Preemption Games with Private Information*

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Abstract

Preemption games are widely used to model patent races, innovation adoption and market entry problems. A previously neglected feature of these problems is that the agents' states (e.g. R&D firms' technological improvements) are kept secret and stochastically change over time. We fully characterize equilibrium in preemption games where private information evolves according to Poisson processes, and provide a strategic rationale for the common wisdom that 'big things happen fast.' In the context of patent races we surprisingly find that strengthening patent rights need not increase innovation disclosure. Furthermore, we clarify a basic welfare tradeoff between duplication costs and preemption: the former likely take place in early stages of the race, and preemption in later stages.

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1 Introduction

The analysis of timing decisions is of prime relevance in economic theory. For example, the timing of patenting determines the rate of disclosure of innovations, and hence welfare assessments of R&D activities and policies. Similarly, the timing of adoption of process innovations determines the basic technological fundamentals of industrial organization problems. The timing of product innovation and market entry is a major concern of both incumbents and potential entrants in contestable markets. Such economics problems are typically represented as 'preemption games': models where each agents' key strategic decision is the timing of a given action, players would be better off if they could jointly commit to postpone their actions, but there is a first-mover advantage in payoffs.[1]

One obviously important feature of these timing problems is *private information:* the agents' states in the game are only privately known and are stochastically changing over time. For example, R&D competitors do not share information about their technological improvements before filing for a patent. R&D results are jealously kept secret and the practice of industrial espionage has developed as a result of this. Similarly, the development of product and process innovations is typically kept secret from potential competitors. *This paper studies preemption games where players' state stochastically changes over time,* and their payoffs depends on both players' states. We derive general theoretical results that underline fundamental structural differences with the previous analyses of timing games. Furthermore, our analysis uncovers novel positive insights on the timing of innovation adoption and patenting as well as novel welfare predictions and policy implications.

Incorporating in the analysis the realistic feature that players' private information states change over time generates novel conceptual obstacles. The only available information to a player is that the opponent did not leave the game yet. How should a player update her beliefs on the opponent's state and hence on the risk of being preempted? If the opponent is still in the game at a late time, should a player believe that likely the opponent will remain longer and take the risk of delaying exit, or should she believe that the opponent is coming close to end the game and leave the race at

[1] Among the earliest examples of preemption games, see for example Reinganum (1981a) and Fudenberg and Tirole (1985). Related to preemption games are their mirror-image games, wars of attritions, i.e. stopping games with a late-mover advantage. Possibly the class of timing games that has been studied in more depth are auction models.

the earliest opportunity? And how should a player react to her own state dynamics? Should she be more willing to leave the game immediately after a large increment, so as to cash this increment in, and avoid being preempted by the competitor and losing everything? Or should she more willing to leave after an unlucky streak with no or minor state increments for fear that the opponent is ahead of her in the game and coming close to end the game?

We provide definite answers to these questions in a stylized continuous time framework. In any instant, each player's state is the sum of past increments that arrive according to i.i.d. Poisson processes. Conditionally on arrival, the value of the state increment is randomly drawn from distributions identical and independent across players. Under mild assumptions on the players' payoffs, we show that (essentially) all equilibria are characterized by a time-dependent threshold function: at each moment in time, an agent ends the game if and only if her state is above a certain threshold. Contrary to simple-minded intuition, these preemption games do not unravel: despite being ignorant of their opponent's state, players do not immediately end the game, even when they would lose everything if preempted.

We then show existence of time-decreasing threshold equilibrium, and we derive in closed-form the ordinary differential equation governing the equilibrium threshold. To illustrate the substantive meaning of our equilibrium, consider for example a patent race where an innovation 'is in the air': more than one firms are working on it. As time advances, the competitors become more and more concerned with the risk of preemption, and less willing to wait for additional results before applying for a patent.[2] From the standpoint of an outside observer, there is an inverse relation between the timing of disclosure and the entity of innovations: we provide a strategic justification for the common wisdom that *big things happen fast.*

Our equilibrium comparative statics analysis surprisingly find that strengthening patent rights does not necessarily lead to more innovation disclosure.[3] In industries where research is more expensive than development, stronger patent rights induce firms to anticipate patenting. When

[2]Even though, in reality, one does not know precisely when the opponent starts competing, our model can be easily extended to account for this by allowing the competitors to randomly entering the race over time.

[3]As well as patent races, our analysis equally applies to sponsored research tournaments (e.g. Aoki (2001)). Venture capitalists, for example, frequently run R&D tournaments when they allow only the best entrepreneur to go to the initial public offering (IPO) market. In some research tournaments (e.g. the recent Federal Communications Commission sponsored tournament to develop the best technology for high-definition television), the firm with the best idea wins an exclusive right for commercializing it.

development is costlier than research, stronger patent protection laws induce firms to procrastinate disclosure.[4] In the context of innovation adoption, elaborating on the model by Reinganum (1981a), we unexpectedly find that, while subsidizing an innovation's adoption makes it happen faster, subsidizing the innovation development bears the perverse effect of delaying the innovation adoption.

Our welfare results, specialized to the patent race problem again, allows us to clarify the interplay of counteracting strategic inefficiency effects known to the R&D race literature. On the one hand, patent races competitors do not want to rush to the patent office too soon and disclose minor innovations, they would rather wait until accumulating a technological edge. This strategic delay may be distorsive because it yields *duplication costs* when different competitors are reinventing the same innovations instead of disclosing them through patents. On the other hand, firms do not want to delay patenting too long, lest a competitor makes a similar discovery and beats them in the race to the patent office. Such strategic *preemption* is distorsive when it is socially wasteful to patent and develop too many incremental innovations.

Our analysis clarifies that *both effects may coexist in a patent race, but excessive duplication costs likely take place in early stages of the race, and excessive preemption in later stages.* If a firm obtains a valuable innovation early in the race, it is not much concerned for the risk of being preemptied, and would delay patenting in the hope of further increments. The social planner dislikes this delay because it internalizes the futile duplication costs borne by the opponent trying to catch up in the race. Later in the race, each firm becomes more and more concerned that her opponent will soon end the race. This fear of preemption feeds on itself in equilibrium and makes the firms willing to preempt each other and patent relatively unprofitable innovations.

This paper is presented as follows. After the literature review, section 3 presents the general model with a few applications, and the equilibrium is characterized in section 4. The social planner's problem is studied in section 5, and section 6 concludes.

[4] Interestingly, development costs can themselves be understood as a policy parameter, e.g. the FDA trials before a patented drug is allowed to be produced and marketed.

2 Literature Review

Timing and preemption games have been studied widely; but their compelling feature that private information changes over time has not been previously incorporated in the analysis. Among the earliest games ever studied is Duel (see for example, Karlin (1959)). In this simple two player preemption game two duellists shoot at each other with efficacy increasing over time.[5] A number of papers model patent races as preemption games. Possibly the earliest one is by Fudenberg et al. (1983). Closer to our work, Weeds (2002) studies a model with symmetric information where the value of the innovation changes stochastically over time. Lambrecht and Perraudin (2003) study a preemption game with Brownian-motion where two firms own options to buy an asset with publicly observable value stochastically changing over time, and where strike prices are private information. Unlike our general analysis, private information is of private value and constant over time. Reinganum (1981a), Fudenberg and Tirole (1985) and Riordan (1992) study technology adoption preemption games. Competitors may decide whether to immediately adopt or delay the adoption of a process innovation disclosed by a 'third party,' that reduces production costs. Their analysis also applies to market entry problems. Trading in a financial bubbles can also be understood as a preemption game (for example Abreu and Brunnermeier (2002)): Everyone wants to sell before the bubble bursts but stay in as long as the bubble lasts.

Several papers on R&D races have highlighted either duplication costs or preemption effects in different models. In the 'Poisson games' framework pioneered by Reinganum, (1981b, 1982), each firm selects its experimentation intensity over time, and this affects the Poisson rate of an innovation arrival. These models' equilibria display duplication costs: firms overinvest in equilibrium. In the "tug-of-war" models following Harris and Vickers (1985), firms take turns in making costly steps towards a "finish line." In the absence of uncertainty, a dramatic preemption effect takes place: Once a firm is ahead in the race, its competitors immediately quit. But this effect disappears when introducing uncertainty in the duration of each step (Harris and Vickers (1987)), and again equilibrium R&D displays duplication costs. Unlike these models, our analysis incorporates private information, and proves that both preemption and duplication effects may be present in a race, but

[5] This model has evident military and economic applications, e.g. Binmore (2004) motivates its study as a model of patent races.

they predominate at different stages of the race.

Related to preemption games are their mirror-image games, wars of attritions, i.e. timing games with a late-mover advantage. Baye, Kovenock and de Vries (1993) apply these games to lobbying. Bulow and Klemperer (1999) generalize the war of attrition (without private information) to the case where the last k out of n players receive a higher payoff, and provide applications to industry dynamics. Fudenberg and Tirole (1986) model a duopoly war with changing demand as a war of attrition with private information on marginal costs. Krishna and Morgan (1997) study general war of attritions with private information. Unlike our contribution, none of these papers study stopping games where private information changes while the race is running. Decamps and Mariotti (2003) study a Poisson war-of-attrition where the cost of the irreversible invesment in an uncertain project is private information. Players have an incentive to wait-and-see, before exercising their investment option. Private information is only of private value, whereas in our game it has interdependent value and changes over time.

Park and Smith (2003) provide a general formulation and solution of stopping games in relation to the payoff ranking of players depending of their exit order, without private information. Their analysis subsumes both preemption games and wars of attrition. Finally, this paper contributes to the literature on learning and experimentation in games, which dates back to Bolton and Harris (1999), and has been further advanced by Bergemann and Valimaki (1997, 2000), Keller and Rady (2003), and Cripps, Rady and Keller (2004).

3 The Game

Two players, A and B are engaged in the following timing game. The state of each player i at any time t is expressed as $x_i(t) \in \mathbb{R}_+$; for example, x_i may be the value of player i's innovation in a R&D race or innovation adoption game. Each player has independent Poisson arrivals of state increments of rate $\rho \geq 0$: at any time t the next arrival $\tau_i \geq t$ is distributed according to the c.d.f. $H(\tau_i|\rho, t) = 1 - e^{-\rho(\tau_i - t)}$. The distribution for the increment $w \in \mathbb{R}_+$ -conditional on arrival- is G. The c.d.f. G admits a density g and has connected support that contains 0. We assume that the distribution of increments is log-concave, i.e.

$$d \log G(w) / dw = g(w) / G(w) \text{ is decreasing in } w.$$

For example, increments may be negative exponentially distributed of parameter λ, e.g. $g(y) = \lambda e^{-\lambda y}$. Increments are independent across players. We let the expected value of an increment be $\bar{w} = \int_0^\infty w dG(w)$. The renewal process we have just described defines implicitly a distribution of the random state x of each player at time t; we denote the associated c.d.f. by $F(t,x)$, and its density by $f(t,x)$. For notational simplicity, we normalize to zero the state $x(0)$ at time zero.

Each player i incurs a flow cost $c(x,t)$ at time t to remain in the game. We assume that c is weakly increasing over time and state and bounded, setting $\underline{c} = c(0,0)$ and $\bar{c} = \lim_{\substack{t\to\infty \\ x\to\infty}} c(x,t)$. At any point in time T, each player i may decide to stop the game and achieve payoff $u(x_i(T), x_j(T))$. This payoff need not be instantaneous, but it may be obtained in a continuation game separate from the timing game; for example, it may be the present discounted value of future market competition. The opponent receives payoff $\underline{u}(x_j(T), x_i(T))$. By definition of preemption game, the first-mover has an advantage: there is a uniform (possibly small) bound ϵ such that $u(x,y) > \underline{u}(x,y) + \epsilon$ for any x, y not smaller than $x(0)$.

Evidently, each player's payoff is increasing in her own state, and we assume that $u_1 > 0$ and $\underline{u}_1 \geq 0$, this weak inequality allows the utility not to depend on one's state when preempted by the opponent. In order to highlight the competitive features of the environment, we also assume that $u_2 \leq 0$ and $\underline{u}_2 \leq 0$: an increment of the opponent state cannot increase a player's payoff. The functions u and $\underline{u}$ are assumed to be $\mathcal{C}^2$.

