

Bidding among friends and enemies*

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Abstract

We consider an auction setting in which bidders, even if they fail to obtain the good, care about the price paid by the winner. We study the impact of these *price externalities* on the first-price auction and the second-price auction in a symmetric information framework. We establish a distinction between price externalities that do not depend on the identity of the winner and price externalities that depend on the identity of the winner. We prove that the outcome of the first-price auction is not affected by the first type of price externalities while the outcome of the second-price auction is. In contrast, the second type of price externalities affects the outcome of both auction formats. In any case, in comparison with the first-price auction, the second-price auction exacerbates the effects of price externalities whatever their types are. The two auction formats are generically not equivalent.

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1 Introduction

In 1999, the *Ligue Nationale de Football (LNF)*, the organism that represents the interests of the French professional soccer teams, auctioned the retransmission rights of the French Soccer Championship for the next four years. Canal+, the leading French pay-TV channel, was among the bidders. Canal+ also owns one of the most important French professional soccer teams, i.e. Paris Saint-Germain. But, the auction revenue was divided among the professional teams including the Paris Saint-Germain. Thus, in its capacity of TV channel, Canal+ wanted to buy these rights for the lowest possible price while in its capacity of owner of Paris Saint-Germain, Canal+ preferred the price to be high. As a result, Canal+, independently from the identity of the winner, was not indifferent to the price paid by the winner. Conditional on losing, it preferred the price to be high.

Eventually, Canal+ won the ascending auction organized by the LNF, for a total amount of more than a billion Euros. The specific interest that Canal+ had in the price conditional on losing the auction is likely to have influenced the strategy of Canal+ during the auction, and indirectly these of the other bidders (TF1, TPS, M6). Was Canal+ more aggressive during the auction process because of this specificity? Was the choice of a first-price auction by the LNF optimal considering the specificity of the situation?

The standard auction theory analysis framework does not allow to answer these questions. As a matter of fact, there is a specific element in this setting that the standard auction framework does not take into consideration. A bidder, here Canal+, cares about the price paid by the winner even if he loses the auction. Such a possibility is generally ruled out. However, this interest in the price paid by another bidder may be a key element in many auction settings. The following examples support this assertion.

Suppose that a bidder owns a fraction of another bidder's capital. Then, the first bidder, conditional on his losing the auction, prefers this other bidder to win the auction and to pay the lowest possible price since he gets back, through his shares, a fraction of the other bidder's profit.

Suppose now that a bidder and the seller form a duopoly on a market

independent from the selling good. Markets are imperfect and firms are budget-constrained.¹ Therefore, this bidder prefers that the seller raises the lowest possible amount of money through the auction, whoever the winner is. As a matter of fact, if the seller receives less money, he will be able to finance less research, marketing or other competitive activities on their common market.

A charity sale: A good is auctioned. All the bidders know that the auction revenue will be used to finance a charitable organization. We may assume that many bidders want the charitable organization to raise as much money as possible. Then, independently from the identity of the buyer, bidders prefer the price to be high. Of course, for the winner, there is a trade-off between the interest he has in the funding of the organization and his preference for keeping his money for other uses. In any case, even for the winner, the money given to the cause is not *lost* as it would be in a standard auction.

In all these cases, at least one bidder, even if he fails to win the auction, cares about the price paid by the winner. We call this concern of losing bidders about the price a *price externality* (PE). This short series of examples highlights few different features of price externalities which will play a key role in the analysis.

There is a fundamental difference between a situation in which a bidder owns a part of another bidder and a charity sale. In the first case, the bidder who owns a fraction of another bidder cares about the price paid when he loses the auction only if this specific other bidder wins the auction. The price externality depends on the identity of the winner. In a charity sale, the amount paid by the winner is used to finance a public good whose existence affects bidders' utilities independently from the identity of the winner. The amount given to the charity organization affects a bidder's utility whoever the winner is including himself. This difference between the two types of situations concerns the essence of these price externalities.

¹The budget constraint may be strict or firms may face an increasing cost for money. This does not matter

Therefore, we define two categories of price externalities depending on whether the identity of the winner matters or not. When the identity of the buyer matters, we will speak of identity dependent price externalities (IDPE). When the identity of the winner does not matter, we will speak of identity independent price externalities (IIPE).

Furthermore, the examples also show that a losing bidder may have a preference for a low or a high price. Price externalities may be decreasing or increasing functions of the price.

In this paper, we examine how both types of price externalities, identity dependent and identity independent, affect the first-price auction and the second-price auction,² focusing on the two-buyers case. This setting is sufficient for the illustration of the effects caused by the presence of price externalities. With more than two bidders, we would have to distinguish between the specific effects of price externalities and the effects of allocative externalities (cf. *infra*). Besides, we show in the final section that the results we obtain in the two-bidders case can be qualitatively extended to the n -bidders case with $n \geq 3$. These results are as follows.

IIPE do not have any effect on the equilibrium of the first-price auction, while they generically have an effect in a second-price auction. IIPE, by definition, do not depend on the identity of the winner. Therefore, they do not affect the price for which bidders are indifferent between winning and losing. In a first-price auction, it turns out that equilibrium bids depend only on these indifference prices. Thus, IIPE do not have any impact on the equilibrium of the first-price auction. On the other hand, in a second-price auction, a losing bidder may fix the price through his bid. If he strictly prefers the price to be the highest (resp: the lowest) possible, he will raise (resp: lower) his bid. As a result, in a second-price auction, IIPE may affect the equilibrium and the two auction formats are not equivalent.

With IDPE, things are slightly different. By definition, IDPE depend on the identity of the winner. As a result, they do affect the price for which bidders are indifferent between losing and winning the auction. As we said before, this indifference price is the only element that matters in a first-price

²Here, the two other standard auction formats, the descending and the ascending auction are equivalent to respectively the first-price auction and the second-price auction.

auction. Therefore, IDPE do affect the equilibrium of the first-price auction. However, even when there are only IDPE, the two auction formats are not equivalent either. The second-price auction is more *sensitive* to IDPE than the first-price auction. Once again, this is due to the very structure of the second-price auction in which the loser, through his bid, determines the price paid by the winner. Moreover, given what we already said regarding IPE, the amplification of the effects of price externalities with a second-price auction remains true with any type of price externalities.

While the literature on auction theory is flourishing, few papers have been published so far on topics related to price externalities considerations. The idea of considering externalities in an auction context was recently introduced. Articles such as Jehiel and Moldovanu (1996) or Jehiel and Moldovanu (2000) first present the possible consequences of allocative externalities in an auction framework. However, in this literature, the key element is not the price but rather the identity of the winner. They assume that a losing bidder may have preferences regarding the identity of the winner. As a result, they observe equilibrium multiplicity and strategic non-participation. In our setting, the main issue is not that bidders care about who wins but rather that they care about how much money is spent by the winner. Thus, bidders have other motivations and we observe qualitatively different results. For instance, contrary to what they obtain in their original setting (see Jehiel and Moldovanu (1996)) without reserve price or entry fees,³ here, the standard auction formats are not equivalent.

Apart from this literature that focuses on allocative externalities, there is no systematic study of auctions with price externalities. However, some specific cases of auctions with price externalities have been examined.

