On the singularity structure of differential equations in the complex plane

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Date

Abstract

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In this dissertation the structure of singularities in the complex plane of solutions of certain classes of ordinary differential equations and systems of equations is studied. The thesis treats two different aspects of this topic. Firstly, we introduce the concept of movable singularities for first and second-order ordinary differential equations. On the one hand the local behaviour of solutions about their movable singularities is investigated. It is shown, for the classes of equations considered, that all movable singularities of all solutions are either poles or algebraic branch points. That means locally, about any movable singularity z_0, the solutions are finitely branched and represented by a convergent Laurent series expansion in a fractional power of $z - z_0$ with finite principle part. This is a generalisation of the Painlevé property under which all solutions have to be single-valued about all their movable singularities.

The second aspect treated in the thesis deals with the global structure of the solutions. In general, the solutions of the equations discussed in the first part have a complicated global behaviour as they will have infinitely many branches. In the second part conditions are discussed for certain equations under the existence of solutions that are globally finite-branched, leading to the notion of algebroid solutions. In order to do so, some concepts from Nevanlinna theory, the value-distribution theory of meromorphic functions and its extension to algebroid functions are introduced. Then, firstly, Malmquist's theorem for first-order rational equations with algebroid solutions is reviewed. Secondly, certain second-order equations are considered and it is examined to what types of equations they can be reduced under the existence of an admissible algebroid solution.

Contents

Chapter 1

Introduction

This thesis is concerned with a natural question in the theory of differential equations in the complex plane: What types of singularities can a local analytic solution develop when one tries to analytically continue it along some curve? An answer to this question will certainly depend on the class of equations we are considering, in particular on the order of the differential equation. One task in this thesis therefore is to list, for certain classes of equations and also systems of equations, the different types of singularities that can occur, usually described by certain series expansions, and to prove that the list is exhaustive. Although answering the initial question is of theoretical interest in complex analysis, the singularity structure of the solutions of a differential equation also plays an important role for the integrability of the equation, as was probably most prominently demonstrated by the work of Sophia Kowalevskaya [22] in the 19th century on the motion of a rigid body around a fixed point. There she showed that the existence of first integrals for the Euler equations is connected to there being a sufficiently large family of Laurent series solutions with finite principle part about every point in the complex plane. In particular, by this method she found, besides the examples of Lagrange and Euler, one other case where the equations are exactly integrable known as the Kowalevskaya top.

When considering singularities of differential equations in the complex plane we distinguish two types: *fixed* and *movable* singularities. The fixed singularities of an equation are a discrete set of points Φ at which the equation itself behaves in a non-generic way, e.g. some coefficient in the equation becomes singular. All other singularities are called movable as their position varies with the integration constants, i.e. the initial conditions of the equation, in a continuous way.

The ideas of Kowalevskaya were taken up by Paul Painlevé and his school to classify equations demanding that all solutions be single-valued about all their movable singularities, a property now known as the Painlevé property. In [40] Painlevé attempted a classification for second-order rational equations $y'' = R(z, y, y')$ with this property. His classification contained some errors and gaps which were successively fixed by Gambier [11] and Fuchs [10]. The result is a list of 50 canonical equations from which any equation with the Painlevé property in this class can be obtained by a Möbius type transformations,

$$z \to \phi(z), \quad y \to \frac{a(z)y + b(z)}{c(z)y + d(z)}.$$

Of these 50 equations all but 6 can be solved in terms of existing classical functions like solutions of second-order linear equations or elliptic functions. The remaining ones are known as the six Painlevé equations,

$$\begin{aligned}
P_I &: y'' = 6y^2 + z \\
P_{II} &: y'' = 2y^3 + zy + \alpha \\
P_{III} &: y'' = \frac{(y')^2}{y} - \frac{y'}{z} + \frac{1}{z}\left(\alpha y^2 + \beta\right) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV} &: y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2 - \alpha)y + \frac{\beta}{y} \\
P_V &: y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{z} + \frac{\delta y(y+1)}{y-1} \\
P_{VI} &: y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y' \\
&\quad + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right),
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ are arbitrary parameters. For a general set of parameters the solutions define new transcendental functions not expressible in terms of formerly known functions, called the Painlevé transcendents. In principle it is possible to apply the methods of Kowalevskaya and Painlevé to higher-order equations and systems of equations. The complexity of the treatment, however, increases enormously and a complete classification for higher-order equations has not been achieved to this date. A partial classification for the third order was carried out by Chazy [1], results for the fourth and fifth order with a special form of the right hand side were obtained by Cosgrove [4, 5].

The methods of Kowalevskaya and Painlevé only give necessary conditions for an equation to have the Painlevé property, meaning that the mere existence of Laurent series solutions with finite principle part at every point does not guarantee that all movable singularities are poles. This was demonstrated by Painlevé with the example

$$y'' = \frac{2y-1}{y^2+1}(y')^2,$$

which, although one can find a one-parameter family of Laurent series solutions at every point, has the general solution

$$y(z) = \tan\left(\log(c(z - z_0))\right),$$

having a logarithmic branch points at z_0. However, Kowalevskaya's method is an important detector for equations with the Painlevé property and if the result is affirmative the equation is said to pass the Painlevé(-Kowalevskaya) test. It thus remained to prove that the six Painlevé equations in fact have the Painlevé property. A proof for the first equation was given by Painlevé himself. It contained, however, some gaps which were only fixed in the published literature by Hinkkanen and Laine [16] in 1999, although some lecture notes by Hukuhara containing a complete proof were already circulating at the University of Tokyo in 1960 which were, however, only published in 2001 by Okamoto and Takano [38]. A proof for the fourth Painlevé equation was given in 2000 by Steinmetz [46]. Proofs of the Painlevé property for all six Painlevé equations were also given by Shimomura [42], see also the book [12].

In this thesis we are concerned with differential equations with a singularity structure more general than imposed by the Painlevé property. In particular, we study classes of equations and systems of equations the solutions of which allow for certain branching at the movable singularities. Namely, we allow for movable algebraic singularities to occur, i.e. the solutions can be expanded, locally about every point $z_0 \in \mathbf{C} \setminus \Phi$, in a Laurent series in fractional powers of $z - z_0$,

$$y(z) = \sum_{k=k_0}^{\infty} c_k (z - z_0)^{\frac{k}{n}}, \quad k_0 \in \mathbf{Z}, \quad n \in \mathbf{N}. \tag{1.1}$$

An equation for which one can find, about every point $z_0 \in \mathbf{C} \setminus \Phi$, a maximal family of formal solutions of the form (1.1), is said to pass the *weak Painlevé test*. It is a main task in this thesis to show, for the classes of equations considered, that passing the weak Painlevé test is equivalent to the fact that all movable singularities of all their solutions are either poles or algebraic branch points. For the proofs of these theorems we will use similar methods as in [16] and [42], in fact our proofs are generalisations of the proofs presented there. Some further difficulties arise when we consider certain Hamiltonian systems in section 3.1. Broadly speaking, the content of this thesis consists of two parts. The first part is the one just mentioned, dealing with the local behaviour of the solutions about their movable singularities. Although the solutions of the equations considered are finite branched about every movable singularity, globally they will in general be infinitely branched, or, expressed differently, the solutions will extend over a Riemann surface with an infinite number of sheets. This is an indicator that these equations are in general non-integrable.

The second part of the thesis is concerned with the global structure of the solutions. In particular we consider conditions for equations with solutions that are also globally finite branched, giving rise to the notion of algebroid solutions, functions that are algebraic over the field of meromorphic functions. First-order equations with algebroid solutions were studied in [24] by Malmquist. The question here is to what possible forms a differential equation can be reduced if we assume the existence of at least one sufficiently complicated meromorphic or algebroid solution. (For example, if the coefficient functions in the equation are rational, sufficiently complicated would mean a transcendental function). In this case one can apply certain tools from Nevanlinna theory, the value-distribution theory of meromorphic functions, which were not developed at the time when Malmquist wrote his first article on this topic. We will review and generalise one of Malmquist's theorems to the notion of *admissible solutions* introduced by I. Laine in [23] using Nevanlinna theory. We then give some results for certain second-order equations with algebroid solutions.

Contents of the thesis

In chapter 2 we start by introducing and explaining the notions of fixed and movable singularities for first and second-order ODEs in section 2.1. After some preliminary lemmata in section 2.2 we review some of the previous work on movable singularities of second-order ODEs in section 2.3. We then present two classes of ODEs for which we show that all their movable singularities are at most algebraic branch points. In section 2.4 we consider a class of scalar second-order equations of the form $y'' = E(z,y)(y')^2 + F(z,y)y' + G(z,y)$, which extends the results by the author in [19]. The other class of equations presented in section 3.1 consists of Hamiltonian systems with polynomial Hamiltonian $H(z,q,p)$ in the two dependent variables q and p, which was treated by the author in [18]. We review Hamiltonian systems with the Painlevé property in section 3.2. In section 3.3 we present a different method of studying the singularity structure of ODEs, the so-called *space of initial conditions* by Okamoto. We construct this space for a system of equations obtained in section 3.1.

Chapters 4 and 5 form the second part of the thesis concerned with the global branching of solutions. In chapter 4 we give a brief introduction to Nevanlinna theory, introducing the Nevanlinna functions in section 4.2 and stating the main results needed for applications to differential equations in section 4.3. We also discuss an extension of Nevanlinna theory to algebroid functions in section 4.4. In Chapter 5 we review Malmquist's results for first-order differential equations. We review one of Malmquist's theorems in his article [24] for algebroid solutions and generalise it to the notion of admissible algebroid solutions in section 5.1 by using Nevanlinna theory. In section 5.2 we prove a theorem of the type of Malmquist's theorem for certain second-order equations.

Chapter 2

ODEs with movable algebraic singularities

2.1 Fixed and movable singularities

As explained in the introduction we have to distinguish between two types of singularities: *fixed* and *movable* singularities. We now give a more accurate definition of these notions. In general, the set of fixed singularities $\Phi \subset \mathbf{C}$ is a set of points in the complex plane at which a solution of the equation may behave in a non-generic way. However, a solution may not have a singularity at all at a point in Φ.

First-order rational equations

For first-order rational equations a definition of fixed singularities was given by P. Painlevé in his Stockholm lectures [39], for discussions thereof we refer to the books by Ince [17] and Hille [15]. Suppose that in the equation

$$y' = \frac{P(z,y)}{Q(z,y)}, \tag{2.1}$$

P and Q are polynomials in y with coefficients in a certain class of functions, for example the field of algebraic functions. We suppose that the right hand side of (2.1) is in reduced terms, in particular the polynomials P and Q have no common factor.

Definition 2.1. *Let Φ_0 be the set of singular points of the coefficients of P and Q so that $D = \mathbf{C} \setminus \Phi_0$ is the largest domain on which all coefficients are analytic. The set of fixed singularities for (2.1) is defined as the union $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \Phi_3$, where*

$$\begin{aligned}
\Phi_1 =& \{\zeta \in D : Q(\zeta, y) \equiv 0\}, \\
\Phi_2 =& \{\zeta \in D : P(\zeta, \eta) = Q(\zeta, \eta) = 0 \text{ for some } \eta \in \mathbf{C}\}, \\
\Phi_3 =& \{\zeta \in D : \tilde{P}(\zeta, 0) = \tilde{Q}(\zeta, 0) = 0\}.
\end{aligned}$$

Here $\tilde{P}$ and $\tilde{Q}$ are polynomials in $u = 1/y$ such that $u' = \frac{\tilde{P}(z,u)}{\tilde{Q}(z,u)}$ where $\tilde{P}$ and $\tilde{Q}$ are again in reduced terms.

A singularity of a solution which is not in the set of fixed singularities is called a movable singularity. Painlevé showed that all movable singularities of any solution of (2.1) are algebraic, i.e. they are either poles or algebraic branch points. This means that in a cut neighbourhood of a movable singularity z_0 the solution is represented by a convergent series expansion

$$y(z) = \sum_{k=k_0}^{\infty} c_k (z - z_0)^{k/n}, \quad k_0 \in \mathbf{Z}, \quad n \in \mathbf{N}.$$

For a proof of this statement see e.g. the textbooks by Hille [15] or Ince [17].

Example. Consider the equation

$$y' = \frac{1 + y^2}{z^2},$$

which has the general solution

$$y(z) = \tan\left(c - \frac{1}{z}\right),$$

$c \in \mathbf{C}$ being the integration constant. The position of the singularity at $z = 0$ does not depend on the initial condition and belongs to the set Φ. The positions of the other singularities, located at $z = \left(c - (2k+1)\frac{\pi}{2}\right)^{-1}$ vary with c and are therefore movable.

Second-order rational equations

For second-order rational equations a description of the set of fixed singularities was given by Kimura [20]. Suppose that in the equation

$$y''(z) = \frac{P(z, y, y')}{Q(z, y, y')}, \tag{2.2}$$

P, Q are polynomials in y and y' in reduced terms. Again, let $D \subset \mathbf{C}$ be the largest domain where all coefficients of P and Q are analytic. To define the set Φ we let

$$\begin{aligned} P(z, y, y') =& \Pi_p(x, y)(y')^p + \cdots + \Pi_0(x, y), \quad (\Pi_p(z, y) \neq 0), \\ Q(z, y, y') =& K_q(z, y)(y')^q + \cdots + K_0(z, y), \quad (K_q(z, y) \neq 0). \end{aligned}$$

Under the transformation $y = 1/u$ this equation is transformed into

$$u''(z) = \frac{2(u')^2}{u} - \frac{u^2 P(z, 1/u, -u'/u^2)}{Q(z, 1/u, -u'/u^2)} = \frac{\tilde{P}(z, u, u')}{\tilde{Q}(z, u, u')}, \tag{2.3}$$

where we have expanded the fraction such that $\tilde{P}$ and $\tilde{Q}$ are again polynomials in u and u' in reduced terms. We extract the highest power of u from $\tilde{Q}$ by writing

$$\tilde{Q}(z, u, u') = u^k \bar{Q}(z, u, u').$$

To decribe the set of fixed singularities one needs to consider a number of cases where the equation may behave in a non-generic way. This may be any point where a coefficient in the equation becomes infinite or the expression on the right hand side of either equation (2.2) or the transformed equation (2.3) becomes indeterminate.

Definition 2.2. *The set $\Phi \subset D$ of fixed singularities for equation (2.2) is given by the union of the following six sets Φ_i, $i = 1, \dots, 6$.*

$$\begin{aligned}
\Phi_1 =& \{\zeta \in D : Q(\zeta, y, y') \equiv 0\},\\
\Phi_2 =& \{\zeta \in D : P(\zeta, y, y') \textit{ and } Q(\zeta, y, y') \textit{ have a common factor}\}\\
\Phi_3 =& \{\zeta \in D : \textit{ the equations } K_i(\zeta, \eta) = 0, i = 0, \dots, q, \textit{ have a common root } \eta\}\\
\Phi_4 =& \begin{cases} \{\zeta \in D : \Pi_p(\zeta, y) \equiv 0\}, & \textit{if } p > q + 3\\ \{\zeta \in D : \Pi_p(\zeta, \eta) = 0, K_q(\zeta, \eta) = 0 \textit{ have a common root } \eta\}, & \textit{if } p = q + 3\\ \{\zeta \in D : K_q(\zeta, y) \equiv 0\}, & \textit{if } p < q + 3 \end{cases}\\
\Phi_5 =& \{\zeta \in D : \bar{Q}(\zeta, 0, u') \equiv 0\}\\
\Phi_6 =& \{\zeta \in D : \tilde{P}(\zeta, 0, 0) = 0 = \tilde{Q}(\zeta, 0, 0)\}.
\end{aligned}$$

Any singularity $z_0 \in D$ of a solution of (2.2) not contained in Φ is called a movable singularity.

In contrast to first-order equations, movable singularities of second-order equations can be more complicated and e.g. essential singularities, logarithmic branch points or transcendental singularities can occur in general. To see this consider the following equations

$$\begin{aligned}
(yy'' - (y')^2)^2 + 4y(y')^3 =& 0,\\
y'' + (y')^2 =& 0.
\end{aligned}$$

The general solutions of these equations are

$$\begin{aligned}
y(z) =& c \exp\left(\frac{1}{z - z_0}\right),\\
y(z) =& \log(z - z_0) + C,
\end{aligned}$$

respectively. One main question addressed in this chapter are conditions under which the movable singularities of certain classes of second-order equation are at most algebraic.

Higher-order equations and systems of ODEs

For equations of higher than second order even more types of movable singularities can occur. In particular, the movable singularities may no longer be isolated. For example, movable natural boundaries are known to exist in the solutions of third-order equations like the Chazy equation [1]

$$y''' = 2yy'' - 3(y')^2.$$

By this we mean a closed curve in the complex plane beyond which the solution cannot be analytically continued. For the Chazy equation this curve is a circle the radius of which depends on the initial conditions for the solution.

For systems of ordinary differential equations of the form

$$\frac{dy_k}{dz} = \frac{P_k(z, y_1, \dots, y_n)}{Q_k(z, y_1, \dots, y_n)}, \quad P_k, Q_k \in \mathcal{O}_D[y_1, \dots, y_m], \quad k = 1, \dots, n,$$

where $\mathcal{O}_D$ is the ring of analytic functions on a domain $D \subset \mathbf{C}$, Y. Murata has described the set of fixed singularities in a precise way. We will only outline the discussion here, the details can be found in his paper [31]. The idea is to extend the system to a *rational compactification* M of the space $\mathbf{C}^n$ of dependent variables. By this we mean an n-dimensional complex manifold with the following properties

- M has an atlas $\{(U_i, \phi_i), i = 1, \ldots, m\}$ consisting of a finite number of charts.
- for each $i = 1, \ldots, m$ we have $U_i \cong \mathbf{C}^n$ and $\phi_1 = \mathrm{id} : U_1 = \mathbf{C}^n \to \mathbf{C}^n$
- $\phi_j \circ \phi_i^{-1} : \left(y_1^{(i)}, \ldots, y_n^{(i)}\right) \mapsto \left(y_1^{(j)}, \ldots, y_n^{(i)}\right)$, where $y_k^{(j)} = R_k^{ji}\left(y_1^{(i)}, \ldots, y_n^{(i)}\right)$ is rational in $y_1^{(i)}, \ldots, y_n^{(i)}$ over $\mathbf{C}$

The easiest example of a rational compactification is the complex projective space $\mathbf{CP}^n$. In each of the charts (U_i, ϕ_i), $i = 1, \ldots, m$, we can re-write the system of equations in the coordinates $\left(y_1^{(i)}, \ldots, y_n^{(i)}\right)$ in the form

$$\frac{dy_k^{(i)}}{dz} = \frac{P_k^{(i)}\left(z, y_1^{(i)}, \ldots, y_n^{(i)}\right)}{Q_k^{(i)}\left(z, y_1^{(i)}, \ldots, y_n^{(i)}\right)}.$$

To define the set of fixed singularities one has to examine the system of equations in each of the charts for points where the expression on the right hand side becomes singular or indeterminate. The set Φ then is the union of all these sets of points.

We will encounter the idea of compactifying the space of dependent variables again in section 3.3 when we discuss the procedure of *blowing up* the space of dependent variables in order to remove certain points at which the equation becomes indeterminate.

2.2 Local existence and uniqueness theorem and analytic continuation

The starting point for our study of movable singularities is Cauchy's local existence and uniqueness theorem which guarantees a unique local analytic solution of an ODE in some neighbourhood of any point $z_0 \notin \Phi$. We formulate the theorem in a general form for a system of first-order differential equations. For the discussion in this section we refer to Ince's book [17].

Theorem 2.3. *Given a system of ordinary differential equations,*

$$\begin{aligned} y_1'(z) &= F_1(z, y_1, \ldots, y_m) \\ &\vdots \\ y_m'(z) &= F_n(z, y_1, \ldots, y_m), \end{aligned}$$

where $F_1, \ldots, F_m$ are analytic functions in a neighbourhood U of $(z_0, \eta_1, \ldots, \eta_m) \in \mathbf{C}^{m+1}$, $U = \{|z - z_0| \leq a,\ |y_k - \eta_k| \leq b,\ k = 1, \ldots, m\}$, there exists a unique analytic solution $(y_1(z), \ldots, y_m(z))$ satisfying $y_k(z_0) = \eta_k$, $k = 1, \ldots, m$, with radius of convergence at least $r = a\left(1 - e^{-\frac{b}{(m+1)Ma}}\right)$. Here $M = \max\{|F_k(z, y_1, \ldots, y_m)| : (z, y_1, \ldots, y_m) \in U, k = 1, \ldots, m\}$.

An immediate consequence of Theorem 2.3 is a lemma by Painlevé regarding the analytic continuation of a solution of a system of ODEs in the complex plane.

Lemma 2.4. *Let $F_k(z, y_1, \ldots, y_m)$, $k = 1, \ldots, m$, be analytic functions in a neighbourhood of a point $(z_*, \eta_1, \ldots, \eta_m) \in \mathbf{C}^{m+1}$. Let γ be a curve with end point z_* and suppose that $(y_1, \ldots, y_m)$ are analytic on $\gamma \setminus \{z_*\}$ and there satisfy*

$$y_k' = F_k(z, y_1, \ldots, y_m), \quad k = 1, \ldots, m.$$

Suppose there is a sequence $(z_n)_{n\in\mathbf{N}} \subset \gamma$ such that $z_n \to z_*$ and $y_k(z_n) \to \eta_k \in \mathbf{C}$ as $n \to \infty$ for all $k = 1, \ldots, m$.

Then the solution $(y_1, \ldots, y_n)$ can be analytically continued to include the point z_*.

Proof. We can choose some r such that all the functions F_k, $k = 1, \ldots, m$ are analytic in the set $D = \{|z - z_*| \leq r, |y_k - \eta_k| \leq r, k = 1, \ldots, m\}$ and define the maximum modulus $M = \max\{|F_k(z, y_1, \ldots, y_m)| : (z, y_1, \ldots, y_m) \in D, k = 1, \ldots, m\}$. For sufficiently large n we have $\{|z - z_n| < r/2, |y_k - y_k(z_n)| < r/2, k = 1, \ldots, m\} \subset D$. By Theorem 2.3, a solution around z_n is defined at least in the disc of radius $\rho = \frac{r}{2}\left(1 - e^{-\frac{1}{(m+1)M}}\right)$. For some n we have $z_* \in B(z_n, \rho)$. □

2.3 Second-order ODEs with movable algebraic singularities

As noted earlier, all movable singularities of solutions of the first-order rational equation (2.1) are algebraic. The aim of this and the following sections is to present classes of second-order equations and systems of equations in two dependent variables for which it is shown that this is likewise the case. Unlike for first-order equations there are however certain obstructions for a second-order equation to have this property. We start with an overview of work that has been done on this topic previously.

In 1953, R. A. Smith [45] proved the following theorem.

Theorem 2.5. *Consider the equation*

$$y'' + f(y)y' + g(y) = P(z), \tag{2.4}$$

where f and g are polynomials of degree n and m, respectively, where $n > m$, and let P be analytic at some point z_0.

1) *There is an infinite family of solutions which have an algebraic branch point at z_0, in a neighbourhood of which the solution can be expressed in the form*

$$y(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^{(j-1)/n}. \tag{2.5}$$

2) *Let Γ be a contour of finite length in $\mathbf{C}$ having z_0 as an end point. If $y(z)$ is a solution of (2.4) which can be continued analytically along Γ as far as z_0 but not over it, then the singularity at z_0 is of the form (2.5).*

3) Let now Γ be any continuous curve with end point z_0 in $\mathbf{C}$. If the singularity at z_0 is not of the algebraic form (2.5) then Γ has infinite length and z_0 is an accumulation point of such algebraic branch points.