For each player $i = A, B$, a history at time h_i^t is a increasing path of states $x_i(\tau)$ for $0 \leq \tau < t$. In general, a pure strategy in this game is a measurable stopping time σ^i function of the history h_i^t, that identifies the earliest moment at which firm i is willing to stop the game given history h_i^t. Hence player i ends the game at time $T_i = \inf\{t : \sigma^i(h_i^t) = t\}$.[6] It is natural to focus on strategies σ^i that depend on the innovation state x and on calendar time t only, and not on the entire history of increments h_i^t.[7] The equilibrium belief of either player with respect to the opponent's state y is

[6] For a general treatment on how to construct stopping time strategies in continuous time games and on their interpretation, see Simon and Stinchcombe (1989).

[7] Because the underlying parameters of the process F are known, the player i does not draw any inference based on her private history h_i^t, additional to the inference based on the state $x(t)$.

denoted by the c.d.f. $\beta(y,t)$. To simplify the analysis, as is customary in the timing-game literature, we shall focus on symmetric equilibria: both players adopt the same equilibrium threshold strategy σ^i and we shall henceforth omit the subscript i.

3.1 Applications

The general preemption game environment described above is of wide applicability. Here we present some simple stylized economic problems that can be analyzed within the framework.

Patent races. In a patent race application, the two players are firms who conduct research activity at a flow cost $c(x,t)$ that depends on the state x and the time t. This research activity improves the value of a patentable innovation over time. The state x_i of firm i corresponds to the value of the innovation. As time goes by, and as the state increases, the research activity does not become cheaper. At any time T, each firm may end the game, patent and develop the innovation $x_i(T)$. The patenting firm receives the payoff $u(x_i(T), x_j(T))$ and the competitor receives the payoff $\underline{u}(x_j(T), x_i(T))$. Evidently, being the first to patent an innovation gives an advantage, ceteris paribus, and hence it is meaningful to state that $u(x_i(T), x_j(T)) > \underline{u}(x_i(T), x_j(T)) + \epsilon$. We single out in the payoffs a cost c_0 for patenting and developing an innovation, and assume that $u(x_i(T), x_j(T)) = \hat{u}(x_i(T), x_j(T)) - c_0$, with $\hat{u}_1 > 0$ and $\hat{u}_2 \leq 0$.

One key feature of the patent institution is that it discloses the innovation and erases any claims of partial ownership by competitors. If the innovation $x_j(T)$ of firm i's opponent j is covered by i's patent, j will lose the benefit of its research achievements. This in turns implies that when $x \geq y$, the payoff $u(x,y)$ for patenting an innovation x does not depend on y, nor does the opponent's payoff $\underline{u}(y,x)$. Hence in the context of patent races we assume that

$$\hat{u}(x,y) = v(x) \text{ and } \underline{u}(y,x) = \underline{v}(x) \text{ if } x \geq y, \quad \text{with } v' > 0 \text{ and } \underline{v}' \leq 0;$$

but we retain the possibility that payoffs depend on both firms states in the case that the patenting firm does not own the most advanced innovation.

In order to assess the effect of patent policy on equilibrium, we shall also introduce the utility specification $u(x,y;\alpha) = \alpha \cdot \hat{u}(x,y)$, where the policy parameter α that measures the extent to which the innovator captures the value of the innovation: for concreteness, α can be understood as the breadth and length of the patent. Finally, we say that $v(0) > c_0 + \epsilon$, so that, as long as one is ahead in the race, the option of dropping out of the patent race is dominated by the option of patenting one's own innovations. This assumption allows us to focus the analysis on the patenting choice.

An innovation adoption game. In the classic preemption game scenario studied by Reinganum (1981) and later perfected by Fudenberg and Tirole (1985), an innovation is disclosed (and patented) by a 'third player' to competitors in a specific industry. It is common knowledge that the innovation may potentially yield process optimizations that reduce production costs. A reduction of production costs yields a competitive edge and tilts the profits in favor of the first firm adopting the innovation. We focus on the case where the first-mover advantage is not immediately wiped out when the second-mover adopts the process technology. Because the choice of adoption by the second mover is a simple one-agent decision problem, we can summarize the problem as a preemption game, where the present discounted values of the continuation profits at the moment of first adoption of the innovation process are Π_1 and Π_2, with $\Pi_1 > \Pi_2$

As in Reinganum (1981) adopting the new process technology is costly. We expand her model to posit adoption costs that depend on the stage of development of the innovation. The technology is not readily implementable in the process and needs to be developed and tailored to each firm's processes. This development and adaptation activity is private to each separate firm. Within our framework, it is natural to represent the stage of development of the innovation by firm i at time t by the state $x_i(t)$, and to posit that $C(x_i(t))$, the cost of adoption of the process innovation by firm i depends solely on firm i's stage of technology development. Thus, we may study utility specifications

$$u(x, y) = \Pi_1 - C(x) \text{ and } \underline{u}(y, x) = \Pi_2 - C(y), \text{ with } C' < 0.$$

For simplicity, we also assume that $c_1 = 0$ and omit the dependence of $c(x, t)$ on x. In order to assess the effect of subsidization policies on equilibrium, we shall introduce the possibility that development and technology adoption be subsidized, specifically that the cost functions $c(x, t)$ and $C(x)$ may be reduced to $\beta c(x, t)$ and $\gamma C(x)$, where $(1 - \beta)$ denotes a innovation development subsidy, and $(1-\gamma)$ a subsidy for innovation adoption.

Investigative Reports. Suppose that different teams of journalists from competing news agencies, such as journals or television broadcasting networks, collect information on a big investigative report, such as one to uncover a government or military scandal. Thanks to the continuing efforts of the journalists, more pieces of information are uncovered over time, and they accumulate in determining the value of the report. We represent the quality of news agency i's report at time t by the state $x_i(t)$. The game ends whenever one of the news agencies publishes or broadcast its investigative report.

The journalists face a basic trade off between anticipating or postponing the publication of their investigative report. If they publish the report too early, they will break the news with a weak story, that may lead in principle to defamation charges and to public opinion dismissal. If they wait too long, they run the risk of being preempted by their competitors, and of losing everything. Because breaking the news is all that matters, we assume that the payoff $\underline{u}(x,y)$ for being second in publishing the report is negligible. Also, we suppose that the payoff for breaking the news depends mostly on the quality of the report and is not much affected by a subsequent competitor's report. In sum, the payoffs for 'breaking the news' and for 'coming second' are:

$$u(x,y) = v(x) \text{ and } \underline{u}(x,y) = 0, \text{ with } v(x) > \epsilon \text{ for all } x \text{, and } v' > 0.$$

4 The Equilibrium

Because strategies depend only on innovation state and calendar time, we let $V(x,t)$ denote the equilibrium value given state x at time t, and V be the value at the start of the game. The equation determining the stopping time is:

$$V(x,t) = \max\left\{\int u(x,y)\,\beta(dy,t), W(x,t)\right\}, \tag{1}$$

where W is the equilibrium flow value for remaining in the race.

We introduce three simple regularity assumptions.

Condition 1 *For any* x, y, $u_1(x,y) \geq \underline{u}_1(x,y)$.

Condition 2 *For any* x, y, $u_{11}(x, y) \leq 0$.

Condition 3 *For any* x, y, $u_{12}(x, y) \leq 0$.

These assumptions are very mild. As an illustration, in the patent race application, they only require that $v'' \leq 0$, and in the innovation adoption game that $C'' \geq 0$. We verify in the Appendix that under these regularity conditions, all the equilibrium strategies σ can be represented as time-dependent threshold functions z. According to such a strategy, a player ends the game at time $T = \inf\{t : x(t) \geq z(t)\}$.

Proposition 1 *Under Conditions 1 - 3, for any* t, *in equilibrium, there is a unique threshold* z *such that* $\int u(x, y) \beta(dy, t) < (>) W(x, t)$ *if and only if* $x < (>) z$. *Hence, any equilibrium strategy* σ *is represented by a time-dependent threshold function* $z(t)$.

To derive the dynamic program equation for W, the flow value of not ending the game, we follow a standard discrete-time approximation approach. The discrete-time version of W is:

$$
\begin{aligned}
W(x,t) &= -c(x,t)\Delta + e^{-r\Delta}\rho\Delta \int_0^\infty W(x+w, t+\Delta) G(dw) \\
&+ \rho\Delta \int_0^{z(t)} \int_{z(t)-y}^{\infty} \underline{u}(x, y+w) G(dw) \beta(dy,t) + e^{-r\Delta} \underline{u}(x, z(t)) \min\{0, \beta(z(t), t) - \beta(z(t+\Delta), t)\} \\
&+ e^{-r\Delta} W(x, t+\Delta) \left[1 - \rho\Delta - \rho\Delta \int_0^{z(t)} [1 - G(z(t) - y)] \beta(dy,t) - \min\{0, \beta(z(t), t) - \beta(z(t+\Delta), t)\}\right].
\end{aligned}
$$

The first term is the flow cost of staying in the game, the second term is the potential capital gain as a result of obtaining one further state increment w. The third and fourth terms identify the payoff when the opponent ends the game: this may happen either because an increment w arrives to the opponent and her state $y + w$ crosses the threshold $z(t)$ from below, or because the opponent's state y crosses the threshold z between time t and $t + \Delta$ without receiving a further increment: the probability of such an event is $\min\{0, \beta(z, t) - \beta(z(t+\Delta), t)\}$. The last term identifies the value for staying in the game if no increments arrive and the opponent does not end the game.

We now study the inference problem with respect to the opponent's innovation state y faced by each player in equilibrium as the game develops over time. In general, a player should condition her inference at time t on the information that the opponent state $y(\tau)$ has been smaller than the threshold $z(\tau)$ at any time $\tau \leq t$. This makes the explicit equilibrium calculation hardly feasible. However, we will now show that the updating of beliefs $\beta(y,t)$ is much simpler in equilibria where the threshold functions z is *decreasing* over time. At any time t, player i may safely ignore all information gathered on the opponent at any previous time $\tau < t$. Because the opponent innovation state is increasing over time, once realized that $y(t) \leq z(t)$, the additional information that $y(\tau) \leq z(\tau)$ for any $\tau < t$ is redundant. Clearly, this major simplification would be incorrect if z were increasing on any non-degenerate time interval.

Lemma 1 *In any time-decreasing threshold equilibrium, the equilibrium belief of either player with respect to the opponent state y is the c.d.f. $\beta(y,t) = F(t,y)/F(t,z(t))$ at any time t in the game.*

Proof. Because the equilibrium threshold function $z(\tau)$ is increasing in time τ, the inequality $y(t) \leq z(t)$ implies that $y(\tau) \leq z(\tau)$ for any $\tau < t$, for any increasing random sample path $y(\tau)$. Thus the equilibrium belief $\beta(y,t)$, given that the opponent did not end the game at any time $t < \tau$ are such that:

$$\Pr(y(t) < y | y(\tau) \leq z(\tau) \text{ for any } \tau \leq t) = \Pr(y(t) < y | y(t) \leq z(t)) = F(t,y)/F(t,z(t)).$$

■

Substituting $\beta(y,t)$ with $F(t,y)/F(t,z(t))$ in light of the previous Lemma, we can derive the continuous-time approximation of W:

$$\lim_{\Delta\to 0} -\frac{e^{-r\Delta}W(x,t+\Delta) - e^{r0}W(x,t)}{\Delta} = \lim_{\Delta\to 0}\frac{\partial}{\partial\Delta}\left(-e^{-r\Delta}W(x,t+\Delta)\right) =$$

$$rW(x,t) - W_2(x,t) = -c(x,t) + \rho\int_0^\infty [W(x+w,t) - W(x,t)]\,G(dw)$$
$$+\rho\int_0^{z(t)}\int_{z-y}^\infty [\underline{u}(x,y+w) - W(x,t)]\,G(dw)\frac{F(t,dy)}{F(t,z(t))} - [\underline{u}(x,z(t)) - W(x,t)]\frac{f(t,z(t))}{F(t,z(t))}z'(t),$$

so that the continuous-time flow value for continuing the game simplifies as:

$$rW(x,t) = V_2(x,t) - c(x,t) + \rho\int_0^\infty [V(x+w,t) - V(x,t)]\,G(dw) \tag{2}$$
$$+\rho\int_0^{z(t)}\int_{z-y}^\infty [\underline{u}(x,y+w) - V(x,t)]\,G(dw)\frac{F(t,dy)}{F(t,z(t))} - [\underline{u}(x,z(t)) - V(x,t)]\frac{f(t,z(t))}{F(t,z(t))}z'(t),$$

where the first term is simply the change in value V because of passing time (which indirectly affects the beliefs over the opponent's state through the distribution F).