In a symmetric information framework, Pitchik and Schotter (1988) study sequential auctions with budget-constrained bidders. They observe that the standard auction formats are not revenue equivalent. This can be reinterpreted as a specific application of our more general results. Benoit and Krishna (2001) also analyze sequential auctions with budget-constrained bidders but with a different perspective. The paper emphasizes matters such as the

³For an analysis of the impact of reserve prices and entry fees in the context of auctions with externalities, see Jehiel and Moldovanu (2000).

best sequencing to sell goods. We are more focused on the situation in which the first seller has a unique good to sell. We take the environment as given and recommend an adequate format to sell his good

In an asymmetric information framework, Bulow, Huang and Klemperer (1999) consider a setting in which bidders own a fraction of the good for sale. They assume that the value of the good is common and derive that small asymmetries among bidders -in terms of fraction of the good they own- may have dramatic effects. Their point is more related to the impact of asymmetries in a common value environment than specifically to price externalities.⁴ Finally, Burkart (1995), Singh (1998), Engelbrecht-Wiggans (1994) Maasland and Onderstal (2001) and Ettinger (2001) study the impact of some types of toeholds in a private value framework and, more recently, Engers and McManus (2001) and Goeree and Turner (2002) study charity auctions. Our results are generally consistent with theirs. Furthermore, the choice to consider a symmetric information framework allows us to study a broader range of situations, to emphasize results that are solely due to price externalities rather than a mix of asymmetric information and price externalities and to identify the differences between identity dependent and identity independent price externalities.

After having explained the interest of the price externality problematic and its large degree of applicability, we introduce in section 2 a general model of auctions with price externalities. In section 3, we show how a wide range of applications fits into the model. In section 4, we study more specifically identity independent price externalities and, in section 5, identity dependent price externalities. At last, in section 6, we explain how the results we obtain can be applied to any specific setting with price externalities, we make recommendations, discuss our hypotheses and present possible extensions.

2 The Model

One good is sold through an auction process to two bidders, 1 and 2. Bidders' preferences depend on the identity of the winner and the price paid by the

⁴See, on this topic, their other papers such as Klemperer (1998) or Bulow and Klemperer (1999).

winner whoever the winner is. For $i = 1, 2$, bidder i 's preference is represented by a utility function U_i . $U_i(k, p)$ stands for the utility of bidder i if the good is bought for a price p by bidder k , with $k = 1, 2$.

Without loss of generality, we can normalize utilities so that if $i \neq j$, $U_i(j, 0) = 0$. If a bidder buys the good for the price zero, the other bidder derives a utility zero. Besides, we assume that utilities functions are common knowledge among bidders and that utilities are non transferable.⁵

For convenience, we introduce v_i , $f_i(p)$ and $g_i(p)$ defined by:

$$\begin{aligned} v_i &= U_i(i, 0) \\ g_i(p) &= U_i(i, p) - (v_i - p) \\ f_i(p) &= U_i(j, p) - g_i(p) \end{aligned}$$

Utility functions can then be written:

$$U_i(i, p) = v_i - p + g_i(p) \tag{1}$$

$$U_i(j, p) = f_i(p) + g_i(p) \tag{2}$$

The functions $g_i(p)$ and $f_i(p)$ are to be interpreted as follows. $g_i(p)$ is the identity independent price externality (IIPE) incurred by bidder i if the good is sold for the price p , whoever the buyer is. $f_i(p)$ is the identity dependent price externality (IDPE) incurred by bidder i if the good is sold for the price p specifically to bidder j . As a matter of fact, whoever the winner is, if the price paid is p , $g_i(p)$ appears in the utility function of bidder i . Besides, if bidder j buys the good for the price p , the utility of bidder i is $g_i(p) + f_i(p)$. $g_i(p)$ is the IIPE incurred by bidder i if the price paid is p . Then, $f_i(p)$ must be the IDPE, the residual price externality that bidder i incurs specifically because this is bidder j who pays p . Everything else corresponds to the standard representation of bidders' utilities, v_i playing the role of bidder i 's valuation for the good. Therefore, whatever the shapes of $U_i(i, p)$ and $U_i(j, p)$ are, we do not lose any generality by representing the different types of price externalities in an additively separable fashion. We also remark that in the case without price externalities i.e. $f_1 = f_2 = g_1 = g_2 = 0$, $U_i(i, p) = v_i - p$

⁵We do not consider the efficiency issue in this paper.

and $U_i(j, p) = 0$ as in the standard case.

We consider two auction formats, the first-price auction and the second-price auction. In both auction formats, each bidder submits simultaneously a bid $b \geq 0$ and the one who submits the highest bid obtains the good. In the first-price auction, the winner pays the amount of his bid. In the second-price auction, he pays the second highest bid which reduces here to the bid of his opponent.

Whatever the auction format is, if both bidders submit the same bid, b , the price paid is b and bidder i obtains the good if $v_i - f_i(b) > v_j - f_j(b)$.⁶ If $v_1 - f_1(b) = v_2 - f_2(b)$, the seller flips a fair coin to choose the winner.

We make the following assumptions. For $i = 1, 2$:

- A1. $U_i(i, 0) > 0$.
- A2. $U_i(i, p), U_i(j, p)$ are continuous and differentiable functions of p .
- A3. for $p \geq 0$, $\frac{\partial U_i(i, p)}{\partial p} < 0$.
- A4. for $p \geq 0$, $\frac{\partial U_i(i, p)}{\partial p} < \frac{\partial U_i(j, p)}{\partial p}$.
- A5. $\nexists p$ such that $U_i(i, p) = U_i(j, p)$ and $U_j(j, p) = U_j(i, p)$ (genericity-assumption).

With the notations we introduced, this can be written:

- A1'. $v_i > 0$.
- A2'. f_i, g_i are continuous and differentiable.
- A3'. for $p \geq 0$, $g'_i < 1$.
- A4'. for $p \geq 0$, $f'_i > -1$.
- A5'. $\nexists p$ such that $v_i - p = f_i(p)$ and $v_j - p = f_j(p)$.

⁶In this kind of situation, the standard hypothesis is that the limit of the discrete case is to allocate the good to bidder i if $v_i > v_j$. Here, what is important for a bidder is not his v_i but his utility difference for his obtaining the good or not $U_i(i, p) - U_i(j, p) = v_i - p - f_i(p)$. Comparing the values of this formula between the two bidders is equivalent to a comparison between $v_1 - f_1(p)$ and $v_2 - f_2(p)$. Hence, the tie-breaking rule.

Assumption A1 is equivalent to a strict preference for buying the good rather than leaving it to the other bidder when the price is equal to zero. Assumptions A3 and A4 suggest some limits to bidders' altruism. They can be interpreted as follows. A3: Bidders have a strict preference for paying the lowest possible price, a limit to the altruism in the direction of the seller. A4: In the neighborhood of any price, for both bidders, the marginal disutility of paying ε more is always strictly higher than the marginal disutility of the other bidder's paying ε more. This is a limit now to the altruism in the direction of the other bidder. A2 and A5 are technical assumptions.

Notation: $v = (v_1, v_2)$, $f = (f_1, f_2)$ and $g = (g_1, g_2)$.

We only consider equilibria with pure and undominated strategies. A strategy is a bid $b \geq 0$ and an equilibrium is a couple (b_1, b_2) .

Eventually, let us define (i, p) , with $i \in \{1, 2\}$ and $p \in R^+$, as an outcome of the auction. An outcome (i, p) is enforceable if and only if there exists an equilibrium of the auction such that the good is allocated with probability 1 to bidder i for the price p . By extension, the price p is enforceable if and only if, there exist an i such that (i, p) is an enforceable outcome. The allocation i is enforceable if and only if there exists a p such that (i, p) is an enforceable outcome.

All the proofs are in the appendix.

3 Illustrations

In this section, we provide some motivations for looking at price externalities. We develop two examples, one with IIPE and the other with IDPE. Through these examples, we also show that our model allows to represent situations with price externalities and that, in standard cases, assumptions A1-A5 are satisfied.