In the following we will look at several other classes of rational second-order differential equations and systems of equations for which a similar statement of this form holds. In [43] and [44], Shimomura considered the following classes of equations, which he denoted to be of P_I and P_{II}-type, respectively:

$$y'' = \frac{2(2k+1)}{(2k-1)^2} y^{2k} + z, \quad k \in \mathbf{N}, \tag{2.6a}$$

$$y'' = \frac{k+1}{k^2} y^{2k+1} + zy + \alpha, \quad k \in \mathbf{N} \setminus \{2\}, \alpha \in \mathbf{C}. \tag{2.6b}$$

He proved that near any movable singularity z_0, which can be reached by analytic continuation of a local analytic solution along a rectifiable curve, the solutions are of the form

$$y(z) = \zeta^{-\frac{2}{2k-1}} - \frac{(2k-1)^2}{12k-2} z_0 \zeta^2 + c\zeta^{\frac{4k}{2k-1}} + \frac{(2k-1)^2}{2(2k-3)(4k-1)} \zeta^3 + \sum_{j \geq 6k-2} c_j \zeta^{\frac{j}{(2k-1)}}$$

for (2.6a), where $\zeta = z - z_0$, and

$$y(z) = \omega_k \zeta^{-1/k} - \frac{\omega_k k z_0}{6} \zeta^{2-1/k} - \frac{k^2 \alpha}{3k+1} \zeta^2 + c\zeta^{2+1/k} + \frac{\omega_k k}{4k-8} \zeta^{3-1/k} + \sum_{j \geq 3k} c_j \zeta^{j/k}$$

for (2.6b), where $\omega_k = 1$ or $e^{i\pi/k}$, the series being convergent in some branched, punctured neighbourhood of z_0. In particular, every movable singularity is a branch point with a fixed number of branches locally. In [7], Filipuk and Halburd generalise these results to a larger class of equations of the form

$$y'' = P(z, y) = \sum_{n=0}^{N} a_n(z) y^n. \tag{2.7}$$

By a simple transformation the equation (2.7) can be normalised so that

$$y'' = \frac{2(N+1)}{(N-1)^2} + \sum_{n=0}^{N-2} a_n(z) y^n. \tag{2.8}$$

When looking for solutions with leading order behaviour

$$y(z) = c_0 (z - z_0)^{-p} + o((z - z_0)^{-p}), \quad \text{as } z \to z_0,$$

one easily finds $p = 2/(N+1)$ and $c_0^{N-1} = 1$. However, in general there do not exist series solutions in fractional powers of $(z - z_0)$ of the form

$$\sum_{j=0}^{\infty} c_j (z - z_0)^{(j-2)/(N-1)}. \tag{2.9}$$

In fact, if one inserts the expansion (2.9) into the equation (2.8), in trying to recursively determine the coefficients c_j, $j = 0, 1, 2, \ldots$, one finds certain restrictions, as shown in the following. To compute the coefficient c_j one obtains the recurrence formula

$$(j + N - 1)(j - 2N - 2) c_j = (N-1)^2 P_j(c_0, c_1, \ldots, c_{j-1}),$$

where the P_j are certain polynomials in all their arguments. Hence, one can compute c_j recursively from $c_0, \ldots, c_{j-1}$ except when $j = 2N+2$, in which case one finds that $P_{2N+2}(c_0, \ldots, c_{2N+1}) = 0$ must hold in order for a series solution of the form (2.9) to exist. This is known as a *resonance condition* and if satisfied, the coefficient c_{2N+2} can be chosen arbitrarily.

In order for all movable singularities of all solutions of equation (2.8) to be represented in the form (2.9), a necessary condition is that for every possible leading order, one can always formally compute the coefficients c_j recursively. The essence of the paper [7] is that the existence of these formal series solutions is also sufficient. This class of equations contains as special cases the first and second Painlevé equation. A class of equations which contains Painlevé's equations $II - VI$ as special cases was studied by the same authors in [9]. In [8] they also give a generalisation of Smith's theorem 2.5, see also the next section.

Letting $y_1 = y$, $y_2 = y'$, equation (2.7) can be seen as a Hamiltonian system

$$y_1' = \frac{\partial H}{\partial y_2}$$
$$y_2' = -\frac{\partial H}{\partial y_1},$$

with Hamiltonian

$$H(z, y_1, y_2) = y_2^2 - \hat{P}(z, y_1),$$

where $\hat{P}(z, y)$ is a polynomial in y such that $\hat{P}_y = P$. In section 3.1 we will extend these results to a more general class of Hamiltonian systems which in general cannot be written as a scalar second-order equation.

2.4 Equations in the class $y'' = E(y')^2 + Fy' + G$

In this section we study a class of equations of the form

$$y'' = E(z,y)(y')^2 + F(z,y)y' + G(z,y), \tag{2.10}$$

where E, F and G are rational functions in y of the form

$$E(z,y) = \sum_{\mu=1}^{M} \frac{l_\mu}{y - \alpha_\mu(z)} = \frac{e(z,y)}{\prod_{\mu=1}^{M}(y - \alpha_\mu(z))},$$
$$F(z,y) = \frac{f(z,y)}{\prod_{\mu=1}^{M}(y - \alpha_\mu(z))^{m_\mu}}, \quad G(z,y) = \frac{g(z,y)}{\prod_{\mu=1}^{M}(y - \alpha_\mu(z))^{n_\mu}}$$

where $\alpha_\mu(z)$, $\mu = 1, \ldots, M$, are analytic functions in z in some common domain $\Omega \subset \mathbf{C}$, $l_\mu \in \mathbf{Q}$, $m_\mu, n_\mu \in \mathbf{N}$ for all $\mu = 1, \ldots, M$ and $e(z,y)$, $f(z,y)$ and $g(z,y)$ are certain polynomials in y with analytic coefficients. For the classes of equations considered we will show that all movable singularities of all their solutions are algebraic branch points, i.e. in some cut neighbourhood of a movable singularity z_∞ the solution can be expressed by a convergent Puisseux series

$$y(x) = \sum_{k=0}^{\infty} c_k (z - z_\infty)^{(k+m)/n}, \quad m \in \mathbf{Z}, n \in \mathbf{N}.$$

In [8] the case was treated where $E \equiv 0$ and F and G are just polynomials in y with $\deg_y G \leq \deg_y F + 1$, equations of so-called Lienard type, which is a generalisation of Smith's Theorem 2.5. A further generalisation of this class was studied by the author in [19] where now $E \neq 0$ and the l_μ, $\mu = 1, \ldots, M$ are integers. In [9], a class of equations was studied with $m_\mu = 1$ for all $\mu \in \{1, \ldots, M\}$ in the denominator of F and the l_μ, $\mu = 1, \ldots, M$ can be integers or half-integers. This class contains as a special case the Painlevé equations $P_{II} - P_{VI}$.

Suppose we have a movable singularity at z_∞ and let $c \in \mathbf{C} \setminus \{\alpha_1(z_\infty), \ldots, \alpha_m(z_\infty)\}$. The proofs of the theorems mentioned above all rely on the introduction of an auxiliary function $W(z, w, w')$, rational in w and polynomial in w', where $w = (y - c)^{-1}$, which is shown to remain bounded as any movable singularity is approached. For the class of equations in [8] and [19], W can be taken to be linear in w'. In [9], the function W is quadratic in w', however the class of equations considered there does not contain [19] as a special case. For the class of equations considered in the following it will be necessary to assume W to be of the form

$$W(z, w, w') = A_N(z, w)(w')^N + \cdots + A_1(z, w)w' + A_0(z, w),$$

where the functions $A_n(z, w)$, $n = 0, \ldots, N$, are to be determined. A main step in the proof will be to show below that W satisfies a first-order linear differential equation of the form

$$W' = P(z, w)W + Q(z, w)w' + R(z, w), \tag{2.11}$$

where P, Q and R are polynomial in w which by the following lemma shows that W remains bounded provided that w is bounded.

Lemma 2.6. *Let Γ be a finite length curve in the complex plane and let $\tilde{P}(z)$, $\tilde{Q}(z)$ and $\tilde{R}(z)$ be bounded functions on Γ. Then any solution of the equation*

$$W' = \tilde{P}W + \tilde{Q}' + \tilde{R}, \tag{2.12}$$

is also bounded on Γ.

Proof. Choosing a point $z_0 \in \Gamma$ the solution can be written as

$$W(z) = \tilde{Q}(z) + I(z)\left(C + \int_{z_0}^{z} (\tilde{R}(\zeta) + \tilde{P}(\zeta)\tilde{Q}(\zeta))I(\zeta)^{-1} d\zeta\right),$$

where $C = W(z_0) - \tilde{Q}(z_0)$ is an integration constant and I is the integrating factor

$$I(z) = \exp\left(\int_{z_0}^{z} \tilde{P}(\zeta) d\zeta\right).$$

Since $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are bounded on Γ and Γ has finite length, $I(z)$ and $I(z)^{-1}$ are bounded and hence $W(z)$ itself is bounded on Γ. □

Remark. To apply Lemma 2.6 to (2.11) choose $\tilde{P}(z) = P(z, w(z))$, $\tilde{Q}(z) = Q(z, w(z))$ and $\tilde{R}(z) = R(z, w(z)) - Q_z(z, w(z))$.

We now state the assumptions we will make on equation (2.10) in the following.

- For all $\mu = 1, \dots, M$ we have $f(z_\infty, \alpha_\mu(z_\infty)) \neq 0$ and also the highest coefficient of f is non-zero at z_∞. For the degrees of f and g we assume

$$\begin{aligned} m_0 :=& \deg_y f - \sum_{\mu=1}^{M} m_\mu > 0, \\ \deg_y g \leq& 1 + \deg_y f - \sum_{\mu=1}^{M} (m_\mu - n_\mu). \end{aligned} \tag{2.13}$$

- Let $l_0 = 2 - \sum_{\mu=1}^{M} l_\mu$. We assume $l_\mu \neq m_\mu - 1$ for all $\mu = 0, \dots, M$.

- For the integers m_μ, n_μ, $\mu = 1, \dots, M$ we have $m_\mu \geq n_\mu \geq 0$. For those $\mu \in \{1, \dots, M\}$ for which $\alpha'_\mu = 0$, i.e. $\alpha_\mu =$ const., we have $m_\mu > n_\mu \geq 0$. For the remaining μ with $\alpha'_\mu \neq 0$ we have $m_\mu = n_\nu > 1$ and the following condition is satisfied identically:

$$G_\mu(z) + \alpha'_\mu(z) F_\mu(z) = 0, \tag{2.14}$$

  where

$$\begin{aligned} F_\mu(z) &= f(z, \alpha_\mu(z)) \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{M} (\alpha_\mu(z) - \alpha_\nu(z))^{-m_\nu} \\ G_\mu(z) &= g(z, \alpha_\mu(z)) \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{M} (\alpha_\mu(z) - \alpha_\nu(z))^{-n_\nu}. \end{aligned}$$

- For all $\mu = 1, \dots, M$, at every $\hat{z} \in \Omega$ there exists a formal series solution

$$y(z) = \alpha_\mu(\hat{z}) + \sum_{k=1}^{\infty} c_k (z - \hat{z})^{k/m_\mu}. \tag{2.15}$$

- At every $\hat{z} \in \Omega$ there exists a formal series solution

$$y(z) = \sum_{k=0}^{\infty} c_k (z - \hat{z})^{(k-1)/m_0}. \tag{2.16}$$

Under these assumptions we are going to prove the following theorem.

Theorem 2.7. *Let $z_\infty \in \Omega$ be such that $\alpha_1(z_\infty), \dots, \alpha_M(z_\infty)$ are pairwise distinct and let Γ be a finite length curve with endpoint z_∞. Let y be a solution of (2.10) under the above assumptions which is analytic on $\Gamma \setminus \{z_\infty\}$ but cannot be further analytically continued to include the point z_∞. Then y is represented, in a cut neighbourhood of $\hat{z} = z_\infty$, either by one of the series (2.15), $\mu \in \{1, \dots, M\}$, or by a series (2.16).*

Remark. The theorem states that all movable singularities of a solution of equation (2.10) are of the form described. The existence of the formal series expansions (2.15) and (2.16) are necessary conditions for this and are equivalent to a number of resonance conditions as explained in section 2.3.

The first step in our proof is choosing a number $c \in \mathbf{C} \setminus \{\alpha_1(z_\infty), \dots, \alpha_M(z_\infty)\}$ and making the transformation $w(z) = (y(z) - c)^{-1}$ in equation (2.10). The form of the equation remains unchanged by this transformation as one obtains

$$w'' = \left(2w^{-1} - w^{-2}E(z, c + w^{-1})\right)(w')^2 + F(z, c + w^{-1})w' - w^2 G(z, c + w^{-1}),$$

which is of the form

$$w'' = \tilde{E}(z, w)(w')^2 + \tilde{F}(z, w)w' + \tilde{G}(z, w), \tag{2.17}$$

where

$$\begin{aligned} \tilde{E}(z, w) &= \sum_{\mu=0}^{M} \frac{l_\mu}{w - \tilde{\alpha}_\mu(z)} = \frac{\tilde{e}(z, w)}{\prod_{\mu=0}^{M}(w - \tilde{\alpha}_\mu(z))}, \\ \tilde{F}(z, w) &= \frac{\tilde{f}(z, w)}{\prod_{\mu=0}^{M}(w - \tilde{\alpha}_\mu(z))^{m_\mu}}, \quad \tilde{G}(z, w) = \frac{\tilde{g}(z, w)}{\prod_{\mu=0}^{M}(w - \tilde{\alpha}_\mu(z))^{n_\mu}} \end{aligned}$$

and we have defined

$$\begin{aligned} \tilde{\alpha}_\mu(z) &= \begin{cases} (\alpha_\mu(z) - c)^{-1} & \text{for } \mu = 1, \dots, M, \\ 0 & \text{for } \mu = 0, \end{cases} \\ l_0 &= 2 - \sum_{\mu=1}^{M} l_\mu, \\ m_0 &= \deg_y f - \sum_{\mu=1}^{M} m_\mu, \\ n_0 &= \deg_y g - 2 - \sum_{\mu=1}^{M} n_\mu. \end{aligned}$$

The conditions (2.13) imply that we have $m_0 > n_0 \geq 0$. Let N be the smallest positive integer such that $N l_\mu \in \mathbf{Z}$ for all $\mu = 0, \dots, M$. The case where $N = 1$ was treated in the article [19]. Here we consider the case where $N \geq 2$.

Lemma 2.8. *There exist functions $A_0(z, w), \dots, A_N(z, w)$ meromorphic in w with analytic coefficients and only possible poles at $w = \tilde{\alpha}_\mu(z)$ such that*

$$W = A_N(z, w)(w')^N + \dots + A_1(z, w)w' + A_0(z, w) \tag{2.18}$$

satisfies a first-order linear differential equation of the form

$$W' = P(z, w)W + Q(z, w)w' + R(z, w), \tag{2.19}$$

where $P(z, w)$ is polynomial in w and $Q(z, w)$ and $R(z, w)$ entire functions in w with coefficients analytic in $z \in \Omega$.

Proof. We differentiate (2.18) with respect to z, replace w'' on the right hand side using the differential equation (2.17) and equate this the with the right hand side of (2.19). One obtains an expression which is polynomial in w' of degree $N+1$. The coefficients of $(w')^n$, $n = 0, \dots, N+1$, are as follows:

$$\begin{aligned} n = N+1 &: NA_N\tilde{E} + (A_N)_w = 0 \\ n = N, \dots, 2 &: nA_n\tilde{F} + (A_n)_z + (A_{n-1})_w + (n-1)A_{n-1}\tilde{E} + (n+1)A_{n+1}\tilde{G} \\ &\quad = PA_n \\ n = 1 &: A_1\tilde{F} + (A_1)_z + (A_0)_w + 2A_2\tilde{G} = PA_1 + Q \\ n = 0 &: (A_0)_z + A_1\tilde{G} = PA_0 + R, \end{aligned} \tag{2.20}$$

where in the second line n runs from 2 to N. For $n = N$, the term involving $\tilde{G}$ is absent which we express by letting $A_{N+1} \equiv 0$. Thus it suffices to determine the functions $A_0, \dots, A_N$ such that the equations (2.20) are satisfied. To satisfy the first equation we choose

$$A_N(z, w) = \prod_{\mu=0}^{M} (w - \tilde{\alpha}_\mu(z))^{-Nl_\mu}. \tag{2.21}$$

For the other functions $A_n(z, w)$, $n = 0, \dots, N-1$, we make an ansatz in form of Laurent series expansions in w with coefficients analytic in z:

$$A_n(z, w) = \sum_{k=-k_n^\mu}^{\infty} a_{n,k}^\mu(z)(w - \tilde{\alpha}_\mu(z))^k.$$

To determine the coefficient functions $a_{n,k}^\mu(z)$, $k = -k_n^\mu, -k_n^\mu + 1, \dots,$ by some recursion we also expand the functions $\tilde{E}, \tilde{F}, \tilde{G}$ as Laurent series in w about $\tilde{\alpha}_\mu(z)$ for each $\mu \in \{0, \dots, M\}$:

$$\begin{aligned} \tilde{E}(z, w) &= \sum_{k=-1}^{\infty} e_k^\mu(z)(w - \tilde{\alpha}_\mu(z))^k, \\ \tilde{F}(z, w) &= \sum_{k=-m_\mu}^{\infty} f_k^\mu(z)(w - \tilde{\alpha}_\mu(z))^k, \quad \tilde{G}(z, w) = \sum_{k=-n_\mu}^{\infty} g_k^\mu(z)(w - \tilde{\alpha}_\mu(z))^k. \end{aligned}$$

For P, Q and R we start with expansions

$$\begin{aligned} P(z, w) &= \sum_{k=0}^{\infty} p_k^\mu(z)(w - \tilde{\alpha}(z))^k, \\ Q(z, w) &= \sum_{k=-\infty}^{\infty} q_k^\mu(z)(w - \tilde{\alpha}(z))^k, \quad R(z, w) = \sum_{k=-\infty}^{\infty} r_k^\mu(z)(w - \tilde{\alpha}(z))^k, \end{aligned}$$

but we will show that in fact all coefficients q_k^μ and r_k^μ can be set to 0 for $k < 0$. We will also see that, in order to satisfy the equations (2.20) we only need to compute a finite number of non-zero coefficients at every $\tilde{\alpha}_\mu$, $\mu = 0, \dots, M$. We write down all the summands in equation (2.20), for $n = 2, \ldots, N$:

$$\begin{aligned}
A_n\tilde{F} &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a^{\mu}_{n,i-k^{\mu}_n} f^{\mu}_{k-i-m_\mu}\right)(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_n-m_\mu}\\
(A_n)_z &= \sum_{k=0}^{\infty}\left((a^{\mu}_{n,k-k^{\mu}_n})' - \tilde{\alpha}'_\mu(z)(k-k^{\mu}_n)a^{\mu}_{n,k-k^{\mu}_n}\right)(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_n-1}\\
(A_{n-1})_w &= \sum_{k=0}^{\infty}(k-k^{\mu}_{n-1})a^{\mu}_{n-1,k-k^{\mu}_{n-1}}(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_{n-1}-1}\\
A_{n-1}\tilde{E} &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a^{\mu}_{n-1,i-k^{\mu}_{n-1}} e^{\mu}_{k-i-1}\right)(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_{n-1}-1}\\
A_{n+1}\tilde{G} &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a^{\mu}_{n+1,i-k^{\mu}_{n+1}} g^{\mu}_{k-i-n_\mu}\right)(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_{n+1}-n_\mu}\\
PA_n &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a^{\mu}_{n,i-k^{\mu}_n} p^{\mu}_{k-i}\right)(w-\tilde{\alpha}_\mu(z))^{k-k^{\mu}_n}
\end{aligned}$$

For $n = N$, in the absence of the term $A_{N+1}\tilde{G}$ we let, in order for the terms in equation (2.20) with lowest power of $(w - \tilde{\alpha}_\mu)$ to balance,

$$k^{\mu}_N + m_\mu = k^{\mu}_{N-1} + 1.$$

Subsequently, for $n = N-1, \ldots, 1$, we can also let

$$k^{\mu}_n + m_\mu = k^{\mu}_{n-1} + 1.$$

Therefore, with $k^{\mu}_N = Nl_\mu$, for $n = 0, \ldots, N$ we find

$$k^{\mu}_n = Nl_\mu + (N-n)(m_\mu - 1).$$

In the following we will compute the coefficients

$$a^{\mu}_{n,k-k^{\mu}_n}, \quad n = 0, \ldots, N-1, \quad k = 0, 1, \ldots,$$

the coefficients $a^{\mu}_{N,k-Nl_\mu}$, $k = 0, 1, \ldots$, being completely determined by (2.21). From the equations (2.20), by comparing powers of $(w-\alpha_\mu)^{k-k^{\mu}_n-m_\mu}$, we find the relations between the coeffients $a^{\mu}_{n,k}$, $n = N, \ldots, 2$:

$$\begin{aligned}
n\sum_{i=0}^{k} a^{\mu}_{n,i-k^{\mu}_n} f^{\mu}_{k-i-m_\mu} + (k-k^{\mu}_{n-1})(a^{\mu}_{n-1,k-k^{\mu}_{n-1}} - \tilde{\alpha}'_\mu a^{\mu}_{n,k-k^{\mu}_{n-1}})\\
+(a^{\mu}_{n,k-k^{\mu}_{n-1}-1})' + (n-1)\sum_{i=0}^{k} a^{\mu}_{n-1,i-k^{\mu}_{n-1}} e^{\mu}_{k-i-1}\\
+(n+1)\sum_{i=0}^{k-2m_\mu+n_\mu+1} a^{\mu}_{n+1,i-k^{\mu}_{n+1}} g^{\mu}_{k-i-n_\mu} = \sum_{i=0}^{k-m_\mu} p^{\mu}_i a^{\mu}_{n,k-k^{\mu}_n-m_\mu-i}.
\end{aligned} \tag{2.22}$$

For $n = 1$ we find

$$\begin{aligned}
\sum_{i=0}^{k} a^{\mu}_{1,i-k^{\mu}_1} f^{\mu}_{k-i-m_\mu} + (k-k^{\mu}_0)(a^{\mu}_{0,k-k^{\mu}_0} - \tilde{\alpha}'_\mu a^{\mu}_{1,k-k^{\mu}_0}) + (a^{\mu}_{1,k-k^{\mu}_0-1})'\\
+2\sum_{i=0}^{k-2m_\mu+n_\mu+1} a^{\mu}_{2,i-k^{\mu}_2} g^{\mu}_{k-i-n_\mu} = \sum_{i=0}^{k-m_\mu} p^{\mu}_i a^{\mu}_{1,k-k^{\mu}_1-m_\mu-i} + q^{\mu}_{k-k^{\mu}_0-1},
\end{aligned} \tag{2.23}$$

and for $n = 0$:

$$\sum_{i=0}^{k-m_\mu+n_\mu} a^\mu_{1,i-k^\mu_1} g^\mu_{k-i-n_\mu} - \tilde{\alpha}'_\mu (k-k^\mu_0) a^\mu_{0,k-k^\mu_0} + (a^\mu_{0,k-k^\mu_0-1})' \\ = \sum_{i=0}^{k-1} p^\mu_i a^\mu_{0,k-k^\mu_0-1-i} + r^\mu_{k-k^\mu_0-1}. \tag{2.24}$$

We can set

$$q^\mu_{k-k^\mu_0-1} = r^\mu_{k-k^\mu_0-1} = 0 \quad \text{for all } k < 0.$$

We will show in the following that most of the remaining coefficients q^μ_j, r^μ_j, for $j < 0$, can also be set to zero. If $k^\mu_0 < 0$ there is nothing to be done. We now distinguish the case of those $\mu \in \{0, 1, \ldots, M\}$ for which $\tilde{\alpha}'_\mu \equiv 0$ and those for which $\tilde{\alpha}'_\mu \neq 0$. We consider first the case $\tilde{\alpha}'_\mu = 0$ where we have $m_\mu > n_\mu \geq 0$. For $k = 0$ equations (2.22), (2.23) and (2.24) become

$$n a^\mu_{n,-k^\mu_n} f^\mu_{-m_\mu} - k^\mu_{n-1} a^\mu_{n-1,-k^\mu_{n-1}} + (n-1) a^\mu_{n-1,-k^\mu_{n-1}} e^\mu_{-1} = 0, \quad n = N, \ldots, 2, \tag{2.25}$$

$$a^\mu_{1,-k^\mu_1} f^\mu_{-m_\mu} - k^\mu_0 a^\mu_{0,-k^\mu_0} = q^\mu_{-k^\mu_0-1}, \tag{2.26}$$

$$0 = r^\mu_{-k^\mu_0-1}. \tag{2.27}$$

$a^\mu_{N,-k^\mu_N}$ being known from (2.21), one can recursively compute $a^\mu_{n-1,-k^\mu_{n-1}}$ from (2.25) for $n = N, \ldots, 2$. (Note that $e^\mu_{-1} = l_\mu$ and therefore the coefficient of $a^\mu_{n-1,-k^\mu_{n-1}}$ is $(n-1)l_\mu - k^\mu_{n-1} = (N-n+1)(l_\mu - m_\mu + 1) \neq 0$ by the second assumption of the theorem). In equation (2.26) one can let $q^\mu_{-k^\mu_0-1} = 0$ and determine

$$a^\mu_{0,-k^\mu_0} = \frac{1}{k^\mu_0} a^\mu_{1,-k^\mu_1} f^\mu_{-m_\mu}, \tag{2.28}$$

unless $k^\mu_0 = 0$, in which case one has $q^\mu_{-1} = a^\mu_{1,-k^\mu_1} f^\mu_{-m_\mu}$.