For any $x \geq z(t)$, Lemma 1 implies that $V(x,t) = \int_0^{z(t)} u(x,y) \frac{F(t,dy)}{F(t,z(t))}$. Substituting this into the flow value equation (2) and applying the condition that $W(z(t),t) = \int_0^{z(t)} u(z(t),y) \frac{F(t,dy)}{F(t,z(t))}$ from equation (1) we derive the ordinary differential equation that governs any equilibrium time-decreasing threshold function z:

$$\begin{aligned} r\int_0^{z(t)} u(z(t),y)\frac{F(t,dy)}{F(t,z(t))} &= -c(x,t) + \rho\int_0^{z(t)}\int_0^{\infty}\left[u(z(t)+w,y) - u(z(t),y)\right]G(dw)\frac{F(t,dy)}{F(t,z(t))} \\ &\quad + \rho\int_0^{z(t)}\int_{z(t)-y}^{\infty}\left[\underline{u}(z(t),y+w) - u(z(t),y)\right]G(dw)\frac{F(t,dy)}{F(t,z(t))} \\ &\quad - \left[\underline{u}(z(t),z(t)) - u(z(t),z(t))\right]\frac{f(t,z(t))}{F(t,z(t))}z'(t). \end{aligned} \quad (3)$$

The above analysis is summarized in the following result.

Theorem 1 *Any time-decreasing equilibrium threshold z solves the Ordinary Differential Equation (3).*

We shall now introduce regularity conditions that guarantee existence of equilibrium with decreasing threshold.

Condition 4 *For any $x \geq y$, $u_2(x,y) \geq \underline{u}_2(x,y)$.*

Condition 5 *For any x, y, $\underline{u}_{22} \leq 0$.*

Condition 6 *For any $x \geq y$, $\int_0^{\infty} (\rho u_2(x+w,y) - (\rho+r) u_2(x,y))\, G(dw) \leq 0$.*

While the first two conditions are of simple interpretation, the last one requires some comments. Condition 6 trivially holds if $u_2 = 0$, so that the first-mover payoff does not depend on the opponent's state. When $u_2 < 0$, it requires that u_{12} is sufficiently negative (relative to u_2), and it more easily satisfied when r is small relative to ρ. The conditions are very mild. As an illustration in our patent race application, they require only that $\underline{v}''(x) \leq 0$, and they are always satisfied in the innovation adoption game.

The final set of conditions, while apparently complex, are in fact fairly innocuous boundary conditions. Condition 7 below only makes sure that the players are willing to enter the game for low values of x, it can be understood as a normalization condition on u and $\underline{u}$. Condition 8 avoids that they remain in the game forever for high values of x; given that $u_{11} \leq 0$ it is a very mild restriction. Lifting these assumptions would only make the presentation of the equilibrium characterization less transparent, with no added substantial value.

Condition 7 $ru(0,0) < -\underline{c} + \rho \int_0^\infty [u(w,0) + \underline{u}(0,w) - 2u(0,0)]\, G(dw)\,.$

Condition 8 *For any* y, $\lim_{x\to\infty} ru(x,y) > -\bar{c} + \lim_{x\to\infty} \rho \int_0^\infty [u(x+w,y) - u(x,y)]\, G(dw)\,.$

In the Appendix, we show time-dependent equilibrium threshold existence, and we characterize equilibrium strategies at early and late stages of the game.

Theorem 2 *Under conditions 1 - 8, there exists an equilibrium with time-decreasing threshold function* z*. As time* t *converges to zero,* $z(t)$ *converges to* $\bar{z}$*, the smallest* z *that solves the equation*

$$ru(z,0) = -\underline{c} + \rho \int_0^\infty [u(z+w,0) - u(z,0)]\, G(dw) + \rho \int_z^\infty [\underline{u}(z, 0+w) - u(z,0)]\, G(dw) \quad (4)$$

For t *large enough,* $z(t)$ *approximates* $\underline{z}$*, the smallest* z *that solves the equation:*

$$ru(z,z) = -\bar{c} + \rho \int_0^\infty [u(z+w,z) - u(z,z)]\, G(dw) + \rho \int_0^\infty [\underline{u}(z, z+w) - u(z,z)]\, G(dw)\,. \quad (5)$$

This characterization results shows that, contrary to simple-minded intuition and unlike for instance the tug-of-war model by Harris and Vickers (1985), these preemption games with private information do not unravel. Despite being ignorant of their opponent's state, players do not immediately end the game. These is true even in the case that they would lose everything if preempted (e.g. $u > \epsilon$ and $\underline{u} = 0$).

We conclude this subsection by showing how the above characterization results at early and late stages extend to any possible time-decreasing threshold equilibria other than the ones for which we proved existence in Theorem 2.

Proposition 2 *Under conditions 1 - 8, any possible time-decreasing equilibrium threshold function z is such that*

$$\lim_{t\to 0} z(t) = \bar{z} \text{ and } \lim_{t\to\infty} z(t) \leq \underline{z}.$$

4.1 Comparative Statics for Patent Races and Innovation Adoption Games

We now study comparative statics of the equilibrium in Theorem 2 in the context of the applications we introduced in the previous section, with respect to a number of variables of interest.

We consider first patent races. In the limits for t small and for t large, the equilibrium threshold z is increasing in the cost of developing the innovation c_0 and decreasing in the flow cost of research $c(x,t)$, which we momentarily assume to be constant in time and state, to make the comparison with c_0 more transparent. These result are intuitive: if the cost of staying in the race increases, each firm patents less significant innovations, whereas if the cost of developing the innovation increases, each firm chooses to accumulate more significant innovations before ending the race.

Quite unexpectedly, making patent rights stronger does not necessarily increase innovation disclosure. Increasing the policy parameters α, representing the appropriability of one's innovation (e.g. the breadth and length of patent) does not necessarily reduce the equilibrium threshold z. This is because an increment in α increases both the value of stopping the race with the current innovation, and the option value for remaining in the race and appropriating of a further innovation increment. As it turns out the comparative statics with respect to α depend on the costs parameters c and c_0. If the development cost c_0 is large relative to the research cost c, then stronger patent rights induce firms to anticipate patenting. On the other hand, if research is the costlier activity, then an increment in breadth and length of a patent induces patent delays.

Proposition 3 *In the limits for t small and for t large, the equilibrium threshold function z uniformly increases in the development cost c_0 and decreases in research cost c. The relation between z and the policy parameters α, the breadth and length of patent, is negative (positive) when c is small (large) enough relative to c_0, unless $\underline{v}$ is too large.*

As far as the innovation game is concerned, in the limits for t small and for t large, the equilibrium threshold z decreases both with the value of innovation Π_2 and with the first-mover advantage $\Pi_1 - \Pi_2$. The result is intuitive: more valuable innovations are adopted earlier. Also intuitive is the result that subsidizing the innovation adoption makes it happen faster. On the other hand, subsidizing the innovation development bears the perverse effect of delaying the innovation adoption.

Proposition 4 *In the limits for t small and for t large, the equilibrium threshold function z uniformly decreases in the value of innovation Π_2 and with the first-mover advantage $\Pi_1 - \Pi_2$. It also decreases when innovation adoption is subsidized, but it increases with subsidies in innovation development.*

5 Social Planner's Problem

We consider the problem of a social planner that does not enjoy any informational advantage over the players. The social planner instructs each player $i = A, B$ with state x at time t to adopt a second-best efficient policy in the form of (symmetric) strategies σ^*. To make the comparison with equilibrium transparent, we study strategies σ^* that depend on x and t only, and not on the entire history of increments h_i^t, for $i = A, B$. Such a second-best efficient policy σ^* maximizes joint expected payoffs, which we denote as $V^*(x, t)$. We denote by $\gamma\,(dy; t)$ the beliefs with respect to the opponent's state y: Because the optimal threshold function need not be decreasing in time, this expression cannot be further simplified.

The equation governing the social planner's stopping problem is:

$$V^*\,(x,t) = \max\left\{ \int \left[u\,(x,y) + \bar{u}\,(y,x)\right]\gamma\,(dy,t)\,, W^*\,(x,t) \right\}.$$

We introduce conditions under which the optimal policy imposed by the social planner is identified by a threshold $z^*\,(t)$ for any time t. Hence, the social planner instructs each player i to patent at the time $T_i = \inf\{t : x_i\,(t) > z^*\,(t)\}$.

Condition 9 *For any x, y, $\underline{u}_{11} \leq 0$.*

Condition 10 *For any x, y, $\underline{u}_{12} \leq 0$.*

Proposition 5 *Under conditions 2, 3, 9, 10, for any* t, *there is a unique threshold* z^* *in the social planner's problem, such that* $\int [u(x,y) + \bar{u}(y,x)] \gamma(dy,t) < (>) W^*(x,t)$ *if and only if* $x < (>) z^*$. *Hence, the optimal policy* σ^* *is described by a time-dependent threshold function* z^*.

The flow equation of the social planner's problem is as follows:

$$rW^*(x,t) = -c(x,t) - \int_0^{z(t)} c(y,t)\gamma(dy,t) + \rho\left(\int_0^\infty V^*(x+w,t)\,G(dw) - V^*(x,t)\right)$$
$$+\rho\int_0^{z(t)}\int_{z(t)-y}^\infty (V^*(w+y,t) - V^*(x,t))G(dw)\,\gamma(dy,t)$$
$$+\frac{d\gamma(z(t),t)}{dz}\min\{0,z'(t)\}\left[u(z,x) + \underline{u}(x,z) - u(x,z) - \underline{u}(z,x)\right] + V_2^*(x,t).$$

We highlight changes with respect to the equilibrium flow equation (2). Clearly, the flow cost of waiting is $c(x,t) + \int_0^{z(t)} c(y,t)\gamma(dy,t)$ rather than $c(x,t)$, because the player is requested to internalize the cost of staying in the game paid by the other player. The third term, representing the arrival of a good draw to the opponent which results in her stopping the game, now induces contribution $V^*(w+y,t)$ to the total value. Similarly, the fourth term, representing the chance that the opponent crosses the threshold without receiving any increment, now induces a contribution $u(z,x) + \bar{u}(x,z)$ to the flow value of the social planner, but this contribution is balanced against the loss $u(x,z) + \underline{u}(z,x)$. Of course, V needs to be replaced by V^* throughout.

Because for any t, $V^*(x,t) = \int [u(x,y) + \bar{u}(y,x)]\gamma(dy,t)$ for any $x \geq z^*(t)$ and $W^*(x,t) = \int [u(x,y) + \bar{u}(y,x)]\gamma(dy,t)$ for $x = z^*(t)$, with appropriate manipulations, we obtain that the optimal threshold $z^*(t)$ is determined by the condition that for any t,

$$r\int_0^{z^*(t)} [u(z^*(t),y) + \bar{u}(y,z^*(t))]\gamma(dy,t;z^*) = -c(x,t) - \int_0^{z(t)} c(y,t)\gamma(dy,t) \tag{6}$$
$$+\rho\int_0^{z^*(t)}\int_0^\infty (u(z^*(t)+\varepsilon,y) + \underline{u}(y,z^*(t)+\varepsilon) - u(z^*(t),y) - \underline{u}(y,z^*(t)))\,G(d\varepsilon)\,\gamma(dy,t;z^*)$$
$$+\rho\int_0^{z^*(t)}\int_{z^*(t)-y}^\infty (\underline{u}(z^*(t),y+\varepsilon) + u(y+\varepsilon,z^*(t)) - u(z^*(t),y) - \underline{u}(y,z^*(t)))\,G(d\varepsilon)\,\gamma(dy,t;z^*).$$

As for the equilibrium threshold z, appropriate boundary conditions make sure that the planner will instruct the players to enter the game and will not force them to remain in the game forever. Specifically, the following conditions make sure that $z^*(t) \in (0,\infty)$ for any t.

Condition 11 $r[u(0,0) + \bar{u}(0,0)] < -2\bar{c} + 2\rho\int_0^\infty (u(\varepsilon,0) + \underline{u}(0,\varepsilon) - u(0,0) - \underline{u}(0,0))\,G(d\varepsilon)$.