Example 1 IIPE. *A good is auctioned by a charitable organization to either bidder 1 or 2. The value of the good for bidder i is V_i with $V_1 < V_2$. Bidder i derives a specific extra utility $u_i(p)$ when the organization receives an amount of money p with u_i continuously differentiable. For $i = 1, 2$, we have $u_i(0) = 0$ and $0 < u'_i < 1$. We can represent agents as if they were maximizing the*

following utility functions:

$$\begin{aligned} U_1(1, p) &= V_1 - p + u_1(p) \quad \text{and} \quad U_1(2, p) = u_1(p) \\ U_2(1, p) &= u_2(p) \quad \text{and} \quad U_2(2, p) = V_2 - p + u_2(p). \end{aligned}$$

Which can also be represented by:

$$v = (V_1, V_2), \quad f_1(p) = f_2(p) = 0, \quad g_1(p) = u_1(p) \quad \text{and} \quad g_2(p) = u_2(p).$$

Both bidders, conditional on losing, prefer the price to be high since $0 < u'_1, u'_2$. Assumptions A1-A5 are verified, and, more specifically, assumption A3 is verified because $u'_1, u'_2 < 1$.

Example 2 IDPE. Two risk-neutral bidders, bidder 1 and 2, are competing in two sequential auctions, first for good A and then for good B. The valuations for both goods are, respectively: $V_A^1 = 70$, $V_B^1 = 100$, $V_A^2 = 80$, $V_B^2 = 100$. Bidder 2 has a strict budget constraint of 100 and bidder 1 has no budget constraint. Good A is sold at date $t = 1$. At date $t = 2$, with a probability β , good B is sold. With a probability $1 - \beta$, it is not sold. After the first auction, before knowing if the second auction will take place or not, the utility that bidder 1 expects to derive from the second auction⁷ is βq , q being the money spent by bidder 2, in the first auction. By backward induction, we can apply our model to the auction of good A. At date $t = 1$, expected utilities depending on the allocation of good A can be written as follows:

$$\begin{aligned} U_1(1, p) &= 70 - p \quad , \quad U_1(2, p) = \beta \min(p, 100) \\ U_2(1, p) &= 0 \quad \text{and} \quad U_2(2, p) = 80 - p \end{aligned}$$

Which can also be represented by:

$$v = (70, 80), \quad f_2(p) = g_1(p) = g_2(p) = 0 \quad \text{and} \quad f_1(p) = \beta \min(p, 100)$$

Bidder 1 prefers that bidder 2 pays a high price, then f_1 is increasing in p . Assumptions A1-A5 are verified and, more specifically, assumption A4

⁷Whether it is a second-price auction or a first-price auction has strictly no incidence.

and A5 are verified because $\beta \geq 0$.

These two examples help us showing how our model allows to represent situations with price externalities. Assumptions A1-A5 do not constrain too much the analysis. We will refer to these examples in the body of our analysis.

4 Identity independent price externalities

In this section, we study how the presence of identity independent price externalities affects the outcome of an auction. We consider a setting in which there are no IDPE, $\forall x \in R^+$, $f_1(x) = f_2(x) = 0$. We observe that IIPE do not affect the outcome of the first-price auction while they generally affect the outcome of the second-price auction.

In order to simplify the presentation of our results, we assume throughout this section, without loss of generality, that $v_1 < v_2$.⁸

4.1 The first-price auction

Proposition 1 *There is a unique equilibrium of the first-price auction: both bidders submit v_1 and bidder 2 buys the good for a price v_1 .*

The equilibrium is exactly the same as in a standard framework without price externalities. Both bidders submit a bid equal to the second highest v . The bidder with the highest v obtains the good for a price equal to the second highest v . Identity price externalities do not affect bidders' equilibrium strategies. If bidders prefer the price paid to the seller to be high or low, independently from the identity of the winner, this has no impact on bidding strategies in a first-price auction.

⁸We cannot have $v_1 = v_2$ since it induces $U_1(1, v_1) = U_2(2, v_1) = g_1(v_1)$ and $U_2(2, v_1) = U_1(1, v_1) = g_2(v_1)$ which is impossible because of assumption A5. We can also rule out the possibility of an equilibrium with both bidders obtaining the good with a probability $\frac{1}{2}$. As a matter of fact, suppose that (b, b) is an equilibrium such that both bidders obtain the good with probability $\frac{1}{2}$. If $b > v_1$, since g_1 is strictly continuous, there always exists an $\varepsilon > 0$ and small enough such that bidder 1 can profitably deviate submitting $b - \varepsilon$. Now if $b < v_2$, since g_2 is strictly continuous, there always exists an $\varepsilon > 0$ and small enough such that bidder 2 can profitably deviate submitting $b + \varepsilon$.

To illustrate and understand this result, let us reconsider example 3.1, the charity sale. There, $v_1 = V_1$ and $v_2 = V_2$ with $V_1 < V_2$. In a first-price auction, at the equilibrium, bidder 2 wins the good and pays a price V_1 . This is equivalent to what would have happened without price externalities.

Ceteris paribus, both bidders would like the charity organization to receive the highest possible amount of money. However each bidder always prefers a dollar in his pocket than a dollar given to the charity organization (in accordance with assumption A4). As a result, bidders are indifferent between winning and losing the auction when the price is equal to their valuations for the good. It is a dominated strategy for bidder 1 to submit more than v_1 . Bidder 2 knows it. As a result, he can win the auction and buy the good for a price equal to v_1 , the second highest valuation, as in the standard case.

We would observe exactly the same phenomenon if bidders preferred the price to be low. In that case, the losing bidder would like the other bidder to win for a low price, below v_1 . However, if bidder 2 were to submit a bid b strictly smaller than v_1 , bidder 1 would always be better off overbidding him and obtaining the good for a price slightly higher than b .

Ex post, a bidder cares about his bid only if he wins the auction. Bidder 1 cannot credibly commit neither to submit more than v_1 nor to let bidder 2 win the auction for a price below v_1 without overbidding him. That is why IPE have no impact on the outcome of the first-price auction.

4.2 The second-price auction

In a second-price auction, with two bidders, the losing bid, by definition, determines the price. As a result, contrary to what we observed with the first-price auction, in a second-price auction, IPE may affect the losing bid and consequently the price. Before presenting a general analysis of this issue, a simple example will illustrate that point and give some intuitions about the differences between the two auction formats.

Example 3 $v = (10, 15)$ and for $p \geq 0$, $f_1(p) = f_2(p) = g_2(p) = 0$. Whatever g_1 is, the equilibrium in a first-price auction is $(10, 10)$ with bidder 2 obtaining the good.

Now, for g_1 defined as follows: $g_1(p) = \min(\frac{p}{k}, 2 - \frac{p}{k})$ with $k \in (1, 15)$, there is a unique equilibrium of the second-price auction: $(k, 15)$.

The equilibrium of the first-price auction is not affected by IIPE. In contrast, the equilibrium price of the second-price auction varies between 1 and 15 depending on the shape of IIPE. In a sense, the losing bidder chooses the price he prefers his opponent to pay in the interval between 0 and the other bidder's bid. Thus, the preferences of the losing bidder regarding the price paid do matter.

Corollary 1 *IIPE may affect the outcome of the second-price auction.*

In fact, we can be more explicit about the extent to which IIPE may have an impact on the outcome of the second-price auction.

