Players with fixed resources in Elimination Tournaments

Alexander Matros*
ELSE, Department of Economics
University College London

November 22, 2001

Abstract. We consider T-round elimination tournaments where players have fixed resources instead of cost functions. We show that players always spend a higher share of their resources in early than in later rounds in a symmetric equilibrium. Equal resource allocation across T rounds takes place only in the winner-take-all case. Applications for career paths, elections, and sports are discussed.

1. Introduction

The aim of this note is to analyze elimination tournaments in which players have fixed resources instead of cost functions. At every round players are matched in pairs for contests and the winner of each contest proceeds to the next round. All losers receive the prize of the current round and are eliminated from the tournament. There is a trade off here. On the one hand, the more resources a player invests in the current round, the higher her chance to win the current contest and continue the tournament. On the other hand, the more resources a player invests in the current round, the less her chance to win the following contests in the tournament. Each player has to allocate optimally her overall resources across all rounds. This strategic problem is different from the problem analyzed in the contest literature, where players must decide how much effort to spend to win the prize(s) in one contest; see, for example, Dixit (1987, 1999), Baik and Shogren (1992), Baye and Shin (1999), and Moldovanu and Sela (2001).

Several real-world phenomena have the structure of such tournaments. A participant in a multi-stage election campaign, who has a fixed total budget, plays an

*This paper is a revised version of Chapter 5 of my Ph.D. thesis at the Stockholm School of Economics. I would like to thank Gian Luigi Albano, Jonas Björnerstedt, Guido Friebel and Dmitry Matros for very useful discussions. I have also benefited from comments by Tatiana Damjanovic, Elena Palzeva, Jörgen Weibull, Karl Wärneryd, Peyton Young, and seminar participants at the Stockholm School of Economics. The support of the Economic and Social Research Council (ESRC) is gratefully acknowledged. The work was part of the programme of the ESRC Research Center for Economic Learning and Social Evolution. Remaining errors are solely due to the author.

elimination tournament with fixed resources. A new worker (an assistant professor) at the beginning of his/her career, when he/she has a limited number of working hours for the whole career, has to distribute optimally those hours. Needless to say, that many sportsmen, for example, tennis players, have to allocate optimally his/her overall energy across all rounds in elimination tournaments, or chess players have to decide in which rounds he/she should use his/her novelties in elimination FIDE World Championship.

The tournament literature has focused theoretically, see, for example, Lazear and Rosen (1981), Rosen (1981, 1986), and empirically, Ehrenberg and Bognanno (1990), Knoeber and Thurman (1994), on players' incentives in tournaments, when players have some costs for exerting effort. Classical papers Lazear and Rosen (1981) and Rosen (1981) show that high differences in prizes in the last round(s) must provide enough incentives for players to insert the same effort in all rounds. We show that the equal resource allocation across all rounds takes place only in the winner-take-all case. Moreover, in the symmetric equilibrium every player will spend more resources in the earlier round that in the later round, if the prize scheme is different from the winner-take-all. The intuition is straightforward: if a player keeps her resources until the very last rounds to get higher prizes, then she will be eliminated long before these very last rounds.

The rest of this note is organized as follows. Section 2 introduces the elimination tournament and brings results. Section 3 provides a discussion. Proof of the main result is in the Appendix.

2. The Model

Consider a T-round elimination tournament with 2^T risk-neutral players fighting for prizes (payoffs). In round 1 all players are matched in pairs for fights/contests, where only the winners of the current round continue to fight for higher payoffs in the following rounds. All losers get payoff Z_0, each of them won 0 contests, and are eliminated from the tournament. In round 2, the winners of the first round, 2^{T-1} players, are matched in pairs for new fights/contests. Every winner proceeds to the next round and all losers receive Z_1, each of them won 1 contest, and are out of the tournament, and so on. Finally, in round T, only two players remain. The winner of the final gets Z_T and the loser receives payoff Z_{T-1}. In other words, in every round $k \in \{1,...,T\}$, there are 2^{T-k+1} players who are matched in pairs for new fights/contests. All losers in round k, each of them won $k-1$ contests, receive payoff Z_{k-1} and are eliminated from the tournament and all winners continue. We make the standard assumption that prizes increase from round to round

$$A1.\ 0 \leq Z_0 \leq Z_1 \leq ... \leq Z_T \tag{1}$$

and at least one of the inequalities $Z_0 \leq Z_1 \leq ... \leq Z_T$ is strict.

