishing the analogue for ω_φ of the statement that we proved in §3.4.3.1 for the conically singular metric of elementary form $\hat{\omega}$. In fact, the proof carries over word by word, where we only have to replace $\tilde{\omega}$ by $\tilde{\omega}_\varphi$ (cf. the proof of Lemma 3.4.12); (3.44) is replaced by the analogue of (3.48), $\Delta_{\omega_\varphi} f'$ is bounded by Lemma 3.3.7 to establish the analogue of (3.49), and (3.45) can be established by observing that we can replace $d^c f'$ by $\sqrt{-1}\sum_{j=2}^n (\partial_{\bar{j}} f' d\bar{z}_j - \partial_j f' dz_j)$, as we did in (3.50).

Thus, arguing exactly as in §3.4.3.1, we get the second item of Corollary 3.1.14.

3.5 Some invariance properties for the log Futaki invariant

3.5.1 Invariance of volume and the average of holomorphy potential for conically singular metrics of elementary form

We first specialise to the conically singular metric of elementary form $\hat{\omega}$. Momentum-constructed conically singular metrics will be discussed in §3.5.3.

We recall that the volume $\mathrm{Vol}(X, \hat{\omega})$ or the average of the integral $\int_X \hat{H} \frac{\hat{\omega}^n}{n!}$ is not necessarily a invariant of the Kähler class, unlike in the smooth case. This is because, as we mentioned in §3.1.3, the singularities of $\hat{\omega}$ mean that we have to work on the noncompact manifold $X \setminus D$, on which we cannot naively use the integration by parts. The aim of this section is to find some conditions under which the boundary integrals vanish, as in the smooth case. We first prove the following lemma.

Lemma 3.5.1. *The volume* $\mathrm{Vol}(X, \hat{\omega})$ *of* X *measured by a conically singular metric with cone angle* $2\pi\beta$ *of elementary form* $\hat{\omega} = \omega + \lambda\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$ *with* $\omega \in c_1(L)$ *is equal to the cohomological* $\int_X c_1(L)^n/n!$ *if* $\beta > 0$.

Proof. Consider a path of metrics $\{\hat{\omega}_t := \omega + t\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}\}$ defined for $0 \le t \le \lambda$

for sufficiently small $\lambda > 0$, and write $\hat{g}_t$ for the metric corresponding to $\hat{\omega}_t$, with $g := \hat{g}_0$. Then we have $\frac{d}{dt}\Big|_{t=T} \hat{\omega}_t^n = n\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta} \wedge \hat{\omega}_T^{n-1} = \Delta_T |s|_h^{2\beta} \hat{\omega}_T^n$, where Δ_T is the (negative $\bar{\partial}$) Laplacian with respect to $\hat{\omega}_T$. If we show that $\frac{d}{dt}\Big|_{t=T} \int_X \hat{\omega}_t^n = \frac{d}{dt}\Big|_{t=T} \int_{X\setminus D} \hat{\omega}_t^n = \int_{X\setminus D} \frac{d}{dt}\Big|_{t=T} \hat{\omega}_t^n = 0$ for any $0 \le T \le \lambda \ll 1$ (where we used the Lebesgue convergence theorem in the second equality), then we will have proved $\mathrm{Vol}(X, \hat{\omega}_T) = \mathrm{Vol}(X, \omega) = \int_X c_1(L)^n/n!$. We thus compute $\int_{X\setminus D} \Delta_T |s|_h^{2\beta} \hat{\omega}_T^n$ for any $0 \le T \le \lambda$. We treat the case $T = 0$ and $T \neq 0$ separately. Note that in both cases, we may reduce to a local computation on $U \subset X$ by applying the partition of unity as we did in the proof of Theorems 3.1.11 and 3.1.12.

First assume $T = 0$. We now choose local holomorphic coordinates $(z_1, \dots, z_n)$ on U so that $D = \{z_1 = 0\}$. Writing $z_1 = re^{\sqrt{-1}\theta}$, we define a local C^∞-tubular neighbourhood D_ε around $D = \{z_1 = 0\}$ by $D_\varepsilon := \{x \in X \mid |s|_h(x) \le \varepsilon\}$. Then we have

$$\begin{aligned}\int_{U\setminus D} \Delta_\omega |s|_h^{2\beta} \omega^n &= \int_{U\setminus D_\varepsilon} \Delta_\omega |s|_h^{2\beta} \omega^n + \int_{D_\varepsilon \setminus D} \Delta_\omega |s|_h^{2\beta} \omega^n \\ &= \int_{U\setminus D_\varepsilon} \Delta_\omega |s|_h^{2\beta} \omega^n + \int_{D_\varepsilon \setminus D} \sum_{i,j} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} |s|_h^{2\beta} \omega^n.\end{aligned}$$

Writing $r = |z_1|$ and noting that $|s|_h = fr$ for some locally defined smooth bounded function f, we can evaluate $\left| \sum_{i,j} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} |s|_h^{2\beta} \right| \le \mathrm{const.}(r^{2\beta-2} + r^{2\beta-1} + r^{2\beta})$. Thus

$$\left| \int_{D_\varepsilon \setminus D} \sum_{i,j} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} |s|_h^{2\beta} \omega^n \right| \le \mathrm{const.} \int_0^\varepsilon (r^{2\beta-2} + r^{2\beta-1} + r^{2\beta}) r dr \to 0$$

as $\varepsilon \to 0$, if $\beta > 0$.

We thus have to show that $\int_{U\setminus D_\varepsilon}$ goes to 0 as $\varepsilon \to 0$. Note that this is reduced to the boundary integral on ∂D_ε by the Stokes theorem (by recalling that we have been assuming $|s|_h^{2\beta}$ is compactly supported in U as a consequence of applying the partition of unity) as $\int_{U\setminus D_\varepsilon} \Delta_\omega |s|_h^{2\beta} \omega^n = \int_{\partial D_\varepsilon} n\sqrt{-1}\bar{\partial}|s|_h^{2\beta} \wedge \omega^{n-1}$. Recalling $d\bar{z}_1 \mid_{|z_1|=r} = (-\sqrt{-1}\cos\theta - \sin\theta) r d\theta$, we may write

$$\bar{\partial}|s|_h^{2\beta} \wedge \omega^{n-1}\Big|_{\partial D_\varepsilon} = \frac{\partial |s|_h^{2\beta}}{\partial \bar{z}_1} F \varepsilon d\theta \wedge \sqrt{-1} dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \sqrt{-1} dz_n \wedge d\bar{z}_n$$

with some smooth function F, in the local coordinates $(z_1, \dots, z_n)$. We thus have

$\left| \int_{\partial D_\varepsilon} n\sqrt{-1}\bar{\partial}|s|_h^{2\beta} \wedge \omega^{n-1} \right| \leq$ const.$\varepsilon^{2\beta-1}\varepsilon \to 0$ as $\varepsilon \to 0$, if $\beta > 0$.

When $T > 0$, note that $\Delta_T |s|_h^{2\beta} = O(1)$ by Lemma 3.4.3. By Lemma 3.4.2, we have $\hat{\omega}_T^n = O(r^{2\beta-1})$, which shows that $\left| \int_{D_\varepsilon \setminus D} \sum_{i,j} \hat{g}_T^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} |s|_h^{2\beta} \hat{\omega}_T^n \right| \leq$ const. $\int_0^\varepsilon r^{2\beta-1} dr \to 0$ as $\varepsilon \to 0$. We are thus reduced to showing that the boundary integral $\int_{U \setminus D_\varepsilon} \Delta_T |s|_h^{2\beta} \hat{\omega}_T^n = \int_{\partial D_\varepsilon} n\sqrt{-1}\bar{\partial}|s|_h^{2\beta} \wedge \hat{\omega}_T^{n-1}$ goes to 0 as $\varepsilon \to 0$. We first evaluate $\int_{\partial D_\varepsilon} n\sqrt{-1}\frac{\partial |s|_h^{2\beta}}{\partial \bar{z}_1} d\bar{z}_1 \wedge \hat{\omega}_T^{n-1}$. By noting $dz_1 \wedge d\bar{z}_1 = 0$ on ∂D_ε, we observe that $d\bar{z}_1 \wedge \hat{\omega}_T^{n-1}|_{\partial D_\varepsilon} = F\varepsilon d\theta \wedge \sqrt{-1}dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n$ for some function F, bounded as $\varepsilon \to 0$, on ∂D_ε. Thus $\left| \int_{\partial D_\varepsilon} n\sqrt{-1}\frac{\partial |s|_h^{2\beta}}{\partial \bar{z}_1} d\bar{z}_1 \wedge \hat{\omega}_T^{n-1} \right| = O(\varepsilon^{2\beta-1}\varepsilon) \to 0$ as $\varepsilon \to 0$ if $\beta > 0$.

Again by noting $dz_1 \wedge d\bar{z}_1 = 0$ on ∂D_ε, we observe that $d\bar{z}_i \wedge \hat{\omega}_T^{n-1}|_{\partial D_\varepsilon} = F\varepsilon d\theta \wedge \sqrt{-1}dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n$ for some function $F = O(\varepsilon^{2\beta-1})$ on ∂D_ε. Thus $\left| \int_{\partial D_\varepsilon} n\sqrt{-1}\frac{\partial |s|_h^{2\beta}}{\partial \bar{z}_i} d\bar{z}_i \wedge \hat{\omega}_T^{n-1} \right| = O(\varepsilon^{2\beta}\varepsilon^{2\beta-1}\varepsilon) \to 0$ as $\varepsilon \to 0$ if $\beta > 0$. □

Lemma 3.5.2. *The average of the holomorphy potential $\int_X \hat{H}\frac{\hat{\omega}^n}{n!}$ in terms of the conically singular metric with cone angle $2\pi\beta$ of elementary form $\hat{\omega} = \omega + \lambda\sqrt{-1}\partial\bar{\partial}|s|_h^{2\beta}$ with $\omega \in c_1(L)$ is equal to the one $\int_X H\frac{\omega^n}{n!}$ measured in terms of the smooth Kähler metric ω, if $\beta > 0$.*

In particular, it is equal to b_0 (in §1.2) of the product test configuration for (X, L) defined by the holomorphic vector field on X generated by H (cf. §2 of [41]), if $\beta > 0$.

Proof. Recall that the holomorphy potential varies as (cf. (3.46)) $\frac{d}{dt}\Big|_{t=T} \hat{H}_t = \hat{g}_T^{i\bar{j}} \left(\frac{\partial}{\partial \bar{z}_j} \hat{H}_T \right) \left(\frac{\partial}{\partial z_i} |s|_h^{2\beta} \right)$. Thus, using the Lebesgue convergence theorem (as in the proof of Lemma 3.5.1), we get

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=T} \int_X \hat{H}_t \hat{\omega}_t^n &= \int_{X \setminus D} \sqrt{-1}n \left(\partial |s|_h^{2\beta} \wedge \bar{\partial}\hat{H}_T + \hat{H}_T \partial\bar{\partial}|s|_h^{2\beta} \right) \wedge \hat{\omega}_T^{n-1} \\ &= -\sqrt{-1}n \int_{X \setminus D} d\left(\hat{H}_T \partial |s|_h^{2\beta} \right) \wedge \hat{\omega}_T^{n-1}. \end{aligned}$$

We proceed as we did above in proving Lemma 3.5.1. When $T = 0$ we evaluate

$$\begin{aligned} &\int_{U \setminus D} d\left(H\partial |s|_h^{2\beta} \right) \wedge \omega^{n-1} \\ &= \lim_{\varepsilon \to 0} \int_{U \setminus D_\varepsilon} d\left(H\partial |s|_h^{2\beta} \right) \wedge \omega^{n-1} + \lim_{\varepsilon \to 0} \int_{D_\varepsilon \setminus D} d\left(H\partial |s|_h^{2\beta} \right) \wedge \omega^{n-1} \end{aligned}$$

Noting that H is a smooth function defined globally on the whole of X, we apply exactly the same argument that we used in proving Lemma 3.5.1 to see that both these terms go to 0 as $\varepsilon \to 0$.

When $T > 0$, we evaluate

$$\int_{U \setminus D} d\left(\hat{H}_T \partial |s|_h^{2\beta}\right) \wedge \hat{\omega}_T^{n-1} = \int_{U \setminus D_\varepsilon} d\left(\hat{H}_T \partial |s|_h^{2\beta}\right) \wedge \hat{\omega}_T^{n-1} + \int_{D_\varepsilon \setminus D} d\left(\hat{H}_T \partial |s|_h^{2\beta}\right) \wedge \hat{\omega}_T^{n-1}.$$

Recalling that $|\hat{H}_T| < \text{const.}(1 + r^{2\beta})$, we can apply exactly the same argument as we used in the proof of Lemma 3.5.1. This means that $\frac{d}{dt}\big|_{t=T} \int_{X \setminus D} \hat{H}_t \hat{\omega}_t^n = 0$ for all $0 \leq T \ll 1$ if $\beta > 0$. □

As a consequence of Corollary 3.1.14 and Lemmas 3.5.1, 3.5.2, we have the following.

Corollary 3.5.3. *If $0 < \beta < 1$, we have*

$$\begin{aligned} \mathrm{Fut}(\Xi, \hat{\omega}) &= \int_X \hat{H}(S(\hat{\omega}) - \bar{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!} \\ &= \int_{X \setminus D} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!} + 2\pi(1-\beta) \left(\int_D H \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)} \int_X H \frac{\omega^n}{n!} \right), \end{aligned}$$

where we note that the last two terms are invariant under changing the Kähler metric $\omega \mapsto \omega + \sqrt{-1}\partial\bar{\partial}\phi$ by $\phi \in C^\infty(X, \mathbb{R})$ (cf. Theorem 3.2.7).

Remark 3.5.4. Note that the "distributional" term

$$2\pi(1-\beta) \left(\int_D H \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)} \int_X H \frac{\omega^n}{n!} \right)$$

in the above formula is precisely the term that appears in the definition of the log Futaki invariant (up to the factor of 2π). Note also that $\mathrm{Vol}(D, \hat{\omega}) = \int_X [D] \wedge \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_D \frac{\omega^{n-1}}{(n-1)!} = \mathrm{Vol}(D, \omega)$ and $\int_D \hat{H} \frac{\hat{\omega}^{n-1}}{(n-1)!} = \int_D H \frac{\omega^{n-1}}{(n-1)!}$ by Lemma 3.4.10 (and its extension given in §3.4.3.1), where $[D]$ is the current of integration over D. This means that, combined with Lemmas 3.5.1 and 3.5.2, we get

$$\int_D \hat{H} \frac{\hat{\omega}^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\hat{\omega})}{\mathrm{Vol}(X,\hat{\omega})} \int_X \hat{H} \frac{\hat{\omega}^n}{n!} = \int_D H \frac{\omega^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D,\omega)}{\mathrm{Vol}(X,\omega)} \int_X H \frac{\omega^n}{n!}.$$

Thus, if we compute the log Futaki invariant $\mathrm{Fut}_{D,\beta}$ in terms of the conically singular metrics of elementary form $\hat{\omega}$, we get $\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega}) = \frac{1}{2\pi} \int_{X \setminus D} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!}$, which will certainly be 0 if $\hat{\omega}$ satisfies $S(\hat{\omega}) = \underline{S}(\hat{\omega})$ on $X \setminus D$, i.e. is cscK as defined in Definition 3.1.4.

3.5.2 Invariance of the Futaki invariant computed with respect to the conically singular metrics of elementary form

We first recall how we prove the invariance of the Futaki invariant in the smooth case, following the exposition given in §4.2 of Székelyhidi's textbook [119]. Write ω for an arbitrarily chosen reference metric in $c_1(L)$ and write $\omega_t := \omega + t\sqrt{-1}\partial\bar{\partial}\psi$ with some $\psi \in C^\infty(X,\mathbb{R})$. Defining $\mathrm{Fut}_t(\Xi) := \int_X H_t(S(\omega_t) - \bar{S})\frac{\omega_t^n}{n!}$, where H_t is the holomorphy potential of Ξ with respect to ω_t, we need to show $\frac{d}{dt}|_{t=0}\mathrm{Fut}_t(\Xi) = 0$.

Arguing as in §4.2, [119], we get

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} \mathrm{Fut}_t(\Xi) \\ &= \int_X \sqrt{-1}n\left((S(\omega) - \bar{S})\partial\psi \wedge \bar{\partial}H - H(\overline{\mathfrak{D}^*_\omega\mathfrak{D}_\omega\psi} - \partial\psi \wedge \bar{\partial}S(\omega)) + H(S(\omega) - \bar{S})\partial\bar{\partial}\psi\right) \wedge \omega^{n-1} \end{aligned}$$

where $\mathfrak{D}^*_\omega\mathfrak{D}_\omega$ is the operator defined in (1.1). We now perform the following integration by parts

$$\begin{aligned} &\int_X (S(\omega) - \bar{S})\partial\psi \wedge \bar{\partial}H \wedge \omega^{n-1} \\ &= -\int_X d\left(H(S(\omega) - \bar{S})\partial\psi \wedge \omega^{n-1}\right) + \int_X H\bar{\partial}S(\omega) \wedge \partial\psi \wedge \omega^{n-1} - \int_X (H(S(\omega) - \bar{S})\partial\bar{\partial}\psi \wedge \omega^{n-1} \\ &= \int_X H\bar{\partial}S(\omega) \wedge \partial\psi \wedge \omega^{n-1} - \int_X (H(S(\omega) - \bar{S})\partial\bar{\partial}\psi \wedge \omega^{n-1} \end{aligned}$$

by using Stokes' theorem. This means

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Fut}_t(\Xi) = -\int_X H\overline{\mathfrak{D}^*_\omega\mathfrak{D}_\omega\psi}\omega^n = -\int_X \psi\mathfrak{D}^*_\omega\mathfrak{D}_\omega H\omega^n = 0$$

as required, again integrating by parts.

We now wish to perform the above calculations when the Kähler metric $\hat{\omega}$ has cone singularities along D. An important point is that, since we are on the noncompact manifold $X \setminus D$, we have to evaluate the boundary integral when we apply Stokes' theorem, and that the remaining integrals may not be finite.

As we did in the proof of Lemma 3.5.1, we apply the partition of unity and reduce to a local computation around an open set U on which the integrand is compactly supported. Writing $\hat{H} = H + H'$ for the holomorphy potential of Ξ with respect to $\hat{\omega}$, as we did in (3.47), we first evaluate

$$\int_{U\setminus D} d\left(\hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega}))\partial\psi \wedge \hat{\omega}^{n-1}\right) = \lim_{\varepsilon\to 0}\int_{U\setminus D_\varepsilon} d\left(\hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega}))\partial\psi \wedge \hat{\omega}^{n-1}\right)$$
$$= \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon} \hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega}))\partial\psi \wedge \hat{\omega}^{n-1}.$$

Note $dz_1 \wedge d\bar{z}_1 = 0$ on ∂D_ε, which implies

$$\partial\psi \wedge \hat{\omega}^{n-1}|_{\partial D_\varepsilon} = \frac{\partial\psi}{\partial z_1} F_1 \varepsilon d\theta \wedge \sqrt{-1}dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n$$
$$+ \sum_{i\neq 1} \frac{\partial\psi}{\partial z_i} F_i \varepsilon d\theta \wedge F_1\sqrt{-1}dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge \sqrt{-1}dz_n \wedge d\bar{z}_n \quad (3.51)$$

where F_1 is bounded as $\varepsilon \to 0$ and F_i ($i \neq 1$) is at most of order $\varepsilon^{2\beta-1}$, we see that $\partial\psi \wedge \hat{\omega}^{n-1}|_{\partial D_\varepsilon} = O(\varepsilon) + O(\varepsilon^{2\beta})$. Recalling $\hat{H} = O(1) + O(|z_1|^{2\beta})$ and $S(\hat{\omega}) = O(1) + O(|z_1|^{2-4\beta})$, we see that the integrand of the above is at most of order $O(\varepsilon^{1+2-4\beta})$. Thus we need $\beta < 3/4$ for the boundary integral to be 0.