Proposition 2 *Any enforceable price of the second-price auction lies in the interval $[0, v_2]$. Conversely, for any $(v_1, v_2) \in R_+^2$ and for any $t \in [0, v_2]$, there always exist a couple (g_1, g_2) such that t is an enforceable price.*

Corollary 2 *In presence of identity independent price externalities, the first-price auction and the second-price auction are not equivalent.*

In a second-price auction, the equilibrium price is not always equal to v_1 . Depending on the shape of price externalities, it may have any value in the interval $[0, v_2]$. In the second-price auction, compared to the first-price auction, the losing bidder has an extra means of action. He can choose the price he prefers his opponent to pay between zero and the bid of his opponent. The second price auction allows the losing bidder to adjust more precisely his bid to his true motivations, hence the sensitivity of this auction format to price externalities.

v_1 and v_2 are no longer a sufficient statistic to determine the equilibrium of the auction. For more precise results regarding the price and the allocation, we must put new constraints on the structure of price externalities. That is why we propose the following assumption that we will consider as verified for the remaining part of this section:

- B1. g_1 and g_2 are strictly monotonic.

Assumption B1 can be interpreted as follows. Independently from the amount of money that he spends, a bidder prefers the price to be the lowest possible or the highest possible. It excludes cases in which a bidder strictly prefers the price to be as high as possible provided that it is lower than $K > 0$ while he prefers it to be the lowest possible if it is higher than K . In this context, we obtain more easily interpretable results.

Proposition 3 *If g_1 and g_2 are both strictly increasing, there is a unique equilibrium: (v_2, v_2) . Bidder 2 obtains the good for a price v_2 .*

If g_1 and g_2 are both strictly decreasing, there is a unique enforceable price: 0. $(2, 0)$ is always an enforceable outcome and $(1, 0)$ is an enforceable outcome if and only if $v_2 - v_1 + g_2(v_1) \leq 0$.

If g_1 is strictly decreasing and g_2 is strictly increasing, $(2, 0)$ is the only enforceable outcome.

If g_1 is strictly increasing and g_2 is strictly decreasing, $(1, 0)$ is always an enforceable outcome and $(2, t)$ is an enforceable outcome if and only if $v_2 - t + g_2(t) \geq 0$ and $t \geq v_1$. There are no other enforceable outcomes.

In order to provide a clearer representation of the results stated in proposition 3, we gathered them in table 1.

$v_1 < v_2$	g_2 strictly increasing	g_2 strictly decreasing
g_1 strictly increasing	$(2, v_2)$	$(1, 0)$ $(2, t)$ iff $v_2 + g_2(t) \geq t$ and $t \geq v_1$
g_1 strictly decreasing	$(2, 0)$	$(2, 0)$ $(1, 0)$ iff $v_2 - v_1 + g_2(v_1) \leq 0$

Table 1. Enforceable prices in the second-price auction for different specifications of g_1 and g_2 .

If both bidders prefer the price to be high, there is a unique equilibrium. Bidder 2 wins the good and pays v_2 . Bidder 1 makes bidder 2 pay the highest possible price for which he prefers buying the good rather than leaving it to his opponent. Thanks to the presence of IIPE, the seller sells the good for the highest amount that he can extract from one bidder. On the other hand, if both bidders prefer the price to be low, the equilibrium price is always equal to zero.

In fact, zero is an enforceable price in many cases. When at least one bidder prefers the price to be low, there always exists an equilibrium such that the price is equal to zero. As a matter of fact, conditional on losing the auction, the best thing to do for a bidder who prefers the price to be low is to bid zero. Therefore, at the equilibrium, if the losing bidder prefers the price to be low, he does bid zero. Since the other bidder knows it, he can take advantage of it and win the auction for a low price. We illustrate this point through the following example.

Example 4 *Two bidders compete in a second-price auction. $v = (5, 20)$, for $p \geq 0$ $f_1(p) = f_2(p) = 0$, $g_1(p) = \frac{p}{10}$ and $g_2(p) = -\frac{p}{10}$. $v_1 \lll v_2$, however $(20, 0)$ is an equilibrium.*

If bidder 1 loses the auction, he strictly prefers bidder 2 to pay a high price. Thus, submitting an extremely high bid is not a dominated strategy for him. He can credibly threaten bidder 2 with submitting a high bid. Bidder 2 knows that, at the equilibrium, bidder 1 does submit a high bid. His best response is to bid zero as he strictly prefers the price to be the lowest possible when he loses the auction.

Bidder 1, the bidder with the lowest valuation can obtain the good for a price zero in the two following situations: (1) Bidder 1 prefers the price to be high and bidder 2 prefers the price to be low. (2) They both prefer the price to be low and bidder 2 prefers that bidder 1 obtains the good for a price zero rather than obtaining it for a price v_1 (v_1 is not *too small* compared to v_2 and bidder 2 *cares enough* about the price being low).

Thus, while IIPE do not affect the first-price auction, they play a key role in a second-price auction. Depending on the shapes of the IIPE, the equilibrium price varies between zero and the highest valuation. Besides, most of the time, the equilibrium price is equal to one of the extremum, zero and v_2 .

This difference between the two auction formats is due to the structure of the second-price auction. As we mentioned it, in a first-price auction, only a winning bid affects the outcome of the auction. In a second-price auction, because of the structure of the format, a losing bid may have a direct impact on the price. Bidders have an extra leverage. When they lose

the auction, they fix the price with their bids. In a standard framework, this has strictly no incidence, since bidders do not care about the price paid by their opponents. Here, bidders do care about the price paid even when they lose and do not pay it. A losing bidder uses this extra leverage with its suitability. Hence, the extremum equilibrium prices.

5 Identity dependent price externalities

In this section, we study the impact of identity dependent price externalities on the outcome of the auction. We consider a setting in which there are no IIPE, $\forall x \in R^+, g_1(x) = g_2(x) = 0$. IDPE affect the outcome of both auction formats. However, the outcome is still generally different with a first-price auction and a second-price auction.

5.1 The first-price auction

First, let us remark that, in this context, v_i is not the price for which bidder i is indifferent between obtaining or not the good for sale. It represents the difference in utility for bidder i between obtaining the good at a price zero and leaving it to the other bidder for a price zero. With the addition of identity dependent price externalities, v_i is no longer the price for which bidder i is indifferent between his buying or the other bidder's buying the good. That is why we introduce e_i , bidder i 's indifference price defined as follows:

$$U_i(i, e_i) = U_i(j, e_i)$$

Bidder i is indifferent between the two events: "Bidder i buys the good for a price e_i " and "Bidder j buys the good for a price e_i ".⁹ The existence and uniqueness of a strictly positive e_i follows from assumptions A1, A2 and A4. Furthermore, the genericity assumption (A5) implies that $e_1 \neq e_2$.¹⁰ Then, without loss of generality, we will assume in this section that $e_1 < e_2$. Notice

⁹In the standard case, without price externalities, this gives us $e_i = v_i$.

¹⁰This allows us to rule out the possibility of an equilibrium with both bidders obtaining the good with a probability $\frac{1}{2}$. As a matter of fact, suppose that (b, b) is an equilibrium such that both bidders obtain the good with probability $\frac{1}{2}$. $e_1 \neq e_2$, then $\exists i$ such that $e_i \neq b$.

that it follows from A4 that for $p < e_i$, bidder i prefers buying the good and for $p > e_i$, he prefers leaving it to bidder j . The equilibrium of the first-price auction derives directly from the status of e_1 and e_2 .

Proposition 4 *If $e_1 < e_2$, there is a unique equilibrium of the first-price auction: both bidders submit e_1 and bidder 2 buys the good for a price e_1 .*

Contrary to what we observed with identity independent price externalities, identity dependent price externalities have an impact on the outcome of the auction. The equilibrium depends on the indifference prices, functions of the price externalities. The bidder with the highest indifference price wins the auction and pays the indifference price of his opponent. This equilibrium derives from two constraints. First, it is a dominated strategy for bidders to submit more than their indifference prices. Second, the winning bid cannot be lower than the indifference price of the loser. Otherwise, the loser could profitably overbid it. Indifference prices play the part that valuations play in a standard setting. In fact, in any context, what really matters is the price for which bidders are indifferent between winning and losing the auction. In the absence of IDPE, with quasi-linear utility functions, this indifference price is equal to the utility of a bidder if he obtains the good for free. That is why, these two notions are usually considered as equivalent. Here, there is a difference between these two notions. As a result, we see more clearly that only indifference prices matter.