Each player i has an initial fixed resource E, and must decide how to allocate this resource in all T rounds. Denote an invested part of player i's resource in round k by x_k^i. If player i chooses to use a part $x_k^i \in [0, E]$ of her resource in round k, when her opponent in round k, player j, invests a part $x_k^j \in [0, E]$, then player i wins this fight with probability

$$\frac{f\left(x_k^i\right)}{f\left(x_k^i\right) + f\left(x_k^j\right)}, \tag{2}$$

where $f(x)$ is a positive, twice differentiable, and increasing function:

$$\text{A2. } f(x) > 0,\ f'(x) \geq 0 \text{ on interval } [0, E]. \tag{3}$$

A pure strategy for player i is a rule $(x_1^i, ..., x_T^i)$, which assigns a part of her resource for every round in the tournament, such that $\sum_{k=1}^{T} x_k^i = E$, $x_k^i \geq 0$ for any $i \in \left\{1, ..., 2^T\right\}$ and $k \in \{1, ..., T\}$.

It will be shown that a symmetric equilibrium in pure strategies is unique if function $f(x)$ is "not very convex":

$$\text{A3. } f(x) f''(x) - \left[f'(x)\right]^2 \leq 0 \text{ on interval } [0, E], \tag{4}$$

and

$$\text{A4. } f'(0) = 0. \tag{5}$$

We will call the following prize structure

$$0 \leq Z_0 = Z_1 = ... = Z_{T-1} < Z_T$$

winner-take-all. The main result of this note can be stated now.

Proposition 1. *Suppose that assumptions (1) and (3)-(5) hold. Then, in symmetric equilibrium $(x_1, ..., x_T)$, it must be $x_1 \geq x_2 \geq ... \geq x_T$, for any prize structure $(Z_0, Z_1, ..., Z_T)$. Equal resource allocation across all rounds $x_1 = x_2 = ... = x_T$ takes place only in the winner-take-all case.*

Proof: See the Appendix.

Corollary 1. *When at least one of the differences in prizes $(Z_1 - Z_0), (Z_2 - Z_1), ..., (Z_{T-1} - Z_{T-2})$ is positive, the players' resource allocation decreases across rounds.*

To get the main result we ask for assumptions (1), (3)-(5). Assumptions (1) and (3) are common and the role of assumptions (4) and (5) can be illustrated by the following example, where function $f(x)$ is "very convex".

Example. Suppose that there are two rounds, $T = 2$, in an elimination tournament, the resource is equal to one, $E = 1$; $f(x) = e^{(x+1)^2}$; and the following prize structure, $Z_0 = 0$, $Z_1 = 1$ and $Z_2 = 3$. Then, in the symmetric equilibrium $(x_1^*, x_2^*) = (0, 1)$, or every player spends all resources in the final round.[1]

3. Discussion

We consider T-round elimination tournaments, where risk-neutral players have fixed resources. In the symmetric equilibrium, all players spend more resources in earlier rounds and less in later rounds. The intuition is straightforward: the expected payoffs are much higher in earlier rounds than in later rounds of the tournament. The same reasoning is valid for Rosen's (1986) model, where players have costs for exerting effort in every round instead of fixed resources.

Rosen (1986) shows that prizes must increase over rounds to provide enough incentives for players to exert the same effort in every round, if players have trade off between costs and expected high future payoffs. In our model, if a principle wants players to allocate resources equally across rounds in the elimination tournament, then he must implement the winner-take-all prize scheme.

Moldovanu and Sela (2001) show that the winner-take-all is the optimal prize scheme, if cost functions are linear or concave in effort. However, they analyze a contest with multiple, nonidentical prizes with deterministic relation between effort and output. Deterministic assumption should be contrasted with stochastic result of a contest in every round of the tournament in Rosen (1986) and this note.