Now consider the case $\tilde{\alpha}'_\mu \neq 0$ where $m_\mu = n_\mu > 1$. Here, for $k = 0$, equations (2.22), (2.23) and (2.24) reduce to

$$n a^\mu_{n,-k^\mu_n} f^\mu_{-m_\mu} - k^\mu_{n-1} (a^\mu_{n-1,-k^\mu_{n-1}} - \tilde{\alpha}'_\mu a^\mu_{n,-k^\mu_{n-1}}) + (n-1) a^\mu_{n-1,-k^\mu_{n-1}} e^\mu_{-1} \\ + (n+1) a^\mu_{n+1,-k^\mu_{n+1}} g^\mu_{-m_\mu} = 0, \tag{2.29}$$

$$a^\mu_{1,-k^\mu_1} f^\mu_{-m_\mu} - k^\mu_0 a^\mu_{0,-k^\mu_0} = q^\mu_{-k^\mu_0-1}, \tag{2.30}$$

$$a^\mu_{1,-k^\mu_1} g^\mu_{-m_\mu} + \tilde{\alpha}'_\mu k^\mu_0 a^\mu_{0,-k^\mu_0} = r^\mu_{-k^\mu_0-1}. \tag{2.31}$$

Again, from (2.29) one can recursively determine the $a^\mu_{n-1,k^\mu_{n-1}}$, $n = N, \ldots, 2$. Letting $q^\mu_{-k^\mu_0-1} = 0$ we obtain as before $a^\mu_{0,-k^\mu_0} = \frac{1}{k^\mu_0} a^\mu_{1,-k^\mu_1} f^\mu_{-m_\mu}$. Now,

$$r^\mu_{-k^\mu_0-1} = a^\mu_{1,-k^\mu_1} \left(g^\mu_{-m_\mu} + \tilde{\alpha}'_\mu f^\mu_{-m_\mu} \right) = 0,$$

by condition (2.14), since, as one can compute,

$$f^\mu_{-m_\mu} = (-1)^{m_\mu} (\alpha_\mu(z) - c)^{-2m_\mu} F_\mu(z),$$
$$g^\mu_{-n_\mu} = (-1)^{n_\mu+1} (\alpha_\mu(z) - c)^{-2n_\mu-2} G_\mu(z),$$

and

$$\tilde{\alpha}'_\mu(z) = -(\alpha_\mu(z) - c)^{-2}\alpha'_\mu(z).$$

If $k_0^\mu \geq 1$ one can now successively determine the coefficients $a^\mu_{n,k-k_n^\mu}$ and p^μ_{k-1} for $k = 1, \ldots, k_n^\mu - 1$ by solving the linear system of equations

$$\begin{aligned}
&\delta_{1,m_\mu} a^\mu_{n,-k_n^\mu} p^\mu_{k-m_\mu} + ((n-1)e^\mu_{k-1} - (k - k^\mu_{n-1}))a^\mu_{n-1,k-k^\mu_{n-1}} = \\
&n \sum_{i=0}^{k} a^\mu_{n,i-k_n^\mu} f^\mu_{k-i-m_\mu} - \tilde{\alpha}'_\mu (k - k^\mu_{n-1}) a^\mu_{n,k-k^\mu_{n-1}} + (a^\mu_{n,k-k^\mu_{n-1}-1})' \\
&+(n-1) \sum_{i=1}^{k} a^\mu_{n-1,i-k^\mu_{n-1}} e^\mu_{k-i-1} - \sum_{i=0}^{k-m_\mu-\delta_{1,m_\mu}} p_i^\mu a^\mu_{n,k-k_n^\mu-m_\mu-i} \\
&\qquad +(n+1) \sum_{i=0}^{k-m_\mu+n_\mu} a^\mu_{n+1,i-k^\mu_{n+1}} g^\mu_{k-i-n_\mu},
\end{aligned} \tag{2.32}$$

for $n = N, \ldots, 2$, as well as

$$\begin{aligned}
&\delta_{1,m_\mu} a^\mu_{1,-k_1^\mu} p^\mu_{k-m_\mu} - (k - k_0^\mu) a^\mu_{0,k-k_0^\mu} = \\
&\sum_{i=0}^{k} a^\mu_{1,i-k_1^\mu} f^\mu_{k-i-m_\mu} - \tilde{\alpha}'_\mu (k - k_0^\mu) a^\mu_{1,k_1^\mu} + (a^\mu_{1,k-k_0^\mu-1})' \\
&+2 \sum_{i=0}^{k-m_\mu+n_\mu} a^\mu_{2,i-k_2^\mu} g^\mu_{k-i-n_\mu} - \sum_{i=0}^{k-m_\mu-\delta_{1,m_\mu}} p_i^\mu a^\mu_{1,k-k_1^\mu-m_\mu-i},
\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}
&a^\mu_{0,-k_0^\mu} p^\mu_{k-1} + \tilde{\alpha}'_\mu (k - k_0^\mu) a^\mu_{0,k-k_0^\mu} = \\
&\sum_{i=0}^{k-m_\mu+n_\mu} a^\mu_{1,i-k_1^\mu} g^\mu_{k-i-n_\mu} + (a^\mu_{k-k_0^\mu-1})' - \sum_{i=0}^{k-2} p_i^\mu a^\mu_{0,k-k_0^\mu-1-i},
\end{aligned} \tag{2.34}$$

where $\delta_{1,m_\mu} = 0$ if $m_\mu > 1$ and $\delta_{1,m_\mu} = 1$ if $m_\mu = 1$. Note that in this last case we have $k_N^\mu = \cdots = k_1^\mu = k_0^\mu$. The coefficient matrix for the system of equations (2.32), (2.33), (2.34) has determinant (developed from the bottom row)

$$\begin{aligned}
&\det \begin{pmatrix}
\delta_{1,m_\mu} a^\mu_{N,-k_N^\mu} & k^\mu_{N-1} - k + (N-1)e^\mu_{k-1} & 0 & \cdots & 0 \\
\delta_{1,m_\mu} a^\mu_{N-1,-k^\mu_{N-1}} & 0 & k^\mu_{N-2} - k + (N-2)e^\mu_{k-1} & 0 & \vdots \\
\vdots & \vdots & 0 & \ddots & 0 \\
\delta_{1,m_\mu} a^\mu_{1,-k_1^\mu} & 0 & \cdots & 0 & -(k-k_0^\mu) \\
a^\mu_{0,-k_0^\mu} & 0 & \cdots & 0 & \tilde{\alpha}'_\mu (k - k_0^\mu)
\end{pmatrix} \\
&= (a^\mu_{0,-k_0^\mu} + \delta_{1,m_\mu} a^\mu_{1,-k_1^\mu} \tilde{\alpha}'_\mu) \cdot (k - k_0^\mu) \cdot \prod_{n=2}^{N} (k^\mu_{n-1} - k + (n-1)e^\mu_{k-1}) \\
&= \frac{1}{k_0^\mu} a^\mu_{1,-k_1^\mu} \left(f^\mu_{-m_\mu} + \delta_{1,m_\mu} k_0^\mu \tilde{\alpha}'_\mu \right) \cdot (k - k_0^\mu) \cdot \prod_{n=2}^{N} (k^\mu_{n-1} - k + (n-1)e^\mu_{k-1}),
\end{aligned}$$

by (2.28), which is non-zero for all $0 \leq k < k_0^\mu$. The matrix being invertible one can thus determine the coefficients p^μ_{k-1} and $a^\mu_{n,k-k_n^\mu}$, $n = 0, \ldots, N-1$, such that $q^\mu_{k-k_0^\mu-1} = r^\mu_{k-k_0^\mu-1} = 0$ for all $0 \leq k < k_0^\mu$. For $k = k_0^\mu$ one can still determine the coefficients $p^\mu_{k_0^\mu-1}$

and $a^\mu_{n,k_0^\mu - k_n^\mu}$ for $n = N-1,\dots,1$ such that $r^\mu_{-1} = 0$. However, q^μ_{-1} will in general be non-zero.

We have thus determined the coefficients $a^\mu_{n,k-k_n^\mu}$ for $n = 0,\dots,N-1$ and $k = 0,\dots,k_0^\mu - 1$, as well as p_k^μ, $k = 0,\dots,k_0^\mu - 1$. We write down the initial parts of the series expansions in w of P about $\tilde{\alpha}_\mu(z)$,

$$p_0^\mu(z) + p_1^\mu(z)(w - \tilde{\alpha}_\mu(z)) + \cdots + p^\mu_{k_0^\mu - 1}(z)(w - \tilde{\alpha}(z))^{k_0^\mu - 1}. \tag{2.35}$$

One can construct a polynomial, of degree at least $D = \sum_{\mu=0}^M k_0^{\mu+} - 1$ in w (where $k_0^{\mu+} := \max\{0, k_0^\mu\}$), which has (2.35) as initial terms in its expansion about $\tilde{\alpha}_\mu$ for all $\mu = 0,\dots,M$. We let $P(z,w)$ be such a polynomial. We now consider again the equations (2.20) for $n = N,\dots,2$. These can be viewed as differential equations for A_{n-1} with respect to w,

$$(A_{n-1})_w + (n-1)\tilde{E}A_{n-1} = PA_n - nA_n\tilde{F} - (A_n)_z - (n+1)A_{n+1}\tilde{G}, \tag{2.36}$$

where, inductively starting at $n = N$, the right hand side is known in every step. As this is a linear equation in A_{n-1}, its solutions can have no other singularities than $w = \tilde{\alpha}_\mu(z)$, $\mu = 0,\dots,M$. Above we have seen that at all of these points A_{n-1} possesses a Laurent series expansion in $w - \tilde{\alpha}_\mu$. Therefore this particular solution, which we then denote by $A_{n-1}(z,w)$, is a meromorphic function in w with poles at most at $\tilde{\alpha}_\mu$, $\mu = 0,\dots,M$. Now consider the last two equations of (2.20) for $n = 1$ and $n = 0$. We have already determined the initial part of the Laurent expansions of A_1 and A_0,

$$\frac{a^\mu_{n,-k_n^\mu}}{(w - \tilde{\alpha}_\mu)^{k_n^\mu}} + \frac{a^\mu_{n,1-k_n^\mu}}{(w - \tilde{\alpha}_\mu)^{k_n^\mu - 1}} + \cdots + a^\mu_{n,k_0^\mu - k_n^\mu - 1}(w - \tilde{\alpha}_\mu)^{k_0^\mu - k_n^\mu - 1}. \tag{2.37}$$

For fixed z, by the Mittag-Leffler theorem there exists a rational function in w which has the initial Laurent expansion (2.37) about every $\tilde{\alpha}_\mu$, $\mu = 0,\dots,M$, and we take $A_0(z,w)$ and $A_1(z,w)$ to be such functions. Given A_0, A_1, A_2 and P, the last two equations of (2.20) determine Q and R which, by this construction are meromorphic functions having at most simple poles at the points $w(z) = \tilde{\alpha}_\mu(z)$, $\mu = 0,\dots,M$.

We now employ the existence of the formal series solutions (2.15) and (2.16) to show that Q and R in fact have no poles. In the variable w, these series correspond to

$$w(z) = \tilde{\alpha}_\mu(\hat{z}) + \sum_{k=1}^\infty \tilde{c}_k (z - \hat{z})^{k/m_\mu}, \quad \mu = 0,\dots,M. \tag{2.38}$$

With the integration factor

$$I(z) = \exp\left(-\int_{z_0}^z P(\zeta, w(\zeta))d\zeta\right)$$

equation (2.19) can be written in the form

$$\frac{d}{dz}(I(z)W(z)) = (Q(z,w)w' + R(z,w))I(z). \tag{2.39}$$

If in the definition of W (equation (2.18)) we substitute for w any of the series expansions (2.38) we see that W has a Laurent series expansion in $(z - \hat{z})^{1/m_\mu}$ with finite principle part. Also, P being a polynomial in w, $I(z)$ has a power series expansion in $(z-\hat{z})^{1/m_\mu}$. Thus the product $I(z)W(z)$ also has a Laurent series expansion in $(z-\hat{z})^{1/m_\mu}$ with finite principle part. Similarly, the right hand side of equation (2.39) has a Laurent series expansion in $(z-\hat{z})^{1/m_\mu}$ in which the coefficient of the term $(z-\hat{z})^{-1}$ must be zero since otherwise integration of both sides would imply that $I(z)W(z)$ contained a term involving $\log(z-\hat{z})$. Q has leading order of the form

$$Q(z,w(z)) \sim \frac{q^\mu_{-1}(z)}{w-\tilde{\alpha}_\mu(z)} \sim q^\mu_{-1}(\hat{z})(z-\hat{z})^{-1/m_\mu}.$$

Therefore, the right hand side of equation (2.39) has leading order

$$\frac{q^\mu_{-1}(\hat{z})}{z-\hat{z}},$$

but by the above argument we must have $q^\mu_{-1}(\hat{z})=0$. Since this condition holds for all $\hat{z}$ in some open neighbourhood of z_∞, we have in fact shown that $q^\mu_{-1}\equiv 0$. In the case where $k^\mu_0=0$ we then also have $r^\mu_{-1}\equiv 0$. This proves that Q and R are in fact entire functions in w. $\square$

To prove Theorem 2.7 we need the following lemma which is similar to an argument by Shimomura [42].

Lemma 2.9. *Let y be a solution of (2.10) under the assumptions in Theorem 2.7. Let c be some complex number not equal to $\alpha_1(z_\infty),\ldots,\alpha_M(z_\infty)$. Then Γ can be continuously deformed, in the region where y is analytic, to a new curve $\tilde{\Gamma}$ with same endpoint and of finite length such that there exists $\epsilon>0$ for which $|y(z)-c|>\epsilon$ for all $z\in\tilde{\Gamma}$.*

By Lemma 2.9 we can continuously deform the curve Γ such that w is bounded on the modified curve $\tilde{\Gamma}$. The following lemma shows that w has a well-determined behaviour as z approaches z_∞ along $\tilde{\Gamma}$.

Lemma 2.10. *Let y be a solution of (2.10) under the assumptions of Theorem 2.7. Let $w=(y-c)^{-1}$ as before. Then, for some $\mu\in\{0,\ldots,M\}$, we have*

$$\lim_{\tilde{\Gamma}\ni z\to z_\infty} w(z)=\tilde{\alpha}_\mu(z_\infty).$$

Proof. In the contrary case there would exist some $\epsilon>0$ and a sequence $(z_n)_{n\in\mathbf{N}}\subset\tilde{\Gamma}$ with $z_n\to z_\infty$ as $n\to\infty$ such that $|w(z_n)-\tilde{\alpha}_\mu(z_\infty)|>\epsilon$ for all $\mu=0,\ldots,M$. Then $A_N=\prod_{\mu=0}^M(w-\tilde{\alpha}_\mu(z))^{-Nl_\mu}$ is bounded and bounded away from 0 on the sequence (z_n). Now w' can be seen as a solution of the algebraic equation

$$A_N(z,w)(w')^N+\cdots+A_1(z,w)w'+A_0(z,w)=W, \tag{2.40}$$

where the coefficient functions A_n, $n=0,\ldots,N$, being meromorphic functions in w with only possible poles at $\tilde{\alpha}_\mu(z)$, are bounded on (z_n). Since the right hand side, W, is bounded, this implies that w' is bounded on the sequence (z_n). Lemma 2.4 applied to the system

$$w'=w_1,\quad w_1'=E(z,w)(w')^2+F(z,w)w'+G(z,w)$$

now shows that w, and therefore y, can be analytically continued to z_∞ in contradiction to the assumption of Theorem 2.7. $\square$

Proof of Theorem 2.7. By Lemma 2.10, suppose that $w(z) = (y(z) - c)^{-1} \to \tilde{\alpha}_\mu(z_\infty)$ as $z \to z_\infty$, where $z \in \tilde{\Gamma}$, for some $\mu \in \{0, \ldots, M\}$. Since W is bounded as $z \to z_\infty$, there exists a sequence $(z_n)_{n\in\mathbf{N}} \subset \tilde{\Gamma}$ with $z_n \to z_\infty$ as $n \to \infty$ such that $W(z_n) \to W_\infty$ for some $W_\infty \in \mathbf{C}$. Any solution of equation (2.40) for w' is of the form

$$w' = K(z, w, W)(w - \tilde{\alpha}_\mu(z_\infty))^{-m_\mu+1},$$

where $K(z, w, W)$ is analytic at $(z_\infty, \tilde{\alpha}_\mu(z_\infty), W_\infty)$ and $K(z_\infty, \tilde{\alpha}(z_\infty), W_\infty) \neq 0$. We now change the role of dependent and independent variables, to obtain

$$\frac{dz}{dw} = K(z, w, W)^{-1}(w - \tilde{\alpha}_\mu(z_\infty))^{m_\mu-1}. \tag{2.41}$$

Together with equation (2.19), rewritten in the form

$$\frac{dW}{dw} = \frac{dW}{dz}\frac{dz}{dw} = Q + (PW + R)K(z, w, W)^{-1}(w - \tilde{\alpha}_\mu(z_\infty))^{m_\mu-1}, \tag{2.42}$$

the two equations (2.41) and (2.42) form a system of differential equations for z and W as functions of w. The right hand sides of equations (2.41) and (2.42) are analytic functions in the variables (w, z, W) in some neighbourhood of the point $(z_\infty, \tilde{\alpha}_\mu(z_\infty), W_\infty)$. Lemma (2.4) applied to this system of equations shows that z and W are analytic functions of w at the point $w = \tilde{\alpha}_\mu(z_\infty)$ and z has a series expansion of the form

$$z - z_\infty = \sum_{k=0}^{\infty} \xi_k (w - \tilde{\alpha}_\mu(z_\infty))^{k+m_\mu},$$

with positive radius of convergence. Taking the m_μ-th root on both sides shows

$$(z - z_\infty)^{1/m_\mu} = \sum_{k=1}^{\infty} \eta_k (w - \tilde{\alpha}_\mu(z_\infty))^k,$$

where the choice of branch can be absorbed into the leading coefficient η_1. Inverting this series one finally obtains

$$w(z) = \tilde{\alpha}_\mu(z_\infty) + \sum_{k=1}^{\infty} \zeta_k (z - z_\infty)^{k/m_\mu},$$

with positive radius of convergence, showing that y is represented either by a series (2.15) for $\mu \in \{1, \ldots, M\}$, or (2.16) for $\mu = 0$. $\square$

Theorem 2.7 shows that singularities obtained by analytic continuation along *finite* length curves are algebraic. In the third part of Theorem 2.5 by Smith the possibility of singularities along curves of infinite length is discussed. A singularity z_* of this kind is an accumulation point of algebraic singularities. In his article [45], he gives an example of a solution for the equation

$$y'' + 4y^3y' + y = 0,$$

which exhibits such behaviour. There, the curve along which the solution is analytically continued winds around the point z_* infinitely often and it is shown that for every $\varepsilon > 0$ there is an algebraic singularity with distance less than ε from the curve and from z_*. In the neighbourhood of such a singularity z_* the solution will be infinitely branched and for the class of equations considered in this section we cannot exclude that such singularities can in general occur.

Chapter 3

Hamiltonian systems in two dependent variables

3.1 Polynomial Hamiltonian systems with movable algebraic singularities

Having studied classes of second-order differential equations in the last chapter we now continue with a class of Hamiltonian systems, with Hamiltonian function polynomial in two dependent variables, for which we show that all movable singularities are algebraic branch points. The Hamiltonian systems studied here can in general not be reduced to scalar second-order equations which brings some further difficulties with it. The work in this section is contained in the preprint [18] by the author. The Hamiltonians we consider are of the form,

$$H(z, y_1, y_2) = \frac{\alpha_{M+1,0}(z)}{M+1} y_1^{M+1} + \frac{\alpha_{0,N+1}(z)}{N+1} y_2^{N+1} + \sum_{(i,j)\in I} \alpha_{ij}(z) y_1^i y_2^j, \tag{3.1}$$

where the set of indices I is defined by

$$I = \{(i,j) \in \mathbf{N}^2 : i(N+1) + j(M+1) < (N+1)(M+1)\}, \tag{3.2}$$

and $\alpha_{ij}, (i,j) \in I \cup \{(M+1,0),(0,N+1)\}$ are analytic functions in some common domain $\Omega \subset \mathbf{C}$. The Hamiltonian equations are given by

$$\begin{aligned} y_1' =& \alpha_{0,N+1}(z) y_2^N + \sum_{(i,j)\in I} j\alpha_{ij}(z) y_1^i y_2^{j-1} \\ y_2' =& -\alpha_{M+1,0}(z) y_1^M - \sum_{(i,j)\in I} i\alpha_{ij}(z) y_1^{i-1} y_2^j. \end{aligned} \tag{3.3}$$

The set of fixed singularities in Ω of the system (3.3) is given by the zeros of the leading coefficients in the equation,

$$\Phi = \{\zeta \in \Omega : \alpha_{M+1,0}(\zeta) = 0 \text{ or } \alpha_{0,N+1}(\zeta) = 0\},$$

where the solutions of the system may behave in a non-generic way.

Theorem 3.1. *Suppose that at every point $\hat{z} \in \Omega\backslash\Phi$ the Hamiltonian system (3.3) admits, for every pair of values $(c_{1,-N-1}, c_{2,-M-1})$ satisfying*

$$
\begin{aligned}
c_{1,-N-1}^{MN-1} &= -\left(\alpha_{0,N+1}(\hat{z})\alpha_{M+1,0}(\hat{z})^N (MN-1)^{N+1}\right)^{-1}, \\
c_{2,-M-1} &= (MN-1)\alpha_{M+1,0}(\hat{z})c_{1,-N-1}^M,
\end{aligned}
$$

formal series solutions of the form

$$
\begin{aligned}
y_1(z) &= \sum_{k=-\frac{N+1}{d}}^{\infty} c_{1,k}(z-\hat{z})^{\frac{kd}{MN-1}}, \\
y_2(z) &= \sum_{k=-\frac{M+1}{d}}^{\infty} c_{2,k}(z-\hat{z})^{\frac{kd}{MN-1}},
\end{aligned}
\tag{3.4}
$$

where $d = \gcd\{M+1, N+1, MN-1\}$. Let $\Gamma \subset \Omega$ be a finite length curve with endpoint $z_\infty \in \Omega \setminus \Phi$ such that a solution (y_1, y_2) can be analytically continued along Γ up to, but not including z_∞. Then the solution is represented by a series (3.4) at $\hat{z} = z_\infty$, convergent in some punctured, branched, neighbourhood of z_∞.