Now, since IDPE affect the equilibrium of the first-price auction, we can examine how a change in these IDPE modifies the equilibrium.

Corollary 3 *If \bar{f}_1 and \underline{f}_1 are such that for $p > 0$, $\bar{f}_1(p) > \underline{f}_1(p)$, then, for any v, f_2 ¹¹, the equilibrium price of $\langle v, (\bar{f}_1, f_2) \rangle$ is lower than the equilibrium price of $\langle v, (\underline{f}_1, f_2) \rangle$.*

In a first-price auction, if $b < e_i$, then $v_i - b + g_i(b) > f_i(b) + g_i(b)$ and, by continuity, $\exists \varepsilon > 0$ small enough such that bidder i is better off bidding $b + \varepsilon$. If $b > e_i$, as $v_i - b + g_i(b) < f_i(b) + g_i(b)$, bidder i is better off bidding e_i .

In a second price auction, if $b < e_i$, then $v_i - b + g_i(b) > f_i(b) + g_i(b)$ and bidder i is better off bidding e_i . If $b > e_i$, as $v_i - b + g_i(b) < f_i(b) + g_i(b)$, by continuity, $\exists \varepsilon > 0$ and small enough such that bidder i is better off bidding $b - \varepsilon$.

¹¹As we do not want to lose any generality, we keep $\bar{e}_1 < e_2$ but we do not impose that $\underline{e}_1 < e_2$.

If for any $p > 0$, $f_1(p)$ increases, it means that the utility bidder 1 derives if bidder 2 buys the good for a price p is higher. Therefore, he is less eager to win the auction since his utility is higher if he loses the auction. His indifference price is lower and he submits a lower bid. Bidder 2 takes these elements into account and submits a lower winning bid. This result is reminiscent of what was observed with fixed allocative externalities (see Jehiel and Moldovanu (1996)). In that case also, with two bidders, the larger the externality that the loser derives conditional on losing, the lower the final price.

To illustrate this result, let us reconsider example 2 in which $v = (70, 80)$, $f_2(p) = g_1(p) = g_2(p) = 0$, $f_1(p) = \beta \min(100, \beta)$. We obtain that $e_1 = \frac{70}{1+\beta}$ and $e_2 = 80$.

At the equilibrium (see Proposition 4), the price paid is $\frac{70}{1+\beta}$ which is indeed a decreasing function of β . If β increases, it is more important for bidder 1 that bidder 2 buys good A for a high price as it becomes more and more likely that the second auction will take place. However, the equilibrium price goes in the opposite direction. The larger β is, the smaller is the price paid by bidder 2 for good A. At the extreme, if β goes to 1, bidder 2 pays 35 for good A while if β goes to 0, bidder 2 pays 70 for good A.

For higher values of β , it is indeed more important for bidder 1 that bidder 2 spends a higher fraction of his budget on the first auction. For any additional dollar spent by bidder 2 in the first auction, the expected gain of bidder 1 in the second auction increases by β dollar. However, this gain exists also if the price is low. Even for a low bid submitted by bidder 2, bidder 1 prefers that bidder 2 obtains the good and spends this amount of money. This second effect dominates. The larger β is, the less credible is bidder 1 if he threatens bidder 2 with submitting a high bid. This would be a dominated strategy because bidder 1 does not want to win the auction unless the price is extremely low. In the last analysis, bidder 1 submits a small bid and bidder 2 submits the smallest necessary bid to overbid him and win the auction. Thus, the equilibrium price of the first auction is decreasing in β .

5.2 The second-price auction

Some elements of the analysis of identity independent price externalities remain true with identity dependent price externalities. In a second-price auction, losing bids directly affect the price. Hence, the equilibrium varies strongly according to the losing bidder preferences regarding the price paid by his opponent.

Even though identity independent price externalities affect the first-price auction, the outcome of the first-price auction and the outcome of the second-price auction still differ in presence of IDPE.

Proposition 5 *Any enforceable price of the second-price auction lies in the interval $[0, e_2]$. Conversely, for any $(t, x_1, x_2) \in R_+^3$ such that $x_1 < x_2$ and $t \leq x_2$, there always exist a couple $\langle v, f \rangle$ such that $(e_1, e_2) = (x_1, x_2)$ and t is an enforceable price.*

Corollary 4 *In presence of identity dependent price externalities, the first-price auction and the second-price auction are not equivalent.*

As in the identity independent price externalities case, with this degree of generality, we cannot obtain more than a higher bound on the equilibrium prices. The equilibrium price may have any value in the interval $[0, e_2]$. Notice that in the second-price also e_1 and e_2 plays the role of v_1 and v_2 . With identity dependent price externalities, whatever the auction format is, what really matters are the indifference prices and not the vector (v_1, v_2) .

However, indifference prices are not a sufficient statistic to determine the equilibrium of the auction. For more precise results regarding the equilibrium price and allocation, we must put more constraints on the structure of price externalities. That is why we propose the following assumption that we will consider as verified for the remaining part of this section:

- B2. f_1 and f_2 are strictly monotonic.

Each bidder is either *benevolent* or *malevolent* towards the other bidder. Either he prefers his opponent to pay the lowest possible price or he prefers him to pay the highest price. In this restricted framework, we are able to give a more precise description of the shape of the equilibria.

Proposition 6 *If f_1 and f_2 are both strictly increasing, there is a unique equilibrium: (e_2, e_2) . Bidder 2 obtains the good for a price e_2 .*

If f_1 and f_2 are both strictly decreasing, there is a unique enforceable price: 0. $(2, 0)$ is always an enforceable outcome and $(1, 0)$ is an enforceable outcome if and only if $v_2 - e_1 \leq 0$.

If f_1 is strictly decreasing and f_2 is strictly increasing, $(2, 0)$ is the only enforceable outcome.

If f_1 is strictly increasing and f_2 is strictly decreasing, $(1, 0)$ is always an enforceable outcome and $(2, t)$ is an enforceable outcome if and only if $t \in [e_1, v_2]$. There are no other enforceable outcomes.

Equilibria have the same qualitative characteristics as with identity independent price externalities. Equilibrium prices are as extreme except that e_2 replaces v_2 (e_2 is higher than v_2 if f_2 is decreasing and lower than v_2 if f_2 is increasing). If both bidders are mutually benevolent, the equilibrium price is always zero and if they are mutually malevolent, the bidder with the highest indifference price wins the good and pays his indifference price. If bidder i is benevolent towards bidder j and bidder j is malevolent towards bidder i , for any values of e_i and e_j , there always exists an equilibrium in which bidder j obtains the good for a price zero. Bidder j can always turn the benevolence of bidder i to his advantage. In fact, with both type of price externalities, the second-price auction exacerbates the effect of price externalities.

Now, since identity dependent price externalities affect both auction formats, we can compare their impact on the equilibria of the two auction formats. At first glance, what is the most striking is the difference between the extreme equilibrium prices in the second-price auction and the intermediate values of the equilibrium price in the first-price auction. Besides, in the second-price auction, unlike in the first-price auction case, if the loser prefers that the winner pays a high (resp: low) price, the winner does not pay a lower (resp: higher) price, quite the reverse. If the loser prefers the price to be high (resp: low), the price is actually at its maximum (resp: minimum). This is the complete reversed as compared to what we observed with the first-price auction in corollary 3.