Condition 12 *For any y,*

$$\lim_{x\to\infty} r\left[u\left(x,y\right)+\bar{u}\left(y,x\right)\right] > -2\underline{c}+\lim_{x\to\infty}\rho\int_0^\infty\left(u\left(x+\varepsilon,y\right)+\underline{u}\left(y,x+\varepsilon\right)-u\left(x,y\right)-\underline{u}\left(y,x\right)\right)G\left(d\varepsilon\right).$$

Explicit computation of the optimal threshold function z^* is severely impeded because the beliefs γ in equation (6) are a function of z^* itself. Nevertheless it is possible to assess welfare properties of equilibrium. Suppose that one's opponent adopts the equilibrium threshold $z(t)$. Then, the socially-optimal best-response is the threshold function (which we denote as z^* with a minor notational violation) that solves the following parametric equation:

$$\begin{aligned} & r\int_0^z\left[u\left(z^*,y\right)+\underline{u}\left(y,z\right)\right]\frac{F\left(t,dy\right)}{F\left(t,z\right)} = \qquad (7)\\ & -c\left(x,t\right)+\rho\int_0^z\int_0^\infty\left(u\left(z^*+\varepsilon,y\right)+\underline{u}\left(y,z+\varepsilon\right)-u\left(z^*,y\right)-\underline{u}\left(y,z\right)\right)G\left(d\varepsilon\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}\\ & -\int_0^{z(t)}c\left(y,t\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}+\rho\int_0^z\int_{z-y}^\infty\left(\underline{u}\left(z^*,y+\varepsilon\right)+u\left(y+\varepsilon,z\right)-u\left(z^*,y\right)-\underline{u}\left(y,z\right)\right)G\left(d\varepsilon\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}. \end{aligned}$$

Welfare properties of the equilibrium can then be assessed by comparing z with z^*. As already anticipated in the Introduction, the welfare analysis is especially meaningful in the context of patent races, as it allows to identify the 'duplication cost' and 'preemption effects' in the race. These negative and positive equilibrium externalities can be assessed by subtracting the equilibrium threshold differential equation (3) from the parametric equation (7), specialized to the patent race application, so as to obtain:

$$\begin{aligned} & (r+\rho)\left[v\left(z^*\right)-v\left(z\right)\right] = -\int_0^{z(t)}c\left(y,t\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}-(r+\rho)\,\underline{v}\left(z^*\right) \qquad (8)\\ & +\rho\int_0^\infty\left[v\left(z^*+\varepsilon\right)-v\left(z^*\right)+\underline{v}\left(z^*+\varepsilon\right)-\underline{v}\left(z^*\right)-v\left(z+\varepsilon\right)+v\left(z\right)\right]G\left(d\varepsilon\right)\\ & +\rho\int_0^z\int_{z-y}^\infty\left[v\left(y+\varepsilon\right)-c_0\right]G\left(d\varepsilon\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}-\left[\left(v\left(z\right)-c_0\right)-\underline{v}\left(z\right)\right]\frac{f\left(t,z\right)}{F\left(t,z\right)}z'\left(t\right) \end{aligned}$$

1 **(Duplication Costs).** On the one hand, the competitive firm does not consider the duplication of costs, hence the first term capturing the expected cost borne by the opponent for duplicating the leading firm's innovation.

2 **(Preemption).** On the other hand, the competitive firm is concerned about preemption. For this reason it has an incentive to anticipate patenting. This shows up in the last two terms which are both positive because $v-c_0>\underline{v}$ and $z'<0$.

The remaining terms are ambiguous, the third term corresponds to the difference in the value of an increment at z or at z^*, where as the second one incorporates the (possibly negative) value for 'losing the race'.

Now we proceed to show that the relation between duplication cost effect and preemption effect crucially depends on calendar time. We will show that, under some regularity conditions, in early stages of the race, the first effect dominates the second and hence that the social planner instructs firms to anticipate patenting with respect to any possible equilibrium solution z. In later stages, the second effect dominates the first and hence the social planner instructs firms to postpone patenting with respect to any possible equilibrium solution.

Proposition 6 *As long as the development cost* c_0 *is not too small and* $\underline{v}(z^*(0))$ *not too negative, the equilibrium threshold* $z(t)$ *is larger than the optimal best-response threshold* $z^*(t)$ *for sufficiently early calendar time* t*.*

Intuitively, if a firm is sufficiently lucky to obtain valuable incremental innovations in a very short time, it is not much concerned for the possibility that the opponent will quickly catch up and win the patent race. As a result, the firm will be willing to drag on experimenting in the attempt to achieve a more profitable innovation. The regulatory agency dislikes this delay in patenting because, unlike the firm, it internalizes the duplication cost borne by the opponent in the most likely futile attempt to catch up. Therefore, it optimally instructs the firm to immediately patent the outcome of its lucky innovation streak.

Proposition 7 *As long as* $\lim_{t\to\infty} \underline{v}(z^*(t))$ *and the discount rate* r *are not too large, the equilibrium threshold* $z(t)$ *is smaller than the optimal best-response threshold* $z^*(t)$ *for sufficiently late calendar time* t*.*

Intuitively, as time goes by, each firm becomes more and more afraid that her opponent will soon end the race and rip the however meager innovations that it has so far achieved. But this fear of prevention feeds on itself in equilibrium and makes the firm willing to preempt and patent relatively unprofitable innovations. The regulatory agency would like the firm instead to persist in experimenting in the expectation of making more progress and eventually patent more valuable innovations.

6 Conclusion

We have presented a general analysis of preemption games with private information changing over time. The analysis of timing decisions is of prime relevance in several economic problems such as patent races, innovation adoption and market entry problems. A previously neglected feature of these preemption games is that the agents' states (e.g. R&D firms' technological improvements) change over time and are kept secret throughout the game. This paper has studied the general class of preemption games with private information randomly changing over time. Under mild conditions, we have proved that all equilibria are described by a time-dependent threshold. A player ends the game if and only if her state is above a certain threshold. We have proved existence of equilibrium described by a time-decreasing threshold, and derived in closed form the expression of the ordinary differential equation governing the equilibrium threshold. This characterization provides a strategic rationale for the common wisdom that 'big things happen fast.'

Our analysis uncovers novel positive and normative insights. In the context of patent races we surprisingly find that making patent rights stronger may not lead to higher innovation rates when development is costlier than research. Most importantly, our analysis makes the classic welfare trade-off between duplication costs and preemption effects precise: Compared with socially optimal rules, equilibrium strategies display excessive duplication costs in early stages of the race, and excessive preemption in later stages. In the context of innovation adoption, we find that subsidizing innovation development may delay its adoption.

A Appendix: Properties of the Distribution $F(t,y)/F(t,z)$.

Before proceeding with the analysis, we introduce the following preliminaries.

Definition 3 *For any two c.d.f.s H_1, H_2, we say that $H_1 \preceq H_2$ if and only if, for all $x \leq z$,*

$$\frac{H_1(x)}{H_1(z)} \geq \frac{H_2(x)}{H_2(z)}.$$

Remark 4 *This definition says that for all z the distributions H_2 conditional on the set $\{x \leq z\}$ is greater than the corresponding distribution for H_1 in first order stochastic dominance.*

Lemma 2 *The c.d.f.s H_1 and H_2 are ordered as $H_1 \preceq H_2$ if and only if for any two decreasing functions f_1, f_2*

$$\frac{\int f_1(x) f_2(x) H_1(dx)}{\int f_2(x) H_1(dx)} \geq \frac{\int f_1(x) f_2(x) H_2(dx)}{\int f_2(x) H_2(dx)}.$$

Proof. Sufficiency is immediate by taking f_2 to be the indicator function of the set $\{x \leq z\}$. To prove necessity, let $f_2 = \sum \alpha_i \chi\{x \leq z_i\}$ for positive scalars α_i, where $\chi\{x \leq z_i\}$ is the indicator function of $\{x \leq z_i\}$. Without loss of generality let $z_i \leq z_{i+1}$.

$$\frac{\int f_1(x) f_2(x) H_1(dx)}{\int f_2(x) H_1(dx)} = \frac{\sum \alpha_i \int^{z_i} f_1(x) H_1(dx)}{\sum \alpha_i H_1(z_i)}$$

$$= \sum \left(\frac{\int^{z_i} f_1(x) H_1(dx)}{H_1(z_i)}\right)\left(\frac{\alpha_i H_1(z_i)}{\sum \alpha_i H_1(z_i)}\right) \geq \sum \left(\frac{\int^{z_i} f_1(x) H_2(dx)}{H_2(z_i)}\right)\left(\frac{\alpha_i H_1(z_i)}{\sum \alpha_i H_1(z_i)}\right)$$

So need to show:

$$\sum \left(\frac{\int^{z_i} f_1(x) H_2(dx)}{H_2(z_i)}\right)\left[\left(\frac{\alpha_i H_1(z_i)}{\sum \alpha_i H_1(z_i)}\right) - \left(\frac{\alpha_i H_2(z_i)}{\sum \alpha_i H_2(z_i)}\right)\right] \geq 0.$$

Observe that the last inequality is a weighted average and since f_1 is a decreasing function, the terms in the first bracket are decreasing in z_i.

Hence, it suffices to show that for all k,

$$\frac{\sum_{i=1}^{k} \alpha_i H_1(z_i)}{\sum \alpha_i H_1(z_i)} \geq \frac{\sum_{i=1}^{k} \alpha_i H_2(z_i)}{\sum \alpha_i H_2(z_i)} \tag{9}$$

Now notice that for any i, since $H_1 \preceq H_2$ and $z_i \leq z_{i+1}$

$$\frac{\alpha_i H_1(z_i)}{\alpha_{i+1} H_1(z_{i+1})} \geq \frac{\alpha_i H_2(z_i)}{\alpha_{i+1} H_2(z_{i+1})}$$

which can easily be shown to imply (9). ■

We can now state our results on the properties of $F(t,y)/F(t,z)$ in their full generality. We start by showing that such a distribution is stochastically increasing in time.

Theorem 5 *If the c.d.f. G of the random increment w is log-concave, i.e. the inverse hazard rate $g(w)/G(w)$ is weakly decreasing in w, then the c.d.f. $F(t,y)/F(t,z)$ is stochastically increasing in t for any z and $y \leq z$.*

Proof. We express the innovation state $X(t)$ c.d.f. at time t, by making the relation with the number of increment arrivals $K(t)$ at time t explicit:

$$F(z,t) = \Pr(X(t) \le z) = \sum_{k=0}^{\infty} F_k(X(t) \le z | K(t) = k) \Pr(K(t) = k) = \sum_{k=0}^{\infty} F_k(z)\, \mu_k(t),$$

where $F_k(z)$ is the innovation state $X(t)$ c.d.f. at time t conditional on k arrivals, and $\mu_k(t)$ is the Poisson distribution with arrival rate ρ, of the number of increment arrivals k at time t, setting $\mu_{-1} = 0$. We shall make use of following properties of the Poisson distribution and arrival rate ρ,

$$\begin{aligned} \mu_{k-1}(t) &= e^{-\rho t}(\rho t)^{k-1}/(k-1)! = \left(\frac{k}{\rho t}\right) e^{-\rho t}(\rho t)^k/k! = \left(\frac{k}{\rho t}\right) \mu_k(t) \\ \dot{\mu}_k(t) &= \rho e^{-\rho t}\left((\rho t)^{k-1}/(k-1)! - (\rho t)^k/k!\right) = \rho\left(\mu_{k-1}(t) - \mu_k(t)\right) \end{aligned} \tag{10}$$

Noting that $F_k(z)$ is independent of calendar time t, we can write:

$$\begin{aligned} & \frac{d}{dt}\left(\frac{F(t,y)}{F(t,z)}\right) = \frac{d}{dt}\left(\frac{\sum_{k=0}^{\infty} F_k(y)\,\mu_k(t)}{\sum_{k=0}^{\infty} F_k(z)\,\mu_k(t)}\right) \\ \propto & \left(\sum_{k=0}^{\infty} F_k(y)\,\dot{\mu}_k(t)\right)\left(\sum_{k=0}^{\infty} F_k(z)\,\mu_k(t)\right) - \left(\sum_{k=0}^{\infty} F_k(z)\,\dot{\mu}_k(t)\right)\left(\sum_{k=0}^{\infty} F_k(y)\,\mu_k(t)\right) \\ = & \left(\sum_{k=0}^{\infty} F_k(y)\,\mu_{k-1}(t)\right)\left(\sum_{k=0}^{\infty} F_k(z)\,\mu_k(t)\right) - \left(\sum_{k=0}^{\infty} F_k(z)\,\mu_{k-1}(t)\right)\left(\sum_{k=0}^{\infty} F_k(y)\,\mu_k(t)\right) \\ \propto & \frac{\sum_{k=0}^{\infty}\left(\frac{F_k(y)}{F_k(z)}\right) F_k(z)\,\mu_{k-1}(t)}{\sum_{k=0}^{\infty} F_k(z)\,\mu_{k-1}(t)} - \frac{\sum_{k=0}^{\infty}\left(\frac{F_k(y)}{F_k(z)}\right) F_k(z)\,\mu_k(t)}{\sum_{k=0}^{\infty} F_k(z)\,\mu_k(t)}. \end{aligned}$$

We now apply Lemma 2 to show that the above quantity is negative. We let

$$f_1(k) = F_k(z), \;\; f_2(k) = \frac{F_k(y)}{F_k(z)}, \;\; H_1(k) = \sum_{j \le k} \mu_{j-1}(t), \;\text{ and } H_2(k) = \sum_{j \le k} \mu_j(t).$$

Because the increments w are positive, $f_1(k)$ is decreasing in k (i.e. $F_k(z)$ is stochastically increasing in k). Hence to apply Lemma 2 we need to verify that $H_1 \preceq H_2$ and that $f_2(k)$ is decreasing in k. This is proved in the next two Lemmata.