We assume in the following and for the rest of this section that $N \geq M$. In the neighbourhood of any movable singularity one can let

$$
\begin{aligned}
\tilde{y}_1(z) &= \left(\alpha_{M+1,0}(z)^N \alpha_{0,N+1}(z)\right)^{\frac{1}{MN-1}} \left(y_1(z) + \frac{\alpha_{M,0(z)}}{\alpha_{M+1,0}(z)}\right), \\
\tilde{y}_2(z) &= \left(\alpha_{M+1,0}(z)\alpha_{0,N+1}(z)^M\right)^{\frac{1}{MN-1}} \left(y_2(z) + \frac{\alpha_{0,N}(z)}{\alpha_{0,N+1}(z)}\right),
\end{aligned}
$$

achieving that the transformed Hamiltonian $\tilde{H}$ is of the same form as in (3.1) but with $\tilde{\alpha}_{M+1} \equiv 1 \equiv \tilde{\alpha}_{0,N+1}$ and $\tilde{\alpha}_{0N} \equiv 0$ (and also $\tilde{\alpha}_{M0} \equiv 0$ if $N = M$). In the following we will assume that the Hamiltonian is already given in this normalised form and readily omit the tildes again,

$$
H(z, y_1, y_2) = y_1^{M+1} + y_2^{N+1} + \sum_{(i,j)\in I'} \alpha_{ij}(z) y_1^i y_2^j, \tag{3.5}
$$

where $I' = I \setminus \{(0, N)\}$, the Hamiltonian equations being

$$
\begin{aligned}
y_1' &= (N+1)y_2^N + \sum_{(i,j)\in I'} j\alpha_{ij}(z) y_1^i y_2^{j-1}, \\
y_2' &= -(M+1)y_1^M - \sum_{(i,j)\in I'} i\alpha_{ij}(z) y_1^{i-1} y_2^j.
\end{aligned}
\tag{3.6}
$$

For $N \geq M$, condition (3.2) in fact implies that $j \leq N-1$ for all $(i,j) \in I'$.

An approximate first integral

In this section we will show the existence of a function W that remains bounded whenever a solution $(y_1(z), y_2(z))$ of (3.6) develops a movable singularity by analytic continuation along a finite length curve. Formally inserting the series expansions (3.4) for y_1 and y_2 into

$$
H' = \frac{dH}{dz} = \frac{\partial H}{\partial z} = \sum_{(i,j)\in I'} \alpha_{ij}'(z) y_1(z)^i y_2(z)^j, \tag{3.7}
$$

yields a formal series expansion for H' in $(z-\hat{z})^{\frac{1}{MN-1}}$. Heuristically, W is constructed from H by adding certain terms, rational in y_1 and y_2, which would cancel all terms of H' with negative powers of $(z-\hat{z})^{\frac{1}{MN-1}}$. Note, however, that terms of power $(z-\hat{z})^{-1}$ cannot be cancelled in this way, since these would correspond to terms of H that are logarithmic in $z-\hat{z}$ and cannot be obtained by rational expressions in y_1 and y_2. We define

$$W(z,y_1,y_2) = y_1^{M+1} + y_2^{N+1} + \sum_{(i,j)\in I'} \alpha_{ij}(z) y_1^i y_2^j + \sum_{(k,l)\in J} \beta_{kl}(z) \frac{y_2^k}{y_1^l}, \tag{3.8}$$

where the $\beta_{kl}(z)$ are certain analytic functions to be determined in terms of the $\alpha_{ij}(z)$ and their derivatives, and the index set J is given by

$$J = \{(k,l) \in \mathbf{N}^2 : 1 \leq k \leq N+1, 1 - MN < k(M+1) - l(N+1) < M+N+2\}.$$

Note that the pairs of indices in the set J are in one-to-one correspondence with the elements of the set $I \setminus \{(0,0)\}$, which can easily be seen by setting $k = j+1$ and $l = M-i$. Thus for every unbounded term $\alpha'_{ij}(z) y_1^i y_2^j$ in (3.7) there is one function β_{kl} to compensate for. However, we will see that not all the functions β_{kl} can be used. The other essential ingredient is the existence of the formal series solutions (3.4), which will ensure that the terms of power $(z-z_0)^{-1}$ vanish identically. To show that the auxiliary function W is bounded we will again use Lemma 2.6. In order to be able to apply it we must first show that, by modification of the curve Γ, we can achieve that certain rational expressions in y_1 and y_2 are bounded along Γ.

Lemma 3.2. *Let (y_1, y_2) be a solution of the system (3.6), analytic on the finite length curve Γ ending in a movable singularity z_∞, such that $\frac{1}{y_1}$ and $\frac{1}{y_2}$ are bounded on Γ. Then, after a possible deformation of Γ in the region where y_1, y_2 are analytic, one can achieve that $\frac{y_2^k}{y_1^l}$ is bounded on $\tilde{\Gamma}$ for all $k, l \geq 0$ for which $l(N+1) - k(M+1) \geq 0$.*

We have put the proof of this lemma, which is rather technical, into the appendix. Assuming that we have modified the curve according to Lemma 3.2, the next lemma shows that the auxiliary function W remains bounded as a movable singularity is approached.

Lemma 3.3. *The coefficients $\beta_{kl}(z)$, $(k,l) \in J$, in (3.8) can be chosen such that the function W is bounded on the curve $\tilde{\Gamma}$.*

Proof. Taking the total z-derivative of (3.8) one obtains

$$\begin{aligned}
W' &= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} + k\beta_{kl} \frac{y_2^{k-1} y_2'}{y_1^l} - l\beta_{kl} \frac{y_2^k y_1'}{y_1^{l+1}} \right) \\
&= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j - \sum_{(i,j)\in I'} \sum_{(k,l)\in J} (ik+jl)\alpha_{ij}\beta_{kl} y_1^{i-l-1} y_2^{k+j-1} \\
&\quad + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} - k(M+1)\beta_{kl} y_1^{M-l} y_2^{k-1} - l(N+1)\beta_{kl} \frac{y_2^{N+k}}{y_1^{l+1}} \right) \\
&= \sum_{(i,j)\in I'} \alpha'_{ij} y_1^i y_2^j + \sum_{(k,l)\in J} (l(N+1) - k(M+1))\beta_{kl} y_1^{M-l} y_2^{k-1} \qquad (3.9) \\
&\quad + \sum_{(k,l)\in J} \left(\beta'_{kl} \frac{y_2^k}{y_1^l} - l(N+1)\beta_{kl} \frac{y_2^{k-1}}{y_1^{l+1}} W \right) \\
&\quad + \sum_{(i,j)\in I'} \sum_{(k,l)\in J} (l(N-j+1) - ik)\alpha_{ij}\beta_{kl} y_1^{i-l-1} y_2^{k+j-1} \\
&\quad + \sum_{(k,l)\in J} \sum_{(k',l')\in J} l(N+1)\beta_{kl}\beta_{k'l'} \frac{y_2^{k+k'-1}}{y_1^{l+l'+1}},
\end{aligned}$$

where we have used (3.8). All terms in (3.9) are now either of the form $y_1^{i_0} y_2^{j_0}$ with $(i_0, j_0) \in I$, or of the form $\frac{y_2^{j_0}}{y_1^{i_0}}$ with $i_0 \geq 1$ and $j_0(M+1) - i_0(N+1) < (M+1)(N+1)$. Note also that for the coefficients $\frac{y_2^{k-1}}{y_1^{l+1}}$ of W, $(k,l) \in J$, we have $(l+1)(N+1) - (k-1)(M+1) \geq 0$, so by Lemma 3.2 these are bounded on $\tilde{\Gamma}$. By repeating the process of replacing powers y_2^{N+1} using (3.8) one can achieve in a finite number of steps that the terms of the form $\frac{y_2^{j_0}}{y_1^{i_0}}$ either have $j_0 \geq N+1$ with $i_0(N+1) - j_0(M+1) \geq 0$ and are therefore bounded by Lemma 3.2, or have $j_0 \leq N$ and $j_0(M+1) - i_0(N+1) \leq MN - 1$, equality holding if and only if $(i_0, j_0) = (1, N)$. We now manipulate the terms of the form $\frac{y_2^{j_0}}{y_1^{i_0}}$, $j_0 \leq N$, in the following way:

$$\begin{aligned}
&(M+1)(j_0+1) \frac{y_2^{j_0}}{y_1^{i_0}} \\
&\quad = -(j_0+1) \frac{y_2' y_2^{j_0}}{y_1^{M+i_0}} - \sum_{(i,j)\in I'} i(j_0+1)\alpha_{ij} \frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\quad = -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}} \right)' - (M+i_0) \frac{y_2^{j_0+1} y_1'}{y_1^{M+i_0+1}} - \sum_{(i,j)\in I'} i(j_0+1)\alpha_{ij} \frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\quad = -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}} \right)' - (N+1)(M+i_0) \frac{y_2^{N+j_0+1}}{y_1^{M+i_0+1}} - \sum_{(i,j)\in I'} (i(j_0+1) + j(M+i_0))\,\alpha_{ij} \frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\quad = -\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}} \right)' - (N+1)(M+i_0) \frac{y_2^{j_0}}{y_1^{M+i_0+1}} W \\
&\qquad + \sum_{(i,j)\in I'} ((N+1)(M+i_0) - j(M+i_0) - i(j_0+1))\,\alpha_{ij} \frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&\qquad + \sum_{(k,l)\in J} (N+1)(M+i_0)\beta_{kl} \frac{y_2^{k+j_0}}{y_1^{M+l+i_0+1}} + (N+1)(M+i_0) \frac{y_2^{j_0}}{y_1^{i_0}}.
\end{aligned} \qquad (3.10)$$

Thus, unless $j_0(M+1) - i_0(N+1) = MN - 1$, one can solve (3.10) for $\frac{y_2^{j_0}}{y_1^{i_0}}$:

$$
\begin{aligned}
\frac{y_2^{j_0}}{y_1^{i_0}} =& \frac{1}{MN-1+i_0(N+1)-j_0(M+1)} \Bigg((N+1)(M+i_0) \frac{y_2^{j_0}}{y_1^{M+i_0+1}} W \\
&+ \sum_{(i,j)\in I'} \left(i(j_0+1) + j(M+i_0) - (N+1)(M+i_0)\right) \alpha_{ij} \frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}} \\
&- \sum_{(k,l)\in J} (N+1)(M+i_0)\beta_{kl} \frac{y_2^{k+j_0}}{y_1^{M+l+i_0+1}} + \left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}} \right)' \Bigg).
\end{aligned}
\tag{3.11}
$$

Again, in (3.11) the coefficient $\frac{y_2^{j_0}}{y_1^{M+i_0+1}}$ of W is bounded by Lemma 3.2 since we have $(M+i_0+1)(N+1) - j_0(M+1) > 0$. Also, the term $\frac{y_2^{j_0+1}}{y_1^{M+i_0}}$ is bounded by Lemma 3.2 since $(M+i_0)(N+1) - (j_0+1)(M+1) > 0$. Therefore, the term $\left(\frac{y_2^{j_0+1}}{y_1^{M+i_0}}\right)'$ is bounded when integrated over the finite length curve $\tilde{\Gamma}$. For the terms of type $\frac{y_2^{k+j_0}}{y_1^{M+l+i_0+1}}$, $(k,l) \in J$, we find

$$(M+l+i_0+1)(N+1) - (k+j_0)(M+1) \geq 0,$$

which are therefore all bounded, and for the terms $\frac{y_2^{j+j_0}}{y_1^{M-i+i_0+1}}$, $(i,j) \in I'$,

$$(j+j_0)(M+1) - (M-i+i_0+1)(N+1) < j_0(M+1) - i_0(N+1).$$

We can thus replace $\frac{y_2^{j_0}}{y_1^{i_0}}$ by terms which are bounded or proportional to W with bounded factor, and a sum of terms of the form $\frac{y_2^{j_1}}{y_1^{i_1}}$ with $j_1 = j + j_0$, $i_1 = M - i + i_0 + 1$, such that the quantity $j_1(M+1) - i_1(N+1)$ is strictly decreasing. Performing this process iteratively a finite number of times we eventually end up only with terms $\frac{y_2^{j_n}}{y_1^{i_n}}$ for which $j_n(M+1) - i_n(N+1) \leq 0$, Lemma 3.2 showing that they are bounded on $\tilde{\Gamma}$.

We thus arrive at a first-order differential equation for W of the form

$$
\begin{aligned}
W' =& P(z, y_1^{-1}, y_2)W + \sum_{(i,j)\in I} \gamma_{ij}(z) y_1^i y_2^j + \gamma_{-1N}(z) \frac{y_2^N}{y_1} \\
&+ Q(z, y_1^{-1}, y_2) + \frac{d}{dz} R(z, y_1^{-1}, y_2),
\end{aligned}
$$

where P, Q and R are polynomial in their last two arguments and for each monomial $\frac{y_2^k}{y_1^l}$ we have $l(N+1) - k(M+1) \geq 0$, so they are bounded on $\tilde{\Gamma}$. We will now show that, by a suitable choice of the β_{kl} and the existence of the formal series solutions (3.4), all the coefficients γ_{ij}, $(i,j) \in I$, as well as γ_{-1N}, are identically 0.

We determine the functions $\beta_{kl} = \beta_{j+1,M-i}$ recursively starting with the pairs $(i,j) \in I$ for which the quantity $i(N+1) + j(M+1)$ is maximal. From (3.9) we see that

$$\gamma_{ij}(z) = \alpha'_{ij}(z) + (MN - 1 - i(N+1) - j(M+1))\beta_{j+1,M-i}(z) + \cdots, \tag{3.12}$$

where the dots stand for expressions involving only terms $\beta_{k'l'} = \beta_{j'+1,M-i'}$ for which $i'(N+1) + j'(M+1)$ is strictly greater than $i(N+1) + j(M+1)$. We can thus determine

$\beta_{kl} = \beta_{j+1,M-i}$ for all pairs $(i,j) \in I$ for which $i(N+1)+j(M+1) > MN-1$. However, when $i(N+1)+j(M+1) = MN-1$, the coeffcient of $\beta_{j+1,M-i}$ in (3.12) vanishes. We now make use of the existence of the formal series solutions (3.4) to show that also $\gamma_{ij} \equiv 0$ in this case.

Let $n = \frac{N+1}{d}$ and $m = \frac{M+1}{d}$ where $d = \gcd\{M+1, N+1\}$. Consider the d terms $\gamma_{-1,N}(z)\frac{y_2^N}{y_1}$, $\gamma_{m-1,N-n}(z)y_1^{m-1}y_2^{N-n}, \ldots, \gamma_{M-m,n-1}(z)y_1^{M-m}y_2^{n-1}$. When one inserts the formal series solutions (3.4) into these expressions they have leading order $(z-\hat{z})^{-1}$. But there are essentially d formal series solutions corresponding to the different choices of the leading coefficients $c_{1,-N-1}, c_{2,-M-1}$ such that $c_{1,-N-1}^{MN-1} = -\frac{1}{(MN-1)^{N+1}}$. Inserting any of the series into (3.8) shows that W has a Laurent series expansion in powers of $(z-\hat{z})^{1/(MN-1)}$. Thus the coefficient of $(z-\hat{z})^{-1}$ in W' vanishes since otherwise W would have terms logarithmic in $z-\hat{z}$ in its expansion. The coefficients of $(z-\hat{z})^{-1}$ in W', for the different choices of $(c_{1,-N-1}, c_{2,-M-1})$, are

$$\begin{aligned}
\frac{-1}{MN-1}\left(\gamma_{-1,N}(\hat{z}) + \omega_1\gamma_{m-1,N-n}(\hat{z}) + \cdots + \omega_1^{d-1}\gamma_{M-m,n-1}(\hat{z})\right) &= 0\\
\frac{-1}{MN-1}\left(\gamma_{-1,N}(\hat{z}) + \omega_2\gamma_{m-1,N-n}(\hat{z}) + \cdots + \omega_2^{d-1}\gamma_{M-m,n-1}(\hat{z})\right) &= 0\\
&\vdots\\
\frac{-1}{MN-1}\left(\gamma_{-1,N}(\hat{z}) + \omega_d\gamma_{m-1,N-n}(\hat{z}) + \cdots + \omega_d^{d-1}\gamma_{M-m,n-1}(\hat{z})\right) &= 0,
\end{aligned}$$

where ω_i, $i = 1, \ldots, d$, are the d distinct roots of $\omega^d = -1$. This system of d equations shows

$$\gamma_{-1,N}(\hat{z}) = \gamma_{m-1,N-n}(\hat{z}) = \cdots = \gamma_{M-m,n-1}(\hat{z}) = 0.$$

However, the formal series expansions exist for all $\hat{z}$ in a neighbourhood of z_∞. Therefore we have shown in fact that

$$\gamma_{-1,N} = \gamma_{m-1,N-n} = \cdots = \gamma_{M-m,n-1} \equiv 0.$$

The functions $\beta_{j+1,M-i}$ with $i(N+1)+j(M+1) = MN-1$ can be chosen arbitrarily and will henceforth be set to 0. The remaining functions $\beta_{j+1,M-i}$ with $i(N+1)+j(M+1) < MN-1$ can now all be determined recursively, so that $\gamma_{ij} \equiv 0$ for all $(i,j) \in I \cup \{(-1,N)\}$. We have thus arrived at a first-order linear differential equation for W of the form

$$W' = P(z, y_1^{-1}, y_2)W + Q(z, y_1^{-1}, y_2) + R'(z, y_1^{-1}, y_2), \tag{3.13}$$

where P, Q and R are bounded on $\tilde{\Gamma}$ near a movable singularity z_∞ of a solution $(y_1(z), y_2(z))$. Lemma 2.6 now shows that W is bounded on $\tilde{\Gamma}$. $\square$

A regular initial value problem

To show that a movable singularity is an algebraic branch point we will now introduce coordinates u and v for which there exists a regular initial value problem. The coordinate u is defined by

$$y_1 = u^{-\frac{N+1}{d}}, \tag{3.14}$$

where a choice of branch is made. We also define

$$w = y_2 u^{\frac{M+1}{d}}. \tag{3.15}$$

From (3.8) one obtains an algebraic equation for w,

$$\begin{aligned} 0 =& w^{N+1} + \sum_{(i,j)\in I'} \alpha_{ij}(z) u^{\frac{(M+1)(N+1)-i(N+1)-j(M+1)}{d}} w^j \\ &+ \sum_{(k,l)\in J} \beta_{kl}(z) u^{\frac{(M+1)(N+1)+l(N+1)-k(M+1)}{d}} w^k + 1 - W u^{\frac{(M+1)(N+1)}{d}}, \end{aligned} \tag{3.16}$$

all the exponents of u being positive integers. The solutions of this equation for w will be denoted by $w_1, \ldots, w_{N+1}$. They are analytic functions of u, z and W in some neighbourhood of $u = 0$, $z = z_\infty$ and $W = W_0$ for any $W_0 \in \mathbf{C}$. We express the w_n as power series in u and W with analytic coefficients in z,

$$w_n = F_n(z, u, W) = \omega_n \sum_{j,k=0}^{\infty} a_{jkn}(z) u^j W^k,$$

where ω_n, $n = 1, \ldots, N+1$, are the distinct roots of $\omega^{N+1} = -1$, $a_{00n} \equiv 1$, and the first monomial containing W is of the form $-\frac{1}{N+1} u^{\frac{(M+1)(N+1)}{d}} W$. We denote $\bar{F}_n(z,u) = \sum_{j=0}^{\frac{(M+1)(N+1)}{d}} a_{j0n}(z) u^j$ and define functions v_n by

$$w_n = \omega_n \left(\bar{F}_n(z,u) - \frac{1}{N+1} u^{\frac{(M+1)(N+1)}{d}} v_n \right), \tag{3.17}$$

so that in the limit $u \to 0$, v_n agrees to leading order with W. From the definiton (3.15) of w we see that the choice of branch for ω_n can partially be absorbed into the original choice of branch for u if $1 < d < M+1$, and completely be absorbed if $d = 1$, so that there are essentially only d inequivalent choices for (u, v_n). From (3.14) and (3.6) we obtain the differential equation satisfied by u:

$$\begin{aligned} u' =& -\frac{d}{N+1} u^{\frac{N+1}{d}+1} \Bigg[(N+1)\omega_n^N \left(u^{-\frac{M+1}{d}} \bar{F}_n(z,u) - \frac{1}{N+1} u^{\frac{(M+1)N}{d}} v_n \right)^N \\ &+ \sum_{(i,j)\in I'} j\alpha_{ij}(z) u^{-i\frac{N+1}{d}} \omega_n^{j-1} \left(u^{-\frac{M+1}{d}} \bar{F}_n(z,u) - \frac{1}{N+1} u^{\frac{(M+1)N}{d}} v_n \right)^{j-1} \Bigg]. \end{aligned} \tag{3.18}$$

Taking the reciprocal of (3.18) and changing the role of the dependent and independent variables u and z one obtains, extracting the highest power of u on the right hand side, an equation of the form,

$$\frac{dz}{du} = u^{\frac{MN-1}{d}-1} A(u, z, v), \tag{3.19}$$

where $A(u,z,v)$ is analytic in (u,z,v) at $(0, z_\infty, v_0)$ for any $v_0 \in \mathbf{C}$, and $A(0, z_\infty, v_0) = \frac{\omega_n}{d}$. We drop the index n from now on. Re-inserting (3.17) into (3.16) yields an expression for W in terms of u and v of the form

$$W = v + G(z, u, v), \tag{3.20}$$

where G is a polynomial in v of degree $N+1$ and analytic in z and u near $u=0$, satisfying $G(z,0,v)=0$. We differentiate (3.20) with respect to z,

$$W' = v' + G_z + G_u u' + G_v v', \tag{3.21}$$

and compare this with equation (3.13), which can be written in the form

$$\begin{aligned} W' &= \tilde{P}(z,u,v)W + \tilde{Q}(z,u,v) + \frac{d}{dz}\tilde{R}(z,u,v) \\ &= \tilde{P}(v+G) + \tilde{Q} + \tilde{R}_z + \tilde{R}_u u' + \tilde{R}_v v', \end{aligned} \tag{3.22}$$

where $\tilde{P}$, $\tilde{Q}$ and $\tilde{R}$ are polynomial in u and v. One can solve (3.21) and (3.22) for v' to obtain an equation of the form

$$v' = B(z,u,v)u' + C(z,u,v), \tag{3.23}$$

where B and C are analytic in their arguments. Multiplying (3.23) by (3.19) one obtains an equation for v as a function of u:

$$\frac{dv}{du} = \frac{dv}{dz}\frac{dz}{du} = B(z,u,v) + u^{\frac{MN-1}{d}-1}A(z,u,v)C(z,u,v). \tag{3.24}$$

Equations (3.19) and (3.24) together form a regular initial value problem for z and v as functions of u near $u=0$ with $z(0)=z_\infty$ and $v(0)=v_0$.

Proof of Theorem 3.1

We can now complete the proof of Theorem 3.1.