Although elimination tournaments are usually associated with sports: tennis, football and chess, for example, there are many applications for hierarchy in a firm, academic career, and election campaigns as well. This simple model helps to explain why an assistant professor must work harder at the beginning of his/her career, ten-

[1] Note that $f'(x) = 2(x+1)e^{(x+1)^2}$. From equation (9) in the Appendix, it follows that in a symmetric equilibrium $(x_1, 1-x_1)$, x_1 is a solution of the following equation

$$4\frac{2(x_1+1)e^{(x_1+1)^2}}{e^{(x_1+1)^2}}\frac{e^{(2-x_1)^2}}{2(2-x_1)e^{(2-x_1)^2}} = 2,$$

or

$$x_1 = 0.$$

Hence, (0, 1) is a symmetric equilibrium.

nis and football players have to exert a lot of effort at the beginning of an elimination tournament, and why new workers spend all day long in their offices.

Some work has been done to test prediction of Lazear and Rosen (1981) theory, see for example Ehrenberg and Bognanno (1990) and Knoeber and Thurman (1994). It will be interesting to test the relationship between prizes/relative prizes and allocation of players' resources in experimental and nonexperimental frameworks.

Appendix.

Proof of Proposition 1: Show first that there exists a symmetric equilibrium in pure strategies. Suppose that all players, but player i, allocate resource E in the same way $(y_1,...,y_T)$. Given the opponents' resource allocation $(y_1,...,y_T)$, the player i's resource allocation decision x_k in the $k-th$ round of the tournament is determined by the solution of:

$$V_k = \max_{x_k} \left[\frac{f(x_k)}{f(x_k)+f(y_k)} V_{k+1} + \frac{f(y_k)}{f(x_k)+f(y_k)} Z_{k-1} \right], \text{ for } k = 1,...,T-1, \tag{6}$$

and in the final round by:

$$V_T = \frac{f(E - x_{T-1} - ... - x_1)}{f(E - x_{T-1} - ... - x_1) + f(E - y_{T-1} - ... - y_1)} Z_T + \frac{f(E - y_{T-1} - ... - y_1)}{f(E - x_{T-1} - ... - x_1) + f(E - y_{T-1} - ... - y_1)} Z_{T-1}, \tag{7}$$

$$\text{such that } 0 \le x_1 + ... + x_{T-1} \le E. \tag{8}$$

The first order condition for problem (6) – (8) is

$$\frac{f'(x_k) f(y_k)}{[f(x_k)+f(y_k)]^2} [V_{k+1} - Z_{k-1}] + \frac{f(x_k)}{f(x_k)+f(y_k)} \times \frac{f(x_{k+1})}{f(x_{k+1})+f(y_{k+1})} \times ... \times \frac{f(x_{T-1})}{f(x_{T-1})+f(y_{T-1})} \times$$

$$\times \left[\frac{-f'(E - x_{T-1} - ... - x_1) f(E - y_{T-1} - ... - y_1)}{[f(E - x_{T-1} - ... - x_1) + f(E - y_{T-1} - ... - y_1)]^2} (Z_T - Z_{T-1}) \right] = 0.$$

In symmetric equilibrium, $x_{T-1} = y_{T-1},...,x_1 = y_1$ and we have

$$\frac{f'(y_k)}{4f(y_k)} \left(\frac{V_{k+2} + Z_k}{2} - Z_{k-1} \right) = \left(\frac{1}{2} \right)^{T-k} \left[\frac{f'(E - y_{T-1} - ... - y_1)}{4f(E - y_{T-1} - ... - y_1)} (Z_T - Z_{T-1}) \right].$$

Finally, we get

$$\frac{f'(y_k)}{f(y_k)}\left[(Z_T - Z_{T-1}) + 2(Z_{T-1} - Z_{T-2}) + 4(Z_{T-2} - Z_{T-3}) + ... + 2^{T-k}(Z_k - Z_{k-1})\right] =$$

$$\frac{f'(E - y_{T-1} - ... - y_1)}{f(E - y_{T-1} - ... - y_1)}(Z_T - Z_{T-1}). \tag{9}$$