We now evaluate $\int_X \hat{H}\bar{\partial}S(\hat{\omega}) \wedge \partial\psi \wedge \hat{\omega}^{n-1}$. Writing

$$\bar{\partial}S(\hat{\omega}) \wedge \partial\psi \wedge \hat{\omega}^{n-1} = \frac{\partial S(\hat{\omega})}{\partial\bar{z}_1} d\bar{z}_1 \wedge \partial\psi \wedge \hat{\omega}^{n-1} + \sum_{i\neq 1}\frac{\partial S(\hat{\omega})}{\partial\bar{z}_i} d\bar{z}_i \wedge \partial\psi \wedge \hat{\omega}^{n-1},$$

we see that the order of the first term is at most $O(|z_1|^{2-4\beta-1}|z_1|^{2\beta-1+1}) = O(|z_1|^{1-2\beta})$, and the second term is at most of order $O(|z_1|^{2-4\beta}|z_1|^{2\beta-1}) = O(|z_1|^{1-2\beta})$, and hence we need $1-2\beta > -1$, i.e. $\beta < 1$ for the integral to be finite, by recalling $\hat{H} = O(1) + O(|z_1|^{2\beta})$. Since the second term $\int_X (\Delta_{\hat{\omega}}\psi)\hat{H}(S(\hat{\omega}) - \underline{S}(\hat{\omega}))\hat{\omega}^n$ is manifestly finite (by Lemma 3.4.3 and Remark 3.4.5), we can perform the integration by parts to have $\frac{d}{dt}\Big|_{t=0}\mathrm{Fut}_t(\Xi) = -\int_{X\setminus D}\hat{H}\overline{\mathfrak{D}^*_{\hat{\omega}}\mathfrak{D}_{\hat{\omega}}\psi}\hat{\omega}^n$ if $0 < \beta < 3/4$. It remains to prove that $\int_{X\setminus D}\hat{H}\overline{\mathfrak{D}^*_{\hat{\omega}}\mathfrak{D}_{\hat{\omega}}\psi}\hat{\omega}^n = \int_{X\setminus D}\psi\mathfrak{D}^*_{\hat{\omega}}\mathfrak{D}_{\hat{\omega}}\hat{H}\hat{\omega}^n = 0$ holds. Recalling $\overline{\mathfrak{D}^*_{\hat{\omega}}\mathfrak{D}_{\hat{\omega}}\psi} = \Delta^2_{\hat{\omega}}\psi + \hat{\nabla}_{\bar{j}}(\mathrm{Ric}(\hat{\omega})^{k\bar{j}}\partial_k\psi)$ (by noting $\bar{\psi} = \psi$ as ψ is a real function), where $\hat{\nabla}$ is the covariant derivative on TX defined by the Levi-Civita connection of $\hat{\omega}$, we first consider

$$\begin{aligned}\int_{U\setminus D}\hat{H}\hat{\nabla}_{\bar{j}}(\mathrm{Ric}(\hat{\omega})^{k\bar{j}}\partial_k\psi)\hat{\omega}^n &= \int_{U\setminus D}\hat{H}(\hat{\nabla}_{\bar{j}}\mathrm{Ric}(\hat{\omega})^{k\bar{j}})\partial_k\psi\hat{\omega}^n + \int_{U\setminus D}\hat{H}\mathrm{Ric}(\hat{\omega})^{k\bar{j}}\bar{\partial}_j\partial_k\psi\hat{\omega}^n \\ &= \sqrt{-1}n\int_{U\setminus D}\hat{H}\partial\psi\wedge\bar{\partial}S(\hat{\omega})\wedge\hat{\omega}^{n-1} + \int_{U\setminus D}\hat{H}S(\hat{\omega})\Delta_{\hat{\omega}}\psi\hat{\omega}^n \\ &\quad - \sqrt{-1}n(n-1)\int_{U\setminus D}\hat{H}\mathrm{Ric}(\hat{\omega})\wedge\partial\bar{\partial}\psi\wedge\hat{\omega}^{n-2}\end{aligned}$$

where we used the Bianchi identity $\hat{\nabla}_{\bar{j}}\mathrm{Ric}(\hat{\omega})^{k\bar{j}} = \hat{g}^{k\bar{j}}\partial_{\bar{j}}S(\hat{\omega})$ and the identity in Lemma 4.7, [119]. We perform the integration by parts for the second and the third term. We re-write the second term as

$$\begin{aligned}\int_{U\setminus D}\hat{H}S(\hat{\omega})\Delta_{\hat{\omega}}\psi\hat{\omega}^n = \sqrt{-1}n\Bigg(&-\int_{U\setminus D}d(\hat{H}S(\hat{\omega})\partial\psi\wedge\hat{\omega}^{n-1}) - \int_{U\setminus D}d(S(\hat{\omega})\bar{\partial}\hat{H}\psi\wedge\hat{\omega}^{n-1}) \\ &+ \int_{U\setminus D}\partial S(\hat{\omega})\wedge\bar{\partial}\hat{H}\wedge\psi\hat{\omega}^{n-1} + \int_{U\setminus D}S(\hat{\omega})\partial\bar{\partial}\hat{H}\wedge\psi\hat{\omega}^{n-1} \\ &+ \int_{U\setminus D}\hat{H}\bar{\partial}S(\hat{\omega})\wedge\partial\psi\wedge\hat{\omega}^{n-1}\Bigg)\end{aligned}$$

and the third term as

$$\begin{aligned}&\int_{U\setminus D}\hat{H}\mathrm{Ric}(\hat{\omega})\wedge\partial\bar{\partial}\psi\wedge\hat{\omega}^{n-2} \\ &= \int_{U\setminus D}d(\hat{H}\mathrm{Ric}(\hat{\omega})\wedge\bar{\partial}\psi\wedge\hat{\omega}^{n-2}) + \int_{U\setminus D}d(\partial\hat{H}\wedge\mathrm{Ric}(\hat{\omega})\wedge\psi\hat{\omega}^{n-2}) \\ &\quad - \int_{U\setminus D}\psi\bar{\partial}\partial\hat{H}\wedge\mathrm{Ric}(\hat{\omega})\wedge\hat{\omega}^{n-2}.\end{aligned}$$

We thus have $\int_{U\setminus D}\hat{H}\hat{\nabla}_{\bar{j}}(\mathrm{Ric}(\hat{\omega})^{k\bar{j}}\partial_k\psi)\hat{\omega}^n = \int_{U\setminus D}\psi\hat{\nabla}_j(\mathrm{Ric}(\hat{\omega})^{\bar{k}j}\partial_{\bar{k}}\hat{H})\hat{\omega}^n - \sqrt{-1}n(n-1)(B_1+B_2) - \sqrt{-1}n(B_3+B_4)$, where the B_i's stand for the boundary integrals $B_1 := \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\hat{H}\mathrm{Ric}(\hat{\omega})\wedge\bar{\partial}\psi\wedge\hat{\omega}^{n-2}$, $B_2 := \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\psi\partial\hat{H}\wedge\mathrm{Ric}(\hat{\omega})\wedge\hat{\omega}^{n-2}$, $B_3 := \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\hat{H}S(\hat{\omega})\partial\psi\wedge\hat{\omega}^{n-1}$, $B_4 := \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon}\psi S(\hat{\omega})\bar{\partial}\hat{H}\wedge\hat{\omega}^{n-1}$, which we now evaluate.

We first evaluate $\int_{\partial D_\varepsilon}\hat{H}\mathrm{Ric}(\hat{\omega})\wedge\bar{\partial}\psi\wedge\hat{\omega}^{n-2}$ in terms of ε. Since $dz_1\wedge d\bar{z}_1 = 0$ on ∂D_ε, we can see that this converges to 0 ($\varepsilon\to 0$) as long as $0<\beta<1$, by recalling Lemma 3.4.4. We thus get $B_1 = 0$.

We then evaluate $\int_{\partial D_\varepsilon}\psi\partial\hat{H}\wedge\mathrm{Ric}(\hat{\omega})\wedge\hat{\omega}^{n-2}$. We see that this converges to 0

$(\varepsilon \to 0)$ as long as $0 < \beta < 1$, exactly as we did before. We thus get $B_2 = 0$.

Now we see that $\int_{\partial D_\varepsilon} \hat{H} S(\hat{\omega}) \partial \psi \wedge \hat{\omega}^{n-1}$ is at most of order $\varepsilon^{3-4\beta}$, since $S(\hat{\omega})$ is at most of order $\varepsilon^{2-4\beta}$ and $\partial \psi \wedge \hat{\omega}^{n-1}$ is of order $O(\varepsilon) + O(\varepsilon^{2\beta})$ (cf. (3.51)), and hence converges to 0 (as $\varepsilon \to 0$) if $\beta < 3/4$. Similarly, we can show that $\int_{\partial D_\varepsilon} \psi S(\hat{\omega}) \bar{\partial} \hat{H} \wedge \hat{\omega}^{n-1}$ converges to 0 if $\beta < 3/4$. Thus, we get $B_3 = B_4 = 0$.

Note that $\int_{U \setminus D} \psi \hat{\nabla}_k (\mathrm{Ric}(\hat{\omega})^{\bar{j}k} (\partial_{\bar{j}} \hat{H})) \hat{\omega}^n$ converges if $0 < \beta < 1$, since Lemma 3.4.4, combined with Lemma 3.4.3, implies $\mathrm{Ric}(\hat{\omega})^{1\bar{1}} = O(|z_1|^{2-2\beta}) + O(|z_1|^{4-4\beta})$, $\mathrm{Ric}(\hat{\omega})^{1\bar{j}} = O(|z_1|) + O(|z_1|^{3-4\beta}) + O(|z_1|^{2-2\beta})$ $(j \neq 1)$, and $\mathrm{Ric}(\hat{\omega})^{i\bar{j}} = O(1) + O(|z_1|^{2\beta}) + O(|z_1|^{2-2\beta})$ $(i, j \neq 1)$. We thus see that we can perform the integration by parts in the above computation if we have $0 < \beta < 3/4$.

We are now left to prove $\int_{X \setminus D} \psi \Delta^2_{\hat{\omega}} \hat{H} \hat{\omega}^n = \int_{X \setminus D} \hat{H} \Delta^2_{\hat{\omega}} \psi \hat{\omega}^n$. We write

$$\begin{aligned}
&\int_{X \setminus D} \hat{H} \Delta^2_{\hat{\omega}} \psi \hat{\omega}^n = \sqrt{-1} n \int_{X \setminus D} \hat{H} \partial \bar{\partial} (\Delta_{\hat{\omega}} \psi) \wedge \hat{\omega}^{n-1} \\
&= \sqrt{-1} n \int_{X \setminus D} d(\hat{H} \bar{\partial} (\Delta_{\hat{\omega}} \psi) \wedge \hat{\omega}^{n-1}) + \sqrt{-1} n \int_{X \setminus D} d(\partial \hat{H} \wedge (\Delta_{\hat{\omega}} \psi) \hat{\omega}^{n-1}) \\
&\quad + \int_{X \setminus D} (\Delta_{\hat{\omega}} \hat{H})(\Delta_{\hat{\omega}} \psi) \hat{\omega}^{n-1}
\end{aligned}$$

and evaluate the boundary integrals $\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \hat{H} \bar{\partial} (\Delta_{\hat{\omega}} \psi) \wedge \hat{\omega}^{n-1}$ and $\lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \partial \hat{H} \wedge (\Delta_{\hat{\omega}} \psi) \hat{\omega}^{n-1}$ which, as before, can be shown to converge to zero as long as $\beta > 0$.

We finally evaluate $\int_{U \setminus D} (\Delta_{\hat{\omega}} \hat{H})(\Delta_{\hat{\omega}} \psi) \hat{\omega}^n$, where we recall from Lemma 3.4.3 that $\Delta_{\hat{\omega}} \hat{H} = O(1) + O(|z_1|^{2-2\beta}) + O(|z_1|^{2\beta})$. Thus, computing as we did above, we see that this is finite.

Summarising the above argument, together with the results in §3.5.1, we have the following. Suppose that we compute the log Futaki invariant

$$\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega}) = \frac{1}{2\pi} \int_X \hat{H} (S(\hat{\omega}) - \bar{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!} - (1-\beta) \left(\int_D \hat{H} \frac{\hat{\omega}^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(D, \hat{\omega})}{\mathrm{Vol}(X, \hat{\omega})} \int_X \hat{H} \frac{\hat{\omega}^n}{n!} \right)$$

with respect to the conically singular metric of elementary form $\hat{\omega}$ for a holomorphic vector field v that preserves the divisor D, with $\hat{H}$ as its holomorphy potential. As we mentioned in Remark 3.5.4, Lemmas 3.5.1, 3.5.2, Corollary 3.5.3, combined with Lemma 3.4.10 (and its extension given in §3.4.3.1), show $\mathrm{Fut}_{D,\beta}(\Xi, \hat{\omega}) = \frac{1}{2\pi} \int_{X \setminus D} \hat{H} (S(\hat{\omega}) - \underline{S}(\hat{\omega})) \frac{\hat{\omega}^n}{n!}$, and the calculations that we did above prove the first item of Theorem 3.1.15.

3.5.3 Invariance of the log Futaki invariant computed with respect to the momentum-constructed conically singular metrics

Now consider the case of momentum-constructed metrics on $X := \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$ with the $\mathbb{P}^1$-fibration structure $p : \mathbb{P}(\mathscr{F} \oplus \mathbb{C}) \to M$ over a Kähler manifold (M, ω_M). In this section, we shall assume that the σ-constancy hypothesis (Definition 3.3.1) is satisfied for our data $\{p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M), I\}$. Let $D \subset \mathbb{P}(\mathscr{F} \oplus \mathbb{C}) = X$ be the ∞-section, as before.

We first prove some lemmas that are well-known for smooth momentum-constructed metrics; the point is that they hold also for conically singular momentum-constructed metrics, since, as we shall see below, the proof applies word by word. We start with the following consequence of Lemma 3.3.9.

Lemma 3.5.5. (Lemma 2.8, [61]) *Suppose that the σ-constancy hypothesis (Definition 3.3.1) is satisfied for our data. For any function $f(\tau)$ of τ, we have*

$$\int_X f(\tau)\frac{\omega_\varphi^n}{n!} = 2\pi \mathrm{Vol}(M, \omega_M)\int_{-b}^{b} f(\tau)Q(\tau)d\tau,$$

where $Q(\tau)$ is as defined in (3.6). In particular, $\int_X f(\tau)\frac{\omega_\varphi^n}{n!}$ does not depend on the choice of φ or the boundary value $\varphi'(\pm b)$.

Proof. σ-constancy hypothesis implies that $Q(\tau) = \omega_M(\tau)^{n-1}/\omega_M^{n-1}$ is a function which depends only on τ. We thus have

$$\int_X f(\tau)\frac{\omega_\varphi^n}{n!} = \int_X \frac{\omega_M^{n-1}}{(n-1)!}\wedge\left(\frac{f(\tau)Q(\tau)}{\varphi}d\tau\wedge d^c\tau\right) = 2\pi\mathrm{Vol}(M,\omega_M)\int_{-b}^{b} f(\tau)Q(\tau)d\tau,$$

by (3.23) in Lemma 3.3.9. □

We summarise what we have obtained as follows.

Lemma 3.5.6. *Suppose that the σ-constancy hypothesis is satisfied for our data. Let $\varphi : [-b, b] \to \mathbb{R}_{\geq 0}$ be a real analytic momentum profile with $\varphi(\pm b) = 0$ and $\varphi(-b) = 2$, $\varphi(-b) = -2\beta$, so that $\omega_\varphi = p^*\omega_M - \tau p^*\gamma + \frac{1}{\varphi}d\tau\wedge d^c\tau$ has cone singularities with cone angle $2\pi\beta$ along the ∞-section. Let $\phi : [-b, b] \to \mathbb{R}_{\geq 0}$ be another momentum profile with $\varphi(\pm b) = 0$ and $\varphi(\pm b) = \mp 2$, so that $\omega_\phi = p^*\omega_M - \tau p^*\gamma + \frac{1}{\phi}d\tau\wedge d^c\tau$ is a smooth momentum-constructed metric. Then we have the following.*

1. $[\omega_\varphi] = [\omega_\phi]$,

2. $\mathrm{Vol}(X, \omega_\varphi) = 2\pi \mathrm{Vol}(M, \omega_M) \int_{-b}^{b} Q(\tau) d\tau = \mathrm{Vol}(X, \omega_\phi)$,

3. $\int_X \tau \frac{\omega_\varphi^n}{n!} = 2\pi \mathrm{Vol}(M, \omega_M) \int_{-b}^{b} \tau Q(\tau) d\tau = \int_X \tau \frac{\omega_\phi^n}{n!}$.

Proof. The first item follows from Lemma 3.3.9, and the second and the third from Lemma 3.5.5. □

The second and the third item of the above lemma shows that the second "distributional" term in Corollary 3.1.14 agrees with the "correction" term in the log Futaki invariant, as we saw in the case of conically singular metrics of elementary form (cf. Corollary 3.5.3 and Remark 3.5.4). We thus get the following result.

Corollary 3.5.7. *Suppose that the σ-constancy hypothesis is satisfied for our data $\{p : (\mathscr{F}, h_{\mathscr{F}}) \to (M, \omega_M), I\}$. Writing* $\mathrm{Fut}(\Xi, \omega_\varphi)$ *for the Futaki invariant computed with respect to the momentum-constructed conically singular metric ω_φ with cone angle $2\pi\beta$ and with real analytic momentum profile φ and $0 < \beta < 1$, evaluated against the generator Ξ of fibrewise $\mathbb{C}^*$-action of $X = \mathbb{P}(\mathscr{F} \oplus \mathbb{C})$, we have*

$$\begin{aligned}\mathrm{Fut}(\Xi, \omega_\varphi) = & \int_{X \setminus D} \tau (S(\omega_\varphi) - \underline{S}(\omega_\varphi)) \frac{\omega_\varphi^n}{n!} \\ & + 2\pi(1-\beta) \left(b \int_M \frac{\omega_M(b)^{n-1}}{(n-1)!} - \frac{\mathrm{Vol}(M, \omega_M(b))}{\mathrm{Vol}(X, \omega_\phi)} \int_X \tau \frac{\omega_\phi^n}{n!} \right)\end{aligned}$$

where ω_ϕ is a smooth momentum-constructed metric in the same Kähler class as ω_φ. In particular,

$$\mathrm{Fut}_{D,\beta}(\Xi, \omega_\varphi) = \int_{X \setminus D} \tau (S(\omega_\varphi) - \underline{S}(\omega_\varphi)) \frac{\omega_\varphi^n}{n!}.$$

We now wish to establish the analogue of the first item of Theorem 3.1.15. We first of all have to estimate the Ricci and scalar curvature of the metric $\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi$ for $\psi \in C^\infty(X, \mathbb{R})$. We show that this is exactly the same as the ones for the conically singular metrics of elementary form.