Proof. By Lemma 3.3, after a possible modification of Γ to $\tilde{\Gamma}$, W is bounded along $\tilde{\Gamma}$. Consider a sequence $(z_n) \subset \tilde{\Gamma}$ such that $z_n \to z_\infty$ as $n \to \infty$. Suppose that the sequence $(y_1(z_n))$ is bounded. Then the functional form of $W(z,y_1,y_2)$ implies that the sequence $(y_2(z_n))$ is also bounded. However, Lemma 2.4 now implies that the solution (y_1,y_2) can be analytically continued to z_∞, contradicting the assumption in the theorem. Therefore, the sequence $(y_1(z_n))$ must tend to infinity since otherwise it would have a bounded subsequence. In the variables u,v introduced in the previous section we therefore have $u(z_n) \to 0$ and $v(z_n)$ is bounded. Hence there exists some subsequence (z_{n_k}) such that $v(z_{n_k}) \to v_0$ for some $v_0 \in \mathbf{C}$. Now Equations (3.19) and (3.24) form a regular initial value problem for z and v as functions of u with initial values z_∞ and v_0 at $u=0$. Lemma 2.4 then shows that z and v are analytic at $u=0$. Since $A(0,z_\infty,v_0) \neq 0$ in (3.19), z has a convergent power series expansion of the form

$$z = z_\infty + \sum_{k=0}^{\infty} \xi_k u^{k+\frac{MN-1}{d}},$$

in a neighbourhood of $u=0$. Taking the $\frac{MN-1}{d}$-th root,

$$(z-z_\infty)^{\frac{d}{MN-1}} = \sum_{k=1}^{\infty} \eta_k u^k,$$

and inverting the power series, one shows that u has a convergent series expansion

$$u = \sum_{k=1}^{\infty} \zeta_k (z - z_\infty)^{\frac{kd}{MN-1}}.$$

By the definition (3.14) of u, one obtains a series expansion for y_1,

$$y_1(z) = \sum_{k=-\frac{N+1}{d}}^{\infty} C_{1,k}(z - z_\infty)^{\frac{kd}{MN-1}},$$

convergent in a branched, punctured neighbourhood of z_∞. Also, from the definition (3.15) we find, since $w \neq 0$ at $z = z_\infty$,

$$y_2(z) = \sum_{k=-\frac{M+1}{d}}^{\infty} C_{2,k}(z - z_\infty)^{\frac{kd}{MN-1}}.$$

$\square$

3.2 Hamiltonian systems of Painlevé type

In the last section we discussed a class of Hamiltonian systems in two dependent variables with movable algebraic singularities. Interestingly, all six Painlevé equations can be written in an equivalent form as Hamiltonian systems with polynomial Hamiltonians $H_J(z, q, p)$, $J = I, \ldots, VI$,

$$\begin{aligned}
H_I &= \frac{1}{2}q^2 - 2p^3 - zp \\
H_{II} &= \frac{1}{2}q^2 - \left(p^2 - \frac{z}{2}\right) q - \kappa p \\
H_{III} &= \frac{1}{z}\left[2q^2p^2 - \left(2\eta_\infty z p^2 + (2\kappa_0 + 1)p - 2\eta_0 z\right) q + \eta_\infty(\kappa_0 + \kappa_\infty) zp\right] \\
H_{IV} &= 2pq^2 - \left(p^2 + 2zp + \kappa_0\right) q + \kappa_\infty p \\
H_V &= \frac{1}{z}\left[p(p-1)^2q^2 - \left(\kappa_0(p-1)^2 + \kappa_t p(p-1) - \eta z p\right) q + \kappa(p-1)\right] \\
H_{VI} &= \frac{1}{z(z-1)}\left[p(p-1)(p-z)q^2 - \left[\kappa_0(p-1)(p-z) + \kappa_1 p(p-z)\right.\right. \\
&\quad \left.\left. +(\kappa_t - 1)p(p-1)\right] q + \kappa(p-t)\right],
\end{aligned}$$

where the various κ's and η's are arbitrary complex parameters. These were already known to Malmquist [26] and later have been studied extensively by Okamoto in a series of four papers [34, 35, 36, 37]. A classification of systems of equations

$$q' = P(z, q, p), \quad p' = Q(z, q, p), \tag{3.25}$$

with the Painlevé property has not been carried out to this date. It has been conjectured in [21], however, that any system (3.25) with the Painlevé property which cannot be reduced to the integration of a first-order equation or a linear second-order equation is equivalent to one of the Hamiltonian systems H_J, $J = I, \ldots, VI$. In [33], Okamoto also constructed, for each of the six Painlevé equations, what he called the *space of initial conditions*. In this space, every point defines a regular initial value problem for the system of equations considered. It is obtained by compactifying the space of dependent variables $(q, p) \in \mathbf{C}^2$ to some rational surface and applying a sequence of *blow-ups* to the space, a certain algebro-geometric construction to remove certain points of indeterminacy of the system. We will construct the space of initial conditions for one of the systems contained in the class of section 3.1 below.

We investigate which of the Hamiltonian systems contained in the class of section 3.1 have the Painlevé property. For $N = 1$ the Hamiltonian can be written as

$$H(z, y_1, y_2) = \frac{1}{2} y_2^2 + y_2 Q(z, y_1) + P(z, y_1),$$

where P is a polynomial in y_1 of degree M and Q a polynomial of degree less than $\frac{M+1}{2}$. The Hamiltonian system of equations is then

$$\begin{aligned} y_1' &= y_2 + Q(z, y_1), \\ y_2' &= -y_2 Q_y(z, y_1) - P_y(z, y_1). \end{aligned}$$

Letting $\tilde{y}_2 = y_2 + Q(z, y_1)$ this reduces to the system

$$\begin{aligned} y_1' &= \tilde{y}_2, \\ \tilde{y}_2' &= y_2' + Q_z(z, y_1) + Q_y(z, y_1)\tilde{y}_2 \\ &= -P_y(z, y_1) + Q_z(z, y_1) - Q(z, y_1) Q_y(z, y_1), \end{aligned}$$

i.e. essentially a second-order differential equation for $y = y_1$:

$$y'' = \tilde{P}(z, y) := -P_y(z, y) + Q_z(z, y) - Q(z, y) Q_y(z, y),$$

where the right hand side is an arbitrary polynomial in y. This is the class of second-order equations treated in [7] which includes as special cases the Painlevé equations *I* and *II*, where $\tilde{P}$ is of third and fourth degree, respectively. The next higher case of Hamiltonian systems in the class of section 3.1 is $M = N = 2$ where we have

$$H(z, y_1, y_2) = \frac{1}{3}\left(y_1^3 + y_2^3\right) + \alpha(z) y_1 y_2 + \beta(z) y_1 + \gamma(z) y_2.$$

The resonance conditions in this case are $\alpha'' \equiv 0$, $\beta' \equiv 0$ and $\gamma' \equiv 0$, that is, α is a linear function in z, $\alpha = az + b$, and β and γ are constants. In case $a = 0$ the system is autonomous and can be integrated by classical methods. If $a \neq 0$, by a linear transformation in z we can accomplish that $\alpha(z) = z$ and are therefore essentially left with the system of equations

$$\begin{aligned} y_1' &= y_2^2 + z y_1 + \gamma, \\ y_2' &= -y_1^2 - z y_2 - \beta. \end{aligned} \tag{3.26}$$

This system has meromorphic solutions with Laurent series expansions,

$$
\begin{aligned}
y_1(z) =& \frac{-\omega^j}{z-z_*} + \frac{\omega^j z_*}{2} + \left(\omega^j\left(1+\frac{z_*^2}{4}\right) - \frac{\alpha}{3} + \frac{2}{3}\omega^{2j}\beta\right)(z-z_*) + h(z-z_*)^2 \\
&+ \sum_{n=3}^{\infty} c_n (z-z_*)^n \\
y_2(z) =& \frac{\omega^{2j}}{z-z_*} + \frac{\omega^{2j} z_*}{2} + \left(\omega^{2j}\left(1-\frac{z_*^2}{4}\right) - \frac{2}{3}\omega^j\alpha + \frac{\beta}{3}\right)(z-z_*) + k(z-z_*)^2 \\
&+ \sum_{n=3}^{\infty} d_n (z-z_*)^n,
\end{aligned}
$$

$j \in \{0,1,2\}$, where $h = c_2$ and $k = d_2$ are coupled by the relation

$$
\omega^j h - k = \left(\frac{5}{4}\omega^{2j} - \frac{\omega^j \alpha}{2} + \frac{\beta}{2}\right) z_*,
$$

and either h or k can be taken to be an arbitrary parameter. Thus there are three possible leading order beahviours for the simples poles of the solutions. The system (3.26) is related to the fourth Painlevé equation as the combination $w = y_1 + y_2 - z$ solves [1]

$$
2ww'' = (w')^2 - w^4 - 4zw^3 - (2\beta + 2\gamma + 3z^2)w^2 - (1-\beta+\gamma)^2.
$$

The combinations $w_1 = \omega y_1 + y_2 - \omega^2 z$ and $w_2 = \omega^2 y_1 + y_2 - \omega z$ both satisfy similar equations and therefore y_1 and y_2 can be expressed completely in terms of the fourth Painlevé transcendents.

For all other Hamiltonian systems in the class of section 3.1 the solutions are branched.

3.3 Okamoto's space of initial conditions

At a singularity z_0 of a solution of the system (3.26) we have by Lemma 2.4,

$$
\lim_{z\to z_0} \max\{|y_1(z)|, |y_2(z)|\} = \infty.
$$

To study the singularities of the system it therefore seems natural to compactify the space $\mathbf{C}^2$ of the variables (y_1, y_2), for example to the complex projective space. $\mathbf{CP}^2$ is covered by the three standard coordinate charts

$$
[1 : y_1 : y_2], \quad [u_1 : 1 : u_2], \quad [v_1 : v_2 : 1].
$$

The set consisting of the points where $u_1 = 0$ or $v_1 = 0$ is called the *line at infinity* which we denote by L. Re-writing the system of equations (3.26) in the other two coordinate charts one obtains certain *base points* where the right hand side of the system becomes indeterminate. For the case of the six Painlevé equations, Okamoto [33] has shown that the base points can be removed by a sequence of blow-ups of these points. Blowing up is an algebro-geometric construction of regularising the points of indeterminacy which will be explained below. Okamoto showed for each of the Painlevé equations that a sequence

[1] I thank Norbert Steinmetz for making me aware of this fact.

of nine blow-ups suffices to obtain a space on which the equation possesses a regular initial value problem at every point. This space is the so-called space of initial conditions. We will now demonstrate the construction of the space of initial conditions for the system (3.26). In the variables (u_1, u_2), respectively (v_1, v_2), the system of equations becomes

$$\begin{aligned} u_1' &= -\gamma u_1^2 - z u_1 - u_2^2 \\ u_2' &= -\frac{\beta u_1^2 + \gamma u_1^2 u_2 + u_2^3 + 2z u_1 u_2 + 1}{u_1}, \\ v_1' &= \beta v_1^2 + z v_1 + v_2^2 \\ v_2' &= \frac{\beta v_1^2 v_2 + \gamma v_1^2 + v_2^3 + 2z v_1 v_2 + 1}{v_1}. \end{aligned} \tag{3.27}$$

Let $\omega = \frac{-1+\sqrt{3}}{2}$ denote the third root of unity. We consider the points $(u_1, u_2), (v_1, v_2) \in \{(0, -1), (0, -\omega), (0, -\omega^2)\}$ where the right hand sides of these systems of equations are indeterminate as they become expressions of the form $\frac{0}{0}$. These are called the *base points* of the compactified system of equations. Note that $(u_1, u_2) = (0, -1)$ and $(v_1, v_2) = (0, -1)$ describe the same point in $\mathbf{CP}^2$. Similarly $(u_1, u_2) = (0, -\omega)$ and $(v_1, v_2) = (0, -\omega^2)$ describe the same point, as well as $(u_1, u_2) = (0, -\omega^2)$ and $(v_1, v_2) = (0, -\omega)$. So the system of equations has three distinct base points. We will now describe the procedure of blowing up the surface at one of these base points. The blow up at the point $p = (p_1, p_2)$ in the coordinate chart (u_1, u_2) is defined as

$$\mathrm{Bl}_p\mathbf{C}^2 = \{((u_1, u_2), [z_1 : z_2]) \in \mathbf{C}^2 \times \mathbf{CP}^1 : (u_1 - p_1)z_2 = (u_2 - p_2)z_1\}.$$

The projection $\pi : \mathrm{Bl}_p(\mathbf{C}^2) \to \mathbf{C}^2$ is given by $\pi : ((u_1, u_2), [z_1 : z_2]) \mapsto (u_1, u_2)$. We can see that for any point $q \neq p$ the pre-image $\pi^{-1}(q)$ is a single point whereas the pre-image of p itself is $\pi^{-1}(p) = (0, 0) \times \mathbf{CP}^1$. This is called the *exceptional curve* in $\mathrm{Bl}_p(\mathbf{C}^2)$. So in some sense we have extended the space where the point p has been blown up to a sphere $\mathbf{CP}^1$. The idea is that the singularity at p is smeared out over this sphere and may eventually disappear. Some of the calculations in this section have been carried out using Mathematica.

To perform the blow-up in the coordinates (u_1, u_2) we introduce two new coordinate charts, $(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ and $(\tilde{u}_{2,1}, \tilde{u}_{2,2})$. The first coordinate chart covers the part of $\mathrm{Bl}_p(\mathbf{C}^2)$ where $z_2 \neq 0$,

$$\tilde{u}_{1,1} = \frac{z_1}{z_2} = (u_1 - p_1)(u_2 - p_2)^{-1}, \quad \tilde{u}_{1,2} = u_2 - p_2.$$

The second coordinate chart covers the part of $\mathrm{Bl}_p(\mathbf{C}^2)$ where $z_1 \neq 0$,

$$\tilde{u}_{2,1} = u_1 - p_1, \quad \tilde{u}_{2,2} = (u_1 - p_1)^{-1}(u_2 - p_2).$$

In these coordinate charts, the system of equations takes the following forms.

$$\begin{aligned} \tilde{u}_{1,1}' &= \frac{2 - 2z\tilde{u}_{1,1} - \tilde{u}_{1,2} + z\tilde{u}_{1,1}\tilde{u}_{1,2} + (\beta - \gamma)\tilde{u}_{1,1}^2\tilde{u}_{1,2}}{\tilde{u}_{1,2}} \\ \tilde{u}_{1,2}' &= \frac{(\gamma - \beta)\tilde{u}_{1,1}^2\tilde{u}_{1,2} - \gamma\tilde{u}_{1,1}^2\tilde{u}_{1,2}^2 + 2z\tilde{u}_{1,1} - 2z\tilde{u}_{1,1}\tilde{u}_{1,2} - \tilde{u}_{1,2}^2 + 3\tilde{u}_{1,2} - 3}{\tilde{u}_{1,1}}, \end{aligned}$$

$$
\begin{aligned}
\tilde{u}'_{2,1} =& -1 - z\tilde{u}_{2,1} - \tilde{u}_{2,1}^2 + 2\tilde{u}_{2,1}\tilde{u}_{2,2} - \tilde{u}_{2,1}^2\tilde{u}_{2,2}^2 \\
\tilde{u}'_{2,2} =& \frac{2z - 2\tilde{u}_{2,2} + (\gamma-\beta)\tilde{u}_{2,1} + \tilde{u}_{2,1}\tilde{u}_{2,2}^2 - z\tilde{u}_{2,1}\tilde{u}_{2,2}}{\tilde{u}_{2,1}}.
\end{aligned}
$$

The exceptional curve introduced by this blow-up wil be denoted by $L_1^{(1)}$ which in the coordinate chart $(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ corresponds to the set $(0, c)$ $c \in \mathbf{C}$ and in the chart $(\tilde{u}_{2,1}, \tilde{u}_{2,2})$ corresponds to the set $(c, 0)$ $c \in \mathbf{C}$. We see that in the coordinate chart $(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ there are still three base points, at $(0, 1-\omega)$, $(0, 1-\omega^2)$ and $(\frac{1}{z}, 0)$. The first two correspond to the base points $(0, -\omega)$ and $(0, -\omega^2)$ in the coordinates (u_1, u_2). The third point lies on the exceptional curve introduced by the blow-up. So we see that despite having performed a blow-up the singular behaviour has not disappeared. We will see, however, that by performing two further blow-ups it will disappear. We perform the blow-up in the coordinate chart $(\tilde{u}_{2,1}, \tilde{u}_{2,2})$ which covers only the base point at $(0, z)$. Again we introduce two new coordinate charts,

$$
\bar{u}_{1,1} = \frac{\tilde{u}_{2,1}}{\tilde{u}_{2,2} - z}, \quad \bar{u}_{1,2} = \tilde{u}_{2,2} - z,
$$

$$
\bar{u}_{2,1} = \tilde{u}_{2,1}, \quad \bar{u}_{2,2} = \frac{\tilde{u}_{2,2} - z}{\tilde{u}_{2,1}}.
$$

In these coordinates, the system of equations takes the following forms.

$$
\begin{aligned}
\bar{u}'_{1,1} =& \frac{1}{\bar{u}_{1,2}}\left(1 + (1+\beta-\gamma)\bar{u}_{1,1} + \bar{u}_{1,1}\bar{u}_{1,2}^2 - (z^2+\gamma)\bar{u}_{1,1}^2\bar{u}_{1,2}^2 - 2z\bar{u}_{1,1}^2\bar{u}_{1,2}^3 - \bar{u}_{1,1}^2\bar{u}_{1,2}^4\right) \\
\bar{u}'_{1,2} =& \frac{1}{\bar{u}_{1,1}}\left(-2 - (1+\beta-\gamma)\bar{u}_{1,1} + z\bar{u}_{1,1}\bar{u}_{1,2} + \bar{u}_{1,1}\bar{u}_{1,2}^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\bar{u}'_{2,1} =& -1 + z\bar{u}_{2,1} - (z^2+\gamma)\bar{u}_{2,1}^2 + 2\bar{u}_{2,1}^2\bar{u}_{2,2} - 2z\bar{u}_{2,1}^3\bar{u}_{2,2} - \bar{u}_{2,1}^4\bar{u}_{2,2}^2 \\
\bar{u}'_{2,2} =& \frac{1}{\bar{u}_{2,1}}\left(-1 - \beta + \gamma - \bar{u}_{2,2} + (z^2+\gamma)\bar{u}_{2,1}^2\bar{u}_{2,2} - \bar{u}_{2,1}^2\bar{u}_{2,2}^2 + 2z\bar{u}_{2,1}^3\bar{u}_{2,2}^2 + \bar{u}_{2,1}^4\bar{u}_{2,2}^3\right).
\end{aligned}
$$

Again we see that the singularity is still there after the second blow-up, at the point $(\bar{u}_{1,1}, \bar{u}_{1,2}) = ((-1-\beta+\gamma)^{-1}, 0)$, or, in terms of the second coordinate chart, $(\bar{u}_{2,1}, \bar{u}_{2,2}) = (0, -1-\beta+\gamma)$. We will see that after a third blow-up, the indeterminacy will disappear and we are left with a regular system of equations. We introduce another two coordinate charts,

$$
\hat{u}_{1,1} = \frac{\bar{u}_{2,1}}{\bar{u}_{2,2} + 1 + \beta - \gamma}, \quad \hat{u}_{1,2} = \bar{u}_{2,2} + 1 + \beta - \gamma,
$$

$$
\hat{u}_{2,1} = \bar{u}_{2,1}, \quad \hat{u}_{2,2} = \frac{\bar{u}_{2,2} + 1 + \beta - \gamma}{\bar{u}_{2,1}}.
$$

In these coordinates the system of equations finally becomes

$$\begin{aligned}
\hat{u}'_{1,1} =& z\hat{u}_{1,1} + (z^2+1+\beta)(1+\beta-\gamma)\hat{u}_{1,1}^2 + 2(z^2-\gamma-2(1+\beta))\hat{u}_{1,1}^2\hat{u}_{2,1} \\
&- 2z(1+\beta-\gamma)^2\hat{u}_{1,1}^3\hat{u}_{2,1} + 3\hat{u}_{1,1}^2\hat{u}_{2,1}^2 + 6z(1+\beta-\gamma)\hat{u}_{1,1}^3\hat{u}_{2,1}^2 \\
&+ (1+\beta-\gamma)^3\hat{u}_{1,1}^4\hat{u}_{2,1}^2 - 4z\hat{u}_{1,1}^3\hat{u}_{2,1}^3 - 4(1+\beta-\gamma)^2\hat{u}_{1,1}^4\hat{u}_{1,2}^3 \\
&+ 5(1+\beta-\gamma)\hat{u}_{1,1}^4\hat{u}_{2,1}^4 - 2\hat{u}_{1,1}^4\hat{u}_{2,1}^5 \\
\hat{u}'_{1,2} =& \frac{-1}{\hat{u}_{1,1}} - (z^2+1+\beta)(1+\beta-\gamma)\hat{u}_{1,1}\hat{u}_{2,1} + 2(z^2-\gamma-2(1+\beta))\hat{u}_{1,1}\hat{u}_{2,1}^2 \\
&+ 2z(1+\beta-\gamma)^2\hat{u}_{1,1}^2\hat{u}_{2,1}^2 - \hat{u}_{1,1}\hat{u}_{2,1}^3 - 4z(1+\beta-\gamma)\hat{u}_{1,1}^2\hat{u}_{2,1}^3 \\
&- (1+\beta-\gamma)^3\hat{u}_{1,1}^3\hat{u}_{2,1}^3 + 2z\hat{u}_{1,1}^2\hat{u}_{2,1}^4 + 3(1+\beta-\gamma)^2\hat{u}_{1,1}^3\hat{u}_{2,1}^4 \\
&- 3(1+\beta-\gamma)^2\hat{u}_{1,1}^3\hat{u}_{2,1}^5 + \hat{u}_{1,1}^3\hat{u}_{1,2}^6,
\end{aligned}$$

$$\begin{aligned}
\hat{u}'_{2,1} =& -1 + z\hat{u}_{2,1} - (z^2-\gamma+2(1+\beta))\hat{u}_{2,1}^2 + 2z(1+\beta-\gamma)\hat{u}_{2,1}^3 + 2\hat{u}_{2,1}^3\hat{u}_{2,2} \\
&- (1+\beta-\gamma)^2\hat{u}_{2,1}^4 - 2z\hat{u}_{2,1}^4\hat{u}_{2,2} + 2(1+\beta-\gamma)\hat{u}_{2,1}^5\hat{u}_{2,2} - \hat{u}_{2,1}^6\hat{u}_{2,2}^2 \\
\hat{u}'_{2,2} =& (z^2+1+\beta)(\gamma-1-\beta) - 2z(1+\beta-\gamma)^2\hat{u}_{2,1} - (1+\beta-\gamma)^3\hat{u}_{2,1}^2 - z\hat{u}_{2,2} \\
&+ 2(z^2-\gamma+2(1+\beta))\hat{u}_{2,1}\hat{u}_{2,2} - 6z(1+\beta-\gamma)\hat{u}_{2,1}^2\hat{u}_{2,2} - 3\hat{u}_{2,1}^2\hat{u}_{2,2}^2 \\
&+ 4(1+\beta-\gamma)^2\hat{u}_{2,1}^3\hat{u}_{2,2} + 4z\hat{u}_{2,1}^3\hat{u}_{2,2}^2 - 5(1+\beta-\gamma)\hat{u}_{2,1}^4\hat{u}_{2,2}^2 + 2\hat{u}_{2,1}^5\hat{u}_{2,2}^3.
\end{aligned}$$

Thus we see that the indeterminacy of the first base point disappears if we enlarge the space on which the system of equations is defined. The other two base points present in the original system of equations compactified on $\mathbf{CP}^2$ can be removed in similar way. The blow-up calculations are essentially the same with various factors of ω or ω^2 inserted. We denote the exceptional curves introduced at every blow-up by $L_i^{(\omega)}$ and $L_i^{(\omega^2)}$, $i=1,2,3$, respectively. Let the compact space, obtained by enlarging $\mathbf{CP}^2$ by these sequences of blow-ups, covered by the various coordinate charts introduced, be donoted by S. Every point of the space

$$\mathcal{I} = S \setminus \left(L \cup L_1^{(1)} \cup L_2^{(1)} \cup L_1^{(\omega)} \cup L_2^{(\omega)} \cup L_1^{(\omega^2)} \cup L_2^{(\omega^2)} \right)$$

describes a regular initial value problem for the system of equations. The space I is Okamoto's space of initial conditions for the system (3.26).

Chapter 4

Nevanlinna Theory applied to differential equations

In this chapter we give a brief introduction to Nevanlinna theory, the value distribution theory of meromorphic functions, to the extent that we will need it for applications to complex differential equations in chapter 5. We define the Nevanlinna functions in section 4.2 and state some of their properties as well as the important *Lemma on the logarithmic derivative*. In section 4.3 we define the notion of *admissible solutions* of a differential equation and state an important result by Clunie. We also briefly discuss an extension of Nevanlinna theory to algebroid functions in section 4.4.