Lemma 3 *If the c.d.f. G is log-concave, then $F_k(y)/F_k(z)$ is decreasing in k for all z, i.e. $F_k \preceq F_{k+1}$, for all k.*

Proof. Since $F_0(0) = 1$, it follows immediately that $F_0 \preceq F_1$. Hence, it is sufficient to prove that

$$F_k \preceq F_{k+1} \Rightarrow F_{k+1} \preceq F_{k+2},$$

and then proceed by induction over k. This follows by Lemma 2 because for any $y \le z$,

$$\frac{F_{k+2}(y)}{F_{k+2}(z)} = \frac{\int_0^y G(y-x)\, F_{k+1}(dx)}{\int_0^z G(z-x)\, F_{k+1}(dx)} = \frac{\int_0^y \frac{G(y-x)}{G(z-x)} G(z-x)\, F_{k+1}(dx)}{\int_0^z G(z-x)\, F_{k+1}(dx)},$$

and because $G(z-x)$ is a decreasing function of x, as long as $G(y-x)/G(z-x)$ is decreasing in x, i.e.

$$\frac{g(z-x)}{G(z-x)} \leq \frac{g(y-x)}{G(y-x)} \text{ for } z \geq y,$$

which is equivalent to log-concavity of G. ■

Lemma 4 *For any pair $k \leq m$,*

$$\frac{\sum_{j\leq k}\mu_{j-1}(t)}{\sum_{j\leq m}\mu_{j-1}(t)} \leq \frac{\sum_{j\leq k}\mu_{j}(t)}{\sum_{j\leq m}\mu_{j}(t)}.$$

Proof. Because for any $l \in \{0,1\}$,

$$\frac{\sum_{j\leq k}\mu_{j-l}(t)}{\sum_{j\leq m}\mu_{j-l}(t)} = \left(\frac{\sum_{j\leq k}\mu_{j-l}(t)}{\sum_{j\leq k+1}\mu_{j-l}(t)}\right)\left(\frac{\sum_{j\leq k+1}\mu_{j-l}(t)}{\sum_{j\leq k+2}\mu_{j-l}(t)}\right)\cdots\left(\frac{\sum_{j\leq m-1}\mu_{j-l}(t)}{\sum_{j\leq m}\mu_{j-l}(t)}\right),$$

it is sufficient to show that:

$$\frac{\sum_{j\leq k}\mu_{j-1}(t)}{\sum_{j\leq k+1}\mu_{j-1}(t)} = \frac{\sum_{j\leq k}j\mu_{j}(t)}{\sum_{j\leq k+1}j\mu_{j}(t)} \leq \frac{\sum_{j\leq k}\mu_{j}(t)}{\sum_{j\leq k+1}\mu_{j}(t)}$$

(by using one of the properties (10) of the Poisson distribution μ_j), or equivalently that

$$\frac{1}{1+\frac{(k+1)\mu_{k+1}(t)}{\sum_{j\leq k}j\mu_j(t)}} \leq \frac{1}{1+\frac{\mu_{k+1}(t)}{\sum_{j\leq k}\mu_j(t)}},$$

which holds because $(k+1)/j > 1$ for any $j \leq k$. ■

This concludes the proof of the Theorem. ■

We now show that the limit distribution of $F(t,\cdot)/F(t,z)$ for $t \to \infty$ converges to a point mass on the upper bound of its support z.

Theorem 6 *If the increment c.d.f. $G(w)$ is differentiable, of connected support that includes $w=0$, then for any z, the c.d.f. $F(t,\cdot)/F(t,z)$ concentrates all mass on z as t goes to infinity, i.e.*

$$\lim_{t\to\infty}\frac{F(t,y)}{F(t,z)} = 0 \textit{ for any } y < z.$$

Proof. For any z, we have shown in Theorem 5 that $\frac{d}{dt}\frac{F(t,y)}{F(t,z)} < 0$ for any y and t. Because $F(t,y)/F(t,z) \geq 0$ for any t, the c.d.f. $F(t,\cdot)/F(t,z)$ must converge pointwise for $t\to\infty$ to a c.d.f. $\tilde{F}(\cdot;z)$, and the time derivative of $F(t,y)/F(t,z)$ must converge to zero as $t\to\infty$. That is, $\tilde{F}(\cdot;z)$ must be a fixed point of the law of motion induced by the time derivative of $F(t,y)/F(t,z)$.

Suppose by contradiction that $\tilde{F}(y;z) \equiv \epsilon > 0$ for some $y > 0$, and hence that $F(t,y)/F(t,z) \geq \epsilon > 0$ for all t. Let

$$y_0 = \inf\left\{y < z | \frac{F(t,y)}{F(t,z)} \geq \epsilon\right\}, \text{ so that } \int_0^z \frac{F(t,z-x)}{F(t,z)} G(dx) \equiv \varepsilon > 0$$

provided that G has connected support. Exploiting the continuity of G, pick $\delta : G(\delta) < \varepsilon$ and consider any $y : y_0 < y < y_0 + \delta$. Then it must be the case that:

$$\int_0^y \frac{F(t,y-x)}{F(t,y)} g(x)\,dx < G(\delta),$$

because $F(t,y-x) < F(t,y)$ for any $x > 0$, and hence that

$$\frac{F(t,y)}{F(t,z)}\left[\int_0^y \frac{F(t,y-x)}{F(t,y)} g(x)\,dx - \int_0^y \frac{F(t,z-x)}{F(t,z)} g(x)\,dx\right] < \frac{F(t,y)}{F(t,z)}\left[G(\delta) - \varepsilon\right] < 0.$$

Because the time t is arbitrary, this is in contradiction with:

$$\begin{aligned}
\lim_{t\to\infty} \frac{d}{dt}\frac{F(t,y)}{F(t,z)} &= \lim_{t\to\infty}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{F(t+\Delta,y)}{F(t+\Delta,z)} - \frac{F(t,y)}{F(t,z)}\right] \\
&= \lim_{t\to\infty}\lim_{\Delta\to 0}\frac{1}{\Delta}\left[\frac{(1-\rho\Delta)F(t,y) + \rho\Delta\int_0^y F(t,y-x)g(x)\,dx}{(1-\rho\Delta)F(t,z) + \rho\Delta\int_0^y F(t,z-x)g(x)\,dx} - \frac{F(t,y)}{F(t,z)}\right] \\
&= \lim_{t\to\infty}\lim_{\Delta\to 0}\frac{1}{\Delta}\frac{F(t,y)}{F(t,z)}\left[\frac{(1-\rho\Delta) + \rho\Delta\int_0^y \frac{F(t,y-x)}{F(t,y)}g(x)\,dx}{(1-\rho\Delta) + \rho\Delta\int_0^y \frac{F(t,z-x)}{F(t,z)}g(x)\,dx} - 1\right] \\
&\propto \lim_{t\to\infty}\frac{F(t,y)}{F(t,z)}\left[\int_0^y \frac{F(t,y-x)}{F(t,y)}g(x)\,dx - \int_0^y \frac{F(t,z-x)}{F(t,z)}g(x)\,dx\right] = 0.
\end{aligned}$$

■

B Appendix. Omitted Proofs.

Proof of Proposition 1. Since W and V are differentiable, we only need to show that for any t and any x such that $\int u(x,y)\beta(dy,t) < W(x,t) = V(x,t)$

$$W_1(x,t) < \int u_1(x,y)\beta(dy,t)$$

because this implies that the functions $W(x,t)$ and $\int u(x,y)\beta(dy,t)$, cross only once. Since $\frac{d}{dx}\left(\int u(x,y)\beta(dy,t)\right) > 0$, it follows that $\int u(x,y)\beta(dy,t)$ dominates $W(x,t)$ for any x larger than the crossing point.

Consider any state (x,t), such that $V(x,t) = W(x,t)$ and hence $\sigma(x,t) > t$ for any optimal strategy σ of either of the two players (say player A); player A optimally chooses to remain in the race at (x,t). For any $x' < x$ and state (x',t), let $V(x',t;\sigma')$ be the value associated to applying the same strategy σ re-scaled by a factor $x - x'$, i.e. the strategy σ' such that

$$\sigma'\left(x''+x',\tau\right)=\sigma\left(x''+x,\tau\right) \text{ for any } \tau\geq t \text{ and } x''\geq 0.$$

Fixing player B's strategy σ_B (which does not depend on x), the optimal payoff at state (x,t) is

$$V(x,t)=\int \mathbb{E}\left[V\left(x+\mathbf{x},y+\mathbf{y},t;\sigma,\sigma_B\right)\right]\beta\left(dy,t\right),$$

and the expected payoff for playing strategy σ' starting at state (x',t) is

$$V\left(x',t;\sigma'\right)=\int \mathbb{E}\left[V\left(x'+\mathbf{x},y+\mathbf{y},t;\sigma',\sigma_B\right)\right]\beta\left(dy,t\right)$$

where the expectation $\mathbb{E}$ is taken with respect to the sample paths $\mathbf{x}=\{x(\tau):\tau\geq t\}$ and $\mathbf{y}=\{y(\tau):\tau\geq t\}$, generated by the i.i.d. innovation processes in our model, and such that $x(t)=0$ and $y(t)=0$.

Because the innovation process is the sum of increments with Poisson arrival, for any pair of sample paths $\mathbf{x}$ and $\mathbf{y}$, the realized time-t payoff of player A is:

$$\begin{aligned}
&V\left(x+\mathbf{x},y+\mathbf{y},t;\sigma,\sigma_B\right)=u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)\chi\left(T_A<T_B\right)e^{-r(T_A-t)}\\
&+\underline{u}\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)\chi\left(T_A>T_B\right)e^{-r(T_B-t)}-\int_0^{\min\{T_A,T_B\}}c\left(x+\mathbf{x}\left(t\right),t\right)e^{-rv}dv,
\end{aligned}$$

where $T_B=\inf\{\tau:\sigma_B\left(y\left(\tau\right)+y,\tau\right)=\tau\}$ and $T_A=\inf\{\tau:\sigma\left(x\left(\tau\right)+x,\tau\right)=\tau\}$, and $\chi\left(T_i<T_j\right)$ denotes the indicator function over the set of paths $(\mathbf{x},\mathbf{y})$ such that $T_i<T_j$.