Lemma 3.5.8. $\mathrm{Ric}(\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi)$ *and $S(\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi)$ satisfy the estimates as given in Lemma 3.4.4.*

Proof. Choose a local coordinate system $(z_1, \dots, z_n)$ around a point in X so that D is locally given by $\{z_1 = 0\}$. Lemma 3.3.6 and the estimate (3.12) imply that we have

$$
\begin{aligned}
\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi =& p^*\omega_M - \tau p^*\gamma + \frac{1}{\varphi} d\tau \wedge d^c\tau + \sqrt{-1}\sum_{i,j=1}^n \frac{\partial^2\psi}{\partial z_i \partial \bar{z}_j}\sqrt{-1} dz_i \wedge d\bar{z}_j \\
=& |z_1|^{2\beta-2}\left(F_{11} + |z_1|^{2-2\beta}\frac{\partial^2\psi}{\partial z_1\partial\bar{z}_1}\right)\sqrt{-1}dz_1 \wedge d\bar{z}_1 \\
&+ \sum_{j=2}^n |z_1|^{2\beta-2}\left(F_{1j}\bar{z}_1 + |z_1|^{2-2\beta}\frac{\partial^2\psi}{\partial z_1\partial\bar{z}_j}\right)\sqrt{-1}dz_1 \wedge d\bar{z}_j + c.c. \\
&+ \sum_{i,j=2}^n \left(F_{ij}|z_1|^{2\beta} + \frac{\partial^2\psi + \psi_M}{\partial z_i \partial\bar{z}_j}\right)\sqrt{-1}dz_i \wedge d\bar{z}_j
\end{aligned}
$$

where F_{ij}'s stand for locally uniformly convergent power series in $|z_1|^{2\beta}$ with coefficients in smooth functions which depend only on the base coordinates $(z_2, \dots, z_n)$. We also wrote ψ_M for the local Kähler potential for $p^*\omega_M$. When we Taylor expand ψ and ψ_M, we thus get $(\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi)^n = |z_1|^{2\beta-2}[O(1) + O(|z_1|^{2\beta}) + O(|z_1|^{2-2\beta})]\prod_{i=1}^n(\sqrt{-1}dz_i \wedge d\bar{z}_i)$. Writing $\omega_0 := \prod_{i=1}^n(\sqrt{-1}dz_i \wedge d\bar{z}_i)$, we thus get

$$
\log\frac{(\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi)^n}{\omega_0^n} = (\beta - 1)\log|z_1|^2 + O(1) + O(|z_1|^{2\beta}) + O(|z_1|^{2-2\beta}).
$$

This is exactly the same as (3.27), from which Lemma 3.4.4 follows (since $\partial\bar{\partial}\log|z_1|^2 = 0$ on $X \setminus D$). □

Since that the holomorphy potential for Ξ with respect to $\omega_\varphi + \sqrt{-1}\partial\bar{\partial}\psi$ is given by $\tau - \sqrt{-1}\Xi(\psi) = O(|z_1|^{2\beta}) + O(1)$ (cf. Lemma 4.10, [119]), it is now straightforward to check that the calculations in §3.5.2 apply word by word. We thus get the second item of Theorem 3.1.15.

Chapter 4

Stability and canonical metrics on $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$

4.1 Introduction

4.1.1 Statement of the results

Consider now the following problem[1].

Problem 4.1.1. Suppose that a Kähler manifold X admits a cscK (resp. extremal) metric. Under what geometric hypotheses does the blowup $\mathrm{Bl}_Y X$ of X along a complex submanifold Y admit a cscK (resp. extremal) metric?

The case $\dim_{\mathbb{C}} Y = 0$ was solved by the theorems of Arezzo–Pacard [6, 7], and Arezzo–Pacard–Singer [8], which will be discussed in detail in §4.1.2, and will provide a background and motivation for considering Problem 4.1.1. The remaining case is $\dim_{\mathbb{C}} Y > 0$, and we assume $\dim_{\mathbb{C}} X \geq 3$ for the blowup to be non-trivial. On the other hand, there seems to be very few results known about Problem 4.1.1 when $\dim_{\mathbb{C}} Y > 0$, and the solution of Problem 4.1.1 in general seems to be out of reach at the moment; see §4.1.3 for the review of previously known results.

We thus decide to focus instead on a particular example, the blowup $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ of $\mathbb{P}^n$ along a line, in the hope that this may serve as a useful example in attacking Problem 4.1.1. The result that we prove is the following.

Theorem 4.1.2. *Let $n \geq 3$ and consider the blowup $\pi : \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n \to \mathbb{P}^n$ of $\mathbb{P}^n$ along a line. $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ is slope unstable (and hence K-unstable, cf. §4.2.1) with respect to any*

[1] This is mentioned, for example, in Székelyhidi's survey [118] in the case $\dim_{\mathbb{C}} Y > 0$.

polarisation; in particular, $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ *cannot admit a cscK metric in any rational Kähler class. However, if we choose* $\varepsilon > 0$ *sufficiently small, there exists an extremal metric in the Kähler class* $\pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1)) - \varepsilon c_1([E])$*, with an explicit formula given in Proposition 4.4.1, where* $[E]$ *is the line bundle associated to the exceptional divisor* E*.*

Notation 4.1.3. In this chapter, given a divisor D in a Kähler manifold X, we write $[D]$ for the line bundle $\mathscr{O}_X(D)$ associated to D. Also, we shall use the additive notation for the tensor product of line bundles, and the multiplicative notation will be reserved for the intersection product of divisors: given n divisors $D_1, \dots, D_n$, we shall write $D_1.D_2.\cdots.D_n$ to mean $\int_X c_1([D_1])c_1([D_2])\dots c_1([D_n])$, and $D_1^i.D_2^{n-i}$ to mean $\int_X c_1([D_1])^i c_1([D_2])^{n-i}$.

Remark 4.1.4. In spite of its apparent simplicity, there has been no known result on $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ in terms of cscK or extremal metrics, to the best of the author's knowledge (cf. §4.1.3). This is perhaps related to the fact that $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ does not admit a structure of a $\mathbb{P}^1$-bundle; see §4.1.3.1 for details.

4.1.2 Blowup of cscK and extremal manifolds at points

We now discuss the background for Theorem 4.1.2, namely Problem 4.1.1 for the case $\dim_{\mathbb{C}} Y = 0$. We prepare some notation before doing so; write $\mathrm{Ham}(\omega, g)$ for the group of Hamiltonian isometries of g, i.e. isometries of (X,g) which are also Hamiltonian diffeomorphisms of ω and let $\mathfrak{ham}$ be its Lie algebra. We observe that $\mathrm{Ham}(\omega, g)$ is a finite dimensional compact Lie group. This allows us to define a moment map $m : X \to \mathfrak{ham}^*$, which we may normalise so that $\int_X \langle m, v \rangle \omega^n / n! = 0$ for all $v \in \mathfrak{ham}$ with $\langle , \rangle$ being the natural duality pairing between $\mathfrak{ham}$ and $\mathfrak{ham}^*$. If X admits a cscK metric, a classical theorem of Matsushima [87] and Lichnerowicz [74] states the following for the Lie algebra of the group $\mathrm{Aut}_0(X,L)$ consisting of the elements in $\mathrm{Aut}(X)$ which lift to the automorphism of L (cf. §1.3).

Theorem 4.1.5. (cf. Theorem 1 in [70], Theorems 6.1 and 9.4 in [67]) *Suppose that* X *admits a cscK metric. Writing* $\mathfrak{aut}(X,L)$ *for* $\mathrm{Lie}\mathrm{Aut}_0(X,L)$*, we have* $\mathfrak{aut}(X,L) = \mathfrak{ham}^{\mathbb{C}}$.

We now consider Problem 4.1.1 for $\dim_{\mathbb{C}} Y = 0$. Suppose that we have a polarised cscK manifold (X,L) which we blow up at points $p_1, \dots, p_l$. We ask if the blown-up manifold $\mathrm{Bl}_{p_1,\dots,p_l} X$ admits a cscK metric in a "perturbed" Kähler class so that the size of the exceptional divisor is small. Solution to this problem is given by the following theorem of Arezzo and Pacard [7], which generalises their previous result in [6].

Theorem 4.1.6. (Arezzo and Pacard [7]) *Let (X,L) be a polarised Kähler manifold with a cscK metric $\omega \in c_1(L)$. Let $p_1,\dots,p_l$ be distinct points in X and $a_1,\dots,a_l$ be positive real numbers. Suppose that the following conditions are satisfied:*

1. *$m(p_1),\dots,m(p_l)$ spans $\mathfrak{ham}^*$,*
2. *$\sum_{i=1}^{l} a_i^{n-1} m(p_i) = 0 \in \mathfrak{ham}^*$.*

Then there exists $\varepsilon_0 > 0$, $c > 0$, and $\theta > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ the blowup $\hat{X} := \mathrm{Bl}_{p_1,\dots,p_l} X$ of X with the blowdown map $\pi : \hat{X} \to X$ admits a cscK metric ω_ε in the perturbed Kähler class

$$\pi^*[\omega] - \varepsilon \sum_{i=1}^{l} \tilde{a}_i c_1([E_i])$$

where $\tilde{a}_i$ depends only on ε and satisfies $|\tilde{a}_i - a_i| \leq c\varepsilon^\theta$ as $\varepsilon \to 0$ and E_i stands for the exceptional divisor corresponding to the blowup at p_i. Moreover, $\omega_\varepsilon \to \omega$ in the C^∞-norm as $\varepsilon \to 0$, away from $p_1,\dots,p_l$.

By Theorem 4.1.5, all of these hypotheses are vacuous if we assume $\mathfrak{aut}(X,L) = 0$. However, in presence of nontrivial holomorphic vector fields on X, we cannot choose the number and positions of $p_1,\dots,p_l$ arbitrarily to get a cscK metric on $\hat{X}$ (cf. Theorem 4.1.8).

Theorem 4.1.6 has many differential-geometric and algebro-geometric applications [32, 44, 108, 111]; we note in particular that it was used to construct an example of asymptotically Chow unstable cscK manifold ([32], cf. Remark 2.1.4), and also to prove the K-stability of cscK manifolds with discrete automorphism group ([111], cf. Theorem 1.2.9).

Even though $\mathfrak{aut}(X,L) \neq 0$ (or more precisely $\mathfrak{ham} \neq 0$) imposes some restrictions on the applicability of Arezzo–Pacard theorem, there is still hope of finding an extremal metric under weaker hypotheses, and moreover, it is natural to expect a version of this theorem for extremal metrics. Such result was indeed proved by Arezzo, Pacard, and Singer (cf. [8], Theorem 2.0.2). Just as Theorem 4.1.6 was used by Stoppa [111] to prove the K-stability of cscK manifolds when $\mathfrak{aut}(X,L) = 0$, this result was used by

Stoppa and Székelyhidi [114] to prove the relative K-stability of Kähler manifolds with an extremal metric. We finally note that Székelyhidi [117, 120] later established a connection to the K-stability of the blowup $\hat{X}$ when X admits an extremal metric.

4.1.3 Comparison to previous results

We now return to the case $X = \mathbb{P}^n$ and $Y = \mathbb{P}^1$, to consider $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$. Our results (Theorem 4.1.2) have much in common with, or more precisely are modelled after, the ones for the blowup $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$ of $\mathbb{P}^n$ at a point. We review some previously known results on $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$ and its generalisations, as well as several nonexistence results that seem to be particularly relevant to Problem 4.1.1.

4.1.3.1 Calabi's work on projectivised bundles and related results

In a seminal paper, Calabi [23] presented the first examples of Kähler manifolds which admit a non-cscK extremal metric. More precisely, he proved the following theorem.

Theorem 4.1.7. (Calabi [23]) *The projective completion $\mathbb{P}(\mathscr{O}_{\mathbb{P}^{n-1}}(-m) \oplus \mathbb{C}) \to \mathbb{P}^{n-1}$ of line bundles $\mathscr{O}_{\mathbb{P}^{n-1}}(-m) \to \mathbb{P}^{n-1}$, for any $m, n \in \mathbb{N}$, admits an extremal metric in each Kähler class.*

We observe that $\mathbb{P}(\mathscr{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathbb{C})$ is simply the blowup $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$ of $\mathbb{P}^n$ at a point; the above theorem thus implies that there exists an extremal metric in each Kähler class on $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$, although Theorem 4.1.8 due to Ross and Thomas shows that none of these extremal metrics can be cscK.

There are two important features of $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$ (or more generally $\mathbb{P}(\mathscr{O}_{\mathbb{P}^{n-1}}(-m) \oplus \mathbb{C})$) that can be used in the construction of extremal metrics; the $\mathbb{P}^1$-bundle structure and the toric structure. We first focus on the $\mathbb{P}^1$-bundle structure. Calabi's original proof exploited this structure, which was later generalised by many mathematicians to various situations. While the reader is referred to §4.5 of [62] for a historical survey, we wish to particularly mention the following case to which this theory applies: suppose that we blow up two skew planes $P_1 \cong \mathbb{P}^k$ and $P_2 \cong \mathbb{P}^{n-k-1}$ in $\mathbb{P}^n$. Then $\mathrm{Bl}_{P_1,P_2}\mathbb{P}^n$ is isomorphic to the total space of the projectivised bundle $\mathbb{P}(\mathscr{O}(1,-1) \oplus \mathbb{C})$ over an exceptional divisor $\mathbb{P}^k \times \mathbb{P}^{n-k-1}$, where $\mathscr{O}(1,-1) = p_1^* \mathscr{O}_{\mathbb{P}^k}(1) \otimes p_2^* \mathscr{O}_{\mathbb{P}^{n-k-1}}(-1)$ and $p_1 : \mathbb{P}^k \times \mathbb{P}^{n-k-1} \to \mathbb{P}^k$ and $p_2 : \mathbb{P}^k \times \mathbb{P}^{n-k-1} \to \mathbb{P}^{n-k-1}$ are the obvious projections. Then, we see that Theorem 3.3.2 by Hwang [61] immediately implies that $\mathrm{Bl}_{P_1,P_2}\mathbb{P}^n$

carries an extremal metric in each Kähler class.[2]

On the other hand, $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ does not have a structure of a $\mathbb{P}^1$-bundle, so the above theorems do not apply. Thus, we now focus on the toric structure of $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$. This was treated in [3] and [99], which (amongst other results) re-established Calabi's theorem using toric methods. This is the approach that we follow for $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$, and will be discussed in greater detail in §4.4.2.1.

4.1.3.2 Nonexistence results

We mention several nonexistence results which seem to be particularly relevant to Problem 4.1.1. There are several approaches to prove the nonexistence of cscK (resp. extremal) metrics. One frequently used approach is to use Theorem 1.2.8 due to Donaldson [42] (resp. Theorem 1.4 in [114] due to Stoppa and Székelyhidi), namely to prove K-instability (resp. relative K-instability) of (X,L). Proving K-instability is often possible by establishing a stronger statement, which is to prove *slope instability* of (X,L); the reader is referred to §4.2.1 for more details on this. Along this line, we recall the following result of Ross and Thomas. We follow their approach very closely in proving the slope instability of $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ (cf. Proposition 4.3.1).

Theorem 4.1.8. (Ross–Thomas [102], Examples 5.27, 5.35.) $\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^n$ *is slope unstable with respect to any polarisation. In particular, it cannot admit a cscK metric in any rational Kähler class.*

On the other hand, in some cases it is still possible to show K-instability directly, without proving slope instability, as in the following theorem due to Della Vedova [33]. They can be regarded as an extension of Stoppa's results [112, 111] to blowing up higher dimensional submanifolds. By defining the notion of "Chow stability" for subschemes inside a general polarised Kähler manifold (cf. Definition 3.5, [33]), he proved the following by showing the K-instability of the blowup.

Theorem 4.1.9. (Della Vedova [33], Theorem 1.5.) *Let (X,L) be a polarised Kähler manifold with a cscK metric in $c_1(L)$. Let $Z_1,\dots,Z_s$ be pairwise disjoint submanifolds of codimension greater than two, and let $\pi:\hat{X}\to X$ be the blowup of X along $Z_1\cup\cdots\cup$*

[2] In fact, some of the above examples admit Kähler–Einstein metrics, as shown by Koiso and Sakane [69], Mabuchi [80], and also Nadel [91].

Z_s with E_j being the exceptional divisor over Z_j. Define the subscheme Z by the ideal sheaf $\mathscr{I}_Z := \mathscr{I}_{Z_1}^{m_1} \cap \cdots \cap \mathscr{I}_{Z_s}^{m_s}$ for $m_1, \dots, m_s \in \mathbb{N}$.

If $Z \hookrightarrow X$ is Chow unstable, then the class $\pi^* c_1(L) - \varepsilon \sum_{j=1}^s m_j c_1([E_j])$ contains no cscK metrics for $0 < \varepsilon \ll 1$.

Della Vedova also proved an analogous statement for the extremal metrics (Theorem 1.7, [33]), by defining "relative Chow stability" for subschemes inside a general polarised Kähler manifold. See Examples 1.6 and 1.11 in [33] for explicit examples in which these results are used.

Remark 4.1.10. Recalling Theorem 4.1.5, we now ask whether the automorphism group of $X = \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ is reductive. It is easy to see that the Lie algebra $\mathfrak{aut}(X)$ of $\mathrm{Aut}(X)$ is equal to the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n+1,\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A, B, C are matrices of size 2×2, $2\times(n-1)$, $(n-1)\times(n-1)$, respectively. Note that X is Fano, and $\mathfrak{aut}(X) = \mathfrak{aut}(X, -K_X)$. Note also that $\mathfrak{aut}(X, -K_X) \cong \mathfrak{aut}(X, L)$ for any ample line bundle L, cf. [67, 70].

It is easy to see that the centre of $\mathfrak{h}$ is trivial, and hence $\mathfrak{h}$ is reductive if and only if it is semisimple. In principle this can be checked e.g. by Cartan's criterion using the Killing form, although in practice it may be a nontrivial task. We can still prove that $\mathfrak{h}$ is not semisimple, and hence nonreductive, as follows. Theorem 4.1.2 shows that we have a non-cscK extremal metric in the polarisation $L := \pi^*\mathscr{O}_{\mathbb{P}^n}(1) - \varepsilon[E]$ if $\varepsilon > 0$ is sufficiently small. This means that the Futaki invariant evaluated against the extremal vector field is not zero (Lemma 1.4.5). However, since the Futaki invariant is a Lie algebra character (Corollary 2.2, [54]), this means that $\mathfrak{h} \cong \mathfrak{aut}(X, L)$ cannot be semisimple. We thus conclude that $\mathfrak{h}$ is not reductive.

4.2 Some technical backgrounds

We briefly recall slope stability in §4.2.1, and toric Kähler geometry in §4.2.2. The aim of these sections is to fix the notation and recall some key facts; the reader is referred to the literature cited in each section for more details.

4.2.1 Slope stability

For the details of what is discussed in the following, the reader is referred to the paper [102] by Ross and Thomas.