4.1 Introduction

Nevanlinna Theory is the value distribution theory of meromorphic functions developed by R. Nevanlinna [32] in the 1920's. Whereas in the value distribution theory of an entire function f the maximum modulus

$$M(r, f) = \max_{|z| \leq r} |f(z)|,$$

is the relevant quantity to describe the growth of the function this cannot be used in the case of meromorphic functions where poles are present. Nevanlinna realised that for a meromorphic function the role of M is best replaced by what is called the characteristic function $T(r, f)$, which consists of two parts,

$$T(r, f) = N(r, f) + m(r, f),$$

where $N(r, f)$ measures the number of poles of f within the disc of radius r around the origin, weighted with a logarithmic measure, and $m(r, f)$ is called the proximity function which measures how big $|f|$ is on average on a circle of radius r. The proper definitions are given below.

4.2 The Nevanlinna functions

This section gives a brief outline of Nevanlinna Theory in the complex plane. We follow standard introductory books on Nevanlinna theory, for example [14] or [2]. After defining the Nevanlinna functions we discuss some main results of Nevanlinna Theory, however, we will restrict ourselves to results that are needed in applications to complex differential equations, e.g. the first main theorem, the Lemma on the logarithmic derivative and Mohon'ko's lemma.

A starting point for the development of Nevanlinna Theory can be taken in Jensen's formula as deduced below. Suppose that F is analytic and nowhere vanishing on the disc of radius r, $D = \{z \in \mathbf{C} | |z| \leq r\}$. Then $\log F(z)$ is holomorphic in D and by Cauchy's integral formula, taking real parts, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|d\theta. \tag{4.1}$$

Now let $f(z)$ be any meromorphic function in $\mathbf{C}$ and denote by $a_1, \dots, a_p$ the zeros and by $b_1, \dots, b_q$ the poles of f within the disc of radius r, each listed according to their multiplicity, but not including the origin if there is a zero or pole. We denote by $\mathrm{ord}_0 f$ the order of a zero or pole of f at 0 ($\mathrm{ord}_0 f > 0$ in case of a zero, $\mathrm{ord}_0 < 0$ in case of a pole). For every zero a_i and every pole b_j we multiply f by a so-called Blaschke factor,

$$B(z, c) = \frac{r^2 - \bar{c}z}{r(z - c)},$$

or its inverse, respectively. Note that $|B(z, c)| = 1$ whenever $|z| = r$. We thus define

$$F(z) = f(z) \cdot z^{-\mathrm{ord}_0 f} \cdot \prod_{i=1}^{p} \frac{r^2 - \bar{a}_i z}{r(z - a_i)} \cdot \prod_{j=1}^{q} \frac{r(z - b_j)}{r^2 - \bar{b}_j z}. \tag{4.2}$$

Note that $F(z)$ has neither zeros nor poles within the disc of radius r. Applying formula (4.1) to (4.2) one obtains

$$\log|\mathrm{ilc}(f, 0)| + \sum_{i=1}^{p} \log\left|\frac{r}{a_i}\right| - \sum_{i=1}^{q} \log\left|\frac{r}{b_i}\right| = \int_0^{2\pi} \log\left|f\left(re^{i\theta}\right)\right|\frac{d\theta}{2\pi} - (\mathrm{ord}_0 f)\log r. \tag{4.3}$$

Here, $\mathrm{ilc}(f, 0)$ denotes the initial Laurent coefficient of the Laurent series of f at $z = 0$. Defining $\log^+ x = \max\{0, \log x\}$ and splitting the expression under the integral using

$$\log x = \log^+ x - \log^+ \frac{1}{x},$$

equation (4.3) can be written in the somewhat symmetric form

$$\begin{aligned}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right|\frac{d\theta}{2\pi} + \sum_{i=1}^{q} \log\left|\frac{r}{b_i}\right| = \int_0^{2\pi} \log^+\left|\frac{1}{f\left(re^{i\theta}\right)}\right|\frac{d\theta}{2\pi} + \sum_{i=1}^{p} \log\left|\frac{r}{a_i}\right| \\ + (\mathrm{ord}_0 f)\log r + \log|\mathrm{ilc}(f, 0)|,\end{aligned} \tag{4.4}$$

where we have grouped on the left hand side the terms that contribute where $|f|$ is large or where f has poles, and on the right hand side the terms that contribute where $|f|$ is small or f has zeros.

The proximity function is defined by

$$m(r,f) = \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| \frac{d\theta}{2\pi}.$$

For any point $a \in \mathbf{C}$ we denote the proximity function with respect to a by

$$m(r,f,a) = m\left(r, \frac{1}{f-a}\right).$$

The counting function $n(r,f)$ denotes the number of poles f has within the disc of radius r, counting every pole according to its multiplicity. Analogously,

$$n(r,f,a) = n\left(r, \frac{1}{f-a}\right)$$

counts the number of a-points of f in D. The integrated counting function is defined by

$$N(r,f,a) = \int_0^r \frac{n(t,f,a) - n(0,f,a)}{t} dt - n(0,f,a)\log r.$$

With this notation, equation (4.4) can be written in the form

$$m(r,f) + N(r,f) = m(r,f,0) + N(r,f,0) + \log|\mathrm{ilc}(f,0)|,$$

or, introducing the Nevanlinna characteristic function by

$$T(r,f) = m(r,f) + N(r,f), \quad T(r,f,a) = T\left(r, \frac{1}{f-a}\right),$$

we have

$$T(r,f) = T(r,f,0) + \log|\mathrm{ilc}(f,0)|.$$

In the Riemann sphere there is nothing special about the points 0 and ∞ and in fact a similar equality holds for any point $a \in \mathbf{C}$, which is the content of the *First Main Theorem of Nevanlinna Theory.*

Theorem 4.1. *Let $a \in \mathbf{C}$ and let $f \not\equiv a, \infty$ be a meromorphic function. Then*

$$T(r,f,a) = T(r,f) + O(1), \quad \text{as } r \to \infty. \tag{4.5}$$

Since $T(r,f,a)$ and $T(r,f)$ only differ by a bounded term by Theorem 4.1, one can work only with $T(r,f)$ as *the* characteristic function. The first main theorem justifies the characteristic function $T(r,f)$ to be the correct quantity to describe the value-distribution of a meromorphic function f, as was realised by R. Nevanlinna. In particular $T(r,f)$ measures the growth of f as $r \to \infty$ and we make the following

Definition 4.2. *The order $\sigma(f)$ of a meromorphic function f is defined by*

$$\sigma(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

We now give some elementary inequalities for the functions m, N and T as they are applied to a sum or product of meromorphic functions, which are easily established. Let $f_1, \ldots, f_p$ be meromorphic functions. Then

$$\begin{aligned}
m\left(r, \sum_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} m(r, f_i) + \log p, \\
m\left(r, \prod_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} m(r, f_i), \\
N\left(r, \sum_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} N(r, f_i), \\
N\left(r, \prod_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} N(r, f_i), \\
T\left(r, \sum_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} T(r, f_i) + \log p, \\
T\left(r, \prod_{i=1}^{p} f_i\right) &\leq \sum_{i=1}^{p} T(r, f_i).
\end{aligned}$$

For technical reasons in Nevanlinna theory it is often necessary to express that an equality or inequality holds for all $r \in \mathbf{R}_+$ outside some *exceptional set* E of finite measure. Comparing $T(r, f)$ with the characteristic $T(r, g)$ of another meromorphic function g enables us to say whether the function f grows faster or slower than g as $r \to \infty$.

Definition 4.3. *Let f and g be meromorphic functions. We shall say that g has small growth compared to f if*

$$T(r, g) = o(T(r, f)), \quad r \to \infty, \tag{4.6}$$

possibly outside an exceptional set E of finite measure. The set of all functions g for which (4.6) holds is denoted by $S(r, f)$ and we use the notation

$$T(r, g) = S(r, f).$$

One of the main tools of Nevanlinna theory needed in applications to complex differential equations is the so-called Lemma on the logarithmic derivative, which expresses that the proximity function of the logarithmic derivative of a transcendental meromorphic function f has small growth compared to f itself.

Lemma 4.4. *Let f be a transcendental meromorphic function. Then we have*

$$m\left(r, \frac{f'}{f}\right) = O(\log T(r, f) + \log r).$$

In particular this means that

$$m\left(r, \frac{f'}{f}\right) = S(r, f).$$

Furthermore, if $\sigma(f) < \infty$ we have

$$m\left(r, \frac{f'}{f}\right) = O(\log r).$$

We will also need an immediate corollary of this,

Corollary 4.5. *Let f be a transcendental meromorphic function and $k \geq 1$ an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

and, if $\sigma(f) < \infty$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

The following Lemma is due to G. Valiron [48] and A. Z. Mohon'ko [29].

Lemma 4.6. *Let f be a meromorphic function and*

$$R(z, f) = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j}$$

an irreducible rational function in f with meromorphic coefficients such that $T(r, a_i) = S(r, f)$ for all $i = 0, \ldots, p$ and $T(r, b_j) = S(r, f)$ for all $j = 0, \ldots, q$. Let $d = \max\{p, q\}$, the degree of R. Then we have

$$T(r, R(z, f)) = dT(r, f) + S(r, f).$$

4.3 Nevanlinna theory and differential equations

Given an algebraic differential equation for the dependent variable $y(z)$,

$$F\left(z, y, y', \ldots, y^{(n)}\right) = 0, \tag{4.7}$$

where F is polynomial in y and its derivatives, suppose that the coefficients $a_\lambda(z)$, $\lambda \in I$ where I is some set of indices, are elements of a certain class of functions, for example the rational functions $\mathbf{C}(z)$. Usually we seek solutions of (4.7) which are more complicated than the functions that define the equation, for example transcendental meromorphic functions. We have the following characterisation of rational functions.

Theorem 4.7. *A meromorphic function f is rational if and only if*

$$T(r, f) = O(\log r).$$

The Nevanlinna characteristic function $T(r, y)$ provides a natural way of selecting subfields of the field of meromorphic functions $\mathcal{M}$. Therefore let $\phi : [0, \infty) \to [0, \infty)$ be a function such that $\log r = O(\phi(r))$ as $r \to \infty$. We denote by $\mathcal{M}_\phi$ the set of all functions $f \in \mathcal{M}$ for which

$$T(r, f) = O(\phi(r)), \quad r \to \infty.$$

If we take $\phi(r) = \log r$ we obtain again the set of rational functions: $\mathcal{M}_{\log} = \mathbf{C}(z)$. This notation also allows us to work with more general classes of functions as coefficients for differential equations, as described by the following definition.

Definition 4.8. *Suppose that $F \in \mathcal{M}_\phi[Y, Y_1, \ldots, Y_n]$. A solution y of (4.7) is called admissible if $y \in \mathcal{M} \setminus \mathcal{M}_\phi$.*

In particular, for an admissible solution y of (4.7) we have $T(r, a_\lambda) = S(r, y)$ for all $\lambda \in I$. One of the most important applications of Nevanlinna theory to differential equations is given by the following lemma by Clunie [3].

Lemma 4.9. *Let f be an admissible solution of the equation*

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$, $Q(z, f)$ are polynomials in f and a finite number of its derivatives with meromorphic coefficients. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is $\leq n$, then we have

$$m(r, P(z, f)) = S(r, f).$$

4.4 Nevanlinna theory for algebroid functions

In chapter 5 we will study differential equations which have solutions that are globally finite-branched over the complex plane. In particular we consider equations with so-called algebroid solutions, i.e. functions algebraic over the field of meromorphic functions. Nevanlinna theory, the value-distribution theory of meromorphic functions, has a generalisation to algebroid functions which was given by Selberg [41] and Ullrich [47].

Definition 4.10. *Suppose a multi-valued function $f(z)$ satisfies the irreducible algebraic equation*

$$f^n + s_1(z) f^{n-1} + \cdots + s_{n-1}(z) f + s_n(z) = 0, \tag{4.8}$$

where $s_1(z), \ldots, s_n(z)$ are meromorphic functions. Then f is called an n-valued algebroid function. If all the functions $s_1, \ldots, s_n$ are rational then f is called algebraic. If at least one of the functions $s_1, \ldots, s_n$ is non-rational then f is called transcendental algebroid.

Over every point $z_0 \in \mathbf{C}$ an algebroid function takes on at most n values and can be expressed by a certain number $i = 1, \ldots, k$ of algebraic series expansions

$$f(z) = a_i + \sum_{j=\tau_i}^{\infty} c_j (z - z_0)^{\frac{j}{\lambda_i}}, \tag{4.9}$$

for a finite value of f, or

$$f(z) = \sum_{j=-\tau_i}^{\infty} c_j (z - z_0)^{\frac{j}{\lambda_i}}, \tag{4.10}$$

called an (algebraic) pole of f. Here it is assumed that the numbers λ_i in each series expansion have no common factor with all the indices j for which $c_j \neq 0$. Here, the numbers λ_i add up to the total number of sheets of f: $\lambda_1 + \cdots + \lambda_k = n$. At any point where f is locally unbranched we have $k = n$ and $\lambda_1 = 1, \cdots, \lambda_n = 1$, i.e. there are n Laurent series expansions for the n sheets of f.

Algebroid Nevanlinna functions

Let f be a n-valued algebroid function. We denote $n(r, f) = \sum_{|z_0|\leq r} \tau_i$, where the sum is over the numbers τ_i of all points z_0 where f has an expansion of the form (4.10). Let $f_1, \ldots, f_n$ denote the n branches of f. The algebroid Nevanlinna functions are then defined as follows:

$$\begin{aligned} N(r, f) &= \frac{1}{n}\int_0^r \frac{n(r, f) - n(0, f)}{r}\mathrm{d}r + \frac{1}{n}n(0, f)\log(r) \\ m(r, f) &= \frac{1}{2\pi n}\sum_{\nu=1}^{n}\int_0^{2\pi} \log^+ \left| f_\nu\left(r\mathrm{e}^{i\phi}\right)\right| \mathrm{d}\phi \\ T(r, f) &= m(r, f) + N(r, f). \end{aligned}$$

In the single-valued (meromorphic) case these functions reduce to the usual Nevanlinna functions. Most of the notation and some standard theorems of Nevanlinna theory carry over to the algebroid case with some modifications, see e.g. [48] for the Lemma on the Logarithmic derivative, and [29, 30] for compositions of algebroid functions.

Chapter 5

Differential equations with algebroid solutions

In chapter 2 we studied the solutions of classes of ordinary differential equations for which all movable singularities are algebraic, i.e. the solutions are locally finite branched. We have remarked that the global structure of the solutions can be very complicated, and in general a solution extends over an infinitely sheeted Riemann surface over the complex plane. An interesting question therefore is whether it is possible to have solutions which are also globally finite branched and if so, what condition the existence of such a solution imposes on the class of equations under consideration.

5.1 Malmquist's Theorem

In 1913, J. Malmquist [24] proved the following theorem about first-order ODEs with algebroid solutions.

Theorem 5.1. *Suppose that the equation*

$$y' = \frac{P(z,y)}{Q(z,y)}, \quad P, Q \in \mathbf{C}(z)[y], \tag{5.1}$$

where P and Q are in reduced terms, has at least one transcendental algebroid solution. Then, by a transformation

$$w = \frac{y^n + \alpha_2 y^{n-2} + \cdots + \alpha_n}{y^{n-1} + \beta_2 y^{n-2} + \cdots + \beta_n}, \quad \alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n \in \mathbf{C}(z),$$

the equation can be reduced to a Riccati equation in w with rational coefficients,

$$w' = a(z)w^2 + b(z)w + c(z).$$

Remark. Often, Malmquist's theorem is quoted as the following statement: 'Suppose that the equation (5.1) has a transcendental meromorphic solution. Then it must already be a Riccati equation.' This may be due to the fact that in 1932 Yosida [49] gave a proof of Malmquist's theorem using Nevanlinna theory, but only for the case of meromorphic solutions. In this form the theorem is therefore also known as the Malmquist-Yosida theorem. Malmquist's paper was written before the advent of Nevanlinna Theory and instead uses certain growth arguments due to Boutroux in the proof. The fact that Malmquist's paper is considerably longer than Yosida's paper must however be attributed to the fact that the case of algebroid solutions is more difficult.

Example. We give an example of Malmquist's theorem for the case where $Q \equiv 1$ and $\deg_y P = 3$. So suppose y is a 2-valued transcendental algebroid solution of the equation

$$y' = a_3 y^3 + a_2 y^2 + a_1 y + a_0, \tag{5.2}$$

satisfying the quadratic equation

$$y^2 + py + q = 0. \tag{5.3}$$

Differentiating (5.3) with respect to z and using (5.2) to replace y' one obtains for p, q the system of equations

$$p' = a_0 p^3 - 3a_0 pq - a_1 p^2 + 2a_1 q + a_2 p - 2a_3 \tag{5.4}$$

$$q' = a_0 p^2 q - 2a_0 q^2 - a_1 pq + 2a_2 q - a_3 p. \tag{5.5}$$

The arguments in the proof of Malmquist's theorem show that in fact p is rational whereas q is transcendental meromorphic. We therefore must have

$$p' - a_0 p^3 + a_1 p^2 - a_2 p + 2a_3 = (2a_1 - 3a_0 p) q \equiv 0, \tag{5.6}$$

since the left hand side is rational. Therefore we have $p = \frac{2a_1}{3a_0}$, which re-inserted into (5.6) yields a condition on the coefficients a_0, a_1, a_2, a_3,

$$a_1' a_0 - a_1 a_0' = -\frac{2}{9} a_1^3 + a_0 a_1 a_2 - \frac{3}{2} a_0^2 a_3,$$

and q satisfies the Riccati equation

$$q' = -2a_0 q^2 + \left(-\frac{2a_1^2}{9a_0} + 2a_2 \right) q - \frac{2a_1 a_3}{3a_0}.$$

The Malmquist-Yosida theorem was generalised to the case of admissible solutions independently by Laine [23] and A. Z. and V. D. Mohon'ko [28]:

Theorem 5.2. *Suppose the first-order equation*

$$y' = \frac{P(z,y)}{Q(z,y)}, \quad P, Q \in \mathcal{M}[y], \tag{5.7}$$

where P and Q are in reduced terms, has at least one admissible meromorphic solution. Then $\deg_y Q = 0$ and $\deg_y P \leq 2$, i.e. (5.7) is a Riccati equation (or linear equation in case $\deg_y P \leq 1$.).

We will now state and prove Malmquist's Theorem 5.1 in the generalised form for the notion of admissible solutions. We denote the field of algebroid functions by $\mathcal{A}$. One can select subfields of $\mathcal{A}$ by using the Nevanlinna characteristic.

Definition 5.3. *Let $\phi : [0, \infty) \to \mathbf{R}_+$ be a function such that $\log r = O(\phi(r))$. The elements $f \in \mathcal{A}$ for which*

$$T(r, f) = O(\phi(r)), \quad r \to \infty,$$

possibly outside some set of finite measure, are called algebroid functions of characteristic ϕ. The set of all such elements is denoted by $\mathcal{A}_\phi$.

One can easily verify that $\mathcal{A}_\phi$ is a subfield of $\mathcal{A}$. If we take for example $\phi(r) = \log r$, then $\mathcal{A}_\phi$ is the field of algebraic functions.

Theorem 5.4. *Let ϕ be as in Definition 5.3. Suppose that the equation*

$$y' = \frac{P(z,y)}{Q(z,y)}, \quad P, Q \in \mathcal{A}_\phi[y] \tag{5.8}$$

has an admissible algebroid solution, $y \in \mathcal{A} \setminus \mathcal{A}_\phi$. Then, by a transformation

$$w = \frac{y^n + \alpha_2 y^{n-2} + \cdots + \alpha_n}{y^{n-1} + \beta_2 y^{n-2} + \cdots + \beta_n},$$

where $\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_n$ are rational expressions in the coefficient functions of $P(z,y)$ and $Q(z,y)$, the equation reduces to a Riccati equation

$$w' = a(z)w^2 + b(z)w + c(z), \quad a, b, c \in \mathcal{A}_\phi,$$

which w satisfies admissibly, i.e. $w \notin \mathcal{A}_\phi$.

Proof. The main ideas for the proof are taken from Malmquist's paper [24], but the arguments due to Boutroux regarding the growth of the solutions are replaced by Nevanlinna theoretic arguments. By a transformation of the form $\tilde{y} = \alpha + y^{-1}$, for some $\alpha \in \mathbf{C}$, one can always achieve that $\deg_y P = \deg_y Q + 2$, which we assume already to be the case in the following. We therefore let $P(z,y) = a_0(z)y^p + \cdots + a_p(z)$ and $Q(z,y) = b_0(z)y^q + \cdots + b_q(z)$, $p = q + 2$.

Let y be an m-valued algebroid solution of equation (5.8) and let z_* be an (algebraic) pole of the solution around which y can be represented by m different series solutions $y_1, \ldots, y_m$ in a fractional power of $z - z_*$. In the following we denote by $s_1, \ldots, s_m$ the elementary symmetric functions in m variables,

$$s_k(y_1, \ldots, y_m) = \sum_{1 \le i_1 < \cdots < i_k \le m} y_{i_1} \cdots y_{i_k}, \quad k = 1, \ldots, m.$$

We denote the branches of the algebroid function defined by the equation $Q(z,y) = 0$, represented by series expansions in fractional powers of $z - z_*$, by $\beta_1(z), \ldots, \beta_{\bar{q}}(z)$, occuring with multiplicities μ_i, $i = 1, \ldots, \bar{q}$. At the point z_* we have

$$\lim_{z \to z_*} y(z) = \beta_i(z_*),$$

for some $i \in \{1, \ldots, \bar{q}\}$. The local series representation of a solution then takes the form

$$y(z) = \beta_i(z_*) + \sum_{j=1}^{\infty} c_j (z - z_*)^{\frac{j}{\mu_i + 1}}. \tag{5.9}$$

Now consider the functions

$$w_{\mu,i} = s_\mu\left(\frac{1}{y_1-\beta_i},\dots,\frac{1}{y_m-\beta_i}\right), \quad \mu=1,\dots,\mu_i, \quad i=1\dots,\bar{q}, \tag{5.10}$$

which are single-valued around z_*, i.e. they can contain no fractional powers of $z-z_*$ in their series expansions about z_*. By the series expansions (5.9) we have $\frac{1}{y_k-\beta_i} \sim (z-z_*)^{-\frac{1}{\mu_i+1}}$, $k=1,\dots,m$. The functions $w_{\mu,i}$, $\mu=1,\dots,\mu_i$ are therefore represented by series expansions

$$(z-z_*)^{-\frac{\mu}{\mu_i+1}}\sum_{j=1}^{\infty}\zeta_j(z-z_*)^{\frac{1}{\mu_i+1}},$$

in which the fractional powers disappear. Since in (5.10) $\mu \le \mu_i$ the negative powers disappear and therefore the functions $w_{\mu,i}$ are analytic at z_*. In terms of Nevanlinna theory this means that $N(r,w_{\mu,i}) = S(r,y)$, as $w_{\mu,i}$ can only have a pole at a fixed singularity of (5.8). We now show that also $m(r,w_{\mu,i}) = S(r,y)$. Writing equation (5.8) in the form

$$y^p = \frac{1}{a_0}\left(\left(b_q y^{q+1}+\dots+b_0 y\right)\frac{y'}{y} - a_1 y^{p-1} - \dots - a_p\right),$$

one obtains by Lemma 4.4,

$$\begin{aligned}
pm(r,y) &= m(r,y^p)\\
&= m(r,y) + m\left(r, b_q y^q + \dots + b_0 - a_1 y^{p-2} - \dots - a_{p-1}\right) + S(r,y)\\
&\ \ \vdots\\
&= (p-1)m(r,y) + S(r,y),
\end{aligned}$$

and therefore $m(r,y) = S(r,y)$. Since the $w_{\mu,i}$ are rational functions of the branches of y we also have $m(r,w_{\mu,i}) = S(r,y)$. In summary we have $T(r,w_{\mu,i}) = S(r,y)$.