We can interpret this difference between the two auction formats in terms of credibility. In both auction formats, one of the two bidders would like to

be able to commit but he is not. For instance, if f_1 is increasing, in the first-price auction, bidder 1, the losing bidder, would like to commit to a bid of e_2 . That way, he would force bidder 2 to bid e_2 and to pay e_2 . In the same case, with a second-price auction, bidder 2 would like to commit to a bid of e_1 . That way, he would force bidder 1 to bid e_1 which would allow bidder 2 to obtain the good for the price e_1 . However, none of these commitments are credible. They require bidders playing dominated strategies. Thus, the ruling out of dominated strategies constrains the losing bidder more in the first-price auction and the winning bidder more in the second-price auction. The burden of the credibility is on a different bidder in each auction format.

6 Conclusion

In this conclusion, we summarize and arrange our results, derive recommendations for specific applications and discuss the hypotheses of our model.

6.1 Summary

We found that price externalities have two major effects.

First, they change the price for which bidders are indifferent between winning and losing the auction. It is no longer equivalent to the utility level of a bidder if he obtains the good for a price zero. This effect only arises with IDPE and has an impact on the equilibrium of both auction formats.

Second, they change the preferences of losing bidders. By definition, this is true with both types of price externalities. However, this consequence of price externalities only affects the second-price auction. Besides, the impact of the preferences of a bidder when he loses the auction is qualitatively the same whether they are due to identity dependent or to identity independent price externalities.

That is why, in order to provide a synthetic representation of our results, we define h_i as $h_i = f_i + g_i$ and gather in table 2 all the enforceable prices for setting with both type of price externalities.

Auction format	Specifications	Enforceable price
First-price	-	e_1
Second-price	-	$[0, e_2]$
Second-price	h_1 and h_2 strictly \nearrow	e_2
Second-price	h_1 and h_2 strictly \searrow	0
Second-price	h_1 stric. \searrow and h_2 stric. \nearrow	0
Second-price	h_1 stric. \nearrow and h_2 stric. \searrow	$0 \cup [e_1, e_2]$

Table 2. Enforceable prices depending on the auction formats and the specifications of h with $e_1 < e_2$.

The essence of our results can be summarized with the four following points:

1. The two auction formats are not revenue equivalent. The difference between equilibrium prices can be large.
2. The equilibrium of the first-price auction does not depend on identity independent price externalities while they do affect the equilibrium of the second-price auction.
3. The burden of credibility is on a different bidder for each auction format. On the loser in the first-price auction, on the winner in the second-price auction. Consequence: in a first-price auction, a losing bidder, if he prefers the price to be the lowest possible, cannot credibly commit to bid less than his indifference price, e_1 . Therefore, the price is e_1 . In a second-price auction, he will bid 0 which will be the final price.¹²
4. Consequence of the former point: the second-price auction magnifies the effect of price externalities while the first-price auction tempers them. The second-price auction which was designed partly in view of his robustness properties (it relies on dominant strategies) is more sensitive than the first-price auction to the introduction of price externalities.

¹²We introduced, in the former section, the symmetric case: the loser prefers the price paid by his opponent to be high. With a second-price auction, the price is e_2 and with a first-price auction, the price is e_1 .

6.2 Recommendations and applications

Across the paper we made the underlying assumption that the seller does not know the exact value of $\langle v, f, g \rangle$. Nevertheless, in general, the seller has, at least, a qualitative perception of the kind of price externalities bidders are facing. Let us consider that he perceives two polar cases. In the first case, the most favorable, price externalities are increasing in the price. In the second case, the most unfavorable, price externalities are decreasing in the price. Thanks to the results we obtained, we can now make some recommendations to the seller. In the first case, in order to take advantage of price externalities, he should choose the second-price auction. In the second case, in order to protect himself from undesirable effects of price externalities, he should choose the first-price auction. Thus, there is no auction format which the seller should choose in general when facing price externalities. The better choice crucially depends on the type of price externalities at stake.

To illustrate these results, we apply them to the situations that we mentioned in the introduction.

In both the charity auction and the Canal+ case, bidders, or at least one of them, have a specific preference for high prices, independently from the identity of the winner. These are typical cases of what we defined as increasing IIPE. The extra motivations do not affect bidders' strategies in a first-price auction. On the other hand, bidders submit higher bids in a second-price auction because of these considerations. As a result, the LNF or a charity organization should choose a second-price auction. This result holds true for the charity organization even if it does not know to which extent bidders care about its funding.

Now, consider the bidder who prefers that the seller raises the lowest amount of money because of future interactions and budget constraints, a case of decreasing IIPE. The first-price auction is not sensitive to the presence of these specific motivations. In contrast, with a second-price auction, the concerned bidder lowers his bid. The seller may ignore the exact motivations of the bidder, how important it is for this bidder to budget-constrain him.¹³ However, the seller is always better off with a first-price auction.

¹³It depends on the bidder anticipations regarding future interactions with the seller.

Finally, when a bidder owns a fraction of another bidder, the seller faces decreasing IDPE. In this case, the outcome of the two auction formats is potentially affected by the presence of these extra motivations. With both auction formats, the concerned bidder may submit lower bids. However, the bid reduction is weaker with a first-price auction than with a second-price auction. Therefore, whatever the size of the toehold is, the seller obtains a higher revenue with a first-price auction.

Our results can be applied to any auction setting with price externalities the same way.

6.3 Discussion and possible extensions

Several limits of our contribution need to be mentioned and discussed.

The first limit of this work concerns our choice to restrain to the first-price and the second-price auction. We made this choice because of the wide use of this two formats.¹⁴ In practice, implementing a new auction format is a costly and long process. Bidders need to learn how to play the equilibrium of this new auction format. Besides, studying these two auction formats allows to compare our results to those already obtained in the auction theory literature which mainly focused on these two formats. Despite these remarks, a natural further step in our research would consist in designing an optimal mechanism in this context. This study would require to define what the seller knows about bidders' preferences and to make some more assumptions.¹⁵

Another limit of our contribution relies on the symmetric information hypothesis. Because of the assumption of common knowledge of preferences among bidders, we obtain extreme results with, in the second-price auction, equilibria varying discontinuously from zero to the highest indifference price. These equilibrium prices should not be considered as a precise description of what the price would be if such an auction was to be ran but rather interpreted in qualitative terms. This hypothesis regarding information distribution among bidders allows to better isolate the specific effects of price

¹⁴Or the dynamic equivalent formats: the descending and the ascending auction.

¹⁵For instance, $\sum_{i=1}^n g_i(x) < 1$ for any $x \geq M$ with $M \in R^+$. Otherwise bidders all together may prefer to pay an infinite amount to the seller

externalities in an auction context. For a more detailed analysis of a particular situation, we may need to consider different assumptions regarding information distribution.

The last limit of the model that we will mention regards the choice to restrain to the two-bidders case. One may wonder to which extent our results can be extended to a setting with more than two bidders. In order to convince that our results do not crucially rely on this assumption, an interesting extension of the model would consist in studying a setting with $n > 2$ bidders.