Let (X,L) be a polarised Kähler manifold. Then for $k \gg 1$,

$$\dim H^0(X,L^{\otimes k}) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}).$$

Now let Z be a subscheme of X. The **Seshadri constant** $\mathrm{Sesh}(Z)$ for $Z \subset X$ (with respect to L) can be defined as follows. Considering the blowup $\pi : \mathrm{Bl}_Z X \to X$ with the exceptional divisor E, we define

$$\mathrm{Sesh}(Z) = \mathrm{Sesh}(Z,X,L) := \sup\{c \mid \pi^* L - cE \text{ is ample on } \mathrm{Bl}_Z X\}.$$

Then, writing $\mathscr{I}_Z$ for the ideal sheaf defining Z, we compute

$$\dim H^0(X, L^{\otimes k}/(L^{\otimes k} \otimes \mathscr{I}_Z^{xk})) = \tilde{a}_0(x) k^n + \tilde{a}_1(x) k^{n-1} + O(k^{n-2})$$

for $k \gg 1$ and $x \in \mathbb{Q}$ such that $kx \in \mathbb{N}$. It is well-known that $\tilde{a}_i(x)$ is a polynomial in x of degree at most $n-i$, and hence can be extended as a continuous function on $\mathbb{R}$ (cf. §3, [102]).

Definition 4.2.1. The **slope** of (X,L) is defined by $\mu(X,L) := a_1/a_0$, and the **quotient slope** of Z with respect to $c \in \mathbb{R}$ is defined by

$$\mu_c(\mathscr{O}_Z, L) := \frac{\int_0^c (\tilde{a}_1(x) + \tilde{a}_0(x)/2)\, dx}{\int_0^c \tilde{a}_0(x) dx}.$$

Definition 4.2.2. (X,L) is said to be **slope semistable with respect to** Z if $\mu(X,L) \leq \mu_c(\mathscr{O}_Z, L)$ for all $c \in (0, \mathrm{Sesh}(Z)]$. (X,L) is said to be **slope semistable** if it is slope semistable with respect to all subschemes Z of X. (X,L) is **slope unstable** if it is not slope semistable.

We remark that, since X is a manifold, the slope can be computed by the Hirzebruch–Riemann–Roch theorem as

$$\mu(X,L) = -\frac{n \int_X c_1(K_X) c_1(L)^{n-1}}{2 \int_X c_1(L)^n}. \tag{4.1}$$

The quotient slope can also be computed in terms of Chern classes when $\mathrm{Bl}_Z X$ is smooth, by noting $\pi_* \mathscr{O}_{\mathrm{Bl}_Z X}(-jE) = \mathscr{I}_Z^j$ for $\pi : \mathrm{Bl}_Z X \to X$ and $j \geq 0$ and again using Hirzebruch–Riemann–Roch. It takes a particularly neat form when Z is a divisor in X.

Theorem 4.2.3. (Ross–Thomas [102], Theorem 5.2.) *Let Z be a divisor in X. Then*

$$\mu_c(\mathscr{O}_Z, L) = \frac{n\left(L^{n-1}.Z - \sum_{j=1}^{n-1}\binom{n-1}{j}\frac{(-c)^j}{j+1}L^{n-1-j}.Z^j.(K_X+Z)\right)}{2\sum_{j=1}^{n}\binom{n}{j}\frac{(-c)^j}{j+1}L^{n-j}.Z^j}, \tag{4.2}$$

where the dot stands for the intersection product (cf. Notation 4.1.3), by identifying line bundles with corresponding divisors.

A fundamental theorem of Ross and Thomas is the following.

Theorem 4.2.4. (Ross–Thomas [102], Theorem 4.2.) *If (X, L) is K-semistable, then it is slope semistable with respect to any smooth subscheme Z.*

Remark 4.2.5. Slope stability is strictly weaker than K-stability; the blowup of $\mathbb{P}^2$ at two distinct points with the anticanonical polarisation is K-unstable, and yet slope stable (Example 7.6 in [94]).

4.2.2 Toric Kähler geometry

In addition to the original papers cited below, we mention [3, 45] and Chapters 27-29 of [26] as particularly useful reviews on the details of what is discussed in this section.

We first of all demand that the symplectic form ω on X be fixed throughout in this section. Recall that an action of a group G on a manifold X is called **effective** if for each $g \in G$, $g \neq \mathrm{id}_G$, there exists $x \in X$ such that $g \cdot x \neq x$. We first define a toric symplectic manifold, by regarding a Kähler manifold (X, ω) merely as a symplectic manifold.

Definition 4.2.6. A **toric symplectic manifold** is a symplectic manifold (X, ω) equipped with an effective Hamiltonian action of an n-torus $T^n := \mathbb{R}^n / 2\pi\mathbb{Z}^n$ with a corresponding moment map $m : X \to \mathbb{R}^n$.

Remark 4.2.7. Recall that a **moment map** for the action $T^n \curvearrowright X$ is a T^n-invariant map $m : X \to \mathrm{Lie}(T^n)^* \cong \mathbb{R}^n$ such that $\iota(v)\omega = -d\langle m, v\rangle$ for all $v \in \mathrm{Lie}(T^n)$. A T^n-action is called **Hamiltonian** if there exists a moment map for the action.

A theorem due to Atiyah [9], and Guillemin and Sternberg [59] states that the image of the moment map m is the convex hull of the images of the fixed points of the Hamiltonian torus action. For a toric symplectic manifold, it is a particular type of convex polytope called a **Delzant polytope**. Delzant [34] showed that we have a one-to-one correspondence between a Delzant polytope $\mathscr{P}$ and a toric symplectic manifold $T^n \curvearrowright (X, \omega)$; Delzant polytopes are *complete invariants* of toric symplectic manifolds. This allows us to confuse a toric symplectic manifold with its associated Delzant polytope $\mathscr{P}$, which is often called the **moment polytope**.

It is well-known that on (the preimage inside X of) the interior $\mathscr{P}^\circ$ of the moment polytope $\mathscr{P}$, the T^n-action is free and we have a coordinate chart $\{(x,y) = (x_1, \dots, x_n, y_1, \dots y_n) \in \mathscr{P}^\circ \times (\mathbb{R}^n/2\pi\mathbb{Z}^n)\}$, called **action-angle coordinates**, on $m^{-1}(\mathscr{P}^\circ)$. Action coordinates $(x_1, \dots, x_n)$ are also called **momentum coordinates**. In action-angle coordinates, the symplectic form can be written as $\omega = \sum_{j=1}^n dx_j \wedge dy_j$ and the moment map can be given by $m(x,y) = x$.

We now consider a complex structure on X to endow (X, ω) with a Kähler structure; the reader is referred, for example, to §3 of [3] or §2 of [45] for more details. We first recall that the way we construct $T^n \curvearrowright (X, \omega)$ from $\mathscr{P}$ [34] shows that any toric symplectic manifold automatically admits a T^n-invariant *complex* structure compatible with ω; a toric symplectic manifold is automatically a toric *Kähler* manifold. Let $\mathscr{S}_n$ be the **Siegel upper half space** consisting of complex symmetric $n \times n$ matrices of the form $Z = R + \sqrt{-1}S$ where R and S are real symmetric matrices and S is in addition assumed to be positive definite. It is known that $\mathscr{S}_n$ is isomorphic to $Sp(2n, \mathbb{R})/U(n)$, and that $\mathscr{S}_n$ bijectively corresponds to the set $\mathscr{J}(\mathbb{R}^{2n}, \omega_{\text{std}})$ of all complex structures on $\mathbb{R}^{2n}$ which are compatible with its standard symplectic form ω_{std}. It follows that in the action-angle coordinates on $m^{-1}(\mathscr{P}^\circ)$, by taking a Darboux chart, any almost complex structure J on (X, ω) can be written as

$$J = \begin{pmatrix} -S^{-1}R & -S^{-1} \\ RS^{-1}R + S & RS^{-1} \end{pmatrix}.$$

If we assume that J is T^n-invariant, we can make R, S depend only on the action coordinates x. Moreover, by a Hamiltonian action generated by a function $f(x)$, given infinitesimally as $y_j \mapsto y_j + \frac{\partial f}{\partial x_j}(x)$, we may choose $R = 0$. Furthermore, if we choose J to be integrable, we can show that there exists a potential function $s(x)$ of S such that $S_{ij} = \frac{\partial^2 s}{\partial x_i \partial x_j}(x)$. Such $s(x)$ is called a **symplectic potential**. Guillemin [60] showed that we can define a canonical complex structure, or canonical symplectic potential on (X, ω), from the data of the moment polytope $\mathscr{P}$.

Theorem 4.2.8. (Guillemin [60]) *Suppose that $\mathscr{P}$ has d facets (i.e. codimension 1 faces) which are defined by the vanishing of affine functions $l_i : \mathbb{R}^n \ni x \mapsto l_i(x) := \langle x, \nu_i \rangle - \lambda_i \in \mathbb{R}$, $i = 1, \dots, d$, where $\nu_i \in \mathbb{Z}^n$ is a primitive inward-pointing normal vector to the i-th facet and $\lambda_i \in \mathbb{R}$. Then in the action-angle coordinates on $m^{-1}(\mathscr{P}^\circ)$, the canonical symplectic potential $s_{\mathscr{P}}(x)$ is given by*

$$s_{\mathscr{P}}(x) := \frac{1}{2} \sum_{i=1}^{d} l_i(x) \log l_i(x).$$

Note that $s_{\mathscr{P}}(x)$ is not smooth at the boundary of the polytope, and this singular behaviour will be important in what follows. Abreu [2] further showed that *all* ω-compatible T^n-invariant complex structures can be obtained by adding a smooth function to the above $s_{\mathscr{P}}(x)$.

Theorem 4.2.9. (Abreu [2], Theorem 2.8) *An ω-compatible T^n-invariant complex structure on a toric Kähler manifold (X, ω) is determined by a symplectic potential of the form $s(x) := s_{\mathscr{P}}(x) + r(x)$ where $r(x)$ is a function which is smooth on the whole of $\mathscr{P}$ such that the Hessian* $\mathrm{Hess}(s)$ *of s is positive definite on the interior of $\mathscr{P}$ and has determinant of the form*

$$\det(\mathrm{Hess}(s)(x)) = \left[\delta(x) \prod_{i=1}^{d} l_i(x) \right]^{-1}, \tag{4.3}$$

with δ being a smooth and strictly positive function on the whole of $\mathscr{P}$. Conversely, any symplectic potential of this form defines an ω-compatible T^n-invariant complex structure on a toric Kähler manifold (X, ω).

The description in terms of the symplectic potential gives the scalar curvature a particularly neat form. Now let g_s be the Riemannian metric defined by ω and the complex structure determined by the symplectic potential $s(x)$. Write $s^{ij}(x)$ for the inverse matrix of the Hessian $\frac{\partial^2 s}{\partial x_i \partial x_j}(x)$. Abreu [1] derived the following equation in the action-angle coordinates.

Theorem 4.2.10. (Abreu [1], Theorem 4.1) *The scalar curvature $S(g_s)$ of g_s can be written as*

$$S(g_s) = -\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 s^{ij}}{\partial x_i \partial x_j}(x). \tag{4.4}$$

Moreover, g_s is extremal if and only if

$$\frac{\partial}{\partial x_k} S(g_s) = \text{const} \tag{4.5}$$

for all $k = 1, \dots, n$.

The equation (4.4) is often called **Abreu's equation**.

4.3 Slope instability of $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$

4.3.1 Statement of the result

We now return to the case where we blow up a line $\mathbb{P}^1$ inside $\mathbb{P}^n$, where we assume $n \geq 3$ for the blowup to be nontrivial. For ease of notation, we write $X := \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ and also write π for the blowdown map $\pi : X \to \mathbb{P}^n$. We re-state the first part of Theorem 4.1.2 as follows.

Proposition 4.3.1. *$X = \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$, $n \geq 3$, is slope unstable with respect to any polarisation. In particular, X cannot admit a cscK metric in any rational Kähler class.*

4.3.2 Proof of Proposition 4.3.1

4.3.2.1 Preliminaries on intersection theory

Observe first of all that any line bundle L on $X = \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ can be written as $L = a\pi^*\mathscr{O}_{\mathbb{P}^n}(1) - b[E]$, with some $a, b \in \mathbb{Z}$, by recalling $\mathrm{Pic}(X) = \mathbb{Z}\pi^*\mathscr{O}_{\mathbb{P}^n}(1) \oplus \mathbb{Z}[E]$. This is ample if and only if $a > b > 0$. Thus, up to an overall scaling, we may say that any ample line bundle on X can be written, as a $\mathbb{Q}$-line bundle, as $L = \pi^*\mathscr{O}_{\mathbb{P}^n}(1) - \varepsilon[E]$ for some $\varepsilon \in \mathbb{Q} \cap (0,1)$.

This also implies $\mathrm{Sesh}(E, X, L) = 1 - \varepsilon$; suppose that we blow up E in X, with the blowdown map $\tilde{\pi} : \mathrm{Bl}_E X \cong X \xrightarrow{\sim} X$. Then $\tilde{\pi}^* L - c[E] = \pi^*\mathscr{O}_{\mathbb{P}^n}(1) - (c + \varepsilon)[E]$, which is ample if and only if $-\varepsilon < c < 1 - \varepsilon$.

Henceforth, to simplify the notation, we write H for the hyperplane in $\mathbb{P}^n$ so that $[H] = \mathscr{O}_{\mathbb{P}^n}(1)$.

Our aim is to show that X is slope unstable with respect to the exceptional divisor E. Since the slope (4.1) and the quotient slope (4.2) can be computed in terms of intersection numbers, we first need to prepare some elementary results on the intersection theory on X; more specifically, we need to compute $\int_X c_1(\pi^*[H])^j c_1([E])^{n-j}$ for $0 \leq j \leq n$.

Recall (e.g. §3, Chapter 3, [58]) the Euler exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^n} \xrightarrow{j} \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0,$$

where the vector bundle homomorphism j takes $1 \in \mathscr{O}_{\mathbb{P}^n}$ to the Euler vector field

$$\sum_{i=0}^{n} Z_i \frac{\partial}{\partial Z_i} \in \bigoplus_{i=0}^{n} \left(\mathscr{O}_{\mathbb{P}^n}(1) \frac{\partial}{\partial Z_i} \right) \cong \mathscr{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)},$$

with $[Z_0 : \cdots : Z_n]$ being the homogeneous coordinates on $\mathbb{P}^n$. Restricting this sequence to a line $\mathbb{P}^1 \subset \mathbb{P}^n$, we get $0 \to \mathscr{O}_{\mathbb{P}^1} \xrightarrow{j} \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n}|_{\mathbb{P}^1} \to 0$. Combining this with the exact sequences $0 \to T_{\mathbb{P}^1} \to T_{\mathbb{P}^n}|_{\mathbb{P}^1} \to N_{\mathbb{P}^1/\mathbb{P}^n} \to 0$ and $0 \to \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to T_{\mathbb{P}^1} \to 0$, we get $N_{\mathbb{P}^1/\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}$. Thus the exceptional divisor $E = \mathbb{P}(N_{\mathbb{P}^1/\mathbb{P}^n})$ is isomorphic to $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}) \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus(n-1)}) \cong \mathbb{P}^1 \times \mathbb{P}^{n-2}$. Note also that the adjunction formula (§1, Chapter 1, [58]) shows $[E]|_E \cong N_{E/X}$, and that $N_{E/X}$ is isomorphic to the tautological bundle $\mathscr{O}_E(-1)$ over $E = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)})$. We observe that $\mathscr{O}_E(-1) \cong p_1^*\mathscr{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathscr{O}_{\mathbb{P}^{n-2}}(-1)$, where p_1 (resp. p_2) is the natural projection from E to $\mathbb{P}^1$ (resp. $\mathbb{P}^{n-2}$), and that $\pi^*[H]|_E \cong p_1^*\mathscr{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathscr{O}_{\mathbb{P}^{n-2}}$.

With these observations, and recalling that $c_1([E])$ is the Poincaré dual of E, we compute

$$\begin{aligned} E^n &= \int_X c_1([E])^n = \int_E c_1([E])^{n-1} \\ &= \int_{\mathbb{P}^1 \times \mathbb{P}^{n-2}} (p_1^* c_1(\mathscr{O}_{\mathbb{P}^1}(1)) - p_2^* c_1(\mathscr{O}_{\mathbb{P}^{n-2}}(1)))^{n-1} \\ &= (-1)^{n-2}(n-1) \end{aligned}$$

and

$$\begin{aligned}\pi^*H.E^{n-1} &= \int_X c_1(\pi^*[H])c_1([E])^{n-1}\\ &= \int_{\mathbb{P}^1\times\mathbb{P}^{n-2}} p_1^*c_1(\mathscr{O}_{\mathbb{P}^1}(1))(p_1^*c_1(\mathscr{O}_{\mathbb{P}^1}(1)) - p_2^*c_1(\mathscr{O}_{\mathbb{P}^{n-2}}(1)))^{n-2}\\ &= (-1)^{n-2}.\end{aligned}$$

If $2 \leq j < n$, we have

$$\begin{aligned}\pi^*H^j.E^{n-j} &= \int_{\mathbb{P}^1\times\mathbb{P}^{n-2}} p_1^*c_1(\mathscr{O}_{\mathbb{P}^1}(1))^j(p_1^*c_1(\mathscr{O}_{\mathbb{P}^1}(1)) - p_2^*c_1(\mathscr{O}_{\mathbb{P}^{n-2}}(1)))^{n-j-1}\\ &= 0\end{aligned}$$

and

$$\pi^*H^n = \int_X \pi^*c_1([H])^n = \int_{\pi(X)} c_1([H])^n = \int_{\mathbb{P}^n} c_1([H])^n = 1.$$

Summarising the above, we get the following lemma.

Lemma 4.3.2. *Writing* $x := c_1(\pi^*[H])$ *and* $y := c_1([E])$, *we have the following rules:*

1. $x^n = 1$,
2. $xy^{n-1} = (-1)^{n-2}$,
3. $y^n = (-1)^{n-2}(n-1)$,
4. $x^jy^{n-j} = 0$ *for* $2 \leq j \leq n-1$.

4.3.2.2 Computation of the slope $\mu(X,L)$

We apply Lemma 4.3.2 to the formula (4.1) for the slope $\mu(X,L)$. Recall first of all that we have $K_X = \pi^*K_{\mathbb{P}^n} + (n-2)[E]$ since we have blown up a complex submanifold of codimension $n-1$ (cf. §6, Chapter 4, [58]). Note that $L = \pi^*[H] - \varepsilon[E]$ and $K_X = \pi^*K_{\mathbb{P}^n} + (n-2)[E]$ implies $c_1(L) = x - \varepsilon y$ and $c_1(K_X) = -(n+1)x + (n-2)y$. We thus get

$$\int_X c_1(K_X)c_1(L)^{n-1} = -(n+1)(1-\varepsilon^{n-1}) + (n-1)(n-2)\varepsilon^{n-2}(1-\varepsilon)$$

by Lemma 4.3.2. Similarly we get $\int_X c_1(L)^n = 1 - n\varepsilon^{n-1} + (n-1)\varepsilon^n$. Hence

$$\mu(X,L) = \frac{n}{2}\frac{(n+1)(1-\varepsilon^{n-1}) - (n-1)(n-2)\varepsilon^{n-2}(1-\varepsilon)}{(n-1)\varepsilon^n - n\varepsilon^{n-1} + 1}.$$

For the later use, we write Den_1 for the denominator and Num_1 for the numerator of the fraction above, so that $\mu(X,L) = \frac{n}{2}\mathrm{Num}_1/\mathrm{Den}_1$.