For functions $w_1,\dots,w_m$, to be determined below, we let

$$\psi(y) = y^m + w_1 y^{m-1} + \dots + w_m.$$

One can easily see that

$$s_\mu\left(\frac{1}{y_1-y},\dots,\frac{1}{y_m-y}\right) = \frac{1}{\mu!}\frac{\psi^{(\mu)}(y)}{\psi(y)},$$

and therefore

$$w_{\mu,i} = \frac{1}{\mu!}\frac{\psi^{(\mu)}(\beta_i)}{\psi(\beta_i)}.$$

Written in the form

$$\mu!\cdot w_{\mu,i}\psi(\beta_i) - \psi^{(\mu)}(\beta_i) = 0, \quad \mu=1,\dots,\mu_i, \quad i=1,\dots,q, \tag{5.11}$$

these can be seen as linear relations between the functions $w_1,\dots,w_m$ with coefficients of small growth $S(r,y)$. We now derive a system of differential equations for the functions $w_1,\dots,w_m$ when $\psi(y)\equiv 0$. By differentiating

$$y^m + w_1(z)y^{m-1} + \dots + w_m(z) = 0, \tag{5.12}$$

one obtains

$$\begin{aligned}(my^{m-1}+(m-1)w_1y^{m-2}+\cdots+w_{m-1})(a_0y^p+\cdots+a_p)\\ +(w_1'y^{m-1}+\cdots+w_m')(b_0y^q+\cdots+b_q)=0.\end{aligned} \tag{5.13}$$

By employing equation (5.12) repeatedly, one can reduce the degree in y of equation (5.13). Effectively this means one has determined functions $B_1,\ldots,B_p$ such that by adding

$$(y^m+w_1y^{m-1}+\cdots+w_m)(B_1y^{p-1}+\cdots+B_p)=0, \tag{5.14}$$

to equation (5.13) this reduces to an equation of degree at most $m-1$,

$$\sum_{\mu=1}^{m}(w_\mu B_p+\cdots+w_{\mu+p-1}B_1+A_\mu)y^{m-\mu}=0, \tag{5.15}$$

where, for $\mu=1,\ldots,m$,

$$\begin{aligned}A_\mu=&(m-\mu+1)a_pw_{\mu-1}+(m-\mu)a_{p-1}w_\mu+\cdots+(m-\mu-p+1)a_0z_{\mu+p-1}\\ &+b_qw_\mu'+\cdots+b_0w_{\mu+q}'.\end{aligned}$$

For example, the first two functions B_1 and B_2 are given by

$$B_1=-na_0,\quad B_2=-na_1+a_0w_1.$$

However, the other functions $B_3,\ldots,B_p$ also involve the derivatives $w_1',\ldots,w_m'$. Since the left hand side of equation (5.15) is of order $m-1$ in y the coefficients of all powers of y must vanish individually, so for $\mu=1,\ldots,m$ we have

$$\begin{aligned}b_qw_\mu'+\cdots+b_0w_{\mu+q}'=&-(m-\mu+1)a_pw_{\mu-1}\\ &-w_\mu(B_p+(m-\mu)a_{p-1})-\cdots-w_{\mu+p-1}(B_1+(m-\mu-p+1)a_0).\end{aligned} \tag{5.16}$$

Since $Q(z,\beta_i)=0$, $i=1,\ldots,q$, adding equations (5.13) and (5.14) and setting $y=\beta_i$, yields the equations

$$B_1\beta_i^{p-1}+\cdots+B_p=-P(z,\beta_i)\frac{\psi'(\beta_i)}{\psi(\beta_i)}.$$

Suppose now first that the β_i, $i=1,\ldots,q=p-2$ are all distinct and define the matrix

$$M=\begin{pmatrix}\beta_1^{q-1} & \beta_1^{q-2} & \ldots & \beta_1 & 1\\ \beta_2^{q-1} & \beta_2^{q-2} & \ldots & \beta_2 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_q^{q-1} & \beta_q^{q-2} & \ldots & \beta_q & 1\end{pmatrix},$$

which is non-singular in this case. Also define the matrices M_ν, $\nu=1,\ldots,q$, which are obtained from M by replacing the νth column by the vector

$$\left(B_1\beta_\nu^{q+1}+B_2\beta_\nu^q+P(z,\beta_\nu)\frac{\psi'(\beta_\nu)}{\psi(\beta_\nu)}\right)_{\nu=1\ldots,q}.$$

Then, by Cramer's rule, one can express the solutions for $B_3,\ldots,B_p$ as

$$B_{\nu+2}=\frac{|M_\nu|}{|M|},\quad \nu=1,\ldots,q.$$

By using standard rules to compute determinants of this kind, one finds

$$B_{\nu+2} = \frac{b_\nu}{b_0}(a_0 w_1 - na_1) + \frac{na_0}{b_0^2}\begin{vmatrix} b_\nu & b_0 \\ b_{\nu+1} & b_1 \end{vmatrix} - R_\nu, \quad \nu = 1, \dots, q, \tag{5.17}$$

where

$$R_v = \frac{|N_\nu|}{|M|}$$

and N_ν is the matrix obtained from M by replacing the νth column by the vector $\left(P(z,\beta_\nu)\frac{\psi'(\beta_\nu)}{\psi(\beta_\nu)}\right)_{\nu=1,\dots,q}$.

If there are multiple roots among β_i, $i = 1, \dots, q$, the matrix M becomes singular and one cannot solve for B_ν, $\nu = 1, \dots, q$ in this way. However, one can use the derivative of the relation (5.13) to obtain additional relations. Essentially one can perturb the β_i, $i = 1, \dots, q$, slightly such that the expression for R_ν is well-defined and take the limit in which the β_i coalesce according to their multiplicity. Inserting the expressions in (5.17) into the equations (5.16) one obtains, solving for $w_1', \dots, w_n'$,

$$w_\nu' = -\frac{a_0}{b_0} w_1 w_\nu + \sum_{\mu=1}^{m}\left(\sum_{i=1}^{q} \alpha_{\mu i}^{(\nu)} R_i + \alpha_\mu^{(\nu)}\right) w_\mu + \alpha^{(\nu)}. \tag{5.18}$$

We can now use the linear relations (5.11) between the functions $w_1, \dots, w_m$,

$$\mu! \cdot w_{\nu,i}(z)\psi(\beta_i) - \psi^{(\mu)}(\beta_i) = 0, \quad \mu_1, \dots, \mu_i, \quad i = 1, \dots, q,$$

to reduce the number of variables in the system of equations (5.18). Also, from any such linear relation,

$$\kappa_1 w_1 + \cdots + \kappa_m w_m = \kappa, \tag{5.19}$$

by differentiating one obtains

$$\sum_{\mu=1}^{m}\left(\frac{d\kappa_\mu}{dz} w_\mu + \kappa_m w_\mu'\right) = \frac{d\kappa}{dz}.$$

Inserting the expressions for w_μ' from (5.18) one can obtain further linear relations of the form (5.19). However, since y is supposed to be an admissible solution of the equation, there can be at most $m-1$ such linear relations. One thus ends up with a system of equations

$$w_{\mu_\nu}' = -\frac{a_0}{b_0} w_{\mu_1} w_{\mu_\nu} + \sum_{\lambda=1}^{\rho} b_{\lambda\nu} w_{\mu_\lambda} + c_\nu, \quad \nu = 1, \dots, \rho, \tag{5.20}$$

where $w_{\mu_1} = w_1$. A system of equations like this can be linearised in the following way

$$w_{\mu_\nu} = \frac{\zeta_\nu}{\zeta}, \quad \nu = 1, \dots, \rho.$$

Let $\zeta, \zeta_1, \dots, \zeta_\rho$ satisfy the linear system of equations

$$\begin{aligned} \zeta' &= \frac{a_0}{b_0}\zeta_1 \\ \zeta_\nu' &= \sum_{\lambda=1}^{\rho} b_{\lambda\nu}\zeta_\lambda + c_\nu \zeta. \end{aligned} \tag{5.21}$$

Let $(\xi_k, \xi_{1k}, \dots, \xi_{\rho k})$, $k = 0, 1, \dots, \rho$, be a fundamental system of solutions for the linear system (5.21). Any solution can therefore be written as

$$\begin{aligned} \zeta =& c_0\xi_0 + \cdots + c_\rho \xi_\rho \\ \zeta_\nu =& c_0\xi_{\nu 0} + \cdots + c_\rho \xi_{\nu\rho}, \quad c_\nu \in \mathbf{C}, \quad \nu = 1, \dots, \rho. \end{aligned} \tag{5.22}$$

We will now show that the system (5.20) in fact reduces to a single Riccati equation. Suppose to the contrary that $\rho > 1$. Denote the numbers $\{1, \dots, m\} \setminus \{\mu_1, \dots, \mu_\rho\}$ by $\mu'_1, \dots, \mu'_{\rho'}$ where $\rho' = m - \rho$. We define $\zeta_{\mu'_\nu} := \zeta w_{\mu'_\nu}$ for $\nu = 1, \dots, \rho'$.

$$\zeta \left(y^m + w_{\mu'_1} y^{m-\mu_1} + \cdots + w_{\mu'_{\rho'}} y^{m-\mu'_{\rho'}} \right) + \zeta_{\mu_1} y^{m-\mu_1} + \cdots + \zeta_{\mu_\rho} y^{m-\mu_\rho} = 0. \tag{5.23}$$

Let $y(z_0) = y_0$ be some initial values. It is always possible to choose $c_0, \dots, c_\rho$ such that $\zeta(z_0) \neq 0$, since otherwise we would have

$$\begin{aligned} 0 &= \zeta_{\mu_1} y^{m-\mu_1} + \cdots + \zeta_{\mu_\rho} y^{m-\mu_\rho} \\ &= \sum_{l=0}^{\rho} \left(\xi_{\mu_1,l} y^{m-\mu_1} + \cdots + \xi_{\mu_\rho,l} y^{m-\mu_\rho} \right) c_l \\ &= \frac{1}{\xi_0} \sum_{l=1}^{\rho} \Big(\xi_0 \left(\xi_{\mu_1,l} y^{m-\mu_1} + \cdots + \xi_{\mu_\rho,l} y^{m-\mu_\rho} \right) - \\ &\qquad\qquad \xi_l \left(\xi_{\mu_1,0} y^{m-\mu_1} + \cdots + \xi_{\mu_\rho,0} y^{m-\mu_\rho} \right) \Big) c_l, \end{aligned}$$

where we have used $c_0 = -\frac{1}{\xi_0}\left(c_1\xi_1 + \cdots + c_\rho\xi_\rho\right)$ in the last step. With the sum vanishing for arbitrary values of the constants c_ν, $\nu = 0, \dots, \rho$, we therefore have

$$\xi_0 \left(\xi_{\mu_1,l} y^{m-\mu_1} + \cdots + \xi_{\mu_\rho,l} y^{m-\mu_\rho} \right) - \xi_l \left(\xi_{\mu_1,0} y^{m-\mu_1} + \cdots + \xi_{\mu_\rho,0} y^{m-\mu_\rho} \right) = 0,$$

for $l = 1, \dots, \rho$. Comparing coefficients of powers of y we thus find

$$\xi_0 \xi_{\mu_\nu,l} = \xi_l \xi_{\mu_\nu,0}, \quad \nu, l = 1, \dots, \rho,$$

which would mean that $\frac{\zeta_\nu}{\zeta}$ were independent of $c_0, \dots, c_\rho$, which is impossible. We can therefore assume that $\zeta \neq 0$ in equation (5.23). If we now add to this equation $y^m + w_1 y^{m-1} + \cdots + w_m = 0$, we obtain

$$\left(\zeta_{\mu_1} - \zeta w_{\mu_1}\right) y^{m-\mu_1} + \cdots + (\zeta_{\mu_\rho} - \zeta w_{\mu_\rho}) y^{m-\mu_\rho} = 0,$$

showing that the w_{μ_ν} are determined by a solution of the system (5.21),

$$w_{\mu_\nu} = \frac{\zeta_{\mu_\nu}}{\zeta}, \quad \nu = 1, \dots, \rho.$$

The solution $y(z)$ is fixed by one integration constant $y(z_0) = y_0$. However the quotients

$$\frac{\zeta_{\mu_\nu}}{\zeta} = \frac{c_0\xi_{\nu 0} + \cdots + c_\rho \xi_{\nu\rho}}{c_0\xi_0 + \cdots + c_\rho\xi_\rho}$$

would depend on arbitrary constants if $\rho > 1$. Therefore we must have $\rho = 1$ which means that the system (5.20) is in fact a single Riccati equation for $w = w_1$,

$$\frac{dw}{dz} = -\frac{a_0}{b_0} w^2 + bw + c.$$

The other coefficients are then linear functions in w,

$$w_\mu = \beta_\mu(z)w - \alpha_\mu(z), \quad \mu = 2, \ldots, m.$$

Inserting these expressions into (5.12),

$$y^m + wy^{m-1} + (\beta_2 w - \alpha_2)y^{m-2} + \cdots + \beta_m w - \alpha_m = 0,$$

showing that w is obtained by a transformation of the form

$$w = \frac{y^m + \alpha_2 y^{m-2} + \cdots + \alpha_m}{y^{m-1} + \beta_2 y^{m-2} + \cdots + \beta_m}.$$

□

Malmquist published two further articles [25, 27] on first-order differential equations with meromorphic or algebroid solutions, extending the result from 1913 to the general algebraic first-order differential equation

$$F(z, y, y') = 0, \tag{5.24}$$

where F is an irreducible polynomial in y and y' with algebraic coefficients. The result in the article [25] concerns meromorphic solutions of equation (5.24). It was generalised by Eremenko [6] to the case of admissible solutions in which it takes the following form. Let $\mathcal{P}_\phi$ denote the smallest field containing $\mathcal{A}_\phi$ and all meromorphic functions, i.e. $\mathcal{P}_\phi$ consists of all algebroid functions with 'few' branch points.

Theorem 5.5. *Suppose the differential equation*

$$F(z, y, y') = 0, \quad F \in \mathcal{A}_\phi[y, y'],$$

has an admissible solution $y \in \mathcal{P}_\phi \backslash \mathcal{A}_\phi$. Then either it can be reduced to a Riccati equation,

$$y' = a(z)y^2 + b(z)y + c(z), \quad a, b, c \in \mathcal{A}_\phi,$$

or to the differential equation satisfied by the Weierstraß elliptic function,

$$\left(\frac{dy}{dz}\right)^2 = a(z)\left(4y^3 + g_2 y + g_3\right), \quad a \in \mathcal{A}_\phi.$$

An extension of both Theorem 5.1 and Theorem 5.5 was given by Malmquist in his article [27].

Theorem 5.6 (Malmquist 1941)**.** *Suppose equation (5.24) has a transcendental algebroid solution. Then it can either be reduced to a Riccati equation by a transformation*

$$y^n + R_1(z, w)y^{n-1} + \cdots + R_n(z, w) = 0,$$

or to an elliptic differential equation

$$\left(\frac{dw}{dz}\right)^2 = a(z)\left(4w^3 + g_2 w + g_3\right)$$

by a transformation

$$y^n + R_1(z, w)y^{n-1} + \cdots + R_n(z, w) + \frac{dw}{dz}\left(S_1(z, w)y^{n-1} + \cdots + S_n(z, w)\right) = 0.$$

Using the main arguments in Malmquist's article [27], together with certain Nevanlinna theoretic arguments, it also should be possible to generalise Theorem 5.6 to the case of admissible solutions, however this will be considered in future work.

5.2 2^{nd}-order equations with algebroid solutions

We now consider equations in the class

$$y'' = \frac{2(N+1)}{(N-1)^2} y^N + \sum_{k=0}^{N-1} a_k(z) y^k, \tag{5.25}$$

the normalisation factor being chosen for convenience. Suppose that (5.25) has an admissible algebroid solution y. Then, rearranging (5.25) and using Lemma 4.4, one obtains

$$\begin{aligned}
Nm(r,y) =& m(r, y^N) \\
=& m(r, y'' - a_{N-1}y^{N-1} - \cdots - a_1 y - a_0) + O(1) \\
\leq& m(r,y) + m\left(r, \frac{y''}{y} - a_{N-1}y^{N-2} - \cdots - a_1\right) + m(r, a_0) + O(1) \\
\leq& 2m(r,y) + m(r,a_0) + m(r,a_1) + m(r, a_{N-1}y^{N-3} - \cdots - a_2) \\
&+ S(r,y) \\
\leq& \cdots \leq (N-1)m(r,y) + \sum_{j=0}^{N-1} m(r,a_j) + S(r,y),
\end{aligned}$$

and therefore, since y is assumed to be admissible,

$$m(r,y) = S(r,y). \tag{5.26}$$

This shows that $N(r,y) \asymp T(r,y)$, the notation meaning that both $T(r,y) = O(N(r,y))$ and $N(r,y) = O(T(r,y))$ as $r \to \infty$. In particular, this means that at least one of the symmetric functions $s_1, \ldots, s_n$ has a number of poles growing like $T(r,y)$.

Example: 2-valued algebroid solutions

We will prove the following theorem, see [13].

Theorem 5.7. *Let y be a solution of the equation*

$$y'' = \frac{3}{4} y^5 + \sum_{k=0}^{4} a_k(z) y^k, \tag{5.27}$$

such that y also satisfies

$$y(z)^2 + s_1(z)y(z) + s_2(z) = 0, \tag{5.28}$$

s_1, s_2, $a_0, \ldots, a_4$ being meromorphic functions such that for some $j \in \{1,2\}$, $T(r,a_k) = S(r,s_j)$ for all $k \in \{0,\ldots,4\}$. Suppose that equation (5.28) is irreducible over the meromorphic functions. Then s_1 is proportional to a_4, and s_2 reduces either to the solution of a Riccati equation with coefficients that are rational expressions in $a_0, \ldots, a_4$ and their derivatives, or to the equation

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4(az+b)w^2 + 2((az+b)^2 - c)w, \tag{5.29}$$

which, in case of $a \neq 0$ is equivalent to a special case of the fourth Painlevé equation and in case of $a = 0$ can be solved in terms of elliptic functions or their degenerations.

Proof. Here s_1 and s_2 are the elementary symmetric functions of the two branches y_1, y_2 of y, i.e.

$$s_1 = -(y_1 + y_2), \quad s_2 = y_1 y_2.$$

It follows from (5.26) that also $m(r, s_1) = S(r, y)$ and $m(r, s_2) = S(r, y)$.

At any singularity z_0 of y, where $a_k(z)$, $k \in \{0, \dots, 4\}$ are analytic, we have $y_1, y_2 \sim (z - z_0)^{-\frac{1}{2}}$. Therefore, since s_1 is single-valued, it has no pole at these points z_0 and hence we have $T(r, s_1) = S(r, y)$. On the other hand, since y is an admissible solution, s_2 must have a number of poles of order $T(r, y)$. Differentiating (5.28) once yields

$$2yy' + s_1' y + s_1 y' + s_2' = 0 \quad \Longrightarrow \quad y' = -\frac{s_1' y + s_2'}{2y + s_1}. \tag{5.30}$$

We differentiate again and insert y' from (5.30) and y'' from (5.27). Multiplying by the common denominator $(2y + s_1)^2$ one obtains an equation polynomial in y, s_1 and s_2 and their first and second derivatives. One can use (5.28) repeatedly to reduce the order in y, and in a finite number of steps one obtains an equation

$$F_1(s_1, s_1', s_1'', s_2, s_2', s_2'')y + F_0(s_1, s_1', s_1'', s_2, s_2', s_2'') = 0.$$

Since (5.28) was assumed to be irreducible, y does not satisfy a linear equation of this kind, i.e. we have in fact shown that $F_1 \equiv F_0 \equiv 0$. For F_1 we have

$$\begin{aligned} 0 = F_1 = \left(4s_2 - s_1^2\right) \Big[& s_1'' - s_1^5 + a_4 s_1^4 - a_3 s_1^3 + a_2 s_1^2 - a_1 s_1 + 2a_0 \\ & + s_2(2a_2 + 3a_3 s_1 - 4a_4 s_1^2 + 5s_1^3) + s_2^2(2a_4 - 5s_1) \Big], \end{aligned}$$

and, since $4s_2 - s_1^2$ is the discriminant of the irreducible quadratic equation (5.28), the expression in the brackets must vanish identically, which yields an equation of the form

$$s_1'' + p(s_1) = s_2 q(s_1) + s_2^2(2a_4 - 5s_1),$$

where p and q are polynomial in s_1. However, the left hand side of this equation is of order $S(r, y)$ whereas the right hand side involves s_2. This is only possible if both sides vanish identically, giving the conditions

$$s_1 = \frac{2}{5}a_4, \quad q(s_1) = 0, \quad s_1'' + p(s_1) = 0. \tag{5.31}$$

By a linear transformation in y we could have set $a_4 = 0$ (and therefore $s_1 = 0$) from the start, which we will assume to be done in the following. The other conditions in (5.31) then become $a_2 = 0$ and $a_0 = 0$. The equation $F_0 = 0$ now yields an equation satisfied by s_2:

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2}s_2^3 - 2a_3(z)s_2^2 + 2a_1(z)s_2. \tag{5.32}$$

We will now examine this equation further which must have an admissible meromorphic solution. At any pole z_0 of s_2, where $a_3(z)$ and $a_1(z)$ are analytic,

$$s_2 \sim \alpha(z - z_0)^p, \quad p \in \mathbf{Z},$$

one easily finds that $p = -1$ and $\alpha = \pm 1$. Inserting the full Laurent series

$$\frac{\alpha}{z - z_0} + \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

into (5.32) one can determine the coefficients c_k, $k = 0, 1, 2, \dots$ recursively and finds the expansion

$$\frac{\alpha}{z - z_0} + \frac{1}{2} a_3(z_0) + \left(\frac{\alpha}{4} a_3(z_0)^2 + \frac{2}{3} a_3'(z_0) - \frac{2\alpha}{3} a_1(z_0) \right) (z - z_0) + h(z - z_0)^2 + \cdots, \quad (5.33)$$

where the coefficient h cannot be determined by the recursion, which breaks down for $k = 2$. Instead one finds the resonance condition

$$\alpha a_3''(z_0) + a_3(z_0) a_3'(z_0) - 2a_1'(z_0) = 0. \quad (5.34)$$

From equation (5.32) one obtains, using Lemma 4.4,

$$\begin{aligned} 2m(r, s_2) =& m(r, s_2^2) \\ \leq& m\left(r, \frac{s_2''}{s_2}\right) + 2m\left(r, \frac{s_2'}{s_2}\right) + m(r, s_2) + m(r, 2a_3) + m(r, 2a_1) + O(1), \\ \Rightarrow m(r, s_2) =& S(r, s_2). \end{aligned}$$

It follows that we must have $N(r, s_2) \asymp T(r, s_2)$. However, it is not certain whether both cases of the leading order behaviour $\alpha = \pm 1$ occur with frequency of order $T(r, s_2)$. We denote the integrated counting function of the number of poles of s_2 with leading order behaviour $\frac{\alpha}{z - z_0}$ by $N_\alpha(r, s_2)$. Essentially we consider two different cases. First suppose that both leading order behaviours at the poles of s_2 occur with the same frequency $N_{\pm 1}(r, s_2) \asymp T(r, s_2)$. We then consider the functions

$$\alpha a_3''(z) + a_3(z) a_3'(z) - 2a_1'(z), \quad \alpha = \pm 1.$$

By (5.34) each of these functions has zeros with frequency of order $T(r, s_2)$. But therefore, since s_2 is admissible, they must both vanish identically and one obtains the two conditions

$$a_3'' \equiv 0, \quad (a_3^2 - 4a_1)' \equiv 0,$$

and letting $a_3(z) = -2(az + b)$ and $a_1(z) = (az + b)^2 - c$, equation (5.32) becomes equation (5.29). In case of $a \neq 0$, equation (5.32) reduces, by a linear transformation in z, to the equation

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2} s_2^3 + 4z s_2^2 + 2(z^2 - c) s_2,$$

which is a special case of the fourth Painlevé equation for which it is known that all solutions are meromorphic functions in the complex plane, see e.g. [46] or the book [12]. Otherwise, in case of $a = 0$, equation (5.32) reduces to

$$s_2'' = \frac{(s_2')^2}{2s_2} + \frac{3}{2} s_2^3 + 4b s_2^2 + 2(b^2 - c) s_2,$$

which can be solved in terms of elliptic functions or their degenerations.