For the same paths $\mathbf{x}$ and $\mathbf{y}$, if player A adopts strategy σ' starting at the state (x',t), the realized time-t payoff is:

$$\begin{aligned}
&V\left(x'+\mathbf{x},y+\mathbf{y},t;\sigma,\sigma_B\right)=u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)\chi\left(T_A<T_B\right)e^{-r(T_A-t)}\\
&+\underline{u}\left(x'+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)\chi\left(T_A>T_B\right)e^{-r(T_B-t)}-\int_0^{\min\{T_A,T_B\}}c\left(x'+\mathbf{x}\left(t\right),t\right)e^{-rv}dv,
\end{aligned}$$

where $T'_A=\inf\{\tau:\sigma'\left(x\left(\tau\right)+x',\tau\right)=\tau\}$ is the stopping time induced by strategy σ' on the path $\mathbf{x}'$. Noting that $T'_A=T_A$, we obtain:

$$\begin{aligned}
&V\left(x+\mathbf{x},y+\mathbf{y},t;\sigma,\sigma_B\right)-V\left(x'+\mathbf{x},y+\mathbf{y},t;\sigma',\sigma_B\right)\\
=\;&\left[u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)-u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)\right]e^{-r(T_A-t)}\chi\left(T_A<T_B\right)\\
&+\left[\underline{u}\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)-\underline{u}\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)\right]e^{-r(T_B-t)}\chi\left(T_B<T_A\right)\\
&-\int_0^{\min\{T_A,T_B\}}\left[c\left(x+\mathbf{x}\left(t\right),t\right)-c\left(x'+\mathbf{x}\left(t\right),t\right)\right]e^{-rv}dv.
\end{aligned}$$

Compounding across sample paths, $(\mathbf{x},\mathbf{y})$, we obtain:

$$\begin{aligned}
& V(x,t) - V\left(x',t;\sigma'\right) \\
= & \int \{\mathbb{E}[\left[u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right) - u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)\right] e^{-r(T_A-t)}\chi\left(T_A<T_B\right) \\
& + \left[\underline{u}\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right) - \underline{u}\left(x'+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)\right] e^{-r(T_B-t)}\chi\left(T_B<T_A\right) \\
& - \int_0^{\min\{T_A,T_B\}} \left[c\left(x+\mathbf{x}(t),t\right) - c\left(x'+\mathbf{x}(t),t\right)\right]\Big]\Big\} \beta(dy,t).
\end{aligned}$$

Notice that

$$W(x,t) - W\left(x',t\right) = V(x,t) - V\left(x',t\right) \leq V(x,t) - V\left(x',t;\sigma'\right),$$

hence

$$\begin{aligned}
& \frac{W(x,t) - W(x',t)}{x-x'} \\
\leq & \int \left\{\mathbb{E}\left[\left[\frac{u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right) - u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)}{x-x'}\right] e^{-r(T_A-t)}\chi\left(T_A<T_B\right)\right.\right. \\
& + \left[\frac{\underline{u}\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right) - \underline{u}\left(x'+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)}{x-x'}\right] e^{-r(T_B-t)}\chi\left(T_B<T_A\right) \\
& \left.\left. - \int_0^{\min\{T_A,T_B\}} \left[\frac{c\left(x+\mathbf{x}(t),t\right) - c\left(x'+\mathbf{x}(t),t\right)}{x-x'}\right] e^{-rv}dv\right]\right\} \beta(dy,t).
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{x-x'}\left[\frac{u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right) - u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)}{x-x'}\right] &= u_1\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right), \\
\lim_{x-x'}\left[\frac{\underline{u}\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right) - \underline{u}\left(x'+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)}{x-x'}\right] &= \underline{u}_1\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right), \\
\lim_{x-x'}\left[\frac{c\left(x+\mathbf{x}(t),t\right) - c\left(x'+\mathbf{x}(t),t\right)}{x-x'}\right] &= c_1\left(x+\mathbf{x}(t),t\right),
\end{aligned}$$

the conditions 1, 2 and 3, together with $c_1 \geq 0$, yield:

$$\begin{aligned}
& \lim_{x-x'\to 0}\frac{W(x,t) - W(x',t)}{x-x'} \leq \lim_{x-x'}\int\left\{\mathbb{E}\left[\left[\frac{u(x,y) - u(x',y)}{x-x'}\right] e^{-r(T_A-t)}\chi\left(T_A<T_B\right)\right.\right. \\
& \left.\left. + \left[\frac{u(x,y) - u(x',y)}{x-x'}\right] e^{-r(T_B-t)}\chi\left(T_B<T_A\right)\right]\right\} \beta(dy,t) \\
= & \lim_{x-x'}\int\left[\frac{u(x,y) - u(x',y)}{x-x'}\right]\{\mathbb{E}[e^{-r(T_A-t)}\chi\left(T_A<T_B\right) + e^{-r(T_B-t)}\chi\left(T_B<T_A\right)]\}\beta(dy,t)
\end{aligned}$$

Because $\mathbb{E}\left[T_A|T_A<T_B\right] > 0$, $\mathbb{E}\left[T_B|T_B<T_A\right] > 0$, and T_A and T_B are independent of x', the quantity $\mathbb{E}[e^{-r(T_A-t)}\chi\left(T_A<T_B\right) + e^{-r(T_B-t)}\chi\left(T_B<T_A\right)]$ is strictly smaller than 1 and constant in x'. Thus:

$$\begin{aligned}
& \lim_{x-x'\to 0}\frac{W(x,t) - W(x',t)}{x-x'} < \lim_{x-x'}\int\left[\frac{u(x,y) - u(x',y)}{x-x'}\right]\beta(dy,t), \text{ and hence} \\
& W_1(x,t) < \frac{d}{dx}\left(\int u(x,y)\,\beta(dy,t)\right).
\end{aligned}$$

■

Proofs of Theorems 2 and Proposition 2. The construction hinges on the comparison of the ODE (3) with the parametric equation

$$\begin{aligned}\phi(z,t) &= r\int_0^z u(z,y)\frac{F(t,dy)}{F(t,z)}+c(x,t)-\rho\int_0^z\int_0^\infty [u(z+w,y)-u(z,y)]\,G(dw)\,\frac{F(t,dy)}{F(t,z)}\\ &\quad -\rho\int_0^z\int_{z-y}^\infty [\underline{u}(z,y+w)-u(z,y)]\,G(dw)\,\frac{F(t,dy)}{F(t,z)}. \qquad (11)\end{aligned}$$

Lemma 5 (Auxiliary Solution) *For any t, the equation $\phi(z,t)=0$ admits a solution z. Under the Conditions 4 - 6, the selection*

$$\hat{z}(t)=\min\{z:\phi(z,t)=0\}$$

is strictly decreasing in t.

Proof. For any t, conditions (7) and (8) make sure that the function $\phi(z,t)$ is negative for z small enough and positive for z large enough. By continuity, ϕ admits a zero z for any t.

To show that $d\phi/dt>0$, because $F_t(dy)$ is increasing in t in FSD sense by Theorem 5 and c is non-decreasing in t, we only need to establish that for all z and $y\le z$,

$$\begin{aligned}0 &< \frac{d}{dy}\left[ru(z,y)-\rho\int_0^\infty (u(z+\varepsilon,y)-u(z,y))\,G(d\varepsilon)-\rho\int_{z-y}^\infty (\bar{u}(z,y+\varepsilon)-u(z,y))\,G(d\varepsilon)\right]\\ &= -\int_0^\infty \rho\,(u_2(z+\varepsilon,y)-\rho u_2(z,y)-ru_2(z,y))\,G(d\varepsilon)-\rho\int_{z-y}^\infty (\bar{u}_2(z,y+\varepsilon)-u_2(z,y))\,G(d\varepsilon)\\ &\quad -\rho\,(\bar{u}(z,z)-u(z,y))\,g(z-y),\end{aligned}$$

this follows from Conditions 6 (applied to the first term), 4 and 5 (applied to the second term), and because $u>\bar{u}$ and $u_2<0$. Therefore the selection $\hat{z}(t)$ is strictly decreasing in t while possibly discontinuous. ■

We now establish that, as long as t is bounded away from zero, the ODE (3) has a well-behaved solution field. Clearly, at $t=0$, the derivative $z'(0)$ is indeterminate because $f(0,z)/F(0,z)=0$ for any $z>0$. We shall complete the solution at zero later on.

Lemma 6 (Existence) *For any small $\bar{\delta}>0$, consider the set $\mathcal{R}(\bar{\delta})=\{(t,z):t>\bar{\delta},z>0\}$. For any initial condition $(\delta,z_\delta)\in\mathcal{R}(\bar{\delta})$, the ODE (3) has a unique (twice-differentiable) solution path $z(t;\delta,z_\delta)$ in $\mathcal{R}(\bar{\delta})$ such that $z(\delta)=z_\delta$.*

Proof. Consider the Cauchy problem described by the ODE (3) together with initial condition (δ, z) in the open rectangle $\mathcal{R}\left(\bar{\delta}\right)$. We write the ODE in the form:

$$\begin{aligned} z' &= q(z,t) \equiv \frac{\int_0^z \left[ru(z,y) - \rho\int_0^\infty \left[u(z+w,y) - u(z,y)\right]G(dw)\right]F(t,dy)/F(t,z)}{\left[u(z,z) - \underline{u}(z,z)\right]f(t,z)/F(t,z)} \\ &+ \frac{-\int_0^z \rho\int_{z-y}^\infty \left[\underline{u}(z,y+w) - u(z,y)\right]G(dw)F(t,dy)/F(t,z) + c(x,t)}{\left[u(z,z) - \underline{u}(z,z)\right]f(t,z)/F(t,z)}. \end{aligned}$$

For any bound B, the function $q(z,t)$ is bounded on the open rectangle $\mathcal{R}\left(\bar{\delta}, B\right) = \{(t,z) : \bar{\delta} < t < B; 0 < z < B\}$ and Lipschitz continuous i.e. there is a uniform bound K such that $|q(z,t) - q(z',t')| \leq K\|(z,t) - (z',t')\|$ for any (z,t) and (z',t') in $\mathcal{R}\left(\bar{\delta}, B\right)$ close to each other. The numerator is bounded because it is an expected value, and the quantity $\left[u(z,z) - \underline{u}(z,z)\right]f(t,z)/F(t,z)$ is bigger than zero, because $u(z,z)$ is bounded above $\underline{u}(z,z)$ and because $f(t,z)/F(t,z(t))$ is bounded away from zero for $t > \bar{\delta}$. The claim then follows by application of the Picard and Lindelhof general existence and uniqueness Theorem (see e.g., Hurewitz, 1963). ■

We are left to show that the ODE (3) has an admissible (i.e. strictly decreasing and non-explosive).

Lemma 7 (Admissibility) *For any small $\delta > \bar{\delta} > 0$, there is a initial state z_δ^* such that the solution path $z(t)$ of the ODE (3) with initial condition (δ, z_δ^*), is strictly decreasing and non-explosive (i.e. well-defined onto the entire range $t > \bar{\delta}$).*

Proof. Take any time $T > \delta$, and consider the solution path $z(t;\delta,z_\delta)$ such that $z(T;\delta,z_\delta) = \hat{z}(T)$, the two solutions coincide at T. By Lemma 6, the solution path $z(t;\delta,z_\delta)$ must be twice differentiable in any open interval $(T-\varepsilon, T)$. We shall now prove that $z(t;\delta,z_\delta) < \hat{z}(t)$ for all $t \in (T-\varepsilon,T)$, with ε small enough. Suppose first that $\hat{z}(t)$ is discontinuous at T. Then there is a $\epsilon > 0$ such that $\hat{z}(t) > z(T;\delta,z_\delta) + \epsilon$ for all $t \in (T-\varepsilon,T)$. Hence $z(t;\delta,z_\delta) < \hat{z}(t)$ for all $t \in (T-\varepsilon,T)$ by continuity of $z(t;\delta,z_\delta)$. Second, suppose that $\hat{z}$ is continuous at T. By the definition of $\hat{z}(t)$, for any point $z \geq \hat{z}(t)$ and z close enough to $\hat{z}(t)$, it must be that $\phi(z,t) \geq 0$, unless $\hat{z}$ is discontinuous at t. Because $\hat{z}$ has only a countable set of discontinuity points, and

$$\frac{f(z(t;\delta,z_\delta),t)}{F(t,z(t;\delta,z_\delta))} > 0$$

we obtain that $z'(t;\delta,z_\delta) \geq 0$ for almost all $t \in (T-\varepsilon,T)$ such that $z(t;\delta,z_\delta) > \hat{z}(t)$. Because $\hat{z}(t)$ is strictly decreasing, it cannot be the case that $z(t;\delta,z_\delta)$ converges to $z(T;\delta,z_\delta) = \hat{z}(T)$ if $z(t;\delta,z_\delta) \geq \hat{z}(t)$ for any t smaller than T and close to T.