4.3.2.3 Computation of the quotient slope $\mu_c(\mathscr{O}_E, L)$

We now compute the quotient slope $\mu_c(\mathscr{O}_E, L)$ with respect to the exceptional divisor E and for $c = \mathrm{Sesh}(E,X,L) = 1-\varepsilon$, by using the formula (4.2).

We write $\mu_{\mathrm{Sesh}(E)}(\mathscr{O}_E, L) = \frac{n}{2}\mathrm{Num}_2/\mathrm{Den}_2$ and compute the denominator Den_2 and the numerator Num_2 separately. We first compute the denominator by using Lemma 4.3.2.

$$\begin{aligned}
\mathrm{Den}_2 &= \sum_{j=1}^{n} \binom{n}{j} \frac{(\varepsilon-1)^j}{j+1}(x-\varepsilon y)^{n-j} y^j \\
&= \sum_{j=1}^{n} \binom{n}{j} \frac{(1-\varepsilon)^j}{j+1}\left(-(n-j)\varepsilon^{n-j-1} + (n-1)\varepsilon^{n-j}\right) \\
&= \varepsilon^{n-1}\left(\left(1+\frac{1-\varepsilon}{\varepsilon}\right)^n - 1\right) + \sum_{j=1}^{n}\binom{n}{j}\varepsilon^{n-1}\left(-\frac{n+1}{j+1}\left(\frac{1-\varepsilon}{\varepsilon}\right)^j + \frac{n-1}{j+1}\varepsilon\left(\frac{1-\varepsilon}{\varepsilon}\right)^j\right).
\end{aligned}$$

We now set $\chi := \frac{1-\varepsilon}{\varepsilon}$ and note the following identity

$$\sum_{j=1}^{n}\binom{n}{j}\frac{\chi^j}{j+1} = \sum_{j=1}^{n}\binom{n}{j}\frac{1}{\chi}\int_0^{\chi} T^j dT = \frac{1}{\chi}\left(\frac{(1+\chi)^{n+1}-1}{n+1} - \chi\right). \tag{4.6}$$

Observing $1+\chi = \varepsilon^{-1}$, we thus get the denominator as

$$\mathrm{Den}_2 = -\frac{1-\varepsilon^n}{1-\varepsilon} + n\varepsilon^{n-1} + \frac{n-1}{n+1}\frac{1-\varepsilon^{n+1}}{1-\varepsilon} - (n-1)\varepsilon^n.$$

We now compute the numerator. Since the first term $L^{n-1}.E$ is equal to $(x-$

$\varepsilon y)^{n-1}y = (n-1)\varepsilon^{n-2}(1-\varepsilon)$, we are left to compute the following second term

$$\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{(\varepsilon-1)^j}{j+1}L^{n-1-j}.E^j.(K_X+E)$$
$$=\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{(\varepsilon-1)^j}{j+1}(x-\varepsilon y)^{n-1-j}y^j(-(n+1)x+(n-2)y+y).$$

By applying Lemma 4.3.2, we can compute each summand as

$$(x-\varepsilon y)^{n-1-j}y^j(-(n+1)x+(n-2)y+y)$$
$$=(-1)^j(n-1)(n-j-1)\varepsilon^{n-j-2}-(-1)^j\varepsilon^{n-j-1}n(n-3).$$

We thus get

$$\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{(\varepsilon-1)^j}{j+1}L^{n-1-j}.E^j.(K_X+E)$$
$$=\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{\varepsilon^{n-2}}{j+1}\left((n-1)(n-j-1)\left(\frac{1-\varepsilon}{\varepsilon}\right)^j-\varepsilon\left(\frac{1-\varepsilon}{\varepsilon}\right)^j n(n-3)\right).$$

Setting $\chi=\frac{1-\varepsilon}{\varepsilon}$ as we did before, the above is equal to

$$\sum_{j=1}^{n-1}\binom{n-1}{j}\varepsilon^{n-2}\left((n-1)^2\frac{\chi^j}{j+1}-(n-1)j\frac{\chi^j}{j+1}-\varepsilon n(n-3)\frac{\chi^j}{j+1}\right)$$
$$=-(n-1)\varepsilon^{n-2}\sum_{j=1}^{n-1}\binom{n-1}{j}\chi^j+\varepsilon^{n-2}\sum_{j=1}^{n-1}\binom{n-1}{j}\left(n(n-1)\frac{\chi^j}{j+1}-\varepsilon n(n-3)\frac{\chi^j}{j+1}\right).$$

Now recalling the identity (4.6), we see that the above is equal to

$$-(n-1)\varepsilon^{n-2}((1+\chi)^{n-1}-1)+n(n-1)\varepsilon^{n-2}\frac{1}{\chi}\left(\frac{(1+\chi)^n-1}{n}-\chi\right)$$
$$-n(n-3)\varepsilon^{n-1}\frac{1}{\chi}\left(\frac{(1+\chi)^n-1}{n}-\chi\right)$$
$$=(n-1)\frac{1-\varepsilon^{n-1}}{1-\varepsilon}-(n-1)^2\varepsilon^{n-2}-(n-3)\frac{1-\varepsilon^n}{1-\varepsilon}+n(n-3)\varepsilon^{n-1}.$$

Thus we find the numerator to be

$$\mathrm{Num}_2=(n-1)\varepsilon^{n-2}(1-\varepsilon)-(n-1)\frac{1-\varepsilon^{n-1}}{1-\varepsilon}+(n-1)^2\varepsilon^{n-2}+(n-3)\frac{1-\varepsilon^n}{1-\varepsilon}-n(n-3)\varepsilon^{n-1}.$$

4.3.2.4 Proof of instability

We now compute $\mu_{\mathrm{Sesh}(E)}(\mathscr{O}_E, L) - \mu(X,L)$. Since $\mu_{\mathrm{Sesh}(E)}(\mathscr{O}_E, L) - \mu(X,L) < 0$ implies that (X,L) is slope unstable with respect to the divisor E (cf. Definition 4.2.2), it suffices to show that

$$\frac{2}{n}(\mu_{\mathrm{Sesh}(E)}(\mathscr{O}_E, L) - \mu(X,L)) = \frac{\mathrm{Num}_2}{\mathrm{Den}_2} - \frac{\mathrm{Num}_1}{\mathrm{Den}_1}$$

is strictly negative for all $0 < \varepsilon < 1$.

Since $\varepsilon^j > \varepsilon^{j+1}$ for any non-negative integer j if $0 < \varepsilon < 1$, we have the following inequalities:

$$\begin{aligned}
\mathrm{Den}_2 &= -\frac{1-\varepsilon^n}{1-\varepsilon} + n\varepsilon^{n-1} + \frac{n-1}{n+1}\frac{1-\varepsilon^{n+1}}{1-\varepsilon} - (n-1)\varepsilon^n \\
&= -\frac{2}{n+1}\sum_{j=0}^{n-1}\varepsilon^j + \frac{n(1-n)}{n+1}\varepsilon^n + n\varepsilon^{n-1} \\
&< -\frac{2n}{n+1}\varepsilon^{n-1} + \frac{n(1-n)}{n+1}\varepsilon^{n-1} + n\varepsilon^{n-1} = 0 \qquad (4.7)
\end{aligned}$$

and

$$\mathrm{Den}_1 = 1 - n\varepsilon^{n-1} + (n-1)\varepsilon^n = (1-\varepsilon)\left(-n\varepsilon^{n-1} + \sum_{j=0}^{n-1}\varepsilon^j\right) > 0,$$

for $0 < \varepsilon < 1$.

Thus, to show slope instability, we are reduced to proving $\mathrm{Num}_2\mathrm{Den}_1 - \mathrm{Num}_1\mathrm{Den}_2 > 0$, or equivalently

$$(1-\varepsilon)(\mathrm{Num}_2\mathrm{Den}_1 - \mathrm{Num}_1\mathrm{Den}_2) > 0$$

for $0 < \varepsilon < 1$.

We first re-write $(1-\varepsilon)\mathrm{Num}_2$ as

$$\begin{aligned}
(1-\varepsilon)\mathrm{Num}_2 = (n-1)\varepsilon^{n-2}(1-\varepsilon)^2 - (n-1)(1-(n-1)\varepsilon^{n-2} + (n-2)\varepsilon^{n-1}) \\
+ (n-3)(1 - n\varepsilon^{n-1} + (n-1)\varepsilon^n).
\end{aligned}$$

Let

$$F_m := 1 - m\varepsilon^{m-1} + (m-1)\varepsilon^m$$

be defined for an integer $m > 1$. We record the following lemma which we shall use later.

Lemma 4.3.3. *The following hold for* F_m*, where* $m > 1$ *is an integer:*

1. $F_m > \frac{m(m-1)}{2}(1-\varepsilon)^2\varepsilon^{m-2} > 0$ *for* $0 < \varepsilon < 1$,

2. $F_m - F_{m-1} = (m-1)\varepsilon^{m-2}(1-\varepsilon)^2 > 0$ *for* $0 < \varepsilon < 1$,

3. $F_n = \mathrm{Den}_1$.

Proof. Observe first of all

$$\begin{aligned} F_m = 1 - m\varepsilon^{m-1} + (m-1)\varepsilon^m &= 1 - \varepsilon^{m-1} - (m-1)\varepsilon^{m-1}(1-\varepsilon) \\ &= (1-\varepsilon)\left(\sum_{j=0}^{m-2}(\varepsilon^j - \varepsilon^{m-1})\right) \\ &= (1-\varepsilon)^2\left(\sum_{j=0}^{m-2}\left(\sum_{k=0}^{m-j-2}\varepsilon^{j+k}\right)\right). \end{aligned}$$

Since $0 < \varepsilon < 1$, we have $\varepsilon^{j+1} < \varepsilon^j$ for any positive integer j. Thus

$$F_m > (1-\varepsilon)^2\left(\sum_{j=0}^{m-2}(m-j-1)\varepsilon^{m-2}\right) = \frac{m(m-1)}{2}(1-\varepsilon)^2\varepsilon^{m-2} > 0,$$

proving the first item of the lemma. The second item follows from a straightforward computation. The third is a tautology. □

Using Lemma 4.3.3, we can write $(1-\varepsilon)\mathrm{Num}_2 = (n-2)F_n - nF_{n-1}$, and hence

$$(1-\varepsilon)\mathrm{Num}_2\mathrm{Den}_1 = (n-2)F_n^2 - nF_{n-1}F_n.$$

Similarly, we compute $\mathrm{Num}_1 = (n-2)F_{n-1} + 3(1-\varepsilon^{n-1})$ and

$$(1-\varepsilon)\mathrm{Den}_2 = -F_n + \frac{n-1}{n+1}F_{n+1}. \tag{4.8}$$

Summarising these calculations, we finally get

$$\begin{aligned}&(1-\varepsilon)\left(\mathrm{Num}_2\mathrm{Den}_1-\mathrm{Num}_1\mathrm{Den}_2\right)\\&=(n-2)F_n^2-nF_{n-1}F_n+[(n-2)F_{n-1}+3(1-\varepsilon^{n-1})]\left(F_n-\frac{n-1}{n+1}F_{n+1}\right),\end{aligned}$$

and our aim now is to show that the right hand side of the above equation is strictly positive for all $0<\varepsilon<1$.

By using Lemma 4.3.3, we first re-write

$$\begin{aligned}(n-2)F_{n-1}+3(1-\varepsilon^{n-1})&=(n-2)F_{n-1}+3(F_n+(n-1)\varepsilon^{n-1}(1-\varepsilon))\\&=(n+1)F_n-(n-1)(n-2)\varepsilon^{n-2}(1-\varepsilon)^2+3(n-1)\varepsilon^{n-1}(1-\varepsilon)\end{aligned}$$

so as to get

$$\begin{aligned}&(n-2)F_n^2-nF_{n-1}F_n+[(n-2)F_{n-1}+3(1-\varepsilon^{n-1})]\left(F_n-\frac{n-1}{n+1}F_{n+1}\right)\\&=F_n\left(-nF_{n-1}+(2n-1)F_n-(n-1)F_{n+1}\right)\\&\qquad+[-(n-1)(n-2)\varepsilon^{n-2}(1-\varepsilon)^2+3(n-1)\varepsilon^{n-1}(1-\varepsilon)]\left(F_n-\frac{n-1}{n+1}F_{n+1}\right).\end{aligned}$$

Now compute

$$\begin{aligned}-nF_{n-1}+(2n-1)F_n-(n-1)F_{n+1}&=-nF_{n-1}+(n-2)F_n+(n+1)F_n-(n-1)F_{n+1}\\&=n(n-1)\varepsilon^{n-2}(1-\varepsilon)^3,\end{aligned}$$

and get

$$\begin{aligned}&(1-\varepsilon)\left(\mathrm{Num}_2\mathrm{Den}_1-\mathrm{Num}_1\mathrm{Den}_2\right)\\&=(n-1)\varepsilon^{n-2}(1-\varepsilon)\left[n(1-\varepsilon)^2F_n+((n+1)\varepsilon-(n-2))\left(F_n-\frac{n-1}{n+1}F_{n+1}\right)\right].\end{aligned}$$

Since $n(1-\varepsilon)^2F_n>0$ by Lemma 4.3.3 and

$$F_n-\frac{n-1}{n+1}F_{n+1}=-(1-\varepsilon)\mathrm{Den}_2>0$$

by recalling (4.8) and (4.7), we see that the above quantity is strictly positive if $(n+1)\varepsilon-(n-2)\geq 0$, i.e. $\frac{n-2}{n+1}\leq\varepsilon<1$. This means that we have proved slope instability for $\frac{n-2}{n+1}\leq\varepsilon<1$.

Thus assume $0<\varepsilon<\frac{n-2}{n+1}$ from now on. Now, again using Lemma 4.3.3, we have

$$\begin{aligned}
&(1-\varepsilon)\left(\mathrm{Num}_2\mathrm{Den}_1-\mathrm{Num}_1\mathrm{Den}_2\right)\\
&=(n-1)\varepsilon^{n-2}(1-\varepsilon)\\
&\quad\times\left[\left(n(1-\varepsilon)^2+2\left(\varepsilon-\frac{n-2}{n+1}\right)\right)F_n+[(n+1)\varepsilon-(n-2)]\left(-\frac{n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2\right)\right].
\end{aligned}$$

Noting $n(1-\varepsilon)^2+2\left(\varepsilon-\frac{n-2}{n+1}\right)=n\left(\varepsilon-\frac{n-1}{n}\right)^2+\frac{5n-1}{n(n+1)}$ and also

$$\begin{aligned}
&[(n+1)\varepsilon-(n-2)]\left(-\frac{n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2\right)\\
&=n(n-1)\varepsilon^{n-1}(1-\varepsilon)^3-\frac{3n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2,
\end{aligned}$$

we are thus reduced to proving that

$$\left[n\left(\varepsilon-\frac{n-1}{n}\right)^2+\frac{5n-1}{n(n+1)}\right]F_n-\frac{3n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2+n(n-1)\varepsilon^{n-1}(1-\varepsilon)^3$$

is strictly positive for $0<\varepsilon<\frac{n-2}{n+1}$.

Observe that $\frac{n-2}{n+1}<\frac{n-1}{n}$, which holds if $n\geq 1$, implies that $\left(\varepsilon-\frac{n-1}{n}\right)^2$ is monotonically decreasing on $0<\varepsilon<\frac{n-2}{n+1}$. Thus

$$n\left(\varepsilon-\frac{n-1}{n}\right)^2+\frac{5n-1}{n(n+1)}>n\left(\frac{n-2}{n+1}-\frac{n-1}{n}\right)^2+\frac{5n-1}{n(n+1)}=\frac{9n}{(n+1)^2}$$

for $0<\varepsilon<\frac{n-2}{n+1}$. Hence, recalling Lemma 4.3.3, we finally have

$$\begin{aligned}
&\left[n\left(\varepsilon-\frac{n-1}{n}\right)^2+\frac{5n-1}{n(n+1)}\right]F_n-\frac{3n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2\\
&>\frac{9n}{(n+1)^2}(1-\varepsilon)^2\varepsilon^{n-2}\frac{n(n-1)}{2}-\frac{3n(n-1)}{n+1}\varepsilon^{n-1}(1-\varepsilon)^2\\
&>(1-\varepsilon)^2\varepsilon^{n-1}\frac{n(n-1)}{n+1}\left(\frac{9n}{2(n+1)}-3\right)>0
\end{aligned}$$

for $0 < \varepsilon < \frac{n-2}{n+1}$, since $n \geq 3$. We have thus proved $(1-\varepsilon)(\mathrm{Num}_2\mathrm{Den}_1 - \mathrm{Num}_1\mathrm{Den}_2) > 0$ both for $0 < \varepsilon < \frac{n-2}{n+1}$ and $\frac{n-2}{n+1} \leq \varepsilon < 1$, finally establishing the slope instability for all $0 < \varepsilon < 1$.

4.4 Extremal metrics on $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$

4.4.1 Statement of the result

Having established the nonexistence of cscK metrics in Proposition 4.3.1, we now discuss the extremal metrics on $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ ($n \geq 3$), with the blowdown map $\pi : \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n \to \mathbb{P}^n$, as mentioned in the second part of Theorem 4.1.2. We write $(x_1, \dots, x_n)$ for the action coordinates on the moment polytope corresponding to $(\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n, \pi^*\mathscr{O}_{\mathbb{P}^n}(1) - \varepsilon[E])$, where the exceptional divisor is defined by $\{\sum_{i=1}^{n-1} x_i = \varepsilon\}$, and we write $r := \sum_{i=1}^{n} x_i$ and $\rho := \sum_{i=1}^{n-1} x_i$; see §4.4.2.1 for more details. We re-state the second part of Theorem 4.1.2 as follows, with an explicit description of the extremal metrics in the action-angle coordinates.