A first step would be to consider situations in which the utility of a losing bidder depends on the price paid by the winner but not on which other bidder obtains the good. In such a case, the utility of a bidder i could be defined just the same as we did by a 3-uplet (v_i, f_i, g_i) .¹⁶ We should also assume that assumptions A1'-A5' are verified for any $i, j^2 \in \{1, 2, \dots, n\}$. e_i defined by $v_i - e_i = f_i(e_i)$ is still the price for which bidder i is indifferent between losing and winning the auction. We should also rearrange bidders so that $i < j$ if and only if $e_i < e_j$. We derive results close to what we obtained with two bidders.¹⁷

- There is a unique enforceable outcome of the first-price auction: (n, e_{n-1}) .
- In a second-price auction:
 - For any $(x_1, x_2, \dots, x_n, t)$ such that $x_1 < x_2 < \dots < x_n$ and $t \leq x_n$, there always exist a $\langle v, f, g \rangle$ such that $(e_1, e_2, \dots, e_n) = (x_1, x_2, \dots, x_n)$ and t is an enforceable price. Conversely, for any $\langle v, f, g \rangle$, there cannot exist an enforceable price strictly higher than e_n .
 - If h_n is not decreasing and if there exists an $i \in \{1, 2, \dots, n-1\}$ such that h_i is strictly increasing, there is a unique enforceable outcome, (n, e_n) .

¹⁶In that case, we define v, f and g by $v = \{v_1, v_2, \dots, v_n\}$, $f = \{f_1, f_2, \dots, f_n\}$ and $g = \{g_1, g_2, \dots, g_n\}$.

¹⁷We use h_i rather than f_i and g_i in order to obtain a shorter presentation of the results.

- If all the h_i are strictly decreasing, the only enforceable price is zero.
- If $\exists(i, j) \in \{1, 2, \dots, n\}^2$ with $i < j$ such that h_i and h_j are strictly increasing, then any enforceable price p must satisfy $p \geq e_j$.

In the first-price auction, at the equilibrium, the good is always allocated to the bidder with the highest indifference price for a price equal to the second highest indifference price. Thus, as in the two-bidders case, in a first-price auction, indifference prices have the same role as valuations in a context without price externalities. IIPE have still no impact on the equilibrium.

This result differs from what happens with allocative externalities that depend on the identity of the winner but not on the price. In that case, with more than two bidders, there may be more than one enforceable outcome with the first-price auction (see Jehiel and Moldovanu (1996)). Here, there is a unique enforceable outcome.

In the second-price auction, as in the two-bidders case, the situation is slightly more complex. The equilibrium depends on both types of price externalities. However, these results also are not qualitatively different from what we observed in the two-bidders case. Depending on the shape of price externalities, the equilibrium may take any value between zero and the highest indifference price. When bidders prefer the price to be high conditional on losing, the equilibrium price is equal to the highest indifference price. When they prefer the price to be low, the equilibrium price is equal to zero. Thus, in the second-price auction also, the results we obtained seem to be robust to an increase of the number of bidders.

Finally, another extension of our model that would be worthwhile is to consider a setting with $n > 2$ bidders whose utilities conditional on losing may depend on both the price paid and the identity of the winner. The study of this broader framework awaits future research.

7 Appendix

7.1 Proof of proposition 1

First, let us remark that (v_1, v_1) with bidder 2 obtaining the good is an equilibrium since no bidder can profitably deviate. Now, let us prove that there are no other equilibria.

We may first note that there cannot exist an equilibrium with the two bidders submitting different bids. In that case, the bidder submitting the highest bid could profitably deviate submitting a lower bid. Therefore, at the equilibrium, the two bidders submit the same bid, b .

Suppose that $b < v_1$. With our tie-breaking rule, bidder 2 obtains the good. By continuity of g_1 , there always exists an $\varepsilon > 0$ small enough so that: $g_1(b) < g_1(b + \varepsilon) + v_1 - b - \varepsilon$. Therefore, bidder 1 can strictly improve his utility submitting $b + \varepsilon$ rather than b and (b, b) cannot be an equilibrium.

Now, for bidder 1 submitting a bid \tilde{b} higher than v_1 is a strategy dominated by bidding v_1 . As a matter of fact, if $b_2 < v_1$, bidder 1 pays less if he bids v_1 rather than \tilde{b} and he prefers paying less (see assumption A3). If $b_2 > \tilde{b}$, submitting v_1 or \tilde{b} is equivalent for bidder 1. If $b_2 \in [v_1, \tilde{b})$, since $\forall t \in [v_1, \tilde{b})$, $g_1(t) > v_1 - \tilde{b} + g_1(\tilde{b})$ (because of assumption A3), bidder 1 is better off submitting v_1 rather than \tilde{b} .

$b \geq v_1$ and $b \leq v_1$, thus, there are no other possible equilibria than (v_1, v_1) . Q.E.D.

7.2 Proof of proposition 2

Suppose that there exists a $p > v_2$ such that p is an enforceable price. This means that there exist an equilibrium in which bidder i obtains the good for a price p . Since $p > v_2 > v_1$, because of the continuity of g_1 and g_2 , for $i = 1, 2$, there always exists an $\varepsilon > 0$ such that $g_i(p - \varepsilon) > g_i(p) + v_i - p$. Thus, bidder i can profitably deviate submitting $p - \varepsilon$ and $p > v_2$ cannot be an enforceable price.

Now, for any $(v_1, v_2) \in R_+^2$ and for any $t \in [0, v_2]$, we can build a g such that t is an enforceable price. As a matter of fact, if g_1 and g_2 are defined as follows: $\forall x \in R$, $g_2(x) = 0$, $\forall x \leq t$, $g_1(x) = \frac{x}{5}$ and $\forall x > t$, $g_1(x) = \frac{x-2t}{5}$,

(t, v_2) is an equilibrium and t is an enforceable price.

Q.E.D.

7.3 Proof of proposition 3

We treat the four cases separately.

g_1 and g_2 strictly increasing.

First, let us remark that there cannot exist an equilibrium with the two bidders submitting different bids. In that case, the bidder submitting the lowest bid could profitably deviate submitting a bid in the opened interval between the value of the two bids. His utility would be higher since he prefers his opponent to pay a higher price. Thus, at the equilibrium, the two bidders submit the same bid b .

Now, for bidder 2, submitting a bid lower than v_2 is dominated by a strategy consisting in submitting v_2 since g_2 is increasing. At last, suppose that $b > v_2$. With our tie-breaking rule, bidder 2 obtains the good. By continuity of g_2 , there always exist an $\varepsilon > 0$ such that: $g_2(b) + v_2 - b < g_2(b - \varepsilon)$. Then, bidder 2 could strictly improve his utility submitting $b - \varepsilon$ rather than b and (b, b) cannot be an equilibrium.

Therefore, (v_2, v_2) is the only possible equilibrium and it is an equilibrium since no bidder can profitably deviate.

g_1 and g_2 strictly decreasing.

The equilibrium price cannot be strictly positive otherwise the losing bidder could profitably deviate submitting 0, since he prefers the price to be low. $(0, v_2)$ is always an equilibrium, thus $(2, 0)$ is always an enforceable outcome. Now, for $t > 0$, $(t, 0)$ is an equilibrium with bidder 1 obtaining the good if and only if $g_2(t) + v_2 - t \leq 0$ (otherwise bidder 2 could profitably deviate) and $t \leq v_1$ (otherwise, bidding t is a dominated strategy for bidder 1). Since $g_2' < 1$, this means that there exists an equilibrium with bidder 1 obtaining the good for a price if and only if $g_2(v_1) + v_2 - v_1 \leq 0$.

g_1 strictly decreasing and g_2 strictly increasing.

Since g_2 is increasing, it is dominated for bidder 2 to submit a bid lower than v_2 . Now, for any bid higher than v_2 , bidder 1 has a unique best response: submitting zero. Therefore, there is a unique enforceable outcome $(2, 0)$.

g_1 strictly increasing and g_2 strictly decreasing.