By the converse argument, for any $t < T$, if $z(t;\delta,z_\delta) < \hat{z}(t)$, then $z'(t;\delta,z_\delta) < 0$, consistently with $\hat{z}'(t) < 0$, continuity of $z(t;\delta,z_\delta)$ and the condition that $z(T;\delta,z_\delta) = \hat{z}(T)$. We conclude by continuity that if the solution path $z(t;\delta,z_\delta)$ coincides with $\hat{z}(t)$ at $t = T$, then it lies below

$\hat{z}$ and is strictly decreasing for all $t < T$; i.e. $z(t;\delta,z_\delta) < \hat{z}(t)$ and $z'(t;\delta,z_\delta) < 0$ for all $t < T$. Furthermore, because $z'(t;\delta,z_\delta) < 0$, the solution path $z(t;\delta,z_\delta)$ lies entirely in the admissible set $\mathcal{R}(\bar{\delta})$.

By Lemma 6, for any point (z,t) in $\mathcal{R}(\bar{\delta})$, the solution path $z(t)$ such that $z(t) = z$ is unique. Hence for any $T' > T$ the solution path $z(T;\delta,z_\delta)$ such that $z(T;\delta,z_\delta) = \hat{z}(T)$ must lie uniformly above the solution $z(T';\delta,z'_\delta)$ such that $z(T';\delta,z'_\delta) = \hat{z}(T')$, i.e. $z(t;\delta,z_\delta) > z(t;\delta,z'_\delta)$ for any t. For any $T > \delta$ we identify by $(\delta, z_{\delta,T})$ the initial condition (δ, z_δ) pinning down the solution path $z(T;\delta,z_\delta)$ such that $z(T;\delta,z_\delta) = \hat{z}(T)$. The initial state $z_{\delta,T}$ is decreasing in T and bounded, because $z(t;\delta,z_{\delta,T}) > z(T;\delta,z_\delta) = \hat{z}(T)$. Hence there exists a limit z^*_δ for $T \to \infty$. By construction, the solution path $z(t;\delta,z^*_\delta)$ is decreasing on the whole range $t > \delta$. Note that $z^*_\delta < \hat{z}(\delta)$ for any $\delta > 0$. ■

The above Lemmata have proved existence only of an admissible solution of the ODE (3) –the one identified by the solution path $z(t;\delta,z^*_\delta)$– there may be other admissible (i.e. decreasing and non-explosive) solution paths $z(t;\delta,z_\delta)$ with $z_\delta < z^*_\delta$. Hence proving the first part of Theorem 2. The next two results apply to all such solutions, hence proving the second part of Theorem 2 together with Proposition 2. The first one completes the construction of the admissible solutions $z(t)$ by taking the limit for $\delta \to 0$ (and hence $\bar{\delta} \to 0$).

Lemma 8 (Solution Completion for $\delta \to 0$) *For any admissible (i.e. decreasing and non-explosive) solution path $z(t)$ of the ODE (3),*

$$\lim_{\delta\to 0^+} \frac{f(z(\delta),\delta)}{F(z(\delta),\delta)} z'(\delta) = 0 \text{ and } \lim_{\delta\to 0^+} z(\delta) \equiv \hat{z}(0).$$

Hence $z(0)$ solves equation (4).

Proof. We inspect again the ODE (3). By continuity of the process describing the opponent state $y(t)$, it must be that

$$\frac{f(z,0)}{F(z,0)} = 0 \text{ for any } z > 0.$$

It follows that (i) the derivative $z'(t)$ is indeterminate at $t = 0$, (ii) the solution $z(0) = \hat{z}(0)$ is a solution of the ODE at $t = 0$, and (iii)

$$\lim_{\delta\to 0^+} z(\delta) = \hat{z}(0), \quad \text{unless} \lim_{\delta\to 0^+} \frac{f(z(\delta),\delta)}{F(z(\delta),\delta)} z'(\delta) \in \mathbb{R}_-.$$

By the properties of Poisson arrival.

$$\lim_{\delta\to 0^+} \frac{f(z(\delta),\delta)}{F(z(\delta),\delta)} z'(\delta) = \lim_{\delta\to 0^+} \frac{\rho\delta g(z(\delta)) z'(\delta)}{1-\rho\delta+\rho\delta G(z(\delta))} = \lim_{\delta\to 0^+} \rho\delta g(z) z'(\delta)$$

but $\lim_{\delta \to 0^+} \rho \delta g(z) z'(\delta) \in \mathbb{R}_-$ is in contradiction with $z(\delta) < \hat{z}(\delta)$ for any δ small enough. In fact, it requires that $z'(\delta) = k/\delta + o(1/\delta)$, where k is a negative constant and $o(1/\delta)$ denotes a term that converges to zero if multiplied by δ when $\delta \to 0$; hence it requires that $\lim_{\delta \to 0^+} z(\delta) = \lim_{\delta \to 0^+} -k \log \delta = +\infty$. This shows that

$$\lim_{\delta \to 0^+} \frac{f(z(\delta), \delta)}{F(z(\delta), \delta)} z'(\delta) = 0 \text{ and hence that } \lim_{\delta \to 0^+} z(\delta) \equiv \hat{z}(0).$$

■

The final Lemma determines the bound for $z \to \infty$.

Lemma 9 (Bound) *For any admissible (i.e. decreasing and non-explosive) solution path $z(t)$ of the ODE (3),*

$$z(t) < \lim_{t \to \infty} z(t) \leq \underline{z}, \text{ for any } t,$$

where $\underline{z}$ solves equation (5). The weak inequality is satisfied as an equality by the solution for which existence is proved in Lemma 7.

Proof. Because by definition $\hat{z}(t) = \min\{z : \phi(z,t) = 0\}$, it must that $\phi(z,t) > 0$ for almost any point $z > \hat{z}(t)$ and z close enough to $\hat{z}(t)$. This implies that $z'(t; \delta, z_\delta) \geq 0$ for the solution path $z(t; \delta, z_\delta)$ such that $z(t; \delta, z_\delta) = z$. Therefore any admissible solution path $z(t)$ must be such that $z(t) < \hat{z}(t)$, thus providing an upper bound. Because by Theorem 6, $\lim_{t \to \infty} F(t, y) / F(t, z) = 0$ for any $z > 0$ and $y < z$, and hence for t large enough, equation (11) is approximated by equation (5). ■ ■

Proof of Proposition 3. As shown in Theorem 2, in the limits for t small and t large, the equilibrium threshold z is approximated by the selection

$$\hat{z}(t) = \min\{z : \phi(z,t) = 0\}, \text{ where}$$

$$\begin{aligned}\phi(z,t) &= c + r[v(z) - c_0] \\ &\quad - \rho \int_0^z \left[\int_0^\infty [v(z+w) - v(z)] G(dw) + \int_{z-y}^\infty [\underline{v}(y+w) - v(z) + c_0] G(dw) \right] \frac{F(t,dy)}{F(t,z)}.\end{aligned}$$

The threshold z uniformly decreases in c because:

$$\frac{\partial}{\partial c} \phi(z,t) = 1 > 0.$$

The threshold z uniformly increases in c_0 because:

$$\frac{\partial}{\partial c_0} \phi(z,t) = -(\rho + r) < 0.$$

Let

$$\begin{aligned}\phi\left(z,t;\alpha\right) &= r\left[\alpha v\left(z\right)-c_{0}\right]+c-\alpha\rho\int_{0}^{\infty}\left[v\left(z+w\right)-v\left(z\right)\right]G\left(dw\right)\\ &\quad +\rho\int_{0}^{z}\int_{z-y}^{\infty}\left[\alpha v\left(z\right)-c_{0}-\underline{v}\left(y+w\right)\right]G\left(dw\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}.\end{aligned}$$

We then calculate:

$$\frac{\partial}{\partial\alpha}\phi(z,t,\alpha)=(r+\rho)v\left(z\right)-\rho\int_{0}^{\infty}\left[v\left(z+w\right)-v\left(z\right)\right]G\left(dw\right)$$

$$\begin{aligned}&= (r+\rho)v\left(z\right)-\frac{1}{\alpha}[r\left[\alpha v\left(z\right)-c_{0}\right]+c\\ &\quad +\rho\int_{0}^{z}\int_{z-y}^{\infty}\left[\alpha v\left(z\right)-c_{0}-\underline{v}\left(y+w\right)\right]G\left(dw\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}]\\ &\propto (r+\rho)c_{0}-c+\rho\int_{0}^{z}\int_{z-y}^{\infty}[\underline{v}\left(y+w\right)G\left(dw\right)\frac{F\left(t,dy\right)}{F\left(t,z\right)}].\end{aligned}$$

This quantity is negative (positive) when c is large (small) enough relative to c_0, as long as $\underline{v}$ is small enough. ■

Proof of Proposition 4. As shown in Theorem 2, in the limits for t small and t large, the equilibrium threshold z is approximated by the selection

$$\hat{z}\left(t\right)=\min\left\{z:\phi(z,t)=0\right\},$$

where, letting $\Pi_1=\Pi_2+\Delta$:

$$\begin{aligned}\phi\left(z,t;\beta,\gamma\right) &= r\left(\Pi_{2}+\Delta-\gamma C\left(z\right)\right)+\beta c\left(x,t\right)-\rho\int_{0}^{\infty}\left[\gamma C\left(z\right)-\gamma C\left(z+w\right)\right]G\left(dw\right)\\ &\quad +\rho\int_{0}^{z}\Delta\left[1-G\left(z-y\right)\right]\frac{F\left(t,dy\right)}{F\left(t,z\right)}.\end{aligned}$$

Because,

$$\frac{\partial}{\partial\Pi_{2}}\phi(z,t;\beta,\gamma)=r>0,\ \ \frac{\partial}{\partial\Delta}\phi(z,t;\beta,\gamma)=r+\rho\int_{0}^{z}\left[1-G\left(z-y\right)\right]\frac{F\left(t,dy\right)}{F\left(t,z\right)}>0,$$

it follows that $\partial z/\partial\Pi_2<0$ and $\partial z/\partial\Delta<0$. Also,

$$\begin{aligned}&\frac{\partial}{\partial\gamma}\phi(z,t;\beta,\gamma)=-rC\left(z\right)-\rho\int_{0}^{\infty}\left[C\left(z\right)-C\left(z+w\right)\right]G\left(dw\right)\\ &= -rC\left(z\right)-\frac{1}{\gamma}\left[r\left(\Pi_{1}-\gamma C\left(z\right)\right)+c\left(x,t\right)+\rho\int_{0}^{z}\left[\Pi_{1}-\Pi_{2}\right]\left[1-G\left(z-y\right)\right]\frac{F\left(t,dy\right)}{F\left(t,z\right)}\right]\\ &= -\frac{1}{\gamma}\left[r\Pi_{1}+c\left(x,t\right)+\rho\int_{0}^{z}\left[\Pi_{1}-\Pi_{2}\right]\left[1-G\left(z-y\right)\right]\frac{F\left(t,dy\right)}{F\left(t,z\right)}\right]<0,\end{aligned}$$

Incrementing the subsidy, reduces γ, makes ϕ increase, and hence reduces z, anticipating technology adoption.

$$\frac{\partial}{\partial\beta}\phi(z,t) = c\left(x,t\right) > 0.$$

Incrementing the subsidy, reduces β, hence making ϕ decrease, and increasing z, delaying technology adoption. ■

Proof of Proposition 5. Since W^* and V^* are differentiable, and $u_1 > 0$, $\bar{u}_1 \geq 0$, we only need to show that for any t, x such that $\int \left[u\left(x,y\right) + \bar{u}\left(y,x\right)\right]\gamma\left(t,dy;z^*\right) < W^*\left(x,t\right)$,

$$W_1^*\left(x,t\right) < \frac{d}{dx}\left(\int \left[u\left(x,y\right) + \bar{u}\left(y,x\right)\right]\gamma\left(t,dy;z^*\right)\right)$$