Proposition 4.4.1. *There exists* $0 < \varepsilon_0 < 1$ *such that* $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ *admits an extremal Kähler metric in the Kähler class* $\pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1)) - \varepsilon c_1([E])$ *for any* $\varepsilon \in (0, \varepsilon_0)$. *Moreover, this metric admits an explicit description in terms of the symplectic potential* $s(x)$ *in the action-angle coordinates as follows:*

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right) \tag{4.9}$$

where $h(\rho)$ *is given as an indefinite integral by*

$$h(\rho) = \int^{\rho} d\rho \int^{\rho} \frac{-1 - \frac{2n+\delta}{n(n-1)} + \rho + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)} + \alpha\rho^{-n} + \beta\rho^{-n+1}}{(1-\rho)\left(1 - \rho\left(1 - \frac{2n+\delta}{n(n-1)} + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)} + \alpha\rho^{-n} + \beta\rho^{-n+1}\right)\right)} d\rho,$$

with

$$\alpha = -1 - \frac{\delta}{n(n+1)} - \frac{\gamma}{(n+1)(n+2)}, \tag{4.10}$$

$$\beta = \frac{n+1}{n-1} + \frac{\delta}{n(n-1)} + \frac{\gamma}{n(n+1)}, \tag{4.11}$$

$$\gamma = \frac{n(n+1)(n+2)\left(\left(\frac{\varepsilon^{n+1}-1}{n(n+1)} + \frac{\varepsilon-\varepsilon^n}{n(n-1)}\right)\delta - 1 + \frac{n+1}{n-1}\varepsilon - \varepsilon^{n-1} + \frac{n-3}{n-1}\varepsilon^n\right)}{-n\varepsilon^{n+2} + (n+2)\varepsilon^{n+1} + n - (n+2)\varepsilon}, \tag{4.12}$$

$$\begin{aligned}\delta = \Bigg(&\varepsilon^{n-2}(1-\varepsilon) - \frac{(-n\varepsilon^{n+1} + (n+1)\varepsilon^n - 1)(n+2)\left(-1 + \frac{n+1}{n-1}\varepsilon - \varepsilon^{n-1} + \frac{n-3}{n-1}\varepsilon^n\right)}{-n\varepsilon^{n+2} + (n+2)\varepsilon^{n+1} + n - (n+2)\varepsilon} \\ &+ \frac{n(n-3)}{n-1}\varepsilon^{n-1} - (n-1)\varepsilon^{n-2} + \frac{n+1}{n-1}\Bigg) \\ &\times \Bigg(\frac{(-n\varepsilon^{n+1} + (n+1)\varepsilon^n - 1)(n+2)}{-n\varepsilon^{n+2} + (n+2)\varepsilon^{n+1} + n - (n+2)\varepsilon}\left(\frac{\varepsilon^{n+1}-1}{n(n+1)} + \frac{\varepsilon - \varepsilon^n}{n(n-1)}\right) \\ &+ \frac{-(n-1)\varepsilon^n + n\varepsilon^{n-1} - 1}{n(n-1)}\Bigg)^{-1}.\end{aligned} \tag{4.13}$$

Remark 4.4.2. Note that the symplectic potential is well-defined up to affine functions, and hence the integration constants in $h(\rho)$ are not significant.

4.4.2 Proof of Proposition 4.4.1

4.4.2.1 Overview of the proof

The basic strategy of the proof, as given in §4.4.2.2 and §4.4.2.3, is exactly the same as in §5 of [3] or §4.2 of [99] for the point blow-up case; the crux of what is presented in the following is to show that the same strategy does indeed work for $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$, with an extra hypothesis $\varepsilon \ll 1$.

We recall that the moment polytope $\mathscr{P}(\mathbb{P}^n)$ for $\mathbb{P}^n$, with the Fubini–Study symplectic form, is the region in $\mathbb{R}^n$ defined by the set of affine inequalities $\mathscr{P}(\mathbb{P}^n) := \{x_1 \geq 0, \dots, x_n \geq 0, \sum_{i=1}^n x_i \leq 1\}$ (cf. Figure 4.1), where $(x_1, \dots, x_n) \in \mathbb{R}^n$ are the action coordinates as defined in §4.2.2. The moment polytope $\mathscr{P}_\varepsilon(X)$ for the blowup $X = \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ is obtained by cutting one edge by ε amount: $\mathscr{P}_\varepsilon(X) := \{x_1 \geq 0, \dots, x_n \geq 0, \sum_{i=1}^n x_i \leq 1, \sum_{i=1}^{n-1} x_i \geq \varepsilon\}$ (cf. Figure 4.2), where the $\mathbb{P}^1$ that is blown up corresponds to the line defined by $\{x_1 = \cdots = x_{n-1} = 0\}$. Note that the symplectic form ω on X is in the cohomology class $\pi^* c_1(\mathscr{O}_{\mathbb{P}^n}(1)) - \varepsilon c_1([E])$ (cf. Theorem 6.3, [60]). We write

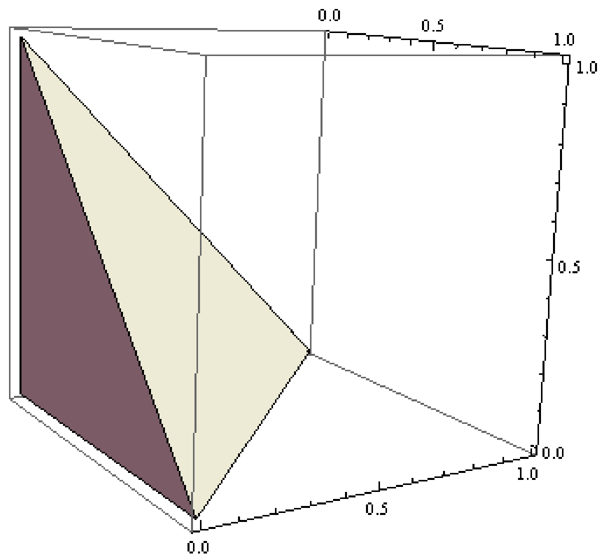

Figure 4.1: The moment polytope $\mathscr{P}(\mathbb{P}^3)$ for $\mathbb{P}^3$.

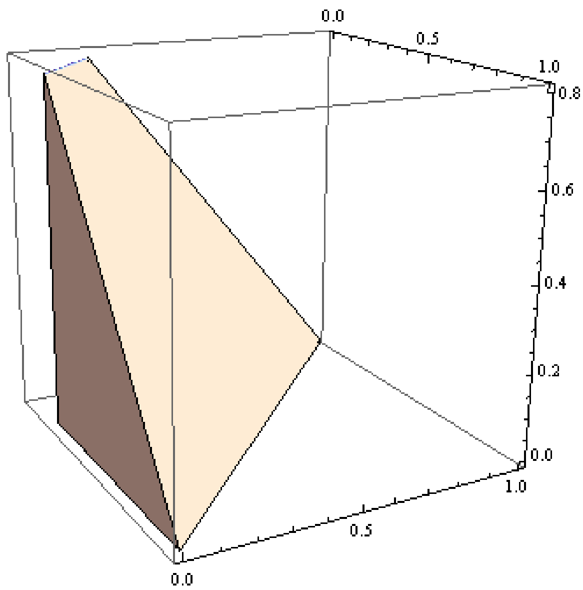

Figure 4.2: The moment polytope $\mathscr{P}_\varepsilon(X)$ for $X = \mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^3$, with $\varepsilon = 0.2$.

$r := \sum_{i=1}^n x_i$ and $\rho := \sum_{i=1}^{n-1} x_i$ for notational convenience. Recall also that we assume $n \geq 3$ for the blow-up to be non-trivial.

Our strategy is to seek a symplectic potential s of the form

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^n x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right), \tag{4.14}$$

where $h(\rho)$ stands for some function of ρ, so that the Riemannian metric g_s given by the symplectic form ω and the complex structure defined by s (cf. Theorem 4.2.9) satisfies the equation

$$S(g_s) = -\gamma\rho - \delta \tag{4.15}$$

for some constants[3] γ and δ. Such a metric g_s would be an extremal metric by Theorem 4.2.10. Our first result is that the equation (4.15) reduces to a second-order linear ODE as given in (4.16), similarly to the case of the point blowup (cf. [3, 99]). The equation (4.16) can be easily solved, and the solution is given in (4.17) with two additional free constants α and β. This is the content of §4.4.2.2.

However, it is not a priori obvious that $s(x)$ as defined in (4.14), with h obtained from (4.17), gives a well-defined symplectic potential. The main technical result (Proposition 4.4.4) that we establish in §4.4.2.3 is that, once we choose α, β, γ, δ as in (4.10), (4.11), (4.12), (4.13) and ε to be sufficiently small, h obtained from (4.17) does satisfy all the regularity hypotheses required in Theorem 4.2.9, so that $s(x)$ is a well-defined symplectic potential. This is the content of §4.4.2.3.

4.4.2.2 Reducing the equation (4.15) to a second order linear ODE

We first compute the Hessian

$$s_{ij} := \frac{\partial^2 s}{\partial x_i \partial x_j}(x)$$

of the symplectic potential $s(x) = \frac{1}{2}\left(\sum_{i=1}^n x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right)$ as follows:

$$s_{ij} = \begin{cases} \frac{1}{2}\left(\frac{\delta_{ij}}{x_i} + \frac{1}{1-r} + h''\right) & \text{if } i, j \neq n, \\ \frac{1}{2}\left(\frac{\delta_{ij}}{x_i} + \frac{1}{1-r}\right) & \text{if } i = n \text{ or } j = n \text{ or both.} \end{cases}$$

[3]The factor of -1 in (4.15) is an artefact to be consistent with the equation (4.16).

By direct computation, we find the inverse matrix s^{ij} of s_{ij} to be

$$s^{ij} = \begin{cases} 2\left(x_i\delta_{ij} - \dfrac{x_ix_j(1+(1-\rho)h'')}{1+\rho(1-\rho)h''}\right) & \text{if } i,j \neq n, \\ -\dfrac{2x_ix_n}{1+\rho(1-\rho)h''} & \text{if } i \neq n \text{ and } j = n, \\ \dfrac{2x_n}{1-\rho}\left(1-\rho-x_n+\dfrac{x_n\rho}{1+\rho(1-\rho)h''}\right) & \text{if } i=j=n. \end{cases}$$

Let A be a function of ρ defined by

$$A(\rho) := \frac{1+(1-\rho)h''}{1+\rho(1-\rho)h''},$$

so that we can re-write the above as

$$s^{ij} = \begin{cases} 2(x_i\delta_{ij} - x_ix_jA) & \text{if } i,j \neq n, \\ -\dfrac{2x_ix_n(1-\rho A)}{1-\rho} & \text{if } i \neq n \text{ and } j = n, \\ \dfrac{2x_n}{1-\rho}\left(1-\rho-x_n+\dfrac{x_n\rho(1-\rho A)}{1-\rho}\right) & \text{if } i=j=n. \end{cases}$$

Thus, by Abreu's equation (4.4) (cf. Theorem 4.2.10), we have

$$\begin{aligned} S(g_s) = &\sum_{i=1}^{n-1}(2A+4x_iA'+x_i^2A'') + 2\sum_{1\leq i<j\leq n-1}(A+x_iA'+x_jA'+x_ix_jA'') \\ &+2\sum_{i=1}^{n-1}\left(\frac{1-\rho A}{1-\rho}+x_i\left(\frac{1-\rho A}{1-\rho}\right)'\right)+\left(\frac{2}{1-\rho}-\frac{2\rho(1-\rho A)}{(1-\rho)^2}\right). \end{aligned}$$

Hence, re-arranging the terms, we find

$$S(g_s) = \rho^2A''+2\left(n-\frac{\rho}{1-\rho}\right)\rho A'+\left(n(n-1)-\frac{2n\rho}{1-\rho}\right)A+\frac{2n}{1-\rho}.$$

Thus the equation (4.15) to be solved can now be written as

$$\rho^2A''+2\left(n-\frac{\rho}{1-\rho}\right)\rho A'+\left(n(n-1)-\frac{2n\rho}{1-\rho}\right)A+\frac{2n}{1-\rho}+\gamma\rho+\delta=0 \tag{4.16}$$

for some constants γ and δ. The general solution to this equation is given by

$$A = \frac{1}{1-\rho}\left(-\frac{2n+\delta}{n(n-1)} + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)} + \alpha\rho^{-n} + \beta\rho^{-n+1}\right),$$

for some constants α and β. Recalling $A = \frac{1+(1-\rho)h''}{1+\rho(1-\rho)h''}$, we can now write h'' as

$$\begin{aligned} h'' &= \frac{A-1}{(1-\rho)(1-\rho A)} \\ &= \frac{-1-\frac{2n+\delta}{n(n-1)} + \rho + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)} + \alpha\rho^{-n} + \beta\rho^{-n+1}}{(1-\rho)\left(1-\rho\left(1-\frac{2n+\delta}{n(n-1)} + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)} + \alpha\rho^{-n} + \beta\rho^{-n+1}\right)\right)}. \end{aligned} \tag{4.17}$$

We have thus solved the equation (4.15), with 4 undetermined parameters α, β, γ, δ. We now have to prove that the function h as obtained above satisfies all the regularity conditions as stated in Theorem 4.2.9, and we claim that this holds once α, β, γ, δ are chosen as in (4.10), (4.11), (4.12), (4.13).

Before discussing the claimed regularity of h'', which we do in §4.4.2.3, we define two polynomials $P(\rho)$ and $Q(\rho)$, with α, β, γ, δ as parameters, as follows. They play an important role in what follows.

Definition 4.4.3. We define a polynomial $P(\rho)$ by

$$P(\rho) := -\frac{2n+\delta}{n(n-1)} + \frac{(\delta-\gamma)\rho}{n(n+1)} + \frac{\gamma\rho^2}{(n+1)(n+2)}$$

and $Q(\rho)$ by

$$\begin{aligned} Q(\rho) &:= \rho^{n-1} - \rho^n - \rho^n P(\rho) - \alpha - \beta\rho \\ &= -\frac{\gamma}{(n+1)(n+2)}\rho^{n+2} - \frac{\delta-\gamma}{n(n+1)}\rho^{n+1} - \left(1-\frac{2n+\delta}{n(n-1)}\right)\rho^n + \rho^{n-1} - \alpha - \beta\rho, \end{aligned}$$

so that we can write

$$h''(\rho) = \frac{\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho}{(1-\rho)\rho Q(\rho)}. \tag{4.18}$$

4.4.2.3 Regularity of h

The main technical result is the following.

Proposition 4.4.4. *For h as given by (4.17), there exists a function $R(\rho)$ which is smooth on the whole of the polytope $\mathscr{P}_\varepsilon(X)$ such that*

$$h(\rho) = (\rho - \varepsilon)\log(\rho - \varepsilon) + R(\rho)$$

and that the Hessian of the symplectic potential

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right)$$

is positive definite over the interior $\mathscr{P}^\circ_\varepsilon(X)$ of the polytope $\mathscr{P}_\varepsilon(X)$, with the determinant of the form required in (4.3), if we choose α, β, γ, δ as in (4.10), (4.11), (4.12), (4.13) and $\varepsilon > 0$ to be sufficiently small.

Proof. Recall from (4.18) that h'' is given by

$$h''(\rho) = \frac{\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho}{(1-\rho)\rho Q(\rho)}.$$

We first need to prove that $\rho = 1$ is a removable singularity. In Lemma 4.4.5, we shall prove that this is indeed the case, once we choose α and β as in (4.10), (4.11).

We then consider the asymptotic behaviour of h'' as $\rho \to \varepsilon$. We now write

$$h''(\rho) = \frac{1}{\rho}\frac{1}{\rho - 1} + \frac{\rho^{n-2}(1-\rho)}{Q(\rho)}$$

and consider the Taylor expansion

$$Q(\rho) = Q_0 + Q_1(\rho - \varepsilon) + \cdots$$

of $Q(\rho)$ around $\rho = \varepsilon$, with some $Q_0, Q_1 \in \mathbb{R}$. Writing now

$$h''(\rho) = \frac{1}{\rho}\frac{1}{\rho - 1} + \frac{\rho^{n-2}(1-\rho)}{Q_0 + Q_1(\rho - \varepsilon) + \cdots}$$

around $\rho = \varepsilon$, our strategy is to show that, for the choice of γ and δ as in (4.12), (4.13), we have a Laurent expansion

$$h''(\rho) = \frac{1}{\rho - \varepsilon} + \hat{Q}_0 + \hat{Q}_1(\rho - \varepsilon) + \cdots \tag{4.19}$$

in $\rho - \varepsilon$, with some $\hat{Q}_0, \hat{Q}_1 \in \mathbb{R}$. This will be proved in Lemma 4.4.6. We shall also prove in Lemma 4.4.9 that $Q(\rho) > 0$ on $(\varepsilon, 1)$, for these choices of α, β, γ, δ and sufficiently small $\varepsilon > 0$. Since $\rho = 1$ is a removable singularity, this means that h'' is smooth on the whole polytope except for a pole of order 1 and residue 1 at $\rho = \varepsilon$.

We now consider a function

$$\tilde{R}(\rho) := h''(\rho) - \frac{1}{\rho - \varepsilon}.$$

This is smooth on the whole of the polytope $\mathscr{P}_\varepsilon(X)$ by the above properties of h'', and hence integrating both sides twice, we get a function $R(\rho)$ that is smooth on the whole polytope which satisfies

$$R(\rho) = h(\rho) - (\rho - \varepsilon)\log(\rho - \varepsilon),$$

as we claimed. Finally, we shall prove in Lemma 4.4.10 that the Hessian of the symplectic potential

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right)$$

is indeed positive definite over the interior $\mathscr{P}_\varepsilon^\circ(X)$ of the polytope $\mathscr{P}_\varepsilon(X)$ and has determinant of the form required in (4.3), for the above choices of α, β, γ, δ and sufficiently small $\varepsilon > 0$.

Therefore, granted Lemmas 4.4.5, 4.4.6, 4.4.9, and 4.4.10 to be proved below, we complete the proof of the proposition.

□

Lemma 4.4.5. *For the choice of α and β as in (4.10), (4.11), the following hold:*

1. *the numerator $\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho$ of h'' has a zero of order at least 3 at $\rho = 1$,*

2. *$Q(1) = Q'(1) = 0$, $Q''(1) = 2$; in particular, Q has a zero of order exactly 2 at $\rho = 1$.*

In particular, $\rho = 1$ is a removable singularity of h'' if we choose α and β as in (4.10), (4.11).

Proof. The numerator $\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho$ of h'' has a zero at $\rho = 1$ if and only if $P(1) + \alpha + \beta = 0$. We thus choose $\beta = -\alpha - P(1)$. The zero is of order at least two if and only if $1 + nP(1) + P'(1) + \beta = 0$, in addition to $\beta = -\alpha - P(1)$. We thus choose α by the equation

$$1 + nP(1) + P'(1) + (-\alpha - P(1)) = 0 \tag{4.20}$$

and β by the equation

$$\beta = -\alpha - P(1) = -1 - (n-1)P(1) - P'(1) - P(1) \tag{4.21}$$

by noting that $P(1)$ and $P'(1)$ depend only on γ and δ.

Finally, we observe

$$\begin{aligned} \left.\frac{d^2}{d\rho^2}\right|_{\rho=1} (\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho) &= 2n + n(n-1)P(1) + 2nP'(1) + P''(1) \\ &= 0 \end{aligned} \tag{4.22}$$

identically for any choice of γ and δ. Thus the numerator $\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho$ of h'' vanishes at $\rho = 1$ with order at least 3, if α, β are chosen as in the equations (4.20), (4.21). We now unravel the equations (4.20) and (4.21), to find that they are exactly as given in (4.10) and (4.11).

We have thus established the first claim in the lemma: the numerator $\rho^{n+1} - \rho^n + \rho^n P(\rho) + \alpha + \beta\rho$ of h'' has a zero of order at least 3 at $\rho = 1$ if α, β are chosen as (4.10) and (4.11).

The second claim of the lemma is an easy consequence of the equations (4.20), (4.21), (4.22): we simply compute $Q(1) = 1 - 1 - P(1) - \alpha - \beta = 0$ and $Q'(1) = (n-1) - n - nP(1) - P'(1) - \beta = 0$, by virtue of (4.20) and (4.21). We finally have $Q''(1) = 2$ by (4.22).