$(v_2 + 1, 0)$ is an equilibrium. The strategy of bidder 1 is not dominated since he prefers the price to be high. Thus, $(1, 0)$ is an enforceable outcome. Besides, no equilibrium can exit with bidder 1 obtaining the good for a strictly positive price otherwise bidder 2 could profitably deviate submitting zero. Now, for any equilibrium in which bidder 2 obtains the good, bidder 1 submits the same bid as bidder 2 since bidder 1 prefers the price to be high. Thus, for $t > 0$, $(2, t)$ is an enforceable outcome if and only if $v_2 - t + g_2(t) \geq 0$ (otherwise bidder 2 could profitably deviate submitting 0) and $g_1(t) \geq v_1 - t + g_1(t)$ equivalent to $t \geq v_1$ (otherwise bidder 1 could profitably deviate submitting a bid higher than t). Q.E.D.

7.4 Proof of proposition 4

(e_1, e_1) with bidder 2 obtaining the good is an equilibrium since no bidder can profitably deviate. Now, let us prove that there are no other equilibria.

We may first note that there cannot exist an equilibrium with the two bidders submitting different bids. The bidder submitting the highest bid could profitably deviate submitting a lower bid, higher than his opponent's. Then, at the equilibrium, the two bidders submit the same bid b .

Suppose that $b < e_1$. With our tie-breaking rule, bidder 2 obtains the good. Since, for any price lower than e_1 , bidder 1 always prefer winning the auction rather than losing it, there always exists an $\varepsilon > 0$ small enough so that: $v_1 - b - \varepsilon > f_1(b + \varepsilon)$. Therefore, bidder 1 can strictly improve his utility by submitting $b + \varepsilon$ rather than b and (b, b) cannot be an equilibrium.

Now, for bidder 1 submitting a bid \tilde{b} higher than e_1 is a strategy dominated by bidding e_1 . As a matter of fact, if $b_2 < e_1$, bidder 1 pays less if he bids e_1 rather than \tilde{b} and he prefers paying less (see assumption A3). If $b_2 > \tilde{b}$, submitting e_1 or \tilde{b} is equivalent for bidder 1. If $b_2 \in [e_1, \tilde{b})$, since

$\forall t \in [e_1, \tilde{b})$, $f_1(t) \geq v_1 - t > v_1 - \tilde{b}$, bidder 1 is better off submitting e_1 rather than \tilde{b} .

$b \geq e_1$ and $b \leq e_1$, then, there are no other possible equilibria than (e_1, e_1) .
Q.E.D.

7.5 Proof of corollary 3

Let us define \bar{e}_1 (resp: \underline{e}_1) as the indifference price for $f_1 = \bar{f}_1$ (resp: \underline{f}_1). From proposition 4, we derive that \bar{e}_1 is the equilibrium price of $\langle v, (\bar{f}_1, f_2) \rangle$. Suppose that $\underline{e}_1 < e_2$, then the equilibrium price of $\langle v, (\underline{f}_1, f_2) \rangle$ is \underline{e}_1 . However, as for $p > 0$ $\bar{f}_1(p) > \underline{f}_1(p)$ and $v_1 - \underline{e}_1 = \underline{f}_1(\underline{e}_1)$, we derive that $v_1 - \underline{e}_1 < \bar{f}_1(\underline{e}_1)$ and $\bar{e}_1 < \underline{e}_1$. Now, suppose that $\underline{e}_1 > e_2$, equilibrium prices are e_2 and \bar{e}_1 and by definition $\bar{e}_1 < e_2$.
Q.E.D.

7.6 Proof of proposition 5

Suppose that there exists a $p > e_2$ such that p is an enforceable price. Thus, there exist an equilibrium in which bidder i obtains the good for a price p . $p > e_2 > e_1$, for $i = 1, 2$, bidder i strictly prefer losing the auction than winning it when the price is higher than his indifference price and f_i is continuous. Then, there always exists an $\varepsilon > 0$ such that $f_i(p - \varepsilon) > v_i - p$. Thus, bidder i can profitably deviate submitting $p - \varepsilon$ and $p > e_2$ cannot be an enforceable price.

Now, let us prove that for any $(t, x_1, x_2) \in R_+^3$ such that $x_1 < x_2$ and $t \leq x_2$, there exists a v and a f such that $(e_1, e_2) = (x_1, x_2)$ and t is an enforceable price. As a matter of fact, if we define f_1 and f_2 as follows: $\forall x \in R$, $f_2(x) = 0$, $\forall x \leq t$, $f_1(x) = \frac{x}{10}$, $\forall x > t$, $f_1(x) = \frac{x-2t}{10}$ and $v_1 = x_1 + f_1(x_1)$, $v_2 = x_2$, then $(e_1, e_2) = (x_1, x_2)$ and (t, v_2) is an equilibrium which means that t is an enforceable price.
Q.E.D.

7.7 Proof of proposition 6

We treat the four cases separately.

f_1 and f_2 strictly increasing.

First, let us remark that there cannot exist an equilibrium with the two bidders submitting different bids. In that case, the bidder submitting the lowest bid could profitably deviate submitting a bid in the opened interval between the value of the two bids. His utility would be higher since he prefers his opponent to pay a higher price. Thus, at the equilibrium, the two bidders submit the same bid b .

Now, for bidder 2, submitting a bid lower than e_2 is dominated by a strategy consisting in submitting e_2 since f_2 is increasing. At last, suppose that $b > e_2$. With our tie-breaking rule, bidder 2 obtains the good. For any strictly higher than e_2 , bidder 2 strictly prefers losing the auction than winning it and f_2 is continuous. Then, there always exist an $\varepsilon > 0$ such that: $v_2 - b < f_2(b - \varepsilon)$. Then, bidder 2 could strictly improve his utility submitting $b - \varepsilon$ rather than b and (b, b) cannot be an equilibrium.

Therefore, (e_2, e_2) is the only possible equilibrium and it is an equilibrium since no bidder can profitably deviate.

f_1 and f_2 strictly decreasing.

The equilibrium price cannot be strictly positive otherwise the losing bidder could profitably deviate submitting 0, since he prefers his opponent to pay the lowest possible price. $(0, e_2)$ is always an equilibrium, thus $(2, 0)$ is always an enforceable outcome. Now, for $t > 0$, $(t, 0)$ is an equilibrium with bidder 1 obtaining the good if and only if $v_2 - t \leq 0$ (otherwise bidder 2 could profitably deviate submitting e_2) and $t \leq e_1$ (otherwise, bidding t is a dominated strategy for bidder 1). This means that there exists an equilibrium with bidder 1 obtaining the good for a price zero if and only if $v_2 - e_1 \leq 0$.

f_1 strictly decreasing and f_2 strictly increasing.

Since f_2 is increasing, it is a dominated for bidder 2 to submit a bid lower than e_2 . Now, for any bid higher than e_2 , bidder 1 has a unique best response: submitting zero. Therefore, there is a unique enforceable outcome $(2, 0)$.

f_1 strictly increasing and f_2 strictly decreasing.

$(e_2 + 1, 0)$ is an equilibrium. The strategy of bidder 1 is not dominated since he prefers bidder 2 to pay the highest possible price. Thus, $(1, 0)$ is an enforceable outcome. Besides, no equilibrium can exist with bidder 1 obtaining the good for a strictly positive price otherwise bidder 2 could profitably deviate submitting zero. Now, for any equilibrium in which bidder 2 obtains the good, bidder 1 submits exactly the same bid as bidder 2 since bidder 1 prefers bidder 2 to pay the highest possible price. Thus, for $t > 0$, $(2, t)$ is an enforceable outcome if and only if $v_2 - t \geq 0$ (otherwise bidder 2 could profitably deviate submitting 0) and $f_1(t) \geq v_1 - t$ equivalent to $t \geq e_1$ (otherwise bidder 1 could profitably deviate submitting a bid higher than t). Therefore, $(2, t)$ is an enforceable outcome if and only if $t \in [e_1, v_2]$. Q.E.D.

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