We proceed as in the proof of Proposition 1: consider any state (x,t) such that $V^*\left(x,t\right) = W^*\left(x,t\right)$, given any optimal strategy σ, we construct the strategy σ' such that $\sigma'\left(x''+x',\tau\right) = \sigma\left(x''+x,\tau\right)$ for any $\tau \geq t$ and $x'' \geq 0$, with expected payoff $V^*\left(x',t;\sigma'\right)$ at state $\left(x',t\right)$. Thus

$$\begin{aligned}
&\lim_{x'\to x}\frac{W^*\left(x,t\right) - W^*\left(x',t\right)}{x-x'} \leq \frac{V^*\left(x,t\right) - V^*\left(x',t;\sigma'\right)}{x-x'} \\
= &\ \frac{\int\left\{\mathbb{E}\left[V^*\left(x+\mathbf{x},y+\mathbf{y},t;\sigma,\sigma_B\right)\right] - \mathbb{E}\left[V^*\left(x'+\mathbf{x},y+\mathbf{y},t;\sigma',\sigma_B\right)\right]\right\}\gamma\left(t,dy;z^*\right)}{x-x'}
\end{aligned}$$

$$\begin{aligned}
= \int\Bigg\{\mathbb{E}\Bigg[&\Bigg[\frac{u\left(x+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right) - u\left(x'+\mathbf{x}\left(T_A\right),y+\mathbf{y}\left(T_A\right)\right)}{x-x'} \\
&+\frac{\underline{u}\left(y+\mathbf{y}\left(T_A\right),x+\mathbf{x}\left(T_A\right)\right) - \underline{u}\left(y+\mathbf{y}\left(T_A\right),x'+\mathbf{x}\left(T_A\right)\right)}{x-x'}\Bigg]e^{-r\left(T_A-t\right)}\chi\left(T_A<T_B\right) \\
&+\Bigg[\frac{\underline{u}\left(x+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right) - \underline{u}\left(x'+\mathbf{x}\left(T_B\right),y+\mathbf{y}\left(T_B\right)\right)}{x-x'} \\
&+\frac{u\left(y+\mathbf{y}\left(T_B\right),x+\mathbf{x}\left(T_B\right)\right) - u\left(y+\mathbf{y}\left(T_B\right),x'+\mathbf{x}\left(T_B\right)\right)}{x-x'}\Bigg]e^{-r\left(T_B-t\right)}\chi\left(T_B<T_A\right) \\
&-\int_0^{\min\{T_A,T_B\}}\left[c\left(x+\mathbf{x}\left(t\right),t\right) - c\left(x'+\mathbf{x}\left(t\right),t\right)\right]e^{-rv}dv\Bigg]\Bigg\}\gamma\left(t,dy;z^*\right)
\end{aligned}$$

where the expectation $\mathbb{E}$, the sample paths $\mathbf{x}$,$\mathbf{y}$ and the stopping times T_A, T_B are as in the proof of Proposition 1.

Because $c_1 \geq$, $\underline{u}_{11} \leq 0$, $u_{11} \leq 0$, $\underline{u}_{12} \leq 0$, and $u_{12} \leq 0$, we thus obtain:

$$\lim_{x-x'\to 0}\frac{W^*\left(x,t\right) - W^*\left(x',t\right)}{x-x'} < \lim_{x-x'}\int\left[\frac{u\left(x,y\right) - u\left(x',y\right) + \underline{u}\left(x,y\right) - \underline{u}\left(x',y\right)}{x-x'}\right]\gamma\left(t,dy;z^*\right).$$

■

Proof of Proposition 6. Because $z(0) > 0$, and because of the continuity of the process describing the opponent state $y(t)$, it follows that for any $y > 0$,

$$\lim_{t \to 0^+} \frac{F(t, y)}{F(t, z(t))} = 1.$$

As we show in the Appendix (Lemma 8),

$$\lim_{t \to 0^+} \frac{f(z(t), t)}{F(z(t), t)} z'(t) = 0,$$

for any possible admissible solution $z(t)$ of the ODE (3). Thus, equation (8) for t small enough is approximated by:

$$\begin{aligned}(r+\rho)\left[v(z^*) - v(z)\right] &= \rho \int_z^\infty \left[v(0+\varepsilon) - c_0\right] G(d\varepsilon) - (r+\rho)\underline{v}(z^*) - \underline{c} \\ &+\rho \int_0^\infty \left[v(z^*+\varepsilon) - v(z^*) + \underline{v}(z^*+\varepsilon) - \underline{v}(z^*) - v(z+\varepsilon) + v(z)\right] G(d\varepsilon).\end{aligned}$$

By contradiction, suppose that $z^* > z$. Then by concavity of v, $\left[v(z^*+\varepsilon) - v(z^*) - (v(z+\varepsilon) - v(z))\right] \leq 0$ for all ε. Evidently, $\rho \int_z^\infty \left[\underline{v}(z^*+\varepsilon) - \underline{v}(z^*)\right] G(d\varepsilon) \leq 0$ because $\underline{v}' \leq 0$, and because the next inequalities will be satisfied with slack, and we assumed $\underline{v}(z^*(0))$ not to be too negative, we are left to show that

$$-\underline{c} + \rho \int_z^\infty \left[v(0+\varepsilon) - c_0\right] G(d\varepsilon) < 0;$$

this quantity is indeed negative for $c_0 \to \infty$ (because $v(0+\varepsilon)$ would fixed and $c_0 \to \infty$, if were z bounded, and because the quantity would converge to something smaller than $-\underline{c}$ if it were that $z \to \infty$). ■

Proof of Proposition 7. As we show in the Appendix (Theorem 6), for any $y < z$,

$$\lim_{t \to \infty} \frac{F(t, y)}{F(t, z)} = 0,$$

Hence equation (8) approximates for large t:

$$\begin{aligned}&(r+\rho)\left[v(z^*) - v(z)\right] = -c(z^*, t) - (r+\rho)\underline{v}(z^*) \\ &+\rho \int_0^\infty \left[v(z^*+\varepsilon) - v(z^*) + \underline{v}(z^*+\varepsilon) - \underline{v}(z^*) - v(z+\varepsilon) + v(z)\right] G(d\varepsilon) \\ &+\rho \int_0^\infty \left[v(z+\varepsilon) - c_0\right] G(d\varepsilon) - \left[(v(z) - c_0) - \underline{v}(z)\right] \frac{f(t, z)}{F(t, z)} z'(t)\end{aligned} \tag{12}$$

By contradiction, suppose that $z > z^*$. Then by concavity of v, $\left[v(z^*+\varepsilon) - v(z^*) - (v(z+\varepsilon) - v(z))\right] \geq 0$ for all ε. Evidently, $-\left[(v(z) - c_0) - \underline{v}(z)\right] z'(t) f(t, z) / F(t, z) \geq 0$ because $z' < 0$ and $v(z) - c_0 > \underline{v}(z)$. Because the next inequalities will be satisfied with slack, and we assumed $r\underline{v}(z^*(0))$ not to be too large, we are left to show to generate a contradiction by showing that

$$-c(z^*, t) + \rho \int_0^\infty \left[\underline{v}(z^*+\varepsilon) - \underline{v}(z^*)\right] G(d\varepsilon) + \rho \int_0^\infty \left[v(z+\varepsilon) - c_0 - \underline{v}(z^*)\right] G(d\varepsilon) > 0.$$

Because $v(z)-c_0 > \underline{v}(z^*)$, and by the conditions $\underline{v}' \leq 0$, $\underline{v}'' \leq 0$, the hypothesis that $z > z^*$ implies that the above expression is strictly larger than

$$\begin{aligned}
& -c(z^*,t)+\rho\int_0^\infty [v(z+\varepsilon)-v(z)+\underline{v}(z+\varepsilon)-\underline{v}(z)]\,G(d\varepsilon) \\
= \quad & -c(z^*,t)+2\rho\int_0^\infty \frac{v(z+\varepsilon)+\underline{v}(z+\varepsilon)}{2}-\frac{v(z)+\underline{v}(z)}{2}G(d\varepsilon) \\
> \quad & -c(z,t)+2\rho\int_0^\infty \left(\frac{v(z+\varepsilon)+\underline{v}(z+\varepsilon)}{2}-v(z)+\frac{c_0}{2}\right)G(d\varepsilon) \\
= \quad & -c(z,t)+\rho\int_0^\infty (v(z+\varepsilon)-v(z))\,G(d\varepsilon)+\rho\int_0^\infty (\underline{v}(z+\varepsilon)-v(z)+c_0)\,G(d\varepsilon)=rv(z)>0.
\end{aligned}$$

where the first inequality follows from $c_1 \geq 0$ and $\underline{v}(z) < v(z)-c_0$, the last equality holds for the equilibrium threshold z and t large enough (see equation 5), and the last inequality follows from $v(z) > c_0$. ■

References

Abreu, D., and M. Brunnermeier (2003): "Bubbles and Crashes," *Econometrica,* 71: 173-204

Aoki, Masahiko (2001), *Towards a Comparative Institutional Analysis,* MIT Press, Cambridge, MA.

Baye Michael, D. Kovenock, and C. de Vries, (1993) "Rigging the lobbying process: An application of the all-pay auction," *American Economic Review* 83, 289 294.

Bergemann, Dirk and Juuso Valimaki (1996): "Learning and Strategic Pricing", *Econometrica,* 64: 1125-1150

Bergemann, Dirk and Juuso Valimaki (1996): "Experimentation in Markets", 2000, *Review of Economic Studies,* 67: 213-234.

Binmore, Ken (2004): *Fun and Games: A Text on Game Theory,* 2nd Ed.

Bolton, Patrick and Chris Harris (1999), "Strategic Experimentation", *Econometrica,* 67, pp. 349-374.

Bulow, Jeremy and Paul Klemperer (1999), "The Generalized War of Attrition" *American Economic Review* 89(1), pp. 175-89.

Cripps, Martin, Godfrey Keller and Sven Rady (2004) "Strategic Experimentation with Exponential Bandits", *Econometrica,* forthcoming.

Décamps J.P. and Thomas Mariotti (2004): "Investment Timing and Learning Externalities," *Journal of Economic Theory,* 118, 80-102.

J.-P. Décamps, Thomas Mariotti and S. Villeneuve (2004): "Investment Timing under Incomplete Information," *Mathematics of Operations Research,* forthcoming.

Fudenberg, Drew, R. Gilbert, Joseph Stiglitz and Jean Tirole (1983), "Preemption, Leapfrogging, and Competition in Patent Races," *European Economic Review*, 22, pp. 3-31.

Fudenberg, Drew and Jean Tirole (1985), "Preemption and Rent Equalization in the Adoption of New Technology", *Review of Economic Studies* 52(3), pp. 383-401

Fudenberg, Drew and Jean Tirole (1986) "A Theory of Exit in Duopoly", *Econometrica* 54(4), pp. 943-60.

Harris, Chris and John Vickers (1985), "Perfect Equilibrium in a Model of a Race", *Review of Economic Studies*, 52(2), pp. 193-209.

Harris, Chris and John Vickers (1987), "Racing with Uncertainty", *Review of Economic Studies*, 54(1), pp. 1-21.

Hurewicz W. *Lectures on Ordinary Differential Equations, 1963* Cambridge, MA: MIT Press

Karlin, S. (1959): *The Theory of Infinite Games,* vol. II. Dover Publications, New York.

Keller Godfrey and Sven Rady (1999): "Optimal Experimentation in a Changing Environment", *Review of Economic Studies* 66, pp. 475-507.

Keller Godfrey and Sven Rady (2003): "Price Dispersion and Learning in a Dynamic Differentiated Goods-Duopoly" *RAND Journal of Economics* 34, pp. 138-165.

Krishna Vijay and John Morgan (1997), "An Analysis of the War of Attrition and the All-Pay Auction", *Journal of Economic Theory*, 72(2), pp. 343-62.

Lambrecht B.M., and W.R.M. Perraudin (2003), "Real Options and preemption under incomplete information", *Journal of Economic Dynamics and Control,* 27(4), pp.619-643

Maynard Smith, John (1982): *Evolution and the Theory of Games,* Cambridge Univ. Press, New York.

Park, Andreas and Lones Smith (2003): "Caller Number Five: Timing Games That Morph From One Form to Another", mimeo *University of Michigan.*

Reinganum, Jennifer (1981a), "Dynamic Games of Innovation", *Journal of Economic Theory*, 25, pp. 21-41.

Reinganum, Jennifer (1981b), "On the diffusion of new technology: A game theoretic approach", *Review of Economic Studies* 48, pp. 395-405.

Reinganum, Jennifer (1982), "A Dynamic Game of R and D: Patent Protection and Competitive Behavior", *Econometrica*, 50(3), pp. 671-88.

Riordan, Michael (1992): "Regulation and preemptive technology adoption," *RAND Journal of Economics* 23, 334-349.

Simon, Leo and Maxwell Stinchcombe (1989), "Extensive Form Games in Continuous Time, Part I: Pure Strategies," *Econometrica* 57(5), pp. 1171-1214

Weeds, Helen (2002): "Strategic Delay in a Real Options Model of R&D Competition," *Review of Economic Studies,* 69, 729-747.