□

Lemma 4.4.6. *We have the expansion (4.19), namely we have the Laurent expansion*

$$h''(\rho) = \frac{1}{\rho - \varepsilon} + \hat{Q}_0 + \hat{Q}_1(\rho - \varepsilon) + \cdots$$

in $\rho - \varepsilon$, *if we choose* α, β, γ, δ *as in (4.10), (4.11), (4.12), (4.13), and if* ε *is sufficiently small.*

Proof. We first consider the Taylor expansion

$$Q(\rho) = Q_0 + Q_1(\rho - \varepsilon) + \cdots \tag{4.23}$$

of $Q(\rho)$ around $\rho = \varepsilon$, with $Q_0, Q_1 \in \mathbb{R}$. When we have α and β as defined in (4.10) and (4.11), we find the 0th order term Q_0, which is equal to $Q(\varepsilon)$, to be

$$\begin{aligned} Q_0 &= -\frac{\gamma}{(n+1)(n+2)}\varepsilon^{n+2} - \frac{\delta - \gamma}{n(n+1)}\varepsilon^{n+1} - \left(1 - \frac{2n+\delta}{n(n-1)}\right)\varepsilon^n + \varepsilon^{n-1} - \alpha - \beta\varepsilon \\ &= \left(\frac{-n\varepsilon^{n+2} + (n+2)\varepsilon^{n+1} + n - (n+2)\varepsilon}{n(n+1)(n+2)}\right)\gamma \\ &\quad + \left(\frac{1-\varepsilon^{n+1}}{n(n+1)} + \frac{\varepsilon^n - \varepsilon}{n(n-1)}\right)\delta + 1 - \frac{n+1}{n-1}\varepsilon + \varepsilon^{n-1} - \frac{n-3}{n-1}\varepsilon^n. \end{aligned}$$

We choose γ as in (4.12), so that $Q_0 = 0$; note that $-n\varepsilon^{n+2} + (n+2)\varepsilon^{n+1} + n - (n+2)\varepsilon \neq 0$ if ε is chosen to be sufficiently small. This means that we can write

$$\begin{aligned} h''(\rho) &= \frac{1}{\rho}\frac{1}{\rho - 1} + \frac{\rho^{n-2}(1-\rho)}{Q(\rho)} \\ &= \frac{\rho^{n-2}(1-\rho)}{Q_1}\frac{1}{\rho - \varepsilon} + \text{power series in } \rho - \varepsilon, \end{aligned}$$

near $\rho = \varepsilon$. In order to prove the stated claim, we need to show that the residue at the pole $\rho = \varepsilon$ of h'' is 1. We prove this by showing $Q_1 = \varepsilon^{n-2}(1-\varepsilon)$ for an appropriate choice of δ, with α, β, and γ as determined in the above.

We thus consider the coefficient Q_1 in the expansion (4.23), which is equal to $\frac{d}{d\rho}\big|_{\rho=\varepsilon} Q(\rho)$, i.e.

$$Q_1 = -\frac{\gamma}{n+1}\varepsilon^{n+1} - \frac{\delta - \gamma}{n}\varepsilon^n - \left(n - \frac{2n+\delta}{n-1}\right)\varepsilon^{n-1} + (n-1)\varepsilon^{n-2} - \beta.$$

For the choice of β and γ as in (4.11) and (4.12), we can re-write this as

$$
\begin{aligned}
Q_1 = & \\
& \left[\frac{(-n\varepsilon^{n+1}+(n+1)\varepsilon^n-1)(n+2)}{-n\varepsilon^{n+2}+(n+2)\varepsilon^{n+1}+n-(n+2)\varepsilon} \left(\frac{\varepsilon^{n+1}-1}{n(n+1)} + \frac{\varepsilon-\varepsilon^n}{n(n-1)} \right) \right. \\
& \qquad \left. + \frac{-(n-1)\varepsilon^n+n\varepsilon^{n-1}-1}{n(n-1)} \right] \delta \\
& + \frac{(-n\varepsilon^{n+1}+(n+1)\varepsilon^n-1)(n+2)}{-n\varepsilon^{n+2}+(n+2)\varepsilon^{n+1}+n-(n+2)\varepsilon} \left(-1 + \frac{n+1}{n-1}\varepsilon - \varepsilon^{n-1} + \frac{n-3}{n-1}\varepsilon^n \right) \\
& \qquad - \frac{n(n-3)}{n-1}\varepsilon^{n-1} + (n-1)\varepsilon^{n-2} - \frac{n+1}{n-1}.
\end{aligned}
\tag{4.24}
$$

The equation $Q_1 = \varepsilon^{n-2}(1-\varepsilon)$ can be solved for δ if and only if the coefficient of δ in the equation (4.24) is not zero, i.e.

$$
\frac{(-n\varepsilon^{n+1}+(n+1)\varepsilon^n-1)(n+2)}{-n\varepsilon^{n+2}+(n+2)\varepsilon^{n+1}+n-(n+2)\varepsilon} \left(\frac{\varepsilon^{n+1}-1}{n(n+1)} + \frac{\varepsilon-\varepsilon^n}{n(n-1)} \right) + \frac{-(n-1)\varepsilon^n+n\varepsilon^{n-1}-1}{n(n-1)} \neq 0.
$$

Note that the left hand side is equal to $\frac{-2}{n^2(n-1)(n+1)} \neq 0$ when $\varepsilon = 0$, and hence this is non-zero for all sufficiently small $\varepsilon > 0$ by continuity. Hence the equation $Q_1 = \varepsilon^{n-2}(1-\varepsilon)$ can be solved for δ, with the solution as given in (4.13), if $\varepsilon > 0$ is sufficiently small. We thus obtain the claimed expansion

$$
h''(\rho) = \frac{1}{\rho-\varepsilon} + \hat{Q}_0 + \hat{Q}_1(\rho-\varepsilon) + \cdots
$$

near $\rho = \varepsilon$, if we choose α, β, γ, δ as in (4.10), (4.11), (4.12), (4.13) and ε to be sufficiently small.

□

Note that $Q_0 = 0$ (resp. $Q_1 = \varepsilon^{n-2}(1-\varepsilon)$) proved in the above is equivalent to saying $Q(\varepsilon) = 0$ (resp. $Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon)$). Together with what was proved in Lemma 4.4.5, we summarise below the properties of the polynomial $Q(\rho)$ that we have established so far.

Lemma 4.4.7. *For the choice of α, β, γ, δ as in (4.10), (4.11), (4.12), (4.13) and* *sufficiently small ε, the polynomial $Q(\rho)$ satisfies the following properties:*

1. $Q(1) = Q'(1) = 0$, $Q''(1) = 2$,

2. $Q(\varepsilon) = 0$, $Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon)$.

We also need the following estimates of α, β, γ, δ in the later argument.

Lemma 4.4.8. *We can estimate $\alpha = O(\varepsilon^2)$, $\beta = O(\varepsilon^2)$, $\gamma = O(\varepsilon^2)$, $\delta = -n(n+1) + O(\varepsilon^2)$, when ε is sufficiently small.*

Proof. The proof is just a straightforward computation; we compute δ as

$$\delta = \frac{\frac{2}{n(n-1)} + \frac{2(n+2)}{n^2(n-1)}\varepsilon + O(\varepsilon^2)}{\frac{-2}{n^2(n-1)(n+1)} - \frac{2(n+2)}{n^3(n-1)(n+1)}\varepsilon + O(\varepsilon^2)} = -n(n+1) + O(\varepsilon^2),$$

and similarly for γ. The claim for α and β follows easily from the definitions (4.10) and (4.11).

□

With these preparations, we now prove that $Q(\rho)$ is non-zero for all $\rho \in (\varepsilon, 1)$.

Lemma 4.4.9. $Q(\rho) > 0$ *for all $\rho \in (\varepsilon, 1)$, if α, β, γ, δ are chosen as in (4.10), (4.11), (4.12), (4.13), and ε is sufficiently small.*

Proof. Note first of all that the second derivative of Q can be computed as

$$Q''(\rho) = \rho^{n-3}\left[-\gamma\rho^3 - (\delta - \gamma)\rho^2 - (n(n-1) - (2n+\delta))\rho + (n-1)(n-2)\right].$$

Re-write the terms in the bracket $[\cdots]$ as

$$\begin{aligned}&-\gamma\rho^3 - (\delta-\gamma)\rho^2 - (n(n-1) - (2n+\delta))\rho + (n-1)(n-2)\\&= \tilde{Q}(\rho) + \varepsilon^2 \tilde{Q}_{\mathrm{rem}}(\rho),\end{aligned}$$

where we defined

$$\tilde{Q}(\rho) := n(n+1)\rho^2 - 2n(n-1)\rho + (n-1)(n-2)$$

and

$$\tilde{Q}_{\mathrm{rem}}(\rho) := \frac{1}{\varepsilon^2}\left(-\gamma\rho^3 - (\delta_0 - \gamma)\rho^2 - \delta_0\rho\right)$$

with $\delta_0 := \delta + n(n+1)$. Recalling $\gamma = O(\varepsilon^2)$ and $\delta_0 = O(\varepsilon^2)$ (cf. Lemma 4.4.8), we see that there exists a constant $\tilde{C}(\varepsilon_1) > 0$, which depends only on (sufficiently small) ε_1 and hence can be chosen uniformly for all ε satisfying $0 < \varepsilon < \varepsilon_1$, such that

$$|\tilde{Q}_{\mathrm{rem}}(\rho)| < \tilde{C}(\varepsilon_1) \tag{4.25}$$

holds for all $\rho \in (0,1)$ and all ε satisfying $0 < \varepsilon < \varepsilon_1$.

Observe now that $\tilde{Q}(\frac{1}{2n}) = \frac{n+1}{4n} + (n-1)(n-3) > 0$ for $n \geq 3$. Observe also that $\tilde{Q}'(\rho) = 2n(n+1)\rho - 2n(n-1)$, meaning that $\tilde{Q}(\rho)$ is monotonically decreasing on $(0, \frac{n-1}{n+1})$. Noting $\frac{1}{2n} < \frac{n-1}{n+1}$ if $n \geq 3$, we hence have

$$\tilde{Q}(\rho) > \tilde{Q}\left(\frac{1}{2n}\right) > \frac{n^2(n-2)+2n+1}{n} \tag{4.26}$$

if $\rho \in (0, \frac{1}{2n})$. The estimates (4.25) and (4.26) imply that, if ε is chosen to be sufficiently small,

$$Q''(\rho) = \rho^{n-3}\left(\tilde{Q}(\rho) + \varepsilon^2\tilde{Q}_{\mathrm{rem}}(\rho)\right) > \rho^{n-3}\left(\frac{n+1}{4n} + (n-1)(n-3)\right) > 0$$

for all $\rho \in (0, \frac{1}{2n})$.

Now recall $Q(\varepsilon) = 0$ and $Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon) > 0$ (cf. Lemma 4.4.7). Since $Q''(\rho)$ is strictly positive for all $\rho \in (0, \frac{1}{2n})$ if ε is chosen to be sufficiently small, $Q'(\rho)$ is strictly monotonically increasing on $(0, \frac{1}{2n})$. Combined with $Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon) > 0$, we thus see that $Q'(\rho) > 0$ for all $\rho \in (\varepsilon, \frac{1}{2n})$. Thus $Q(\rho)$ is strictly monotonically increasing on $(\varepsilon, \frac{1}{2n})$ if ε is chosen to be sufficiently small, but recalling $Q(\varepsilon) = 0$, we see that $Q(\rho)$ is strictly positive for all $\rho \in (\varepsilon, \frac{1}{2n})$ if ε is chosen to be sufficiently small.

Having established $Q(\rho) > 0$ for all $\rho \in (\varepsilon, \frac{1}{2n})$, we are now reduced to proving the positivity of $Q(\rho)$ for all $\rho \in [\frac{1}{2n}, 1)$ when ε is sufficiently small. We need some preparations (i.e. the estimate (4.29)) before doing so.

We now recall that $\delta_0 = \delta + n(n+1)$ is of order ε^2 by Lemma 4.4.8, and write

$$Q(\rho) = \rho^{n-1}(\rho-1)^2 - \frac{\gamma}{(n+1)(n+2)}\rho^{n+2} - \frac{\delta_0 - \gamma}{n(n+1)}\rho^{n+1} + \frac{\delta_0}{n(n-1)}\rho^n - \alpha - \beta\rho. \tag{4.27}$$

Note that, by Lemma 4.4.8, there exist real constants $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ (when ε is sufficiently small) which remain bounded as $\varepsilon \to 0$ such that $\alpha = \tilde{\alpha}\varepsilon^2$, $\beta = \tilde{\beta}\varepsilon^2$, $\gamma = \tilde{\gamma}\varepsilon^2$, $\delta_0 = \tilde{\delta}\varepsilon^2$. We can thus write

$$Q(\rho) = \rho^{n-1}(\rho-1)^2 - \varepsilon^2\left(\frac{\tilde{\gamma}}{(n+1)(n+2)}\rho^{n+2} + \frac{\tilde{\delta}-\tilde{\gamma}}{n(n+1)}\rho^{n+1} - \frac{\tilde{\delta}}{n(n-1)}\rho^n + \tilde{\alpha} + \tilde{\beta}\rho\right).$$

Suppose that we write

$$\tilde{F}_0(\rho) := \frac{\tilde{\gamma}}{(n+1)(n+2)}\rho^{n+2} + \frac{\tilde{\delta}+\tilde{\gamma}}{n(n+1)}\rho^{n+1} - \frac{\tilde{\delta}}{n(n-1)}\rho^n + \tilde{\alpha} + \tilde{\beta}\rho$$

for the terms in the bracket. Now recall that $Q(\rho)$ has a zero of order exactly 2 at $\rho = 1$ by Lemma 4.4.7. This means that $\tilde{F}_0$ must have a zero of order at least 2 at $\rho = 1$, and hence we can factorise

$$\tilde{F}_0(\rho) = (\rho-1)^2\tilde{F}_1(\rho)$$

for some polynomial $\tilde{F}_1(\rho)$. Observe that this implies

$$Q(\rho) = (\rho-1)^2(\rho^{n-1} - \varepsilon^2\tilde{F}_1(\rho)). \tag{4.28}$$

Note that, since $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are uniformly bounded for all sufficiently small $\varepsilon > 0$, there exists a constant $\tilde{C}_1(\varepsilon_1) > 0$, which depends only on (sufficiently small) ε_1 and hence can be chosen uniformly for all ε satisfying $0 < \varepsilon < \varepsilon_1$, such that

$$|\tilde{F}_1(\rho)| < \tilde{C}_1(\varepsilon_1) \tag{4.29}$$

holds for all $\rho \in (0,1)$ and all ε satisfying $0 < \varepsilon < \varepsilon_1$.

Now consider the equation (4.28) for $\rho \in [\frac{1}{2n}, 1)$. Suppose $Q(\rho_0) = 0$ at some $\rho_0 \in [\frac{1}{2n}, 1)$. We would then have $\rho_0^{n-1} - \varepsilon^2\tilde{F}_1(\rho_0) = 0$. However, since $\rho_0^{n-1} \geq (2n)^{-n+1}$ and $\tilde{F}_1$ is uniformly bounded on $[\frac{1}{2n}, 1)$ (as given in (4.29)), we have $\varepsilon^2\tilde{F}_1 \to 0$ uniformly on $[\frac{1}{2n}, 1)$ as $\varepsilon \to 0$, and hence the equation $\rho_0^{n-1} - \varepsilon^2\tilde{F}_1(\rho_0) = 0$ cannot hold if we take

ε to be sufficiently small. We thus get $Q(\rho) \neq 0$ for all $\rho \in [\frac{1}{2n}, 1)$. Since $Q(\rho) > 0$ on $(\varepsilon, \frac{1}{2n})$, we get $Q(\rho) > 0$ for all $\rho \in [\frac{1}{2n}, 1)$ by continuity, and finally establish $Q(\rho) > 0$ for all $\rho \in (\varepsilon, 1)$ and all sufficiently small $\varepsilon > 0$.

□

We shall finally prove the positive-definiteness of the Hessian of the symplectic potential

$$s(x) = \frac{1}{2}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r) + h(\rho)\right). \tag{4.30}$$

Writing s_{ij}^{FS} for the Hessian of the symplectic potential corresponding to the Fubini–Study metric on $\mathbb{P}^n$, i.e.

$$\begin{aligned} s_{ij}^{FS} &:= \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\left(\sum_{i=1}^{n} x_i \log x_i + (1-r)\log(1-r)\right) \\ &= \frac{1}{2}\begin{pmatrix} x_1^{-1} & 0 & 0 & \dots & 0 \\ 0 & x_2^{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_n^{-1} \end{pmatrix} + \frac{1}{2}\frac{1}{1-r}\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}, \end{aligned} \tag{4.31}$$

we can write the Hessian s_{ij} of s as

$$s_{ij} = s_{ij}^{FS} + \frac{h''}{2} T_{ij},$$

where T is a matrix defined by

$$T := \begin{pmatrix} 1 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Observe that T is positive semi-definite.

Lemma 4.4.10. *s_{ij} is positive definite on the interior $\mathscr{P}_\varepsilon^\circ(X)$ of the polytope $\mathscr{P}_\varepsilon(X)$ and has the determinant of the form (4.3), if α, β, γ, δ are chosen as in (4.10), (4.11), (4.12), (4.13), and ε is sufficiently small.*

Proof. Observe first of all that, since s_{ij}^{FS} is positive definite (as given in (4.31)) and T is positive semi-definite, it suffices to prove that there exists a constant $C(\varepsilon_1) > 0$, which depends only on some (small) $\varepsilon_1 > 0$ and hence can be chosen uniformly for all ε satisfying $0 < \varepsilon < \varepsilon_1$, such that

$$h''(\rho) > -\varepsilon C(\varepsilon_1) \tag{4.32}$$

holds for all $\rho \in (\varepsilon, 1)$ and all ε satisfying $0 < \varepsilon < \varepsilon_1$; the claimed positive-definiteness would then follow by taking ε to be sufficiently small.

The inequality (4.32) also implies that $\det(s_{ij})$ is of the form required in (4.3); by a straightforward computation, representing s_{ij} with respect to the following basis

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} e_2 = \begin{pmatrix} x_1^{-1} \\ -x_2^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_{n-1} = \begin{pmatrix} x_1^{-1} \\ 0 \\ \vdots \\ -x_{n-1}^{-1} \\ 0 \end{pmatrix}, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

we see that

$$\begin{aligned}\det(s_{ij}) &= \frac{1}{2^n}\prod_{i=1}^n x_i^{-1} \det \begin{pmatrix} 1+h''+\frac{\rho}{1-r} & 0 & \cdots & 0 & \frac{x_n}{1-r} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{\rho}{1-r} & 0 & \cdots & 0 & 1+\frac{x_n}{1-r} \end{pmatrix} \\ &= \frac{1}{2^n}\prod_{i=1}^n x_i^{-1} \frac{1}{1-r}\left((1+h'')(1-\rho)+\rho\right).\end{aligned}$$

Granted (4.32), we thus see that $\det(s_{ij})$ is of the form required in (4.3), by taking $\varepsilon > 0$ to be sufficiently small and also by recalling Lemmas 4.4.5, 4.4.6, and 4.4.9.