For the second case suppose $N_\alpha(r, s_2) \asymp T(r, s_2)$, but $N_{-\alpha}(r, s_2) = S(r, s_2)$, for $\alpha = 1$ or $\alpha = -1$. We will show that in this case s_2 is an admissible solution of a Riccati equation

$$s_2' = -\alpha s_2^2 + u(z)s_2 + v(z). \tag{5.35}$$

Differentiating (5.35) and equating with the right hand side of (5.32) yields the following conditions by comparing coefficients of powers of s_2:

$$u = \alpha a_3, \quad 2\alpha v = 2\alpha a_3' + a_3^2 - 4a_1 \equiv 0.$$

Suppose now that s_2 does not satisfy any Riccati equation admissibly. Then define the function

$$w = s_2' + \alpha s_2^2 - \alpha a_3 s_2, \tag{5.36}$$

which has proximity function $m(r, w) = S(r, s_2)$. At any pole z_0 of s_2 with leading order $\frac{\alpha}{z-z_0}$, by employing the expansion (5.33), w is regular. Therefore w can have poles only where s_2 has a pole with leading order $\frac{-\alpha}{z-z_0}$, i.e. we also have $N(r, w) = S(r, s_2)$. But that means that $T(r, w) = S(r, s_2)$, so (5.36) is a Riccati equation for which s_2 is an admissible solution in contradiction to the assumption. We have therefore proved Theorem 5.7. $\square$

Appendix A

Proof of Lemma 3.2

In this appendix, taken from [18], we show that the curve Γ leading up to a singularity of a solution (y_1, y_2) of the system (3.6) can be modified to a curve $\tilde{\Gamma}$, still of finite length, such that it avoids the zeros of (y_1, y_2). This is a technical necessity to show that the auxiliary function W, constructed in section 3.1, is bounded. The lemma proved here is a generalisation of a lemma by S. Shimomura in [42], where he showed a similar statement for a solution $y(z)$ of a second-order ODE of the form $y'' = E(z,y)(y')^2 + F(z,y)y' + G(z,y)$.

Consider a differential system of two equations in y_1 and y_2 of the form

$$\begin{aligned} y_1' =& F_1(z, y_1, y_2) \\ y_2' =& F_2(z, y_1, y_2) \end{aligned} \tag{A.1}$$

where $F_1, F_2 \in \mathcal{O}_D[y_1, y_2]$ are polynomials in y_1, y_2 with coefficients analytic in some domain D which we take to be a disc $D = \{z \in \mathbf{C} : |z - a| \leq R_0\}$. We assume that F_1, F_2 are of the form

$$\begin{aligned} F_1(z, y_1, y_2) =& \alpha_{10N_1} y_2^{N_1} + \sum_{j=0}^{M_1} \sum_{k=0}^{N_1 - 1} \alpha_{1jk}(z) y_1^j y_k^k, \\ F_2(z, y_1, y_2) =& \alpha_{2M_20} y_1^{M_2} + \sum_{j=0}^{M_2-1} \sum_{k=0}^{N_2} \alpha_{2jk}(z) y_1^j y_k^k, \end{aligned}$$

where $N_1 \geq N_2$, $M_2 \geq M_1$ and $\alpha_{10N_1}, \alpha_{2M_20}$ are constants with $|\alpha_{10N_1}| \geq 1$, $|\alpha_{2M_20}| \geq 1$. Let $K > 1$ be a constant so that $|\alpha_{ijk}(z)| < K$ for all i, j, k and $z \in D$. Also, let $N_1 := N, M_2 := M$ and $C := 2^{N+1}(M+1)(N+1)K$.

Lemma A.1. *Let $0 < \Delta < 1$ and $\theta := \min\{\frac{\Delta}{C}, R_0\}$. Let (y_1, y_2) be a solution of (A.1) analytic at a point c for which $|c - a| < \frac{R_0}{2}$. Suppose that $|y_1(c)| < \frac{\theta}{8}$ and $|y_2(c)| > C$. Then $(y_1(z), y_2(z))$ is analytic on the disc $|z - c| < \frac{\theta}{|y_2(c)|}$ and satisfies $|y_1(z)| \geq \frac{\theta}{8}$ and $|y_2(z)| \geq 1$ on the circle $|z - c| = \frac{\theta}{2|y_2(c)|}$.*

Proof. Let $\rho = y_2(c)^N$, $\zeta = \rho(z - c)$ and define $\eta_i(\zeta) := y_i(z)$, $i = 1, 2$. Denoting the derivative with respect to ζ by a dot we have $\dot{\eta}_i(\zeta) = \rho^{-1} y_i'(z)$ and

$$\eta_i(\zeta) = \eta_i(0) + \int_0^\zeta \dot{\eta}_i(\tilde{\zeta}) d\tilde{\zeta},$$

where $\eta_i(0) = y_i(c)$. Define the functions $M_i(r) = \max_{|\zeta|\leq r} |\eta_i(\zeta)|$, $i = 1, 2$, and let $r_0 = \sup\{r : M_1(r) < \Delta, M_2(r) < 2|\rho|^{1/N}\}$. Clearly we have $r_0 > 0$. For $|\zeta| < \min\{r_0, R_0\}$ we have, since $|z - a| \leq |z - c| + |c - a| < \frac{R_0}{|\rho|} + \frac{R_0}{2} \leq R_0$,

$$|\eta_i(\zeta)| \leq |y_i(c)| + |\rho|^{-1}|\zeta| \sum_{j=0}^{M_i} \sum_{k=0}^{N_i} K\Delta^j 2^k |\rho|^{\frac{k}{N}} \leq |y_i(c)| + |\zeta| 2^N K(N+1)(M+1). \quad \text{(A.2)}$$

Now suppose that $r_0 < \theta$. Then, for $|\zeta| < r_0 < R_0$ we have the estimates

$$\begin{aligned} |\eta_1(\zeta)| <& \theta(1/8 + 2^N(N+1)(M+1)K) < \Delta, \\ |\eta_2(\zeta)| <& |y_2(c)| + \theta 2^N(M+1)(N+1)K < 2|y_2(c)|, \end{aligned}$$

in contradiction to the definition of r_0. Therefore we must have $r_0 \geq \theta$, showing that (A.2), $i = 1, 2$, is valid for $|\zeta| < \theta$ and therefore that η_1 and η_2 are analytic for $|\zeta| < \theta$. We now obtain estimates for η_1 and η_2 in the opposite direction on the circle $|\zeta| = \frac{\theta}{2}$:

$$\begin{aligned} |\eta_1(\zeta)| \geq& \left| \int_0^\zeta \rho^{-1} \alpha_{10N} \eta_2(\tilde{\zeta})^N \mathrm{d}\tilde{\zeta} \right| - \left| \int_0^\zeta \rho^{-1} \sum_{i=0}^{M_1} \sum_{j=0}^{N-1} \alpha_{1ij}(z) \eta_1^i \eta_2^j \mathrm{d}\tilde{\zeta} \right| - |\eta_1(0)| \\ \geq& \left| \int_0^\zeta \left(1 + \frac{\eta_2(\tilde{\zeta}) - \eta_2(0)}{\eta_2(0)} \right)^N \mathrm{d}\tilde{\zeta} \right| - \frac{\theta}{2} |\rho|^{-\frac{1}{N}} 2^{N-1}(M+1)NK - \frac{\theta}{8} \\ \geq& \left| \int_0^\zeta \left(1 + \sum_{n=1}^N \binom{N}{n} \left(\frac{\eta_2(\tilde{\zeta}) - \eta_2(0)}{\eta_2(0)} \right)^n \right) \mathrm{d}\tilde{\zeta} \right| - \frac{\theta}{4} \\ \geq& \frac{\theta}{2} - \frac{\theta}{2} \sum_{n=1}^N \binom{N}{n} \left(\frac{\Delta}{C} \right)^n - \frac{\theta}{4} \\ \geq& \frac{\theta}{8}, \\ |\eta_2(\zeta)| \geq& |y_2(c)| - \theta 2^N(M+1)(N+1)K \\ \geq& 1. \end{aligned}$$

□

Remark A.2. *In Lemma A.1 the role of y_1 and y_2 can be interchanged if in every expression one simultaneously replaces $M \leftrightarrow N$.*

Using Lemma A.1 and Remark A.2 we can now show that a curve ending in a movable singularity of a solution (y_1, y_2) of the system (A.1) can be modified by arcs of circles in such a way that both y_1 and y_2 are bounded away from 0 on the modified curve.

Lemma A.3 (1st curve modification)**.** *Suppose (y_1, y_2) is a solution of (A.1), analytic on a finite length curve $\Gamma \subset D$ up to, but not including its endpoint $z_\infty \in D$. Then we can deform Γ, if necessary, in the region where (y_1, y_2) is analytic, to a curve $\tilde{\Gamma}$, still of finite length, such that y_1 and y_2 are bounded away from 0 on $\tilde{\Gamma}$ in a neighbourhood of z_∞.*

Proof. Let Γ be parametrised by arclength such that $\Gamma(0) = z_0$, $\Gamma(l) = z_\infty$ where l is the length of Γ. Define the two sets

$$S_i := \{s : 0 < s < l \text{ and } |y_i(\Gamma(s))| \leq \theta/8\}, \quad i = 1, 2.$$

We assume that $\liminf_{s\to l^-} \min\{|y_1|, |y_2|\} = 0$, otherwise there is nothing to show. Therefore the union $S_1 \cup S_2$ contains values arbitrarily close to l. There now exists some number $0 < s_0 < l$ with the following two properties: (i) $S_1 \cap S_2 \cap [s_0, l) = \emptyset$, (ii) whenever $s \in S_i$, $s > s_0$, we have $|y_{3-i}(\Gamma(s))| > C$. Namely, if this was not the case we could find a sequence $z_i = \Gamma(s_i)$, $s_i \to l$, such that $(y_1(z_i), y_2(z_i))$ is bounded and hence, by Lemma 2.4, the solution could be analytically continued to z_∞ in contradiction to the assumption. Denote $S = (S_1 \cup S_2) \cap [s_0, l)$ and let $s_1 = \inf\{s \in S : s > s_0\}$. Suppose that $s_1 \in S_i$ and let $r_1 = \frac{\theta}{2|y_{3-i}(\Gamma(s_1))|}$. Lemma A.1 now shows that that y_1 and y_2 are analytic for $|z - \Gamma(s_1)| < 2r_1$ and that $|y_i(z)| \geq \theta/8$ and $|y_{3-i}(z)| \geq 1$ on the circle $C_1 = \{z : |z - \Gamma(s_1)| = r_1\}$. We now recursively define a sequence of points s_n and circles C_n with radii r_n as follows: Let $s_{n+1} = \inf\{s \in S : s > s_n + r_n\}$. If $s_{n+1} \in S_i$ ($i = 1$ or 2), then let $r_{n+1} = \frac{\theta}{2|y_{3-i}(\Gamma(s_n))|}$.

By Lemma A.1, for every circle C_n, $n = 1, 2, \ldots,$ we have $|y_1(z)|, |y_2(z)| \geq \frac{\theta}{8}$ for all $z \in C_n$. Also, $\sum_{n=1}^\infty r_n \leq \sum_{n=1}^\infty |s_{n+1} - s_n| \leq l$ which implies $r_n \to 0$ as $n \to \infty$. The centres s_n of the circles accumulate at z_∞: If this was not the case we would have $s_n \to s_\infty$ for some $s_\infty < l$, but then

$$\lim_{n\to\infty} \max\{|y_1(\Gamma(s_n))|, |y_2(\Gamma(s_n))|\} \geq \lim_{n\to\infty} \frac{\theta}{2r_n} = \infty,$$

in contradiction to the fact that $(y_1(z), y_2(z))$ is analytic on $\Gamma \setminus \{z_\infty\}$. We now define $\tilde{\Gamma}$ in the following way. Suppose for convenience that Γ has no self-intersections (otherwise we could shorten Γ by omitting pieces between self-intersections). Let Γ_{ext} be an infinite non-intersecting extension of Γ such that $\Gamma_{\text{ext}}(s) \to \infty$ for $s \to \pm\infty$ which divides the complex plane into parts $\mathbf{C}_+$ and $\mathbf{C}_-$ such that $\mathbf{C}_+$, Γ_{ext} and $\mathbf{C}_-$ are pairwise disjoint and $\mathbf{C}_+ \cup \Gamma_{\text{ext}} \cup \mathbf{C}_- = \mathbf{C}$. Now let $D = \Gamma \cup \bigcup_{n=1}^\infty D_n$ where $D_n = \{z : |z - \Gamma(s_n)| \leq r_n\}$ and define $\tilde{\Gamma} = \partial D \cap (\mathbf{C}_+ \cup \Gamma_{\text{ext}})$. Then (y_1, y_2) is analytic on $\tilde{\Gamma}$ and $|y_1(z)|, |y_2(z)| \geq \frac{\theta}{8}$ for all $z \in \tilde{\Gamma}$. Furthermore, $\tilde{\Gamma}$ has length less than $(1 + 2\pi)l$. $\square$

We will now specialise the results obtained so far in this section to the Hamiltonian system (3.6) which is of the form (A.1) with $N_1 = N$, $M_2 = M$. Lemma A.3 is not quite enough to show that the auxiliary function W in section 3.1, rational in y_1 and y_2, is bounded. We need to show that certain terms of the form $\frac{y_2^k}{y_1^l}$ are bounded. To do so we will apply a second curve modification where we can now make use of the fact that y_1 and y_2 are already bounded away from 0 on Γ. We rewrite the system of equations (3.6) in the variables $u_1 = y_1 \cdot y_2^{-\frac{N+1}{M+1}}$ and $u_2 = y_2$ for some branch of $y_2^{\frac{1}{M+1}}$.

The system of equations in the variables u_1, u_2 becomes

$$\begin{aligned} u_1' =& (N+1)u_2^{N-\frac{N+1}{M+1}}\left(1 + u_1^{M+1}\right) + \sum_{(i,j)\in I'} \left(j + i\frac{N+1}{M+1}\right)\alpha_{ij}u_1^i u_2^{(i-1)\frac{N+1}{M+1}+j-1} \\ u_2' =& -(M+1)u_1^M u_2^{M\frac{N+1}{M+1}} - \sum_{(i,j)\in I'} i\alpha_{ij}u_1^{i-1}u_2^{(i-1)\frac{N+1}{M+1}+j}. \end{aligned} \tag{A.3}$$

Let $K > 1$ be a constant such that $|i\alpha_{ij}(z)| < K$ and $\left|\left(j + i\frac{N+1}{M+1}\right)\alpha_{ij}(z)\right| < K$ for all $(i,j) \in \tilde{I} = I' \cup \{(M+1, 0), (0, N+1)\}$, $z \in D$. As before let $C = 2^{N+1}K(M+1)(N+1)$.

Suppose $(u_1(z), u_2(z))$ is a solution of (A.3), corresponding to a solution $(y_1(z), y_2(z))$ of (3.6) on a curve Γ, which by Lemma A.3 we assume to be such that y_1 and $y_2 = u_2$ are bounded away from 0 on Γ. The following Lemma is somewhat similar to Lemma A.1, the proof, however, requires some modifications.

Lemma A.4. *Let* $0 < \Delta < 2^{-N-2}(N+1)^{-1} < 1$ *and* $\theta := \min\{\frac{\Delta}{C}, R_0\}$. *Let* (u_1, u_2) *be a solution of (A.3) analytic at* c *with* $|c-a| \le \frac{R_0}{2}$ *and suppose that* $|u_1(c)| < \frac{\theta}{8}$ *and* $|u_2(c)| > (4C)^{M+1}$. *Then* $(u_1(z), u_2(z))$ *is analytic in the disc* $|z-c| < \frac{\theta}{|u_2(c)|}$ *and on the circle* $|z-c| = \frac{\theta}{2|u_2(c)|}$ *we have* $|u_1(c)| \ge \frac{\theta}{8}$ *and* $|u_2(c)| \ge 1$.

Proof. Let $\rho = u_2(c)^L$, where $L = N - \frac{N+1}{M+1} \le N-1$. For $i = 1, 2$ let $\eta_i(\zeta) := u_i(z)$, where $\zeta = \rho(z-c)$, and define $M_i(r) = \max_{|\zeta| \le r} |\eta_i(\zeta)|$, $m_i(r) = \min_{|\zeta| \le r} |\eta_i(\zeta)|$. Let

$$r_0 = \sup\left\{ r : M_1(r) < \Delta, M_2(r) < 2|\rho|^{1/L}, m_2(r) > \frac{1}{2}|\rho|^{1/L} \right\}, \tag{A.4}$$

which is positive as $|\eta_1(0)| < \Delta$ and $|\eta_2(0)| = |\rho|^{1/L}$. We have

$$\eta_i(\zeta) = \eta_i(0) + \int_0^\zeta \dot{\eta}_i(\zeta) d\zeta,$$

where $\eta_i(0) = u_i(c)$ and $\dot{\eta}_i(\zeta) = \rho^{-1} u_i'(z)$. For $|\zeta| < \min\{r_0, R_0\}$ we have, since $|z-a| \le |z-c| + |c-a| < \frac{R_0}{|\rho|} + \frac{R_0}{2} < R_0$,

$$\begin{aligned} |\eta_1(\zeta)| \le & |u_1(c)| + |\rho|^{-1}|\zeta| \sum_{(i,j) \in \tilde{I} \setminus \{(0,0)\}} K\Delta^i 2^{|(i-1)\frac{N+1}{M+1}+j-1|} |\rho|^{((i-1)\frac{N+1}{M+1}+j-1)/L} \\ \le & |u_1(c)| + |\zeta| 2^N K(M+1)(N+1), \end{aligned} \tag{A.5}$$

$$\begin{aligned} |\eta_2(\zeta)| \le & |u_2(c)| + |\rho|^{-1}|\zeta| \sum_{\substack{(i,j) \in \tilde{I} \\ i \ne 0}} K\Delta^{i-1} 2^{|(i-1)\frac{N+1}{M+1}+j|} |\rho|^{((i-1)\frac{N+1}{M+1}+j)/L} \\ \le & |u_2(c)| \left(1 + |\zeta| 2^N K(M+1)(N+1)\right), \\ |\eta_2(\zeta)| \ge & |u_2(c)| \left(1 - |\zeta| 2^N K(M+1)(N+1)\right), \end{aligned} \tag{A.6}$$

where we have used condition (3.2) which implies $(i-1)\frac{N+1}{M+1} + j - 1 \le L$ for $(i,j) \in \tilde{I} \setminus \{(0,0)\}$ and therefore $\left|(i-1)\frac{N+1}{M+1} + j - 1\right| \le N$. Now supposing that $r_0 < \theta$ one would obtain the estimates

$$\begin{aligned} |\eta_1(\zeta)| \le & \theta(1/8 + 2^N K(M+1)(N+1)) < \Delta, \\ |\eta_2(\zeta)| \le & |u_2(c)| \left(1 + \theta 2^N K(M+1)(N+1)\right) < 2|\rho|^{1/L}, \\ |\eta_2(\zeta)| \ge & |u_2(c)| \left(1 - \theta 2^N K(M+1)(N+1)\right) > \frac{1}{2}|\rho|^{1/L}, \end{aligned}$$

in contradiction to the definition (A.4) of r_0. Therefore we must have $r_0 \ge \theta$, implying that the estimates (A.5), (A.6) are valid for $|\zeta| < \theta$ and that u_1, u_2 are analytic for $|\zeta| < \theta$.

On the circle $|\zeta| = \frac{\theta}{2}$ we now have

$$\begin{aligned}
|\eta_1(\zeta)| \geq &(N+1)\left|\int_0^\zeta \rho^{-1}\eta_2(\tilde\zeta)^L d\tilde\zeta\right| - \left|\int_0^\zeta \rho^{-1}(N+1)\eta_1^{M+1}\eta_2^{N-\frac{N+1}{M+1}} d\tilde\zeta\right| \\
&- \left|\int_0^\zeta \rho^{-1}\sum_{(i,j)\in I'}\left(j + i\frac{N+1}{M+1}\right)\alpha_{ij}\eta_1^i\eta_2^{(i-1)\frac{N+1}{M+1}+j-1} d\tilde\zeta\right| - |\eta_1(0)| \\
\geq &(N+1)\left|\int_0^\zeta \left(1 + \frac{\eta_2(\tilde\zeta)-\eta_2(0)}{\eta_2(0)}\right)^L d\tilde\zeta\right| - \frac{\theta}{2}(N+1)\Delta^{M+1}2^L \\
&- \frac{\theta}{2}|\rho|^{-\frac{1}{L(M+1)}}2^N K(M+1)(N+1) - \frac{\theta}{8} \\
\geq &\left|\int_0^\zeta d\tilde\zeta\right| - \left|\int_0^\zeta \left(\left(1 + \frac{\eta_2(\tilde\zeta)-\eta_2(0)}{\eta_2(0)}\right)^L - 1\right) d\tilde\zeta\right| - \frac{\theta}{4} \\
\geq &\frac{\theta}{4} - \frac{\theta}{2}\sum_{n=1}^N \binom{N}{n}\left(\frac{\Delta}{4C}\right)^n \geq \frac{\theta}{8}, \\
|\eta_2(\zeta)| \geq &\frac{1}{2}|\rho|^{1/L} > 1.
\end{aligned}$$

□

The final lemma of this appendix is Lemma 3.2 of section 3.1.

Lemma A.5 (2nd curve modification). *Let (y_1, y_2) be a solution of the system (3.6), analytic on the finite length curve Γ ending in a movable singularity z_∞, such that $\frac{1}{y_1}$ and $\frac{1}{y_2}$ are bounded on Γ. Then, after a possible deformation of Γ in the region where y_1, y_2 are analytic, one can achieve that $\frac{y_2^k}{y_1^l}$ is bounded on $\tilde\Gamma$ for all $k, l \geq 0$ for which $l(N+1) - k(M+1) \geq 0$.*

Proof. Define the set $S = \{s : 0 < s < l \text{ and } |u_1(\Gamma(s))| \leq \theta/8\}$. There exists some s_0, $0 < s_0 < l$, such that on $S \cap [s_0, l]$ one has $|u_2(z)| > (4C)^{M+1}$. For, if this was not the case, one would have a sequence of points (z_n) on Γ with $z_n \to z_\infty$ as $n \to \infty$ such that $u_1(z_n)$ is bounded and $u_2(z_n)$ is bounded and bounded away from zero. Lemma 2.4 applied to the system (A.3) would then imply that u_1, u_2 are analytic at z_∞ in contradiction to the assumption. By the same method as in the proof of Lemma A.3 one can now deform the curve Γ by arcs of circles such that u_1 and u_2 are bounded away from 0 on the modified curve $\tilde\Gamma$, that is, $u_1^{-(M+1)} = \frac{y_2^{N+1}}{y_1^{M+1}}$ and $u_2^{-1} = \frac{1}{y_2}$ are bounded on $\tilde\Gamma$. By writing

$$\frac{y_2^k}{y_1^l} = \left(\left(\frac{y_2^{N+1}}{y_1^{M+1}}\right)^l \cdot \frac{1}{y_2^{l(N+1)-k(M+1)}}\right)^{1/(M+1)},$$

one can conclude that $\frac{y_2^k}{y_1^l}$ is bounded on $\tilde\Gamma$ if $l(N+1) - k(M+1) \geq 0$. □

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