Assumption (4) guarantees that the left-hand side (LHS) in equation (9) is a strictly decreasing function of y_k on the interval $[0, E]$, and the right-hand side (RHS) in the same equation is a strictly increasing function of y_k on the interval $[0, E]$. Denote $y = y_{T-1}+...+y_{k+1} + y_{k-1}+...+y_1$. Player i must allocate resource part $(E - y)$ between period k and the last period. Hence, because of assumption (5), equation (9) has a unique solution y_k^* inside of the interval $(0, E - y)$, since it defines the intersection of a decreasing and an increasing continuous functions and $LHS(x_k = 0) = 0 < RHS(x_k = 0)$, $LHS(x_k = E - x) > 0 = RHS(x_k = E - x)$. Note that $y_k \geq y_T$ if and only if $LHS\left(\frac{E-y}{2}\right) \geq RHS\left(\frac{E-y}{2}\right)$. If y_k is equal to $\frac{E-y}{2}$, or allocated resource parts in period k and the last period are equal, then

$$LHS\left(y_k = \frac{E-y}{2}\right) =$$

$$\frac{f'\left(\frac{E-y}{2}\right)}{f\left(\frac{E-y}{2}\right)}\left[(Z_T - Z_{T-1}) + 2(Z_{T-1} - Z_{T-2}) + 4(Z_{T-2} - Z_{T-3}) + ... + 2^{T-k}(Z_k - Z_{k-1})\right]$$

and

$$RHS\left(y_k = \frac{E-y}{2}\right) = \frac{f'\left(\frac{E-y}{2}\right)}{f\left(\frac{E-y}{2}\right)}(Z_T - Z_{T-1}).$$

Note that from assumption (1)

$$(Z_T - Z_{T-1}) + 2(Z_{T-1} - Z_{T-2}) + 4(Z_{T-2} - Z_{T-3}) + ... + 2^{T-k}(Z_k - Z_{k-1}) \geq (Z_T - Z_{T-1}),$$

with equality if and only if $Z_{T-1} = ... = Z_{k-1}$, for any $k = 1,...,T-1$. It means that if other players, but player i choose $(y_1,...,y_T)$, then player i's best reply is also to play $(y_1,...,y_T)$. All players are in the same situation, hence, there exists a symmetric equilibrium $(x_1, x_2,...,x_T)$ for any prize scheme $(Z_0, Z_1,...,Z_T)$, and $x_k \geq x_T$, with equality if and only if $Z_{T-1} = ... = Z_0$.

Using the same logic and fixing an allocation choice in the last period x_T, it can be shown that $x_k \geq x_{T-1}$, for any $k < T-2$ and for any prize scheme $(Z_0, Z_1,...,Z_T)$, with equality if and only if $Z_{T-1} = ... = Z_0$, and so on. Finally, the optimal resource allocation in the symmetric equilibrium must be $x_1 \geq x_2 \geq ... \geq x_T$, for any prize structure $(Z_0, Z_1, ..., Z_T)$. Equal resource allocation across all rounds $x_1 = x_2 = ... = x_T$ takes place only in the winner-take-all case. End of proof.

References

[1] Baik, Kyung H., and Shogren, Jason F. "Strategic Behavior in Contests: Comment." A.E.R. 82 (March 1992): 359-62.

[2] Baye, Michael R., and Shin, Onsong. "Strategic Behavior in Contests: Comment." A.E.R. 89 (June 1999): 691-93.

[3] Dixit, Avinash. "Strategic Behavior in Contests." A.E.R. 77 (December 1987): 891-98.

[4] Dixit, Avinash. "Strategic Behavior in Contests: Reply." A.E.R. 89 (June 1999): 694.

[5] Ehrenberg, Ronald G., and Bognanno, Michael L. "Do Tournaments Have Incentives Effects?" J.P.E. 98 (December 1990): 1307-24.

[6] Lazear, Edward P., and Rosen, Sherwin. "Rank-Order Tournaments as Optimum Labor Contracts." J.P.E. 89 (October 1981): 841-64.

[7] Knoeber, Charles R., and Thurman, Walter N. "Testing the Theory of Tournaments: An Empirical Analysis of Broiler Production." J. Labor Econ. 12 (April 1994): 155-79.

[8] Moldovanu, Benny, and Sela, Aner. "The Optimal Allocation of Prizes in Contests." A.E.R. 91 (June 2001): 542-58.

[9] Rosen, Sherwin. "The Economics of Superstars." A.E.R. 71 (December 1981): 845-58.

[10] Rosen, Sherwin. "Prizes and Incentives in Elimination Tournaments." A.E.R 76. (September 1986): 701-15.