We now prove (4.32). Throughout in the proof, $C(\varepsilon_1)$ will denote a constant which depends only on ε_1 (and not on ε) which varies from line to line.

Now define

$$\tilde{F}_2(\rho) := -\frac{\gamma}{(n+1)(n+2)}\rho^{n+2} - \frac{\delta_0 - \gamma}{n(n+1)}\rho^{n+1} + \frac{\delta_0}{n(n-1)}\rho^n - \alpha - \beta\rho$$

so that

$$Q(\rho) = \rho^{n-1}(1-\rho)^2 + \tilde{F}_2(\rho).$$

Observe first that $Q(\varepsilon) = 0$ (cf. Lemma 4.4.7) is equivalent to $\tilde{F}_2(\varepsilon) = -\varepsilon^{n-1}(1-\varepsilon)^2$. On the other hand, $\gamma = \delta_0 = O(\varepsilon^2)$ (cf. Lemma 4.4.8) implies $\tilde{F}_2(\varepsilon) = O(\varepsilon^{n+2}) - \alpha - \beta\varepsilon$. Thus we get

$$-\alpha - \beta\varepsilon = -\varepsilon^{n-1}(1-\varepsilon)^2 + O(\varepsilon^{n+2}),$$

and hence

$$-\alpha - \beta\rho = -\alpha - \beta\varepsilon - \beta(\rho - \varepsilon) = -\varepsilon^{n-1}(1-\varepsilon)^2 + O(\varepsilon^{n+2}) - \beta(\rho - \varepsilon).$$

On the other hand, since $Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon)$ (cf. Lemma 4.4.7), we have

$$Q'(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon)[(n-1)(1-\varepsilon) - 2\varepsilon] - \beta + O(\varepsilon^{n+1}) = \varepsilon^{n-2}(1-\varepsilon),$$

by differentiating (4.27) and recalling Lemma 4.4.8. We thus get

$$\begin{aligned}\beta &= -\varepsilon^{n-2}(1-\varepsilon) + \varepsilon^{n-2}(1-\varepsilon)[(n-1)(1-\varepsilon) - 2\varepsilon] + O(\varepsilon^{n+1}) \\ &= \varepsilon^{n-2}(1-\varepsilon)[(n-1)(1-\varepsilon) - 1 - 2\varepsilon] + O(\varepsilon^{n+1}).\end{aligned}$$

Define a constant

$$\bar{C}_\varepsilon := (n-1)(1-\varepsilon) - 1 - 2\varepsilon$$

and observe that it satisfies the following bound

$$\frac{10n^2 - 21n - 1}{10n} \leq \bar{C}_\varepsilon < n - 2 \tag{4.33}$$

for all $0 < \varepsilon < \frac{1}{10n}$ say, where we note $10n^2 - 21n - 1 > 0$ if $n \geq 3$; $\bar{C}_\varepsilon$ can be bounded from above and below by a positive constant, uniformly of (all small enough) ε. Then we can write $\beta = \bar{C}_\varepsilon \varepsilon^{n-2}(1-\varepsilon) + O(\varepsilon^{n+1})$, and hence

$$\begin{aligned}-\alpha - \beta\rho &= -\varepsilon^{n-1}(1-\varepsilon)^2 + O(\varepsilon^{n+2}) - \beta(\rho - \varepsilon)\\ &= -\varepsilon^{n-2}(1-\varepsilon)[\varepsilon(1-\varepsilon) + (\bar{C}_\varepsilon + O(\varepsilon^3))(\rho-\varepsilon) + O(\varepsilon^4)].\end{aligned}$$

We now write

$$\frac{\rho^{n-1}(1-\rho)^2}{Q(\rho)} - 1 = \frac{1}{1+\varepsilon^2\tilde{F}_3(\rho) - \tilde{F}_4(\rho)} - 1$$

where we defined

$$\tilde{F}_3(\rho) := \frac{1}{\varepsilon^2(1-\rho)^2}\left(-\frac{\gamma}{(n+1)(n+2)}\rho^3 - \frac{\delta_0-\gamma}{n(n+1)}\rho^2 + \frac{\delta_0}{n(n-1)}\rho\right)$$

and

$$\tilde{F}_4(\rho) := \frac{(\varepsilon/\rho)^{n-2}(1-\varepsilon)}{(1-\rho)^2}[(\varepsilon/\rho)(1-\varepsilon+O(\varepsilon^3)) + (\bar{C}_\varepsilon + O(\varepsilon^3))(1-\varepsilon/\rho)].$$

Arguing as we did in (4.25) and (4.29), we use $\delta_0 = O(\varepsilon^2)$ and $\gamma = O(\varepsilon^2)$ (cf. Lemma 4.4.8) to see that $\tilde{F}_3(\rho)$ satisfies

$$|\tilde{F}_3(\rho)| < C(\varepsilon_1) \tag{4.34}$$

for all $\rho \in (0, \frac{1}{2})$ say, with a constant $C(\varepsilon_1) > 0$ which depends only on (sufficiently small) ε_1 and hence can be chosen uniformly for all ε satisfying $0 < \varepsilon < \varepsilon_1$. Note also that the estimate (4.33) implies

$$\tilde{F}_4(\rho) = \frac{(\varepsilon/\rho)^{n-2}(1-\varepsilon)}{(1-\rho)^2}[(\varepsilon/\rho)(1-\varepsilon+O(\varepsilon^3)) + (\bar{C}_\varepsilon + O(\varepsilon^3))(1-\varepsilon/\rho)] > 0$$

for all $\rho \in (\varepsilon, 1)$ if ε is small enough. Finally, observe that

$$\frac{\rho^{n-1}(1-\rho)^2}{Q(\rho)} = \frac{1}{1+\varepsilon^2\tilde{F}_3(\rho) - \tilde{F}_4(\rho)}$$

and that $Q(\rho) > 0$ for $\rho \in (\varepsilon, 1)$ (cf. Lemma 4.4.9) imply $1 + \varepsilon^2\tilde{F}_3(\rho) - \tilde{F}_4(\rho) > 0$ for all $\rho \in (\varepsilon, 1)$. We thus have

$$0 < \tilde{F}_4(\rho) < 1 + \varepsilon^2 \tilde{F}_3(\rho)$$

for all $\rho \in (\varepsilon, 1)$ if ε is small enough.

Hence we have

$$\begin{aligned} \frac{\rho^{n-1}(1-\rho)^2}{Q(\rho)} - 1 &= \frac{1}{1 + \varepsilon^2 \tilde{F}_3(\rho) - \tilde{F}_4(\rho)} - 1 \\ &> \frac{1}{1 + \varepsilon^2 \tilde{F}_3(\rho)} - 1, \end{aligned}$$

for all $\rho \in (\varepsilon, 1)$. In particular, recalling the estimate (4.34), there exists a constant $C(\varepsilon_1) > 0$ independent of ε such that

$$\begin{aligned} h''(\rho) &= \frac{1}{\rho} \frac{1}{1-\rho} \left(\frac{\rho^{n-1}(1-\rho)^2}{Q(\rho)} - 1 \right) \\ &> \frac{1}{\rho} \frac{1}{1-\rho} \left(\frac{1}{1 + \varepsilon^2 \tilde{F}_3(\rho)} - 1 \right) \\ &> -\frac{\varepsilon}{2} \left| \tilde{F}_3(\rho) \right| \left(1 + \varepsilon^2 \left| \tilde{F}_3(\rho) \right| + \cdots \right) \\ &> -\varepsilon C(\varepsilon_1) \end{aligned}$$

for all $\rho \in (\varepsilon, \frac{1}{2})$, if ε is chosen to be sufficiently small.

Having established the claim for all $\rho \in (\varepsilon, \frac{1}{2})$, we now treat the case $\rho \in [\frac{1}{2}, 1)$. Using the polynomial $\tilde{F}_1$ as given in (4.28), we can write

$$h''(\rho) = \frac{1}{\rho} \frac{1}{1-\rho} \left(\frac{\rho^{n-1}}{\rho^{n-1} - \varepsilon^2 \tilde{F}_1(\rho)} - 1 \right).$$

We thus find that we have a power series expansion of h'' in $\varepsilon^2 \rho^{-n+1} \tilde{F}_1$ as

$$h''(\rho) = \frac{1}{\rho} \frac{1}{1-\rho} \left(\frac{1}{1 - \varepsilon^2 \rho^{-n+1} \tilde{F}_1(\rho)} - 1 \right) = \varepsilon^2 \frac{\rho^{-n} \tilde{F}_1(\rho)}{1-\rho} \left(1 + \varepsilon^2 \rho^{-n+1} \tilde{F}_1(\rho) + \cdots \right)$$

where the series in the bracket is uniformly convergent on $[\frac{1}{2}, 1)$ for all $0 < \varepsilon < \varepsilon_1$ if ε_1 is chosen to be sufficiently small, by noting

$$|\rho^{-n+1} \tilde{F}_1(\rho)| < C(\varepsilon_1)$$

for $\rho \in [\frac{1}{2}, 1)$, following from the estimate (4.29). We thus find

$$|h''(\rho)| < \varepsilon^2 C(\varepsilon_1) \left| \frac{\rho^{-n}\tilde{F}_1(\rho)}{1-\rho} \right|,$$

for a constant $C(\varepsilon_1) > 0$ which does not depend on ε. Recall now that $Q''(1) = 2$ (cf. Lemma 4.4.7) and $Q(\rho) = (\rho - 1)^2(\rho^{n-1} - \varepsilon^2 \tilde{F}_1(\rho))$ (cf. equation (4.28)) imply $\tilde{F}_1(1) = 0$. We thus see that $\frac{\tilde{F}_1(\rho)}{1-\rho}$ is in fact a polynomial, and hence by arguing as we did in (4.25) and (4.29), we get

$$\left| \frac{\rho^{-n}\tilde{F}_1(\rho)}{1-\rho} \right| < C(\varepsilon_1)$$

uniformly on $[\frac{1}{2}, 1)$ and for all ε satisfying $0 < \varepsilon < \varepsilon_1$, if ε_1 is sufficiently small. We can thus evaluate $|h''(\rho)| < \varepsilon^2 C(\varepsilon_1)$ for all $\rho \in [\frac{1}{2}, 1)$, which finally establishes $h''(\rho) > -\varepsilon C(\varepsilon_1)$ for all $\rho \in (\varepsilon, 1)$ and all ε satisfying $0 < \varepsilon < \varepsilon_1$, where ε_1 is chosen to be sufficiently small.

□

4.4.3 Potential extension of Proposition 4.4.1

As we saw in the above, the hypothesis $\varepsilon \ll 1$ is essential in establishing the regularity (Proposition 4.4.4) of the symplectic potential. However, as in the point blowup case (Theorem 4.1.7), it is natural to expect that the extremal metrics exist in each Kähler class.

Question 4.4.11. Does Proposition 4.4.4 hold for any $0 < \varepsilon < 1$? In other words, does $\mathrm{Bl}_{\mathbb{P}^1}\mathbb{P}^n$ admit an extremal metric in each Kähler class?

Some numerical results obtained by a computer experiment seem to suggest that the answer to this question should be affirmative.

Appendix A

Some results on the Lichnerowicz operator used in §2.3.2

Lemma A.0.12. *For any $F \in C^\infty(X,\mathbb{R})$, there exists $F_1 \in C^\infty(X,\mathbb{R})$, $F_2 \in C^\infty(X,\mathbb{R})$ such that $\mathscr{D}^*_\omega \mathscr{D}_\omega F_1 = F + F_2$ with $\mathscr{D}^*_\omega \mathscr{D}_\omega F_2 = 0$. Moreover, writing $\mathrm{pr}_\omega : C^\infty(X,\mathbb{R}) \twoheadrightarrow \ker \mathscr{D}^*_\omega \mathscr{D}_\omega$ by recalling the L^2-orthogonal direct sum decomposition $C^\infty(X,\mathbb{R}) \cong \mathrm{im} \mathscr{D}^*_\omega \mathscr{D}_\omega \oplus \ker \mathscr{D}^*_\omega \mathscr{D}_\omega$, F_2 is in fact $F_2 = -\mathrm{pr}_\omega(F)$.*

Proof. This is a well-known result, which follows from the self-adjointness and the elliptic regularity of $\mathscr{D}^*_\omega \mathscr{D}_\omega$. □

Lemma A.0.13. *Let $\{F_k\}$ be a family of smooth functions parametrised by k, converging to a smooth function F_∞ in C^∞ as $k \to \infty$, and $(\phi_{1,k}, \dots, \phi_{m,k})$ be smooth functions, each of which converges to a smooth function $\phi_{i,\infty}$ as $k \to \infty$. Write $\omega_{(m)} := \omega + \sqrt{-1}\partial\bar{\partial}(\sum_{i=1}^m \phi_{i,k}/k^i)$. Let $\mathrm{pr}_\omega : C^\infty(X,\mathbb{R}) \to \ker \mathscr{D}^*_\omega \mathscr{D}_\omega$ and $\mathrm{pr}_{(m)} : C^\infty(X,\mathbb{R}) \to \ker \mathscr{D}^*_{(m)} \mathscr{D}_{(m)}$ be the projection to $\ker \mathscr{D}^*_\omega \mathscr{D}_\omega$ and $\ker \mathscr{D}^*_{(m)} \mathscr{D}_{(m)}$, respectively. Then, $\mathrm{pr}_{(m)} F_k$ converges to $\mathrm{pr}_\omega F_\infty$ in C^∞.*

Proof. Note that we can write $\mathscr{D}^*_{(m)} \mathscr{D}_{(m)} = \mathscr{D}^*_\omega \mathscr{D}_\omega + D/k$ for some differential operator D of order at most 4, which depends on ω and $(\phi_{1,k}, \dots, \phi_{m,k})$. Since we know that each $\phi_{i,k}$ converges to a smooth function $\phi_{i,\infty}$ in C^∞, the operator norm of D can be controlled by a constant which depends only on ω and $(\phi_{1,\infty}, \dots, \phi_{m,\infty})$ but not on k. Thus, $||\mathrm{pr}_{(m)} F - \mathrm{pr}_\omega F||_{C^\infty} \to 0$ for any fixed $F \in C^\infty(X,\mathbb{R})$ as $k \to \infty$. On the other hand, $||\mathrm{pr}_{(m)} F_k - \mathrm{pr}_{(m)} F_\infty||_{C^\infty} \to 0$ since F_k converges to F_∞ in C^∞. Combining these estimates,

$$||\mathrm{pr}_{(m)}F_k - \mathrm{pr}_\omega F_\infty||_{C^\infty} \leq ||\mathrm{pr}_{(m)}(F_k - F_\infty)||_{C^\infty} + ||\mathrm{pr}_{(m)}F_\infty - \mathrm{pr}_\omega F_\infty||_{C^\infty} \to 0$$

as $k \to \infty$. □

Lemma A.0.14. *Suppose that the following four conditions hold for an arbitrary but fixed $m \geq 1$.*

1. *$(\phi_{1,k}, \dots, \phi_{m,k})$ are smooth functions parametrised by k such that each $\phi_{i,k}$ converges to a smooth function $\phi_{i,\infty}$ in C^∞ as $k \to \infty$, so that $\omega_{(m)} := \omega + \sqrt{-1}\partial\bar{\partial}(\sum_{i=1}^m \phi_{i,k}/k^i)$ converges to ω in C^∞,*
2. *$\{G_k\}$ is a family of smooth functions on X parametrised by k such that it converges to a smooth function G_∞ in C^∞ as $k \to \infty$,*
3. *$\{F_k\}$ is another family of smooth functions on X parametrised by k, each of which is the solution to the equation*

   $$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_k = G_k,$$

   with the minimum L^2-norm,
4. *there exists a smooth function F_∞ which is the solution to the equation*

   $$\mathscr{D}^*_\omega \mathscr{D}_\omega F_\infty = G_\infty$$

   with the minimum L^2-norm.

Then F_k converges to F_∞ in C^∞ as $k \to \infty$.

Proof. Consider the equation

$$\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}(F_\infty - F_k) = \mathscr{D}^*_\omega \mathscr{D}_\omega F_\infty + O(1/k) - G_k = G_\infty - G_k + O(1/k)$$

in $C^\infty(X,\mathbb{R})$. Recalling the L^2-orthogonal direct sum decomposition $C^\infty(X,\mathbb{R}) = \ker \mathscr{D}^*_\omega \mathscr{D}_\omega \oplus \mathrm{im} \mathscr{D}^*_\omega \mathscr{D}_\omega$ (and hence $\mathrm{im}\mathscr{D}^*_\omega \mathscr{D}_\omega = \ker \mathscr{D}^*_\omega \mathscr{D}^\perp_\omega$), we write $(F_\infty - F_k)^\perp$ for the $\ker \mathscr{D}^*_{(m)}\mathscr{D}^{\perp}_{(m)}$-component of $F_\infty - F_k$. By the standard elliptic estimate, we have

$$||(F_\infty - F_k)^\perp||_{L^2_{p+4}} \leq C_{1,p}(\omega, \{\phi_{i,k}\})||G_\infty - G_k + O(1/k)||_{L^2_p} \to 0$$

Recalling also $\mathrm{im}\mathscr{D}^*_\omega\mathscr{D}_\omega = \ker \mathscr{D}^*_\omega\mathscr{D}^\perp_\omega$, the hypothesis 4 implies $F_\infty \in \mathrm{im}\mathscr{D}^*_\omega\mathscr{D}_\omega$, and hence there exists a function $F' \in C^\infty(X,\mathbb{R})$ such that $F_\infty = \mathscr{D}^*_\omega\mathscr{D}_\omega F'$ with the estimate

$$||F'||_{L^2_p} \leq C_{2,p}(\omega)||F_\infty||_{L^2_{p-4}}$$

following from the standard elliptic regularity. On the other hand,

$$F_\infty = \mathscr{D}^*_\omega\mathscr{D}_\omega F' = \mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F' + \frac{1}{k}D(F'),$$

with some differential operator D of order at most 4 which depends on ω and $(\phi_{1,k},\dots,\phi_{m,k})$. This means that $F_\infty - D(F')/k \in \ker \mathscr{D}^*_{(m)}\mathscr{D}^\perp_{(m)}$, and hence

$$\begin{aligned} ||F_\infty - (F_\infty)^\perp||_{L^2_{p+4}} < ||D(F')||_{L^2_{p+4}}/k < C_{3,p}(\omega, \{\phi_{i,k}\})||F'||_{L^2_p}/k \\ < C_{4,p}(\omega, \{\phi_{i,k}\})||F_\infty||_{L^2_{p-4}}/k \to 0 \end{aligned}$$

as $k \to \infty$, where we used the fact that ϕ_i's are the functions that converge to some smooth function as $k \to \infty$, so that $C_4(\omega, \{\phi_{i,k}\})$ stays bounded when k goes to infinity. Thus, recalling that F_k is the solution to $\mathscr{D}^*_{(m)}\mathscr{D}_{(m)}F_k = G_k$ with the minimum L^2-norm (implying $(F_k)^\perp = F_k$), we have

$$||F_\infty - F_k||_{L^2_{p+4}} \leq ||(F_\infty - F_k)^\perp||_{L^2_{p+4}} + ||F_\infty - (F_\infty)^\perp||_{L^2_{p+4}} \to 0$$

as $k \to \infty$.

Since the above argument holds for all large enough p, we see that F_k converges to F_∞ in C^∞.

□

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